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Mathematical Model of Non-stationary Heat Conduction in the Wall: Asymmetric Problem with the Boundary Conditions of Imperfect Heat Transfer

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Abstract. This paper deals with the study of a non-stationary heat conduction in the solid wall. It is focused on mathematical modelling of its asymmetric heating and cooling by imperfect heat transfer to both sides of the wall. It describes method used for deriving of the long time analytical solution describing temperature distribution in the heated (cooled) wall by use Laplace transform and verification of its validity by numerical calculation with COMSOL Multiphysics software. In the tested example, the maximum difference between analytical and numerical solution was about 3.5 % considering the possible maximum and minimum temperatures in the wall under the given conditions.

1 Introduction

Non-stationary heat conduction in a wall made of the solid material is a phenomenon that can occur all around us. It is part of many technological operations, but also also takes place within the walls of heated or cooled building structures, in components of many electronic devices, in heating and cooling equipment etc.

Knowledge of these processes course is necessary to ensure optimal heat supplying or heat dissipation and the associated avoidance of damage to materials or reduce energy intensity etc.[1].

But course of heating and cooling depends on many factors such as the dimensions and material properties of the walls but also the intensity of heat transfer between the wall surface and the surroundings [1–4].

To assess the course of non-stationary phenomena is often advantageous to use a combination of an experimental testing and computer simulation study done for the required initial and boundary conditions. In the paper [5] we dealt a mathematical model of non-stationary heat conduction under the condition of symmetric heating or cooling and also with asymmetric heating or cooling under the conditions of perfect heat transfer between surface of the wall and ambient liquid on the both sides of the wall.

In this paper we also deal with mathematical modelling of a heat conduction in the solid wall during its asymmetric heating and cooling but we will suppose conditions of the imperfect heat transfer to both sides of the wall, which significantly influences time course of the studied process.

2 Description of a Studied Model

In the following text we describe a mathematical model of the transient heat conduction in a solid wall which is made of isotropic material. Geometrical sketch of the studied model is depicts in Figure 1.

The length and width of the wall are order of magnitude larger than its thickness. The wall of the initial temperature t_p, being subjected to a sudden thermal effect of surrounding liquid, wherein the constant ambient temperature t_{o1} exerted on the left side of the wall is different than the constant ambient temperature t_{o2} on the right side of the wall. Moreover, the heat transfer coefficients α_1, α_2 on both sides of the wall are mutually different. Temperatures of both the surrounding environments are independent of time and they are different from the initial temperature considered wall.

We will suppose the following cases of heating or cooling of the wall:

$$
\begin{aligned}
&t_p < t_{o1} < t_{o2} \wedge \alpha_1 \neq \alpha_2,\\
&t_p < t_{o2} < t_{o1} \wedge \alpha_1 \neq \alpha_2,\\
&t_p > t_{o1} > t_{o2} \wedge \alpha_1 \neq \alpha_2,\\
&t_p > t_{o2} > t_{o1} \wedge \alpha_1 \neq \alpha_2.
\end{aligned}
$$

Mathematical model of the studied problem is represented by a linear parabolic partial differential equation (1) [6], with initial condition (2) and boundary conditions (3) - (4) of imperfect heat transfer between surfaces of the wall and the surroundings:

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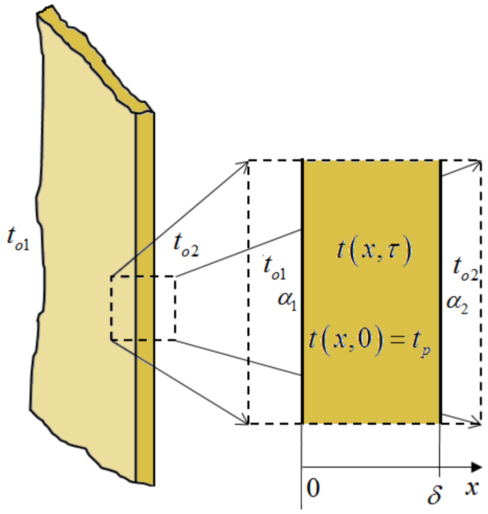

Figure 1. Geometrical sketch of the studied model of non-stationary heat conduction in the solid wall.

$$\frac{\partial t(x,\tau)}{\partial \tau} = a\frac{\partial^2 t(x,\tau)}{\partial x^2}, \quad 0 \le x \le \delta, \ \tau > 0 \tag{1}$$

$$t(x,0) = t_P \tag{2}$$

$$-\lambda \frac{\partial t(0,\tau)}{\partial x} = -\alpha_1 (t(0,\tau) - t_{o1}) \tag{3}$$

$$-\lambda \frac{\partial t(\delta,\tau)}{\partial x} = \alpha_2 (t(\delta,\tau) - t_{o2}) \tag{4}$$

The symbols used in equations (1) - (4) mean:
a - thermal diffusivity, [m^2.s^{-1}];
t - time dependent temperature of the wall, [°C];
t_p - initial temperature of the wall, [°C];
t_{o1} - ambient temperature on the left side of the wall, [°C];
t_{o2} - ambient temperature on the right side of the wall, [°C];
x - space coordinate, [m];
α_1 - heat transfer coefficient on the left side of the wall, [W.m^{-2}.K^{-1}];
α_2 - heat transfer coefficient on the right side of the wall, [W.m^{-2}.K^{-1}];
δ - thickness of the wall, [m];
λ - thermal conductivity of the wall, [W.m^{-1}.K^{-1}];
τ - time, [s].

3 Method Used for the Model Solving

For solving of the above described mathematical model we applied the Laplace transform technique for parabolic partial differential equation described in literature [7], [8].

The Laplace transform of a function $f(\tau)$, defined for all real numbers $\tau \ge 0$, is the function $F(s)$, which is a unilateral transform defined by equation (5):

$$F(s) = \int_0^\infty \exp^{-s\tau} d\tau. \tag{5}$$

According to [7], the long time solution can be obtained by expansion theorem in the Laplace domain as $F(s)$:

$$F(s) = \frac{p(s)}{q(s)}. \tag{6}$$

In our case, a polynomial $q(s)$ has an infinite number of roots. If $s = \mu_0$, $n = 1\ldots\infty$ are the distinct roots of $q(s)$, $q(s)$ can be factorized as:

$$q(s) = (s-\mu_1)(s-\mu_2)\ldots(s-\mu_n)\ldots(s-\mu_\infty). \tag{7}$$

Assuming that order of polynomial $q(s)$ is greater than order of $p(s)$, the equation (6) can be converted to partial fractions as:

$$F(s) = \frac{B_1}{s-\mu_0} + \frac{B_2}{(s-\mu_0)^2} + \sum_{n=1}^{\infty} \frac{A_n}{s-\mu_n}. \tag{8}$$

Where

$$B_1 = \lim_{s\to\mu_0}\left(\frac{d}{ds}\left(\frac{p(s)(s-\mu_0)^2}{q(s)}\right)\right), \tag{9}$$

$$B_2 = \lim_{s\to\mu_0}\left(\frac{p(s)(s-\mu_0)^2}{q(s)}\right), \tag{10}$$

$$A_n = \frac{p(\mu_n)}{q'(\mu_n)}. \tag{11}$$

The analytical solution in time domain can be obtained by inverse Laplace transform as [7]:

$$f(s) = \mathcal{L}^{-1}(F(s)) = B_1 \exp(\mu_0\tau) + B_2\tau\exp(\mu_0\tau) + \sum_{n=1}^{\infty} A_n \exp(\mu_n\tau). \tag{12}$$

The procedure described by equations (6) - (12) we programmed in the user interface of MAPLE software. By calculation with the software Maple we derived analytical solution in a relatively complicated form. A simplified form of the analytical solution is described by the equation (13):

$$t(x,\tau) = Q_1 + \sum_{n=0}^{\infty} 2\,\frac{e^{-Q_2\tau}(Q_3+Q_4+Q_5-Q_6)}{Q_7}. \tag{13}$$

Symbols $Q_1 - Q_7$ represent the following parts of the equation (13):

$$Q_1 = \frac{\alpha_2 t_{o2}\lambda + \alpha_1\lambda t_{o1} + \alpha_1\alpha_2(t_{o2}x - t_{o1}x + t_{o1}\delta)}{\alpha_2\lambda + \alpha_1\lambda + \alpha_1\alpha_2\delta}, \tag{14}$$

$$Q_2 = \frac{{q_n}^2 a}{\delta^2}, \tag{15}$$

$$Q_3 = (t_P - t_{o2}) \sin\left(\frac{q_n x}{\delta}\right) \alpha_2 \alpha_1 a^2 q_n + \tag{16}$$
$$(t_{oI} - t_P) \cos(q_n) \sin\left(\frac{q_n x}{\delta}\right) \alpha_2 \alpha_1 a^2 q_n,$$

$$Q_4 = \left(-t_P + \cos\left(\frac{q_n x}{\delta}\right)(-t_{oI} + t_P)\right) \sin(q_n)\, \alpha_2, \alpha_1 a^2 q_n \tag{17}$$

$$Q_5 = i\left(-\frac{{q_n}^2 a}{\delta^2}\right)^{3/2} \lambda \sqrt{a} \sin(q_n) - \tag{18}$$
$$\frac{a^2 {q_n}^2 (\alpha_2 + \alpha_1) \cos(q_n)\, t_P \lambda}{\delta},$$

$$Q_6 = \left(t_{oI} - t_p\right) \alpha_1 \sin\left(\frac{q_n x}{\delta}\right) \sin(q_n)\, {q_n}^2 a^2 \lambda \delta^{-1} \tag{19}$$
$$+ ((-t_P + t_{o2})\, \alpha_2)\, {q_n}^2 a^2 \lambda \delta^{-1}$$
$$+ (t_{oI} - t_P)\, \alpha_1 \cos(q_n) \cos\left(\frac{q_n x}{\delta}\right) {q_n}^2 a^2 \lambda \delta^{-1},$$

$$Q_7 = \frac{{q_n}^4 a^2 \lambda^2 \cos(q_n)}{\delta^2} + \tag{20}$$
$$i\sqrt{a}\left(-\frac{{q_n}^2 a}{\delta^2}\right)^{\frac{3}{2}} \delta \lambda \sin(q_n)\,(5\lambda + \alpha_1 \delta + \alpha_2 \delta) +$$
$$i a^{\frac{3}{2}} \sqrt{-\frac{{q_n}^2 a}{\delta^2}}\, (3\alpha_1 \alpha_2 \delta \sin(q_n)) +$$
$$i a^{\frac{3}{2}} \sqrt{-\frac{{q_n}^2 a}{\delta^2}}\, (q_n \cos(q_n)\,(4\lambda(\alpha_2 + \alpha_1) + \alpha_1 \alpha_2 \delta)).$$

The positive roots q can be calculated from a transcendental equation (21):

$$q = \frac{ia\left(\alpha_2 \cos(q)\, q \lambda \delta - \lambda^2 q^2\right)}{\left(\sqrt{-\frac{q^2 a}{\delta^2}}\right)\delta^2} +$$
$$\frac{ia\left(\sin(q)\,\delta^2 + \alpha_1 \lambda q \cos(q)\,\delta\right)}{\left(\sqrt{-\frac{q^2 a}{\delta^2}}\right)\delta^2}. \tag{21}$$

In the Figure 2 is shown computed temperature distribution in the wall during its heating. Figure 2a depicts 3D temperature distribution, Figure 2b depicts temperature curves in required times of cooling.

Temperature distribution in the wall in case of the cooling is shown in the Figure 3.

4 Verification of the Model Solution Validity

For the verification of the derived solution accuracy we compared data obtained by analytical solution (13) using MAPLE with results of computer simulation of the same model by COMSOL Multiphysics software, which performs calculations of physical processes numerically by finite element method.

Results based on analytical solution (13) and computer simulation are shown in Figure 4. The temperature fields during heating and cooling of the wall calculated by both procedures were similar. In the example shown in Figure 4, the maximum difference was about 0.7 °C, which makes the difference of 3.5 % relative to the possible maximum and minimum temperatures in the wall under the given conditions.

5 Discussion

The above-described method which was used for the derivation of analytical solution (13) is suitable for long time solution heating (cooling) of the wall [7]. In this case, solution appears to be sufficiently accurate. To derive an analytical solution in case of short time heating (cooling), the used mathematical method should be modified [7].

The derived analytical solution is valid for asymmetric heating, provided that both ambient temperatures are higher than the initial temperature of the wall or one of the ambient temperatures may be equal to the initial wall temperature. Similarly, when cooling is required, both to ambient temperatures were lower than the initial temperature of the wall. Alternatively, one of ambient temperatures may be equal to the initial temperature of the wall.

Special case of a transient heat conduction in the wall can occur if the wall is heated from one side and cooled simultaneously from the other side heated (i.e. for $t_{o1} < t_p < t_{o2}$ or $t_{o1} > t_p > t_{o2}$).

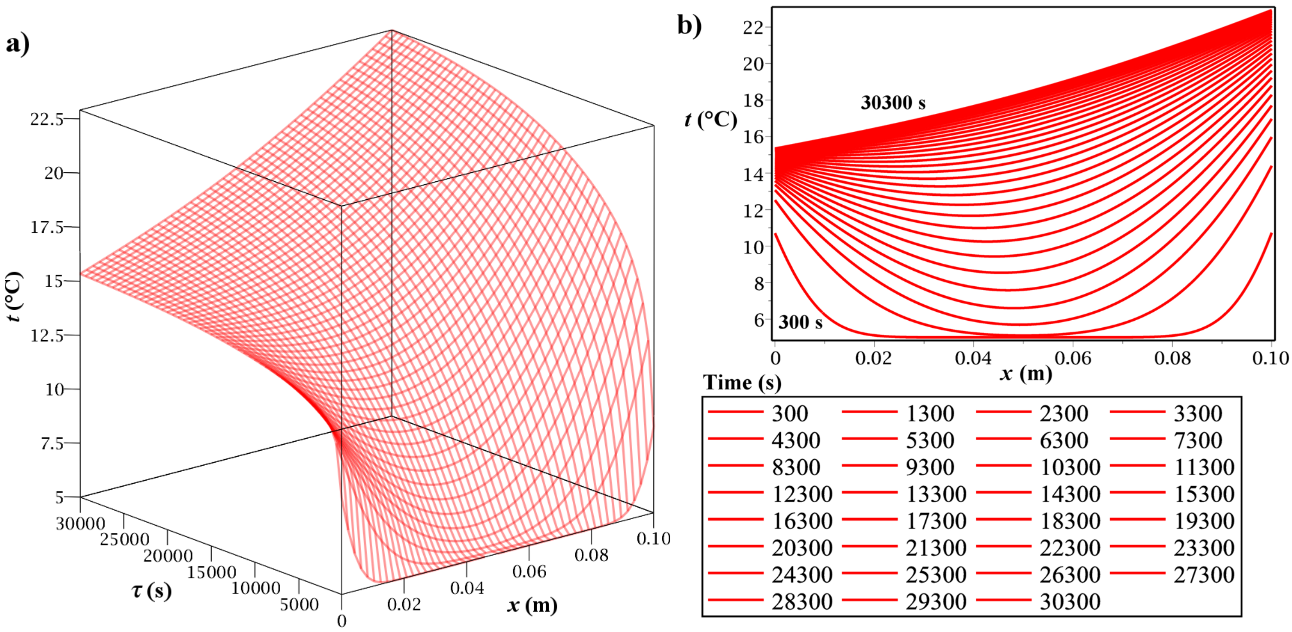

Figure 2. Heating of the wall; a) 3D temperature distribution, b) temperature curves in required times of heating. The used parameters: t_p = 5 °C, t_{o1} = 15 °C, t_{o2} = 25 °C, δ = 0.1 m, α_1 = 30 W.m^{-2}.K^{-1}, α_2 = 10 W.m^{-2}.K^{-1}, λ = 0.2 W.m^{-1}.K^{-1}, ϱ = 1140 kg.m^{-3}, c_p = 1200 J.kg^{-1}.K^{-1}.

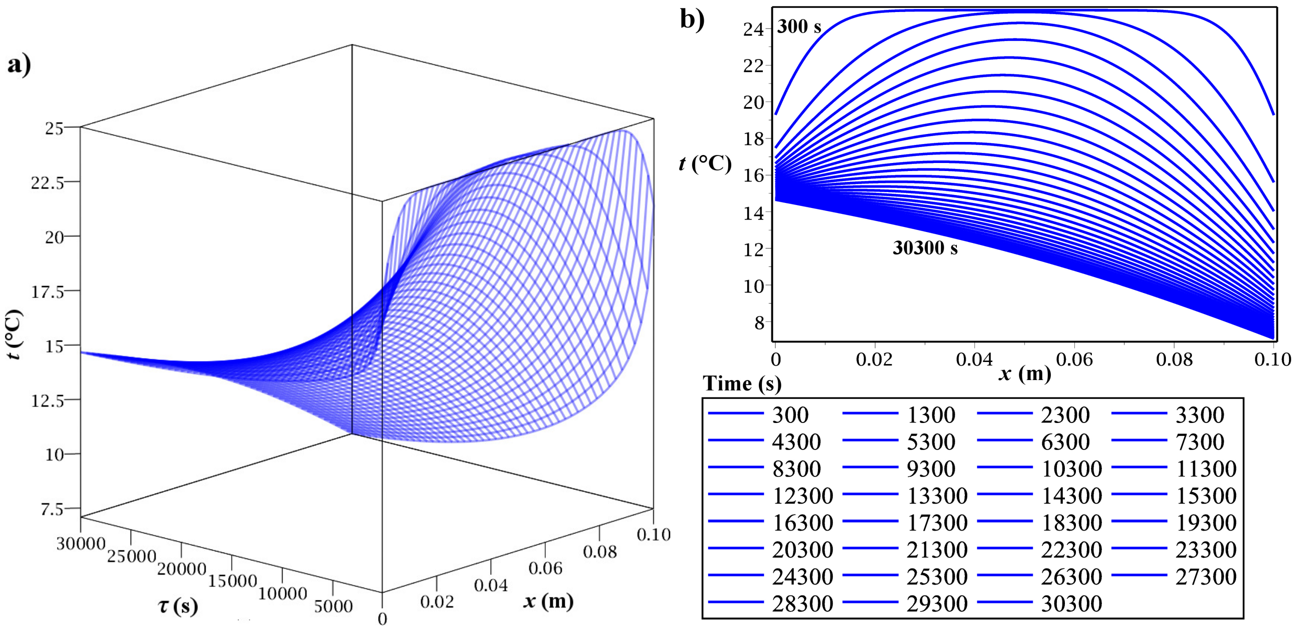

Figure 3. Cooling of the wall; a) 3D temperature distribution, b) temperature curves in required times of cooling. The used parameters: t_p = 25 °C, t_{o1} = 15 °C, t_{o2} = 5 °C, δ = 0.1 m, α_1 = 30 W.m^{-2}.K^{-1}, α_2 = 10 W.m^{-2}.K^{-1}, λ = 0.2 W.m^{-1}.K^{-1}, ϱ = 1140 kg.m^{-3}, c_p = 1200 J.kg^{-1}.K^{-1}.

6 Conclusion

This paper presented mathematical model of a heat conduction in the solid Wall during its asymmetric heating and cooling by imperfect heat transfer to both sides of the wall. The analytical solution of the studied model was derived by using Laplace transform and its validity was verified by numerical calculation with COMSOL Multiphysics software.

The derived analytical solution is suitable for the description of long time processes. Therefore, further studies will focus on the derivation of an analytical solution for the short time processes.

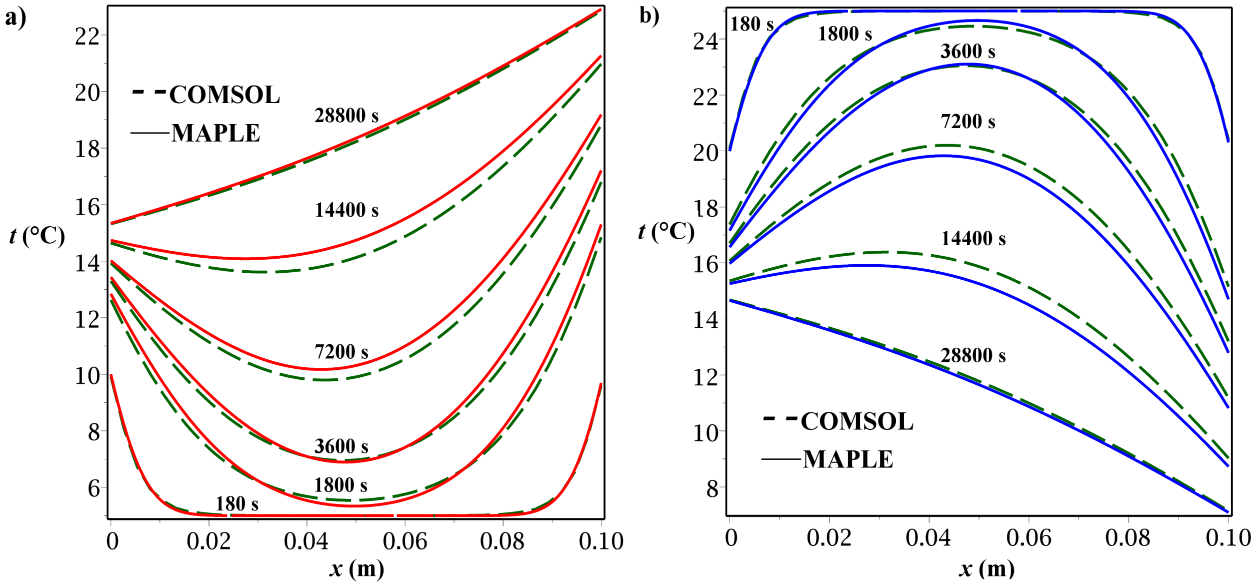

Figure 4. Comparing of analytical solution for model of transient heat conduction solved in Maple and numerical solution obtained by the simulation with software COMSOL Multiphysics; temperature field a) for heating, b) for cooling of the wall. The used parameters: a) t_p = 5 °C, t_{o1} = 15 °C, t_{o2} = 25 °C, δ = 0.1 m, α_1 = 30 $\mathrm{W.m^{-2}.K^{-1}}$, α_2 = 10 $\mathrm{W.m^{-2}.K^{-1}}$, λ = 0.2 $\mathrm{W.m^{-1}.K^{-1}}$, ϱ = 1140 $\mathrm{kg.m^{-3}}$, c_p = 1200 $\mathrm{J.kg^{-1}.K^{-1}}$; b) t_p = 25 °C, t_{o1} = 15 °C, t_{o2} = 5 °C, δ = 0.1 m, α_1 = 30 $\mathrm{W.m^{-2}.K^{-1}}$, α_2 = 10 $\mathrm{W.m^{-2}.K^{-1}}$, λ = 0.2 $\mathrm{W.m^{-1}.K^{-1}}$, ϱ = 1140 $\mathrm{kg.m^{-3}}$, c_p = 1200 $\mathrm{J.kg^{-1}.K^{-1}}$.

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