IEEE Transactions on Automatic Control, Vol. 63, Issue 3, March 2018, pp. 811-818
DOI:10.1109/TAC.2017.2737324

Composite Model Reference Adaptive Control with Parameter Convergence under Finite Excitation

Namhoon Cho, Hyo-Sang Shin*, Youdan Kim, and Antonios Tsourdos

***Abstract*—A new parameter estimation method is proposed in the framework of composite model reference adaptive control for improved parameter convergence without persistent excitation. The regressor filtering scheme is adopted to perform the parameter estimation with signals that can be obtained easily. A new framework for residual signal construction is proposed. The incoming data is first accumulated to build the information matrix, and then its quality is evaluated with respect to a chosen measure to select and store the best one. The information matrix is built to have full rank after sufficient but not persistent excitation. In this way, the exponential convergence of both tracking error and parameter estimation error can be guaranteed without persistent oscillation in the external command which drives the system. Numerical simulations are performed to verify the theoretical findings and to demonstrate the advantages of the proposed adaptation law over the standard direct adaptation law.**

I. Introduction

The adaptive control system aims to maintain the performance of the control system remaining close to the nominal performance under uncertainties. In order to achieve the aim, adaptive control generally includes an approximate model that captures uncertainty and *adaptive augmentation* that complements the nominal controller with the approximate uncertainty model. The uncertainty model is generally a function approximator, and the regression algorithm working for better approximation is called the *adaptation law*.

In order to best maintain the nominal performance, the approximated uncertainty model should be as close as possible to the actual uncertainty. If the chosen uncertainty model has a parametric form, it is widely accepted that there exists an *ideal* value of parameter with which the error between the model and actual uncertainty is minimized over a domain. Accordingly, it is desirable to design an adaptation law drives parameter estimate to the ideal value.

The issue is that the parameter estimation accuracy is not the only consideration that should be taken into account in the design of adaptation laws. Note that the parameter estimation error dynamics closely interacts with the tracking error dynamics in the overall closed-loop system. In order to improve the transient bound of tracking error or to improve the learning rate for fast adaptation, one might adopt a simple adaptation law with a high gain. However, it could induce undesirable behaviours including abnormally abrupt control response, unwished amplification of the high frequency component of unmodelled dynamics. Furthermore, instability phenomena such as parameter drift [1] and bursting [2] could occur in the absence of Persistent Excitation (PE). It is therefore required to consider not only the accuracy of the parameter estimate, but also control response and tracking error performance in the design of adaptation laws.

There have been extensive studies to address the issue arising from the interaction between adaptation and tracking. Most of the previous studies focused on overcoming the shortages of the simple adaptation laws. They can be classified into two categories; 1) robust adaptive control, and 2) composite adaptive control.

The robust adaptive control approach focused on developing modified adaptation laws or new architectures that provide more robust closed-loop system. The modified adaptation laws are usually given by sum of the standard direct adaptation term and a modification term. Various methods are proposed for design of the modification terms; σ-modification [3], e-modification [4], Q-modification [5], Kalman filter modification [6], adaptive loop transfer recovery [7], low-frequency learning [8], etc. The adaptive controller architectures such as the $\mathcal{L}_1$ adaptive control [9] and the derivative-free adaptive control [10] are proposed for robust adaptive control. These methods in general utilize the leading principle that the robustness margin can be enhanced by including additional damping or raising the order of system. However, most of these methods guarantee only the boundedness of parameter estimation error and asymptotic convergence of the tracking error. Also, for the structured uncertainty case, parameter convergence is not guaranteed unless the PE condition is satisfied, in the modification methods such as [3]–[9]. Moreover, these methods perform only adjustment of parameters rather than parameter 'estimation'. Therefore, the existing robust adaptive control methods lack the accuracy of the parameter estimate, while they improve control response and tracking error performance.

The composite adaptive control approach focused on including model prediction error in the adaptation law to take the benefits of combining direct and indirect approaches. The idea of combined direct and indirect adaptation is presented in [11]–[13]. On a similar basis, the composite adaptation laws are developed and applied to robot manipulator control in [14], [15]. A state feedback composite model reference adaptive control system which utilized the regressor filtering scheme is proposed in [16], and it extended the design of [11]–[15]. A locally weighted learning scheme called receptive

Namhoon Cho is with the Automation and Systems Research Institute, Seoul National University and Youdan Kim is with the Department of Mechanical and Aerospace Engineering, Institute of Advanced Aerospace Technology, Seoul National University, Seoul, 08826, Republic of Korea. e-mail: (nhcho91@snu.ac.kr, ydkim@snu.ac.kr)

Hyo-Sang Shin and Antonios Tsourdos are with the School of Aerospace, Transport and Manufacturing, Cranfield University, Cranfield, MK43 0AL, United Kingdom. e-mail: (h.shin@cranfield.ac.uk, a.tsourdos@cranfield.ac.uk)

* Corresponding author

field weighted regression is adopted as the learning algorithm for composite adaptive control in [17]. Many other studies suggest the use of composite approach in adaptive control for various classes of systems [18], [19] and for various control design methods [20]–[22]. Existing methods using the composite approach are based on online parameter estimation schemes. The transient adaptation and control response can be smoothed, the tracking error performance can be improved as a result, and also the robustness of closed-loop system can be improved in comparison to the simple direct adaptation law. However, these methods often require PE for exponential parameter convergence, particularly in the linearly parameterized structured uncertainty case. Furthermore, as the least squares optimal approach used in parameter estimator design results in a time-varying adaptation gain, the tuning is complicated and less flexible. Hence, the existing composite adaptive control methods are incomplete, while they perform parameter estimation and improve control response and tracking error performance.

It is evident that there is no previous study handling the three considerations all together in a well-balanced manner. Our observation is that the PE requirement for parameter convergence is the main obstacle in achieving the balance between the three considerations; the PE requirement could introduce continuous oscillatory behaviour of the state and control, which is not practical.

To this end, this paper aims to develop a new parameter-estimation-based adaptation law that can handle these considerations in a well-balanced way. The main focus of this study is to relax the PE requirement to achieve this. For this purpose, 1) the regressor filtering scheme is adopted, and 2) a new residual design is proposed. The regressor filtering scheme is adopted to get a well-posed linear parameter estimation problem with easily-obtainable signals. After that, a new framework for constructing the residual signal is proposed. The residual is designed to take the form of multiplication of a real symmetric information matrix and the parameter estimation error. Since the rank deficiency of information matrix is the fundamental cause that necessitates PE, the key idea in our study is to build the information matrix explicitly so that it can have full rank after Finite Excitation (FE).

It is proven that the proposed framework provides analytical guarantees on the stability of both tracking error and parameter estimation error for the structured uncertainty case. The exponential stability can be guaranteed under the assumption of FE, not PE. The transient performance can be adjusted by gain tuning. Since the proposed method is basically a composite adaptive control and the exponential stability guarantee is beneficial for the robustness of the closed-loop system, certain enhancement of robustness can be expected as an accompanying benefit.

Note that the proposed method is complementary to the concurrent learning in [23]–[25] and the composite learning in [26]. The adaptation method designed in this paper shares some similarities with these existing methods which are recently developed on a sound basis to relax PE condition. However, the algorithm developed for residual signal construction in this paper is new and novel, and its mechanism is substantially different from the existing methods. Implementation of the proposed method is simpler, because the continuous update procedure can be done with forward integration.

The rest of the paper is organized as follows: The preliminaries and problem formulation are given in Section II. Before moving onto the design and analysis of the adaptation law, the regressor filtering scheme is explained in Section III. In Section IV, the new adaptation law is designed and its stability is analyzed under the assumption of FE for the case of structured uncertainty. In Section V, numerical simulations are performed to verify the theoretical finding such as the exponential convergence guarantee under FE. Conclusions are summarized in Section VI.

II. Preliminaries and Problem Formulation

In this section, the notation and the definitions on the degree of signal excitation are given as preliminaries, and a state feedback Model Reference Adaptive Control (MRAC) problem is formulated.

A. Preliminaries

For the notation throughout this study, vectors and matrices will be written in boldface. Also, $\|\cdot\|$ and $\|\cdot\|_F$ denote the induced 2-norm and the Frobenius norm of a matrix, respectively. In addition, $\vec{(\cdot)}$, $\lambda_{\min}(\cdot)$, and $\lambda_{\max}(\cdot)$ denote the columnwise vectorization, the minimum eigenvalue, and the maximum eigenvalue of a matrix, respectively.

For convenience, the mathematical basics that are used repeatedly throughout this study are summarized in Lemma 1.

Lemma 1 (Basic Facts about Matrix-Vector Algebra)**.**

- *For any matrix* $\mathbf{A} \in \mathbb{R}^{n\times m}$,

$$\left\| \vec{\mathbf{A}} \right\| = \|\mathbf{A}\|_F \tag{1}$$

- *For any co-dimensional column vectors* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^{n\times 1}$,

$$\operatorname{tr}\left(\mathbf{u}\mathbf{v}^T\right) = \mathbf{v}^T\mathbf{u} \tag{2}$$

- *For any co-dimensional real symmetric matrices* $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$,

$$\begin{aligned} 0 \le \lambda_{\min}(\mathbf{A}) + \lambda_{\min}(\mathbf{B}) &\le \lambda_{\min}(\mathbf{A}+\mathbf{B}) \\ \le \lambda_{\max}(\mathbf{A}+\mathbf{B}) &\le \lambda_{\max}(\mathbf{A}) + \lambda_{\max}(\mathbf{B}) \end{aligned} \tag{3}$$

- *For any matrix* $\mathbf{B} \in \mathbb{R}^{n\times m}$ *and any positive (semi-)definite matrix* $\mathbf{A} \in \mathbb{R}^{n\times n}$,

$$\lambda_{\min}(\mathbf{A}) \|\mathbf{B}\|_F^{\,2} \le \operatorname{tr}\left(\mathbf{B}^T\mathbf{A}\mathbf{B}\right) \le \lambda_{\max}(\mathbf{A}) \|\mathbf{B}\|_F^{\,2} \tag{4}$$

The significance and implication of PE in an adaptive control is investigated in [2], [27]–[32]. The main objective of this study is to relax the dependence of parameter convergence on PE condition. Therefore, the degree of signal excitation should be defined. The FE and the PE of a signal are defined as in [33].

Definition 1 (Finite Excitation of a Signal)**.**
A bounded vector signal $\mathbf{v}(t)$ *has Finite Excitation (FE) over*

a finite time interval $[t_s, t_s+T]$, if there exist $T>0$, $t_s \geq t_0$, and $\gamma > 0$ such that

$$\int_{t_s}^{t_s+T} \mathbf{v}(\tau)\mathbf{v}^T(\tau)\, d\tau \geq \gamma \mathbf{I} > \mathbf{0} \tag{5}$$

Definition 2 (Persistent Excitation of a Signal)**.**
A bounded vector signal $\mathbf{v}(t)$ has Persistent Excitation (PE), if there exist $T>0$ and $\gamma>0$ such that

$$\int_{t}^{t+T} \mathbf{v}(\tau)\mathbf{v}^T(\tau)\, d\tau \geq \gamma \mathbf{I} \quad \textit{for } \forall t \geq t_0 \tag{6}$$

B. Problem Formulation

1) System Dynamics: Consider a class of Multi-Input Multi-Output (MIMO) uncertain system given by

$$\begin{aligned} \dot{\mathbf{x}}_p(t) &= \mathbf{A}_p\mathbf{x}_p(t) + \mathbf{B}_p\left(\mathbf{u}(t) + \mathbf{\Delta}(\mathbf{x}_p(t))\right) \\ \mathbf{z}(t) &= \mathbf{H}_p\mathbf{x}_p(t) \end{aligned} \tag{7}$$

where $\mathbf{x}_p(t) \in \mathbb{R}^{n_p\times 1}$ is the fully measureable state vector, $\mathbf{u}(t) \in \mathbb{R}^{m\times 1}$ is the control input vector, $\mathbf{z}(t) \in \mathbf{R}^{m\times 1}$ is the performance output vector, and $\mathbf{\Delta}(\mathbf{x}_p(t)) \in \mathbb{R}^{m\times 1}$ is the state-dependent matched uncertainty. Also, $\mathbf{A}_p \in \mathbb{R}^{n_p\times n_p}$, $\mathbf{B}_p \in \mathbb{R}^{n_p\times m}$, and $\mathbf{H}_p \in \mathbb{R}^{m\times n_p}$ in Eq. (7) are known constant matrices, and assume that $(\mathbf{A}_p, \mathbf{B}_p)$ is controllable. Moreover, assume that the columns of $\mathbf{B}_p$ are linearly independent.

The objective is to design a control $\mathbf{u}(t)$ such that the performance output $\mathbf{z}(t)$ tracks a given bounded piecewise continuous command $\mathbf{z}_{\text{cmd}}(t) \in \mathbb{R}^{m\times 1}$. For this purpose, the integral feedback will be added in the design. Let $\mathbf{e}_{z_I}(t) \triangleq \int_{t_0}^{t}(\mathbf{z}(\tau) - \mathbf{z}_{\text{cmd}}(\tau))\, d\tau$ denote the integrated output tracking error. Augmenting Eq. (7) with the integrated output tracking error yields the extended system as follows

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\left(\mathbf{u}(t) + \mathbf{\Delta}(\mathbf{x}_p(t))\right) + \mathbf{B}_r\mathbf{z}_{\text{cmd}}(t) \\ \mathbf{z}(t) &= \mathbf{H}\mathbf{x}(t) \end{aligned} \tag{8}$$

where $\mathbf{x} \triangleq \left[\mathbf{x}_p,\ \mathbf{e}_{z_I}\right]^T \in \mathbb{R}^{n\times 1}$ $(n = n_p + m)$ is the extended state vector and

$$\begin{aligned} \mathbf{A} &\triangleq \begin{bmatrix} \mathbf{A}_p & \mathbf{0}_{n_p\times m} \\ \mathbf{H}_p & \mathbf{0}_{m\times m} \end{bmatrix} \in \mathbb{R}^{n\times n}, \quad \mathbf{B} \triangleq \begin{bmatrix} \mathbf{B}_p \\ \mathbf{0}_{m\times m} \end{bmatrix} \in \mathbb{R}^{n\times m} \\ \mathbf{B}_r &\triangleq \begin{bmatrix} \mathbf{0}_{n_p\times m} \\ -\mathbf{I}_{m\times m} \end{bmatrix} \in \mathbb{R}^{n\times m}, \qquad \mathbf{H} \triangleq \begin{bmatrix} \mathbf{H}_p & \mathbf{0}_{m\times m} \end{bmatrix} \in \mathbb{R}^{m\times n} \end{aligned} \tag{9}$$

Note that $(\mathbf{A}, \mathbf{B})$ is required to be controllable, and it is the case if and only if $(\mathbf{A}_p, \mathbf{B}_p)$ is controllable and $\det\left(\begin{bmatrix} \mathbf{A}_p & \mathbf{B}_p \\ \mathbf{H}_p & \mathbf{0}_{m\times m} \end{bmatrix}\right) \neq 0$.

2) Uncertainty Model: Uncertainty models can be distinguished by the presence or the lack of knowledge on the parametric structure of the uncertainty $\mathbf{\Delta}(\mathbf{x}_p(t))$. In this study, the uncertainty considered is of a linearly parameterized structure with a known nonlinear basis function vector. The following is assumed for the *structured uncertainty*.

Assumption 1 (Structured Uncertainty)**.**
The uncertainty $\mathbf{\Delta}(\mathbf{x}_p(t)) \in \mathbb{R}^{m\times 1}$ in the model of the system dynamics can be linearly parameterized and structured, that is, there exist a unique constant ideal parameter $\mathbf{W}^* \in \mathbb{R}^{q\times m}$ and a vector of continuously differentiable regressor functions $\mathbf{\Phi}(\mathbf{x}_p) = \begin{bmatrix} \phi_1(\mathbf{x}_p) & \cdots & \phi_q(\mathbf{x}_p) \end{bmatrix}^T \in \mathbb{R}^{q\times 1}$ such that

$$\mathbf{\Delta}(\mathbf{x}_p(t)) = \mathbf{W}^{*T}\mathbf{\Phi}(\mathbf{x}_p(t)) \tag{10}$$

3) Model Tracking Error Dynamics: The main strategy of the MRAC design is to make the states of a system follow those of a reference model system that characterizes the desired closed-loop response. A reference model system can be explained as the *ideal* closed-loop system obtainable with the nominal control if there is no uncertainty in the system. In this regard, it is first assumed that given a Hurwitz closed-loop state matrix $\mathbf{A}_r$, there exists a nominal baseline full-state feedback control $\mathbf{u}_{\text{base}} = -\mathbf{K}\mathbf{x}$ such that the gain $\mathbf{K}$ satisfies $\mathbf{A}_r = \mathbf{A} - \mathbf{B}\mathbf{K}$. Then, the reference model can be represented as,

$$\begin{aligned} \dot{\mathbf{x}}_r(t) &= \mathbf{A}_r\mathbf{x}_r(t) + \mathbf{B}_r\mathbf{z}_{\text{cmd}}(t) \\ \mathbf{z}_r(t) &= \mathbf{H}\mathbf{x}_r(t) \end{aligned} \tag{11}$$

Note that, given $\mathbf{A}_r$ is Hurwitz, for any positive definite symmetric matrix $\mathbf{Q} \in \mathbb{R}^{n\times n}$, there exists a positive definite symmetric matrix $\mathbf{P} \in \mathbb{R}^{n\times n}$ satisfying the following Lyapunov equation.

$$\mathbf{A}_r{}^T\mathbf{P} + \mathbf{P}\mathbf{A}_r + \mathbf{Q} = \mathbf{0} \tag{12}$$

Now, the control law for the uncertain system of Eq. (8) can be designed as,

$$\mathbf{u} = \mathbf{u}_{\text{base}} - \mathbf{u}_{\text{ad}} = -\mathbf{K}\mathbf{x} - \mathbf{u}_{\text{ad}} \tag{13}$$

where $\mathbf{u}_{\text{base}}$ is the *nominal baseline control*, and $\mathbf{u}_{\text{ad}}$ is the *adaptive augmentation*. Then, for the *model tracking error* defined as $\mathbf{e}(t) \triangleq \mathbf{x}_r(t) - \mathbf{x}(t)$, the tracking error dynamics can be written as follows:

$$\dot{\mathbf{e}}(t) = \mathbf{A}_r\mathbf{e}(t) + \mathbf{B}\boldsymbol{\epsilon}(t) \tag{14}$$

where $\boldsymbol{\epsilon}(t) = \mathbf{u}_{\text{ad}}(t) - \mathbf{\Delta}(\mathbf{x}_p(t)) \in \mathbb{R}^{m\times 1}$ is the *adaptation error*. The purpose of introducing the adaptive augmentation in the control law is to cancel out the effect of uncertainty from the tracking error dynamics. The adaptive augmentation can be designed as,

$$\mathbf{u}_{\text{ad}}(t) = \hat{\mathbf{\Delta}}(\mathbf{x}_p(t)) = \hat{\mathbf{W}}^T(t)\mathbf{\Phi}(\mathbf{x}_p(t)) \tag{15}$$

where $\hat{\mathbf{W}}$ is the estimate of the ideal parameter. To cancel out the uncertainty as much as possible, it is desirable to design the adaptive augmentation as the best possible approximation of the uncertainty. In other words, the estimate $\hat{\mathbf{W}}$ should be as close as possible to the ideal value $\mathbf{W}^*$. Let $\tilde{\mathbf{W}}(t) \triangleq \hat{\mathbf{W}}(t) - \mathbf{W}^*$ denote the *parameter estimation error*, and note that $\dot{\tilde{\mathbf{W}}} = \dot{\hat{\mathbf{W}}}$. The model tracking error dynamics given in Eq. (14) can be rewritten as follows:

$$\dot{\mathbf{e}}(t) = \mathbf{A}_r\mathbf{e}(t) + \mathbf{B}\tilde{\mathbf{W}}^T(t)\mathbf{\Phi}(\mathbf{x}_p(t)) \tag{16}$$

It can be observed from Eq. (16) that the parameter estimation error enters into the tracking error dynamics. It is obvious that the ultimate objective for the design of an adaptation law is to keep the parameter estimation error as small as possible, to maintain the nominal control performance even under uncertainty. This might imply that $\hat{\mathbf{W}}$ must be optimally determined in the sense of the minimal mean-squared error or similar. However, note that since the parameter estimation error has its own dynamics which is determined by the adaptation law, the transient behaviour of $\mathbf{e}$ and $\tilde{\mathbf{W}}$ are coupled. Therefore, they must be carefully addressed together.

III. Filtered System Dynamics

This section describes the regressor filtering scheme and the resultant low-pass-filtered system dynamics before proceding to the adaptation law design. The main purpose of the regressor filtering is to avoid the usage of error derivative estimates in parameter adaptation which could complicate the design and implementation of the adaptive control system. It is more advantageous than using a fixed-point/lag Kalman smoother for error derivative estimation, in terms of the implementation simplicity, the computational cost, and the rigor of further analysis. The developments of this section is similar to the regressor filtering scheme described in [16].

Since $\mathbf{B}_p$ is assumed to have linearly independent columns, $\mathbf{B}$ has full column rank. Then, the following equation can be obtained from Eq. (14)

$$\mathbf{u}_{\mathrm{ad}}(t) - \mathbf{B}^{\dagger}\left[\dot{\mathbf{e}}(t) - \mathbf{A}_r \mathbf{e}(t)\right] = \boldsymbol{\Delta}(\mathbf{x}_p(t)) = \mathbf{W}^{*T}\boldsymbol{\Phi}(\mathbf{x}_p(t)) \tag{17}$$

where $(\cdot)^{\dagger}$ denotes the Moore-Penrose pseudoinverse so that $\mathbf{B}^{\dagger} = \left(\mathbf{B}^T\mathbf{B}\right)^{-1}\mathbf{B}^T$. Note that the pseudoinverse provides a solution in least squares sense, when Eq. (14) is looked at as a system of linear equations. Without loss of generality, the initial tracking error $\mathbf{e}(t_0)$ is set to be zero. Then, the Laplace transform of Eq. (17) yields

$$\mathbf{u}_{\mathrm{ad}}(s) - \mathbf{B}^{\dagger}\left(s\mathbf{I}_{n\times n} - \mathbf{A}_r\right)\mathbf{e}(s) = \boldsymbol{\Delta}(\mathbf{x}_p) = \mathbf{W}^{*T}\boldsymbol{\Phi}(s) \tag{18}$$

Consider a stable linear first-order low-pass filter of which the transfer function is given by $F(s) = \frac{1}{\tau_f s+1}$ where $\tau_f > 0$ is the time-constant of the filter. Note that $sF(s) = \frac{1}{\tau_f}(1 - F(s))$. Multiplying each side of Eq. (18) by $F(s)$ gives the filtered system dynamics of uncertainty in the s-domain as follows

$$\begin{aligned}\mathbf{u}_{\mathrm{ad}_f}(s) - \mathbf{B}^{\dagger}\left[\frac{1}{\tau_f}\mathbf{e}(s) - \left(\frac{1}{\tau_f}\mathbf{I}_{n\times n} + \mathbf{A}_r\right)\mathbf{e}_f(s)\right] \\ = \boldsymbol{\Delta}_f(\mathbf{x}_p) = \mathbf{W}^{*T}\boldsymbol{\Phi}_f(s)\end{aligned} \tag{19}$$

where the subscript f is for a signal filtered by $F(s)$, i.e., $\boldsymbol{\alpha}_f(s) = F(s)\boldsymbol{\alpha}(s)$. Therefore, the inverse Laplace transform of Eq. (19) yields the filtered system dynamics in the t-domain:

$$\boldsymbol{\chi}(t) \triangleq \boldsymbol{\xi}(t) - \frac{1}{\tau_f}\mathbf{B}^{\dagger}\mathbf{e}(t) = \mathbf{W}^{*T}\boldsymbol{\eta}(t) \tag{20}$$

$$\dot{\boldsymbol{\xi}}(t) = \frac{1}{\tau_f}\left[\mathbf{u}_{\mathrm{ad}}(t) + \mathbf{B}^{\dagger}\left(\frac{1}{\tau_f}\mathbf{I}_{n\times n} + \mathbf{A}_r\right)\mathbf{e}(t) - \boldsymbol{\xi}(t)\right] \tag{21}$$

$$\dot{\boldsymbol{\eta}}(t) = \frac{1}{\tau_f}\left(\boldsymbol{\Phi}(\mathbf{x}_p(t)) - \boldsymbol{\eta}(t)\right) \tag{22}$$

where $\boldsymbol{\xi}(t_0) = \mathbf{0}_{m\times 1}$ and $\boldsymbol{\eta}(t_0) = \mathbf{0}_{q\times 1}$. The output $\boldsymbol{\chi}(t)$ can be computed at every time instance using the known signals $\boldsymbol{\xi}(t)$ and $\mathbf{e}(t)$. The filtered regressor $\boldsymbol{\eta}(t)$ is also a known signal. Therefore, from Eqs. (20)-(22), it is clear that the unknown parameter can be estimated from the information of the known signals as linear regression. Note that the signal $\boldsymbol{\chi}(t)$ will act as the measurement for parameter estimation in the proposed adaptation law, since it contains information about the ideal parameter.

For further development, following assumption on the degree of excitation in the filtered regressor is required.

Assumption 2 (Finite Excitation of Filtered Regressor)**.** *There exist $t_s \geq t_0$ and $t_e > t_s$ such that filtered regressor $\boldsymbol{\eta}(t)$ has FE over $[t_s, t_e]$.*

The meaning of Assumption 2 will be discussed in the following section.

IV. Adaptation Law for Parameter Convergence Without Persistent Excitation

In this section, a new adaptation law is first proposed for the case of structured uncertainty to improve parameter convergence property without requiring persistent excitation. Then, stability and performance analysis of the overall closed-loop system with the proposed scheme is performed based on Lyapunov stability theory.

A. Design of Adaptation Law

In this case where $\boldsymbol{\delta} \equiv \mathbf{0}$, the unknown parameter $\mathbf{W}^*$ can be estimated from the knowledge of the measurable signals $\boldsymbol{\chi}(t)$ and $\boldsymbol{\eta}(t)$. A continuous-time parameter estimation law for linear regression generally takes the following form

$$\dot{\hat{\mathbf{W}}}(t) = -\boldsymbol{\Gamma}(t)\boldsymbol{\varphi}(t) = -\boldsymbol{\Gamma}(t)\boldsymbol{\Omega}(t)\tilde{\mathbf{W}}(t) \tag{23}$$

where $\boldsymbol{\Gamma}(t)$ is a square and positive *gain matrix*, $\boldsymbol{\varphi}(t) = \boldsymbol{\Omega}(t)\tilde{\mathbf{W}}(t)$ is a *residual matrix*, and $\boldsymbol{\Omega}(t)$ is a symmetric *information matrix*. A natural observation from Eq. (23) is that the convergence properties and the performance of a parameter estimator significantly depend on two factors: how 1) the gain matrix and 2) the residual matrix are designed.

A residual is the signal containing the information on the parameter estimation error. There are various ways to construct a residual. The simplest one is $\boldsymbol{\varphi}(t) \triangleq \boldsymbol{\eta}(t)\left[\hat{\mathbf{W}}^T(t)\boldsymbol{\eta}(t) - \boldsymbol{\chi}(t)\right]^T = \boldsymbol{\eta}(t)\boldsymbol{\eta}^T(t)\tilde{\mathbf{W}}(t)$. However, the corresponding information matrix $\boldsymbol{\Omega}(t) = \boldsymbol{\eta}(t)\boldsymbol{\eta}^T(t)$ is always at most rank 1, and thus only positive semidefinite. The rank deficiency of the information matrix in this case is the fundamental cause of the PE requirement for the parameter convergence. Therefore, to mitigate the PE requirement, the corresponding information matrix of the residual should be designed to possess full rank.

In this regard, a new way of residual design is proposed. First, the *information matrix* $\boldsymbol{\Omega}(t)$ and the *auxiliary matrix* $\mathbf{M}(t)$ are designed as follows:

$$\begin{aligned}\dot{\boldsymbol{\Omega}}(t) &= -k(t)\boldsymbol{\Omega}(t) + \boldsymbol{\eta}(t)\boldsymbol{\eta}^T(t) \qquad &\boldsymbol{\Omega}(t_0) = \mathbf{0}_{q\times q} \\ \dot{\mathbf{M}}(t) &= -k(t)\mathbf{M}(t) + \boldsymbol{\eta}(t)\boldsymbol{\chi}^T(t) \qquad &\mathbf{M}(t_0) = \mathbf{0}_{q\times m}\end{aligned} \tag{24}$$

where $k(t)$ is a forgetting factor bounded by positive constants, i.e., $0 < k_L \le k(t) \le k_U$. Because $k(t)$ is a scalar, the solution of Eq. (24) can be written as

$$\begin{aligned}\mathbf{\Omega}(t) &= \int_{t_0}^{t} \exp\left(-\int_{\tau}^{t} k(\nu)\,d\nu\right)\boldsymbol{\eta}(\tau)\,\boldsymbol{\eta}^T(\tau)\,d\tau \\ \mathbf{M}(t) &= \int_{t_0}^{t} \exp\left(-\int_{\tau}^{t} k(\nu)\,d\nu\right)\boldsymbol{\eta}(\tau)\,\boldsymbol{\chi}^T(\tau)\,d\tau = \mathbf{\Omega}(t)\,\mathbf{W}^*\end{aligned} \tag{25}$$

Equation (25) shows that the rank of the information matrix can be populated to the full rank over time, if the direction of $\boldsymbol{\eta}(\tau)$ varies sufficiently, not persistently. The information matrix will have full rank after a certain moment, unless the vector $\boldsymbol{\eta}(\tau)$ lies on an affine hyperplane for entire time interval $[t_0, t]$. The sufficiency of finite direction change is the implication of Assumption 2, and having a full rank information matrix relaxes the requirement for parameter convergence from PE to FE.

The forgetting term in Eq. (24) is to prevent the degeneration of information update in some direction by putting more weight on the recent data, and also to make the information matrix be upper bounded in its norm. In addition, it is advantageous to increase the weight on the data that contain richer information by increasing $k(t)$. This can be done by putting a larger weight for the data comes from faster variation in $\boldsymbol{\eta}$. Hence, this paper proposes a sigmoidal design:

$$k(t) = k_L + (k_U - k_L)\tanh(\vartheta\,\|\dot{\boldsymbol{\eta}}\|) \tag{26}$$

where $\vartheta > 0$ is a constant design parameter, and $\dot{\boldsymbol{\eta}}$ is of Eq. (22).

Next, the proposed method utilizes the accumulated information in a selective manner. The information matrix at a time instance t is a weighted accumulation of all incoming data from t_0 up to t. However, if the excitation in $\boldsymbol{\eta}$ is only finite, the information matrix will diminish after the end of excitation, due to the forgetting design. This implies degradation of the information quality. Therefore, in the case of FE, using whole information of the entire time interval is not preferred for good and consistent parameter estimation performance. To this end, the *adequate information matrix* $\mathbf{\Omega}_a(t)$ and the *adequate auxiliary matrix* $\mathbf{M}_a(t)$ are designed as follows:

$$\begin{gathered} t_a \triangleq \max\left\{\underset{\tau\in[t_0,t]}{\arg\max}\,\mathcal{F}(\mathbf{\Omega}(\tau))\right\} \\ \mathbf{\Omega}_a(t) \triangleq \mathbf{\Omega}(t_a),\ \mathbf{M}_a(t) \triangleq \mathbf{M}(t_a)\end{gathered} \tag{27}$$

where $\mathcal{F}(\cdot)$ is a measure for the quality of information. It can be inferred from Eq. (27) that $\frac{d\mathcal{F}(\mathbf{\Omega}_a(t))}{dt} \ge 0$ for $\forall t \ge t_0$. The choice of the information measure $\mathcal{F}(\cdot)$ will determine the update direction of $\mathbf{\Omega}_a$. At this point, suppose that $\mathcal{F}(\cdot) = \lambda_{\min}(\cdot)$ for simplicity of analysis. Note that the minimum eigenvalue $\lambda_{\min}(\mathbf{\Omega})$ is an indicator for the evenness of excitation over all eigenvectors of $\mathbf{\Omega}$. It is obvious from Eqs. (25) and (27) that

$$\mathbf{M}_a(t) = \mathbf{\Omega}_a(t)\,\mathbf{W}^* \tag{28}$$

Finally, this paper proposes a new adaptation law as

$$\dot{\hat{\mathbf{W}}}(t) = -\mathbf{\Gamma}_w\left[\mathbf{\Phi}(\mathbf{x}_p(t))\,\mathbf{e}^T(t)\,\mathbf{PB} + R\left(\mathbf{\Omega}_a(t)\,\hat{\mathbf{W}}(t) - \mathbf{M}_a(t)\right)\right] \tag{29}$$

where $\mathbf{\Gamma}_w > 0$ is a constant adaptation gain matrix, $R > 0$ is a scalar relative weight on the parameter-estimation-based modification term, and $\mathbf{P} = \mathbf{P}^T > 0$ is the solution of Eq. (12) for a given $\mathbf{Q} = \mathbf{Q}^T > 0$. Note that a constant adaptation gain is used in the proposed method to make the overall control performance adjustable by gain tuning and to reduce the computational load resulting from gain calculation.

B. Stability and Performance Analysis

From Eqs. (16), (28), and (29), the closed-loop system dynamics of the tracking error $\mathbf{e}$ and the parameter estimation error $\tilde{\mathbf{W}}$ can be written as

$$\begin{aligned}\dot{\mathbf{e}} &= \mathbf{A}_r\mathbf{e} + \mathbf{B}\tilde{\mathbf{W}}^T\mathbf{\Phi}(\mathbf{x}_p), & \mathbf{e}(t_0) = \mathbf{0} \\ \dot{\tilde{\mathbf{W}}} &= -\mathbf{\Gamma}_w\left[\mathbf{\Phi}(\mathbf{x}_p)\,\mathbf{e}^T\mathbf{PB} + R\mathbf{\Omega}_a\tilde{\mathbf{W}}\right]\end{aligned} \tag{30}$$

The equilibrium point of Eq. (30) is $\left(\mathbf{e}, \vec{\tilde{\mathbf{W}}}\right) = (\mathbf{0}, \mathbf{0})$.

In Lemma 2, the adequate information matrix is shown to be positive definite after FE. Using this result, the stability of the equilibrium point is shown in Theorem 1, and the transient performance guarantee is given in Corollary 1.

Lemma 2 (Positive Definiteness and Minimum Eigenvalue of Adequate Information Matrix)**.**
With the FE condition as stated in Assumption 2 and the choice of $\mathcal{F}(\cdot)$ by $\lambda_{\min}(\cdot)$,

- $\mathbf{\Omega}_a(t) \ge 0$ *for* $\forall t \ge t_0$.
- $\mathbf{\Omega}_a(t) > 0$ *for* $\forall t \ge t_e$.
- $\lambda_{\min}(\mathbf{\Omega}_a(t)) \ge \lambda_{\min}(\mathbf{\Omega}_a(t_e)) > 0$ *for* $\forall t \ge t_e$.

Proof: Consider the following quadratic form related to the real symmetric $\mathbf{\Omega}(t)$ given in Eq. (25)

$$\begin{aligned}\mathbf{v}^T\mathbf{\Omega}(t)\,\mathbf{v} &= \mathbf{v}^T\int_{t_0}^{t} K(t,\tau)\,\boldsymbol{\eta}(\tau)\,\boldsymbol{\eta}^T(\tau)\,d\tau\mathbf{v} \\ &= \int_{t_0}^{t}\left[\sqrt{K(t,\tau)}\mathbf{v}\cdot\boldsymbol{\eta}(\tau)\right]^2 d\tau\end{aligned} \tag{31}$$

where $\mathbf{v} \in \mathbb{R}^{q\times 1}$ and $0 < K(t,\tau) = \exp\left(-\int_{\tau}^{t} k(\nu)\,d\nu\right) \le 1$. It is obvious that $\mathbf{\Omega}(t) \ge 0$ for $\forall t \ge t_0$, because $\mathbf{v}^T\mathbf{\Omega}(t)\,\mathbf{v} \ge 0$ for $\forall\mathbf{v}$ and $\forall t \ge t_0$. Therefore, $\mathbf{\Omega}_a(t) \ge 0$ for $\forall t \ge t_0$.

If $\boldsymbol{\eta}$ has FE as stated in Assumption 2, then $\int_{t_s}^{t_e}\boldsymbol{\eta}(\tau)\,\boldsymbol{\eta}^T(\tau)\,d\tau > 0$. According to Eq. (3), if $\int_{t_s}^{t_e}\boldsymbol{\eta}(\tau)\,\boldsymbol{\eta}^T(\tau)\,d\tau > 0$, then $\int_{t_0}^{t}\boldsymbol{\eta}(\tau)\,\boldsymbol{\eta}^T(\tau)\,d\tau > 0$ for $\forall t \ge t_e$. This is equivalent to the nonexistence of $\mathbf{v} \ne \mathbf{0}$ such that $\mathbf{v}\cdot\boldsymbol{\eta}(\tau) \equiv 0$ for $\forall\tau \in [t_0, t]$ where $t \ge t_e$. Consequently, Eq. (31) implies that $\mathbf{\Omega}(t) > 0$ for $\forall t \ge t_e$, because $\sqrt{K(t,\tau)}$ is a strictly positive scalar. If $\mathcal{F}(\cdot)$ in Eq. (27) is chosen by $\lambda_{\min}(\cdot)$, $\mathbf{\Omega}_a(t)$ will be updated at least once at some $t \in [t_0, t_e]$ to have nonzero minimum eigenvalue. Also, $\mathbf{\Omega}_a(t)$ will be updated only if there is any chance of increase in $\lambda_{\min}(\mathbf{\Omega}(t))$. Therefore, $\mathbf{\Omega}_a(t) > 0$ for $\forall t \ge t_e$, and $\lambda_{\min}(\mathbf{\Omega}_a(t)) \ge \lambda_{\min}(\mathbf{\Omega}_a(t_e)) > 0$ for $\forall t \ge t_e$. ∎

Theorem 1 (Global Exponential Stability for the Case of Structured Uncertainty)**.**

With the control law given by Eqs. (13) and (15), the adaptation law given by Eq. (29), and the FE condition as stated in Assumption 2,

- *The trajectory* $\mathbf{e}(t)$ *and* $\vec{\tilde{\mathbf{W}}}(t)$ *are bounded for* $\forall t \geq t_0$*:*
- *The equilibrium point* $\left(\mathbf{e}, \vec{\tilde{\mathbf{W}}}\right) \equiv (\mathbf{0}, \mathbf{0})$ *is globally exponentially stable for* $\forall t \geq t_e$*.*

Proof: Consider the following positive definite and radially unbounded Lyapunov candidate function.

$$V\left(\mathbf{e}, \tilde{\mathbf{W}}\right) = \frac{1}{2}\mathbf{e}^T\mathbf{P}\mathbf{e} + \frac{1}{2}\operatorname{tr}\left(\tilde{\mathbf{W}}^T{\mathbf{\Gamma}_w}^{-1}\tilde{\mathbf{W}}\right) \tag{32}$$

Note that $V(\mathbf{0}, \mathbf{0}) = 0$, and $V\left(\mathbf{e}, \tilde{\mathbf{W}}\right) > 0$ for $\forall \left(\mathbf{e}, \tilde{\mathbf{W}}\right) \neq (\mathbf{0}, \mathbf{0})$. Let $\boldsymbol{\xi} \triangleq \left[\mathbf{e}^T \quad \vec{\tilde{\mathbf{W}}}^T\right]^T$, then the Lyapunov candidate function given by Eq. (32) is bounded from below and above as follows:

$$\begin{aligned}\frac{1}{2}\min\left\{\lambda_{\min}(\mathbf{P}), \lambda_{\min}\left({\mathbf{\Gamma}_w}^{-1}\right)\right\}\|\boldsymbol{\xi}\|^2 &\leq V\left(\mathbf{e}, \tilde{\mathbf{W}}\right)\\ &\leq \frac{1}{2}\max\left\{\lambda_{\max}(\mathbf{P}), \lambda_{\max}\left({\mathbf{\Gamma}_w}^{-1}\right)\right\}\|\boldsymbol{\xi}\|^2\end{aligned} \tag{33}$$

Using Eqs. (2), (12), and (30), the time derivative of Eq. (32) of the closed-loop system along the trajectory can be obtained as follows:

$$\begin{aligned}\dot{V}\left(\mathbf{e}, \tilde{\mathbf{W}}\right) &= \mathbf{e}^T\mathbf{P}\left(\mathbf{A}_r\mathbf{e} + \mathbf{B}\tilde{\mathbf{W}}^T\mathbf{\Phi}(\mathbf{x}_p)\right)\\ &\quad - \operatorname{tr}\left(\tilde{\mathbf{W}}^T\left[\mathbf{\Phi}(\mathbf{x}_p)\mathbf{e}^T\mathbf{P}\mathbf{B} + R\mathbf{\Omega}_a\tilde{\mathbf{W}}\right]\right)\\ &= -\frac{1}{2}\mathbf{e}^T\mathbf{Q}\mathbf{e} - R\operatorname{tr}\left(\tilde{\mathbf{W}}^T\mathbf{\Omega}_a\tilde{\mathbf{W}}\right)\end{aligned} \tag{34}$$

From Eq. (34), $\dot{V}\left(\mathbf{e}, \tilde{\mathbf{W}}\right) \leq 0$ for $\forall t \geq t_0$, because $\mathbf{\Omega}_a(t) \geq 0$ for $\forall t \geq t_0$. Therefore, $\mathbf{e}(t)$ and $\vec{\tilde{\mathbf{W}}}(t)$ are bounded for all $t \geq t_0$.

Next, suppose that the FE condition of Assumption 2 is met and $\mathcal{F}(\cdot) = \lambda_{\min}(\cdot)$. Then, the following inequality can be obtained from Eq. (34) using Eqs. (1), (4), and Lemma 2.

$$\begin{aligned}\dot{V}\left(\mathbf{e}, \tilde{\mathbf{W}}\right) &\leq -\frac{1}{2}\lambda_{\min}(\mathbf{Q})\|\mathbf{e}\|^2 - R\lambda_{\min}(\mathbf{\Omega}_a(t_e))\left\|\vec{\tilde{\mathbf{W}}}\right\|^2\\ &\leq -\frac{1}{2}\min\left\{\lambda_{\min}(\mathbf{Q}), 2R\lambda_{\min}(\mathbf{\Omega}_a(t_e))\right\}\|\boldsymbol{\xi}\|^2\end{aligned} \tag{35}$$

From Eqs. (33) and (35), we have

$$\dot{V}\left(\mathbf{e}, \tilde{\mathbf{W}}\right) \leq -\alpha V\left(\mathbf{e}, \tilde{\mathbf{W}}\right) \tag{36}$$

where $\alpha \triangleq \frac{\min\{\lambda_{\min}(\mathbf{Q}), 2R\lambda_{\min}(\mathbf{\Omega}_a(t_e))\}}{\max\left\{\lambda_{\max}(\mathbf{P}), \lambda_{\max}\left({\mathbf{\Gamma}_w}^{-1}\right)\right\}} > 0$. By the comparison lemma, it can be concluded that $V \to 0$ uniformly exponentially fast, and therefore the equilibrium point $\left(\mathbf{e}, \vec{\tilde{\mathbf{W}}}\right) \equiv (\mathbf{0}, \mathbf{0})$ is globally exponentially stable. ■

Corollary 1 (Performance Guarantee for the Case of Structured Uncertainty)**.**

The bounds for the tracking error and the parameter estimation error can be derived as follows:

$$\begin{aligned}\|\mathbf{e}(t)\| &\leq \begin{cases}\sqrt{\dfrac{\lambda_{\max}\left({\mathbf{\Gamma}_w}^{-1}\right)}{\lambda_{\min}(\mathbf{P})}}\left\|\tilde{\mathbf{W}}(t_0)\right\|_F & \text{for } t_0 \leq t \leq t_e\\ \sqrt{\dfrac{\lambda_{\max}\left({\mathbf{\Gamma}_w}^{-1}\right)}{\lambda_{\min}(\mathbf{P})}\exp(-\alpha(t - t_e))}\left\|\tilde{\mathbf{W}}(t_0)\right\|_F & \text{for } t \geq t_e\end{cases}\\ \left\|\tilde{\mathbf{W}}(t)\right\|_F &\leq \begin{cases}\sqrt{\dfrac{\lambda_{\max}\left({\mathbf{\Gamma}_w}^{-1}\right)}{\lambda_{\min}\left({\mathbf{\Gamma}_w}^{-1}\right)}}\left\|\tilde{\mathbf{W}}(t_0)\right\|_F & \text{for } t_0 \leq t \leq t_e\\ \sqrt{\dfrac{\lambda_{\max}\left({\mathbf{\Gamma}_w}^{-1}\right)}{\lambda_{\min}\left({\mathbf{\Gamma}_w}^{-1}\right)}\exp(-\alpha(t - t_e))}\left\|\tilde{\mathbf{W}}(t_0)\right\|_F & \text{for } t \geq t_e\end{cases}\end{aligned} \tag{37}$$

Proof: For simplicity of writing, let $V(t)$ denotes $V\left(\mathbf{e}(t), \tilde{\mathbf{W}}(t)\right)$. First of all, the following is true about the Lyapunov function defined in Eq. (32) for all $t \geq t_0$.

$$\begin{aligned}\frac{1}{2}\lambda_{\min}(\mathbf{P})\|\mathbf{e}(t)\|^2 &\leq \frac{1}{2}\mathbf{e}^T\mathbf{P}\mathbf{e} \leq V(t)\\ \frac{1}{2}\lambda_{\min}\left({\mathbf{\Gamma}_w}^{-1}\right)\left\|\tilde{\mathbf{W}}(t)\right\|_F^2 &\leq \frac{1}{2}\operatorname{tr}\left(\tilde{\mathbf{W}}^T{\mathbf{\Gamma}_w}^{-1}\tilde{\mathbf{W}}\right) \leq V(t)\end{aligned} \tag{38}$$

Note that $\lambda_{\min}(\mathbf{P}) > 0$ and $\lambda_{\min}\left({\mathbf{\Gamma}_w}^{-1}\right) > 0$, since $\mathbf{P}$ and $\mathbf{\Gamma}_w$ are positive definite. Rewriting Eq. (38) yields

$$\|\mathbf{e}(t)\| \leq \sqrt{\frac{2V(t)}{\lambda_{\min}(\mathbf{P})}}\ , \quad \left\|\tilde{\mathbf{W}}(t)\right\|_F \leq \sqrt{\frac{2V(t)}{\lambda_{\min}\left({\mathbf{\Gamma}_w}^{-1}\right)}} \tag{39}$$

Consider first the time interval in which $\mathbf{\Omega}_a$ is rank deficient, namely $t \in [t_0, t_e]$. Because $\dot{V}(t) \leq 0$ as shown in Eq. (34) regardless of whether $\operatorname{rank}(\mathbf{\Omega}_a)$ is full or not, we have

$$\begin{aligned}V(t_e) \leq V(t) \leq V(t_0) &= \frac{1}{2}\operatorname{tr}\left(\tilde{\mathbf{W}}^T(t_0){\mathbf{\Gamma}_w}^{-1}\tilde{\mathbf{W}}(t_0)\right)\\ &\leq \frac{1}{2}\lambda_{\max}\left({\mathbf{\Gamma}_w}^{-1}\right)\left\|\tilde{\mathbf{W}}(t_0)\right\|_F^2\end{aligned} \tag{40}$$

Therefore, Eq. (37) for $t_0 \leq t \leq t_e$ can be obtained from Eqs. (39) and (40).

Next, consider the right-infinite time interval in which $\mathbf{\Omega}_a$ is full rank, namely $t \geq t_e$. In this case, $\dot{V}(t) \leq -\alpha V(t)$ as shown in Eq. (36). By the comparison lemma and Eq. (40), we have

$$\begin{aligned}V(t) &\leq V(t_e)\exp(-\alpha(t - t_e)) \leq V(t_0)\exp(-\alpha(t - t_e))\\ &= \frac{1}{2}\operatorname{tr}\left(\tilde{\mathbf{W}}^T(t_0){\mathbf{\Gamma}_w}^{-1}\tilde{\mathbf{W}}(t_0)\right)\exp(-\alpha(t - t_e))\\ &\leq \frac{1}{2}\lambda_{\max}\left({\mathbf{\Gamma}_w}^{-1}\right)\left\|\tilde{\mathbf{W}}(t_0)\right\|_F^2\exp(-\alpha(t - t_e))\end{aligned} \tag{41}$$

Therefore, Eq. (37) for $t \geq t_e$ can be obtained from Eqs. (39) and (41). ■

Remark 1 (Interpretation of Corollary 1)**.**
Generally, it is hard to predict the exact time when the information matrix will be full rank (and thus positive definite) in practice, and therefore the time instance t_e *is difficult to be known a priori. Nevertheless, Corollary 1 guarantees the boundedness of the tracking error and the parameter estimation error during the transition, and exponential convergence to zero after once the information matrix becomes full rank.*

Remark 2 (Relation Between Transient Performance and Adaptation Gain)**.**
It can be concluded from Eq. (37) that a higher adaptation gain results in a smaller upper bound of the tracking error.

However, this is an ideal statement without considering any extra disturbance, noise, and time delay in the system. In a real system, there is a tradeoff between the tracking error performance and the robustness.

Remark 3 (Discussions on Adequate Information Matrix Update)**.**
The procedure of Eq. (27) can be implemented as a simple comparison of the current information measure and the previous one. If the current one is better than the previous one with respect to a chosen information measure, then update $\mathbf{\Omega}_a$ by the current one, otherwise keep the previous one. Thus, the update direction is determined by types of information quality measure $\mathcal{F}(\cdot)$.

In the previous section, the minimum eigenvalue was chosen as the measure of information quality for the simplicity of stability analysis. Other choices such as the determinant, trace, or reciprocal of condition number are available as the options for $\mathcal{F}(\cdot)$. If the determinant or the trace is used, then $\mathbf{\Omega}_a$ will be updated if the 'volume' of $\mathbf{\Omega}$ increases in any eigen-direction. If the reciprocal of condition number is used, then $\mathbf{\Omega}_a$ will be updated if the 'uniformity' of $\mathbf{\Omega}$ over all eigenvectors is improved. It is hard to quantitatively analyze the effect of different choices of $\mathcal{F}(\cdot)$ on the convergence characteristics.

C. Comparison with Existing Adaptation Schemes

1) Comparison with the Simple Direct Adaptation Law: The simple direct adaptation law was proposed as follows:

$$\dot{\hat{\mathbf{W}}}(t) = -\mathbf{\Gamma}_w \mathbf{\Phi}(\mathbf{x}_p(t))\, \mathbf{e}^T(t)\, \mathbf{PB} \tag{42}$$

The control law of Eqs. (13) and (15) together with the adaptation law of Eq. (42) guarantees the asymptotic stability of the tracking error. However, there is no guarantee on parameter convergence. It is because the simple direct adaptation law only performs canceling of the coupling between the tracking error dynamics and the parameter estimation error dynamics. On the other hand, the proposed adaptation law of Eq. (29) is a *composite* adaptation law consisting of the direct adaptive term and the parameter-estimation-based modification term. After FE, the proposed adaptation law can guarantee exponential stability of both tracking error and parameter estimation error.

The parameter estimation error dynamics under the simple direct adaptation law is a pure integrator. Given a high adaptation gain, if the uncertainty is significant or there is an abrupt change in the plant, then the parameter estimation error may violently oscillate due to the gradient nature of the simple direct adaptation law. High-frequency oscillations in the parameter estimation error lead to high-frequency oscillations in the control input. Subsequently, violent oscillations in the control input could degrade stability of the overall system and induce unwanted high-frequency excitation of unmodelled dynamics. On the contrary, the parameter estimation error dynamics under the proposed adaptation law has a low-pass filter form as can be seen in Eq. (30). Therefore, a higher adaptation gain could be used in the proposed adaptation law with less sacrifice of robustness, compared with the simple direct adaptation law.

2) Comparison with the Weighted Least Squares Estimation-based Composite Adaptation Law: In [15], [34], various online parameter estimators for composite adaptive control were proposed, and their characteristics were compared. The Weighted Least Squares (WLS) estimator was derived by minimizing the following L^2 regression error

$$J = \int_{t_0}^{t} \exp\left(-\int_{\tau}^{t} k(\nu)\, d\nu\right) \left\|\boldsymbol{\chi}(\tau) - \hat{\mathbf{W}}^T(t)\, \boldsymbol{\eta}(\tau)\right\|^2 d\tau \tag{43}$$

with respect to $\hat{\mathbf{W}}(t)$. The WLS-estimation-based composite adaptation law can be written as follows

$$\begin{aligned} \dot{\hat{\mathbf{W}}} &= -\mathbf{\Gamma}_w(t)\left[\mathbf{\Phi}(\mathbf{x}_p(t))\, \mathbf{e}^T(t)\, \mathbf{PB} + R\boldsymbol{\eta}(t)\left(\boldsymbol{\eta}^T(t)\, \hat{\mathbf{W}}(t) - \boldsymbol{\chi}^T(t)\right)\right] \\ \dot{\mathbf{\Gamma}}_w(t) &= k(t)\, \mathbf{\Gamma}_w(t) - \mathbf{\Gamma}_w(t)\, \boldsymbol{\eta}(t)\, \boldsymbol{\eta}^T(t)\, \mathbf{\Gamma}_w(t) \end{aligned} \tag{44}$$

The proposed method of Eq. (29) and the WLS method of Eq. (44) have similar structure. In the weighted least squares method, the data from the past to the present is reflected into the time-varying adaptation gain and the information matrix is at most rank 1. For this reason, the PE condition is required for parameter convergence in the WLS-estimation-based composite adaptation law. In comparison, in the proposed method, the data from the past to the present is reflected into the rank-populating information matrix and the constant gain is used. These differences make the proposed method more practical from the perspective that considers control as the primary objective, because the PE is not required and the overall performance can be adjusted by the designer.

3) Comparison with Concurrent Learning Adaptation Law: In [25], the concurrent learning adaptation law was proposed as follows:

$$\dot{\hat{\mathbf{W}}} = -\mathbf{\Gamma}_w \left[\mathbf{\Phi}(\mathbf{x}_p(t))\, \mathbf{e}^T(t)\, \mathbf{PB} + \sum_{i=1}^{h} \mathbf{\Phi}(\mathbf{x}_p(t_i))\, {\boldsymbol{\rho}_i}^T(t)\right] \tag{45}$$

where $\boldsymbol{\rho}_i(t) = \hat{\mathbf{W}}^T(t)\, \mathbf{\Phi}(\mathbf{x}_p(t_i)) - \Big\{\hat{\mathbf{W}}^T(t_i)\, \mathbf{\Phi}(\mathbf{x}_p(t_i)) - \mathbf{B}^{\dagger}\left[\dot{\mathbf{e}}(t_i) - \mathbf{A}_r \mathbf{e}(t_i)\right]\Big\}$, t_i is the time instance when the i-th data point is stored, and $h \geq q$ is the size of history stack. In both concurrent learning and proposed scheme, the information matrix can have full rank after sufficient finite excitation, and therefore, the PE requirement for parameter convergence can be relaxed. However, the concurrent learning method utilizes a different way of constructing the residual signal. Unlike the proposed adaptation scheme which accumulates all incoming data, the concurrent learning method picks and stores only a prespecified number of data. A new data can be incorporated into the history stack of a finite size only if the quality of information matrix can be improved with that data.

The concurrent learning adaptation requires knowledge about $\dot{\mathbf{e}}$. All the theoretical analysis such as stability is performed under the assumption of perfect knowledge of $\dot{\mathbf{e}}$ in the concurrent learning adaptation. Since $\dot{\mathbf{e}}$ cannot be perfectly known or measured in practice, the concurrent learning adaptation obtains its estimate, $\hat{\dot{\mathbf{e}}}(t_i)$, using the Kalman fixed-point smoother. This is disadvantageous in terms of rigor of analysis, simplicity of implementation, and computational load. In contrast, the proposed adaptation scheme adopted regressor filtering to remove the need for derivative estimation, and therefore it requires only forward integration. In this way, the stability and performance of the proposed adaptive control system can be theoretically investigated without any assumption on the perfect knowledge of $\dot{\mathbf{e}}$, unlike in the concurrent learning. Also, the adequate information matrix update is based on a simple comparison between the current step and the previous step, whereas the data quality test performed at every step in the concurrent learning adaptation can be exhaustive.

V. Numerical Simulation

This section presents numerical simulation results to demonstrate the performance of the proposed method compared to the standard direct adaptation law.

A virtual-control-augmented model for the wing-rock phenomenon in the roll motion of slender delta wings studied in [35] is used for numerical simulation in this study. In this study, we multiplied 1000 to the value of ideal parameters given in [35] intentionally to increase the amount of uncertainty effect. Note that the time t, state $\mathbf{x}$, and control $\mathbf{u}$ of the model are nondimensional. The simulation model is given by

$$\begin{bmatrix} \dot{x}_{p_1} \\ \dot{x}_{p_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{p_1} \\ x_{p_2} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (\mathbf{u} + \boldsymbol{\Delta}(\mathbf{x}_p)) \tag{46}$$
$$\mathbf{z} = x_{p_1}$$

where x_{p_1} is the roll angle, x_{p_2} is the roll rate, $\mathbf{u}$ is a virtual control input, and $\boldsymbol{\Delta}(\mathbf{x}_p) = \mathbf{W}^{*T}\boldsymbol{\Phi}(\mathbf{x}_p)$ is a structured uncertainty. By multiplying 1000 to the specific model obtained at the angle of attack of 25deg in [35], the ideal parameter and basis of the uncertainty are given by

$$\mathbf{W}^* = \begin{bmatrix} -18.59521 \\ 15.162375 \\ -62.45153 \\ 9.54708 \\ 21.45291 \end{bmatrix}, \quad \boldsymbol{\Phi}(\mathbf{x}_p) = \begin{bmatrix} x_{p_1} \\ x_{p_2} \\ |x_{p_1}| x_{p_2} \\ |x_{p_2}| x_{p_2} \\ x_{p_1}{}^3 \end{bmatrix} \tag{47}$$

The nominal baseline controller $\mathbf{K}$ is designed by the infinite-horizon linear quadratic regulator which minimizes the following performance index.

$$J = \int_0^\infty \left(\mathbf{x}^T \mathbf{Q}_{\text{base}} \mathbf{x} + \mathbf{u}^T \mathbf{R}_{\text{base}} \mathbf{u}\right) d\tau \tag{48}$$

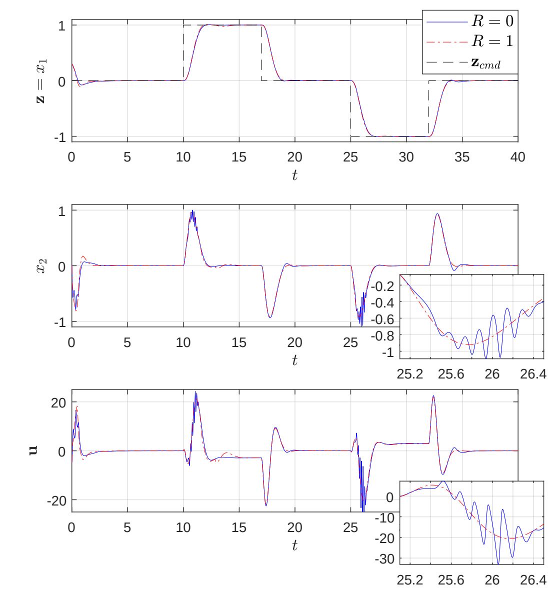
The forgetting factor proposed in Eq. (26) is used. Simulation parameters and initial conditions are summarized in Table I. Note that the proposed adaptation law of Eq. (29) with $R = 0$ is identical to the standard direct adaptation law of Eq. (42), and a high adaptation gain is used in both cases namely $R = 0$ and $R = 1$.

The output and control response of the closed-loop system, the tracking error and parameter estimation error history, and the Lyapunov function and information measure history are shown in Fig. 1. The performance of the standard direct adaptation law and the proposed method are summarized in Table II in terms of average tracking error and average parameter estimation error.

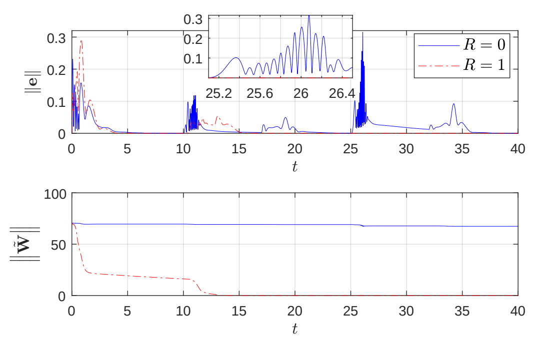
Figure 1(a) shows that the undesirable high-frequency oscillation in state and control response of the standard direct

TABLE I
Simulation Parameters

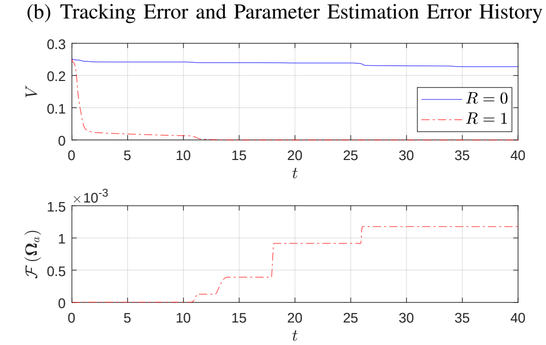
Parameter	Value	Parameter	Value
$\mathbf{x}(t_0)$, $\mathbf{x}_r(t_0)$	$\begin{bmatrix} 0.3 & 0 & 0 \end{bmatrix}^T$	$\mathbf{Q}$	$\mathbf{I}_{3\times3}$
$\hat{\mathbf{W}}(t_0)$	$\mathbf{0}_{5\times1}$	k_L	0.1
$\mathbf{Q}_{\text{base}}$	$\text{diag}(2800, 1, 15000)$	k_U	10
$\mathbf{R}_{\text{base}}$	50	ϑ	0.1
$\mathbf{z}_{\text{cmd}}(t)$	$\begin{cases} 1 & 10 \le t \le 17 \\ -1 & 25 \le t \le 32 \\ 0 & \text{else} \end{cases}$	R	0, 1
		$\boldsymbol{\Gamma}_w$	10^4
		τ_f	10^{-4}
		$\mathcal{F}(\cdot)$	$\lambda_{\min}(\cdot)$



(a) Output and Control Response



(b) Tracking Error and Parameter Estimation Error History



(c) Lyapunov Function and Information Measure History

Fig. 1. Numerical Simulation Result

TABLE II
PERFORMANCE COMPARISON

Parameter		$\frac{1}{t_f - t_0}\int_{t_0}^{t_f}\|\mathbf{e}\|\,d\tau$	$\frac{1}{t_f - t_0}\int_{t_0}^{t_f}\left\|\tilde{\mathbf{W}}\right\|\,d\tau$
R	0	0.0185	68.7379
	1	0.0107	6.2735

adaptation law is avoided in the proposed adaptation law. Unlike the existing method, parameter estimation error $\tilde{\mathbf{W}}$ converges to zero without persistently exciting external command $\mathbf{z}_{\text{cmd}}(t)$ with the proposed method as shown in Fig. 1(b). The plot of $\|\mathbf{e}\|$ for the proposed method shows that the peak values diminish. That is, the tracking error performance is gradually improved as the parameter estimation error converges to zero after several changes in the external command. This is consistent with Corollary 1. Figure 1(c) confirms that the Lyapunov function V monotonically decreases toward zero with the proposed adaptation law. In the initial transient phase and in the intermediate transient phases after each external step command, the information measure increases. This implies that alteration of external command induces more excitation, and it increases the chance of updating the adequate information matrix $\mathbf{\Omega}_a$. Also, the rate of convergence of V is improved as $\mathbf{\Omega}_a$ is updated. Table II also shows that both tracking error and parameter estimation error are reduced in the proposed adaptation law. In summary, the proposed adaptation law can achieve improved parameter estimation accuracy and tracking error performance under a finite amount of excitation without inducing oscillatory behaviour in the control response.

VI. CONCLUSIONS

A new parameter-estimation-based adaptation law for a model reference adaptive control system is proposed to improve convergence of uncertain parameters without requiring persistent excitation. For this purpose, a novel method of constructing a parameter estimation residual is designed considering the linear-in-parameter structure of uncertainty. The residual can be represented as the product of a real symmetric matrix called adequate information matrix and parameter estimation error. The adequate information matrix is formed using weighted accumulation of data to have full rank after finite amount of signal excitation. The proposed composite adaptation law is established by augmenting the standard gradient-based direct adaptation term with the constructed residual.

The exponential stability after finite excitation for the structured uncertainty case is guaranteed by closed-loop stability analysis. Numerical simulation result confirmed the analytical findings and showed convergence of both tracking error and parameter estimation error. In the proposed framework, parameter estimation error converges without persistent excitation. The tracking error and control response can be improved with better parameter estimate while avoiding persistent oscillatory behaviour of the system. Therefore, the proposed composite model reference adaptive control framework can achieve a balance between parameter estimation accuracy, tracking error performance, and control response characteristics.

ACKNOWLEDGEMENT

This material is based upon the work supported by the Ministry of Trade, Industry & Energy (MOTIE, Korea) under Industrial Technology Innovation Program. No. 10066055, 'Development of an SMPC Flap Module for Morphing Wings'.

REFERENCES

[1] P. A. Ioannou and J. Sun, *Robust Adaptive Control*, 1st ed. Upper Saddle River, NJ: Prentice-Hall, 1996, pp. 545–547.

[2] B. D. O. Anderson, "Adaptive Systems, Lack of Persistency of Excitation and Bursting Phenomena," *Automatica*, vol. 21, no. 3, pp. 247–258, 1985.

[3] P. A. Ioannou and P. V. Kokotovic, "Instability Analysis and Improvement of Robustness of Adaptive Control," *Automatica*, vol. 20, no. 5, pp. 583–594, 1984.

[4] K. S. Narendra and A. M. Annaswamy, "A New Adaptive Law for Robust Adaptation Without Persistent Excitation," *IEEE Transactions on Automatic Control*, vol. 32, no. 2, pp. 134–145, 1987.

[5] K. Y. Volyanskyy, W. M. Haddad, and A. J. Calise, "A New Neuroadaptive Control Architecture for Nonlinear Uncertain Dynamical Systems: Beyond σ- and e-Modifications," *IEEE Transactions on Neural Networks*, vol. 20, no. 11, pp. 1707–1723, 2009.

[6] T. Yucelen and A. J. Calise, "Kalman Filter Modification in Adaptive Control," *Journal of Guidance, Control, and Dynamics*, vol. 33, no. 2, pp. 426–439, 2010.

[7] A. J. Calise and T. Yucelen, "Adaptive Loop Transfer Recovery," *Journal of Guidance, Control, and Dynamics*, vol. 35, no. 3, pp. 807–815, 2012.

[8] T. Yucelen and W. M. Haddad, "Low-Frequency Learning and Fast Adaptation in Model Reference Adaptive Control," *IEEE Transactions on Automatic Control*, vol. 58, no. 4, pp. 1080–1085, 2013.

[9] C. Cao and N. Hovakimyan, "Design and Analysis of a Novel $\mathcal{L}_1$ Adaptive Control Architecture With Guaranteed Transient Performance," *IEEE Transactions on Automatic Control*, vol. 53, no. 2, pp. 586–591, 2008.

[10] T. Yucelen and A. J. Calise, "Derivative-Free Model Reference Adaptive Control," *Journal of Guidance, Control, and Dynamics*, vol. 34, no. 4, pp. 933–950, 2011.

[11] M. A. Duarte and K. S. Narendra, "Combined Direct and Indirect Approach to Adaptive Control," *IEEE Transactions on Automatic Control*, vol. 34, no. 10, pp. 1071–1075, 1989.

[12] ——, "A New Approach to Model Reference Adaptive Control," *International Journal of Adaptive Control and Signal Processing*, vol. 3, no. 1, pp. 53–73, 1989.

[13] K. S. Narendra and M. A. Duarte, "Application of Robust Adaptive Control Using Combined Direct and Indirect Methods," *International Journal of Adaptive Control and Signal Processing*, vol. 3, no. 2, pp. 131–142, 1989.

[14] J.-J. E. Slotine and W. Li, "On the Adaptive Control of Robot Manipulators," *International Journal of Robotics Research*, vol. 6, no. 3, pp. 49–59, 1987.

[15] ——, "Composite Adaptive Control of Robot Manipulators," *Automatica*, vol. 25, no. 4, pp. 509–519, 1989.

[16] E. Lavretsky, "Combined/Composite Model Reference Adaptive Control," *IEEE Transactions on Automatic Control*, vol. 54, no. 11, pp. 2692–2697, 2009.

[17] J. Nakanishi, J. A. Farrell, and S. Schaal, "Composite Adaptive Control with Locally Weighted Statistical Learning," *Neural Networks*, vol. 18, no. 1, pp. 71–90, 2005.

[18] M. K. Ciliz, "Combined Direct and Indirect Adaptive Control for a Class of Nonlinear Systems," *IET Control Theory & Applications*, vol. 3, no. 1, pp. 151–159, 2009.

[19] P. M. Patre, W. MacKunis, M. Johnson, and W. E. Dixon, "Composite Adaptive Control for Euler-Lagrange Systems with Additive Disturbances," *Automatica*, vol. 46, no. 1, pp. 140–147, 2010.

[20] M. K. Ciliz, "Adaptive Backstepping Control Using Combined Direct and Indirect Adaptation," *Circuits, Systems & Signal Processing*, vol. 26, no. 6, pp. 911–939, 2007.

[21] P. M. Patre, S. Bhasin, Z. D. Wilcox, and W. E. Dixon, "Composite Adaptation for Neural Network-Based Controllers," *IEEE Transactions on Automatic Control*, vol. 55, no. 4, pp. 944–950, 2010.

[22] Y. Pan, Y. Liu, and H. Yu, "Online Data-Driven Composite Adaptive Backstepping Control with Exact Differentiators," *International Journal of Adaptive Control and Signal Processing*, vol. 30, no. 5, pp. 779–789, 2016.

[23] G. Chowdhary and E. N. Johnson, “Theory and Flight-Test Validation of a Concurrent-Learning Adaptive Controller,” *Journal of Guidance, Control, and Dynamics*, vol. 34, no. 2, pp. 592–607, 2011.

[24] G. Chowdhary, T. Yucelen, M. Mühlegg, and E. N. Johnson, “Concurrent Learning Adaptive Control of Linear Systems with Exponentially Convergent Bounds,” *International Journal of Adaptive Control and Signal Processing*, vol. 27, no. 4, pp. 280–301, 2013.

[25] G. Chowdhary, M. Mühlegg, and E. N. Johnson, “Exponential Parameter and Tracking Error Convergence Guarantees for Adaptive Controllers without Persistency of Excitation,” *International Journal of Control*, vol. 87, no. 8, pp. 1583–1603, 2014.

[26] Y. Pan and H. Yu, “Composite Learning From Adaptive Dynamic Surface Control,” *IEEE Transactions on Automatic Control*, vol. Early, no. NA, p. NA, 2016.

[27] B. D. O. Anderson and C. R. Johnson Jr., “Exponential Convergence of Adaptive Identification and Control Algorithms,” *Automatica*, vol. 18, no. 1, pp. 1–13, 1982.

[28] S. Boyd and S. S. Sastry, “On Parameter Convergence in Adaptive Control,” *Systems & Control Letters*, vol. 3, no. 6, pp. 311–319, 1983.

[29] I. Mareels, “Sufficiency of Excitation,” *Systems & Control Letters*, vol. 5, no. 3, pp. 159–163, 1984.

[30] S. Boyd and S. S. Sastry, “Necessary and Sufficient Conditions for Parameter Convergence in Adaptive Control,” *Automatica*, vol. 22, no. 6, pp. 629–639, 1986.

[31] K. S. Narendra and A. M. Annaswamy, “Persistent Excitation in Adaptive Systems,” *International Journal of Control*, vol. 45, no. 1, pp. 127–160, 1987.

[32] C. Cao, A. M. Annaswamy, and A. Kojic, “Parameter Convergence in Nonlinearly Parameterized Systems,” *IEEE Transactions on Automatic Control*, vol. 48, no. 3, pp. 397–412, 2003.

[33] G. Tao, *Adaptive Control Design and Analysis*. Hoboken, NJ: Wiley-Interscience, 2003, p. 108.

[34] W. Li, “Adaptive Control of Robot Motion,” Ph.D. dissertation, Department of Aeronautics and Astronautics, Massachusetts Institute of Technology, 1990.

[35] J. M. Elzebda, A. H. Nayfeh, and D. T. Mook, “Development of an Analytical Model of Wing Rock for Slender Delta Wings,” *Journal of Aircraft*, vol. 26, no. 8, pp. 737–743, 1989.