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Multi-firm voluntary disclosures for correlated operations

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In memoriam Sudipto Bhattacharya (1951-2012)

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Abstract

We study the no-arbitrage theory of voluntary disclosure (Dye (1985), Ostaszewski and Gietzmann (2008)), generalized to the setting of n firms, simultaneously and voluntarily, releasing at the interim-report date 'partial' information concerning their 'common operating conditions'. Each of the firms has, as in the Dye model, some (known) probability of observing a signal of their end of period performance, but here this signal includes noise determined by a firm-specific precision parameter. The co-dependency of the firms results entirely from their common operating conditions. Each firm has a disclosure cutoff, which is a best response to the cutoffs employed by the remaining firms. To characterize these equilibrium cutoffs explicitly, we introduce n new *hypothetical* firms, related to the corresponding actual firms, which are operationally independent, but are assigned *refined* precision parameters and amended means. This impounds all existing correlations arising from conditioning on the other potentially available sources of information. In the model the actual firms' equilibrium cutoffs are geometric weighted averages of these hypothetical firms. We uncover two countervailing effects. Firstly, there is a *bandwagon effect*, whereby the presence of other firms raises each individual cutoff relative to what it would have been in the absence of other firms. Secondly, there is an *estimator-quality effect*, whereby individual cutoffs are lowered, unless the individual precision is above average.

Keywords: voluntary disclosure, disclosure strategy, correlation structure.

1 Introduction

The 1985 Dye theory of a single firm relates how private (certain) information about end of period value, acquired 'partially', i.e. with a (known ex-ante) probability q, is voluntarily released as an investor relations (IR) announcement to investors at a (known ex-ante) interim-report date with the aim of achieving a valuation upgrade. It establishes that in equilibrium the firm adopts a simple "cutoff strategy", when deciding whether or not to release that private information. This research explores how the simple cutoff strategy is revised when there are n firms with correlated economic activity, partially acquiring *noisy* private information in advance of a common (simultaneous) interim-report date. The current setting is quite different from that of the lone firm, since now the voluntary release of information by the i^{th}-firm tells investors something about the other $n-1$ firms.

A further novel feature here is that the firms face varying levels of noise in their observations and have differing (known) precision, so there are competing disclosures of varying precision. There are now two new general effects at work in this multi-firm setting; good news about the (common) operating environment released by one firm implies other (correlated) firms are also facing "good" conditions. Secondly, since an investor may now receive multiple disclosures about the common operating environment, the investor can choose to assign greater weight to those disclosures that are more precise, and hence this may in turn influence firms with imprecise observations to increase disclosure (when their investor weighting is "low"). Thus, interestingly, the effect of moving from the lone to the n firm setting shows that, while some firms may rationally choose to disclose less (adopt higher cutoffs), others may choose to disclose more (adopt lower cutoffs). Hence the multi-firm model allows us to understand how variation in the precision, with which firms estimate their common operating environment, has an important effect on how the simple Dye cutoff needs to be modified.

A related class of models is based on costly state verification (CSV); the idea, due to Townsend (1979), modeled via an Arrow-Debreu pure exchange economy, identifies a cutoff for verified disclosures about some numeraire, typically describing either consumption or an insurance claim and so – after verification – known with certainty (in contrast to our noisy, albeit truthful, disclosure). As the name indicates the CSV cutoff, while analogous to the Dye IR cutoff, is instead determined by costs of disclosure. The original multi-agent version in Townsend (1979) was extended in Krasa and Villamil (1994) which also analyses, though in a different setting, agent co-dependency. Pursuing the IR approach, Dye and Sridhar (1995) studied disclosure in a multi-firm setting with a correlation between the information endowments of the firms, created by an unobservable event, acting as a probabilistic coordination device. Since in this IR setting the information endowments are independent and the unobservable effect contains no information about the economic activity of the firms, that model "precludes any common 'industry effects' regarding the firms' cash flows", as the authors acknowledge (page 161, footnote 9). Indeed, by introducing a second interim reporting date, that paper's focus is instead on how early voluntary disclosures precipitate subsequent additional disclosures.

Recently the Dye-Sridhar model has been revisited by Acharya, DeMarzo and Kremer (2011). These authors return to the single firm paradigm, but with a second interim report date. They consider a further (noisy) public signal of the firm's true value, modeled to be equivalent to the firm's earlier private observation, but with its firm-specific noise removed rather than filtered. (So this signal is correlated with the possible earlier signal.) They then study the public-news effect on disclosure strategy. Near the end of their paper, they suggest that the arrival of such a public signal may be interpreted as a second firm's information release and propose conjectures, concerning bad news from the second firm precipitating disclosure by the first. (See their Section IV.B.) They report that "the construction of the [two-firm] equilibrium presents a significant computational challenge."

By contrast this research, building on the no-arbitrage disclosure theory established in Ostaszewski and Gietzmann (2008), constructs explicitly a natural, equilibrium, n-firm extension (the constituent firms defining an *industrial sector*) of the Dye 1985 theory, with noisy information, in which the firm equity-values[1] are modeled as one-period log-normal distributions (consistently with the Black-Scholes benchmark); a further co-dependency between the firms capturing the common "industrial sector effects" (or com-

[1] The Dye theory is suited to an equity-valuation focus, as use of the Dye cutoff can be justified by no-arbitrage arguments – as discussed in Section 2.1, especially equation (8). This distinguishes the approach from the alternative focus on how disclosure costs determine a cutoff – see Bayer et al. (2010) for an overview of this literature.

mon operating conditions) is again modeled in a Black-Scholes framework (i.e. log-normally). We will view a firm's end of period return as composed of an idiosyncratic contribution (independent of other firms) and a sector contribution, moderated in the case of firm i by a firm-specific 'loading' coefficient (loading index), α_i, assumed non-zero. We refer to this common contribution simply as the *sector effect*. (It is useful to regard it mathematically as a shared resource, appearing as if it were an $(n+1)$-st asset/firm.) At a common interim date the n firms partially acquire private information about the end of period sector effect. Hence, as above, a disclosure by any one firm, permits inferences about the sector and thereby also about other firms. Consequently, the incremental effect of adding any one firm into an industrial sector formed from the remaining $n-1$ firms will be a dominant feature of the n equilibrium-cutoffs (see the *incremental inclusion effect* below). Our main concern is how this co-dependency determines the nature of the n firms' cutoffs for disclosing their privately observed (noisy) 'sector information'. Indeed, we trace the influence on cutoff levels of the firms' estimate of the sector effect, observed partially (by firm i with probability q_i) with noise a log-normal multiplicative input, having an underlying Gaussian of mean zero and standard deviation σ_i (equivalently, precision $p_i := 1/\sigma_i^2$).

At the heart of our approach is a decoupling theorem which replaces the n co-dependent firms with n independent firms, where critically these new *hypothetical firms* are assigned a modified (refined) precision and an adjusted mean, which subsume all co-dependency effects, and result from partial correlation analysis (cf. Kendall and Stuart (1976)), for which see Appendix 5. Furthermore, the n hypothetical firms allow development of intuition for moving from the lone to the n-firm industry sector.

The relative ease with which the n-firm cutoff can then be explicitly calculated (see (1)) allows one to check how varying firm-precision affects the likelihood of non-release of information, which is of itself price-sensitive, since investors revise expectations, when no news is released. Firstly, when the correlation is positive (all $\alpha_i > 0$), a good-news bandwagon effect is shown to hold: ceteris paribus firms *all* choose a higher cutoff (relative to the lone firm case) reducing the probability that they will release private news. Secondly, there is an intuitively clear estimator-quality effect which leads to firms being *partitioned* into below- and above-average precision (over the n-firm population). Those with below-average precision are shown to adopt a lower cutoff (relative to the single firm case), and thus ceteris paribus increase the probability that they will release private news, with the converse holding for the above-average quality effect.

The i-th firm announces its observed value provided it exceeds its own observation cutoff γ_i, whose logarithm is given by the following 'weighted sum' of all the hypothetical firm log-cutoffs ("hyp-firm" below). We call this weighted sum the i-th *multi-agent induced cutoff.*

$$\begin{aligned} & \text{log-cutoff}_{\text{firm } i} \\ = \; & \text{load-adjusted\_precision-weight}_i \times \text{log-cutoff}_{\text{hyp-firm } i} \\ & + \sum_{j=1}^{n} \text{load-adjusted\_competitive-precision-weight}_j \times \text{log-cutoff}_{\text{hyp-firm } j} \end{aligned}$$

More precisely, denoting below by g_j the lone-firm Dye-cutoff of the hypothetical firm j, one has

$$\log \gamma_i = \frac{\log g_i}{\alpha_i \kappa_{-i}} + \sum\nolimits_{j=1}^{n} \frac{\kappa_j}{\kappa_0} \frac{\log g_j}{\alpha_j \kappa_{-j}},$$

or, combining the weights, one has:

$$\log \gamma_i = \frac{1}{\kappa_0} \left(\frac{\log g_i}{\alpha_i \cdot \left(\frac{\kappa_{-i}}{\kappa_0 + \kappa_i} \right)} + \sum\nolimits_{j \neq i} \frac{\log g_j}{\alpha_j \cdot \frac{\kappa_{-j}}{\kappa_j}} \right). \tag{1}$$

Here, as above, α_j is the power-loading coefficient reflecting the dependence of firm-j returns on the sector (factor) returns X, taken functionally to be X^{α_j}. The weights in this sum, constructed from coefficients κ_i and κ_{-i}, shortly to be defined below in (2) and (3), include the proportionate effects of a firm's precision p_i both as compared with its competitors and also as compared against the sector-effect's precision p_0. The proportionate ratio is determined by two familiar estimates (regression coefficients) that are used by investors to infer the underlying (hidden) sector value conditional upon the observed disclosures made by all the firms in the sector. The two sets of coefficients are as follows:

i) One set consists of

$$\kappa_i = p_i/p, \text{ where } p = p_0 + p_1 + ... + p_n, \tag{2}$$

constructed from the entire set of firms (all the firms plus the sector – viewed as an additional asset with its own variance, or precision p_0).

ii) The other set consists of

$$\kappa_{-i} = p_i/(p - p_i), \tag{3}$$

where the minus signifies omission, is constructed so that this coefficient for firm i is obtained by excluding/omitting that firm from the sector.

Referring to a firm's competitors as its 'competition', call the ratio:

$$\kappa_{-i}/\kappa_i$$

the *firm-competition inclusion effect* (f_iCI); this measures a firm's incremental effect on total forecast precision. The related coefficient

$$\kappa_{-i}/(\kappa_0 + \kappa_i),$$

will be referred to as the *firm-sector inclusion effect* (f_iSI). A slight rearrangement allows us to see the intuition for the terminology above:

$$\frac{\kappa_{-i}}{\kappa_i} = \frac{p_i/(p - p_i)}{p_i/p} = \frac{p}{p - p_i}, \text{ or equivalently } \frac{p_{-i} + p_i}{p_{-i}}, \text{ where } p_{-i} = p - p_i,$$

where p_{-i} (minus for omission, again) refers to summing over the competitors of firm i. Likewise,

$$\frac{\kappa_{-i}}{\kappa_0 + \kappa_i} = \frac{p}{p - p_i} \Big/ \frac{p_i + p_0}{p_i}$$

is, on the right-hand side, the ratio of the previously introduced firm-competition inclusion effect (f_iCI) to a correlation (i.e. sector-on-firm) effect on a firm's precision.

Thus, for instance when $n = 2$, one obtains a familiar regression coefficient:

$$\kappa_1 = \frac{p_1}{p_0 + p_1 + p_2} = \frac{\sigma_1^{-2}}{\sigma_0^{-2} + \sigma_1^{-2} + \sigma_2^{-2}}, \tag{4}$$

and its associate

$$\kappa_{-1} = \frac{p_1}{p_0 + p_2} = \frac{\sigma_1^{-2}}{\sigma_0^{-2} + \sigma_2^{-2}} \text{ (i.e. denominator omits firm } i = 1), \tag{5}$$

giving

$$f_iCI = \frac{\kappa_{-i}}{\kappa_i} = \frac{p_0 + p_1 + p_2}{p_0 + p_2}.$$

The paper is developed as follows. In subsection 2.1 we briefly review the Dye voluntary disclosure cutoff strategy for a single firm and a single investor, when a manager observes operations with certainty. Here the optimal, voluntary, upper-tailed disclosure strategy of firm i (disclosing high enough value, say above some cutoff γ_i) is termed a γ_i strategy. In subsection 2.2 we introduce greater reality into the above setting by assuming the manager can only observe the realization of some random variable T_i that is a value-relevant signal, based on common operating conditions in the industrial sector.

Since investors are concerned with their equity stake in a firm, we investigate the Dye theory as it applies to the bench-mark Black-Scholes log-normal model of risky-asset values albeit in a one-period setting (a framework inspired by the CAPM). The corresponding disclosure cutoff relates to investors' estimates of equity value. This translates to a disclosure cutoff for signals about returns. We find that passing back and forth (via logarithms) between the additive arithmetic averaging of classical linear regression in respect of normal returns and its log-normal counterpart – a multiplicative geometric averaging of asset values – is straightforward and intuitive. The non-linearity of the logarithm turns out to be highly tractable.

To help this back and forth argumentation, we follow the notation convention that log-normally distributed random variables are denoted by upper case letters and related underlying normally distributed variables by lower case letters. Consistently with this, we use Φ_{N} and Φ_{LN} and generally Φ_F to distinguish between various probability laws (distribution functions): normal, log-normal, general.

Section 3 introduces the 'factor model' of a sector first informally and then in general terms in order to identify the equilibrium conditions for the cutoffs.

Section 4 specializes firm values and signals about values to be log-normal; the driver here is one 'sectorial' factor, denoted X, common to all firms in the sector, which accounts for the entire correlation structure of the model (albeit varying across firms via the loading index α_i) and so characterizes the 'state of the sector'. Working in the n-firm setting, Section 5 gives all the general results concerning the cutoffs. We summarize conclusions in Section 6.

There follows a more technical account in the appendices. Appendices 1-7 are devoted to developing some routine Black-Scholes type calculations. We derive the form of the relevant log-normal regression function (a geometric average), from which we deduce a key conditional hemi-mean formula. These permit us to prove our main Existence Theorem, on the existence and uniqueness of cutoffs for the signals. This relies on some partial covariance calculations which relate more directly to the "precision matrix" of Section 4 than to the standard covariance matrix. Appendix 8 derives the Indifference Principle characterizing a firm's cutoff conditional on the cutoff behaviour of the remaining firms.

2 The Disclosure Environment

We begin with a brief review of the Dye voluntary disclosure theory in its original form (possible private information on the next mandatory report of firm value) and its generalization to the context of a noisy signal of value, and then we introduce multi-firm disclosure cutoffs - the main point here is that the equilibrium disclosure of one firm will now depend on parameters of other firms.

2.1 The Dye Cutoff: noiseless scenario

The Dye disclosure model assumes three distinctive times, which we label $\theta = -1, 0, 1$: ex-ante, interim-report date (e.g. a conference call) and terminal date (e.g. 'end of year'). (The dates $\theta = \pm 1$ may also be interpreted as timings just before and just after a known moment in time when the manager of a firm may make a voluntary disclosure.) In the model a random variable F, relating to firm valuation , has density $\varphi_F(x)$, an associated distribution function $\Phi_F(x)$ and an ex-ante (i.e. at time $\theta = -1$) expected value m_F. A realization of the random variable is observed by management at the interim time with a probability q, drawn independently of F, and known to the market. Management's decision whether or not to disclose an observed realization of firm value x is a voluntary (strategic) decision. Dye (1985) establishes that under continuity and positivity of φ_F there exists a unique threshold value γ at which management will be indifferent between disclosure or non-disclosure. Here γ will be called the *Dye cutoff*. The indifference point is characterized by equality between a credibly disclosed value γ and the valuation formed by investors when they face the non-disclosure event (ND), denoted formally $\mathbb{E}[F|ND(\gamma)]$, since investors at equilibrium all conjecture that the cutoff policy is determined by γ. The latter is the computed expected value of the firm, conditioned on the absence of information (non-disclosure). This expression is a consequence of Dye's assumption that "investors cannot discern whether [the manager] has received information but chosen not to release it or whether the manager has not received information" (Dye 1985, §3). That is, the indifference is described by what we term the *Dye indifference equilibrium* equation, or more briefly the *Dye Equation*:

$$\gamma = \mathbb{E}[F|ND(\gamma)]. \tag{6}$$

Under the assumptions above, this implicit definition of a cutoff value γ in fact determines it uniquely.

We retain Dye's (1985) assumptions (see his §3) that "the current shareholders prefer a disclosure policy which maximizes the [interim]-period [i.e. $\theta = 0$] price of the firm" and that "this disclosure policy is adopted" by management[2].

Based on the assumption of a rational expectations equilibrium (in respect of a conjectural threshold value γ for the manager's cutoff), Jung and Kwon (1988) derive (their equation (7)) the equation satisfied by γ to be

$$\frac{1-q}{q}(m_F - \gamma) = H_F(\gamma), \tag{7}$$

where

$$H_F(t) := \mathbb{E}[(t-F)^+] = \int (t-x)^+ d\Phi_F(x) = \int_{x \le t} \Phi_F(x)dx.$$

$H_F(t)$ is the 'lower first partial moment below a target t', well-known in risk management[3]. As this function is central to the Dye calculus, in our analysis we refer to it briefly as the *hemi-mean function*.

Henceforth this paper takes the distribution Φ_F of the Dye framework above as one that fully reflects the market price of risk at the three times $\theta = -1, 0, 1$, as above. That is, any contingent contract traded on the market is priced by computing an expectation of the claim under this distribution Φ_F. This presumes the so-called *complete market hypothesis* to the extent

[2] The alignment of managerial and investor interests in respect of truthful disclosure is arranged in Townsend (1979) and in Krasa & Villamil (1994) through the inclusion of incentive compatibility conditions.

[3] See for example McNeil, Frey and Embrechts (2005), Section 2.2.4.

of asserting that the distribution itself is an observable, i.e. there is a sufficient range of traded instruments to select a distribution from a proposed parameterized family, and so to identify the density of what is called the *risk-neutral measure* (for which see Bingham & Kiesel (2004)). We regard the risk-neutral measure as a summary of an underlying equilibrium market model such as is described by Dana-Jeanblanc (2003).

The equilibrium choice of γ dictated by the Dye equation is then equivalent to the selection of the unique exercise-value γ of the put $\mathbb{E}[(\gamma - F)^+]$ consistent with the no-arbitrage valuation of the firm at $\mathbb{E}[F]$ on the ex-ante date, when its manager is known to use some disclosure cut-off γ at the later interim date (for which see Ostaszewski & Gietzmann 2008 and also Gietzmann & Ostaszewski 2011). This is because (6) is equivalent to

$$\mathbb{E}[F] = \tau_D \mathbb{E}[F|D] + (1 - \tau_D)\gamma, \tag{8}$$

where $D = D(\gamma)$ is the disclosure event complementary to $ND(\gamma)$ above and τ_D is the probability of $D(\gamma)$.

The equilibrium choice of γ is also characterized by the *Minimum Principle of Valuation* established in Ostaszewski & Gietzmann (2008) in the form that investors "discount the value of the firm down to the lowest possible value consistent with whatever discretionary disclosure is made", which is a re-interpretation of the Grossman and Hart (1980) unravelling result.

The Dye cutoff is thus an optimal put-strike (cutoff) in the no-arbitrage sense (i.e. in the 'risk-neutral valuation' sense of finance). Indeed, the Dye model creates a formal 'option to disclose' with its cutoff playing a role analogous to that of a financial put's strike price. This justifies referring to the Dye cutoff as an *optimal cutoff in the no-arbitrage sense*. Moreover, recognition of this equivalence allows us to trace the dependence of the Dye disclosure option through the dependence of a put on its strike price (cutoff).

The hemi-mean function has the traditional hockey-stick shaped payoff (in view of put-call duality): it is call-like. The standard Black-Scholes put option is thus a central tool. (See Appendices 2 and 7.)

As a guide to intuition, the following result identifies the statics of the cutoff in a simple context; this generalizes an observation of Penno (1997) for the special Gaussian case.

Location-scale cutoff standardization theorem[4]. *Let* $\Phi_F(x)$ *be an*

[4]Proof available from the authors.

arbitrary zero-mean, unit-variance, cumulative distribution defined on $\mathbb{R}$*. For the location and scale family of distributions* $\Phi_F(\frac{x-\mu}{\sigma})$*, with mean* μ *and variance* σ^2*, the Dye cutoff* $\gamma(\mu, \sigma, \lambda)$ *satisfies*

$$\gamma(\mu, \sigma, \lambda) = \mu - \sigma\xi(\lambda), \text{ where } \lambda = \frac{1-q}{q},$$

so that

$$\xi(\lambda) = -\gamma(0, 1, \lambda) < 0$$

is the cutoff when standardizing to zero mean and unit variance and is a function only of the odds λ*. The standardized cutoff* $\xi(\lambda)$ *is a convex and decreasing function of* λ *satisfying*

$$\lambda = H_F(-\xi)/\xi,$$

where $H_F(x) = \int_{-\infty}^{x} \Phi_F(t)dt$ *is the corresponding hemi-mean function.*

Thus, ceteris paribus, the larger is the precision (equivalently, the smaller is the variance σ^2) the closer the cutoff γ is to the mean μ.

2.2 Modified Dye Cutoff: noisy scenario

In a multi-firm environment there is scope to study differences in disclosure strategy arising from underlying differences between the firms. We model the differences by choosing to assume that if the manager of firm i observes a signal T_i of firm value he/she does so with noise and the primary source of difference between managers will be the manager-specific noise. We follow the standard Dye assumption that all disclosures are truthful – what differs here is the "quality" of those (forecast) disclosures, since some may have relatively low precision. We also note that we implicitly assume that the underlying *information endowment variables* I_i which take the value 0 or 1 according as firm i does not/does receive its signal T_i are also independent of all of the preceding random variables.

The noisy signal setting can fortunately be analyzed with a straightforward modification to the Dye calculus requiring two steps. First, we observe that the single-firm framework above can embrace observation of a noisy signal by a re-interpretation of F; that is, F may validly be replaced by a noisy signal of the true value F, say by $T = T(F, \varepsilon)$, where ε models noise. Then one may deduce the existence of a cutoff γ_T above which a noisy signal

T would, in equilibrium, be voluntarily disclosed. Given a disclosure in such an environment, investors would then form expectations conditioning on the reported noisy signal, and the market values the firm as $\mathbb{E}[F|T]$ rather than as $\mathbb{E}[F]$. That is, referring to the *regression function* $\mu_F(t) := \mathbb{E}[F|T = t]$, the valuation assigned to the firm is given by the estimator $F^{\text{est}} := \mu_F(T)$. If, however, no disclosure occurs, then the market valuation is $\mu_F(\gamma_T)$. The classic Dye disclosure calculus remains valid in this more complex noisy setting, provided the F in Dye's model is re-interpreted, not as the true firm value, but as F^{est}, the estimated firm value given T. This requires that $\mu_F(t)$ be a *strictly increasing* function[5]. Similarly, all that needs doing in the Jung and Kwon equation is to replace H_F by another, related, hemi-mean function $H_{F^{\text{est}}}$; this yields a cutoff $\gamma_{F^{\text{est}}}$ for the estimator F^{est}, defined implicitly by the amended Jung and Kwon equation. Then $\gamma_{F^{\text{est}}} = \mu_F(\gamma_T)$ with γ_T the disclosure cutoff for the actual signal T.

3 Simultaneous Dye equations with multiple conditioning

In this section we consider a sector of n firms and apply to it the Dye theory with the noisy signals of the last sub-section, to yield a set of n simultaneous Dye equations in n Dye cutoffs. We assume a common time structure for the firms: the interim-report date is identical for all companies. That is, their voluntary disclosure/non-disclosure occurs simultaneously.

We begin informally (see below for a formalization) and assume that each firm i in the sector has a probability q_i of seeing a noisy signal T_i of what is the common 'sector value' X. Each firm i selects a disclosure cutoff γ_i for T_i and we consider the equilibrium profile of choices $\gamma = (\gamma_1, \gamma_2, ..., \gamma_n)$ of cutoffs which leads to each manager i being indifferent between disclosing and not disclosing an observation of $T_i = \gamma_i$ conditional on all the other firms j observing $T_j = \gamma_j$ and making a disclosure of γ_j.

In our model it will be the case – detailed in the next section – that equity value is proportional to sector value. Hence, the equilibrium condition, generalizing (6), is that for each i :

$$\mathbb{E}[X|T_j = \gamma_j \text{ for all } j] = \mathbb{E}[X|ND_i(\gamma)]. \tag{9}$$

[5]This establishes a condition validating the replacement of X by $E[X|T]$, suggested also by Acharya et al. (2011) in their footnote 2.

See Appendix 8 for a formal approach. Here $ND_i(\gamma)$ denotes non-disclosure by firm i while all the remaining firms j (i.e. $\forall j \neq i$) disclose the received signal $T_j = \gamma_j$. That is , the valuation is unaltered if firm i switches from disclosing its cutoff γ_i to non-disclosure while the remaining firms continue to disclose their Dye cutoffs.

Denote by $1_{T_1<\gamma_1}$ the indicator of the event that $T_1 < \gamma_1$ and introduce the regression function

$$\mu_X(t_1, .., t_n) := \mathbb{E}[X|T_i = t_i \ (\forall i)\},$$

so that the left-hand side of equation (9) is then $\mu_X(\gamma_1, ..., \gamma_n)$. Since

$$\Pr(T_1 < \gamma_1 | ND_1(\gamma)) = \mathbb{E}[1_{T_1<\gamma_1} | T_j = \gamma_j(\forall j > 1)],$$

the conditional expected value on the right in that equation reduces for $i = 1$ to

$$\frac{(1-q_1)\mathbb{E}[\mu_X(T_1, \gamma_2, ..., \gamma_n)|ND_1^0(\gamma)] + q_1\mathbb{E}[\mu_X(T_1, \gamma_2, ..., \gamma_n)1_{T_1<\gamma_1}|ND_1^1(\gamma)]}{(1-q_1) + q_1\mathbb{E}[1_{T_1<\gamma_1}|ND_1(\gamma)]},$$

where the superscript indicates conditioning on the information endowment variable I_i being 0 or 1 (cf. Sect. 2.2). Here we have used the conditional mean formula (law of iterated expectation) to obtain

$$\begin{aligned}\mathbb{E}[X|T_j &= \gamma_j(\forall j > 1)] = \mathbb{E}[\mathbb{E}[X|(T_j = \gamma_j(\forall j > 1)), T_1|T_j = \gamma_j(\forall j > 1)] \\ &= \mathbb{E}[\mu_X(T_1, \gamma_2, ..., \gamma_n)|T_j = \gamma_j \ \forall j > 1].\end{aligned}$$

Setting

$$\mu_1(\gamma_2, ..., \gamma_n) := \mathbb{E}[\mu_X(T_1, \gamma_2, ..., \gamma_n)|ND_1(\gamma)], \tag{10}$$

after a simple manipulation the Dye equation takes the form

$$\begin{aligned}&\frac{1-q_1}{q_1}\left(\mu_1(\gamma_2, ..., \gamma_n) - \mu_X(\gamma_1, ..., \gamma_n)\right) \qquad (11)\\ &= \int_{t_1<\gamma_1} (\mu_X(\gamma_1, ..., \gamma_n) - \mu_X(t_1, \gamma_2..., \gamma_n))d\Phi_{T_1}(t_1|\gamma_2..., \gamma_n).\end{aligned}$$

Here the the right-hand side may be viewed as a generalized lower partial moment, or briefly the hemi-mean function of the estimator $X_1 = \mu_X(T_1, \gamma_2, ..., \gamma_n)$, assuming that the regression function μ_X is strictly monotone in each variable. Hence,

$$\mu_1(\gamma_2, ..., \gamma_n) = \mathbb{E}[X_1|(T_j = \gamma_j(\forall j > 1))] = \mathbb{E}[\mu_X(T_1, \gamma_2, ..., \gamma_n)|ND_1(\gamma)].$$

The change of variable $t_1 \to x_1$ given by $x_1 = \mu_X(t_1, \gamma_2, ..., \gamma_n)$, transforms the Dye equation for γ_1 (as a function of $\gamma_2, ..., \gamma_n$) to an equation for the transform γ_{X_1} of γ_1 in the original Dye format of Section 2.1:

$$\frac{1-q_1}{q_1}\left(\mu_1(\gamma_2, ..., \gamma_n) - x_1\right) = H_{X_1}(x_1),$$

with solution $x_1 = \mu_X(\gamma_1(\gamma_2, ..., \gamma_n), \gamma_2, ..., \gamma_n)$, where

$$H_{X_1}(t) := \int_{x_1<t} (t - x_1) d\Phi_{X_1}(x_1|\gamma_2..., \gamma_n).$$

The transformed equation has a unique solution x below $\mu_1(\gamma_2, ..., \gamma_n)$, since $H_{X_1}(x_1)$ is positive, increasing and convex ('call-shaped'), whereas the linear expression on the left-hand side is decreasing in x_1. Thus, just as before, determination of the cutoff reduces to determination of the hemi-mean function.

The approach above, specialized to a log-normal setting in the next section, allows us to reduce the hemi-mean function of T_1 to the standard Black-Scholes put option associated with X_1; we note that the hemi-mean function of T_1 would be recognized by Fishburn (1977) as a general risk-measure (for below-target t risk).

4 Modeling inter-firm correlation

So far the analysis has been conducted in a general setting. We turn now to the benchmark model of mathematical finance, the Black-Scholes model, based on Brownian motion as driving noise, so on an underlying normal/Gaussian error structure and so log-normal distributions; see e.g. Bingham & Kiesel (2004) Ch. 5-6. Other workable exponential variates that permit the benefits of a multiplicative structure could be considered.

The model setting of a sector comprises firms $i = 1, 2, ..., n$ and their values at terminal time $\theta = 1$ are represented at the ex-ante time $\theta = -1$ by the random variables $F_1, F_2, ..., F_n$, which are decomposed into two factors. One factor captures the 'sector' correlation effect and the other a firm-specific effect so that, on incorporating a firm-specific sector *loading index* α_i, one has

$$F_i := f_i X^{\alpha_i} Z_i,$$

where f_i is a scale factor[6] and $X, Z_1, Z_2, ..., Z_n$ are log-normal independent random variables with unit mean so that $\mathbb{E}[F_i]$ is the ex-ante expected terminal value. (So by Appendix 2, $\mathbb{E}[F_i] = f_i \exp[\frac{1}{2}\alpha_i(\alpha_i - 1)\sigma_0^2)]$.)

Hence, at the interim date, since X and Z_i are independent, investors value firm i at

$$\mathbb{E}[F_i|\text{all disclosures}] = \mathbb{E}[f_i X^{\alpha_i} Z_i|...] = f_i \mathbb{E}[X^{\alpha_i}|...], \tag{12}$$

i.e. proportionally to their estimate of X^{α_i}.

In modeling the varying disclosures by firms, there could be multiple sources of variation driving differences in disclosure (cf. Section 2.2). In order to provide stepwise development of intuition, we specialize our focus to the case where the firm-specific factors Z_j are all independent (so uncorrelated), and thus investors are concerned about the correlation between the signals T_j to the extent that they provide (conditioning) information on sectorial performance. Within this framework, we are able to derive some remarkably clear intuitive results, which we suggest provide solid foundations for analysis of environments with more complex correlations between all the variables.

Formally, assume the manager of firm i receives with probability q_i a signal T_i about the terminal value of the sector condition, X. We introduce manager-specific noise in the form of a multiplicative factor Y_i so that:

$$T_i = XY_i,$$

where X (also referred to below as Y_0, for notational convenience), Y_i, Z_j are all independent[7]. The value of T_i, if disclosed (that is to say: observed and above the optimal cutoff), is then conditioned upon in equation (12). We assume that X, Y_i are also log-normal with unit mean and that

$$Y_i = e^{\sigma_i v_i - \frac{1}{2}\sigma_i^2}, \text{ for } i = 0, 1, 2, ..., n,$$

with $v_0, v_1, ..., v_n$ independent $N(0, 1)$ variables. That is, the co-dependence of the signals $T_1, ..., T_n$ is explained by v_0 (which corresponds to a 'sector-wide' effect). Equations such as these are to be viewed as defining a transformation, which provides a natural link between our work in the log-normal

[6]This scale factor allows us to study firms standardized to unit mean.

[7]So T_i is standardized to have unit mean. For $\alpha_i \neq 0$, this is equivalent to a signal generated from X^{α_i} by multiplication with noise; it suffices to replace such a signal by an alternative version obtained by a suitable power and scaling transformation .

domain and standard regression theory, for which see Bingham & Fry (2010). We refer to the parameters σ_i as *volatilities*. This is informed by viewing Y_i as being the time $\theta = 1$ discounted value of a *Black-Scholes asset* with dynamic

$$Y_i(\theta) = e^{\sigma_i v_i(\theta) - \frac{1}{2}\sigma_i^2 \theta},$$

where θ measures time and $v_i(\theta)$ is a standard Wiener process (Brownian motion). Thus the earlier symbol v_i is interpreted as $v_i(1)$, i.e. the time $\theta = 1$ sampled-value of the process. This view entitles us to interpret the parameters σ_i as volatilities of the corresponding returns $dY_i(\theta)/Y_i(\theta)$ evaluated at time $\theta = 1$.

It follows from our assumptions that

$$T_i = Y_0 Y_i = e^{\sigma_0 v_0 - \frac{1}{2}\sigma_0^2} e^{\sigma_i v_i - \frac{1}{2}\sigma_i^2} = e^{\sigma_{0i} w_i - \frac{1}{2}\sigma_{0i}^2},$$

where

$$\sigma_{0i} w_i = \sigma_0 v_0 + \sigma_i v_i \tag{13}$$

and $w_i \sim N(0,1)$. We will need to know that the correlation ρ_{ij} corresponding to the covariance $cov(w_i, w_j)$ is (as in Lemma A1.1)

$$\rho_{ij} = cov(\sigma_0 v_0 + \sigma_i v_i, \sigma_0 v_0 + \sigma_j v_j) = \frac{\sigma_0^2}{\sigma_{0i}\sigma_{0j}}. \tag{14}$$

Under our modeling of co-dependance the correlation matrix $(\rho_{ij})_{i,j \le n}$, which would appear to introduce a standard but intractable formalism, turns out to be equivalent to a very simple *precision matrix:*

$$\begin{bmatrix} p_1 + p_0 & p_2 & \cdots & p_n \\ p_1 & p_2 + p_0 & & p_n \\ \vdots & & \ddots & \vdots \\ p_1 & p_2 & \cdots & p_n + p_0 \end{bmatrix}.$$

See Appendix 4 for further details.

5 Log-normal Simultaneous Dye equations

Having now modeled the correlation structure of the signals received by the firms, we next begin an analysis of the simultaneous system of Dye equations.

Since these refer to conditioning on the signals of other firms, we are first led to considering the form of the multi-firm regression function.

For the above log-normal noisy signals model it is straightforward to show (see Appendix 3) that the regression function introduced in Section 4 has a simple multiplicative power format:

$$\mu_X(t_1, t_2, ..., t_n) := \mathbb{E}[X|T_1 = t_1, ..., T_n = t_n] = Kt_1^{\kappa_1}...t_n^{\kappa_n},$$

and, more generally, after inclusion of an α loading exponent:

$$\mu_X^\alpha(t_1, t_2, ..., t_n) := \mathbb{E}[X^\alpha|T_1 = t_1, ..., T_n = t_n] = K_\alpha t_1^{\alpha\kappa_1}...t_n^{\alpha\kappa_n},$$

for some explicitly derived constants K, K_α (given in (16) and (17) in Prop. A3.3), i.e. these are independent of the variables t_i (with the κ_i being the classical linear regression coefficients for the underlying normal random variables, as per (2) in Section 1). Likewise $\mu_1(.)$, defined in Section 4 by equation (10), also has a power format; more generally, with α replacing the loading index α_1 of firm 1, one has:

$$\begin{aligned} \mu_1^\alpha(\gamma_2, ..., \gamma_n) \quad &: \quad = \mathbb{E}[K_\alpha T_1^{\alpha\kappa_1}...T_n^{\alpha\kappa_n}|T_2 = \gamma_2, ..., T_n = \gamma_n] \\ &= \quad K_\alpha \gamma_2^{\alpha\kappa_2}...\gamma_n^{\alpha\kappa_n}\mathbb{E}[T_1^{\alpha\kappa_1}|T_2 = \gamma_2, ..., T_n = \gamma_n] \\ &= \quad K_\alpha \gamma_2^{\alpha\kappa_2}...\gamma_n^{\alpha\kappa_n} L_{-1}\gamma_2^{\alpha\kappa_{-1}^2\kappa_1}...\gamma_n^{\alpha\kappa_{-1}^n\kappa_1}. \end{aligned}$$

Here again the further constants L_{-i} are explicitly derived (in Appendix 6), whereas the κ_{-i}^j are again classical linear regression coefficients corresponding to omitting firm i, as per (3) in Section 1 – see equation (15) in Appendix 1 below. Thus the Dye equations may be shown to have a tractable form after transforming t_1 and T_1 to s_1 and S_1 via

$$s_1 = \mu_{X^\alpha}(t_1, \gamma_2, ...) = K_\alpha t_1^{\alpha\kappa_1}\gamma_2^{\alpha\kappa_2}...\gamma_n^{\alpha\kappa_n}.$$

Here $\gamma_2, ..., \gamma_n$ are multi-firm equilibrium Dye cutoffs and t_1 is a free variable. Then

$$\gamma_{S_1} = \gamma_{S_1}(\gamma_2, ..., \gamma_n) = K_\alpha(\gamma_1(\gamma_2, ...\gamma_n))^{\alpha\kappa_1}\gamma_2^{\alpha\kappa_2}...\gamma_n^{\alpha\kappa_n},$$

where $\gamma_1(...)$ is the Dye cutoff for firm 1 when the remaining firms follow a γ_j cutoff strategy. From these explicit formulas we develop the simultaneous log-normal Dye equations and solve them in Appendix 7.

6 Hypothetical-firm induced cutoffs

This section introduces the lone n hypothetical firms of Section 1 as a tool in deriving and interpreting the multi-agent induced cutoffs. We then deduce the bandwagon and estimator-quality effects that sector correlation has on the actual individual firm cutoffs.

We write w for the vector $(w_1, ..., w_n)$ of the underlying normal random variables defined by (13) and w_{-i} for the same vector with i-th component omitted. After some analysis conducted in Appendix 7, it transpires that the following change of variable:

$$y_1 = \gamma_1^{\alpha_1\kappa_1} / \left(L_{-1}\gamma_2^{\alpha_1(\kappa_{-1}^2 - \kappa_2)} .. \gamma_n^{\alpha_1(\kappa_{-1}^n - \kappa_n)} \right), \text{... etc., mutatis mutandis,}$$

transforms and uncouples the Dye equations of Section 3 in the log-normal setting to a system in new variables y_i:

$$\frac{1-q_i}{q_i}(1-y_i) = H_{\text{LN}}\left(y_i, \alpha_i\kappa_i\sigma_{0i}\sqrt{1-\rho_i^2}\right), \text{ with } \sigma_{0i}^2 = \sigma_0^2 + \sigma_i^2,$$

where $1-\rho_i^2$ is the partial covariance of w_i given $(w)_{-i}$, as defined in Appendix 5 (and given by an explicit formula there) and H_{LN} is the log-normal hemi-mean function given explicitly by:

$$H_{\text{LN}}(t,\sigma) := t\Phi\left(\frac{\ln t + \frac{1}{2}\sigma^2}{\sigma}\right) - \Phi\left(\frac{\ln t - \frac{1}{2}\sigma^2}{\sigma}\right).$$

We call $\kappa_i\sigma_{0i}\sqrt{1-\rho_i^2}$ the *refined* volatility and refer to the corresponding precision as the *refined precision.* To standardize, we let $\hat{\gamma} = \hat{\gamma}_{\text{LN}}(\lambda, \sigma)$ solve the single Dye equation

$$\lambda(1-\hat{\gamma}) = H_{\text{LN}}(\hat{\gamma}, \sigma), \text{ with } \lambda := \frac{1-q}{q}.$$

Thus the decoupled system has solution

$$\hat{\gamma}_i := \hat{\gamma}_{\text{LN}}\left(\lambda_i, \kappa_i\sigma_{0i}\sqrt{1-\rho_i^2}\right).$$

From here one deduces the following result which yields the i-th multi-agent induced cutoff of the Introduction; see Appendix 7 for the derivation.

The Existence Theorem (Multi-firm Dye Equations). *In the setting of this section, the simultaneous Dye equations have a unique solution and the disclosure cutoff γ_i for the signal T_i is given by*

$$\log \gamma_i = \frac{\log g_i}{\alpha_i \kappa_{-i}} + \frac{1}{\kappa_0}\left(\frac{\kappa_1}{\alpha_1 \kappa_{-1}} \log g_1 + \frac{\kappa_2}{\alpha_2 \kappa_{-2}} \log g_2 + ... + \frac{\kappa_n}{\alpha_n \kappa_{-n}} \log g_n\right),$$

where

$$\begin{aligned} g_i &= \hat{\gamma}_{\mathrm{LN}}(\lambda_i, \alpha_i \kappa_i \sigma_{0i} \sqrt{1-\rho_i^2}) L_{-i} \text{ and } \lambda_i = \frac{1-q_i}{q_i} \text{ (the odds)}, \\ L_{-i} &= \exp\left(\frac{(n-1)\alpha_i + \alpha_i(\alpha_i - 1)}{2(p-p_i)}\right) \exp\left(-\frac{n\alpha_i + \alpha_i(\alpha_i-1)}{2p}\right), \\ & \quad \text{(the `amended mean' -- an adjustment coefficient for the cutoff)}, \end{aligned}$$

and where:
$\hat{\gamma}_{\mathrm{LN}}(\lambda, \sigma)$ *denotes the solution of the following equation in* y :

$$\lambda(1-y) = H_{\mathrm{LN}}(y, \sigma),$$

$\kappa_i = p_i/p$ *(the regression coefficient of* w_i*),*
$\kappa_{-i} = p_i/(p - p_i)$ *(the regression coefficient resulting from the removal of firm-i's contribution from the total precision),*
$1 - \rho_i^2$ *is the partial covariance of* w_i *given the remaining variates* w_j*.*

We can now trace the effect of sector correlation on multi-firm cutoffs via the partial covariance and the mean-adjustment coefficients. In view of the cutoffs g_i appearing in the theorem, we refer to a firm with refined volatility $\kappa_i \sigma_{0i} \sqrt{1-\rho_i^2}$ and with *amended mean* L_{-i} as firm-i's *related hypothetical firm.*

Bandwagon Inflator Theorem. *The presence of correlation increases the precision parameter of the cutoff and hence raises the cutoff:*

$$\hat{\gamma}_{\mathrm{LN}}(\lambda_i, \sigma_{0i}) < \hat{\gamma}_{\mathrm{LN}}(\lambda_i, \kappa_i \sigma_{0i}) < \hat{\gamma}_{\mathrm{LN}}\left(\lambda_i, \kappa_i \sigma_{0i} \sqrt{1-\rho_i^2}\right).$$

Proof. Indeed, $\sigma_{0i} > \kappa_i \sigma_{0i}$ (as $\kappa_i < 1$) and also ρ_i^2 is increasing in p_i (see Appendix 5). The result now follows, since $\hat{\gamma}_{\mathrm{LN}}(\lambda, \sigma)$ is decreasing in σ (by the cited result of Jung and Kwon (1988)). $\square$

When the correlation is positive, there is also a counter-veiling precision effect on the related hypothetical firm's cutoff when the actual firm has below-average precision.

Estimator-Quality Theorem. *Suppose that $n \geq 2$ and $\alpha_i > 0$ for all i. The amended mean of the hypothetical firm i is increasing in p_i with bounds given by*

$$\exp\left(-\frac{\alpha_i}{2(p-p_i)}\right) < L_{-i} < \exp\left(\frac{\alpha_i\left(1+\frac{\alpha_i-1}{n-1}\right)}{2p_{\mathrm{av},-i}}\right), \text{ where } p_{\mathrm{av},-i} := \frac{p-p_i}{n-1},$$

and in particular if the loading index is identical for all firms, then

$$L_{-i} < L_{-j} \text{ iff } p_i < p_j.$$

Otherwise, if $0 < \alpha_i < \alpha_j$ and $p_i < p_j$, then also $L_{-i} < L_{-j}$.

The amended mean is a strict deflator, i.e. $L_{-i} < 1$, iff p_i is below the loading-adjusted competitor average, i.e.

$$p_i < \frac{p}{n-1+\alpha_i} := p_{\mathrm{av},i}$$

so that for $\alpha_i = 1$ one has $p_{\text{av,-}i} = p/n$.

Proof. The first claim is clear as the factor $\exp\left(\frac{(n-1)\alpha_i+\alpha_i(\alpha_i-1)}{2(p-p_i)}\right)$ is independent of p_i and in the second factor the expression $\alpha_i(\alpha_i+n-1)$ is positive for $\alpha_i > 0$ (for $n \geq 1$). The substitutions $p_i = 0$ and the limit as $p_i \to +\infty$ yield the bounds quoted.

Noting (for $p_j > 0$ all j) that

$$\begin{aligned} 2\log L_{-i} &= \frac{(n-1)\alpha_i+\alpha_i(\alpha_i-1)}{(p-p_i)} - \frac{n\alpha_i+\alpha_i(\alpha_i-1)}{p} \\ &= \alpha_i\frac{p(n-1)+p(\alpha_i-1)-(p-p_i)[n+(\alpha_i-1)]}{p(p-p_i)} \\ &= \alpha_i\frac{p_i[n-1+\alpha_i]-p}{p(p-p_i)}, \end{aligned}$$

one has, since $\alpha_i > 0$, that $L_{-i} < 1$ iff

$$p_i < \frac{p}{n-1+\alpha_i} := p_{\mathrm{av},i}.$$

Furthermore, the same calculation shows that $L_{-i} < L_{-j}$ iff

$$\alpha_i \frac{p_i[n-1+\alpha_i]-p}{p(p-p_i)} < \alpha_j \frac{p_j[n-1+\alpha_j]-p}{p(p-p_j)}$$

iff

$$\alpha_i(p-p_j)\left(p_i[n-1+\alpha_i]-p\right) < \alpha_j(p-p_i)\left(p_j[n-1+\alpha_j]-p\right).$$

Thus if $0 < \alpha_i \leq \alpha_j$ and $p_i < p_j$ then one has $\alpha_i(p-p_j) < \alpha_j(p-p_i)$ and also as $n \geq 1$ that $p_i[n-1+\alpha_i]-p < p_j[n-1+\alpha_j]-p$, so that $L_{-i} < L_{-j}$.

If $0 < \alpha_i = \alpha_j = \alpha$, writing p_{av} for $p_{\text{av},i}$ the inequality reduces to

$$(p-p_j)\left(p_i-p_{\text{av}}\right) < (p-p_i)\left(p_j-p_{\text{av}}\right)$$

or, on expansion, to

$$p_{\text{av}}(p_j-p_i) < p(p_j-p_i).$$

But, $n \geq 2$ so one has $1 < n-1+\alpha$ and so $p_{\text{av}} < p$. Thus $L_{-i} < L_{-j}$ iff $p_i < p_j$.

Remark. For $p_j > 0$, the expression $\alpha_j(p-p_i)\left(p_j[n-1+\alpha_j]-p\right)$ is increasing in α_j. Thus it is possible to have $L_{-i} > L_{-j}$ holding when $p_j > p_i$, but for α_j sufficiently small and positive.

7 Conclusion

Moving from the case where an investor evaluates only one signal from a firm operating in a given sector, to a case where the investor evaluates multiple signals from n firms all operating in the same sector affects the nature of the optimal disclosure policy. In terms of the type of information that is disclosed in this one-shot game, the first effect that we identify is referred to as the bandwagon effect. With multiple firms the chance of any one firm disclosing good news has the effect on all the firms of increasing their disclosure cutoffs (i.e. the cutoffs below which the firms do not disclose); this is because of the sector correlation.

The second effect arises because we model the managers of the firms as disclosing 'competing' estimates of the common sector conditions, though with different precisions. In such a case of multiple signals of differing precision we show how the investor will direct attention to the more precise estimates that are disclosed. The key differentiating level of precision is shown to be the average precision across all firms; thus it is natural to refer to above average-precision firms, having (ceteris paribus) a higher cutoff and disclosing less, and to below-average firms, having a lower cutoff and needing to disclose more.

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A1. Some parameter interrelations

Below we collect a number of useful relations between various parameters. Recall that κ_i and κ_{-i} were introduced by equations (2) and (3) in §1. We will also need

$$\kappa_{-i}^{j} := p_j/(p - p_i), \tag{15}$$

so that $\kappa_{-i} = \kappa_{-i}^{i}$.

Lemma A1.1. *For $\sigma_{0i}^2 := \sigma_0^2 + \sigma_i^2$ as above, one has for i, j distinct*

$$T_i = e^{\sigma_{0i} w_i - \frac{1}{2}\sigma_{0i}^2} \text{ with } \rho_{ij} := cov(w_i, w_j) = \frac{\sigma_0^2}{\sigma_{0i}\sigma_{0j}},$$

where w_i is of zero mean and unit variance.

Proof. Since $T_i = XY_i = e^{\sigma_0 v_0 - \frac{1}{2}\sigma_0^2} e^{\sigma_i v_i - \frac{1}{2}\sigma_i^2}$ (for $i = 1, 2$), we may write

$$T_i = e^{\sigma_{0i} w_i - \frac{1}{2}\sigma_{0i}^2}, \text{ with } \sigma_{0i}^2 := \sigma_0^2 + \sigma_i^2,$$

where $w_i = (\sigma_0 v_0 + \sigma_i v_i)/\sigma_{0i}$. Now w_i has mean zero and unit variance, as $var(\sigma_0 v_0 + \sigma_i v_i) = \sigma_0^2 + \sigma_i^2 = \sigma_{0i}^2$. Hencefor distinct i, j

$$\rho_{ij} := \mathbb{E}[w_i w_j] = cov(w_i, w_j) = cov(\frac{\sigma_0 v_0 + \sigma_i v_i}{\sigma_{0i}}, \frac{\sigma_0 v_0 + \sigma_j v_j}{\sigma_{0j}}) = \frac{\sigma_0^2}{\sigma_{0i}\sigma_{0j}}.$$

Lemma A1.2. *One has*

$$\kappa_{-i}^j - \kappa_j = \kappa_{-i}^j \kappa_i.$$

Proof. Indeed, one has

$$\kappa_{-i}^j - \kappa_j = \frac{p_j}{p - p_i} - \frac{p_j}{p} = p_j \frac{p - (p - p_i)}{p(p - p_i)} = \frac{p_i p_j}{p(p - p_i)} = \kappa_{-i}^j \kappa_i.$$

For the $n = 2$ case, write temporarily $h_1 := \kappa_{-1}$ and $h_2 := \kappa_{-1}^2 = p_2/(p_0 + p_2)$; then we have:

Lemma A1.3. *In the $n = 2$ case with $\rho = \rho_{12}$ as in Lemma A1.1,*

$$\frac{\rho \sigma_{01}}{\sigma_{02}} = h_2.$$

Proof. One has, on dividing by $\sigma_0^2 \sigma_2^2$ in the last step, that

$$\frac{\rho \sigma_{01}}{\sigma_{02}} = \frac{\sigma_0^2}{\sigma_{02}^2} = \frac{\sigma_0^2}{\sigma_0^2 + \sigma_2^2} = \frac{p_2}{p_0 + p_2}.$$

In what follows note that, since $h_i \leq 1$, one has $h_1 h_2 = 1$ iff $h_1 = h_2 = 1$ in which case either $\sigma_2^2 = 0$ or $p_0 = 0$. These correspond to one of the two degenerate cases of either infinite precision of managerial vision (no noise in the observation), or infinite variance in the sectorial factor.

Lemma A1.4. *In the $n = 2$ case, one has*

$$1 - \rho^2 = 1 - h_1 h_2,$$

so in particular the two signals are perfectly correlated iff $h_1 h_2 = 1$ *iff* $p_0 = 0$.

Proof. Noting that

$$\sigma_{01}^2 = \frac{1}{p_0} + \frac{1}{p_1} = \frac{p_0 + p_1}{p_0 p_1},$$

one has

$$\rho^2 = \frac{\sigma_0^4}{\sigma_{01}^2 \sigma_{02}^2} = \frac{p_1}{p_0 + p_1} \cdot \frac{p_2}{p_0 + p_2} = h_1 h_2.$$

So that $\rho^2 = 1$ iff $p_1 p_2 = p_0^2 + p_0 p_2 + p_1 p_0 + p_1 p_2$ iff $0 = p_0(p_0 + p_1 + p_2)$.

Lemma A1.5. *For* i, j *distinct one has:*

$$\frac{\sigma_0^2}{\sigma_{0i}^2} = \frac{p_i}{(p_0 + p_i)} \text{ and } \rho_{ij}^2 = \frac{p_i}{(p_0 + p_i)} \frac{p_j}{(p_0 + p_j)}.$$

Proof. By Lemma A1.1

$$\frac{\sigma_0^2}{\sigma_{0i}^2} = \frac{1/p_0}{(1/p_0) + (1/p_i)} = \frac{1}{1 + \frac{p_0}{p_i}} = \frac{p_i}{(p_0 + p_i)},$$

hence in particular

$$\rho_{ij}^2 = \frac{\sigma_0^2}{\sigma_{0i}^2} \frac{\sigma_0^2}{\sigma_{0j}^2} = \frac{p_i}{(p_0 + p_i)} \frac{p_j}{(p_0 + p_j)}.$$

Lemma A1.6 *When* $n = 1$ *one has*

$$e^{\frac{1}{2}\kappa_1(\kappa_1 - 1)\sigma_{01}^2} K_1 = e^{\frac{1}{2}\kappa_1(\kappa_1 - 1)\sigma_{01}^2} \exp\left(\frac{1}{2(p_0 + p_1)}\right) = 1.$$

Proof. Since

$$\sigma_{01}^2 = \sigma_0^2 + \sigma_1^2 = \frac{1}{p_0} + \frac{1}{p_1} = \frac{p_0 + p_1}{p_0 p_1},$$

one has

$$\begin{aligned} \kappa_1(\kappa_1 - 1)\sigma_{01}^2 &= -\frac{p_1}{p_0 + p_1} \frac{p_0}{p_0 + p_1} \frac{p_0 + p_1}{p_0 p_1} \\ &= -\frac{1}{p_0 + p_1}. \end{aligned}$$

A2. Log-normal preliminaries

We will always represent a unit-mean log-normal variable in the form $X = e^{\sigma u - \frac{1}{2}\sigma^2}$ with u standard normal. Then

$$\Pr(X \le x) = \Phi_{\mathrm{LN}}(x, \sigma) := \Phi_{\mathrm{N}}\left(\frac{\log(x) + \frac{1}{2}\sigma^2}{\sigma}\right),$$

where Φ_{N} is the standard normal (cumulative) distribution and Φ_{LN} the log-normal. It follows from the Black-Scholes formula for a put with strike x and unit time to expiry on the underlying $X_t = e^{\sigma w_t - \frac{1}{2}\sigma^2 t}$ that

$$\begin{aligned} H_{\mathrm{LN}}(x, \sigma) &= \int_{t<x} \Phi_{\mathrm{LN}}(t, \sigma) dt = \int_{t<x} (x - t) d\Phi_{\mathrm{LN}}(t, \sigma) \\ &= \mathbb{E}[(x - X)^+] \\ &= x\Phi_{\mathrm{N}}\left(\frac{\log(x) + \frac{1}{2}\sigma^2}{\sigma}\right) - \Phi_{\mathrm{N}}\left(\frac{\log(x) - \frac{1}{2}\sigma^2}{\sigma}\right). \end{aligned}$$

Consider now the power transformation $Y = X^\kappa$ for $0 < \kappa < 1$; then with $s = \kappa\sigma$,

$$\begin{aligned} Y &= e^{\kappa\sigma u - \frac{1}{2}\kappa\sigma^2} = e^{-\frac{1}{2}\kappa(1-\kappa)\sigma^2} e^{su - \frac{1}{2}s^2} \\ &= e^{-\frac{1}{2}\kappa(1-\kappa)\sigma^2} Z. \end{aligned}$$

That is, the new variable has reduced mean

$$m = m(\kappa, \sigma) := e^{\frac{1}{2}\kappa(\kappa-1)\sigma^2},$$

and is the product of this new mean by a log-normal Z with mean 1 and log-variance $\kappa^2\sigma^2$. Now $\log Y = \frac{1}{2}\kappa(\kappa - 1)\sigma^2 + \log Z$, so

$$\begin{aligned} \Pr(Y &\le x) = \Pr\left(\kappa\sigma u - \frac{1}{2}\kappa\sigma^2 \le \log x\right) \\ &= \Pr\left(u \le \frac{\log(x) + \frac{1}{2}\kappa\sigma^2}{\kappa\sigma}\right) = \Pr\left(u \le \frac{\log(x/m) - \frac{1}{2}\kappa^2\sigma^2}{\kappa\sigma}\right) \\ &= \Phi_{\mathrm{LN}}(x/m, \kappa\sigma). \end{aligned}$$

Thus one should bear in mind that if $Y = X^\kappa$ for X log-normal with unit mean and log-variance σ^2, then

$$\Phi_Y(x) = \Phi_{\mathrm{LN}}(xe^{\frac{1}{2}\kappa(1-\kappa)\sigma^2}, \kappa\sigma).$$

Finally, we must find $H_Y(\xi) = \int_{x<\xi} \Phi_Y(x)dx = \int_{x<\xi} \Phi_{\text{LN}}(x/m, \kappa\sigma)dx$. Put $t = x/m$, then

$$\begin{aligned} H_Y(\xi) \quad : &= \int_{x<\xi} \Phi_Y(x)dx = \int_{x<\xi} \Phi_{\text{LN}}(x/m, \kappa\sigma)dx \\ &= m \int_{t<\xi/m} \Phi_{\text{LN}}(t, \kappa\sigma)dt = mH_{\text{LN}}(\xi/m, \kappa\sigma). \end{aligned}$$

We record this result as:

Proposition A2.1 (Exponent effect). *If X is log-normal with volatility σ, then X^κ has volatility $\kappa\sigma$ and hemi-mean function*

$$H_{X^\kappa}(t) = mH_{\text{LN}}(t/m, \kappa\sigma), \text{ for } m = e^{-\frac{1}{2}\kappa(1-\kappa)\sigma^2}.$$

A3. Log-normal Regression

Recall from Appendix 1 that $T_i = e^{\sigma_{0i}w_i - \frac{1}{2}\sigma_{0i}^2}$. To study conditioning on T_i, we take logarithms to pass to the underlying normal variates: $\tau_i := \log T_i + \frac{1}{2}\sigma_{0i}^2 = \sigma_{0i}w_i = \sigma_0 v_0 + \sigma_i v_i$, and likewise pass from $X = Y_0 = e^{\sigma_0 v_0 - \frac{1}{2}\sigma_0^2}$ to $\xi := \log X + \frac{1}{2}\sigma_0^2 = \sigma_0 v_0$. For completeness and to explain our methodology here, we derive a classical regression result concerning these underlying normal variates. We write

$$\xi^{\text{est}} := \mathbb{E}[\xi|\tau_1, ..., \tau_n] = \kappa_1\tau_1 + ... + \kappa_n\tau_n, = \kappa_1(\sigma_0 v_0 + \sigma_1 v_1) + ... + \kappa_n(\sigma_0 v_0 + \sigma_n v_n),$$

and our first result confirms the coefficients κ_i as validly given in the Introduction.

Proposition A3.1. *For the coefficient κ_i defined in (2) it is the case that*

$$\mathbb{E}[\xi|\tau_1, .., \tau_n] = \kappa_1\tau_1 + ... + \kappa_n\tau_n.$$

Proof. Indeed,

$$\xi^{\text{est}} := \mathbb{E}[\xi|\tau_1, ..., \tau_n] = \kappa_1\tau_1 + ... + \kappa_n\tau_n.$$

Hence

$$\begin{aligned} \mathbb{E}[\tau_1\xi^{\text{est}}] &= \mathbb{E}[\kappa_1(\sigma_0 v_0 + \sigma_1 v_1)(\sigma_0 v_0 + \sigma_1 v_1) + \\ &\quad + \kappa_2(\sigma_0 v_0 + \sigma_1 v_1)(\sigma_0 v_0 + \sigma_2 v_2) + ... + \kappa_n(\sigma_0 v_0 + \sigma_1 v_1)(\sigma_0 v_0 + \sigma_n v_n)] \\ &= \kappa_1(\sigma_0^2 + \sigma_1^2) + \kappa_2\sigma_0^2 + ... + \kappa_n\sigma_0^2. \end{aligned}$$

But, by the conditional mean formula,

$$\begin{aligned}\mathbb{E}[\tau_1\xi^{\mathrm{est}}] &= \mathbb{E}[\tau_1\mathbb{E}[\xi|\tau_1,...,\tau_n]] = \mathbb{E}[\mathbb{E}[\tau_1\xi|\tau_1,...,\tau_n]] \\ &= \mathbb{E}[\tau_1\xi] = \mathbb{E}[(\sigma_0 v_0+\sigma_1 v_1)\sigma_0 v_0] = \sigma_0^2.\end{aligned}$$

Comparing, we obtain

$$\kappa_1(\sigma_0^2+\sigma_1^2)+\kappa_2\sigma_0^2+...+\kappa_n\sigma_0^2=\sigma_0^2.$$

Dividing by σ_0^2 and setting $k_i=\kappa_i/p_i$, we obtain

$$k_1(p_0+p_1)+k_2p_2+...+k_np_n=1.$$

More generally, for each i

$$k_1p_1+...+k_i(p_0+p_i)+...+k_np_n=1,$$

with solution $k_i=1/p$, as asserted.

Proposition A3.2 *It is the case that*

$$\mathbb{E}[X|T_1=t_1,...,T_n=t_n]=Kt_1^{\kappa_1}...t_n^{\kappa_n},$$

where

$$K=\exp\left(\frac{1}{2p_{\mathrm{av}}}\right),\ \text{with } p_{\mathrm{av}}:=\frac{p_0+...+p_n}{n}. \tag{16}$$

Proof. Taking logarithms of the corresponding normal regression formula, we observe that the regression function $\mathbb{E}[X|T_1,...,T_n]$ has the form:

$$KT_1{}^{\kappa_1}...T_n{}^{\kappa_n},$$

for some constant K or, since $T_i=XY_i$ with Y_0 for X,

$$KY_0^{\kappa_1+...+\kappa_n}Y_1^{\kappa_1}...Y_n^{\kappa_n}=KY_0^{1-\kappa_0}Y_1^{\kappa_1}...Y_n^{\kappa_n}.$$

Since $\mathbb{E}[Y_i^{\gamma}]=\mathbb{E}[e^{\gamma v_i-\frac{1}{2}\gamma\sigma_i^2}]=e^{\frac{1}{2}\sigma_i^2\gamma(\gamma-1)}=e^{\frac{1}{2}\gamma(\gamma-1)/p_i}$, by independence and the conditional mean formula, one has

$$\begin{aligned}&1\\ =\ &\mathbb{E}[X]=\mathbb{E}[\mathbb{E}[X|T_1,...,T_n]],\\ =\ &\mathbb{E}[KT_1^{\kappa_1}...T_n^{\kappa_n}],\ \text{using } \mathbb{E}[X|T_1,...,T_n]=KT_1^{\kappa_1}...T_n^{\kappa_n}\\ =\ &K\mathbb{E}[Y_0^{1-\kappa_0}]\mathbb{E}[Y_1^{\kappa_1}]...\mathbb{E}[Y_n^{\kappa_n}],\ \text{using } T_1^{\kappa_1}...T_n^{\kappa_n}=Y_0^{\kappa_1+..+\kappa_n}Y_1^{\kappa_1}...Y_n^{\kappa_n},\\ =\ &K\exp\frac{1}{2}\left[\frac{(\kappa_0-1)\kappa_0}{p_0}+\frac{\kappa_1(\kappa_1-1)}{p_1}+...+\frac{\kappa_n(\kappa_n-1)}{p_n}\right],\end{aligned}$$

since $\kappa_0+\kappa_1+...+\kappa_n = 1$, so that $(\kappa_1+...+\kappa_n)(\kappa_1+...+\kappa_n-1) = -\kappa_0(1-\kappa_0)$. But, since $\kappa_i = p_i/p$,

$$\frac{\kappa_i(\kappa_i - 1)}{p_i} = \frac{1}{p}\left(\frac{p_i}{p} - 1\right) = \frac{p_i - p}{p^2},$$

one has

$$\begin{aligned} & \frac{(\kappa_0 - 1)\kappa_0}{p_0} + \frac{\kappa_1(\kappa_1 - 1)}{p_1} + ... + \frac{\kappa_n(\kappa_n - 1)}{p_n} \\ = \quad & \frac{p_0 - p}{p^2} + \frac{p_1 - p}{p^2} + ... + \frac{p_n - p}{p^2} = \frac{p - (n+1)p}{p^2} = \frac{-np}{p^2} = -\frac{n}{p}. \end{aligned}$$

So, we may now identify K from:

$$1 = K \exp\left(-\frac{n}{2p}\right).$$

In fact we have also shown

$$Ke^{\frac{1}{2}(\kappa_1+...+\kappa_n)(\kappa_1+...+\kappa_n-1)\sigma_0^2} \prod\nolimits_i e^{\frac{1}{2}\kappa_i(\kappa_i-1)\sigma_i^2} = 1.$$

In particular for $n = 1$ one has

$$Ke^{\frac{1}{2}\kappa_1(\kappa_1-1)\sigma_{0i}^2} = 1.$$

More generally one has the following.

Proposition A3.3 *It is the case that*

$$\mathbb{E}[X^\alpha | T_1 = t_1, ..., T_n = t_n] = K_\alpha t_1^{\alpha\kappa_1} ... t_n^{\alpha\kappa_n}$$

where

$$K_\alpha = \exp\left(\frac{\alpha + \alpha(\alpha - 1)/n}{2p_{\text{av}}}\right), \text{ with } p_{\text{av}} := \frac{p_0 + ... + p_n}{n}, \text{ as before.} \quad (17)$$

Proof. As before, one has

$$\alpha\mathbb{E}[\xi|\tau_1, ..., \tau_n] = \mathbb{E}[\alpha\xi|\tau_1, ..., \tau_n] = \alpha\kappa_1\tau_1 + ... + \alpha\kappa_n\tau_n.$$

Taking logarithms of the corresponding normal regression formula, we observe that the regression function $\mathbb{E}[X^\alpha|T_1,...,T_n]$ has the form, for K_α some constant:

$$K_\alpha {T_1}^{\alpha\kappa_1}...{T_n}^{\alpha\kappa_n},$$

or, as $T_i = XY_i$ and writing Y_0 for X,

$$K_\alpha Y_0^{\alpha\kappa_1+..+\alpha\kappa_n} Y_1^{\alpha\kappa_1}...Y_n^{\alpha\kappa_n} = K_\alpha Y_0^{\alpha(1-\kappa_0)} Y_1^{\alpha\kappa_1}...Y_n^{\alpha\kappa_n}.$$

Note that

$$\alpha\kappa_i(\alpha\kappa_i - 1) = \frac{\alpha p_i}{p}\frac{\alpha p_i - p}{p} \text{ or } \frac{\alpha\kappa_i(\alpha\kappa_i - 1)}{p_i} = \alpha\frac{\alpha p_i - p}{p^2}.$$

Since $\mathbb{E}[Y_i^\gamma] = \mathbb{E}[e^{\gamma v_i - \frac{1}{2}\gamma\sigma_i^2}] = e^{\frac{1}{2}\sigma_i^2\gamma(\gamma-1)} = e^{\frac{1}{2}\gamma(\gamma-1)/p_i}$, one has

$$\mathbb{E}[X^\alpha] = e^{\frac{1}{2}\alpha(\alpha-1)/p_0},$$

and so, by independence and the conditional mean formula,

$$\begin{aligned}
& e^{\frac{1}{2}\alpha(\alpha-1)/p_0} \\
= \;& \mathbb{E}[X^\alpha] = \mathbb{E}[\mathbb{E}[X^\alpha|T_1,...,T_n]], \\
= \;& \mathbb{E}[K_\alpha T_1^{\alpha\kappa_1}...T_n^{\alpha\kappa_n}], \text{ using } \mathbb{E}[X^\alpha|T_1,...,T_n] = K_\alpha T_1^{\alpha\kappa_1}...T_n^{\alpha\kappa_n} \\
= \;& K_\alpha \mathbb{E}[Y_0^{\alpha(1-\kappa_0)}]\mathbb{E}[Y_1^{\alpha\kappa_1}]...\mathbb{E}[Y_n^{\alpha\kappa_n}], \text{ using } T_1^{\alpha\kappa_1}...T_n^{\alpha\kappa_n} = Y_0^{\alpha\kappa_1+...+\alpha\kappa_n} Y_1^{\alpha\kappa_1}...Y_n^{\alpha\kappa_n}, \\
= \;& K_\alpha \exp\frac{\alpha}{2}\left[\frac{(\kappa_1+...+\kappa_n)(\alpha\kappa_1+...+\alpha\kappa_n-1)}{p_0} + \frac{\kappa_1(\alpha\kappa_1-1)}{p_1} + ...\right. \\
& \left. + \frac{\kappa_n(\alpha\kappa_n-1)}{p_n}\right],
\end{aligned}$$

Now

$$\begin{aligned}
& \left[\frac{(p_1+...+p_n)(\alpha\kappa_1+...+\alpha\kappa_n-1)}{pp_0} + \frac{\kappa_1(\alpha\kappa_1-1)}{p_1} + ... + \frac{\kappa_n(\alpha\kappa_n-1)}{p_n}\right] \\
= \;& \left[\frac{(p-p_0)}{p_0}\frac{(\alpha p_1+...+\alpha p_n - p)}{p^2} + \frac{(\alpha p_1 - p)}{p^2} + ... + \frac{(\alpha p_n - p)}{p^2}\right] \\
= \;& \frac{1}{p^2 p_0}\left[(p-p_0)(\alpha p_1+...+\alpha p_n - p) + p_0(\alpha p_1 - p) + ... + p_0(\alpha p_n - p)\right] \\
= \;& \frac{1}{p^2 p_0}\left[(p-p_0)(\alpha p_1+...+\alpha p_n - p) + p_0(\alpha p_1 - p) + ... + p_0(\alpha p_n - p)\right]
\end{aligned}$$

But

$$
\begin{aligned}
& (p-p_0)(\alpha p_1+...+\alpha p_n-p)+p_0(\alpha p_1-p)+...+p_0(\alpha p_n-p) \\
= \; & p_0(-\alpha p_1...-\alpha p_n+p+\alpha p_1-p+...+\alpha p_n-p)+p(\alpha p_1+...+\alpha p_n-p) \\
= \; & p_0(p-p...-p)+p(\alpha p_1+...+\alpha p_n-p)=p(p_0(1-n)+\alpha(p-p_0)-p) \\
= \; & p[p_0(1-n)+p(\alpha-1)-\alpha p_0]=p^2(\alpha-1)+pp_0((1-n)-\alpha).
\end{aligned}
$$

So

$$
\begin{aligned}
\exp\frac{1}{2}\frac{\alpha(\alpha-1)}{p_0} &= K_\alpha \exp\frac{\alpha}{2p^2p_0}\{p^2(\alpha-1)-pp_0(n-1+\alpha)\} \\
&= K_\alpha \exp\frac{\alpha}{2}\left(\frac{\alpha-1}{p_0}-\frac{n-1+\alpha}{p}\right).
\end{aligned}
$$

Finally, we have identified K_α as

$$
\begin{aligned}
K_\alpha &= \exp\frac{\alpha}{2}\left(\frac{\alpha-1}{p_0}-\left(\frac{\alpha-1}{p_0}-\frac{(n-1)+\alpha}{p}\right)\right) \\
&= \exp\frac{n\alpha+\alpha(\alpha-1)}{2p}=\exp\frac{\alpha+\alpha(\alpha-1)/n}{2(p/n)}.
\end{aligned}
$$

A4. The precision matrix

In each of the next two appendices, we will refer to the matrix

$$P_n(x)=P_n-xI,$$

or $P_{n-1}(x)$, the principal sub-matrix omitting the last row and column, where

$$
P_n:=\begin{bmatrix} p_1 & p_2 & \cdots & p_n \\ p_1 & p_2 & \cdots & p_n \\ \vdots & \vdots & \ddots & \vdots \\ p_1 & p_2 & \cdots & p_n \end{bmatrix}.
$$

In applications we shall have $p_i>0$ for all i, since p_i will be the precision parameter $1/\sigma_i^2$. In one context, we shall see that $P_n(-p_0)$ is related to the covariance matrix $Q_n=(\rho_{ij})_{i,j\le n}$. In a further context we shall see that $P_n(p)$, with $p=p_0+...+p_n$ the total precision, is at the heart of the equilibrium conditions for the multiple Dye cutoffs. It is easy to solve the equation

$P_n(-q)x = s$, but to see that the solution is unique we first check when such a system is non-singular.

Proposition A4.1. *For any n, the characteristic function of the matrix P_n is*

$$\det(P_n - xI) = (-1)^n x^{n-1}(x - p_1 - ... - p_n),$$

equivalently

$$\det(P_n + xI) = x^{n-1}(x + p_1 + ... + p_n).$$

Proof. As P is the (singular) matrix all of whose rows are $(p_1, ..., p_n)$, it has rank 1 and so nullity $n-1$, hence has only one non-zero eigenvalue, say $\bar{\lambda}$, the others being zero (with multiplicity $n-1$, since the null space is the eigenspace to the eigenvalue 0). Hence, since the trace of P is the sum of the eigenvalues,

$$\bar{\lambda} = \mathrm{tr}(P) = p_1 + ... + p_n.$$

Thus the characteristic polynomial of P is seen to be

$$\det(P - xI) = (-1)^n x^{n-1}(x - \bar{\lambda}) = (-1)^n x^{n-1}(x - p_1 - ... - p_n).$$

The leading coefficient is verified by a comparison of both sides (which also identifies $\bar{\lambda}$ as the trace of P). $\square$

The following result, used several times in the paper, is an immediate corollary.

Proposition A4.2. *The simultaneous system of equations*

$$p_1x_1 + ... + (p_i + q)x_i + ... + p_nx_n = s_i,$$

has, for any non-zero parameter q such that $p_q := q + p_1 + ... + p_n \neq 0$, the unique solution

$$x_i = \frac{s_i}{q} - c, \text{ where } c = \frac{1}{qp_q}(p_1s_1 + ... + p_ns_n).$$

Proof. The solution formula is easily checked. By Proposition 1, the solution is unique as $\det(P_n + qI) = q^{n-1}(p_1 + ... + p_n + q) \neq 0$. $\square$

In particular we have:

Proposition A4.3. *For $p = p_0 + p_1 + ... + p_n \neq 0$ and $p_{-n} = p - p_n \neq 0$, one has*

$$P_n(-p_0)^{-1} = \begin{bmatrix} \frac{p-p_1}{p_0 p} & -\frac{p_2}{p_0 p} & \cdots & -\frac{p_n}{p_0 p} \\ \vdots & & \ddots & \vdots \\ -\frac{p_1}{p_0 p} & & \cdots & \frac{p-p_n}{p_0 p} \end{bmatrix}, \text{ and likewise}$$

$$P_{n-1}(-p_0)^{-1} = \begin{bmatrix} \frac{p_{-n}-p_1}{p_0 p_{-n}} & -\frac{p_2}{p_0 p_{-n}} & \cdots & -\frac{p_{n-1}}{p_0 p_{-n}} \\ \vdots & & \ddots & \vdots \\ -\frac{p_1}{p_0 p_{-n}} & & \cdots & \frac{p_{-n}-p_{n-1}}{p_0 p_{-n}} \end{bmatrix}.$$

Proof. For the purposes of this proof only, let m denote one of the numbers n or $n-1$. Correspondingly, let $\bar{p} := p_0 + p_1 + ... + p_m$, which denotes p or p_{-n}, as the case may be. Inversion of $P_m(-p_0) = P_m + p_0 I$ is equivalent to solving $(P_m + p_0 I)x = s$ specializing s one by one to the natural base vectors. Writing $s = (s_1, ..., s_m)^T$, we solve the equations:

$$p_1 x_1 + ... + (p_i + p_0)x_i + ... + p_m x_m = s_i.$$

Putting

$$x_i = \frac{s_i}{p_0} + c,$$

one finds that c must satisfy for each i the equation

$$\bar{p}c + s_i + \frac{1}{p_0}(p_1 s_1 + ... + p_m s_m) = s_i,$$

implying that

$$\begin{aligned} c &= -\frac{(p_1 s_1 + ... + p_n s_n)}{\bar{p} p_0}, \text{ and} \\ x_i &= \frac{s_i - ((p_1/\bar{p})s_1 + ... + (p_n/\bar{p})s_n)}{p_0}. \end{aligned}$$

Now specialize s to each of $e_j = (0, ...1, 0, ..., 0)^T$ (viewed as columns of the identity matrix). Fixing on e_j, take s with $s_i = 0$ unless $i = j$, and $s_j = 1$, which yields

$$\begin{aligned} x_i &= -\frac{p_j}{p_0 \bar{p}}, \text{ for } i \neq j, \\ x_j &= \frac{1}{p_0} - \frac{p_j}{p_0 \bar{p}} = \frac{\bar{p} - p_j}{p_0 \bar{p}}. \end{aligned}$$

Thus, the inverse matrix is

$$\begin{bmatrix} \frac{p-p_1}{p_0\bar{p}} & -\frac{p_2}{p_0\bar{p}} & \cdots & -\frac{p_m}{p_0\bar{p}} \\ \vdots & & \ddots & \vdots \\ -\frac{p_1}{p_0\bar{p}} & & \cdots & \frac{p-p_m}{p_0\bar{p}} \end{bmatrix}.$$

A5. Partial covariance - Schur complement

We establish in this appendix an explicit formula for the variance of the distribution of any of the signals T_i conditional on the remaining signals T_j for $j \neq i$, known in Statistics as the *partial covariance* of the signal T_i given the remaining signals T_j for $j \neq i$. For the general, matrix, partial covariance see Bingham & Fry (2010), Note 4.27, p.120 (cf. Kendall & Stuart (1979) Vol. 2 Ch. 27 and Kendall & Stuart (1974), Vol. 3 §§46.26-28). We briefly recall that the (symmetric) covariance matrix $Q = (\rho_{ij})_{i,j \leq n}$, represents the covariance structure of the Hilbert space spanned by the normal random variables $w_1, ..., w_n$ (with covariance as the inner product). The conditional distribution of w_i given w_j for $j \neq i$ is normal with variance given by the partial covariance matrix, known in Linear Algebra as the *Schur complement*, which is a scalar here.

Let Q_{-i} be obtained from Q by omitting the i-th row and column, and let $\vec{\rho}_{-i}$ denote the i-th row $(\rho_{i1}, ..., \rho_{i,n})$ with its i-th entry omitted. Then the Schur complement (of Q_{-i} in Q) is given by he expression

$$\rho_{ii} - \vec{\rho}_{-i} Q_{-i}^{-1} \vec{\rho}_{-i}^{T} = 1 - \vec{\rho}_{-i} Q_{-i}^{-1} \vec{\rho}_{-i}^{T},$$

so that putting

$$\rho_i := \sqrt{\vec{\rho}_{-i} Q_{-i}^{-1} \vec{\rho}_{-i}^{T}},$$

the Schur complement becomes

$$1 - \rho_i^2.$$

(In this notation the specialization to the $n = 2$ case yields $Q_{-i} = (1)$ and $\vec{\rho}_{-i} = (\rho)$, so that $\rho_i = \rho = \rho_{12}$, for $i = 1, 2$, as per Lemma A1.1.) The *conditional mean* of the conditional distribution of w_i is $\bar{\rho}_i Q_{-i}^{-1} w_{-i}$, where w_{-i} denotes the column $(w_1, ..., w_n)^T$ with i-the entry omitted. The inverse matrix Q_{-i}^{-1} exists, since here Q is positive definite (see the Theorem A5.1 below). Furthermore, since Q_{-i}^{-1} is positive definite, $x^T Q_{-i}^{-1} x > 0$ for non-zero vectors x and so $\rho_i^2 < 1$. So one might thus expect ρ_i^2 to be decreasing in p_j for any $j \neq i$; such is the case – again see Theorem A5.1.

In the current situation it will be enough (by symmetry) to consider the distribution of $\mathbb{E}[T_n|T_1, ..., T_{n-1}]$, or equivalently that of $\mathbb{E}[w_n|w_1, ..., w_{n-1}]$, where we recall that $T_i = e^{\sigma_{0i}w_i - \frac{1}{2}\sigma_{0i}^2}$, with $\sigma_{0i}w_i = \sigma_0 w_0 + \sigma_i v_i$. Put

$$w_n^{n-1} := \mathbb{E}[w_n|w_1, ..., w_{n-1}] = \sum\nolimits_{j<n} \beta_j w_j.$$

Then, by definition and by the conditional mean formula,

$$\begin{aligned} \rho_{in} &= \mathbb{E}[w_i w_n] = \mathbb{E}[\mathbb{E}[w_i w_n|w_1, ..., w_{n-1}]] = \mathbb{E}[w_i \mathbb{E}[w_n|w_1, ..., w_{n-1}]] \\ &= \mathbb{E}[w_i w_n^{n-1}] = \sum\nolimits_{j<n} \beta_j \rho_{ij}. \end{aligned}$$

We solve the system of $n-1$ equations in the β_j for $i < n$

$$\sum\nolimits_{j<n} \rho_{ij}\beta_j = \rho_{in},$$

which in matrix form is $Q_{-n}\beta = \vec{\rho}_{-n}$ by computing explicitly $\beta = Q_{-n}^{-1}\vec{\rho}_{-n}$. (Note that $\vec{\rho}_{-n} := (\rho_{1n}, ..., \rho_{n-1,n})$.) Here, as before Q_{-n} denotes the principal submatrix of the covariance matrix Q_n with n-th row and column omitted:

$$Q_{-n} := (\rho_{ij})_{i,j<n}.$$

Theorem A5.1. *Provided all the precisions p_i are finite and positive, the regression equation*

$$\mathbb{E}[w_n|w_1, ..., w_{n-1}] = \beta_1 w_1 + ... + \beta_{n-1} w_{n-1},$$

which is equivalent to the solution of the system $Q_{-n}\beta = \vec{\rho}_{-n}$, has non-singular matrix Q_{-n} and the equivalent system of equations, for $i = 1, 2, ..., n-1$,

$$\rho_{i1}\beta_1 + ... + \beta_i + ... = \rho_{in},$$

has the unique solution:

$$\beta_i = \frac{p_i + p_0}{p}\rho_{in}.$$

Moreover, the partial covariance corresponding to conditioning w_n on $w_1, .., w_{n-1}$ is

$$1 - \rho_n^2,$$

where

$$\rho_n^2 = \frac{p_{n-1}}{p_0 p_{-n}(p_0 + p_{n-1})}\left[\sum\nolimits_{i=1}^{n-1} p_i(p_{-n} - p_i) + \sum\nolimits_{i<j<n}(p_i + p_j)\sqrt{\frac{p_i p_j}{(p_0 + p_i)(p_0 + p_j)}}\right],$$

and

$$p_{-n} = p_0 + p_1 + ... + p_{n-1} = p - p_n.$$

The expression for ρ_n^2 is increasing in p_i for each $i < n$, and so the partial covariance itself decreases with p_i.

Proof. We use Proposition A4.2 to solve the system of equations. In a later step we will need the equivalent matrix formulation to obtain the partial covariance, so we develop a matrix notation in parallel. Recall from Lemma A1.1 that

$$\rho_{ij} = \frac{\sigma_0^2}{\sigma_{0i}\sigma_{0j}}, \text{ for } i \neq j, \text{ and } \rho_{ii} = 1.$$

Multiply the i-th equation through by σ_{0i} and (with n regarded as fixed) set

$$\sigma = \frac{\sigma_0^2}{\sigma_{0n}} = \sigma_{0i}\rho_{in}.$$

Then the i-th equation becomes

$$\frac{\sigma_0^2}{\sigma_{01}}\beta_1 + ... + \sigma_{0i}\beta_i + ... = \sigma.$$

The corresponding elementary matrix S (which reduces $Q\beta = s$, to $SQ\beta = Ss$) has the diagonal format

$$S = \begin{bmatrix} \sigma_{01} & & & \\ & \sigma_{02} & & \\ & & \ddots & \\ & & & \sigma_{0m} \end{bmatrix}.$$

Now put $\beta_i = b_i/\sigma_{0i}$, to obtain

$$\frac{\sigma_0^2}{\sigma_{01}^2}b_1 + ... + b_i + ... = \sigma.$$

Equivalently, we have in matrix format $SQS^{-1}b = Ss$, where

$$\beta = S^{-1}b = \begin{bmatrix} 1/\sigma_{01} & & & \\ & 1/\sigma_{02} & & \\ & & \ddots & \\ & & & 1/\sigma_{0m} \end{bmatrix} b.$$

But, by Lemma A1.5, $\sigma_0^2/\sigma_{0i}^2 = p_i/(p_i + p_0)$, so putting $x_i = b_i/(p_i + p_0)$, we have for each $i < n$ that

$$p_1 x_1 + ... + (p_i + p_0)x_i + ... = \sigma.$$

Equivalently, in matrix form we have $P'x = SQS^{-1}Rx = Ss$, where

$$b = Rx := \begin{bmatrix} p_1 + p_0 & & & \\ & p_2 + p_0 & & \\ & & \ddots & \\ & & & p_m + p_0 \end{bmatrix} x,$$

and

$$P' = P_m(-p_0) := P_m + p_0 I = \begin{bmatrix} p_1 + p_0 & p_2 & ... & p_m \\ \vdots & & \ddots & \vdots \\ p_1 & & ... & p_m + p_0 \end{bmatrix}$$

As in Proposition A4.2, we spot the obvious constant solution $x_i = \xi$ to these equations with ξ satisfying

$$(p_0 + ... + p_{n-1})\xi = \sigma.$$

So $\xi = \sigma/p_{-n}$, where $p_{-n} = p_0 + ... + p_{n-1} = p - p_n$, and so, by Lemma A1.1

$$\beta_i = \frac{b_i}{\sigma_{0i}} = \frac{p_i + p_0}{p_{-n}} \frac{\sigma}{\sigma_{0i}} = \frac{p_i + p_0}{p_{-n}} \frac{\sigma_0^2}{\sigma_{0i}\sigma_{0n}} = \frac{p_i + p_0}{p_{-n}} \rho_{in}.$$

As for uniqueness, the coefficient matrix of the equations in x_i has determinant

$$\det[P + p_0 I] = p_{-n} p_0^{m-1} > 0,$$

(by Proposition A4.1), so this is non-singular for $0 < p_i < \infty$.

Note also that

$$\det Q = (p_0 + p_1)...(p_0 + p_{n-1}) \det[P + p_0 I] = p_{-n} p_0^{m-1} (p_0 + p_1)...(p_0 + p_m).$$

From Proposition A4.3 the inverse matrix of $P := P_{n-1} + p_0 I$ is

$$P^{-1} = \begin{bmatrix} \frac{p_{-n}-p_1}{p_0 p_{-n}} & -\frac{p_2}{p_0 p_{-n}} & \cdots & -\frac{p_{n-1}}{p_0 p_{-n}} \\ \vdots & & \ddots & \vdots \\ -\frac{p_1}{p_0 p_{-n}} & & \cdots & \frac{p_{-n}-p_{n-1}}{p_0 p_{-n}} \end{bmatrix}.$$

Hence, since $P = S Q_{-n} S^{-1} R$, we may now invert Q_{-n} explicitly as

$$Q_{-n}^{-1} = S^{-1} R P^{-1} S,$$

and this equals

$$\begin{bmatrix} \frac{1}{p_1+p_0} & & & \\ & \frac{1}{p_2+p_0} & & \\ \cdots & & \ddots & \\ & & & \frac{1}{p_{n-1}+p_0} \end{bmatrix} \begin{bmatrix} \frac{p_{-n}-p_1}{p_0 p_{-n}} & -\frac{p_2}{p_0 p_{-n}} & & -\frac{p_{n-1}}{p_0 p_{-n}} \\ -\frac{p_1}{p_0 p_{-n}} & \frac{p_{-n}-p_2}{p_0 p_{-n}} & & \\ \vdots & & \ddots & \vdots \\ -\frac{p_1}{p_0 p_{-n}} & & & \frac{p_{-n}-p_{n-1}}{p_0 p_{-n}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{(p_{-n}-p_1)(p_1+p_0)}{p_0 p_{-n}} & -\frac{p_2}{p_0 p_{-n}} & & -\frac{p_m}{p_0 p_{-n}} \\ -\frac{p_1}{p_0 p_{-n}} & \frac{(p_{-n}-p_2)(p_2+p_0)}{p_0 p_{-n}} & & \\ \vdots & & \ddots & \vdots \\ -\frac{p_1}{p_0 p_{-n}} & & & \frac{(p_{-n}-p_m)(p_{n-1}+p_0)}{p_0 p_{-n}} \end{bmatrix}$$

We now compute the Schur complement (see below for some substitutions) to be

$$1-\begin{bmatrix} \frac{\rho_{1n}}{\sigma_{01}} & \cdots & \cdots & \frac{\rho_{mn}}{\sigma_{0m}} \end{bmatrix} \begin{bmatrix} \frac{(p_{-n}-p_1)(p_1+p_0)}{p_0p_{-n}} & -\frac{p_2}{p_0p_{-n}} & & -\frac{p_m}{p_0p_{-n}} \\ -\frac{p_1}{p_0p_{-n}} & & & \\ \vdots & & \ddots & \\ -\frac{p_1}{p_0p_{-n}} & & & \frac{(p_{-n}-p_{n-1})(p_{n-1}+p_0)}{p_0p_{-n}} \end{bmatrix} \begin{bmatrix} \sigma_{01}\rho_{1n} \\ \vdots \\ \vdots \\ \sigma_{0m}\rho_{mn} \end{bmatrix}$$

$$\begin{aligned}
&= 1-\sum\nolimits_i \frac{(p_{-n}-p_i)(p_i+p_0)}{p_0p_{-n}}\rho_{in}^2-\sum\nolimits_{i<j}\rho_{in}\rho_{jn}\left(\frac{p_i}{p_0p_{-n}}+\frac{p_j}{p_0p_{-n}}\right) \\
&= 1-\sum\nolimits_i \frac{(p_{-n}-p_i)(p_i+p_0)}{p_0p_{-n}}\frac{\sigma_0^4}{\sigma_{0i}^2\sigma_{0n}^2}-\sum\nolimits_{i<j}\frac{\sigma_0^4}{\sigma_{0i}\sigma_{0j}\sigma_{0n}^2}\left(\frac{p_i+p_j}{p_0p_{-n}}\right) \\
&= 1-\sum\nolimits_i \frac{(p_{-n}-p_i)(p_i+p_0)}{p_0p_{-n}}\frac{p_ip_{n-1}}{(p_0+p_i)(p_0+p_{n-1})}-\sum\nolimits_{i<j}\rho_{ij}\frac{\sigma_0^2}{\sigma_{0n}^2}\left(\frac{p_i+p_j}{p_0p_{-n}}\right) \\
&= 1-\sum\nolimits_i \frac{(p_{-n}-p_i)}{(p_0+p_{n-1})}\frac{p_ip_{n-1}}{p_0p_{-n}}-\sum\nolimits_{i<j}\rho_{ij}\frac{p_{n-1}}{(p_0+p_{n-1})}\left(\frac{p_i+p_j}{p_0p_{-n}}\right) \\
&= 1-\frac{p_m}{p_0p_{-n}(p_0+p_m)}\left[\sum\nolimits_{i=1}^{n-1}p_i(p_{-n}-p_i)+\sum\nolimits_{i<j<n}(p_i+p_j)\sqrt{\frac{p_ip_j}{(p_0+p_i)(p_0+p_j)}}\right].
\end{aligned}$$

Thus, setting

$$\rho_n^2=\frac{p_m}{p_0\bar{p}(p_0+p_m)}\left[\sum\nolimits_{i=1}^{m}p_i(\bar{p}-p_i)+\sum\nolimits_{i<j\leq m}(p_i+p_j)\sqrt{\frac{p_ip_j}{(p_0+p_i)(p_0+p_j)}}\right],$$

yields the partial covariance corresponding conditioning w_n on $w_1,..,w_{n-1}$ as

$$1-\rho_n^2.$$

Above in our calculations, we used Lemma A1.5. Finally, we note that as $p_0>0$ and $p_j>0$ both of

$$\frac{p_i}{(p_0+p_i)}=1-\frac{p_0}{(p_0+p_i)} \text{ and } \frac{p_ip_j}{(p_0+p_i)(p_0+p_j)}$$

are increasing in p_i for $i<n$. Hence, so is the expression

$$p_i(\bar{p}-p_i)+(p_i+p_j)\sqrt{\frac{p_ip_j}{(p_0+p_i)(p_0+p_j)}},$$

as $(p_{-n}-p_i)$ is positive and independent of p_i.

A6. Conditional hemi-mean formula

We use the results of the preceding subsections to derive the form of the hemi-mean function corresponding to the conditional distribution of one signal T_i given the value of the other signals T_j. The particular case cited in the result below will be important in later research.

Theorem A6.1 (Conditional hemi-mean formula). *One has*

$$\begin{aligned}
\mathbb{E}[T_1^{\alpha_1\kappa_1}|T_2,...,T_n] &= L_{-1}T_2^{\alpha_1(\kappa_{-1}^2-\kappa_2)}...T_n^{\alpha_1(\kappa_{-1}^n-\kappa_n)}, \\
\text{where } L_{-1} &= \exp\left(\frac{(n-1)\alpha_1+\alpha_1(\alpha_1-1)}{2(p-p_1)}\right)\exp\left(-\frac{n\alpha_1+\alpha_1(\alpha_1-1)}{2p}\right), \\
\text{and } \kappa_{-1}^j &= \frac{p_j}{p-p_1}, \text{ for } j>1.
\end{aligned}$$

More generally,

$$\begin{aligned}
\mathbb{E}[T_i^{\alpha_i\kappa_i}|(T)_{-i}] &= L_{-i}\prod\nolimits_{j\neq i}T_j^{\alpha_i(\kappa_{-i}^j-\kappa_j)}, \\
\text{where } L_{-i} &= \exp\left(\frac{(n-1)\alpha_i+\alpha_i(\alpha_i-1)}{2(p-p_i)}\right)\exp\left(-\frac{n\alpha_i+\alpha_i(\alpha_i-1)}{2p}\right), \\
\text{and } \kappa_{-i}^j &= \frac{p_j}{p-p_i}.
\end{aligned}$$

Hence, for any γ,

$$\mathbb{E}[T_i^{\alpha_i\kappa_i}1_{T_i<\gamma}|(T)_{-i}] = L_{-i}\prod\nolimits_{j\neq i}T_j^{\alpha_i(\kappa_{-i}^j-\kappa_j)}\Phi_{\mathrm{LN}}(\gamma^{\alpha_i\kappa_i}/L_{-i}\prod\nolimits_{j\neq i}T_j^{\alpha_i(\kappa_{-i}^j-\kappa_j)},\alpha_i\kappa_i\sigma_{0i}\sqrt{1-\rho_i^2}),$$

and in particular, since $\kappa_{-i}^j-\kappa_j=\kappa_{-i}^j\kappa_i$,

$$\mathbb{E}[(\gamma_i^{\alpha_i\kappa_i}-T_i^{\alpha_i\kappa_i})1_{T_i<\gamma}|(T)_{-i}] = \gamma_i^{\kappa_i}\Phi_{\mathrm{N}}\left(G_i^+(...,T_j^{\kappa_{-i}^j},...)\right)-L_{-i}\prod\nolimits_{j\neq i}T_j^{\alpha_i\kappa_{-i}^j\kappa_i}\Phi_{\mathrm{N}}\left(G_i^-(...,T_j^{\kappa_{-i}^j},...)\right),$$

where, for $t=(t_1,...,t_n)$

$$G_i^{\pm}(t) = \frac{\sum_{j\neq i}\log(\gamma_i^{\alpha_i\kappa_i}/L_{-i}t_j^{\alpha_i\kappa_i})\pm\frac{1}{2}\alpha_i^2\kappa_i^2\sigma_{0i}^2\rho_i}{\alpha_i\kappa_i\sigma_{0i}\sqrt{1-\rho_i^2}}.$$

Proof. For any $\kappa>0$, the random variable $S=T_1^\kappa$ has mean $m=m(\kappa,\sigma_{01})$ and volatility $\kappa\sigma_{01}$. Hence, by Prop. A2.1 (Exponent effect),

$$H_S(\gamma^\kappa) = mH_{\mathrm{LN}}(\gamma^\kappa/m,\kappa\sigma_{01}).$$

The distribution of S conditional on $T_2 = t_2, ..., T_2 = t_n$ (for any $t_2, ..., t_n$) has a mean $\xi = \xi_{-1}$ (depending on $t_2, ..., t_n$ to be determined below) and a volatility $\kappa\sigma_{01}\sqrt{1-\rho_1^2}$, with $1-\rho_1^2$ the partial covariance of T_1 on $(T_2, .., T_n)$, because that is the effect on normal variates of conditioning (see Appendix A5). Thus putting $\eta = \eta_{-1} := m\xi_{-1}$ we have for any $\gamma > 0$ that

$$\begin{aligned} H_{S|t_2...}(\gamma^\kappa) &= \mathbb{E}[(\gamma^\kappa - T_1^\kappa)1_{T_1<\gamma}|T_2 = t_2, ..., T_n = t_n] \\ &= m\xi H_{\mathrm{LN}}(\gamma^\kappa/m\xi, \kappa\sigma_{01}\sqrt{1-\rho_1^2}) \\ &= \gamma^\kappa\Phi_{\mathrm{N}}\left(\frac{\log(\gamma^\kappa/\eta) + \frac{1}{2}\kappa^2\sigma_{01}^2\rho_1}{\kappa\sigma_{01}\sqrt{1-\rho_1^2}}\right) \\ &\quad -\eta\Phi_{\mathrm{N}}\left(\frac{\log(\gamma^\kappa/\eta) - \frac{1}{2}\kappa^2\sigma_{01}^2\rho_1}{\kappa\sigma_{01}\sqrt{1-\rho_1^2}}\right). \end{aligned} \tag{18}$$

This leaves open the determination of the 'constant' $\eta = \eta_{-1}$. But minus the second term has the value

$$\mathbb{E}[T_1^\kappa 1_{T_1<\gamma}|T_2 = t_2, ..., T_n = t_n].$$

So, taking the limit as $\gamma \to +\infty$, we obtain

$$\eta = \eta_{-1} = \mathbb{E}[T_1^\kappa|T_2 = t_2, ..., T_n = t_n].$$

Writing $\alpha\kappa$ for κ, now apply Proposition A3.3 to the $n-1$ firms $2, 3, ...n$ to obtain ***some kind of typo here; maybe: X^α under the expectation, or maybe asserting the intended result with a"writing of $\alpha\kappa_1$ for κ_1 ***

$$\mathbb{E}[T_1^{\alpha\kappa}|T_2 = t_2, ..., T_n = t_n] = A_{-1}t_2^{\alpha\kappa_{-1}^2}...t_n^{\alpha\kappa_{-1}^n},$$

where we recall the regression weights referred to above are $\kappa_{-1}^j = p_i/\left(p-p_1\right)$ and the constant A_{-1} is the $n-1$ firm analogue of the constant K_α derived for Proposition A3.3 so is

$$A_{-1} = \exp\left(\frac{(n-1)\alpha_1 + \alpha_1(\alpha_1 - 1)}{2(p_0 + p_2 + ... + p_n)}\right).$$

Now, by the conditional mean formula, **and Proposition A3.3 to the $n-1$

firms $2, 3, ...n$ **

$$\begin{aligned} A_{-1}t_2^{\alpha\kappa_{-1}^2}...t_n^{\alpha\kappa_{-1}^n} &= \mathbb{E}[X^\alpha|T_2 = t_2, ..., T_n = t_n] \\ &= \mathbb{E}[\mathbb{E}[X^\alpha|T_1, T_2 = t_2, ..., T_n = t_n]|T_2 = t_2, ..., T_n = t_n] \\ &= \mathbb{E}[K_\alpha T_1^{\alpha\kappa_1} t_2^{\alpha\kappa_2}...t_n^{\alpha\kappa_n}|T_2 = t_2, ..., T_n = t_n] \\ &= K_\alpha t_2^{\alpha\kappa_2}...t_n^{\alpha\kappa_n}\mathbb{E}[T_1^{\alpha\kappa_1}|T_2 = t_2, ..., T_n = t_n] \end{aligned}$$

and so ** with κ replaced by $\alpha\kappa_1$, and taking $\alpha = \alpha_1$ ** ** note t_n below was t_2 in error **

$$\begin{aligned} \eta_{-1} &= (A_{-1}K_{\alpha_1}^{-1})t_2^{\alpha_1(\kappa_{-1}^2-\kappa_2)}...t_n^{\alpha_1(\kappa_{-1}^n-\kappa_n)} \\ &= \exp\left(\frac{(n-1)\alpha_1 + \alpha_1(\alpha_1-1)}{2(p_0+p_2+...+p_n)}\right)\exp\left(-\frac{n\alpha_1+\alpha_1(\alpha_1-1)}{2p}\right)t_2^{\alpha_1(\kappa_{-1}^2-\kappa_2)}...t_n^{\alpha_1(\kappa_{-1}^n-\kappa_n)} \\ &= \exp\left(\frac{(n-1)\alpha_1 + \alpha_1(\alpha_1-1)}{2(p-p_1)}\right)\exp\left(-\frac{n\alpha_1+\alpha_1(\alpha_1-1)}{2p}\right)t_2^{\alpha_1(\kappa_{-1}^2-\kappa_2)}...t_n^{\alpha_1(\kappa_{-1}^n-\kappa_n)}, \end{aligned}$$

as required. The rests is now clear from (18) above.

A7. Existence theorem

With the notation of Appendix A6 we first prove a straightforward result (obtained by a series of canceling factorizations). This is where the *related hypothetical firms* arise.

Theorem A7.1 (Uncoupling Theorem). *The substitution*

$$y_1 = \gamma_1^{\alpha_1\kappa_1}/L_{-1}\gamma_2^{\alpha_1(\kappa_{-1}^2-\kappa_2)}..\gamma_n^{\alpha_1(\kappa_{-1}^n-\kappa_n)}$$

reduces the Dye equation (11), namely

$$\begin{aligned} &\lambda_1\left(\mu_1^{\alpha_1}(\gamma_2,...,\gamma_n) - \mu_X^{\alpha_1}(\gamma_1(\gamma_2),\gamma_2,...,\gamma_n)\right) \\ = &\int_{t_1<\gamma_1}(\mu_X^{\alpha_1}(\gamma_1,\gamma_2,...,\gamma_n) - \mu_X^{\alpha_1}(t_1,\gamma_2,...,\gamma_n))d\Phi_{T_1}(t_1|\gamma_2,...,\gamma_n), \end{aligned}$$

to the standard form

$$\lambda_1(1-y_1) = H_{\mathrm{LN}}(y_1, \alpha_1\kappa_1\sigma_{01}\sqrt{1-\rho_1^2}),$$

where $1-\rho_1^2$ *is the partial covariance of* w_1 *on* $w_2, ..., w_n$.

Proof. Recall that $\mu_X^\alpha(t_1, ..., t_n) = K_\alpha t_1^{\alpha\kappa_1} ... t_n^{\alpha\kappa_n}$, so that we should transform the Dye equation for $\alpha = \alpha_1$ using

$$\begin{aligned} X_1 &= \mu_X^\alpha(T_1, \gamma_2, .., \gamma_n) = K_\alpha T_1^{\alpha\kappa_1}\gamma_2^{\alpha\kappa_2}...\gamma_n^{\alpha\kappa_n}, \text{ and} \\ x_1 &= \mu_X(t_1, \gamma_2, ..., \gamma_n) = K_\alpha t_1^{\alpha\kappa_1}\gamma_2^{\alpha\kappa_2}...\gamma_n^{\alpha\kappa_n}. \end{aligned}$$

Furthermore, in the notation of Section 3

$$\mathbb{E}[T_1^{\alpha\kappa_1}|T_2 = \gamma_2, ..., T_n = \gamma_n] = \eta_{-1} = L_{-1}\gamma_2^{\alpha(\kappa_{-1}^2 - \kappa_1)}...\gamma_n^{\alpha(\kappa_{-1}^n - \kappa_1)}.$$

Hence,

$$\mu_1^\alpha(\gamma_2, .., \gamma_n) := \mathbb{E}[K_\alpha T_1^{\alpha\kappa_1}\gamma_2^{\alpha\kappa_2}...\gamma_n^{\alpha\kappa_n}|T_2 = \gamma_2, ..., T_n = \gamma_n] = K_\alpha\eta_{-1}\gamma_2^{\alpha\kappa_2}......\gamma_n^{\alpha\kappa_n}.$$

The transformations above yield

$$\lambda_1\left(\mu_1(\gamma_2) - x_1\right) = H_{X_1}(x_1).$$

However, since the factor $K_\alpha\gamma_2^{\alpha\kappa_2}......\gamma_n^{\alpha\kappa_n}$ is in fact common, it is preferable to use the transformations

$$S_1 = T_1^{\alpha\kappa_1} \text{ and } s_1 = t_1^{\alpha\kappa_1}.$$

Indeed, on substitution we obtain

$$\begin{aligned} &\lambda_1\left(K_\alpha\gamma_2^{\alpha\kappa_2}...\gamma_n^{\alpha\kappa_1}\eta_{-1}^\alpha - K_\alpha\gamma_1^{\alpha\kappa_1}...\gamma_n^{\alpha\kappa_n}\right) \\ = &\int_{t_1<\gamma_1}[K_\alpha\gamma_1^{\alpha\kappa_1}...\gamma_n^{\alpha\kappa_n} - K_\alpha t_1^{\alpha\kappa_1}\gamma_2^{\alpha\kappa_2}...\gamma_n^{\alpha\kappa_n}]d\Phi_{T_1}(t_1|\gamma_2), \end{aligned}$$

so that after cancelling by $K_\alpha\gamma_2^{\alpha\kappa_2}...\gamma_n^{\alpha\kappa_n}$ the Dye equation reads:

$$\begin{aligned} &\lambda_1\left(\eta_{-1} - \gamma_1^{\alpha\kappa_1}\right) \\ = &\int_{t_1<\gamma_1}[\gamma_1^{\alpha\kappa_1} - t_1^{\alpha\kappa_1}]d\Phi_{T_1}(t_1|\gamma_2) = H_{S_1|\gamma_2...}(\gamma_1^{\alpha\kappa_1}). \end{aligned}$$

But

$$\begin{aligned} H_{S_1|\gamma_2...}(\gamma_1^{\alpha\kappa_1}) &= \gamma_1^{\alpha\kappa_1}\Phi_{\mathrm{N}}\left(\frac{\log[\gamma_1^{\alpha\kappa_1}/\eta_{-1}] + \frac{1}{2}\alpha^2\kappa_1^2\sigma_{01}^2(1-\rho_1^2)}{\alpha\kappa_1\sigma_{01}\sqrt{1-\rho_1^2}}\right) \\ &\quad -\eta_{-1}\Phi_{\mathrm{N}}\left(\frac{\log[\gamma_1^{\alpha\kappa_1}/\eta_{-1}] - \frac{1}{2}\alpha^2\kappa_1^2\sigma_{01}^2(1-\rho_1^2)}{\alpha\kappa_1\sigma_{01}\sqrt{1-\rho_1^2}}\right). \end{aligned}$$

So putting

$$y_1 = \gamma_1^{\alpha\kappa_1} / L_{-1} \gamma_2^{\alpha(\kappa_{-1}^2 - \kappa_2)} .. \gamma_n^{\alpha(\kappa_{-1}^n - \kappa_n)},$$

we obtain

$$\begin{aligned} & \lambda_1 \left(\eta_{-1} - \gamma_1^{\alpha\kappa_1} \right) \\ = \; & y_1 \eta_{-1} \Phi_{\mathrm{N}} \left(\frac{\log y_1 + \frac{1}{2}\alpha^2 \kappa_1^2 \sigma_{01}^2 \rho_1}{\alpha\kappa_1 \sigma_{01} \sqrt{1 - \rho_1^2}} \right) - \eta_{-1} \Phi \left(\frac{\log y_1 - \frac{1}{2}\alpha^2 \kappa_1^2 \sigma_{01}^2 \rho_1}{\alpha\kappa_1 \sigma_{01} \sqrt{1 - \rho_1^2}} \right). \end{aligned}$$

Dividing by η_{-1} yields

$$\lambda_1 (1 - y_1) = H_{\mathrm{LN}} \left(y_1, \alpha\kappa_1 \sigma_{01} \sqrt{1 - \rho_1^2} \right).$$

More generally:

$$\lambda_i (1 - y_i) = H_{\mathrm{LN}} \left(y_i, \alpha\kappa_i \sigma_{0i} \sqrt{1 - \rho_i^2} \right).$$

Above $1 - \rho_i^2$ denotes as usual the partial covariance appropriate to conditioning of w_i on the remaining variables w_j for $j \neq i$ (equivalently: conditioning T_i on the remaining signals, i.e. on T_j for $j \neq i$). (In the two-firm case $1 - \rho_i^2 = 1 - \rho^2$.) For the general case, recall that since

$$T_i = e^{\sigma_{0i} w_i - \frac{1}{2}\sigma_{0i}^2}, \text{ with } \sigma_{0i} w_i = \sigma_0 v_0 + \sigma_i v_i, \sigma_{0i}^2 = \sigma_0^2 + \sigma_i^2$$

one has, as before,

$$\rho_{ij} = cov(w_i, w_j) = cov\left(\frac{\sigma_0 v_0 + \sigma_i v_i}{\sigma_{0i}}, \frac{\sigma_0 v_0 + \sigma_j v_j}{\sigma_{0j}}\right) = \frac{\sigma_0^2}{\sigma_{0i}\sigma_{0j}}, \; i \neq j,$$

and of course

$$\rho_{ii} = cov(w_i, w_i) = cov\left(\frac{\sigma_0 v_0 + \sigma_i v_i}{\sigma_{0i}}, \frac{\sigma_0 v_0 + \sigma_i v_i}{\sigma_{0i}}\right) = \frac{\sigma_0^2 + \sigma_i^2}{\sigma_{0i}\sigma_{0i}} = 1.$$

(This specializes in the $n = 2$ case, to $\rho_{12} = \rho$.)

In our multi-firm model our starting point are the n equations, which follow (by cross-multiplication) from the preceding theorem:

$$\gamma_i^{\alpha_i \kappa_i} = \hat{\gamma}_i L_{-i} \prod_{j \neq i} \gamma_j^{\alpha_i (\kappa_{-i}^j - \kappa_j)} = \hat{\gamma}_i L_{-i} \prod_{j \neq i} \gamma_j^{\alpha_i \kappa_i \kappa_{-i}^j}, \text{ for } i = 1, ..., n.$$

On taking logarithms, these reduce to the log-linear system:

$$x_i - \sum\nolimits_{j \neq i} \kappa_{-i}^{j}\ x_j = A_i := \frac{1}{\alpha_i \kappa_i} \log\left(\hat{\gamma}_i L_{-i}\right) = \frac{p}{\alpha_i p_i} \log\left(\hat{\gamma}_i L_{-i}\right),$$

with $x_i = \log \gamma_i$, in view of Lemma A1.2 (as in Proposition A4.3). The more revealing re-statement is obtained by recalling that as in (15)

$$\kappa_{-i}^{j} = p_j / \left(p - p_i\right),$$

so that substituting this and then cross-multiplying by $(p_i - p)/p = \kappa_i - 1$ yields

$$(\kappa_i - 1)x_i + \sum\nolimits_{j \neq i} \kappa_j x_j = b_i := \frac{(p_i - p)}{\alpha_i p_i} \log\left(\hat{\gamma}_i L_{-i}\right).$$

On the left-hand side this presents a weighted average of the unknowns x_i, pointing towards a solution that is also a weighted average of the right-hand side constants, the b_i's. These are the cutoffs of the *related hypothetical firms* mentioned in the introduction.

We show that the system has non-singular coefficient matrix iff the model is non-degenerate for $0 < p_0 < \infty$ (assuming all other p_i are finite and positive), so that as before one may solve for $\log \gamma_i$.

Put $h_i^i = 1, h_j^i = -\kappa_{-i}^{j} = p_j/(p_i - p)$, then $H = (h_j^i)$ is the coefficient matrix, above. We assume none of the parameters p_i is infinite.

Theorem A7.2 (Regression-adjusted univariate cutoffs)

$$\begin{aligned} \det H &= \frac{p_0 p^{n-1}}{(p - p_1)(p - p_2)...(p - p_n)} = \frac{\kappa_0 \kappa_{-1} ... \kappa_{-n}}{\kappa_1 ... \kappa_n}, \\ \text{with } \kappa_{-i} &= p_i/(p - p_i). \end{aligned}$$

Hence H is finite iff each p_i is finite for $i \geq 0$ and $\sum_{j=0,\, j \neq i}^{n} p_j > 0$ for each $i > 0$. Under these circumstances it is non-singular iff $p_0 > 0$, in which case the unique solution is given by the κ_i-weighted averaging of the terms $(\log g_i)/\alpha_i \kappa_{-i}$:

$$\log \gamma_i = \frac{\log g_i}{\alpha_i \kappa_{-i}} + \frac{1}{\kappa_0} \left(\frac{\kappa_1}{\alpha_1 \kappa_{-1}} \log g_1 + \frac{\kappa_2}{\alpha_2 \kappa_{-2}} \log g_2 + ... + \frac{\kappa_n}{\alpha_n \kappa_{-n}} \log g_n \right),$$

where the loading α_i reflects the dependence of firm f_i on the operating environment. That is, the Dye disclosure cutoff for the signal T_i is

$$\begin{aligned}
\gamma_i &= g_i^{(1+\kappa_i)/\kappa_{-i}} \prod\nolimits_{j\neq i}^{n} g_j^{\kappa_j/\kappa_0\kappa_{-j}}, \text{ where } \kappa_{-i} = \frac{p_i}{p-p_i} \text{ and} \\
g_i &= \hat{\gamma}_{\mathrm{LN}}\left(\lambda_i, \alpha_i\kappa_i\sigma_{0i}\sqrt{1-\rho_i^2}\right) L_{-i}, \ \lambda_i = \frac{1-q_i}{q_i}, \\
L_{-i} &= \exp\left(\frac{(n-1)\alpha_i + \alpha_i(\alpha_i-1)}{2(p-p_i)}\right)\exp\left(-\frac{n\alpha_i + \alpha_i(\alpha_i-1)}{2p}\right),
\end{aligned}$$

and where:
κ_i *is the regression coefficient of* w_i
κ_{-i} *is a regression coefficient resulting from the removal of firm-i's contribution to total precision*
$1-\rho_i^2$ *is the partial covariance of* w_i *on the remaining variates* w_j*, and*
L_{-i} *is the relative adjustment coefficient while* $\gamma = \hat{\gamma}_{\mathrm{LN}}(\lambda,\sigma)$ *solves*

$$\lambda(1-\gamma) = H_{\mathrm{LN}}(\gamma,\sigma).$$

Remarks: 1. For each i the weights applied to $(\log g_j)/\alpha_j\kappa_{-j}$ have the same sum:

$$1 + \frac{1-\kappa_0}{\kappa_0} = \frac{1}{\kappa_0} = \frac{p}{p_0}.$$

2. Note that

$$\begin{aligned}
\frac{p_i}{p}\frac{p-p_i}{\alpha_i p_i}\log L_{-i} &= \frac{p_i}{p}\frac{p-p_i}{\alpha_i p_i}\cdot\frac{\alpha_i}{2}\left(\frac{\alpha_i + (n-2) - p_i/(p-p_i)}{p}\right) \\
&= \left(\frac{(p-p_i)[\alpha_i + (n-2)] + p - p_i - p}{p^2}\right) \\
&= \frac{(p-p_i)[\alpha_i + (n-1)] - p}{p^2}
\end{aligned}$$

Proof. Rewriting the given system $i = 1, ..., n$ with $n \geq 2$,

$$x_i + \sum\nolimits_{j\neq i} \frac{p_j}{p_i - p} x_j = \frac{p}{\alpha_i p_i}\log g_i, \text{ with } g_i = \hat{\gamma}_i L_{-i},$$

this time in the form

$$(p_i - p)x_i + \sum\nolimits_{j\neq i} p_j x_j = B_i := \frac{p(p_i-p)}{\alpha_i p_i}\log g_i,$$

we see that the coefficient matrix is now $P_n - pI$, where P_n is the (full) precision matrix. By Proposition A4.1,

$$\det H = \frac{(-1)^n p^{n-1}(p - p_1 - \dots - p_n)}{(p - p_1)(p_2 - p)\dots(p_n - p)} = \frac{p_0 p^{n-1}}{(p_1 - p)(p_2 - p)\dots(p_n - p)},$$

as asserted.

Finally, by Proposition A4.2, as $p_q := -p + p_1 + \dots + p_n = -p_0 \neq 0$, the solution format is

$$x_i = \log \gamma_i = -\frac{B_i}{p} - c = \frac{(p - p_i)}{\alpha_i p_i} \log(g_i) - c = \frac{\log g_i}{\alpha_i \kappa_{-i}} - c$$

with

$$\begin{aligned} -c &= -\frac{1}{q p_q}(p_1 s_1 + \dots + p_n s_n) = \frac{1}{p p_0}(p_1 B_1 + \dots + p_n B_n) \\ &= \frac{1}{p p_0}(\dots + \frac{p(p_i - p)}{\alpha_i} \log(g_i) + ..) = \frac{1}{p_0}(\dots + \frac{(p_i - p)}{\alpha_i} \log(g_i) + \dots) \\ &= \frac{p}{p_0}(\dots + \frac{(p_i - p)/p_i}{\alpha_i p / p_i} \log(g_i) + ..) = \frac{1}{\kappa_0}(\dots + \frac{\kappa_i}{\alpha_i \kappa_{-i}} \log(g_i) + \dots) \end{aligned}$$

i.e.

$$x_i = \log \gamma_i = \frac{\log g_i}{\alpha_i \kappa_{-i}} - c, \;\; i = 1, \dots, n,$$

with

$$-c = \frac{1}{\kappa_0} \left(\frac{\kappa_1}{\alpha_1 \kappa_{-1}} \log g_1 + \frac{\kappa_2}{\alpha_2 \kappa_{-2}} \log g_2 + \dots + \frac{\kappa_n}{\alpha_n \kappa_{-n}} \log g_n \right),$$

which is indeed a constant (i.e. independent of i).

A8. The indifference condition

In this Appendix we deduce the Indifference Principle which we used in Section 3 as the equilibrium condition identifying the firms' cutoffs γ_i.

Definition. For the manager of firm i and any random variable T, consider the experiment that with probability q_i the random variable T may be observed, and let $ND_i(T, \delta)$ denote the event that either T is not observed by manager i or T is observed but not disclosed by i because $T \leq \delta$. We write below $ND_i(\gamma_i)$ for $ND_i(T_i, \gamma_i)$.

Definition. Say that given γ_{-i} the cutoff γ_i is *rational for firm* i if

$$\mathbb{E}[X|(\forall i \neq j)ND_i(\gamma_i), T_j = \gamma_j] = \mathbb{E}[X|(\forall i)ND_i(\gamma_i)].$$

We show below that given γ_{-i} there is a unique $\gamma_i = \gamma_i(\gamma_{-i})$ which is rational for firm i. Granted this, we say that the cutoff profile $\gamma = (\gamma_1, ...\gamma_n)$ is *individually rational* if γ_j for all j satisfies

$$\gamma_j = \gamma_j(\gamma_{-j}).$$

Theorem (Indifference Principle). *Suppose that each regression function $\mu_j(t) := \mathbb{E}[X|(\forall i \neq j)ND_i(\gamma_i), T_j = t]$ is strictly increasing and that $S_j := \mathbb{E}[X|(\forall i \neq j)ND_i(\gamma_i), T_j]$ has a strictly positive, continuous density function. Then for each γ_{-i} there exists a unique value $\gamma_i := \gamma_i(\gamma_{-i})$ such that*

$$\mathbb{E}[X|(\forall i \neq j)ND_i(\gamma_i), T_j = \gamma_j(\gamma_{-j})] = \mathbb{E}[X|(\forall i)ND_i(\gamma_i)].$$

Suppose that $\gamma = (\gamma_1, ...\gamma_n)$ is individually rational, i.e. satisfies for all j

$$\gamma_j = \gamma_j(\gamma_{-j}).$$

Then for each i one has

$$\mathbb{E}[X|ND_i(\gamma_i), T_{-i} = \gamma_{-i}] = \mathbb{E}[X|T = \gamma], \text{ where } T = (T_1, ..., T_n).$$

Proof. Define

$$V(\gamma_1, ..., \gamma_n) := \mathbb{E}[X|(\forall i)ND_i(\gamma_i)]. \tag{19}$$

The expression will also be written as

$$V(\gamma_i; \gamma_{-i}).$$

Furthermore, put

$$\mu_j(t_j, \gamma_{-j}) := \mathbb{E}[X|(\forall i \neq j)ND_i(\gamma_i), T_j = t_j].$$

We have assumed (similarly as in the main part of the paper) that $\mu_j(t_j, \gamma_{-j})$ is strictly increasing in t_j. We put

$$S_j := \mu_j(T_j, \gamma_{-j}) = \mathbb{E}[X|(\forall i \neq j)ND_i(\gamma_i), T_j]. \tag{20}$$

We have assumed that S_j has a strictly positive, continuous density function. Hence, Theorem 3 (The Minimum Principle) of Ostaszewski and Gietzmann (2008) – see Section 2.1 – may be applied to the (single) random variable S_j, and so there exists for any γ_{-j} a unique

$$\gamma_{S_j} = \gamma_{S_j}(\gamma_{-j})$$

satisfying both

$$\gamma_{S_j} = \arg\min_s \mathbb{E}[S_j|ND_j(S_j, s)],$$

and additionally

$$\gamma_{S_j} = \mathbb{E}[S_j|ND_j(S_j, \gamma_{S_j})]. \tag{21}$$

We may now define γ_j (and express its dependence on γ_{-j} by writing $\gamma_j(\gamma_{-j})$) by setting

$$\gamma_{S_j} = \mu_j(\gamma_j, \gamma_{-j}). \tag{22}$$

Then $t_j \leq \gamma_j$ iff $s_j \leq \gamma_{S_j}$, where $s_j := \mu_j(t_j, \gamma_{-j})$, because $\mu_j(., \gamma_{-j})$ is assumed increasing. So

$$S_j \leq \gamma_{S_j} \text{ iff } (\forall i \neq j)ND_i(\gamma_i) \text{ and } T_j \leq \gamma_j,$$

and so

$$ND_j(S_j, \gamma_{S_j}) \text{ iff } (\forall i \neq j)ND_i(\gamma_i) \text{ and } ND_j[T_j, \gamma_j],$$

i.e.

$$ND_j(S_j, \gamma_{S_j}) \text{ iff } (\forall i)ND_i(\gamma_i). \tag{23}$$

So, using the conditional expectation formula ('tower law') in the last but one line below, one has

$$\begin{aligned}
\mu_j(\gamma_j, \gamma_{-j}) &= \gamma_{S_j}, \text{ by (22)}, \\
&= \mathbb{E}[S_j|ND_j(S_j, \gamma_{S_j})], \text{ by (21)}, \\
&= \mathbb{E}[S_j|(\forall i)ND_i(\gamma_i)], \text{ by (23)}, \\
&= \mathbb{E}[\mathbb{E}[X|(\forall i \neq j)ND_i(\gamma_i), T_j]|(\forall i)ND_i(\gamma_i)], \text{ by (20)}, \\
&= \mathbb{E}[X|(\forall i)ND_i(\gamma_i)] \text{ (tower law)} \\
&= V(\gamma_j(\gamma_{-j}); \gamma_{-j}), \text{ by (19)}.
\end{aligned}$$

Combining the first and last elements of this chain of equation, for given γ_{-j} the cutoff γ_j identified above satisfies

$$\mu_j(\gamma_j, \gamma_{-j}) = V(\gamma_j(\gamma_{-j}); \gamma_{-j}),$$

or, using their definitions,

$$\mathbb{E}[X|(\forall i \neq j)ND_i(\gamma_i), T_j = \gamma_j] = \mathbb{E}[X|(\forall i)ND_i(\gamma_i)]. \tag{24}$$

We now consider a profile $\gamma = (\gamma_1, ..., \gamma_n)$ satisfying *individual rationality*, that is for each j one demands that

$$\gamma_j = \gamma_j(\gamma_{-j}).$$

Assuming such a profile γ exists, and assuming $T_j = \gamma_j$ is observed by all j, one may unravel the non-disclosure conditioning by fixing the conditions $T_j = \gamma_j$ one by one and using (24) inductively for multifirm environments with successively fewer firms. This yields:

$$\begin{aligned}
& V(\gamma_1, ..., \gamma_n) \\
= \ & \mathbb{E}[X|(\forall i > 1)ND_i(\gamma_i), T_1 = \gamma_1] \\
= \ & \mathbb{E}[X|(\forall i > 2)ND_i(\gamma_i), T_2 = \gamma_2, T_1 = \gamma_1] \\
= \ & ... \\
= \ & \mathbb{E}[X|ND_n(\gamma_n), ..., T_2 = \gamma_2, T_1 = \gamma_1] \\
= \ & \mathbb{E}[X|T_n = \gamma_n, ..., T_2 = \gamma_2, T_1 = \gamma_1].
\end{aligned}$$

In particular, we notice from the last two lines that

$$\mathbb{E}[X|ND_n(\gamma_n), ..., T_2 = \gamma_2, T_1 = \gamma_1] = \mathbb{E}[X|T_n = \gamma_n, ..., T_2 = \gamma_2, T_1 = \gamma_1],$$

with analogous switches between disclosing and non-disclosing holding for other firms (by symmetry). But this is just the indifference principle (9) ('equilibrium condition') of Section 3. $\square$