Mahavier completeness and classifying diagrams

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Abstract

Generalised inverse limits of compacta were introduced by Ingram and Mahavier in 2006. The main difference between ordinary inverse limits and their generalised cousins is that the former concerns diagrams of singlevalued functions while the latter permits multivalued functions. However, generalised inverse limits are not merely limits in the Kleisli category of a hyperspace monad, a fact that independently motivated each of the authors of this article to come up with the same formalism which restores the link with category theory through the concept of Mahavier limit of an order diagram in an order extension of a category $\mathscr{B}$. Mahavier limits of diagrams in $\mathscr{B}$ coincide with ordinary limits in $\mathscr{B}$, and so Mahavier limits are an extension of ordinary limits along the functor that views an ordinary diagram as a diagram in the extension. Within that context it is natural to consider Mahavier completeness, namely when all small diagrams admit Mahavier limits, as well as classifying diagrams, namely the existence of a right adjoint to the mentioned functor on diagrams. In this work we show that these two conditions are equivalent, and we study some of the properties of classifying diagrams and of the adjunction.

Keywords: Generalised inverse limit, Mahavier limit, classifying diagram, inverse system, generalised inverse system, category with order, generalised categorical limit, multivalued function, upper semicontinuous function
2010 MSC: 54C60, 18A05, 18D99, 54B30,

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1. Introduction

Generalised inverse limits of compacta were introduced by Ingram and Mahavier in 2006 in [1] and have since received much attention (e.g., [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24]). Recall that an inverse limit of a sequence

$$\cdots \longrightarrow X_{n+1} \xrightarrow{f_n} X_n \xrightarrow{f_{n-1}} \cdots \longrightarrow X_2 \xrightarrow{f_1} X_1$$

Post-print version to appear in Topol. Appl. *September 2, 2017*

of spaces and continuous functions is the space $X = \{\boldsymbol{x} \in \prod X_n \mid \boldsymbol{x}_n = f_n(\boldsymbol{x}_{n+1})\}$, viewed as a subspace of the product space. The passage to generalised inverse limits occurs by allowing the bonding functions $f_n \colon X_{n+1} \to X_n$ to be upper semicontinuous set-valued functions $f_n \colon X_{n+1} \rightsquigarrow X_n$, and by altering the definition of the space X to become $X = \{\boldsymbol{x} \in \prod X_n \mid \boldsymbol{x}_n \in f_n(\boldsymbol{x}_{n+1})\}$. The formal resemblance to inverse limits makes the generalised version very palatable. The hoard of interesting spaces that arise as generalised inverse limits of very simple diagrams with multivalued bonding functions of compacta (see [11, 25] for detailed examples), together with highly non-trivial ramification of the subtle change in definition from singlevaluedness to multivaluedness, and from equality to membership, contribute even more to the appeal of this relatively new area of research.

Of course, inverse limits of spaces are nothing but categorical limits in the category **Top** of topological spaces and continuous mappings, and it is natural to ask whether the slogan generalises. Results addressing some categorical aspects of generalised inverse limits directly can be found in [4, 26], but they were only partially successful in fully restoring the link with category theory, and the difficulty can be traced to the following phenomenon. Consider the functor $T \colon \mathbf{Top} \to \mathbf{Top}$ which maps a space X to $T(X)$, the space of all subsets of X, endowed with the upper Vietoris topology. This hyperspace functor has a natural structure of a monad whose multiplication is given by taking unions. Let $\mathbf{Top}_T$ be the Kleisli category of T, i.e., the objects of $\mathbf{Top}_T$ are all spaces and a morphism $X \rightsquigarrow Y$ is a continuous function $X \to T(Y)$. It is easily seen that these are precisely the upper semicontinuous functions. In other words, the diagrams for generalised inverse limits of spaces are precisely diagrams in $\mathbf{Top}_T$. However, generalised inverse limits in **Top** are not simply limits in $\mathbf{Top}_T$ (an expected reality since limits in Kleisli categories are notoriously ill-behaved ([27]), while generalised inverse limits are much more tame).

The authors of this article independently found the same categorical formalism to fully restore the link between generalised inverse limits of spaces and category theory. In [18] the first named author developed a notion of limit in the category of compacta and upper semicontinuous set-valued functions in such a way that the slogan above is recovered. In [28] the second named author developed a formalism in full generality, allowing for generalised inverse limits to be considered beyond the scope of topology, which specialises to generalised inverse limits of spaces when interpreted in the context of $\mathbf{Top} \subseteq \mathbf{Top}_T$.

The aim of this work is summarised in the diagram

$$\begin{tikzcd}
{[\mathscr{D}, \mathscr{B}]} \arrow[rr, shift left, "i_{\mathscr{D}}"] \arrow[dr, shift left, "\varprojlim"] & & {[\mathscr{D}, \mathscr{C}]_{\mathscr{B}}} \arrow[ll, shift left, "i^*_{\mathscr{D}}"] \arrow[dl, shift left, "\varprojlim^{\mathrm{M}}_{\mathscr{B}}"] \\
& \mathscr{B} \arrow[ul, shift left, "\Delta"] \arrow[ur, shift left, "i_{\mathscr{D}} \circ \Delta"] &
\end{tikzcd}$$

which we briefly explain (all concepts are detailed below). Let $\mathscr{B}, \mathscr{C}$, and $\mathscr{D}$ be categories, assume that $\mathscr{B}$ is a subcategory of $\mathscr{C}$, that $\mathrm{ob}(\mathscr{B}) = \mathrm{ob}(\mathscr{C})$, and moreover that each hom-set in $\mathscr{C}$ is endowed with an ordering, with some conditions. We call $\mathscr{C}$ an *order extension* of $\mathscr{B}$. The ordering allows one to define order variants of functors and of natural transformations by suitably replacing $=$ by $\leq$. One obtains in this way the category $[\mathscr{D}, \mathscr{C}]$ of all order functors $\mathscr{D} \to \mathscr{C}$ and order natural transformations between them. An order natural transformation whose components are morphisms in $\mathscr{B}$ is said to be an *order natural transformation relative* to $\mathscr{B}$, and we then denote by $[\mathscr{D}, \mathscr{C}]_{\mathscr{B}}$ the subcategory of $[\mathscr{D}, \mathscr{C}]$ obtained by restricting to the relative order natural transformations. Let $[\mathscr{D}, \mathscr{B}]$ be the usual category of functors $\mathscr{D} \to \mathscr{B}$ and natural transformations. Since $\mathscr{B}$ is a subcategory of $\mathscr{C}$ there is an inclusion functor $i_{\mathscr{D}} \colon [\mathscr{D}, \mathscr{B}] \to [\mathscr{D}, \mathscr{C}]_{\mathscr{B}}$, depicted at the top of the diagram above. On the left side of the diagram are the diagonal functor $\Delta \colon \mathscr{B} \to [\mathscr{D}, \mathscr{B}]$, mapping an object B to the constantly B functor, and its right adjoint, the functor $\varprojlim$, namely taking limits, provided $\mathscr{D}$-shaped limits in $\mathscr{B}$ exist, e.g., if $\mathscr{B}$ is complete.

On the right side of the diagram is the functor $i_{\mathscr{D}} \circ \Delta$ and its right adjoint $\varprojlim^{\mathrm{M}}_{\mathscr{B}}$ which maps an order diagram $F \colon \mathscr{D} \to \mathscr{C}$ to $\varprojlim^{\mathrm{M}}_{\mathscr{B}} F$, its *Mahavier limit relative* to $\mathscr{B}$, provided these limits exist, e.g., if $\mathscr{C}$ is Mahavier complete relative to $\mathscr{B}$ (i.e., Mahavier limits exist for all small order diagrams). Obviously, the triangle of left adjoints commutes. The result we prove below is that $\mathscr{C}$ is Mahavier complete relative to $\mathscr{B}$ if, and only if, $\mathscr{B}$ is complete and $i_{\mathscr{D}}$ has a right adjoint $i^*_{\mathscr{D}}$, for all small categories $\mathscr{D}$. In that case, the triangle of right adjoints commutes up to a natural isomorphism, and thus $i^*_{\mathscr{D}}$ computes Mahavier limits in the sense that there is a natural isomorphism $\varprojlim^{\mathrm{M}}_{\mathscr{B}} \cong \varprojlim \circ i^*_{\mathscr{D}}$.

The plan of the paper is as follows. Section 2 briefly introduces the terminology above, and Section 3 presents the main result. Properties of the adjunction are studied in Section 4, together with some applications. Finally, Section 5 revisits classical generalised inverse limits, exhibiting, in a rather informal fashion, how categorical Mahavier limit theory meshes with the existing interests and problems in the field.

2. Preliminaries

We briefly present the concepts required for the definition of Mahavier limits. For a much more detailed exposition, stressing motivation and applicability, the reader is referred to [28]. The reader more interested in applications to compacta is referred to [18].

2.1. Order extensions

A main ingredient in the categorical formalisation we consider for generalised inverse limits is the ordering on the hom-sets of $\mathbf{Top}_T$, turning it into an order-enriched category. An ordered category is thus a special form of 2-category, and thus the well-developed theory of 2-categories (see, e.g., [29]) can be applied. To site just a couple of examples where the 2-categorical machinery works very well for particular order-enriched categories we mention [30, 31, 32], which involves a translation of a 2-categorical notion to a condition on a monad known as the Kock-Zöberlein condition, and [33] in the area of ordered universal algebra. However, as noted generally already in [34], the standard 2-categorical constructions yield the 'wrong' results in certain ordered categories arising in computer science. The situation with generalised inverse limits in topology is another case where the 2-categorical notions are inadequate in a particular scenario. Interestingly, even though the motivations are very different, there are some similarities between our notion of Mahavier limits and some of the material in [34], where the notion of near limit is introduced and various lax conditions are given, in the study of partial functions in computer science, capturing some aspects that go back to [35].

Since our motivation is in securing a categorical home for generalised inverse limits in topology, we feel free to deviate from the 2-categorical doctrine. In particular, what we call 'ordered category' is the same as 'order enriched category' but our notion of 'order functor' is not the enriched notion. We make the more permisive choice in order to address even the most esoteric of generalised diagrams considered in the literature on generalised inverse limits. We mention that some aspects of the theory become more 2-categorical if one takes the enriched notion of functor, and there may be good reasons to prefer that. However, the main notion, that of Mahavier limit, remains non-2-categorical. For that reason, we simply spell out the relevant notions, rather than obtain some of them as special cases.

An *ordered category* is a category $\mathscr{C}$ together with an ordering $\leq$ on each hom-set $\mathscr{C}(C, C')$ such that composition is monotone in each variable, i.e., the conditions $c_1 \leq c_2$ and $c_3 \leq c_4$ imply $c_1 \circ c_3 \leq c_2 \circ c_4$, for all morphisms c_1, c_2, c_3, c_4 for which the compositions are defined. An *order functor* $F\colon \mathscr{C} \to \mathscr{C}'$ between ordered categories consists of the same ingredients as a functor, namely an object part and a morphism part, but the preservation of composition is weakened to merely requiring that $F(c_1 \circ c_2) \leq F(c_1) \circ F(c_2)$, for all morphisms $c_1, c_2 \in \mathscr{C}$ for which the composition is defined (though we still demand that $F(\mathrm{id}_C) = \mathrm{id}_{FC}$). Every category $\mathscr{D}$ shall be viewed as an ordered category by endowing each hom-set $\mathscr{D}(D, D')$ with the trivial ordering, namely the identity relation. An order functor $\mathscr{D} \to \mathscr{C}$ is also referred to as an *order diagram* of *shape* $\mathscr{D}$ in $\mathscr{C}$. Given order functors $F_1, F_2\colon \mathscr{D} \to \mathscr{C}$, an *order natural transformation* $\alpha\colon F_1 \to F_2$ is a family $\{\alpha_D\}_{D \in \mathscr{D}}$ of morphisms in $\mathscr{C}$ with the property that the inequality $\alpha_{D'} \circ F_1(d) \leq F_2(d) \circ \alpha_D$ holds for all morphisms $d\colon D \to D'$ in $\mathscr{D}$. It is easy to see that the usual vertical composition of natural transformations extends to order natural transformations. In more detail, if $\alpha\colon F_1 \to F_2$ and $\beta\colon F_2 \to F_3$ are order natural transformations between order functors $F_1, F_2, F_3\colon \mathscr{D} \to \mathscr{C}$, then $\beta \circ \alpha\colon F_1 \to F_3$ is the order natural transformation whose component at D is $\beta_D \circ \alpha_D$. For a fixed category $\mathscr{D}$ and an ordered category $\mathscr{C}$ let $[\mathscr{D}, \mathscr{C}]$ denote the category of all order functors $F\colon \mathscr{D} \to \mathscr{C}$ as objects and all order natural transformations as morphisms.

Let $\mathscr{B}$ be a category. An *order extension* of $\mathscr{B}$ is an ordered category $\mathscr{C}$ of which $\mathscr{B}$ is a subcategory, with $\mathrm{ob}(\mathscr{B}) = \mathrm{ob}(\mathscr{C})$ and such that the ordering on

$\mathscr{B}(C,C')$ induced by the ordering on $\mathscr{C}(C,C')$ is the identity relation, for all objects C,C'. In the context of an order extension $\mathscr{B} \subseteq \mathscr{C}$, morphisms in $\mathscr{C}$ are denoted by $C \rightsquigarrow C'$, and to stress that a morphism is in $\mathscr{B}$ we write $C \to C'$. Further, if $\mathscr{D}$ is a category, we say that an order natural transformation $\alpha \in [\mathscr{D},\mathscr{C}]$ is an order natural transformation *relative to* $\mathscr{B}$ if all of the components α_D belong to $\mathscr{B}$, and we denote the subcategory of $[\mathscr{D},\mathscr{C}]$ consisting of the relative natural transformations by $[\mathscr{D},\mathscr{C}]_{\mathscr{B}}$. Since every functor $F\colon \mathscr{D} \to \mathscr{C}$ is automatically an order functor, and since any natural transformation in $[\mathscr{D},\mathscr{B}]$ is automatically an order natural transformation, we obtain an inclusion functor $i_{\mathscr{D}}\colon [\mathscr{D},\mathscr{B}] \to [\mathscr{D},\mathscr{C}]_{\mathscr{B}}$.

2.2. *Mahavier limits*

Let us fix an order extension $\mathscr{B} \subseteq \mathscr{C}$ and a category $\mathscr{D}$. Recall the diagonal functor $\Delta\colon \mathscr{B} \to [\mathscr{D},\mathscr{B}]$, where $\Delta(B)\colon \mathscr{D} \to \mathscr{B}$ maps every object D to B and every morphism d to id_B. Given a functor $F\colon \mathscr{D} \to \mathscr{B}$, a cone from B to F is precisely a natural transformation $\Delta(B) \to F$, and a limit of F is a universal cone. Extending to $\mathscr{C}$ we consider the extended diagonal functor $i_{\mathscr{D}} \circ \Delta\colon \mathscr{B} \to [\mathscr{D},\mathscr{C}]_{\mathscr{B}}$. Then, given an order diagram $F\colon \mathscr{D} \to \mathscr{C}$, an *order cone* from an object B to F is an order natural transformation $(i_{\mathscr{D}} \circ \Delta)(B) \to F$ relative to $\mathscr{B}$, and a Mahavier limit is a universal such order cone (to stress the role of $\mathscr{B}$ we may refer to an order cone relative to $\mathscr{B}$ or a Mahavier limit relative to $\mathscr{B}$). In more detail, an order cone from B to F is a family $\{\pi_D\colon B \to F(D)\}_{D\in\mathscr{D}}$ of morphisms in $\mathscr{B}$ as in the diagram

$$\begin{tikzcd}
F(D) \arrow[rr, squiggly, "F(d)"] & & F(D') \\
& B \arrow[ul, "\pi_D"] \arrow[ur, "\pi_{D'}"'] &
\end{tikzcd}$$

satisfying $\pi_{D'} \leq F(d) \circ \pi_D$, for all $d\colon D \to D'$ in $\mathscr{D}$. Such an order cone is universal if for any other order cone $\{\psi_D\colon B' \to F(D)\}_{D\in\mathscr{D}}$ from an object $B' \in \mathscr{B}$ to F, there exists a unique morphism $b\colon B' \to B$ in $\mathscr{B}$ with $\psi_D = \pi_D \circ b$. It is obvious that if $F\colon \mathscr{D} \to \mathscr{B}$ is a diagram, then any cone to F is also an order cone to F, and that any limit of F in the usual sense is a Mahavier limit of $i_{\mathscr{D}} \circ F$ relative to $\mathscr{B}$, and vice versa. Of course a Mahavier limit need not exist, and if it exists it is easily seen to satisfy the same uniqueness up to isomorphism property that the usual limit satisfies. We thus write $\varprojlim^{\mathrm{M}}_{\mathscr{B}}(F)$ to denote a Mahavier limit of an order diagram F, with the same ambiguity accepted by the notation $\varprojlim F$ for the limit of a digram. In particular, there is a natural isomorphism $\varprojlim \cong \varprojlim^{\mathrm{M}} \circ i_{\mathscr{D}}$ as functors $[\mathscr{D},\mathscr{B}] \to \mathscr{B}$.

We say that the order extension $\mathscr{B} \subseteq \mathscr{C}$ is *Mahavier complete* if every small order diagram $F\colon \mathscr{D} \to \mathscr{C}$ has a Mahavier limit relative to $\mathscr{B}$. We also say that $\mathscr{C}$ is Mahavier complete relative to $\mathscr{B}$. It is immediate that if $\mathscr{C}$ is Mahavier complete relative to $\mathscr{B}$, then $\mathscr{B}$ is complete in the ordinary sense.

The following are the properties of Mahavier limits which we require below.

- If B and B' are Mahavier limiting objects of the same order cone F, then $B \cong B'$ as objects in $\mathscr{B}$, and there is a unique isomorphism factorising one limiting cone through the other.
- If $\mathscr{C}$ is Mahavier complete relative to $\mathscr{B}$, then for every small category $\mathscr{D}$, any arbitrary choice of Mahavier limiting object $\varprojlim^{\mathrm{M}}_{\mathscr{B}} F$, for each order functor $F \colon \mathscr{D} \to \mathscr{C}$, extends canonically to a functor $\varprojlim^{\mathrm{M}}_{\mathscr{B}} \colon [\mathscr{D}, \mathscr{C}]_{\mathscr{B}} \to \mathscr{B}$.
- If $S \colon \mathscr{D}' \to \mathscr{D}$ is a functor between categories and $F \colon \mathscr{D} \to \mathscr{C}$ is an order diagram, then there is a canonical shape change morphism $\varprojlim^{\mathrm{M}}_{\mathscr{B}} F \to \varprojlim^{\mathrm{M}}_{\mathscr{B}} (F \circ S)$, assuming the Mahavier limits exist.

Proofs can be found in [28].

2.3. Classifying diagrams

Given an order extension $\mathscr{B} \subseteq \mathscr{C}$ and a category $\mathscr{D}$, if the functor $i_{\mathscr{D}} \colon [\mathscr{D}, \mathscr{B}] \to [\mathscr{D}, \mathscr{C}]_{\mathscr{B}}$ has a right adjoint $i^*_{\mathscr{D}}$, then we say that the order extension admits classification of diagrams of shape $\mathscr{D}$. We refer to $i^*_{\mathscr{D}}(F)$, for an order diagram $F \colon \mathscr{D} \to \mathscr{C}$, as the *classifying diagram* of F. We say $\mathscr{B} \subseteq \mathscr{C}$ admits classification of diagrams if it admits classification of diagrams of all small shapes $\mathscr{D}$.

Expectedly, the classification of diagrams is related to the size of the order extension $\mathscr{B} \subseteq \mathscr{C}$, and in a sense their behaviour is a qualitative measurement of it. For a terminal category $\mathscr{D} = \star$, the requirement that $\mathrm{ob}(\mathscr{B}) = \mathrm{ob}(\mathscr{C})$ implies at once that $i_\star$ is the identity, and thus classification of diagrams of shape $\star$ is automatic in any order extension. More interestingly, for the free-living morphism $\mathscr{D} = \{\bullet \to \circ\}$, the category $[\{\bullet \to \circ\}, \mathscr{C}]_{\mathscr{B}}$ has as objects the morphisms c in $\mathscr{C}$, and as morphisms squares

$$\begin{tikzcd}
C_1 \arrow[r, rightsquigarrow, "c"] \arrow[d, "f"] & C_2 \arrow[d, "g"] \\
C_3 \arrow[r, rightsquigarrow, "c'"] & C_4
\end{tikzcd}$$

satisfying $g \circ c \leq c' \circ f$. The category $[\{\bullet \to \circ\}, \mathscr{B}]$ is the usual category of morphisms of $\mathscr{B}$. Thus, for $\mathscr{B} \subseteq \mathscr{C}$ to admit classification of diagrams of shape $\{\bullet \to \circ\}$ entails that with every morphisms $c \in \mathscr{C}$ there is associated a morphism $i^*_{\{\bullet \to \circ\}}(c) \in \mathscr{B}$ such that there is, for all $b \in \mathscr{B}$ and $c \in \mathscr{C}$, a natural bijection

$$\begin{tikzcd}
B \arrow[r, "b"] \arrow[d, "f"'] & B' \arrow[d, "g"'] \\
C \arrow[r, rightsquigarrow, "c"] & C'
\end{tikzcd}
\Longleftrightarrow
\begin{tikzcd}
B \arrow[r, "b"] \arrow[d, "f_\sharp"] & B' \arrow[d, "g_\sharp"] \\
\bullet \arrow[r, "i^*_{\{\bullet \to \circ\}}(c)"] & \bullet
\end{tikzcd}$$

between morphisms in $[\{\bullet \to \circ\}, \mathscr{C}]_{\mathscr{B}}$ on the left, i.e., $g \circ b \leq c \circ f$, and morphisms in $[\{\bullet \to \circ\}, \mathscr{B}]$ on the right, i.e., $g_\sharp \circ b = i^*_{\{\bullet \to \circ\}}(c) \circ f_\sharp$.

Example 1. *For an illustrative example which underlies similar situations in order extensions of the same nature as* $\mathbf{Top} \subseteq \mathbf{Top}_T$*, consider the category* $\mathbf{Set}$ *of sets and functions, and the category* $\mathbf{Set}_T$*, the Kleisli category of the covariant non-empty power set monad* $T\colon \mathbf{Set} \to \mathbf{Set}$*. In simple terms, the objects of* $\mathbf{Set}_T$ *are all sets, and a morphism* $f\colon C \rightsquigarrow C'$ *is a function* $C \to T(C')$*, i.e., a multivalued (total) function from* C *to* C'*. Ordering the hom-sets in* $\mathbf{Set}_T$ *pointwise, namely, for all* $f, g\colon C \rightsquigarrow C'$*, declare that* $f \leq g$ *if* $f(x) \subseteq g(x)$ *for all* $x \in C$*, yields an ordered category, and* $\mathbf{Set} \subseteq \mathbf{Set}_T$ *is an order extension. Given a multivalued function* $c\colon C \rightsquigarrow C'$*, let* $\mathrm{Gr}(c) = \{(c, c') \in C \times C' \mid c' \in f(c)\}$ *be the graph of* c*, and let* $\hat{c}\colon \mathrm{Gr}(c) \to C'$ *be the obvious projection. Suppose that functions* $f\colon B \to C$ *and* $g\colon B' \to C'$ *are given, satisfying* $g \circ b \leq c \circ f$*. Then* $f_\sharp\colon B \to \mathrm{Gr}(c)$*, given by* $f_\sharp(x) = (fx, gbx)$*, is well-defined, and together with* $g_\sharp = g$ *it is easily seen that* $i^*_{\{\bullet \to \circ\}}(c\colon C \to C') = \mathrm{Gr}(c) \xrightarrow{\hat{c}} C'$ *is a classification of diagrams of shape* $\{\bullet \to \circ\}$*. Loosely speaking, the graph of a multivalued function is its classifying diagram.*

Remark 1. *Given a functor* $S\colon \mathscr{D}' \to \mathscr{D}$*, consider the diagram*

$$\begin{tikzcd}
{[\mathscr{D}, \mathscr{B}]} \arrow[r, "S^*"] \arrow[d, shift right, "i_{\mathscr{D}}"'] & {[\mathscr{D}', \mathscr{B}]} \arrow[d, shift right, "i_{\mathscr{D}'}"'] \\
{[\mathscr{D}, \mathscr{C}]_{\mathscr{B}}} \arrow[r, "S^*"] \arrow[u, shift right, "i^*_{\mathscr{D}}"'] & {[\mathscr{D}', \mathscr{C}]_{\mathscr{B}}} \arrow[u, shift right, "i^*_{\mathscr{D}'}"']
\end{tikzcd}$$

where S^* *denotes pre-composition with* S*. It is obvious that the square involving the left adjoints commutes. However, generally, the right adjoints, even when they exist, are not compatible along* S*, namely the square involving the right adjoints typically does not commute. This is seen by the example above for the simple case where* $S\colon \star \to \{\bullet \to \circ\}$*, with* $S(\star) = \bullet$*.*

3. The main result

Recall that for a category $\mathscr{D}$ and an object $D \in \mathscr{D}$ the *slice category* $D/\mathscr{D}$ consists of all morphisms $d_0\colon D \to D_0$ as its objects (where D_0 ranges over all objects of $\mathscr{D}$) and with morphisms $d_1 \to d_2$, for $d_k\colon D \to D_k$, $k = 1, 2$, those morphisms $d\colon D_1 \to D_2$ with $d \circ d_1 = d_2$. Further, with every fixed morphism $d\colon D \to D'$ there is an associated functor $d^*\colon D'/\mathscr{D} \to D/\mathscr{D}$ given on objects $d_0\colon D' \to D_0$ by $d^*(d_0) = d_0 \circ d$, and trivially on morphisms. There is a forgetful functor $\pi_D\colon D/\mathscr{D} \to \mathscr{D}$, mapping an object to its codomain, and acting trivially on morphisms. Obviously, $\pi_D \circ d^* = \pi_{D'}$ holds for all $d\colon D \to D'$. Given an order functor $F\colon \mathscr{D} \to \mathscr{C}$ and an object $D \in \mathscr{D}$ we write $F_D\colon D/\mathscr{D} \to \mathscr{C}$ for the functor $F \circ \pi_D$, and we note that $F_D \circ d^* = F_{D'}$ holds for all $d\colon D \to D'$.

Theorem 1. *Let* $\mathscr{B} \subseteq \mathscr{C}$ *be an order extension. Then* $\mathscr{C}$ *is Mahavier complete relative to* $\mathscr{B}$ *if, and only if,* $\mathscr{B}$ *is complete and* $\mathscr{B} \subseteq \mathscr{C}$ *admits classification of diagrams.*

Proof. Assume that $\mathscr{B} \subseteq \mathscr{C}$ is Mahavier complete. We already noted that it is automatic that $\mathscr{B}$ is then complete, and thus we turn to construct classifying diagrams for some fixed small category $\mathscr{D}$. To construct the functor $i^*_{\mathscr{D}} \colon [\mathscr{D}, \mathscr{C}]_{\mathscr{B}} \to [\mathscr{D}, \mathscr{B}]$, consider a fixed order functor $F \colon \mathscr{D} \to \mathscr{C}$. Let $F_*(D) = \varprojlim^{\mathrm{M}}_{\mathscr{B}}(F_D)$ be an arbitrarily chosen Mahavier limiting object for F_D, which comes equipped with a Mahavier limiting cone $\{\pi_{d_0} \colon F_*(D) \to F_D(d_0)\}_{d_0 \in D/\mathscr{D}}$, where d_0 ranges over all morphisms in $\mathscr{D}$ whose domain is D. In more detail, for every commuting triangle

$$\begin{tikzcd}
D_1 \arrow[rr, "d"] & & D_2 & & F(D_1) \arrow[rr, rightsquigarrow, "F(d)"] & & F(D_2) \\
& D \arrow[ul, "d_1"] \arrow[ur, "d_2"'] & & & & F_*(D) \arrow[ul, "\pi_{d_1}"] \arrow[ur, "\pi_{d_2}"'] &
\end{tikzcd}$$

on the left there corresponds an order commuting triangle on the right, and the universal property holds. For a morphism $d \colon D \to D'$ we obtain a morphism $d_* \colon F_*(D) \to F_*(D')$, the canonical shape change morphism $\varprojlim^{\mathrm{M}}_{\mathscr{B}} F_D \to \varprojlim^{\mathrm{M}}_{\mathscr{B}}(F_D \circ d^*)$, i.e., $d_* \colon F_*(D) \to F_*(D')$ is the unique morphism in $\mathscr{B}$ with the property that $\pi_{d_0 \circ d} = \pi_{d_0} \circ d_*$ holds for all $d_0 \in D'/\mathscr{D}$. Defining $F_*(d) = d_*$ is easily seen to be functorial. We now define $i^*_{\mathscr{D}}(F) = F_*$, obtaining the object part of the functor $i^*_{\mathscr{D}} \colon [\mathscr{D}, \mathscr{C}]_{\mathscr{B}} \to [\mathscr{D}, \mathscr{B}]$, and we now tend to the morphism part of it.

Let $F, F' \colon \mathscr{D} \to \mathscr{C}$ be order functors, and $\alpha \colon F \to F'$ an order natural transformation relative to $\mathscr{B}$, for which we are to construct a natural transformation $\alpha_* \colon F_* \to F'_*$. To obtain the component $(\alpha_*)_D$ at an object $D \in \mathscr{D}$, consider, for an arbitrary commuting triangle

$$\begin{tikzcd}
D_1 \arrow[rr, "d"] & & D_2 \\
& D \arrow[ul, "d_1"] \arrow[ur, "d_2"'] &
\end{tikzcd}$$

the diagram

$$\begin{tikzcd}
& F_*(D) \arrow[dl, "\pi_{d_1}"'] \arrow[dr, "\pi_{d_2}"] \arrow[ddd, dashed, bend left, "(\alpha_*)_D"] & \\
F(D_1) \arrow[rr, rightsquigarrow, "F(d)"] \arrow[d, "\alpha_{D_1}"] & & F(D_2) \arrow[d, "\alpha_{D_2}"] \\
F'(D_1) \arrow[rr, rightsquigarrow, "F'(d)"'] & & F'(D_2) \\
& F'_*(D) \arrow[ul, "\pi'_{d_1}"] \arrow[ur, "\pi'_{d_2}"'] &
\end{tikzcd}$$

where the triangles are the respective Mahavier limiting cones. From the top triangle we have $\pi_{d_2} \leq F(d) \circ \pi_{d_1}$ and from the rectangle, since α is an order natural transformation, we have $\alpha_{D_2} \circ F(d) \leq F'(d) \circ \alpha_{D_1}$. It follows that $\alpha_{D_2} \circ \pi_{d_2} \leq \alpha_{D_2} \circ F(d) \circ \pi_{d_1} \leq F'(d) \circ \alpha_{D_1} \circ \pi_{d_1}$, and thus that $\{\alpha_{D_i} \circ \pi_{d_i}\}_{d_i \colon D \to D_i}$ is an order cone to F'_D. The universal property of the Mahavier limiting cone in the bottom triangle yields the desired dashed morphism. It follows easily that α_* is a natural transformation.

Setting $i^*_{\mathscr{D}}(\alpha) = \alpha_*$, for all order natural transformations $\alpha \in [\mathscr{D}, \mathscr{C}]_{\mathscr{B}}$, concludes the construction of the object and morphism part of $i^*_{\mathscr{D}} \colon [\mathscr{D}, \mathscr{C}]_{\mathscr{B}} \to [\mathscr{D}, \mathscr{B}]$. The functoriality of $i^*_{\mathscr{D}}$ is easily verified.

It now remains to show that $i^*_{\mathscr{D}}$ is right adjoint to $i_{\mathscr{D}}$, for which we construct a bijection $[\mathscr{D}, \mathscr{C}]_{\mathscr{B}}(G, F) \to [\mathscr{D}, \mathscr{B}](G, F_*)$, for some fixed functor $G \colon \mathscr{D} \to \mathscr{B}$ and order functor $F \colon \mathscr{D} \to \mathscr{C}$. Let $\alpha \colon G \to F$ be an order natural transformation, and we must construct a corresponding natural transformation $\alpha^\sharp \colon G \to F_*$. For each $D \in \mathscr{D}$ and every commuting triangle $d_2 = d \circ d_1$ as above, consider the diagram

$$\begin{tikzcd}
 & G(D) \arrow[dl, "G(d_1)"'] \arrow[dr, "G(d_2)"] \arrow[ddd, dashed, "\alpha^\sharp_D"] & \\
G(D_1) \arrow[rr, "G(d)"] \arrow[d, "\alpha_{D_1}"] & & G(D_2) \arrow[d, "\alpha_{D_2}"] \\
F(D_1) \arrow[rr, squiggly, "F(d)"] & & F(D_2) \\
 & F_*(D) \arrow[ul, "\pi_{d_1}"] \arrow[ur, "\pi_{d_2}"'] &
\end{tikzcd}$$

of solid arrows. Since G is a functor and α is an order natural transformation we obtain that $\alpha_{D_2} \circ G(d_2) = \alpha_{D_2} \circ G(d) \circ G(d_1) \leq F(d) \circ \alpha_{D_1} \circ G(d_1)$. Consequently, $\{\alpha_{D_i} \circ G(d_i) \colon G(D) \to F(D_i)\}_{d_i \colon D \to D_i \in D/\mathscr{D}}$ is an order cone to F_D, and thus, by the universal property of $F_*(D)$, the dashed morphism exists, and it is the unique morphism $\alpha^\sharp_D \colon G(D) \to F_*(D)$ in $\mathscr{B}$ satisfying $\alpha_{D_i} \circ G(d_i) = \pi_{d_i} \circ \alpha^\sharp_D$, for all $d_i \colon D \to D_i$. Verifying that we thus obtain a natural transformation is straightforward.

In the other direction, if $\beta \colon G \to F_*$ is a natural transformation, then we construct an order natural transformation $\beta^\flat \colon G \to F$ by considering the diagram

$$\begin{tikzcd}
G(D) \arrow[r, "G(d)"] \arrow[d, "\beta_D"] \arrow[dd, bend right=60, "\beta^\flat_D"'] & G(D') \arrow[d, "\beta_{D'}"] \arrow[dd, bend left=60, "\beta^\flat_{D'}"] \\
F_*(D) \arrow[r, "d_*"] \arrow[d, "\pi_{\mathrm{id}_D}"] \arrow[dr, "\pi_d"] & F_*(D') \arrow[d, "\pi_{\mathrm{id}_{D'}}"] \\
F(D) \arrow[r, squiggly, "F(d)"] & F(D')
\end{tikzcd}$$

where the bent morphisms are defined to be the composition of the vertical morphisms, the top square commutes by naturality of β, the top triangle commutes by definition of d_*, and the bottom triangle order commutes as it is part of the order cone defining $F_*(D)$. A simple diagram chase shows that the morphisms $\beta^\flat_D$ constitute the components of an order natural transformation, as required.

Verifying that each construction is the inverse of the other is routine, as is the verification of the naturality of the constructions.

For the converse, assume that $\mathscr{B}$ is complete and that for each small category $\mathscr{D}$, the adjunction $i_{\mathscr{D}} \dashv i^*_{\mathscr{D}}$ exists. We then have the diagram

$$\begin{tikzcd}
\mathscr{B} \arrow[r, shift left, "\Delta"] \arrow[rr, bend left=50, "i_{\mathscr{D}}\circ\Delta"] & {[\mathscr{D},\mathscr{B}]} \arrow[l, shift left, "\varprojlim"] \arrow[r, shift left, "i_{\mathscr{D}}"] & {[\mathscr{D},\mathscr{C}]_{\mathscr{B}}} \arrow[l, shift left, "i^*_{\mathscr{D}}"] \arrow[ll, bend left=50, "\varprojlim\circ i^*_{\mathscr{D}}"]
\end{tikzcd}$$

in which $i_{\mathscr{D}} \dashv i^*_{\mathscr{D}}$ and $\Delta \dashv \varprojlim$. Since the compositions of left (respectively right) adjoints is a left (respectively right) adjoint it follows that $\varprojlim \circ i^*_{\mathscr{D}}$ is right adjoint to $i_{\mathscr{D}} \circ \Delta$, but this precisely defines Mahavier limits of shape $\mathscr{D}$, so that $\varprojlim^{\mathrm{M}}_{\mathscr{B}} F \cong \varprojlim(i^*_{\mathscr{D}}(F))$, and in particular $\mathscr{C}$ is Mahavier complete relative to $\mathscr{B}$.

In fact, it is evident from the proof that the above result can be stated more accurately as follows.

Theorem 2. *Let $\mathscr{B} \subseteq \mathscr{C}$ be an order extension and $\mathscr{D}$ a small category. If $\mathscr{B}$ has limits of shape $\mathscr{D}$, and $i_{\mathscr{D}} \colon [\mathscr{D},\mathscr{B}] \to [\mathscr{D},\mathscr{C}]_{\mathscr{B}}$ has a right adjoint, then $\mathscr{C}$ has all Mahavier limits relative to $\mathscr{B}$ of shape $\mathscr{D}$. If $\mathscr{C}$ has all Mahavier limits of shapes $D/\mathscr{D}$, for all $D \in \mathscr{D}$, then $i_{\mathscr{D}}$ has a right adjoint.*

For emphasis, we also make a note of the following.

Theorem 3. *Let $\mathscr{B} \subseteq \mathscr{C}$ be an order extension and $\mathscr{D}$ a small category. If $i^*_{\mathscr{D}}$ exists, then, as functors $[\mathscr{D},\mathscr{C}]_{\mathscr{B}} \to \mathscr{B}$, there is a natural isomorphism $\varprojlim^{\mathrm{M}}_{\mathscr{B}} \cong \varprojlim \circ i^*_{\mathscr{D}}$.*

We can now give a definition of the classifying diagram of an order diagram independently of the existence of the adjunction $i_{\mathscr{D}} \dashv i^*_{\mathscr{D}}$.

Definition 1. *Let $\mathscr{B} \subseteq \mathscr{C}$ be an order extension and $F \colon \mathscr{D} \to \mathscr{C}$ an order diagram. The* classifying diagram *of F is the diagram F_* as constructed in the proof of the main theorem, provided the auxiliary Mahavier limits exist.*

Due to the choice of Mahavier limiting objects, a classifying diagram, if it exists, is defined up to isomorphism in $[\mathscr{D},\mathscr{B}]$. Obviously, if F has a classifying digram, then it computes the Mahavier limit of F in the evident sense.

Classifying diagrams are only of interest for order diagrams F that take at least one value outside of $\mathscr{B}$, as the following result clarifies.

Theorem 4. *Let $\mathscr{B} \subseteq \mathscr{C}$ be an order extension and $F \colon \mathscr{D} \to \mathscr{B}$ a diagram. If the classifying diagram F_* of F exists, then $F \cong F_*$ in $[\mathscr{D},\mathscr{B}]$.*

Proof. When F takes values in $\mathscr{B}$ all of the auxiliary Mahavier limits are computed in the degenerate order extension $\mathscr{B} \subseteq \mathscr{B}$, which thus reduce to ordinary categorical limits in $\mathscr{B}$. Then, $F_*(D) \cong \varprojlim^{\mathrm{M}}(F_D) = \varprojlim(F_D)$, and since $\mathrm{id}_D \colon D \to D$ is an initial object in $D/\mathscr{D}$, it follows that $F_*(D) \cong F_D(\mathrm{id}_D) \cong F(D)$. The argument for the morphisms is straightforward, showing that $F_* \cong F$.

Remark 2. *When $\mathscr{B} \subseteq \mathscr{C}$ is Mahavier complete, we thus obtain a reduction of the study of Mahavier limits to the study of the combination of ordinary limits and the adjunctions $i_{\mathscr{D}} \dashv i^*_{\mathscr{D}}$. However, as the proof of Theorem 1 reveals, the generic definition of $i^*_{\mathscr{D}}$ makes use of Mahavier limits, and so the reduction is more of a theoretical tool than an effective computational one. Moreover, the right adjoints $i^*_{\mathscr{D}}$ may be very complicated, even for relatively simple diagrams, and so we propose the following point-of-view. Studies of Mahavier complete order extensions $\mathscr{B} \subseteq \mathscr{C}$ (of which a prominent example is the study of generalised inverse limits of spaces) are, in disguise, studies of the adjunctions $i_{\mathscr{D}} \dashv i^*_{\mathscr{D}}$ within the usual category theoretic framework of limits. One may choose whether to work with these functors directly, if one can compute them, or whether to employ the techniques of [28] as a means to gain information about the functors of interest, and aid in their computation. This idea is reflected in the results given in the following section.*

4. Properties and applications

For an order extension $\mathscr{B} \subseteq \mathscr{C}$ and a small category $\mathscr{D}$, it is trivial that $i_{\mathscr{D}}$ is faithful. The condition that the ordering on each hom-set $\mathscr{C}(C, C')$ induces the identity relation when restricted to $\mathscr{B}(C, C')$ implies that $i_{\mathscr{D}}$ is also full, and thus if the adjunction $i_{\mathscr{D}} \dashv i^*_{\mathscr{D}}$ exists, its unit $\eta \colon \mathrm{id}_{[\mathscr{D},\mathscr{B}]} \to i^*_{\mathscr{D}} \circ i_{\mathscr{D}}$ is a natural isomorphism. In other words, $[\mathscr{D}, \mathscr{B}]$ is a coreflective subcategory of $[\mathscr{D}, \mathscr{C}]_{\mathscr{B}}$. As for the counit $\varepsilon \colon i_{\mathscr{D}} \circ i^*_{\mathscr{D}} \to \mathrm{id}_{[\mathscr{D},\mathscr{C}]_{\mathscr{B}}}$, if the second condition in Theorem 2 is met, then its component at an order diagram F is the order natural transformation $\varepsilon_F \colon i^*_{\mathscr{D}}(F) \to F$, whose components $(\varepsilon_F)_D \colon \varprojlim^{\mathrm{M}}_{\mathscr{B}}(F_D) \to F(D)$ are given by π_{id_D}, the canonical projection from the Mahavier limit to $F_D(\mathrm{id}_D) = F(D)$.

We now turn to consider some relations between properties of classifying diagrams and properties of the order extension.

4.1. Classifying diagrams as a measurement of the size of the order extension

We already remarked above that classifying diagrams are a proxy to the behaviour of the order extension in terms of size; the ability to classify an order diagram in the extension by an ordinary diagram is already an indication that the extension is not too wild. We now look at this phenomenon in more detail. Recall that we write $\{\bullet \to \circ\}$ for the free-living morphism category.

Theorem 5. *For an order extension $\mathscr{B} \subseteq \mathscr{C}$, the following conditions are equivalent*

1. $i^*_{\mathscr{D}}$ exists for all categories $\mathscr{D}$, and $i_{\mathscr{D}} \dashv i^*_{\mathscr{D}}$ is an adjoint equivalence.
2. $i^*_{\{\bullet\to\circ\}}$ exists, and $i_{\{\bullet\to\circ\}} \dashv i^*_{\{\bullet\to\circ\}}$ is an adjoint equivalence.
3. $i_{\{\bullet\to\circ\}}$ is essentially surjective.
4. $\mathscr{B} = \mathscr{C}$.

Proof. Condition 1 trivially implies condition 2, and condition 2 immediately implies condition 3. If condition 4 is met then obviously $i_{\mathscr{D}}$ is the identity, so the validity of condition 1 is clear. We now show that condition 3 implies condition 4. Let $c\colon C \rightsquigarrow C'$ be an arbitrary morphism in $\mathscr{C}$, thought of as an order diagram $F\colon \{\bullet \to \circ\} \to \mathscr{C}$. There is then a diagram $G\colon \{\bullet \to \circ\} \to \mathscr{B}$, corresponding to a morphism $\hat{c}\colon B \to B'$ in $\mathscr{B}$, and an order natural transformation $\alpha\colon G \to F$, together with its inverse, an order natural transformation $\beta\colon F \to G$. Since composition of order natural transformations is component-wise, it follows at once that the components of β are the inverses of the components of α. In more detail, the components of α are thus isomorphisms $b\colon C \to B$ and $b'\colon C' \to B'$ in $\mathscr{B}$ satisfying $b' \circ \hat{c} \le c \circ b$, from which $b' \circ \hat{c} \circ b^{-1} \le c$ follows. Moreover, $(b')^{-1} \circ c \le \hat{c} \circ b^{-1}$, from which $c \le b' \circ \hat{c} \circ b^{-1}$ follows. We may now conclude that $c = (b') \circ \hat{c} \circ b^{-1}$, a composition of morphisms in $\mathscr{B}$, and thus that $c \in \mathscr{B}$, completing the argument.

The above result stems from the following simple observation. While for ordinary natural transformations α, invertibility of α is equivalent to the invertibility of each of its components, the same does not hold for order natural transformation; if an order natural transformation is invertible, then each of its components is too, but the converse may fail. To obtain a somewhat more refined result, we introduce the following concepts for an order extension $\mathscr{B} \subseteq \mathscr{C}$. We say that $\mathscr{C}$ is *nearly equal* to $\mathscr{B}$ if for every $c \in \mathscr{C}$ there is a unique $\hat{c} \in \mathscr{B}$ with $\hat{c} \le c$. $\mathscr{C}$ is *functorially nearly equal* to $\mathscr{B}$ if $\mathscr{C}$ is nearly equal to $\mathscr{B}$ and the assignment $c \mapsto \hat{c}$ is functorial. Finally, we say that $i_{\mathscr{D}}$ is *nearly essentially surjective* if for all order diagrams $F\colon \mathscr{D} \to \mathscr{C}$, there exists a diagram $G\colon \mathscr{D} \to \mathscr{B}$ and an order natural transformation $\alpha\colon i_{\mathscr{D}}(G) \to F$ with each component an isomorphism in $\mathscr{B}$, and such that, for all $d\colon D \to D' \in \mathscr{D}$, the inequality $\alpha_{D'} \circ x \circ \alpha_D^{-1} \le F(d)$ has a unique solution in $\mathscr{B}(G(D), G(D'))$, and that solution is $G(d)$. The adjunction $i_{\mathscr{D}} \dashv i^*_{\mathscr{D}}$ is *nearly an equivalence* if $i_{\mathscr{D}}$ is nearly essentially surjective.

Theorem 6. *For an order extension $\mathscr{B} \subseteq \mathscr{C}$, the following conditions are equivalent*

1. For all categories $\mathscr{D}$, there exists a right adjoint $i^*_{\mathscr{D}}$ such that $i_{\mathscr{D}} \dashv i^*_{\mathscr{D}}$ is nearly an adjoint equivalence.
2. There exists a right adjoint $i^*_{\{\bullet\to\circ\}}$ such that $i_{\{\bullet\to\circ\}} \dashv i^*_{\{\bullet\to\circ\}}$ is nearly an adjoint equivalence.
3. $i_{\{\bullet\to\circ\}}$ is nearly essentially surjective.
4. $\mathscr{C}$ is nearly equal to $\mathscr{B}$.
5. $\mathscr{C}$ is functorially nearly equal to $\mathscr{B}$.

Proof. Condition 1 trivially implies condition 2, and condition 2 equally trivially implies condition 3. Assume now that condition 3 holds, and let $c\colon C\to C'$ be a morphism in $\mathscr{C}$, thought of as an order diagram $F\colon \{\bullet\to\circ\}\to\mathscr{C}$. Since $i_{\{\bullet\to\circ\}}$ is nearly essentially surjective, there exists a morphism $b\colon B\to B'\in\mathscr{B}$ and isomorphisms $\alpha_C\colon B\to C$ and $\alpha_{C'}\colon B'\to C'$, such that $\hat{c}=\alpha_{C'}\circ b\circ\alpha_C^{-1}\le c$. If y is any other morphism in $\mathscr{B}$ with $y\le c$, then $\alpha_{C'}^{-1}\circ y\circ\alpha_C$ is a solution of $\alpha_{C'}\circ x\circ\alpha_C^{-1}\le c$, and thus $b=\alpha_{C'}^{-1}\circ y\circ\alpha_C$, namely $y=\hat{c}$. In other words, for all $c\in\mathscr{C}$ there is a unique $\hat{c}\in\mathscr{B}$ with $\hat{c}\le c$, as required. Showing that condition 4 implies condition 5 is a simple observation which we do not detail. Assume now that condition 5 holds, so that $c\mapsto\hat{c}$ is a functor. Given a category $\mathscr{D}$, if $F\colon\mathscr{D}\to\mathscr{C}$ is an order diagram, then let $\hat{F}\colon\mathscr{D}\to\mathscr{B}$ be given by $\hat{F}(D)=F(D)$ for all $D\in\mathscr{D}$, and $\hat{F}(d)=\widehat{F(d)}$ for all $d\in\mathscr{D}$. Noting that generally $c_1\le c_2$ in $\mathscr{C}$ implies $\widehat{c_1}=\widehat{c_2}$, it follows easily that $\hat{F}$ is a functor. Similarly, the components of an order natural transformation $\alpha\colon F_1\to F_2$ are also the components of a natural transformation $\hat{\alpha}\colon\widehat{F_1}\to\widehat{F_2}$. In short, we may define $i^*_{\mathscr{D}}(F)=\hat{F}$ and $i^*_{\mathscr{D}}(\alpha)=\hat{\alpha}$, which is then easily seen to be a right adjoint of $i_{\mathscr{D}}$. Further, for any order diagram $F\colon\mathscr{D}\to\mathscr{C}$, taking $\alpha\colon i_{\mathscr{D}}(i^*_{\mathscr{D}}(F))\to F$ to have components $\alpha_D=\mathrm{id}_D$, for all $D\in\mathscr{D}$, shows that $i_{\mathscr{D}}$ is nearly essentially surjective, and completing the proof.

4.2. Classifying diagrams and initial functors

For ordinary categorical limits recall that if $S\colon\mathscr{D}_0\to\mathscr{D}$ is an initial functor (which sometimes, confusingly, is also called a final functor, see, e.g., [36], also for further details if needed) then the shape change morphism $\varprojlim S\colon\varprojlim F\to\varprojlim(F\circ S)$ is an isomorphism. This foundational result of category theory, one that is used extensively in applications of inverse limits in topology and algebra prior to the formulation of category theory, is well-known not to hold for generalised inverse limits of spaces. Phrased in the context of diagrams indexed by the integers, the problem was coined as the "subsequence theorem problem" and is one of the earliest driving forces of research efforts in the theory of generalised inverse limits of spaces, calling for conditions under which the shape change morphism as above between the generalised limits is an isomorphism. The subsequence theorem problem is discussed in [37]. Recently, Greenwood and Youl ([10]) presented a subsequence theorem for generalised inverse limits of compacta with a single multivalued bonding function, when the latter is constructed out of a finite family of singlevalued functions satisfying some rather strong fixed-point conditions.

Let us phrase the problem in the context of an arbitrary order extension $\mathscr{B}\subseteq\mathscr{C}$. Let $\mathcal{F}$ be a family of initial functors $S\colon\mathscr{D}\to\mathscr{D}'$, where $\mathscr{D}$ and $\mathscr{D}'$ are allowed to range over all small categories. We shall say that $\mathscr{B}\subseteq\mathscr{C}$ is $\mathcal{F}$-*conservative* if for all order diagrams $F\colon\mathscr{D}\to\mathscr{C}$ such that the Mahavier limits exist, the shape change morphism $\varprojlim^{\mathrm{M}}_{\mathscr{B}}(F)\to\varprojlim^{\mathrm{M}}_{\mathscr{B}}(F\circ S)$ is an isomorphism.

Two extreme cases where conservatism is guaranteed are the following. Firstly, if $\mathcal{F}$ consists only of isomorphisms, then any order extension $\mathscr{B}\subseteq\mathscr{C}$ is $\mathcal{F}$-conservative (trivially so). On the other hand, if $\mathscr{B}=\mathscr{C}$, in which case Mahavier limits are simply ordinary limits, then $\mathcal{F}$-conservatism holds for all collections $\mathcal{F}$ as above. This is nothing but a re-iteration of the opening line of this subsection, which is well-known, but not quite as trivial as the former condition.

Theorem 7. *Let $\mathscr{B} \subseteq \mathscr{C}$ be an order extension. If $\mathscr{C}$ is nearly equal to $\mathscr{B}$, then $\mathscr{B} \subseteq \mathscr{C}$ is $\mathcal{F}$-conservative for all families $\mathcal{F}$ of initial functors.*

Proof. Referring to Theorem 6, let $c \mapsto \hat{c}$ be the unique functor $\mathscr{C} \to \mathscr{B}$ with $\hat{c} \leq c$ for all $c \in \mathscr{C}$. Let $S\colon \mathscr{D}' \to \mathscr{D}$ be an initial functor, and $F\colon \mathscr{D} \to \mathscr{C}$ an arbitrary order diagram, and assume the Mahavier limits $\varprojlim^{\mathrm{M}}_{\mathscr{B}}(F)$ and $\varprojlim^{\mathrm{M}}_{\mathscr{B}}(F \circ S)$ exist. Recall from the proof of Theorem 6 that $i^*_{\mathscr{D}}(F) = \hat{F}$ and that $i^*_{\mathscr{D}'}(F \circ S) = \widehat{F \circ S} = \hat{F} \circ S$. By Theorem 3 we have the vertical canonical isomorphisms in the diagram

$$\begin{tikzcd}
\varprojlim^{\mathrm{M}}_{\mathscr{B}}(F) \arrow[r, dashed] \arrow[d, leftrightarrow, "\cong"] & \varprojlim^{\mathrm{M}}_{\mathscr{B}}(F \circ S) \arrow[d, leftrightarrow, "\cong"] \\
\varprojlim(\hat{F}) \arrow[r, "\cong"] & \varprojlim(\hat{F} \circ S)
\end{tikzcd}$$

while the horizontal arrow is the shape change morphism for ordinary limits, which, since S is initial, is an isomorphism. Composing the three morphisms yields the shape change morphism for the Mahavier limits, which is thus an isomorphism, as required.

Theorem 8. *Let $\mathscr{B} \subseteq \mathscr{C}$ be an order extension, and $\mathcal{F}$ a collection of initial functors containing the functor $S\colon \star \to \{\bullet \to \circ\}$ with $S(\star) = \bullet$. Then if $\mathscr{B} \subseteq \mathscr{C}$ has Mahavier limits of shape $\{\bullet \to \circ\}$ and is $\mathcal{F}$-conservative, then $\mathscr{C}$ is nearly equal to $\mathscr{B}$.*

Proof. Let $c\colon C \to C'$ be a morphism, thought of as an order diagram $F\colon \{\bullet \to \circ\} \to \mathscr{C}$ with $F(\bullet) = C$. By assumption, the canonical morphism $\varprojlim^{\mathrm{M}}_{\mathscr{B}}(F) \to \varprojlim^{\mathrm{M}}_{\mathscr{B}}(F \circ S)$ is an isomorphism. Obviously, we may take $\varprojlim^{\mathrm{M}}_{\mathscr{B}}(F \circ S) = C$, and thus also $\varprojlim^{\mathrm{M}}_{\mathscr{B}}(F) = C$. The universal order cone for that Mahavier limit consists of morphisms $\mathrm{id}_C\colon C \to C$ and $\hat{c}\colon C \to C'$, and its order commutativity is the claim that $\hat{c} \leq c$. Any other morphism b with $b \leq c$ yields, together with id_C, another order cone to F, which thus factorises through the universal one, necessarily trivially, and thus $b = \hat{c}$.

Qualitatively, given an order extension $\mathscr{B} \subseteq \mathscr{C}$, the largest class $\mathcal{F}$ of initial functors with respect to which the extension is conservative may be called the *conservatism degree* of the extension. The collection of all adjunctions $i_{\mathscr{D}} \dashv i^*_{\mathscr{D}}$, for all small categories $\mathscr{D}$, may be called the *classifying degree* of the extension. These degrees measure different aspects of the extension, and the results above reveal that there is a tension between these two aspects: an order extension with a large conservatism degree (in the sense that $\mathcal{F}$ is a large collection of initial functors) tends to have a small classifying degree (in the sense that the adjunctions exhibit simple behaviour), and vice versa.

Remark 3. *Phrased in the formalism we developed, a very general formulation of the subsequence theorem problem in the theory of generalised inverse limits of spaces can be stated as follows. Identify an intermediate order extension* $\mathbf{Top} \subseteq \boldsymbol{X} \subseteq \mathbf{Top}_T$ *which is* $\mathcal{F}$*-conservative for a sufficiently interesting class of initial functors* $\mathcal{F}$*. Classically,* $\mathcal{F}$ *consists of all initial functors between categories isomorphic to the poset of natural numbers, or to more general posets. Of course, one would like* $\boldsymbol{X}$ *to be a significant portion of* $\mathbf{Top}_T$*, namely* $\mathbf{Top} \subseteq \boldsymbol{X}$ *should have a large classifying degree, and, since* $\mathcal{F}$ *should be a useful collection of initial functors, at the same time have a large conservatism degree. As seen above, attaining both degrees to be large is impossible, thus explaining the difficulty in resolving the subsequence theorem problem. This observation, and the results above, are of importance in further framing the subsequence theorem problem of generalised inverse limits of spaces, and, perhaps most importantly, in setting realistic expectations from any possible solution of it.*

4.3. Classifying diagrams and Mahavier limits in terms of ordinary limits and colimits

This final subsection is an interesting consequence of Theorem 3, though we are unaware of practical applications of it. It is well-known that small limits can be constructed from small products and equalisers, and a version of that result for Mahavier limits is given in [28]. It is also well-known that limits can sometimes be constructed in terms of colimits. For instance, a join complete lattice is automatically meet complete. A systematic approach is to consider, given categories $\mathscr{B}, \mathscr{D}$, the diagonal functor $\Delta\colon \mathscr{B} \to [\mathscr{D}, \mathscr{B}]$, and construct its right adjoint by means of (any particular version of) the adjoint functor theorem.

Obviously, given an order extension $\mathscr{B} \subseteq \mathscr{C}$, the same approach can be applied to $i_{\mathscr{D}} \circ \Delta$, but another approach is also possible, namely to apply the adjoint functor theorem to the functor $i_{\mathscr{D}}$. When we constructed $i^*_{\mathscr{D}}$ above, Mahavier limits were explicitly used in the construction, and thus that proof is of limited use in computing Mahavier limits. But, if the solution set condition can be established, then the adjoint functor theorem can circumvent the need to directly use Mahavier limits. We phrase this observation using Freyd's adjoint functor theorem.

Theorem 9. *Let* $\mathscr{B} \subseteq \mathscr{C}$ *be an order extension. If* $\mathscr{B}$ *is complete and cocomplete, and for each small category* $\mathscr{D}$ *the functor* $i_{\mathscr{D}}\colon [\mathscr{D}, \mathscr{B}] \to [\mathscr{D}, \mathscr{C}]_{\mathscr{B}}$ *is cocontinuous and satisfies the solution set condition, then* $\mathscr{C}$ *is Mahavier complete relative to* $\mathscr{B}$*.*

Proof. When $\mathscr{B}$ is cocomplete, so is $[\mathscr{D}, \mathscr{B}]$, and thus, by the adjoint functor theorem, $i^*_{\mathscr{D}}$ exists. Since $\mathscr{B}$ is complete, $i^*_{\mathscr{D}}$ computes Mahavier limits of shape $\mathscr{D}$, so, in particular, they exist.

In as much as the solution set condition can be established without recourse to any explicit Mahavier limits, we obtain a construction of Mahavier limits in terms of limits and colimits in $\mathscr{B}$.

5. Revisiting classical generalised inverse limits

We conclude this work with a fresh look at the elements of generalised inverse limits in topology through the lens of the formalism presented above. We keep the discussion somewhat informal, and we only touch upon a few topics, for the sake of brevity. Where details and proofs can be found in [18] or [28], we will simply omit any arguments.

The ambient order extension for considering classical generalised inverse limits is $\mathbf{Top} \subseteq \mathbf{Top}_T$, where $\mathbf{Top}_T$ is the category of all topological spces with morphisms the upper semicontinuous functions, with each hom-set ordered by inclusion. An important order sub-extension is $\mathbf{Comp} \subseteq \mathbf{Comp}_T$, where $\mathbf{Comp}$ is the full subcategory of $\mathbf{Top}$ spanned by the compact Huasdorff spaces, and $\mathbf{Comp}_T$ has the same objects but restricts to the closed valued mappings.

The first examples of generalised inverse limits were of sequences of spaces indexed by $\mathbb{N}$ or $\mathbb{Z}$. Later on, more general shapes were allowed, but all still made use of *thin* diagrams, namely having at most one bonding function between any two spaces. We note first that Mahavier limits of the most general (small) shapes in $\mathbf{Top} \subseteq \mathbf{Top}_T$ exist.

Theorem 10. *The order extension* $\mathbf{Top} \subseteq \mathbf{Top}_T$ *is Mahavier complete.*

In particular, for different indexing categories $\mathscr{D}$, all of the notions of generalised inverse limits of spaces considered in the literature are obtained. In conjunction with the order extension $\mathbf{Top} \subseteq \mathbf{Top}_T$ it is natural to consider the order extension $\mathbf{Set} \subseteq \mathbf{Set}_T$, where $\mathbf{Set}_T$ is the Kleisli category of the covariant power set monad on $\mathbf{Set}$, in other words, the category of all sets and whose morphisms are the multivalued functions, endowed with the evident order structure induced by set inclusion. Obviously, $\mathbf{Set} \subseteq \mathbf{Set}_T$ is isomorphic (in a suitable category of order extensions) to the full sub-order extension of $\mathbf{Top} \subseteq \mathbf{Top}_T$ spanned by the discrete spaces.

Theorem 11. *The order extension* $\mathbf{Set} \subseteq \mathbf{Set}_T$ *is Mahavier complete.*

Proof. Discretise the details of any proof of Theorem 10.

The next result addresses the compacta oriented needs of the theory.

Theorem 12. *The order extension* $\mathbf{Comp} \subseteq \mathbf{Comp}_T$ *is Mahavier complete.*

Proof. The requirement that the morphisms in $\mathbf{Comp}_T$ are closed valued allows for an adaptation of the proof of Theorem 10.

One now obtains the diagram

$$\begin{tikzcd}
\mathbf{Set} \arrow[r, shift left=2] \arrow[d, hook] & \mathbf{Top} \arrow[l] \arrow[l, shift left=2, leftarrow] \arrow[d, hook] & \mathbf{Comp} \arrow[l] \arrow[d, hook] \\
\mathbf{Set}_T \arrow[r, shift left=2] & \mathbf{Top}_T \arrow[l] \arrow[l, shift left=2, leftarrow] & \mathbf{Comp}_T \arrow[l]
\end{tikzcd}$$

exhibiting the three main order extensions of interest, together with forgetful functors leading from right to left, and, where depicted, their left and right adjoints.

Theorem 13. *A right adjoint* $G\colon \mathscr{B} \to \mathscr{B}'$ *which extends, as in the above diagram, to a right adjoint of order extensions* $G\colon \mathscr{C} \to \mathscr{C}'$ *is Mahavier continuous, i.e., it preserves all Mahavier limits that exist in* $\mathscr{C}$*:* $G(\varprojlim^{\mathrm{M}}_{\mathscr{B}})(F) \cong \varprojlim^{\mathrm{M}}_{\mathscr{B}}(G \circ F)$*, for all order diagrams* $F\colon \mathscr{D} \to \mathscr{C}$*.*

In particular computations in topology one often finds it convenient to change perspective and move around the top part of the diagram, e.g., ignoring the topology and concentrating on the underlying sets. With the above result the same tools are at one's disposal when considering generalised inverse limits, allowing the use of standrd arguments to deduce various properties, for instance surjectivity of induced mappings between generalised inverse limits by applying the forgetful functor to sets.

Obviously, the categorical formalism we consider is a unifying mechanism, but in a somewhat stronger manner than the immediate labour saving consequence of treating different notions of generalised inverse limits as instances of a single concept. To see how, recall that the *graph* of a multivalued function $f\colon X \to Y$ is $\mathrm{Gr}(f) = \{(x,y) \in X \times Y) \mid y \in f(x)\}$, endowed with the subspace topology (in case topologies are involved). Unlike the case of singlevalued functions, the graph of f is typically not homeomorphic, or even in bijection, with the domain of f. More generally, given any finite sequence $\boldsymbol{f} = X_1 \xrightarrow{f_1} X_2 \to \cdots \to X_{n-1} \xrightarrow{f_{n-1}} X_n$, define its graph to be $\mathrm{Gr}(\boldsymbol{f}) = \{\boldsymbol{x} \in X_1 \times \cdots \times X_n \mid \boldsymbol{x}_{k+1} \in f_k(\boldsymbol{x}_k), \forall 1 \leq k < n\}$. We notice at once that these graphs are nothing but an instance of Mahavier limit.

Proposition 1. *The graph* $\mathrm{Gr}(\boldsymbol{f})$ *of a finite sequence of functions, with the evident projections, is the Mahavier limit of* $\boldsymbol{f}$ *considered as a finite diagram:* $\mathrm{Gr}(\boldsymbol{f}) \cong \varprojlim^{\mathrm{M}}_{\mathbf{Top}}(X_1 \xrightarrow{f_1} X_2 \to \cdots \to X_{n-1} \xrightarrow{f_{n-1}} X_n)$*.*

Proof. Verification of the universal property is immediate.

Graphs appear prominently in the study of generalised inverse limits of sequences indexed by $\mathbb{N}$, in fact as an instance of the main result of this article. In more detail, let $\mathscr{D}$ be the category $\mathbb{N}$, i.e., the natural numbers with morphisms corresponding to $m \geq n$, and let $F\colon \mathscr{D} \to \mathbf{Top}_T$ be an order diagram, i.e., a generalised inverse system $\boldsymbol{f}$. According to the main result $\varprojlim^{\mathrm{M}}_{\mathbf{Top}}(F) \cong \varprojlim(i^*_{\mathscr{D}}(F))$, and the Mahavier limits arising in the proof correspond to slices of $\mathscr{D}$. But all such slices are finite sequences, yielding the formula $\varprojlim^{\mathrm{M}}_{\mathbf{Top}}(F) \cong \varprojlim \mathrm{Gr}(\boldsymbol{f})$, where $\mathrm{Gr}(\boldsymbol{f})\colon \mathscr{D} \to \mathbf{Top}$ is the functor sending n to the graph of the initial segment of $\boldsymbol{f}$ of length n.

Further along this line, but no longer under the restriction on the shape of the diagrams to be sequential, the following is a trivial observation, indeed merely a tautology. Let P be a property applicable to a diagram $\mathscr{D} \to \mathbf{Top}$ of spaces with ordinary singlevalued continuous functions. We say that an order diagram $F\colon \mathscr{D} \to \mathbf{Top}_T$ is Mahavier P if the classifying diagram $i^*_{\mathscr{D}}(F)$ is P. Note that by Theorem 4 if a diagram in **Top** is P, then when viewed as a sequence in $\mathbf{Top}_T$ it is automatically Mahavier P.

Theorem 14. *Let P be a property of diagrams in* **Top** *and Q a property of spaces. If it is true that whenever a diagram $\mathscr{D} \to \mathbf{Top}$ is P the limit $\varprojlim(F)$ is Q, then it also holds that whenever an order diagram $\mathscr{D} \to \mathbf{Top}_T$ is Mahavier P the Mahavier limit $\varprojlim^{\mathrm{M}}(F)$ is Q.*

Obviously, the challenge for a fruitful application of this principle is in identifying, for a given property P, conditions verifiable directly on an order diagram $\mathscr{D} \to \mathbf{Top}_T$ that render it Mahavier P. But even in the absence of such criteria, ad-hoc criteria can be obtained. It is precisely this principle that is applied, e.g., when studying generalised inverse limits of a sequence of spaces by means of graphs.

As a final note, the discussion above is meant to extract the essence of some of the most fundamental tools and techniques of generalised inverse limits and portray them categorically, focusing on the relationship between Mahavier limits and classifying diagrams. Although much more can be said, with [18] and [28] already containing significant theory and detail, we remain brief and conclude the work here.

Acknowledgements

The first named author wishes to acknowledge the support of the *Professor Peter Brown Memorial Bursary* from Clare Hall and the *International Macquarie University Research Training Program Scholarship (No. 2017127)* from Macquarie University.

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