Parity of ranks of Jacobians of hyperelliptic curves of genus 2

by

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Contents

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Declarations

The results presented in this thesis are, to the best of the author's knowledge, entirely new, unless otherwise stated. Section 3.4 in Chapter 3 is a joint work with T. and V. Dokchitser and A. Morgan. The appendix is a work of A. Morgan.

Abstract

A consequence of the Birch and Swinnerton-Dyer conjecture is that the parity of the rank of abelian varieties is expected to be given by their global root numbers. This is known as the parity conjecture. Assuming the finiteness of the Shafarevich-Tate groups, the parity conjecture is equivalent to the p-parity conjecture for all prime p, that is the p^∞ Selmer rank is expected to be given by the global root number.

In this thesis we study the parity of the 2^∞ Selmer rank of Jacobians of hyperelliptic curves of genus 2 defined over number fields. This forces us to assume the existence of a Richelot isogeny (the analogue of a 2-isogeny for elliptic curves) to provide an expression for the parity of their 2^∞ Selmer rank as a sum of local factors Λ_v modulo 2. Based on a joint work with T. and V. Dokchitser and A. Morgan on arithmetic of hyperelliptic curves over local fields, this makes the parity of the 2^∞ Selmer rank of such semistable Jacobians computable in practice.

By introducing a set of polynomial invariants in the roots of the defining polynomials of the underlying curves of a specific family of Jacobians, we provide an expression for the local discrepancy existing between the local factors Λ_v and the local root numbers, and prove the 2-parity conjecture in this case.

One outcome of this result it that, using the theory of regulator constants, one can lift the assumption on the existence of an isogeny and prove the parity conjecture for a class of semistable Jacobians of genus 2 curves assuming finiteness of their Shafarevich-Tate group (see [17]).

Chapter 1

Introduction

This thesis studies the parity of ranks of Jacobians of genus 2 curves defined over a number field $\mathcal{K}$ that admit a Richelot isogeny. Curves of genus 2 can be written as

$$C/\mathcal{K} : y^2 = f(x),$$

where $f(x)$ is a separable polynomial in $\mathcal{K}[x]$ of degree 6. A Richelot isogeny is the analogue of a 2-isogeny for elliptic curves, and the condition for the Jacobian J of C to admit such an isogeny is that the Galois group of $f(x)$ preserves a factorization into three quadratics. This work involves investigating the arithmetic of such curves and their Jacobians over both local fields and number fields.

By the Mordell-Weil theorem, the group of rational points of an abelian variety A defined over a number field $\mathcal{K}$ is finitely generated. The main arithmetic invariant of $A/\mathcal{K}$ is its rank, denoted $rk(A/\mathcal{K})$ and defined to be the number of generators of infinite order in this group. At present, computing the rank of abelian varieties in general remains an open problem. However, the Birch and Swinnerton-Dyer conjecture, formulated in the 1960's, predicts that the rank of an abelian variety is given by the order of vanishing of its L-function at $s = 1$.

Conjecture 1.0.1 (Birch and Swinnerton-Dyer conjecture)**.** *Granting analytic continuation of $L(A/\mathcal{K}, s)$ to the whole of $\mathbb{C}$,*

$$rk(A/\mathcal{K}) = ord_{s=1}L(A/\mathcal{K}, s).$$

Now, the conjectural functional equation for the L-function of $A/\mathcal{K}$ relates $L(A/\mathcal{K}, s)$ to $L(A/\mathcal{K}, 2 - s)$ in such a way that $L(A/\mathcal{K}, s)$ is either (essentially) symmetric or anti-symmetric about $s = 1$. In particular, the sign in the functional equation controls the parity of the order of vanishing of $L(A/\mathcal{K}, s)$ at $s = 1$. This yields the "Birch and Swinnerton-Dyer conjecture modulo 2", that is the parity of $rk(A/\mathcal{K})$ is given by this sign. In addition, it is expected that this sign equals the global root number $\omega(A)$ of A, an invariant defined independently of any conjecture. The Birch and Swinnerton-Dyer conjecture modulo 2 then yields the parity conjecture.

Conjecture 1.0.2 (Parity conjecture)**.**

$$(-1)^{rk(A/\mathcal{K})} = \omega(A).$$

One outcome of this work is that if the Shafarevich-Tate group of J is finite, then the parity conjecture holds whenever C is semistable with some extra conditions. In an ongoing work, we use the theory of regulator constants to remove the assumption that J admits a Richelot isogeny. We also expect to be able to remove the extra technical conditions on C.

Assuming finiteness of the Shafarevich-Tate group of A, for each prime p, the rank of $A/\mathcal{K}$ is equal to its p^∞ Selmer rank, denoted $rk_p(A/\mathcal{K})$ (the rank expected from knowing all the p^n Selmer groups for $n \geq 1$). In particular, for all prime p, the parity conjecture is then equivalent to a more accessible conjecture: the p-parity conjecture.

Conjecture 1.0.3 (p-parity conjecture)**.**

$$(-1)^{rk_p(A/\mathcal{K})} = \omega(A/\mathcal{K}).$$

The two central results of this thesis are an explicit formula for the parity of the 2^∞ Selmer rank (Theorem 3.2.16) and the proof of the 2-parity conjecture for a class of $C_2 \times D_4$ Jacobians (Corollary 4.4.12). These are Jacobians of curves C such that the Galois group of $f(x)$ defining C is a subgroup of $C_2 \times D_4$. The latter amounts to the Galois group of $f(x)$ preserving a factorization into three quadratics and fixing one of the quadratic factors.

Principally polarized abelian surfaces are either Jacobians of genus 2 hyperelliptic curves, products of two elliptic curves or Weil restrictions of an elliptic curve defined over a quadratic field extension (see Theorem 3.1. in [18]). In the last two cases, the parity of their rank is given by that of the underlying elliptic curves, which has been thoroughly studied by Monsky, Dokchitser, Dokchitser, Nekovar and Cesnavicius among other contributors. This is why we restrict our attention to Jacobians of genus 2 curves.

Our approach to control the parity of the 2^∞ Selmer rank follows that of Cassels and Fisher (see [16][Appendix] and [2]). Assuming finiteness of the Shafarevich-Tate groups of abelian varieties, Cassels-Tate-Milne showed (see [25] §1.7) that both statements in the Birch and Swinnerton-Dyer conjecture are invariant under isogeny. As a result, if the abelian variety considered admits an isogeny, it is sometimes possible to show that its rank is at least one, and more generally to determine its parity (see §1.3 in [8] for an example using elliptic curves). In particular, the same result can be achieved unconditionally of the finiteness of $Ш(A/\mathcal{K})$ for the p^∞ Selmer rank of abelian varieties admitting a suitable isogeny.

In order to study the 2^∞ Selmer rank of Jacobians of genus 2 curves using this approach, the following features associated to a Richelot isogeny make it the right candidate. Similarly to the case of a 2-isogeny on an elliptic curve, a Richelot isogeny splits multiplication by 2 at the level of the Jacobian and its existence can be checked from the Galois group of the polynomial defining the underlying curve. Namely, if $Gal(f) \subseteq C_2^3 \rtimes S_3$ then J admits a Richelot isogeny. In addition, the codomain $\hat{J}$ of J is also the Jacobian of a genus 2 curve $\hat{C}$, and a model for $\hat{C}$ is given by the Richelot construction. Since unlike for elliptic curves, we need to distinguish the curve C from its Jacobian J, these features are very relevant for us as our method to study the arithmetic of Jacobians of genus 2 curves is via the study of that of their underlying curves. More precisely, we will express all the arithmetic invariants of J that we study in terms of the polynomial $f(x)$ defining C (see Sections 3.3, 3.4, 3.5).

Considering Jacobians admitting a Richelot isogeny, our first step is to provide a formula that computes the parity of their 2^∞ Selmer rank. The main obstacle here is the order of the finite part of their Shafarevich-Tate group. For principally polarized abelian varieties of dimension > 1, this order is either a square or twice a square as pointed out by Poonen and Stoll in [32]. Fortunately, in this article, the authors also provide a way to tell these two cases appart when the variety is a Jacobian. Using their result, we express the parity of $rk_2(J)$ as the following sum over places v of $\mathcal{K}$

$$rk_2(J/\mathcal{K}) \equiv \sum_v \Lambda_v \bmod 2,$$

where Λ_v involves invariants of both J and $\hat{J}$ such as their Tamagawa numbers and the deficiency of C and $\hat{C}$ (see Definition 3.2.7).

At odd finite places and for semistable Jacobians, the local factors Λ_v can be computed thanks to a joint work with T. and V. Dokchitser and A. Morgan presented in [14]. For finite places above 2, we take a slightly different approach and use a result of A. Morgan presented in the appendix. While it provides a way to compute the parity of the 2^∞ Selmer rank in practice, this formula can also be used to prove theoretical results. In particular, we use it to prove the 2-parity conjecture in this case.

Since the global root number is defined as a product of local root numbers, the 2-parity conjecture now becomes equivalent to the statement

$$\prod_v \lambda_v = \prod_v \omega_v, \text{ where } \lambda_v = (-1)^{\Lambda_v} \text{ and } \prod_v \omega_v(J/\mathcal{K}) = \omega(J/\mathcal{K}).$$

It does not come as a surprise to find that the local terms λ_v and ω_v are not equal in general but differ by a term which vanishes when taking product over all places. This was already the case for elliptic curves and abelian varieties for odd primes. However, it follows that finding the exact expression for this local discrepancy represents the crux in the proof of the 2-parity conjecture.

As suggested by the proof of the 2-parity conjecture for elliptic curves, using a product of Hilbert Symbols involving invariant polynomials in the roots of the defining polynomial of the underlying curve seemed like a possible answer to match the local discrepancy, particularly thanks to the product formula for Hilbert Symbols. Only, λ_v and ω_v are sensitive to reduction types and specific Galois actions on the special fibre of J (this is by definition of local root numbers, Tamagawa numbers and deficiency, see Section 3.4). Therefore the sought invariants are expected to respond accordingly to different reduction types. However, in the case of a semistable elliptic curve, there are essentially five cases to consider (excluding infinite places for simplicity): good reduction, split multiplicative and non-split multiplicative reduction, where the last two cases also depend on the particular 2-isogeny chosen. One can then study these cases carefully and find the three invariant polynomials in the roots of the defining polynomial of the elliptic curve that control the local discrepancy (see §7 in [9]). But for a Jacobian of dimension 2, there are roughly 150 cases to consider, excluding infinite places, which complicate the hunt. We tried several methods and proceeded via trial and error using specific families of genus 2 curves and examining infinite places and finite places alternatively. Eventually, under the condition that the Galois group of $f(x)$ be a subgroup of $C_2 \times D_4 \subseteq C_2^3 \rtimes S_3$, we found a set of invariant polynomials $A_1, .., A_{17}$ (see Section 4.3) which, paired in Hilbert Symbols to form the local term E_v as shown below, correctly match the local discrepancy between λ_v and ω_v at all infinite places and finite places of semistable reduction (with a specific condition at 2-adic places):

$$E_v = (-1, A_1)(A_2, A_3)(A_4, A_5)(A_6, A_7)(A_8, A_9)(A_{10}, A_{11})(A_{12}, A_{13})(A_{14}, A_{15})(A_{16}, A_{17}).$$

Moreover, the condition $Gal(f) \subseteq C_2 \times D_4$ was acceptable for us since $C_2 \times D_4$ is the 2-Sylow subgroup of S_6 which was the requirement for our application toward the parity conjecture (see Chapter 7).

Lastly, having produced these invariants experimentally, it remained to prove that they indeed match the discrepancy at all places in order to claim that the 2-parity conjecture holds for this family. Unfortunately at the moment, the "conceptual" meaning of these invariants is still mysterious. We therefore found the algebraic relations that they satisfy in order to, using a case by case analysis, compute and compare the local factor, local discrepancy and local root number at each place and for each reduction type. The exhaustivity of our list of cases and the fact that we found the product of local factor and Hilbert Symbols to be equal to the local root number in each case, constitutes our proof of the 2-parity conjecture.

This work is presented as follows. In Chapter 2, we start by recalling some background material on hyperelliptic curves and their Jacobians. After recalling the definition of Richelot isogenies and the Richelot construction, we give an overview of the p-parity and parity conjectures followed by a summary of the known cases.

The first part of Chapter 3 then relates the parity of the 2^∞ Selmer rank to the existence of an isogeny and provides a local factorization for the 2^∞ Selmer rank of the Jacobian of a genus 2 curve admitting a Richelot isogeny. In its second part, Chapter 3 presents how to express each local factor in terms of the polynomial defining the underlying curve, considering infinite places, odd finite places and finite places above 2. Each computation is illustrated by an example and, combining all local computations, this chapter ends by providing an example of the computation of the parity of the 2^∞ Selmer rank for an explicit Jacobian.

In Chapter 4, we introduce the specific family of $C_2 \times D_4$ Jacobians and define the invariants that form the term of local discrepancy. We then tabulate the local computations at each place and for each reduction type that compose our proof of the 2-parity for $C_2 \times D_4$ Jacobians.

Chapters 5 and 6 provide a detailed proof of these local computations. Chapter 5 addresses the computation of each local factor and root number while Chapter 6 deals with the computation of the local discrepancy in each case.

Chapter 7 concludes this work and is followed by an appendix containing a result of A. Morgan concerning isogenies between abelian varieties with split totally toric reduction which is used in Chapter 5 when considering computations at finite places above 2.

Notation

The following notation will be used throughout the entire thesis.

$\mathcal{K}$	number field
$\overline{\mathcal{K}}$	algebraic closure of $\mathcal{K}$
$M_{\mathcal{K}}$	set of places of $\mathcal{K}$
v	a place in $M_{\mathcal{K}}$
$\mathcal{K}_v$	completion of $\mathcal{K}$ at v
p	a prime in $\mathbb{Q}$
K	finite extension of $\mathbb{Q}_p$
$\mathcal{O}_K$	ring of integers of K
$\overline{K}$	algebraic closure of K
K^{nr}	maximal unramified extenstion of K
π	uniformizer of K
v	normalized valuation of K
k	residue field of K
$\bar{k}$	algebraic closure of k

Some sections introduce a large number of new definitions and conventions. For clarity, each introduction of chapters is followed by the list of specific notations used in the chapter.

Chapter 2

Background Material

2.1 Hyperelliptic curves and their Jacobians

2.1.1 Hyperelliptic curves

By a hyperelliptic curve C over a number field $\mathcal{K}$ given by $C/\mathcal{K} : y^2 = f(x)$ of genus g, where $f(x) \in \mathcal{K}[x]$ is of degree $2g+1$ or $2g+2$ with no multiple roots, we mean the pair of affine patches

$$U_x : y^2 = f(x), \qquad U_t : v^2 = t^{2g+2} f(\frac{1}{t}),$$

glued together along the maps $x = \frac{1}{t}$ and $y = \frac{v}{t^{g+1}}$. We refer to the *points at infinity* (i.e. $C \setminus U_x$) for the points with $t = 0$ on U_t. Explicitly, denoting $c \in \mathcal{K}^\times$ the leading term of $f(x)$, if $f(x)$ is of degree $2g+1$ then

$$U_x : y^2 = c \prod_{i=1}^{2g+1} (x - r_i), \qquad U_t : v^2 = tc \prod_{i=1}^{2g+1} (tr_i - 1)$$

and we denote $P_\infty = (0, 1)$ the only point on C' with $t = 0$. Otherwise if $f(x)$ is of degree $2g+2$ then

$$U_x : y^2 = c \prod_{i=1}^{2g+2} (x - r_i), \qquad U_t : v^2 = c \prod_{i=1}^{2g+2} (tr_i - 1)$$

and we denote $P_\infty^\pm = (0, \pm\sqrt{c})$ the two points on U_t with $t = 0$. We refer to [35] for an introduction to the arithmetic of Hyperelliptic curves.

2.1.2 Jacobians of hyperelliptic curves

Let C be a hyperelliptic curve of genus g defined over $\mathcal{K}$ by

$$C : y^2 = f(x),$$

where $f(x) \in \mathcal{K}[x]$ is a polynomial of degree $2g+1$ or $2g+2$ with no multiple root.

Definition 2.1.1 (Hyperelliptic involution)**.** Let $P = (x_P, y_P)$ be a point on $C(\overline{\mathcal{K}})$. We call $\overline{P} = (x_P, -y_P)$ its conjugate under the hyperelliptic involution.

Definition 2.1.2 (Divisors)**.** A divisor D on C is a formal sum

$$\sum_{P \in C} n_P P,$$

where $n_P \in \mathbb{Z}$ and $n_P = 0$ for all but finitely many points $P \in C(\overline{\mathcal{K}})$. The integer n_P is called the *multiplicity* of P in D and $deg(D) = \sum_{P \in C} n_P$ denotes the *degree* of D.

Divisors of a curve C are elements of the free abelian group on the set of points $P \in C(\overline{\mathcal{K}})$. We denote $Div(C)$ the group of divisors on C.

Definition 2.1.3 (Rational divisors)**.** A divisor

$$D = \sum_{P \in C(\mathcal{F})} n_p P,$$

for some finite Galois extension $\mathcal{F}/\mathcal{K}$, is said to be $\mathcal{K}$-rational or defined over $\mathcal{K}$ if

$$D^\sigma = D \text{ for all } \sigma \in Gal(\mathcal{F}/\mathcal{K}).$$

Definition 2.1.4 (Principal Divisors)**.** Let f be a non zero rational function on C. Define the divisor of f

$$[f] = \sum_{P \in C} ord_P(f) P,$$

where the multiplicity of P in $[f]$ is given by the order of vanishing of f at P in terms of a local uniformizer. Divisors given by a function f on C are called *principal divisors* and we denote by $Princ(C)$ the subgroup of principal divisors. As a rational function, f has as many zeroes as it has poles. Consequently, principal divisors have degree 0.

Definition 2.1.5 (Picard group)**.** The *Picard group* is defined to be

$$Pic(C) = Div(C)/Princ(C).$$

In particular, two divisors differing by a principal divisor are in the same divisor class and are said to be linearly equivalent. The Picard group inherits a notion of degree from $Div(C)$ and we denote by $Pic^j(C)$ the set of elements of $Pic(C)$ of degree j.

Definition 2.1.6 (Jacobian of C)**.** The Jacobian of C is defined as

$$J = Pic^0(C).$$

Points on J are classes of divisors of degree 0 on C.

Theorem 2.1.7 ([26], Theorem 1.1, Proposition 2.1)**.** *The Jacobian of a curve of genus g is an abelian variety of dimension g.*

Theorem 2.1.8 (Mordell-Weil)**.** *The group of $\mathcal{K}$-rational points of J is finitely generated. Hence*

$$J(\mathcal{K}) \simeq \mathbb{Z}^{rk(J/\mathcal{K})} \oplus J(\mathcal{K})_{tors},$$

where $rk(J/\mathcal{K}) < \infty$ is called the rank *of $J/\mathcal{K}$ and the group of torsion $J(\mathcal{K})_{tors}$ is finite.*

Our main goal is to compute the parity of $rk(J/\mathcal{K})$ in the case of Jacobians of dimension 2. Our approach involves the use of an isogeny on J whose kernel is composed of 2-torsion elements. We therefore review briefly how to construct 2-torsion points on J from the roots of the defining polynomial of C. For a detailed exposition on torsion elements on Jacobians of genus 2 curves we refer to [5], Chapter 8.

2.1.3 Jacobians of hyperelliptic curves of genus 2

Let C be a hyperelliptic curve of genus 2 defined over $\mathcal{K}$ by

$$C : y^2 = f(x),$$

where $f(x) \in \mathcal{K}[x]$ is a polynomial of degree 6[1] with no multiple root.

Points on $C(\overline{\mathcal{K}})$ and $J(\overline{\mathcal{K}})$

A point $D \in J(\overline{\mathcal{K}})$ can be given as a divisor on C of the form

$$D = P + Q - P_\infty^+ - P_\infty^-,$$

[1] If C is defined by a polynomial of degree 5, it is always possible after a change of variable, to consider a model for C with a defining polynomial of degree 6

for some $P, Q \in C(\overline{\mathcal{K}})$ and where P_∞^+, P_∞^- denote the two points at infinity of C. We will use the notation $D = [P, Q]$ to denote the point D on $J(\overline{\mathcal{K}})$. Note that a point $P \in C(\overline{\mathcal{K}})$ and its image $\overline{P}$ through the hyperelliptic involution are the points of intersection of a vertical line and $f(x)$. In particular, the principal divisor $D = [P, \overline{P}]$ can be chosen as a representative for the class of $0 \in J(\overline{\mathcal{K}})$.

Addition on J

Choose four points P, P', Q, Q' in general position on $C(\overline{\mathcal{K}})$. Then there exists a curve $y = p(x)$, where $p(x)$ is a polynomial of degree 3, passing through P, P', Q, Q' and intersecting C in two more points S, S' (see Figure 2.1).

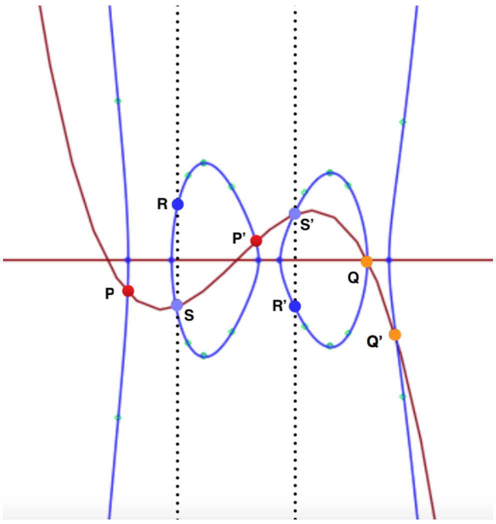


Figure 2.1: $y^2 = f(x)$ and $y = p(x)$

The principal divisor

$$[y - p(x)] = P + P' + Q + Q' + S + S' - 3P_\infty^+ - 3P_\infty^-$$

yields

$$[P, P'] + [Q, Q'] = -[S, S'].$$

Since $[S, R] = [S', R'] = 0$, we obtain

$$[P, P'] + [Q, Q'] = [R, R'].$$

This prompts the following Lemma.

Lemma 2.1.9. *Each non zero element of $J(\overline{\mathcal{K}})[2]$ may be uniquely represented by the following pairs of points of $C(\overline{\mathcal{K}})$. Let $x_1, .., x_6 \in \overline{\mathcal{K}}$ be the roots of $f(x)$, then*

$$J(\overline{\mathcal{K}})[2] = \{[T_i, T_k], i \neq k\} \cup \{0\}, \qquad T_i = (x_i, 0) \in C(\overline{\mathcal{K}}).$$

Proof. See [5], §8.1 and proof of Lemma 8.1.3. in [34]. □

The sequent sections and results rely on the existence of an isogeny on J whose kernel is a subgroup of $J(\overline{\mathcal{K}})[2]$. Using this last Lemma, we conveniently avoid considering an algebraic model for J and simply define its 2-torsion elements from the roots of the defining polynomial of its underlying curve C. Moreover as shown in Chapter 3, this identification allows us to study the local arithmetic of J from that of C.

2.2 Richelot Isogenies and Richelot Construction

We recall here the notion of a Richelot isogeny. Defined for Jacobians of genus 2 curves, they split multiplication by 2 at the level of the Jacobian. Their codomain is the Jacobian of a curve, a model of which is conveniently given by the Richelot construction. Similarly to the case of elliptic curves, their kernel consists of a subgroup of 2-torsion and we show using Lemma 2.1.9 how to guarantee the existence of such an isogeny from the Galois group of the defining polynomial of the underlying curve. Our exposition follows that of [4] to which we refer for the proofs. Other expositions of Richelot isogenies can be found in [5] and [34].

Richelot and Richelot dual polynomials

Definition 2.2.10. 1) Given two polynomials $P(x), Q(x) \in \mathcal{K}[x]$ of degree at most 2, we define the *Richelot operator* $[\ ,\ ]$ by

$$[P(x), Q(x)] = P'(x)Q(x) - Q'(x)P(x),$$

where $'$ refers to the differentiation with respect to x.
2) We say that a polynomial $G(x) \in \mathcal{K}[x]$ of degree 5 or 6 is a *Richelot polynomial over* $\mathcal{K}$ if we can fix a factorization

$$G(x) = G_1(x)G_2(x)G_3(x),$$

where for $i = 1, 2, 3$, the polynomials $G_i(x)$ are of degree at most 2, defined over $\overline{\mathcal{K}}$ individually and over $\mathcal{K}$ as a set[2].
3) If $G(x)$ is a Richelot polynomial over $\mathcal{K}$, write

$$G_i(x) = g_{i2}x^2 + g_{i1}x + g_{i0} = g_i(x - \alpha_i)(x - \beta_i), \quad i = 1, 2, 3,$$

for its factorization over $\overline{\mathcal{K}}$ and define

$$\Delta_G = \det\left((g_{ij})_{0 \le i,j \le 2}\right).$$

Definition 2.2.11. 1) To a Richelot polynomial $G(x)$ with fixed factorization $G(x) = G_1(x)G_2(x)G_3(x)$ such that $\Delta_G \neq 0$, we associate its *Richelot dual polynomial* $F(x)$ given by

$$F(x) = \prod_{i=1}^{3} F_i(x), \qquad \text{with } F_i(x) = \frac{1}{\Delta_G}[G_{i+1}(x), G_{i+2}(x)],$$

[2] i.e. the set $\{G_i(x), i = 1, 2, 3\}$ is preserved by the action of $Gal(\overline{\mathcal{K}}/\mathcal{K})$

where addition of indices is performed modulo 3. Note that by construction, $F(x)$ is a Richelot polynomial over $\mathcal{K}$ with given factorization $F_1(x)F_2(x)F_3(x)$.
2) Write

$$F_i(x) = f_{i2}x^2 + f_{i1}x + f_{i0} = f_i(x - A_i)(x - B_i), \quad i = 1, 2, 3,$$

for the factorization of $F(x)$ over $\overline{\mathcal{K}}$ and define

$$\Delta_F = \det((f_{ij})_{0 \le i,j \le 2}).$$

We will keep the above notation for the roots of $G_i(x)$ and $F_i(x)$ throughout the entire thesis. The quantity Δ_G is essential to the Richelot construction due to the following definitions. We also note that although Δ_G might not be defined over $\mathcal{K}$, its square Δ_G^2 is.

Richelot and Richelot dual curves

Definition 2.2.12.

1. We say that a hyperelliptic curve $C/\mathcal{K}$ of genus 2 is a *Richelot curve over* $\mathcal{K}$ if it is given by

   $$C : y^2 = G(x), \quad \text{together with the factorization } G(x) = G_1(x)G_2(x)G_3(x)$$

   as a Richelot polynomial over $\mathcal{K}$ such that $\Delta_G \neq 0$.

2. To a Richelot curve $C/\mathcal{K}$, we associate its *Richelot dual curve* $\hat{C}/\mathcal{K}$ given by

   $$\hat{C} : \Delta_G^2 y^2 = F(x),$$

   where $F(x)$ is the Richelot dual polynomial of $G(x)$ with respect to the given factorization.

Remark 2.2.13 (Richelot dual polynomial)**.** Our definition of the Richelot dual polynomial differs from that of [5] and [34] but agrees with that of [4]. This is because we insist that $F(x) = F_1(x)F_2(x)F_3(x)$ be a Richelot polynomial, that is, the set $\{F_1(x), F_2(x), F_3(x)\}$ be Galois stable. However, this has no incidence on the definition of the dual curve $\hat{C}$ as the following change of variable $x \mapsto x, \quad y \mapsto \frac{y}{\Delta_G^2}$ is an isomorphism over $\mathcal{K}$ between our curve $\hat{C}$ and that given in [5].

Remark 2.2.14 (Galois group of Richelot polynomials)**.** Let $G(x) \in \mathcal{K}[x]$ a polynomial of degree 5 or 6. Denote by $\mathcal{K}_G$ its splitting field. Then the conditions for $G(x)$ to be a Richelot polynomial over $\mathcal{K}$ (and hence for C to be a Richelot curve) can be rephrased as

$$Gal(\mathcal{K}_G/\mathcal{K}) \subseteq C_2^3 \rtimes S_3 \subset S_6,$$

where the three copies of C_2 are given by the permutations of roots of quadratic factors $G_i(x)$, themselves acted upon by S_3.

Proposition 2.2.15. *Let $C/\mathcal{K}$ be a Richelot curve with fixed quadratic factorization $G(x) = G_1(x)G_2(x)G_3(x)$. Let J be its Jacobian. Keeping notation as in Definition 2.2.11, for each pair of points $(P_i, Q_i) \in C(\overline{\mathcal{K}})^2$, with $P_i = (\alpha_i, 0), Q_i = (\beta_i, 0), i = 1, 2, 3,$ consider the associated 2-torsion elements of $J(\overline{\mathcal{K}})$*

$$D_i = [P_i, Q_i] \in J(\overline{\mathcal{K}})[2].$$

Then
i) the subgroup $H = \{0, D_1, D_2, D_3\} \subset J(\overline{\mathcal{K}})[2]$ is defined over $\mathcal{K}$,
ii) H is a maximal isotropic subrgroup of $J(\mathcal{K})[2]$ with respect to the 2-Weil pairing.

Proof. (i) Follows from the definition of a Richelot polynomial.
(ii) A subgroup H of $J(\mathcal{K})[2]$ is maximal isotropic with respect to the 2-Weil pairing if and only if it is composed of 2-torsion points of J given by two distinct Weierstrass points of C (see [34] Lemma 8.1.4). The result follows from the definition of D_1, D_2, D_3 and the fact that C being a hyperelliptic curve implies that the $G_i(x)$s are necessarily coprime. □

Richelot isogenies

Definition 2.2.16. Let $C/\mathcal{K}$ be a Richelot curve with fixed quadratic factorization $G(x) = G_1(x)G_2(x)G_3(x)$. Let J be its Jacobian. Consider the 2-torsion points of $J(\overline{\mathcal{K}})$ defined by the quadratic factorization of $G(x)$

$$D_i = [P_i, Q_i] \in J(\overline{\mathcal{K}})[2],$$

where $P_i = (\alpha_i, 0), Q_i = (\beta_i, 0), i = 1, 2, 3$ as in Definition 2.2.11. Then the isogeny over $\mathcal{K}$ for J whose kernel is the subgroup $H = \{0, D_1, D_2, D_3\} \subseteq J(\mathcal{K})[2]$ is called a Richelot isogeny.
We say that a Jacobian admits a Richelot isogeny over $\mathcal{K}$ if its underlying curve is a Richelot curve over $\mathcal{K}$.

Theorem 2.2.17. *Let $C/\mathcal{K}$ be a Richelot curve with fixed factorization $G(x) = G_1(x)G_2(x)G_3(x)$ for its Richelot polynomial. Let $\hat{C}/\mathcal{K}$ be its Richelot dual curve and let ϕ denote the associated Richelot isogeny on J. Then*

$$\phi : J \to \hat{J},$$

where $\hat{J}$ is the Jacobian of $\hat{C}$ and, denoting ϕ^t the dual isogeny of ϕ, we have $\phi \circ \phi^t$ is multiplication by 2.

Proof. This is Definition 8.4.10, Corollary 8.4.9 and Theorem 8.4.11 of [34]. □

2.3 Parity and p-parity conjectures

We refer to [8] for an overview of the parity and p-parity conjectures for elliptic curves and present here a review of the situation for abelian varieties. Complementary references include [36] and [37]. Let $A/\mathcal{K}$ be an abelian variety over a number field. By the Mordell-Weil theorem, the group of rational points $A(\mathcal{K})$ of A is finitely generated, so that

$$A(\mathcal{K}) \simeq \mathbb{Z}^{rk(A/\mathcal{K})} \oplus A(\mathcal{K})_{tors},$$

where $rk(A/\mathcal{K}) < \infty$ is called the *rank* of $A/\mathcal{K}$ and the group of torsion $A(\mathcal{K})_{tors}$ is finite. The rank of A is predicted by the following conjecture.

Conjecture 2.3.18 (Birch and Swinnerton-Dyer [3], Tate [36])**.** *Granting analytic continuation of the L-function of $A/\mathcal{K}$ to $\mathbb{C}$,*

$$rk(A/\mathcal{K}) = ord_{s=1} L(A/\mathcal{K}, s),$$

where $ord_{s=1}$ denotes the order of vanishing of $L(A/\mathcal{K}, s)$ at $s = 1$.

Once the analytic continuation of the L-function of $A/\mathcal{K}$ to $\mathbb{C}$ is assumed, it is also predicted that the completed L-function of $A/\mathcal{K}$ satisfies the following functional equation.

Conjecture 2.3.19.

$$L^*(A/\mathcal{K}, s) = W(A)\ L^*(A/\mathcal{K}, 2 - s), \quad W(A) \in \{\pm 1\}.$$

A direct consequence of this functional equation is that the parity of the order of vanishing of $L(A/\mathcal{K}, s)$ at $s = 1$ is given by the sign $W(A)$, i.e.

$$(-1)^{ord_{s=1} L(A/\mathcal{K},s)} = W(A).$$

Therefore, combining with the prediction on the rank of $A/\mathcal{K}$ given by the Birch and Swinnerton-Dyer conjecture, we obtain the following prediction for the parity of $rk(A/\mathcal{K})$.

Conjecture 2.3.20 (Birch and Swinnerton Dyer Modulo 2)**.**

$$(-1)^{rk(A/\mathcal{K})} = W(A).$$

Moreover, it is known for elliptic curves in some cases that the sign in the functional equation above equals the global root number of the curve. In general, this fact remains a conjecture.

Conjecture 2.3.21. *The sign* $W(A)$ *in the conjectural functional equation of* $L^*(A/\mathcal{K})$ *is equal to the* global root number $\omega(A)$ *of* A*:*

$$W(A) = \omega(A) \underset{def}{=} \prod_v \omega_v(A),$$

where $\omega_v(A)$ *denotes the local root number of* A *at the place* v *of* $M_{\mathcal{K}}$*.*

This justifies the prediction on the parity of $rk(A/\mathcal{K})$ given by the parity conjecture.

Conjecture 2.3.22 (Parity conjecture)**.** *The parity of the rank of an abelian variety is given by its global root number:*

$$(-1)^{rk(A/\mathcal{K})} = \omega(A).$$

Definition 2.3.23 (p^∞ Selmer rank)**.** Recall that for each prime p, the p-primary part of $Ш(A/\mathcal{K})$ can be written as

$$Ш[p^\infty] = (\mathbb{Q}_p/\mathbb{Z}_p)^{\delta_p} \times Ш_0[p^\infty], \quad |Ш_0[p^\infty]| < \infty.$$

We define the p^∞ *Selmer rank* of $A/\mathcal{K}$ to be

$$rk_p(A/\mathcal{K}) = rk(A/\mathcal{K}) + \delta_p.$$

In particular, assuming that $Ш(A/\mathcal{K})$ is finite, the rank of A/K equals its p^∞ Selmer rank for all primes p. This yields the p-parity conjecture.

Conjecture 2.3.24 (p-parity conjecture)**.**

$$(-1)^{rk_p(A/\mathcal{K})} = \omega(A).$$

Thanks to the work of Monsky, Nekovar, Dokchitser, Dokchitser, Cesnavicius, Coates, Fukaya, Kato, Sujatha, Kramer, Tunnell and Morgan among other contributors, on the p-parity conjecture, the following cases of the parity conjecture have been proven:

1. for elliptic curves $E/\mathbb{Q}$ assuming $Ш(E)[p^\infty]$ finite for some p (see [27] for $p = 2$, [29], [21], [11] for odd primes),
2. for elliptic curves $E/\mathcal{K}$, where $\mathcal{K}$ is a totally real field, assuming $Ш(E)[p^\infty]$ finite for some p and mild constraints (see [30] and [12]),

3. for elliptic curves $E/\mathcal{K}$ admitting a p-isogeny, assuming $Ш(E)[p^\infty]$ finite (see [9] and [6]),

4. for elliptic curves $E/\mathcal{K}(\sqrt{d})$, E defined over $\mathcal{K}$, $d \in \mathcal{K}^\times \backslash \mathcal{K}^{\times 2}$, $Ш(E/\mathcal{K}(\sqrt{d}))[2^\infty]$ finite (see [23], [22] and [12]),

5. for elliptic curves $E/\mathcal{K}$ assuming $Ш(E/F)[2^\infty], Ш(E/F)[3^\infty]$ finite, where $F = \mathcal{K}(E[2])$ (see [11] and [12]),

6. for Jacobian varieties $A/\mathcal{K}(\sqrt{d})$, $A = Jac(C)$, C semistable hyperelliptic curve over $\mathcal{K}$, $d \in \mathcal{K}^\times \backslash \mathcal{K}^{\times 2}$, $Ш(A/\mathcal{K}(\sqrt{d}))[2^\infty]$ finite, and mild constraints (see [28]),

7. for principally polarized abelian varieties $A/\mathcal{K}$ admitting an isogeny $\Phi : A \to A'$ s.t. $\Phi^*\Phi = [p]$, assuming $Ш(A)[p^\infty]$ finite, p *odd* and mild constraints (see [7]).

One of the main motivation for this work was the proof of the fifth case. It relies on the fact that the 2-parity conjecture holds for elliptic curves over $\mathcal{K}(E[2])$, and make use of the theory of regulator constants to prove a result on the parity of the rank of $E/\mathcal{K}$. The same method can be applied to Jacobians of dimension 2 but first, it is crucial to prove the 2-parity conjecture for those.

Chapter 3

Parity of the 2^∞ Selmer rank

3.1 Introduction

In this chapter we develop an explicit formula for the parity of the 2^∞ Selmer rank of Jacobians of genus 2 curves admitting a Richelot isogeny. The first section explains how to use the existence of a Richelot isogeny to express the parity of the 2^∞ Selmer rank in terms of local factors as follows.

Theorem 3.1.1 (Theorem 3.2.16). *Let $C/\mathcal{K}$ be a hyperelliptic curve of genus 2 such that its Jacobian J admits a Richelot isogeny ϕ. Denote by $\hat{C}$ and $\hat{J}$ the corresponding curve and isogenous Jacobian. Denote ω_v^o, $\hat{\omega}_v^o$ the Néron exterior forms at the place v of $\mathcal{K}$ for J and $\hat{J}$ respectively. Then*

$$rk_2(J/\mathcal{K}) \equiv \sum_{v \in M_{\mathcal{K}}} \Lambda_v \ mod\ 2 \qquad with$$

$$\Lambda_v = ord_2\left(\frac{\hat{n} \cdot m_v}{|ker(\varphi)| \cdot n \cdot \hat{m}_v}\right) \ for\ v \mid \infty, \quad \Lambda_{v\nmid\infty} = ord_2\left(\frac{c_v \cdot m_v}{\hat{c}_v \cdot \hat{m}_v}\left|\frac{\phi^*\hat{\omega}_v^o}{\omega_v^o}\right|_v\right) \ for\ v \nmid \infty,$$

where n, $\hat{n}$ are the number of $\mathcal{K}_v$-connected components of J and $\hat{J}$, φ is the restriction of ϕ to the identity component of $J(\mathcal{K}_v)$, c_v and $\hat{c}_v$ the Tamagawa numbers of J and $\hat{J}$, and $m_v = 2$ if C is deficient at v (see Definition 3.2.7), $m_v = 1$ otherwise, and similarly for $\hat{m}_v$.

In the second section, we describe how to express the invariants involved in Theorem 3.2.16 above in terms of the polynomial defining C, when the given Jacobian is semistable. Part of this work is joint with T. and V. Dokchitser and A. Morgan and is presented in the articles [15] and [14]. We only provide here a summary of the results needed and refer to the articles for the proofs. It follows that the parity of the 2^∞ Selmer rank is computable in this case and an example of such a computation is given in the last section.

List of notation for this chapter

C	Richelot curve defined over $\mathcal{K}$ with the given factorization $y^2 = G(x) = G_1(x)G_2(x)G_3(x)$
c	leading term of $G(x)$
J	Jacobian of C admitting a Richelot isogeny ϕ
$\hat{C}$	Richelot dual curve defined over $\mathcal{K}$ by $\Delta_G^2 y^2 = F(x) = F_1(x)F_2(x)F_3(x)$ (see Section 2.2)
ℓ	leading term of $F(x)$
$\hat{J}$	Jacobian of $\hat{C}$
δ_i, $i = 1, 2, 3$	discriminant of $G_i(x)$
$\hat{\delta}_i$, $i = 1, 2, 3$	discriminant of $F_i(x)$
α_i, β_i, $i = 1, 2, 3$	roots of $G_i(x)$
A_i, B_i, $i = 1, 2, 3$	roots of $F_i(x)$
P_i, Q_i, $i = 1, 2, 3$	Weierstrass points $P_i = (\alpha_i, 0), Q_i = (\beta_i, 0)$
D_i, $i = 1, 2, 3$	$D_i = [P_i, Q_i] \in ker(\phi)$,
$n_v, \hat{n}_v$,	number of connected components of $J(\mathcal{K}_v)$ and $\hat{J}(\mathcal{K}_v)$ when $v \mid \infty$
m_v	$m_v = 2$ if C is deficient at v, $m_v = 1$ otherwise (see Definition 3.2.7)
$\phi_{\mathbb{R}}$	map induced by ϕ on $J(\mathcal{K}_v)$ when $\mathcal{K}_v \cong \mathbb{R}$
φ	restriction of $\phi_{\mathbb{R}}$ to the identity component of $J(K_v)$ when $K_v \cong \mathbb{R}$
$c_v, \hat{c}_v$	Tamagawa numbers of J and $\hat{J}$ at v when $v \nmid \infty$
$\hat{\omega}$	a fixed choice of a non-zero exterior form for $\hat{J}$
ω	the pullback of $\hat{\omega}$ through $\phi : \omega = \phi^*\hat{\omega}$

3.2 Parity of the 2^∞ Selmer rank of Richelot Jacobians

Recall that for an elliptic curve $E/\mathcal{K}$ with an isogeny ϕ of degree p over $\mathcal{K}$, one can express the parity of its p^∞ Selmer rank $(-1)^{rk_p(E/\mathcal{K})}$ as the product of power of p in

$$\frac{|coker(\phi_v : E(\mathcal{K}_v) \to E(\mathcal{K}_v))|}{|ker(\phi_v : E(\mathcal{K}_v) \to E(\mathcal{K}_v))|}$$

for all places v of $\mathcal{K}$ as shown in [16][Appendix]. In [11] and [7], Dokchitser-Dokchitser and Coates-Fukaya-Kato-Sujatha gave generalizations of the above to abelian varieties. In the latter, the authors prove that, for odd primes p and under mild restrictions at places v dividing p, the p-parity conjecture holds for principally polarized abelian varieties A admitting a suitable isogeny. When $p = 2$ however, the key problem to extend their result is that the order of Ш(A) could be twice a square. In [32], Poonen and Stoll provide a way to compute whether the order of the Shafarevich-Tate group of Jacobians is a square or twice a square in terms of local invariants. This is why the use of Richelot isogenies on the Jacobians J that

we study is crucial. Since it guarantees that the codomain of J is also a Jacobian, it let us control the parity of the 2^∞ Selmer rank in this case.
In this section, we recall how to express the parity of the 2^∞ Selmer rank of an abelian variety of dimension g admitting an isogeny of degree 2^g to then provide a formula for the parity of their 2^∞ Selmer rank.

3.2.1 Parity of the 2^∞ Selmer rank and isogenies

Let A, B be abelian varieties over $\mathcal{K}$. Recall the following definitions and results presented in [11] (Section 4.1) concerning the isogeny invariance of the BSD-quotient for Selmer groups.

Definition 3.2.2. For an isogeny $\phi : A \to B$ over $\mathcal{K}$, let

$$Q(\phi) = |coker(\phi : A(\mathcal{K})/A(\mathcal{K})_{tors} \to B(\mathcal{K})/B(\mathcal{K})_{tors})| \times |ker(\phi : Ш(A)_{div} \to Ш(B)_{div})|,$$

where $Ш(A)_{div}$ denotes the divisible part of $Ш(A)$.

Lemma 3.2.3. *$Q(\phi)$ as defined above satisfies the following properties:*

1. *If $\phi : A \to B$ and $\phi' : B \to C$ are isogenies then $Q(\phi'\phi) = Q(\phi')Q(\phi)$,*
2. *If $\phi : A \to A$ is multiplication by p then $Q(\phi) = p^{rk_p(A/\mathcal{K})}$.*

Theorem 3.2.4. *Let $A, B/\mathcal{K}$ be abelian varieties given with exterior forms ω_A, ω_B. Suppose $\phi : A \to B$ is an isogeny and $\phi^t : B^t \to A^t$ its dual. Let $Ш_0(A/\mathcal{K})$ be $Ш(A/\mathcal{K})$ modulo its divisible part and*

$$\Omega_A = \prod_{v|\infty_{real}} \int_{A(\mathcal{K}_v)} |\omega_A|. \prod_{v|\infty_{complex}} \int_{A(\mathcal{K}_v)} |\omega_A \wedge \bar{\omega}_A|.$$

For two exterior forms ω_1, ω_2, writing $\frac{\omega_1}{\omega_2} = \lambda \in \mathcal{K}$ whenever $\omega_1 = \lambda\omega_2$, set

$$C(A/\mathcal{K}) = \prod_{v\nmid\infty} c_v \left|\frac{\omega}{\omega_v^o}\right|_v,$$

where c_v and ω_v^o are respectively the local Tamagawa number and the Néron exterior form at a finite place v of $\mathcal{K}$, and similarly for B, then:

$$\frac{|B(\mathcal{K})_{tors}||B^t(\mathcal{K})_{tors}|C(A/\mathcal{K})\Omega_A}{|A(\mathcal{K})_{tors}||A^t(\mathcal{K})_{tors}|C(B/\mathcal{K})\Omega_B} \prod_{p|deg\phi} \frac{|Ш_0(A)[p^\infty]|}{|Ш_0(B)[p^\infty]|} = \frac{Q(\phi^t)}{Q(\phi)}.$$

Lemma 3.2.5. *Let $A/\mathcal{K}$ be a principally polarized abelian variety admitting an isogeny $\phi : A \to \hat{A}$ so that $\hat{A}$ is principally polarized. Write ϕ^t for the dual of ϕ and suppose that $\phi\phi^t = [p]$. Then, keeping notation as in Theorem 3.2.4,*

$$(-1)^{rk_p(A)} = \lambda(A), \qquad \textit{where } \lambda(A) = (-1)^{ord_p\left(\frac{C(A)\Omega_A}{C(\hat{A})\Omega_{\hat{A}}} \prod_{p|\deg\phi} \frac{|Ш_0(A)[p^\infty]|}{|Ш_0(\hat{A})[p^\infty]|}\right)}.$$

Proof. Using properties 1 and 2 of Lemma 3.2.3 we have

$$\frac{Q(\phi^t)}{Q(\phi)} = \frac{Q(\phi^t)Q(\phi)}{Q(\phi)^2} = \frac{Q(\phi^t\phi)}{Q(\phi)^2} = \frac{p^{rk_p(A)}}{Q(\phi)^2}.$$

Hence by Theorem 2.23. this yields

$$\frac{p^{rk_p(A)}}{Q(\phi)^2} = \frac{|\hat{A}_{tors}||\hat{A}^t_{tors}|C(A)\Omega_A}{|A_{tors}||A^t_{tors}|C(\hat{A})\Omega_{\hat{A}}} \prod_{p|deg\phi} \frac{|Ш_0(A)[p^\infty]|}{|Ш_0(\hat{A})[p^\infty]|}.$$

The result follows since $|\hat{A}_{tors}| = |\hat{A}^t_{tors}|$ and $|A_{tors}| = |A^t_{tors}|$. □

The conditions required by Lemma 3.2.5 are satisfied by Jacobians of genus 2 curves admitting a Richelot isogeny. Richelot isogenies are such that their codomain is the Jacobian of a genus 2 curve, a model of which is given explicitly (see Section 2.2). In particular, Richelot isogenies have principally polarized codomains and are of degree 4. It therefore follows from Lemma 3.2.5 that the parity of the 2^∞ Selmer rank of a Jacobian admitting a Richelot isogeny can be expressed as follows.

Theorem 3.2.6. *Let $A/\mathcal{K}$ be the Jacobian of a genus 2 curve admitting a Richelot isogeny. Then*

$$(-1)^{rk_2(A)} = \lambda(A), \qquad \textit{where } \lambda(A) = (-1)^{ord_2\left(\frac{C(A)\Omega_A}{C(\hat{A})\Omega_{\hat{A}}} \frac{|Ш_0(A)[2^\infty]|}{|Ш_0(\hat{A})[2^\infty]|}\right)}.$$

Proof. Follows from Lemma 3.2.5 and by definition of Richelot isogenies (see Theorem 2.2.17). □

3.2.2 Local factorization of the parity of the 2^∞ Selmer rank

Let $J/\mathcal{K}$ be a Jacobian admitting a Richelot isogeny and denote by $\hat{J}$ its codomain. Then by Theorem 3.2.6 we have

$$(-1)^{rk_2(J)} = \lambda(J), \qquad \text{where } \lambda(J) = (-1)^{ord_2\left(\frac{C(J)\Omega_J}{C(\hat{J})\Omega_{\hat{J}}} \frac{|Ш_0(J)[2^\infty]|}{|Ш_0(\hat{J})[2^\infty]|}\right)}.$$

We now wish to express $\lambda(J)$ as a product of local terms. Note that Theorem 3.2.4 already gives a partial result by defining $C(J)\Omega_J$ and $C(\hat{J})\Omega_{\hat{J}}$ in this way. Using a results of Poonen and Stoll presented in [32], we factor the term $\frac{|Ш_0(J)[2^\infty]|}{|Ш_0(\hat{J})[2^\infty]|}$ in $\lambda(J)$ as a product of local factors. We then refine the formula obtained for $\lambda(J)$ using local invariants of both J and $\hat{J}$ as well as their underlying curves C and $\hat{C}$.

Odd Jacobians and deficient places

Recall that for an abelian variety $A/\mathcal{K}$, $Ш(A/\mathcal{K})[p^\infty]$ is of the form $(\mathbb{Q}_p/\mathbb{Z}_p)^{\delta_p} \times Ш_0(A/\mathcal{K})[p^\infty]$ where $Ш_0(A/\mathcal{K})[p^\infty]$ is a finite p-group (see e.g. [8]). A consequence of the Cassels-Tate pairing for elliptic curves $E/\mathcal{K}$ is that $Ш_0(E/\mathcal{K})[p^\infty]$ is of square order. For general principally polarized abelian varieties however, its order could be twice a square.
In particular, a Jacobian J of a curve is said to be odd if the finite part of $Ш(J)$ has order twice a square, is said to be even otherwise. Thanks to a result of Poonen and Stoll in [32] it is possible to know whether J is odd or even by studying the deficiency of its underlying curve.

Definition 3.2.7. If C is a curve of genus g over a local field $\mathcal{K}_v$, we say that C is deficient if it has no $\mathcal{K}_v$-rational divisor of degree $g-1$.
If C is a curve of genus g over a global field $\mathcal{K}$, then a place v of $\mathcal{K}$ is called deficient if $C/\mathcal{K}_v$ is deficient.

Remark 3.2.8. For genus 2 curves, this is equivalent to saying that C does not have a $\mathcal{L}$-rational point in any extension $\mathcal{L}/\mathcal{K}$ of odd degree.

Definition 3.2.9. For a curve C and a local field $\mathcal{K}_v$ as above, we define

$$m_v(C) = \begin{cases} 2 & \text{if } v \text{ is deficient for } C, \\ 1 & \text{otherwise.} \end{cases}$$

Theorem 3.2.10. *[32][Corollaries 9, 12]*
If N is the number of deficient places of C then $|Ш_0(J/\mathcal{K})| = 2^N \cdot r = \prod_v m_v \cdot r$, where r is a square integer.

This prompts the following factorization for $\lambda(J)$.

Lemma 3.2.11. *Let $J/\mathcal{K}$ be a Jacobian admitting a Richelot isogeny ϕ over $\mathcal{K}$ and denote by $\hat{J}$ its codomain. Then*

$$\lambda(J) = (-1)^{ord_2\left(\frac{C(J)\Omega_J}{C(\hat{J})\Omega_{\hat{J}}}\right)} \prod_{v \in M_\mathcal{K}} (-1)^{ord_2\left(\frac{m_v(C)}{m_v(\hat{C})}\right)}.$$

Proof. This follows from Definition 3.2.9, Theorems 3.2.6 and 3.2.10 as they yield

$$\frac{|Ш_0(J)[2^\infty]|}{|Ш_0(\hat{J})[2^\infty]|} = \prod_{v \in M_\mathcal{K}} \frac{m_v(C)}{m_v(\hat{C})} r, \quad r \in \mathbb{Q}^{\times 2}.$$

□

Infinite places

Definition 3.2.12. Let $J/\mathcal{K}$ be a Jacobian admitting a Richelot isogeny ϕ over $\mathcal{K}$. For $v \in M_{\mathcal{K}}$ such that $v \mid \infty$, we let ϕ_v denote the map induced by ϕ on $J(\mathcal{K}_v)$ and define

$$\varphi : J(\mathcal{K}_v)^0 \to \hat{J}(\mathcal{K}_v)^0,$$

the restriction of the map ϕ_v to the identity component $J(\mathcal{K}_v)^0$ of $J(\mathcal{K}_v)$.

Lemma 3.2.13. *Let $J/\mathcal{K}$ be a Jacobian admitting a Richelot isogeny ϕ over $\mathcal{K}$. Let $\hat{\omega}$ be a choice of exterior form for $\hat{J}$ and choose $\omega = \phi^*\hat{\omega}$ as an exterior form for J. Keeping notation as in Theorem 3.2.4, we have*

$$\frac{\Omega_J}{\Omega_{\hat{J}}} = \prod_{v|\infty} \frac{n(\hat{J}(\mathcal{K}_v))}{|ker(\varphi)|n(J(\mathcal{K}_v))},$$

where $n(J(\mathcal{K}_v))$ and $n(\hat{J}(\mathcal{K}_v))$ denote the number of connected components of $J(\mathcal{K}_v)$ and $\hat{J}(\mathcal{K}_v)$ respectively.

Proof. As in Lemma 7.4 in [13], we have

$$\frac{\Omega_J}{\Omega_{\hat{J}}} = \prod_{v|\infty} \frac{|ker(\phi : J(\mathcal{K}_v) \to \hat{J}(\mathcal{K}_v))|}{|coker(\phi : J(\mathcal{K}_v) \to \hat{J}(\mathcal{K}_v))|} \left|\frac{\omega}{\phi^*\hat{\omega}}\right|_v$$

$$= \prod_{v|\infty} \frac{|ker(\phi : J(\mathcal{K}_v) \to \hat{J}(\mathcal{K}_v))|}{|coker(\phi : J(\mathcal{K}_v) \to \hat{J}(\mathcal{K}_v))|},$$

by our choice of exterior forms. Now, denote $\psi : J(\mathcal{K}_v)/J(\mathcal{K}_v)^0 \to \hat{J}(\mathcal{K}_v)/\hat{J}(\mathcal{K}_v)^0$ the map induced by ϕ_v on $J(\mathcal{K}_v)/J(\mathcal{K}_v)^0$ and consider the following morphism of short exact sequences.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & ker(\varphi) & & ker(\phi_v) & & ker(\psi) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & J(\mathcal{K}_v)^0 & \longrightarrow & J(\mathcal{K}_v) & \longrightarrow & J(\mathcal{K}_v)/J(\mathcal{K}_v)^0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \hat{J}(\mathcal{K}_v)^0 & \longrightarrow & \hat{J}(\mathcal{K}_v) & \longrightarrow & \hat{J}(\mathcal{K}_v)/\hat{J}(\mathcal{K}_v)^0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & coker(\varphi) & & coker(\phi_v) & & coker(\psi) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

As a consequence of ϕ having finite kernel and cokernel, and by the Snake Lemma we have

$$\frac{|ker(\varphi)||ker(\psi)||coker(\phi_v)|}{|ker(\phi_v)||coker(\varphi)||coker(\psi)|} = 1, \text{ and hence } \frac{|coker(\phi_v)|}{|ker(\phi_v)|} = \frac{|coker(\varphi)||coker(\psi)|}{|ker(\varphi)||ker(\psi)|}.$$

Now denoting $n(J(\mathcal{K}_v)) = |J(\mathcal{K}_v)/J(\mathcal{K}_v)^0|$ and $n(\hat{J}(\mathcal{K}_v)) = |\hat{J}(\mathcal{K}_v)/\hat{J}(\mathcal{K}_v)^0|$ the number of connected components of $J(\mathcal{K}_v)$ and $\hat{J}(\mathcal{K}_v)$ respectively, the third column yields :

$$\frac{|ker(\psi)|n(\hat{J}(\mathcal{K}_v))}{n(J(\mathcal{K}_v))|coker(\psi)|} = 1.$$

Hence

$$\frac{|coker(\phi_v)|}{|ker(\phi_v)|} = \frac{|coker(\varphi)|n(\hat{J}(\mathcal{K}_v))}{|ker(\varphi)|n(J(\mathcal{K}_v))}.$$

The result follows since $\mathrm{Im}(\varphi)$ is both open and closed in $\hat{J}(\mathcal{K}_v)^0$ therefore φ is surjective. □

Finite places

Let v be a finite place of $\mathcal{K}$, $\hat{\omega}$ be a choice of exterior form for $\hat{J}$ and choose $\omega = \phi^*\hat{\omega}$ as an exterior form for J. Denote ω_v^o and $\hat{\omega}_v^o$ their associated Néron exterior form at v. For two exterior forms ω_1, ω_2, write $\frac{\omega_1}{\omega_2} = \lambda \in \mathcal{K}$ whenever $\omega_1 = \lambda\omega_2$.

Lemma 3.2.14. *For the choices of exterior form as above*

$$|\frac{\omega}{\omega_v^o}|_v |\frac{\hat{\omega}_v^o}{\hat{\omega}}|_v = |\frac{\phi^*\hat{\omega}_v^o}{\omega_v^o}|_v.$$

Proof. There exists $r \in \mathcal{K}$ such that $\hat{\omega}_v^o = r\hat{\omega}$, hence

$$|\frac{\omega}{\omega_v^o}|_v|\frac{\hat{\omega}_v^o}{\hat{\omega}}|_v = |r|_v|\frac{\phi^*\hat{\omega}}{\omega_v^o}|_v = |r|_v\frac{|\phi^*\frac{1}{r}\hat{\omega}_v^o|_v}{|\omega_v^o|_v} = |\frac{\phi^*\hat{\omega}_v^o}{\omega_v^o}|_v.$$

□

Lemma 3.2.15. *Let $J/\mathcal{K}$ be a Jacobian admitting a Richelot isogeny ϕ over $\mathcal{K}$ and denote $\hat{J}$ its codomain. Fix $\hat{\omega}$ as a choice of exterior form for $\hat{J}$ and choose $\omega = \phi^*\hat{\omega}$ as an exterior form for J. Then keeping notation as in Theorem 3.2.4,*

$$\frac{C(J)}{C(\hat{J})} = \prod_{v\nmid\infty}\frac{c_v}{\hat{c}_v}|\frac{\phi^*\hat{\omega}_v^o}{\omega_v^o}|_v,$$

where c_v and $\hat{c}_v$ denote the Tamagawa numbers of J and $\hat{J}$ at the finite place $v \in M_{\mathcal{K}}$.

Using Theorem 3.2.6 together with Lemmata 3.2.11, 3.2.13 and 3.2.15, we obtain the following factorization for the parity of the 2^∞ Selmer rank of Jacobians admitting a Richelot isogeny.

Theorem 3.2.16. *Let $C/\mathcal{K}$ be a hyperelliptic curve of genus 2 such that its Jacobian J admits a Richelot isogeny ϕ. Denote by $\hat{C}$ and $\hat{J}$ the corresponding curve and isogenous Jacobian. Denote $\hat{\omega}_v^o$, ω_v^o the Néron exterior forms at the place v of $\mathcal{K}$ for J and $\hat{J}$ respectively. Then*

$$rk_2(J/\mathcal{K}) \equiv \sum_{v\in M_{\mathcal{K}}}\Lambda_v \ mod\ 2,$$

with

$$\Lambda_v = \begin{cases} ord_2(\frac{\hat{n}\cdot m_v}{|ker(\varphi)|n\cdot\hat{m}_v}) \text{ for } v \mid \infty, \\ ord_2(\frac{c_v\cdot m_v}{\hat{c}_v\cdot\hat{m}_v}) \text{ for } v \nmid 2\infty, \\ ord_2(\frac{c_v\cdot m_v}{\hat{c}_v\cdot\hat{m}_v}|\frac{\phi^*\hat{\omega}_v^o}{\omega_v^o}|_v) \text{ for } v \mid 2, \end{cases}$$

where n, $\hat{n}$ are the number of $\mathcal{K}_v$-connected components of J and $\hat{J}$, φ is the restriction of ϕ to the identity component of $J(\mathcal{K}_v)$, c_v and $\hat{c}_v$ the Tamagawa numbers of J and $\hat{J}$, and $m_v = 2$ if C is deficient at v, $m = 1$ otherwise, and similarly for $\hat{m}_v$.

Proof. This is clear from previous Lemmata and since $ord_2(|a|_v) = 1$ for all places $v \nmid 2$ and all $a \in \mathcal{K}$. □

Corollary 3.2.17. *Let $C/\mathcal{K}$ be a hyperelliptic curve of genus 2 such that its Jacobian J admits a Richelot isogeny ϕ. Denote by $\hat{C}$ and $\hat{J}$ the corresponding curve and isogenous Jacobian. Using notation as in Theorem 3.2.16, we have*

$$(-1)^{rk_2(J/\mathcal{K})} = \prod_{v\in M_{\mathcal{K}}}\lambda_v, \text{ with } \lambda_v = (-1)^{\Lambda_v}.$$

3.3 Computation of local invariants at infinite places

In this section we discuss how to express the local factor λ_v given by Theorem 3.2.16 in terms of basic properties of the defining polynomial of C when v is an infinite place. In this case, the local factor λ_v is given by

$$\lambda_v = (-1)^{ord_2\left(\frac{\hat{n}\cdot m_v}{|ker(\varphi)|n\cdot\hat{m}_v}\right)},$$

where $n, \hat{n}$ denote the number of connected components of $J(\mathcal{K}_v)$ and $\hat{J}(\mathcal{K}_v)$, φ is the restriction of ϕ to the identity component of $J(K_v)$, $m_v = 2$ if C is deficient for v, $m_v = 1$ otherwise and similarly for $\hat{m}_v$.

Case $\mathcal{K}_v = \mathbb{C}$

Lemma 3.3.18. *Let C be a Richelot curve and denote J its Jacobian. Then $\lambda_v = 1$ for places v of $\mathcal{K}$ with $\mathcal{K}_v = \mathbb{C}$.*

Proof. In this case $\lambda_v = 1$ trivially as $n = \hat{n} = m_v = \hat{m}_v = 1$ and $|ker(\varphi)| = 4$. □

Case $\mathcal{K}_v = \mathbb{R}$

Using a result of Gross and Harris in [19], we compute the number of real connected components of J and $\hat{J}$ from the number of real connected components of their underlying curves. The computation of m_v and $\hat{m}_v$ follows from the definition of the deficiency of a curve at the place v (see Remark 3.2.8). Finally we explain how to compute $|ker\varphi|$ from the defining polynomial of C.

Proposition 3.3.19. *[19][Proposition 3.2.2 and 3.3]*
Let $n(C(\mathbb{R}))$ denote the number of connected components on $C(\mathbb{R})$. Then

$$n(J(\mathbb{R})) = \begin{cases} 2^{n(C(\mathbb{R}))-1} & \text{if } n(C(\mathbb{R})) > 0 \\ 1 & \text{if } n(C(\mathbb{R})) = 0. \end{cases}$$

Proposition 3.3.20. *Let C be a Richelot curve with given quadratic factorization $G(x) = G_1(x)G_2(x)G_3(x)$ and denote by c the leading term of $G(x)$. Let $\delta_1, \delta_2, \delta_3$ denote the discriminants of the quadratic factors $G_1(x), G_2(x), G_3(x)$ respectively. Then C is deficient over $\mathbb{R}$, i.e. $m_{\mathbb{R}} = 2$ if and only if either*

- $\delta_i \in \mathbb{R}$, $\delta_i < 0$, $\forall i \in \{1, 2, 3\}$ *and* $c < 0$ *or,*
- *up to reordering,* $\delta_1 < 0$, $\delta_2 = \overline{\delta_3}$ *and* $c < 0$.

Proof. This is clear from the definition of deficiency (see Definition 3.2.7), since a curve C of genus 2 is deficient over $\mathbb{R}$ if and only if $C(\mathbb{R}) = \emptyset$ (see Remark 3.2.8). □

Proposition 3.3.21. *A divisor* $D_i = [P_i, Q_i] \in ker(\phi)$ *is in* $ker(\varphi)$ *if and only if the points* $P_i, Q_i \in C$ *satisfy either*
i) $P_i = \overline{Q_i}$, *or*
ii) P_i, Q_i *lie on the same connected component of* $C(\mathbb{R})$.

Proof. Recall that $\varphi : J(\mathbb{R})^0 \to \hat{J}(\mathbb{R})^0$ denotes the restriction of $\phi_{\mathbb{R}}$ to the identity component of $J(\mathbb{R})$. In particular, a divisor $D \in J(\mathbb{R})$ belongs to $ker(\varphi)$ if $D \in ker(\phi) \cap J(\mathbb{R})^0$. But since

$$ker(\phi) = \{0, D_1 = [P_1, Q_1], D_2 = [P_2, Q_2], D_3 = [P_3, Q_3]\},$$

it follows that a divisor $D_i = [P_i, Q_i] \in ker(\varphi)$ if D_i and 0 share the same connected component on $J(\mathbb{R})$. Equivalently, $D_i = [P_i, Q_i] \in ker(\varphi)$ if P_i, Q_i can be deformed continuously into two points on C defining $0 \in J$ (those two points being the intersections of $C(\mathbb{R})$ with a vertical line). This is clearly the case when P_i, Q_i are real points and both lie on the same component on $C(\mathbb{R})$. On the other hand, consider the case $P_i = \overline{Q_i}$. If C has no real point then $J(\mathbb{R})$ has only one component and we are done. Otherwise we can assume that $f(x)$ has at least a real root, say r and let $T = (r, 0)$ be the associated real point on $C(\mathbb{R})$. Pick any continuous path from P_i to T on $C(\mathbb{C})$, i.e. a continuous function $F : [0, 1] \to C(\mathbb{C})$, with $F(0) = P_i$ and $F(1) = T$. Then $G(t) = \overline{F(t)}$ is a continuous path on $C(\mathbb{C})$ from $\bar{P}_i$ to T. Hence $D(t) = [F(t), G(t)]$ is a continuous path on $J(\mathbb{R})$ from D_i to $[T, T] = 0$. Lastly, if P_i, Q_i are real points but lie on different components of $C(\mathbb{R})$, then there is no such continuous path $D(t)$ from D_i to 0 and hence $D_i \notin ker(\varphi)$.

□

Computation of $\hat{n}$ and $\hat{m}$

It remains to compute the number of real connected components of $\hat{J}$ as well as the real deficiency of $\hat{C}$. From Propositions 3.3.19 and 3.2.9, it is enough to compute the discriminants of the quadratic factors of the defining polynomial of $\hat{C}$. For that purpose, let C be a Richelot curve with given quadratic factorization $G(x) = G_1(x)G_2(x)G_3(x)$ and denote by $\delta_1, \delta_2, \delta_3$ the discriminants of the quadratic factors $G_1(x), G_2(x), G_3(x)$ respectively. Let α_i, β_i denote the roots of $G_i(x)$.
Consider the dual curve $\hat{C}$ with quadratic factorization $F(x) = F_1(x)F_2(x)F_3(x)$ and denote by $\hat{\delta_1}, \hat{\delta_2}, \hat{\delta_3}$ the discriminants of $F_1(x), F_2(x), F_3(x)$ respectively. Henceforth, addition of indices is performed modulo 3.

Definition 3.3.22. We define the *cross-ratio* of the four real numbers $\alpha_i, \beta_i, \alpha_j, \beta_j \in \mathbb{R}$ to be

$$\kappa_{i,j} = \frac{(\alpha_i - \alpha_j)(\beta_i - \beta_j)}{(\beta_i - \alpha_j)(\alpha_i - \beta_j)}.$$

Proposition 3.3.23. *The number of real roots of $F(x)$ is given as follows.*
i) if $\delta_i \in \mathbb{R}$ and $\delta_{i+1}, \delta_{i+2} \notin \mathbb{R}$, i.e. $\delta_{i+1} = \overline{\delta_{i+2}}$ then

$$\hat{\delta}_i \in \mathbb{R}, \quad \hat{\delta}_{i+1}, \hat{\delta}_{i+2} \notin \mathbb{R} \text{ with } \hat{\delta}_{i+1} = \overline{\hat{\delta}_{i+2}},$$

ii) if $\delta_i, \delta_{i+1} \in \mathbb{R}$ then

$$\hat{\delta}_{i+2} \in \mathbb{R} \text{ and } \hat{\delta}_{i+2} < 0 \Leftrightarrow \kappa_{i,i+1} < 0.$$

Proof. This follows directly from the formal computation of the discriminants $\hat{\delta}_1, \hat{\delta}_2, \hat{\delta}_3$ which gives

$$\hat{\delta}_i = \frac{4}{\Delta_G^2}(\alpha_{i+1} - \alpha_{i+2})(\alpha_{i+1} - \beta_{i+2})(\beta_{i+1} - \alpha_{i+2})(\beta_{i+1} - \beta_{i+2}), \quad i = 1, 2, 3.$$

□

Convention

Let C be a Richelot curve with given quadratic factorization $G(x) = G_1(x)G_2(x)G_3(x)$. In view of Propositions 3.3.19, 3.3.20 and 3.3.21, in order to compute $\lambda_{\mathbb{R}}$ we need to understand how the roots of $G(x)$ are distributed on the real line. We use the following convention to represent real roots of $G(x)$ on the real line: each dot  represents a root of $G(x)$ with the following shaping/colouring: red circles represent the real roots of $G_1(x)$ (respectively $F_1(x)$), blue diamonds that of $G_2(x)$ (respectively $F_2(x)$) and purple stars the real roots of $G_3(x)$ (respectively $F_3(x)$). A line between two roots α, β means that the points $P = (\alpha, 0)$ and $Q = (\beta, 0)$ belong to the same connected component of $C(\mathbb{R})$. It is understood that the broken lines on the outside of the roots meet at infinity so that exterior roots belongs to the same real component.

Remark 3.3.24. 1) The condition on the cross ratio of roots in the second case of Proposition 3.3.23 can be easily seen from their real picture as it involves their interlacing on the real line. As an example, if the real roots of $G_1(x), G_2(x)$ are distributed as follows  or  then $\kappa_{1,2} > 0$ so that $\hat{\delta}_3 > 0$. Otherwise, if they are as follows  then $\kappa_{1,2} < 0$ so that $\hat{\delta}_3 < 0$.
2) The order of $|ker(\varphi)|$ is immediate from the real picture as it suffices to count the number of same colour/shape roots that are linked by a line.

Example 3.3.25. Let $G_1(x) = x^2 - 16$, $G_2(x) = x^2 + x + \frac{17}{4}$ and $G_3(x) = x^2 - 2x + 9$ so that we have the following Richelot curve over $\mathbb{Q}$.

$$C : y^2 = G(x) = (x^2 - 16)(x^2 + x + \frac{17}{4})(x^2 - 2x + 9).$$

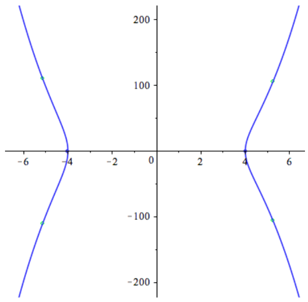


Figure 3.1: $y^2 = G(x)$

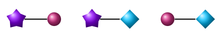
Here $\delta_1 = 64, \delta_2 = -16, \delta_3 = -32$ so that C has only one real connected component. Hence C is not deficient over $\mathbb{R}$. The real roots of $G(x)$ in Figure 3.1 are only that of $G_1(x)$ so that they necessarily share the unique real connected component. The corresponding picture is the simple one  .

Therefore the divisor D_1 corresponding to the quadratic $G_1(x)$ is on the real identity component of J and hence in $ker(\varphi)$. Finally, since $\delta_2, \delta_3 \in \mathbb{R}$, it follows that $\beta_2 = \overline{\alpha_2}, \beta_3 = \overline{\alpha_3}$ so that $D_2, D_3 \in ker(\varphi)$ and $|ker(\varphi)| = 4$.

Example 3.3.26. Let $G_1(x) = \frac{6}{131}x^2 - \frac{19}{2}x - \frac{35}{2}$, $G_2(x) = 2x^2 - 50x + 32$ and $G_3(x) = x^2 + \frac{81}{2}x + 16$ so that we have the following Richelot curve over $\mathbb{Q}$.

$$C : y^2 = G(x) = (\frac{6}{131}x^2 - \frac{19}{2}x - \frac{35}{2})(2x^2 - 50x + 32)(x^2 + \frac{81}{2}x + 16).$$

Here $\delta_1 = \frac{1201}{17161}, \delta_2 = 2244, \delta_3 = \frac{6305}{4}$ so that C has three real connected components. In particular, C is not deficient over $\mathbb{R}$. Note that the leading coefficient of $G(x)$ is positive. Computing the real roots of $G(x)$ we obtain the following picture  . Hence, in this example, none of the divisors defined by the quadratic polynomials of $G(x)$ share the same real component with the identity of J. Therefore $|ker(\varphi)| = 1$.

Remark 3.3.27. The sign of the leading term of $G(x)$ determines how the Weierstrass points of C are paired on real connected components. As an example, consider the same polynomial as in Example 3.3.26 but change the sign of its leading term to be negative. The picture becomes  .

Computation of $\lambda_{\mathbb{R}}$: Example

As in Example 3.3.25, let $G_1(x) = x^2 - 16$, $G_2(x) = x^2 + x + \frac{17}{4}$ and $G_3(x) = x^2 - 2x + 9$ so that we have the following Richelot curve over $\mathbb{Q}$.

$$C : y^2 = G(x) = (x^2 - 16)(x^2 + x + \frac{17}{4})(x^2 - 2x + 9).$$

We have seen that $\delta_1, \delta_2, \delta_3 \in \mathbb{R}$ with $\delta_1 > 0$ and $\delta_2, \delta_3 < 0$. Hence $n(C(\mathbb{R})) = 1$ so that by Proposition 3.3.19, $n(J(\mathbb{R})) = 1$ and by Proposition 3.3.20, $m_{\mathbb{R}} = 1$. From Proposition 3.3.23 it follows that $\hat{\delta_1}, \hat{\delta_2}, \hat{\delta_3} \in \mathbb{R}$ with $\hat{\delta_1}, \hat{\delta_2}, \hat{\delta_3} > 0$ so that $n(\hat{C}(\mathbb{R})) = 3$. Therefore $n(\hat{J}(\mathbb{R})) = 4$ and $\hat{m}_{\mathbb{R}} = 1$. Finally, we found that $|ker(\varphi)| = 4$ so that

$$\lambda_{\mathbb{R}} = (-1)^{ord_2(\frac{1\cdot 1}{4\cdot 4\cdot 1})} = 1.$$

3.4 Computation of local invariants at semistable finite places $v \nmid 2$

In this section, considering odd finite places, we explain how to extract a criterion for the semistability of C, and when C is semistable, the Tamagawa number of J and the deficiency of C from the p-adic properties of the roots of the defining polynomial of C. Since it will be used in later chapters, we also include how to extract the local root number of J. The results presented in this section are primarily a summary of existing results of V. Dokchitser and A. Morgan in [1], T. and V. Dokchitser in [10] and a joint work with T. and V. Dokchitser and A. Morgan in [14] and [15]. We refer to the articles for the proofs.

In the latter article, working over a field K with a discrete valuation, we discuss the semistable types of hyperelliptic curves and introduce the notion of a *cluster picture* for those. We also propose a naming convention which extends that of Namikawa-Ueno for genus 2 curves in [38] that will be used here. In [14], we discuss the arithmetic of hyperelliptic curves over local fields of odd residue characteristics from their cluster pictures. Results of particular interest for us include a criterion to decide whether a hyperelliptic curve C is semistable and an explicit description of the special fibre of its minimal regular model including the action of Galois on its components. Our computation of deficiency for C follows from that description. Table 3.1 below lists the reduction types of a class of semistable genus 2 curves [1], including the computation of Tamagawa numbers, deficiency and local root numbers. This table will be used as a reference in later sections when computing invariants of semistable Richelot curves and Jacobians. We end this section by providing examples to illustrate the use of these results.

[1] the list is not complete as we are not considering the types $1 \times I_n$ and $I_n \times I_m$ in this work. A complete version of this list can be found in [15].

Notation

In this section, K will denote a finite extension of $\mathbb{Q}_p$ for an odd prime p. We write $v(x)$ for the normalized p-adic valuation of $x \in \overline{\mathbb{Q}}_p$, π for a choice of uniformiser of K, k for the residue field of K and $G_K = Gal(\overline{\mathbb{Q}_p}/K)$. Let C/K be a hyperelliptic curve of genus g given by

$$C : y^2 = f(x) = c \prod_{r \in R} (x - r),$$

where

$$R \subset \overline{\mathbb{Q}_p}, \quad c \in K^\times, \quad deg(f) = 2g + 1 \text{ or } 2g + 2 \quad (\neq 1, 2, 4).$$

and write J for its Jacobian.

Cluster pictures

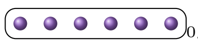
Definition 3.4.28. A *cluster* of roots $\mathfrak{s}$ is a non-empty subset of R of the form

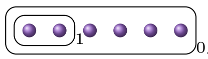
$$\mathfrak{s} = \{r \in R \mid v(r - z_\mathfrak{s}) \geq d\} = R \cap Disc(z_\mathfrak{s}, d),$$

with $z_\mathfrak{s} \in \overline{\mathbb{Q}_p}, d \in \mathbb{Q}$. We call $d_\mathfrak{s} = \min_{r,r' \in \mathfrak{s}} v(r - r')$ the *depth* of $\mathfrak{s}$. The set of clusters for the roots of $f(x)$ is called the *Cluster picture* of C.

As a convention, we draw the roots $r \in R$ by means of the little symbol •, and clusters by circling the roots (we do not circle single roots). We indicate the depth of a cluster at the bottom right of the circle.

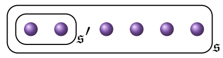
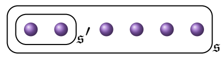
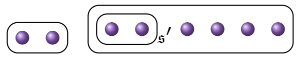
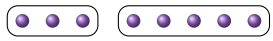
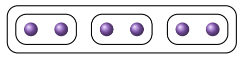
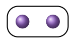
Examples

1) Let $C : y^2 = x(x-1)(x-2)(x-3)(x-4)(x-5)$ and choose $p = 13$. Then $R = \{0, 1, 2, 3, 4, 5\}$ and the only cluster of roots for C is R itself. It has depth 0 and is represented by .

2) Let $C : y^2 = x(x-13)(x-2)(x-3)(x-4)(x-5)$ and choose $p = 13$. Then $R = \{0, 13, 2, 3, 4, 5\}$ and the cluster picture of C consists of 2 clusters: R of depth 0 and $\{0, 13\}$ of depth 1. The Cluster picture of C is .

There is terminology associated to cluster pictures which we will use in this work. We include a little glossary for future reference.

Glossary

root	cluster of size 1	
child of $\mathfrak{s}$	$\mathfrak{s}' < \mathfrak{s}$ maximal subcluster of $\mathfrak{s}$	
parent of $\mathfrak{s}'$	cluster $\mathfrak{s}$ in which $\mathfrak{s}'$ is maximal	
even	cluster of even size	
odd	cluster of odd size	
übereven	even cluster with only even children	
twin	cluster of size 2	
principal $\mathfrak{s}$	$\lvert\mathfrak{s}\rvert > 2$ and if $\lvert\mathfrak{s}\rvert = 2g+2$ then $\mathfrak{s}$ is not the union of children of size $2, 2g$ or $1, 1, 2g$ or odd,odd	

Semistability criterion

Theorem 3.4.29. *The curve C/K is semistable if and only if the following hold:*
(i) $K(R)/K$ has ramification degree at most 2,
(ii) Every cluster of size > 1 is inertia invariant,
(iii) Every principal cluster $\mathfrak{s}$ has $d_{\mathfrak{s}} \in \mathbb{Z}$ and $\nu_{\mathfrak{s}} \in 2\mathbb{Z}$, where

$$\nu_{\mathfrak{s}} = v(c) + |\mathfrak{s}| d_{\mathfrak{s}} + \sum_{r \notin \mathfrak{s}} v(z_{\mathfrak{s}} - r).$$

Semistable reduction of genus 2 curves, Tamagawa numbers, deficiency and local root numbers

Table 3.1 tabulates a class of semistable genus 2 curves by reduction types. Precisely, we are considering curves of genus 2 given by $C/K : y^2 = G(x)$, where $G(x)$ has good reduction $\bmod \pi$ or has one, two or three double roots, or two triple roots within which no deeper double roots occur (the latter corresponding to the case of the Jacobian of C having good reduction while C has bad reduction). These cases essentially cover the semistable cases of C except for the missing cases of having double roots inside triple roots.

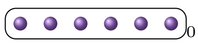
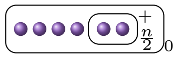
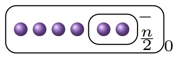
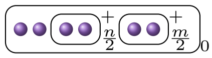
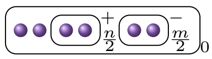
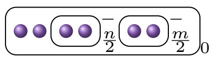
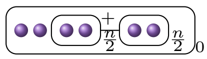
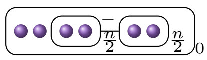
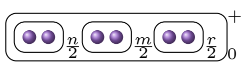
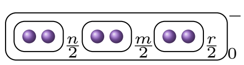
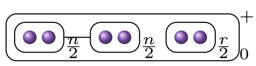
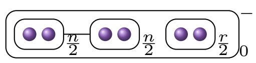
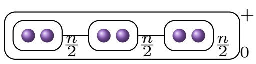
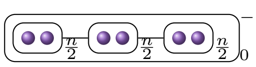
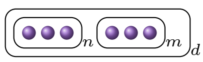
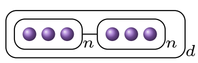
Each type is given using a cluster picture together with its action of G_K, the Tamagawa number of the corresponding Jacobian, the local deficiency and the local root number.

Table 3.1: Semistable reduction type in genus 2

Notation: $n, m, r \in \mathbb{Z}$, and in the last two rows $d \equiv n \equiv m \bmod 2$.

In the third column: $\tilde{n} = 2$ if $2|n$ and $\tilde{n} = 1$ if $2 \nmid n$,

$D = \gcd(m, n, r)$; $N = nm + nr + mr$.

Type	C	c_v	m_v	ω_v
2		1	1	1
1_n^+		n	1	-1
1_n^-		$\tilde{n}$	1	1
$I_{n,m}^{+,+}$		nm	1	1
$I_{n,m}^{+,-}$		$n\tilde{m}$	1	-1
$I_{n,m}^{-,-}$		$\tilde{n}\tilde{m}$	1	1
$I_{n\text{-}n}^{+}$		n	1	-1
$I_{n\text{-}n}^{-}$		$\tilde{n}$	1	1
$U_{n,m,r}^{+}$		N	1	1
$U_{n,m,r}^{-}$		$\widetilde{N/D} \times \widetilde{D}$	$\begin{cases} 2 & n,m,r \text{ odd} \\ 1 & \text{else} \end{cases}$	1
$U_{n\text{-}n,r}^{+}$		$n+2r$	1	-1
$U_{n\text{-}n,r}^{-}$		n	$\begin{cases} 2 & r \text{ odd} \\ 1 & \text{else} \end{cases}$	-1
$U_{n\text{-}n\text{-}n}^{+}$		3	1	1
$U_{n\text{-}n\text{-}n}^{-}$		1	$\begin{cases} 2 & n \text{ odd} \\ 1 & \text{else} \end{cases}$	1
$1 \times_{\frac{n+m}{2}} 1$		1	1	1
$1 \tilde{\times}_n 1$		1	$\begin{cases} 2 & n \text{ odd} \\ 1 & \text{else} \end{cases}$	1

Convention

This table can be used as follows. Consider a semistable genus 2 curve $C/K : y^2 = G(x)$. Denote $c \in K$ the leading term of $G(x)$. By the semistability criterion 3.4.29, either $v(c) \in 2\mathbb{Z}$ or, if $v(c)$ is odd, then C corresponds to the last two rows of the table[2]. In the former case we apply a change of variable if necessary to obtain $v(c) = 0$. Then reducing $G(x)$ mod π, we find C to be in the first block of Table 3.1 if $G(x)$ has good reduction; second, third and fourth block if $G(x)$ has one, two or three double roots mod π respectively. Lastly, if $G(x)$ has two triple roots then the cluster picture of C corresponds to the last block of Table 3.1[3]. In each case, it remains to find the specific row corresponding to C. In a cluster picture, we draw a line between two clusters when they are permuted by G_K. This action can be identified from the field of definition of the roots contained in the clusters. Finally, the sign on top of the clusters are given by Proposition 3.4.30. These signs indicate the action of G_K on the components of the special fibre of the minimal regular model of C corresponding to the nodes on the reduced curve given by the twin clusters, or the action of G_K on the two components with three transversal intersections in the case of three double roots. A "$+$" on the top right corner of a twin means that G_K acts trivially on its corresponding components and a "$-$" means that it permutes them. This is the equivalent of having split or non-split multiplicative reduction on an elliptic curve. Finally, in the last two rows of Table 3.1, the depths displayed are "relative depths", i.e. the valuations of the differences of the roots inside the size three clusters are $n + d$ and $m + d$.

Proposition 3.4.30. *Let C be a semistable hyperelliptic curve of genus 2. Let $r_1, ..., r_6 \in \overline{\mathbb{Q}}_p$ be the roots of $G(x) \in K[x]$ defining C and $c \in K$ be the leading term of $G(x)$.*
i) If C is of type $1_n^{\pm}$ or $I_{n,m}^{\pm,\pm}$ ($I_{n\tilde{~}n}^{\pm,\pm}$ resp.), let $\mathfrak{t}$ be a twin cluster in the cluster picture of C and choose a root r in $\mathfrak{t}$. Then the sign of $\mathfrak{t}$ is $+$ if and only if

$$T_r = c \prod_{r_i \notin \mathfrak{t}} (r - r_i) \in K^{\times 2} \ (F^{\times 2} \textit{ resp.}),$$

where F/K is a quadratic unramified extension.
ii) If C is of type $U_{n,m,r}^{\pm}$, then C is of type $U_{n,m,r}^{+}$ if and only if $c \in K^{\times 2}$.

Proof. This is a reformulation of Theorem 5.6 in [14] adapted to the semistable cases of genus 2 curves. □

[2] or to one of the missing cases that can be found in Table 9 in [15]
[3] See Remark 3.4.31 if the reduction of $G(x)$ is different from that given in Table 3.1

Deficiency of C

It follows from Remark 1 in [32] that C is deficient at a finite place if and only if the order of the $Gal(\bar{k}/k)$-orbit of each irreducible component of the special fibre of the minimal regular model of C is even. In [14], starting with the cluster picture of a semistable curve C/K, we give an explicit description of the special fibre of the minimal regular model of C as well as the Galois action on its components. We refer to the article for the precise statements and proof but we note that this is enough to recover the deficiency of C at that place.

Tamagawa numbers for J

Tamagawa numbers for Jacobians of genus 2 curves at a semistable place are computed in [1]. In the last block of Table 3.1, J has good reduction and hence its Tamagawa number is 1.

Local root number

The local root numbers for semistable abelian varieties are explicitly given from their Weil-Deligne representation in [10][Proposition 3.23]. In particular $\omega_v(J/K) = (-1)^t$, where t is the multiplicity of 1 as an eigenvalue of Frobenius on the toric part of the Galois representation[4]. The Weil-Deligne representation for semistable Jacobians of hyperelliptic curves of genus g at odd places are computed in [14].

Remark 3.4.31. The cluster picture associated to a hyperelliptic curve $C : y^2 = f(x)$ is not canonical. Indeed, applying a Mobius transformation to the roots of $f(x)$ might change their p-adic configuration and hence the new model of $f(x)$ would produce a different cluster picture. However, as shown in [15], these different pictures share the same equivalence class for which there is a canonical representative (the only cluster picture such that the maximal cluster is of size $2g + 2$ and is the only cluster of size $> g + 1$, and there are either 0 or 2 clusters of size $g + 1$). This representative is called a *balanced* cluster picture (see Definition 3.37 in [15]).

In particular, two semistable curves

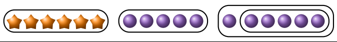
$$C_1 : y^2 = c_1 \prod_{i=1,..,6} (x - r_i), \quad C_2 : y^2 = c_2 \prod_{i=1,..,6} (x - r_i'),$$

such that the cluster pictures of C_1 and C_2 are equivalent, share the same special fibre, Tamagawa number, deficiency and root number. For genus 2, Table 3.2 gives a list of all possible cluster pictures in each class as well as its balanced representative (orange star cluster pictures are balanced). Strictly speaking, the process of

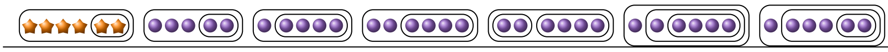
[4]when the cluster picture is balanced and the maximal cluster is not übereven, this amounts to counting the number of + on top of twins

rebalancing also takes into consideration the depth of the clusters. We chose not to display the depths of clusters in Table 3.2 in order to keep its length reasonable.

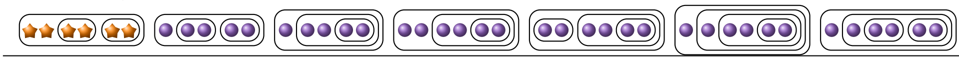
Type 2



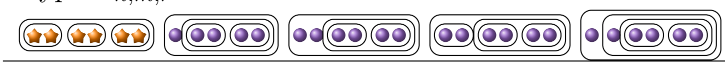
Type I_n



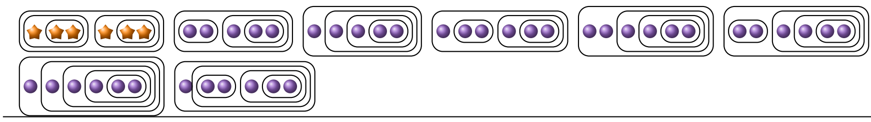
Type $I_{n,m}$



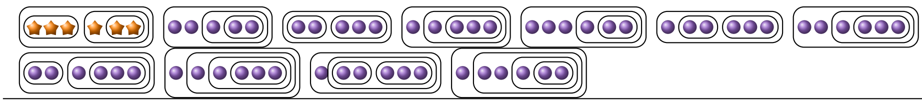
Type $U_{n,m,r}$



Type $I_n \times I_m$



Type $1 \times I_n$



Type 1×1

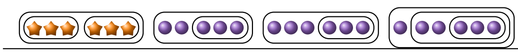
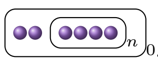


Table 3.2: Cluster pictures for semistable types of genus 2 curves with *balanced* representative

Balanced cluster pictures: example

Let $p > 5, n > 0$ and consider the curve $C/\mathbb{Q}_p$ given by

$$y^2 = (x-1)(x-2)(x-5+p^n)(x-5+2p^n)(x-5+3p^n)(x-5+4p^n).$$

The cluster picture associated to C is . Apply the following changes of variables:

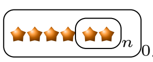
1) $x_1 = x + 5 \Rightarrow C : y^2 = (x_1+4)(x_1-3)(x_1+p^n)(x_1+2p^n)(x_1+3p^n)(x_1+4p^n)$

2) $x_1 = p^n x_2 \Rightarrow C : y^2 = p^{4n}(p^n x_2+4)(p^n x_2-3)(x_2+1)(x_2+2)(x_3+3)(x_2+4)$

3) $y = p^{2n} y_1 \Rightarrow C : y_1^2 = (p^n x_2+4)(p^n x_2-3)(x_2+1)(x_2+2)(x_3+3)(x_2+4)$

4) $x_2 = \frac{1}{x_3} \Rightarrow C : y_1^2 = (p^n \frac{1}{x_3}+4)(p^n \frac{1}{x_3}-3)(\frac{1}{x_3}+1)(\frac{1}{x_3}+2)(\frac{1}{x_3}+3)(\frac{1}{x_3}+4)$

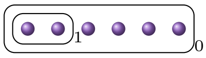
5) $y_2 = y_1 x_3^3 \Rightarrow C : y_2^2 = (4x_3+p^n)(3x_3+p^n)(x_3+1)(x_3+2)(x_3+3)(x_3+4)$,

to find that the new cluster picture for C is .

Semistability criterion: examples

1) Let $C : y^2 = x(x-p)(x-2)(x-3)(x-4)(x-5)$ and choose $p > 5$. (i) and (ii) in Theorem 3.4.29 are trivially satisfied. The only principal cluster in the cluster picture of C is R and

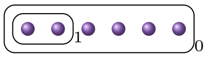
$$\nu_R = v(1) + 6 \times 0 + 0 = 0,$$



hence C semistable.

2) Let $C : y^2 = px(x-p)(x-2)(x-3)(x-4)(x-5)$ and choose $p > 5$. This is similar to our first example, except

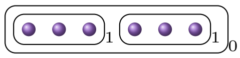
$$\nu_R = v(p) + 6 \times 0 + 0 = 1,$$



hence C is not semistable.

3) Let $C : y^2 = x(x-p)(x+p)(x-1+p)(x-1+p^2)(x-1+2p^2)$ and choose $p > 5$. (i) and (ii) in Theorem 3.4.29 are trivially satisfied. Clusters in the cluster picture of C are R of depth 1, $\mathfrak{s}_1 = \{0, p, -p\}$ and $\mathfrak{s}_2 = \{1-p, 1-p^2, 1-2p^2\}$ both of depth 1. Only $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are principal and

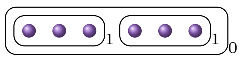
$$\nu_{\mathfrak{s}_1} = v(1) + 3 \times 1 + 0 = 3, \qquad \nu_{\mathfrak{s}_2} = v(1) + 3 \times 1 + 0 = 3,$$



hence C is not semistable.

4) Let $C : y^2 = px(x-p)(x+p)(x-1+p)(x-1+p^2)(x-1+2p^2)$ and choose $p > 5$. This is similar to Example 3) except

$$\nu_{\mathfrak{s}_1} = v(p) + 3 \times 1 + 0 = 4, \qquad \nu_{\mathfrak{s}_2} = v(p) + 3 \times 1 + 0 = 4,$$

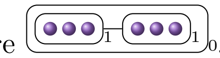


hence C is semistable.

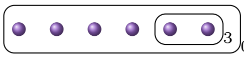
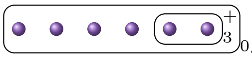
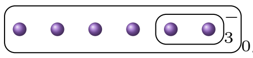
Cluster pictures: examples

1) Consider the curve defined over $\mathbb{Q}_{131}$ as

$$C : y^2 = f(x) = \frac{-131}{2}(-3x^2 + \frac{19}{2}x + \frac{35}{2})(2x^2 - 50x + 32)(x^2 + \frac{81}{2}x + 16),$$

Computing the roots of $f(x)$ gives the following cluster picture . In particular since $v(c) = 1$, C is semistable. The line between the two clusters of size 3 indicates that the roots of $f(x)$ are defined over an unramified extension of $\mathbb{Q}_{131}$ so that their respective clusters are permuted by Frobenius.
2) Let $p > 5$ and consider the hyperelliptic curve

$$C : y^2 = x(x-p^3)(x-1)(x-2)(x-3)(x-4).$$

The reduced curve is $\tilde{C} : \tilde{y}^2 = \tilde{x}^2(\tilde{x}-1)(\tilde{x}-2)(\tilde{x}-3)(\tilde{x}-4)$ so that the cluster picture of C without signs is .
The special fibre of the minimal regular model of C consists of a hexagon of $\mathbb{P}_1$'s, obtained by blowing up the singularity at the node of $\tilde{C}$. It follows from Proposition 3.4.30 that if $24 \in \mathbb{Q}_p^{\times 2}$ then $Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ acts trivially on them (equivalently, both slopes of the tangents at the node of $\tilde{C}$ are defined over $\mathbb{F}_p$) and the cluster picture is .
Otherwise, $Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ permutes the components (both slopes are defined over $\mathbb{F}_{p^2}$) and the cluster picture is .

Computation of λ_v : Example

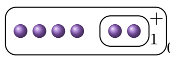
Keeping the same example as in Section 3.3, consider the following Richelot curve defined over $\mathbb{Q}$ by

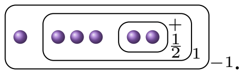
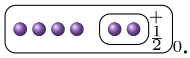
$$C : y^2 = (x^2-16)(x^2+x+\frac{17}{4})(x^2-2x+9),$$

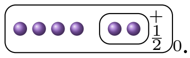
and its Richelot dual

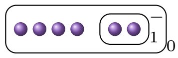
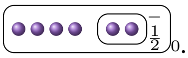
$$\hat{C} : y^2 = \frac{-131}{2}(-3x^2+\frac{19}{2}x+\frac{35}{2})(2x^2-50x+32)(x^2+\frac{81}{2}x+16).$$

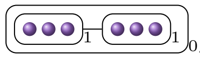
In order to determine the parity $rk_2(J/\mathbb{Q})$ using Theorem 3.2.16, we need to compute λ_p for $J/\mathbb{Q}$ at each prime p. By first computing both discriminants of C and $\hat{C}$ we find that the odd finite places of bad reduction of C are $p = 3, 5, 11, 13, 17, 97, 1201$ and similarly for $\hat{C}$ with the addition of $p = 131$. Hence, outside of this set of primes, $\lambda_p = 1$. For the primes of bad reduction, we compute the cluster pictures of C and $\hat{C}$, and use the results above to compute $c_p, m_p, \hat{c}_p$ and $\hat{m}_p$.

For $p = 3, 17$, we find that the cluster picture of C is .

For $p = 3$ we find the cluster picture of $\hat{C}$ to be . As in Remark 3.4.31, this is equivalent to the cluster picture .

For $p = 17$, the cluster picture for $\hat{C}$ is .
It follows from Table 3.1 that $c_p = 2, m_p = 1$ and $\hat{c}_p = 1$, $\hat{m}_p = 1$ so that $\lambda_p = -1$.

For $p = 5, 11, 13, 97, 1201$, we find that the cluster picture of C is 
and that of $\hat{C}$ is .
It follows that $c_p = 2, m_p = 1$ and $\hat{c}_p = 1$, $\hat{m}_p = 1$ so that $\lambda_p = -1$.

For $p = 131$, C has good reduction while the cluster picture of $\hat{C}$ is .
It follows that $c_p = 1, m_p = 1$ and $\hat{c}_p = 1$, $\hat{m}_p = 2$ so that $\lambda_p = -1$.

3.5 Computation of local invariants at finite places $v|2$

In the case of places v dividing 2, we take a different approach to control λ_v. Indeed, at these places, it is very difficult to compute the Tamagawa numbers and the minimality of the Néron differential for all reduction types. Rather, we come back to the initial definition of λ_v and compute

$$ord_2 \frac{|coker(\phi_v : J(K_v) \to J(K_v))|}{|ker(\phi_v : J(K_v) \to J(K_v))|} \frac{m_v(C)}{m_v(\hat{C})}.$$

First we note that if the curve C has totally split toric reduction at v then it is not deficient (see Section 3.4) and similarly for $\hat{C}$. We therefore construct a family of curves having totally split toric reduction at v and prove, using a result of A. Morgan presented in the appendix, that $\lambda_v = 1$ for their Jacobians.

Lemma 3.5.32. *Let $K/\mathbb{Q}_2$ be a finite extension and let C/K be a hyperelliptic curve of genus 2 given by*

$$C : y^2 + yh(x) = f(x), \text{ with}$$

$$h(x) = h_2x^2 + h_1x + h_0, \quad f(x) = c_6x^6 + c_5x^5 + c_4x^4 + c_3x^3 + c_2x^2 + c_1x + c_0,$$

for $h_i, c_i \in \mathcal{O}_K$, $0 \le i \le 6$. If

$$c_0 \equiv c_1 \equiv c_2 \equiv h_0 \equiv 0 \mod \pi, \text{ and } \quad h_1 \equiv h_2 \equiv c_4 \equiv c_6 \equiv c_5 \equiv c_3 \equiv 1 \mod \pi,$$

then J has totally split toric reduction at v.

Proof. This follows from computing partial derivatives of $g(x,y) = y^2 - yh(x) - f(x)$ to find two singularities on C at the points $(0,0)$ and $(1,0)$. Since $g(x,y) \equiv (y+x(x+1)^2)(y+x^2(x+1)) \bmod \pi$, it follows that the reduction of this chart consists of two genus 0 components intersecting at the two points $(0,0)$ and $(1,0)$. Computing the Taylor series of $g(x,y) \bmod \pi$ around singularities $(0,0),(1,0)$, one finds $xy + y^2 +$ higher order terms and $y^2 + (x+1)y +$ higher order terms respectively. Considering the second affine chart by letting $x = \frac{1}{t}$ and $y = \frac{v}{t^3}$ and computing partial derivatives as above, we find that the point at infinity given by $t = 0, v = 1$ is singular. As above, the reduction of this chart is given by two genus 0 curves intersecting at two points, one of them being the point at infinity. It follows that the reduction of the model of the curve given by these two charts consists of two $\mathbb{P}_1$s intersecting transversally in the three points $(0,0),(1,0)$ and $(t = 0, v = 1)$. Moreover, Frobenius acts trivially on the homology of the special fibre and hence the reduction is totally split toric. □

Corollary 3.5.33. *The curve*

$$C_2 : y^2+y(x^2+x+2) = f_2(x), \qquad f_2(x) = x^6+x^5-135x^4+821x^3-4414x^2+2988x+734$$

admits a Richelot isogeny and has totally split toric reduction at 2.

Proof. The curve C_2 has totally split toric reduction at 2 from Lemma 3.5.32. In order to show that C_2 admits a Richelot isogeny, we perform the following change of variable over $\mathbb{Q}_2$:

$$X = x, \qquad Y = y - \frac{1}{2}h(X), \quad h(x) = x^2 + x + 2,$$

to obtain

$$C_2 : Y^2 = (X-1)(X+15)(X^2 - 9X - \frac{7}{4})(X^2 - 4X + 28)$$

and the existence of a Richelot isogeny follows from the factorization of the right hand side. □

Proposition 3.5.34. *Let $n, m, d, k, r \in \mathcal{O}_K$ and define the following family of curves*

$$\mathcal{C} : y^2 = f(x) = G_1(x)G_2(x)G_3(x), \quad \text{with } G_1(x) = x^2 - (8+4n)^2,$$

$$G_2(x) = x^2+x(-2m-23)+\frac{441}{4}-2d+14m, \quad G_3(x) = x^2+x(-8k-18)+105+8r+56k.$$

Then the curves in $\mathcal{C}$ have totally split toric reduction.

Proof. Follows from performing the change of variable $X = x + 7$, $Y = y + \frac{1}{2}h(X)$

with $h(x) = x^2+x+2$ to find that reducing $\mod \pi$, $f(x) \equiv f_2(x)$ so that all curves $C \in \mathcal{C}$ have totally split toric reduction by Corollary 3.5.33. $\square$

Corollary 3.5.35. *The family $\mathcal{C}$ can be given by mean of congruences on coefficients as follows:*

$$\mathcal{F} : y^2 = (x^2 - (4t_1)^2)(x^2 + t_2x + t_3)(x^2 + t_4x + t_5),$$

where

$$t_1 \in \mathcal{O}_K, \quad t_2 \equiv 1 \ mod \ 2, \quad t_3 - \frac{1}{4} \equiv 0 \ mod \ 2, \quad t_4 = -2 \ mod \ 8, \quad t_5 \equiv 1 \ mod \ 8.$$

Proof. This follows from letting

$$t_1 = \frac{8+4n}{4}, t_2 = -2m - 23, t_3 = \frac{441}{4} - 2d + 14m,$$

$$t_4 = -8k - 18, t_5 = 105 + 8r + 56k.$$

One recovers $n, m, d, k, r \in \mathcal{O}_K$ as follows

$$n = t_1 - 2, \quad m = \frac{t_2+23}{2}, \quad d = \frac{t_3}{2} - \frac{1}{8} - 55 - 7m, \quad k = \frac{t_4+18}{8}, \quad r = \frac{t_5-1}{8} - 13 - 7k.$$

$\square$

Theorem 3.5.36. *Let $K/\mathbb{Q}_2$ be a finite extension and suppose that $C \in \mathcal{C}$ is given by*

$$C : y^2 = G_1(x)G_2(x)G_3(x),$$

such that $G_2(x), G_3(x)$ are both irreducible in K. Then $\lambda_v = 1$ for C.

Proof. From Corollary 3.5.33 and Proposition 3.5.34, all curves $C \in \mathcal{C}$ have totally split toric reduction and admit a Richelot isogeny. Therefore, since from Remark 1 in [32], it follows that C is deficient at a finite place if and only if the order of the $Gal(\bar{k}/k)$-orbit of each irreducible component of the special fibre of the minimal regular model of C is even, none of the curves C are deficient over K. Similarly for $\hat{C}$ since they have isogenous Jacobians so that $\hat{C}$ also has totally split toric reduction. It follows by definition of λ_v that

$$\lambda_v = (-1)^{ord_2 \frac{|coker(\phi_v : J(K_v) \to J(K_v))|}{|ker(\phi_v : J(K_v) \to J(K_v))|}}.$$

Moreover, from Proposition A.0.9 and Remark A.0.11, we have that

$$\frac{|coker(\phi_v : J(K_v) \to J(K_v))|}{|ker(\phi_v : J(K_v) \to J(K_v))|} = 2^{2[K:\mathbb{Q}_2]},$$

since when $G_2(x), G_3(x)$ are irreducible over K we necessarily have $|J(K)[2]| = 4 =$

$|ker(\phi)|$.

Therefore $\lambda_v = 1$. $\square$

Computation of λ_v : Example

It is readily verified that our curve

$$C : y^2 = (x^2 - 16)(x^2 + x + \frac{17}{4})(x^2 - 2x + 9),$$

belongs to the family $\mathcal{F}$ above and that both $G_2(x)$ and $G_3(x)$ are irreducible over $\mathbb{Q}_2$. Therefore $\lambda_2(J) = 1$ by Theorem 3.5.36.

3.6 Example of computation of the parity of the 2^∞ Selmer rank

We now compute the parity of the 2^∞ Selmer rank of the Jacobian of

$$C : y^2 = (x^2 - 16)(x^2 + x + \frac{17}{4})(x^2 - 2x + 9)$$

using Theorem 3.2.16. Let $S = \{3, 5, 11, 13, 17, 97, 1201, 131\}$ be the set of odd primes of bad reduction for C and $\hat{C}$. It follows from the set of examples above that

$$\lambda_{\mathbb{R}}(J) = 1, \quad \lambda_p = 1 \ \forall p \notin S, \quad \lambda_p = -1 \ \forall p \in S.$$

Therefore

$$(-1)^{rk_2(J)} = 1,$$

so that $rk_2(J)$ is even.

Chapter 4

2-parity conjecture for $C_2 \times D_4$ Jacobians

4.1 2-parity theorem

In this chapter we prove the 2-parity conjecture for a particular family of semistable Richelot Jacobians. Namely for these we show

$$(-1)^{rk_2(J)} = \omega(J),$$

where $\omega(J)$ denotes the global root number (see Conjecture 2.3.21). From the results of Chapter 3, we are now able to compute the parity of the 2^∞ Selmer rank for semistable Richelot Jacobians (with conditions at 2-adic places as in Section 3.5). This is achieved by factorizing $(-1)^{rk_2(J)}$ as a product of computable local terms λ_v. In order to prove the 2-parity conjecture, one would hope to prove that at each place v, the local term λ_v equals the local root number ω_v, so that the conjecture follows by taking product over all places. However, as suggested by the proof of the 2-parity conjecture for elliptic curves in [9], these terms do not agree locally but their discrepancy is given by a product of Hilbert Symbols involving some specific invariant polynomials in the roots of the defining polynomial of the curve. The main step of the proof is therefore to find the suitable set of invariants which, correctly paired in Hilbert Symbols, match the local discrepancy between λ_v and ω_v. We found such a set under the condition that the Galois group of the Richelot polynomial of C is a subgroup of $C_2 \times D_4$. In this case we conjecture the following.

Conjecture 4.1.1 (Conjecture 4.4.10)**.** *Let $C/\mathcal{K}$ be a Richelot curve given by*

$$C : y^2 = f(x), \quad \textit{such that} \quad Gal(f) \subseteq C_2 \times D_4,$$

and denote J its Jacobian. Then for all places v of $\mathcal{K}$

$$\lambda_v(J) = E_v(J) \cdot \omega_v(J),$$

for the explicit product of Hilbert Symbols $E_v(J)$ given in Definition 4.3.9, and whenever the invariants involved in E_v are non-zero.

Definition 4.1.2 ($C_2 \times D_4$ curve)**.** A Richelot polynomial of the form

$$G(x) = G_1(x)G_2(x)G_3(x) \in \mathcal{K}[x], \text{ where } \qquad G_1(x) = c(x - \alpha_1)(x + \alpha_1) \in \mathcal{K}[x]$$

is called a $C_2 \times D_4$ polynomial[1]. This factorization is equivalent to

$$Gal(G(x)) \subseteq C_2 \times D_4 \subset C_2^3 \rtimes S_3 \subset S_6.$$

A Richelot curve given by a $C_2 \times D_4$ polynomial is called a $C_2 \times D_4$ curve and its Jacobian a $C_2 \times D_4$ Jacobian.

Theorem 4.1.3 (Theorem 4.4.11)**.** *Let $C/\mathcal{K}$ be a Richelot curve and let J denote its Jacobian. Suppose the following:*
i) C is a $C_2 \times D_4$ curve,
ii) the cluster picture of C at odd finite places is one of Table 3.1,
iii) for $v \mid 2$, $C \in \mathcal{C}$ as in Section 3.5,
iv) none of the $C_2 \times D_4$ invariants defined in Definition 4.3.9 for C are zero.
Then Conjecture 4.4.10 is true for C at all places v of $\mathcal{K}$. In particular, in this case

$$\prod_{v \in M_{\mathcal{K}}} \lambda_v(J) = \prod_{v \in M_{\mathcal{K}}} \omega_v(J),$$

hence the 2-parity conjecture holds.

Corollary 4.1.4. *Let C be hyperelliptic curve of genus 2 defined over $\mathbb{Q}$ given by*

$$C : y^2 = f(x) = (x^2 - 4a)g(x), \qquad a \in \mathbb{Z}, \quad g(x) \in \frac{\mathbb{Z}}{2}[x].$$

such that $Gal(g) \subseteq D_4$ and preserves the factorization $g(x) = (x^2 + t_2x + t_3)(x^2 + t_4x + t_5)$. Suppose that
i) for every odd prime p, the reduction of $f(x)$ mod p has no root of multiplicity ≥ 3,

[1] It is possible to centre the roots of a quadratic polynomial with a simple change of variable. Hence a Richelot polynomial can be made into a $C_2 \times D_4$ by insisting that $G_1(x) \in \mathcal{K}[x]$ and performing that change of variable.

ii) at $p = 2$, the two quadratics are irreducible over $\mathbb{Q}_2$ and $t_i \in \mathbb{Q}_2, i = 2,3,4,5$ with

$$t_2 \equiv 1 \ mod\ 2, \quad t_3 - \frac{1}{4} \equiv 0 \ mod\ 2, \quad t_4 = -2 \ mod\ 8, \quad t_5 \equiv 1 \ mod\ 8.$$

Then

$$(-1)^{rk_2(J)} = \omega(J).$$

Proof. C is a $C_2 \times D_4$ curve (see Definition 4.1.2), at odd places the cluster picture of C is one of Table 3.1, at $v \mid 2$, $C \in \mathcal{C}$ as in Proposition 3.5.34 and finally because $C \in \mathcal{C}$, none of the $C_2 \times D_4$ invariants are zero (see Section 6.5). □

In the first section, we define the dual curve of a $C_2 \times D_4$ curve and set up our notation. We then introduce the set of invariants which form the term of discrepancy $E_v(J)$ in the second section. For a lack of a more efficient method to prove that $\lambda_v = E_v\omega_v$ for all places v of $\mathcal{K}$, we proceed with a case by case analysis. We use the results of Sections 3.3, 3.4, Theorem 3.5.36 and Section 3.4 to compute and tabulate λ_v, ω_v for infinite places, odd finite places and places above 2 respectively. We also compute E_v at all places using the definitions of the invariants and properties of Hilbert Symbols. The proof of Theorem 3.2.6 is then immediate from these computations as it shows that $\lambda_v = E_v\omega_v$ in all cases. The number of cases is however quite significant and moreover, the definitions of λ_v and E_v in terms of invariants make their computations heavy on notation and rather tedious. Consequently, we chose to place them in separate subsequent chapters for clarity.

List of notation for this chapter

$G_1(x)$	$c(x-\alpha_1)(x+\alpha_1)$ with $\alpha_1^2 \in K$
$G_i(x), i=2,3$	$(x-\alpha_i)(x-\beta_i)$
$G(x)$	$G_1(x)G_2(x)G_3(x)$
$\mathcal{L}$	Splitting field of $G(x)$
c	leading term of $G(x)$
C	$C_2 \times D_4$ curve defined over K with the given factorization $y^2 = G(x) = G_1(x)G_2(x)G_3(x)$
J	$C_2 \times D_4$ Jacobian of C
ϕ	Richelot isogeny on J given by the factorization of $G(x)$
$L_1(x)$	$L_1(x) = \frac{1}{\Delta_G}[G_2(x), G_3(x)] = \ell_1(x-A_1)(x-B_1)$
$\ell_1 = \frac{u_1}{\Delta_G}$	leading term of $L_1(x)$ with $u_1 = \alpha_2 + \beta_2 - \alpha_3 - \beta_3$
$L_i(x), i=2,3$	$L_i(x) = [G_{i+1}(x), G_{i+2}(x)] = \ell_i(x-A_i)(x-B_i)$
$L(x)$	$L_1(x)L_2(x)L_3(x)$
$\ell = \ell_1\ell_2\ell_3$	leading term of $L(x)$
$\hat{C}$	dual curve of C defined over K with the given factorization $y^2 = L(x) = L_1(x)L_2(x)L_3(x)$
$\hat{J}$	$C_2 \times D_4$ Jacobian of $\hat{C}$
δ_i, $i=1,2,3$	discriminant of $G_i(x)$
$\hat{\delta}_i$, $i=1,2,3$	discriminant of $L_i(x)$
α_i, β_i, $i=1,2,3$	roots of $G_i(x)$
A_i, B_i, $i=1,2,3$	roots of $L_i(x)$
P_i, Q_i, $i=1,2,3$	Weierstrass points $P_i = (\alpha_i, 0), Q_i = (\beta_i, 0)$
D_i, $i=1,2,3$	Element of $ker(\phi)$, $D_i = [P_i, Q_i]$
$n_v, \hat{n}_v$,	number of connected components of $J(\mathcal{K}_v)$ and $\hat{J}(\mathcal{K}_v)$ when $v \mid \infty$
m_v	$m_v = 2$ if C is deficient at v, $m_v = 1$ otherwise
$\hat{m}_v$	$\hat{m}_v = 2$ if $\hat{C}$ is deficient at v, $\hat{m}_v = 1$ otherwise
$\phi_{\mathbb{R}}$	map induced by ϕ on $J(\mathbb{R})$
φ	restriction of $\phi_{\mathbb{R}}$ to the identity component of $J(\mathbb{R})$
$c_v, \hat{c}_v$	Tamagawa numbers of J and $\hat{J}$ at v when $v \nmid \infty$
$(.,.)_v$	Hilbert Symbol at v
● ●	α_1, β_1 in the cluster picture of C (A_1, B_1 is that of $\hat{C}$)
◆ ◆	α_2, β_2 in the cluster picture of C (A_2, B_2 is that of $\hat{C}$)
★ ★	α_3, β_3 in the cluster picture of C (A_3, B_3 is that of $\hat{C}$)

4.2 $C_2 \times D_4$ Richelot curves and Jacobians

Via the Richelot construction described in Section 2.2, we form the $C_2 \times D_4$ dual curve $\hat{C}$ of a $C_2 \times D_4$ curve.

Definition 4.2.5. [$C_2 \times D_4$ dual curve] The $C_2 \times D_4$ dual curve of a $C_2 \times D_4$ curve is given by

$$\hat{C} \;:\; y^2 = L(x) = L_1(x)L_2(x)L_3(x),$$

where

$$L_1(x) = \frac{1}{\Delta_G}[G_2(x), G_3(x)], \quad L_2(x) = [G_3(x), G_1(x)], \quad L_3(x) = [G_1(x), G_2(x)].$$

Remark 4.2.6. The definition of the $L_i(x)$s above slightly differs from that of the $F_i(x)$s in Definition 2.2.11. We chose the above construction for $C_2 \times D_4$ curves since it eases the computations performed in the remaining of this thesis. This is without loss of generality, since as noted in Remark 2.2.13, both definitions yields isomorphic curves over $\mathcal{K}$. However, the set $\{L_1(x), L_2(x), L_3(x)\}$ is no longer Galois stable. Indeed if for $\sigma \in Gal(\overline{\mathcal{K}}/\mathcal{K})$, $\sigma(G_2(x)) = G_3(x)$, then $\sigma(L_2(x)) = -L_3(x)$. This could be fixed by letting

$$L_1(x) = -\frac{1}{\Delta_G}[G_2(x), G_3(x)], \quad L_2(x) = [G_1(x), G_3(x)], \quad L_3(x) = [G_1(x), G_2(x)].$$

We chose not to do so for computational reasons.

Remark 4.2.7. The choice of $G_1(x)$ to be defined over $\mathcal{K}$ is of course arbitrary. However, it follows that for the remaining of this work, the roots of $G_1(x)$ will be considered as "special" (and so will be their corresponding red circle symbols in the cluster pictures of C). Indeed, we broke the symmetry of the quadratic factorization of $G(x)$ by centering the roots of $G_1(x)$ around 0. The action of $Gal(\mathcal{L}/\mathcal{K})$ on them will be limited compared to that on the roots of $G_2(x)$ and $G_3(x)$ (represented by blue diamond and purple star symbols).

4.3 $C_2 \times D_4$ invariant polynomials

Definition 4.3.8. Let $G(x)$ be a $C_2 \times D_4$ polynomial over $\mathcal{K}$ and $\mathcal{L}/\mathcal{K}$ its splitting field. In addition to the leading terms of $G(x)$ and $L(x)$, c and $\ell = \ell_1\ell_2\ell_3$, we associate the following set of $Gal(\mathcal{L}/\mathcal{K})$-invariant polynomials in the roots of $G(x)$:

$I_{20} = \frac{1}{2^3}\left(\delta_2 + \delta_3\right),$

$I_{21} = (\alpha_2 + \beta_2)(\alpha_3 + \beta_3),$

$I_{22} = \frac{1}{2}(\Delta_G^2 \ell_1^2 - \delta_2 - \delta_3) = (\alpha_2 - \alpha_3)(\beta_2 - \beta_3) + (\beta_2 - \alpha_3)(\alpha_2 - \beta_3),$

$I_{23} = 4c^2\alpha_1^2,$

$I_{40} = \frac{1}{2^6}(\delta_2 - \delta_3)^2,$
$I_{41} = 16\left(\alpha_2\beta_2\alpha_3\beta_3 + \alpha_1^2\left(\alpha_1^2 + \alpha_2\beta_2 + \alpha_3\beta_3 + (\alpha_2+\beta_2)(\beta_3+\alpha_3)\right)\right),$
$I_{42} = 4(2\alpha_1^2 - \alpha_2^2 - \beta_2^2)(2\alpha_1^2 - \alpha_3^2 - \beta_3^2),$
$I_{43} = \delta_2\left(\alpha_2^2 + \beta_2^2 - 2\alpha_1^2\right) + \delta_3\left(\alpha_3^2 + \beta_3^2 - 2\alpha_1^2\right)\right),$
$I_{44} = \delta_2\delta_3 = (\alpha_2 - \beta_2)^2(\alpha_3 - \beta_3)^2,$
$I_{45} = 4(\beta_3 - \beta_2)(\alpha_3 - \beta_2)(\alpha_2 - \beta_3)(\alpha_2 - \alpha_3),$
$I_{60} = 4\hat{\delta_3}\left(\alpha_2^2 + \beta_2^2 - 2\alpha_1^2\right) + 4\hat{\delta_2}\left(\alpha_3^2 + \beta_3^2 - 2\alpha_1^2\right)\right),$
$I_{80} = \frac{1}{c^4}\hat{\delta_2}\hat{\delta_3},$
Each invariant is of the form $I_{i,j}$, where i denotes the degree of $I_{i,j}$ in the roots of $G(x)$ and j indicates the number of this invariant of degree i.

Definition 4.3.9 (Local discrepancy)**.** Let $(.,.)_v$ denote the Hilbert Symbol at a place v of $\mathcal{K}$. For each place $v \in M_{\mathcal{K}}$, let

$$H_1 = (-1, I_{22}I_{41}I_{43}I_{60})_v, \quad H_2 = (I_{20}, -I_{40}I_{44})_v, \quad H_3 = (I_{40}, \ell I_{60}I_{43})_v,$$

$$H_4 = (c, I_{23}I_{44}I_{80})_v, \quad H_5 = (I_{23}, I_{41})_v, \quad H_6 = (I_{45}, -\ell I_{22}I_{21})_v,$$

$$H_7 = (I_{44}, 2I_{22}I_{42}I_{43})_v, \quad H_8 = (I_{80}, -2I_{41}I_{42}I_{60})_v, \quad H_9 = (I_{42}, -I_{60}I_{43})_v,$$

and define

$$E_v = \prod_{i=1}^{9} H_i.$$

4.4 A conjecture on local discrepancy for $C_2 \times D_4$ Jacobians

Conjecture 4.4.10 (Local discrepancy conjecture)**.** *Let $C/\mathcal{K}$ be a $C_2 \times D_4$ curve and J its Jacobian. Then at all places v of $\mathcal{K}$*

$$\lambda_v(J) = E_v(J) \cdot \omega_v(J),$$

where λ_v is given in Corollary 3.2.17, E_v in Definition 4.3.9, and $\omega_v(J)$ denotes the local root number of J, whenever the invariants involved in E_v are non-zero.

Theorem 4.4.11. *Let $C/\mathcal{K}$ be a Richelot curve and let J denote its Jacobian. Suppose the following:*
i) C is a $C_2 \times D_4$ curve,
ii) the cluster picture of C at odd finite places is one of Table 3.1,
iii) for $v \mid 2$, $C \in \mathcal{C}$ as in Section 3.5,
iv) none of the $C_2 \times D_4$ invariants for C are zero (see Definitions 4.3.8 and 4.3.9).
Then the local discrepancy conjecture 4.4.10 is true for J.

Corollary 4.4.12. *The 2-parity conjecture holds for Jacobians satisfying the conditions of Theorem 4.4.11.*

Proof. This is immediate from Theorems 3.2.16 and 4.4.11 since $\prod_{v\in M_{\mathcal{K}}} E_v = 1$ by the product formula for Hilbert Symbols. □

The proof of Theorem 4.4.11 consists of a case by case analysis. Starting with infinite places of K, we consider all possible configurations of the real roots of $G(x)$ as in Section 3.3 and compute λ_v, E_v and ω_v. Then considering finite places v of $\mathcal{K}$ such that $v \mid 2$, we prove that $\lambda_v = E_v\omega_v$ for $C \in \mathcal{C}$. Finally, for odd finite places $v \in M_{\mathcal{K}}$, we consider all reduction types for C at v using cluster pictures as in Section 3.4 with all possible Richelot isogenies (equivalently, $C_2 \times D_4$ quadratic factorization for $G(x)$) associated to this reduction type. For each case, we compute λ_v, E_v and ω_v. The results are presented in the next sections via a set of tables and one can readily see that in all cases $\lambda_v = E_v\omega_v$ as required. Chapters 5 and 6 consist of the proofs of the computations presented in the tables.

4.5 $C_2 \times D_4$ curves at infinite places

Lemma 4.5.13. *Let C be a Richelot curve and denote by J its Jacobian. Then $\omega_v = 1$ for places v of $\mathcal{K}$ with $\mathcal{K}_v \simeq \mathbb{R}$ or $\mathcal{K} \simeq \mathbb{C}$.*

Proof. This follows from Lemma 2.1 in [33] since C is of genus 2 so that J is of dimension 2. □

Theorem 4.5.14. *Let C be a $C_2 \times D_4$ curve and let J be its Jacobian. Then Conjecture 4.4.10 holds for complex places of K.*

Proof. At complex places, $\omega_v = 1$ by Lemma 4.5.13 and $E_v = 1$ trivially. The result follows since $\lambda_v = 1$ by Lemma 3.3.18. □

Lemma 4.5.15. *Let C be a $C_2 \times D_4$ curve given by $C : y^2 = G(x)$ and let J be its Jacobian. Then, for real places v of $\mathcal{K}$, λ_v, E_v and ω_v are invariant under the change of variable $x \mapsto -x$.*

Proof. Let $r_1, .., r_6$ be the roots of $G(x)$. Applying the above change of variable yields $r_i \mapsto -r_i, i = 1, 2, 3, 4, 5, 6$. It follows from Propositions 3.3.19, 3.3.23, 3.3.20 and 3.3.21 that λ_v is invariant under this change of variable. ω_v is trivially invariant from Lemma 4.5.13 and so is E_v since all $C_2 \times D_4$ invariants involved in its definition are of even degrees. □

Notation

In this section, we fix a real place $v \in M_{\mathcal{K}}$. For a given $C_2 \times D_4$ curve C, we wish to compute $\lambda_{\mathbb{R}}, E_{\mathbb{R}}$. In particular, we need to compute the number of real connected components of J and $\hat{J}$, the real deficiency of C and $\hat{C}$ as well as $|ker(\varphi)|$. From Section 3.3, we know that this is possible once we know the real/complex configuration of the roots of $G(x)$. We therefore consider all possible configurations and compute $\lambda_{\mathbb{R}}, E_{\mathbb{R}}$ in each case. The results are tabulated below. Except for $E_{\mathbb{R}}$ whose proof is presented in Section 6.4, the computations are clear from the results of Section 3.3.

Table convention

In a given table, each row corresponds to a particular configuration of the real/complex roots of $G(x)$. The first column names the case considered, the second column gives the configuration of the roots. We used the symmetry between the roots of $G_2(x)$ and $G_3(x)$ (blue diamonds and purple stars in the real picture), and without loss of generality, always placed the roots of $G_2(x)$ on the left of that of $G_3(x)$ in the real pictures. The third, fourth, fifth and sixth columns specify the signs of the leading terms of $G(x)$ and $L(x)$ as well as the signs of $I_{23} = \delta_1$, $I_{44} = \delta_2\delta_3$, $I_{45} = \Delta_G^2\hat{\delta_1}$ and $I_{80} = \hat{\delta_2}\hat{\delta_3}$. The vertical double lines represent the end of the input data. We note that the double lines do not occur at the same place in all tables. This is because in some cases, the leading term of $L(x)$ and I_{45} are part of the input data, while in some other cases, they are fixed by the real configuration of the roots of $G(x)$. The next columns after the vertical double lines give the number of real connected components of J and $\hat{J}$, the order of $ker(\varphi)$ and the real deficiency of C and $\hat{C}$. Finally the three last columns list $\lambda_{\mathbb{R}}, E_{\mathbb{R}}$ and $\omega_{\mathbb{R}}$.

Naming convention

Tables are indexed by real/complex roots configurations. The names start with the number of complex roots: 6C, 4C, 2C, 6R for 6, 4, 2, complex roots and 6 real roots respectively. The number indicates different configurations within the same case, the capital letters A, B vary with the sign of the leading term c of $G(x)$ and the small letter vary with the sign of other invariants.

Theorem 4.5.16. *Let C be a $C_2 \times D_4$ curve. Then Conjecture 4.4.10 holds for real places of K whenever the $C_2 \times D_4$ invariants involved in E_v are non-zero.*

Proof. Follows from Tables 4.1, 4.2 and 4.3 since in all cases we find $\lambda_v = E_v\omega_v$. The exhaustivity of the cases addressed in these tables follows from Lemma 4.5.15. □

Table 4.1: $G(x)$ has 6,4,2 complex roots.

Isogeny	$C(\mathbb{R})$	c	ℓ	I_{23}	I_{44}	I_{45}	I_{80}	n	$\hat{n}$	$\lvert ker(\varphi)\rvert$	$m_{\mathbb{R}}$	$\hat{m}_{\mathbb{R}}$	$\lambda_{\mathbb{R}}$	$\omega_{\mathbb{R}}$	$E_{\mathbb{R}}$
6C1A	$\delta_1, \delta_2, \delta_3 \in \mathbb{R}_{<0}$	$+$	$\pm$	$-$	$+$	$+$	$+$	1	4	4	1	1	1	1	1
6C1B	$\delta_1, \delta_2, \delta_3 \in \mathbb{R}_{<0}$	$-$	$\pm$	$-$	$+$	$+$	$+$	1	4	4	2	1	-1	1	-1
6C2Aa	$\delta_1 \in \mathbb{R}_{<0}, \quad \delta_2 = \overline{\delta_3}$	$+$	$\pm$	$-$	$+$	$-$	$+$	1	1	2	1	1	-1	1	-1
6C2Ab	$\delta_1 \in \mathbb{R}_{<0}, \quad \delta_2 = \overline{\delta_3}$	$+$	$+$	$-$	$+$	$+$	$+$	1	1	2	1	1	-1	1	-1
6C2Ac	$\delta_1 \in \mathbb{R}_{<0}, \quad \delta_2 = \overline{\delta_3}$	$+$	$-$	$-$	$+$	$+$	$+$	1	1	2	1	2	1	1	1
6C2Ba	$\delta_1 \in \mathbb{R}_{<0}, \quad \delta_2 = \overline{\delta_3}$	$-$	$\pm$	$-$	$+$	$-$	$+$	1	1	2	2	1	1	1	1
6C2Bb	$\delta_1 \in \mathbb{R}_{<0}, \quad \delta_2 = \overline{\delta_3}$	$-$	$+$	$-$	$+$	$+$	$+$	1	1	2	2	1	1	1	1
6C2Bc	$\delta_1 \in \mathbb{R}_{<0}, \quad \delta_2 = \overline{\delta_3}$	$-$	$-$	$-$	$+$	$+$	$+$	1	1	2	2	2	-1	1	-1
4C1A	$\delta_2, \delta_3 \in \mathbb{R}_{<0}$	$+$	$\pm$	$+$	$+$	$+$	$+$	1	4	4	1	1	1	1	1
4C1B	$\delta_2, \delta_3 \in \mathbb{R}_{<0}$	$-$	$\pm$	$+$	$+$	$+$	$+$	1	4	4	1	1	1	1	1
4C2A	$\delta_1, \delta_3 \in \mathbb{R}_{<0}$	$+$	$\pm$	$-$	$-$	$+$	$+$	1	4	4	1	1	1	1	1
4C2B	$\delta_1, \delta_3 \in \mathbb{R}_{<0}$	$-$	$\pm$	$-$	$-$	$+$	$+$	1	4	4	1	1	1	1	1
4C3a	$\delta_2 = \overline{\delta_3}$	$\pm$	$\pm$	$+$	$+$	$-$	$+$	1	1	2	1	1	-1	1	-1
4C3b	$\delta_2 = \overline{\delta_3}$	$\pm$	$+$	$+$	$+$	$+$	$+$	1	1	2	1	1	-1	1	-1
4C3c	$\delta_2 = \overline{\delta_3}$	$\pm$	$-$	$+$	$+$	$+$	$+$	1	1	2	1	2	1	1	1
2C1A	$\delta_1 < 0$	$+$	$\pm$	$-$	$+$	$+$	$+$	2	4	2	1	1	1	1	1
2C1B	$\delta_1 < 0$	$-$	$\pm$	$-$	$+$	$+$	$+$	2	4	4	1	1	-1	1	-1
2C2A	$\delta_1 < 0$	$+$	$\pm$	$-$	$+$	$-$	$+$	2	2	2	1	1	-1	1	-1
2C2B	$\delta_1 < 0$	$-$	$\pm$	$-$	$+$	$-$	$+$	2	2	2	1	1	-1	1	-1
2C3A	$\delta_1 < 0$	$+$	$\pm$	$-$	$+$	$+$	$+$	2	4	4	1	1	-1	1	-1
2C3B	$\delta_1 < 0$	$-$	$\pm$	$-$	$+$	$+$	$+$	2	4	2	1	1	1	1	1
2C4A	$\delta_3 < 0$	$+$	$\pm$	$+$	$-$	$+$	$+$	2	4	2	1	1	1	1	1
2C4B	$\delta_3 < 0$	$-$	$\pm$	$+$	$-$	$+$	$+$	2	4	4	1	1	-1	1	-1
2C5A	$\delta_3 < 0$	$+$	$\pm$	$+$	$-$	$+$	$-$	2	2	2	1	1	-1	1	-1
2C5B	$\delta_3 < 0$	$-$	$\pm$	$+$	$-$	$+$	$-$	2	2	2	1	1	-1	1	-1
2C6aA	$\delta_3 < 0$	$+$	$\pm$	$+$	$-$	$+$	$+$	2	4	4	1	1	-1	1	-1
2C6aB	$\delta_3 < 0$	$-$	$\pm$	$+$	$-$	$+$	$+$	2	4	2	1	1	1	1	1
2C6bA	$\delta_3 < 0$	$+$	$\pm$	$+$	$-$	$+$	$+$	2	4	4	1	1	-1	1	-1
2C6bB	$\delta_3 < 0$	$-$	$\pm$	$+$	$-$	$+$	$+$	2	4	2	1	1	1	1	1

Table 4.2: $G(x)$ has 6 real roots

Isogeny	$C(\mathbb{R})$	c	ℓ	I_{23}	I_{44}	I_{45}	I_{80}	n	$\hat{n}$	$\lvert ker(\varphi)\rvert$	$m_{\mathbb{R}}$	$\hat{m}_{\mathbb{R}}$	$\lambda_{\mathbb{R}}$	$\omega_{\mathbb{R}}$	$E_{\mathbb{R}}$
6R1A		+	±	+	+	+	+	4	4	1	1	1	1	1	1
6R1B		−	±	+	+	+	+	4	4	4	1	1	1	1	1
6R2A		+	±	+	+	−	+	4	2	1	1	1	-1	1	-1
6R2B		−	±	+	+	−	+	4	2	2	1	1	1	1	1
6R3A		+	±	+	+	+	+	4	4	2	1	1	-1	1	-1
6R3B		−	±	+	+	+	+	4	4	2	1	1	-1	1	-1
6R4A		+	±	+	+	+	−	4	2	1	1	1	-1	1	-1
6R4B		−	±	+	+	+	−	4	2	2	1	1	1	1	1
6R5A		+	±	+	+	−	−	4	1	1	1	1	1	1	1
6R5B		−	±	+	+	−	−	4	1	1	1	1	1	1	1
6R6A		+	±	+	+	+	−	4	2	2	1	1	1	1	1
6R6B		−	±	+	+	+	−	4	2	1	1	1	-1	1	-1
6R7A		+	±	+	+	+	+	4	4	2	1	1	-1	1	-1
6R7B		−	±	+	+	+	+	4	4	2	1	1	-1	1	-1
6R8aA		+	+	+	+	−	−	4	1	1	1	1	1	1	1
6R8aB		−	+	+	+	−	−	4	1	1	1	1	1	1	1
6R8bA		+	−	+	+	−	−	4	1	1	1	2	-1	1	-1
6R8bB		−	−	+	+	−	−	4	1	1	1	2	-1	1	-1
6R9A		+	±	+	+	+	+	4	1	1	1	1	1	1	1
6R9B		−	±	+	+	+	+	4	1	1	1	1	1	1	1
6R10A		+	±	+	+	+	−	4	2	2	1	1	1	1	1
6R10B		−	±	+	+	+	−	4	2	1	1	1	-1	1	-1
6R11A		+	±	+	+	−	−	4	1	1	1	1	1	1	1
6R11B		−	±	+	+	−	−	4	1	1	1	1	1	1	1
6R12A		+	±	+	+	+	−	4	2	1	1	1	-1	1	-1
6R12B		−	±	+	+	+	−	4	2	2	1	1	1	1	1
6R13A		+	±	+	+	+	+	4	4	4	1	1	1	1	1
6R13B		−	±	+	+	+	+	4	4	1	1	1	1	1	1
6R14A		+	±	+	+	−	+	4	2	2	1	1	1	1	1

Table 4.3: $G(x)$ has 6 real roots

Isogeny	$C(\mathbb{R})$	c	ℓ	I_{23}	I_{44}	I_{45}	I_{80}	n	$\hat{n}$	$\lvert ker(\varphi)\rvert$	$m_{\mathbb{R}}$	$\hat{m}_{\mathbb{R}}$	$\lambda_{\mathbb{R}}$	$\omega_{\mathbb{R}}$	$E_{\mathbb{R}}$
6R14B		−	±	+	+	−	+	4	2	1	1	1	-1	1	-1
6R15A		+	±	+	+	+	−	4	2	1	1	1	-1	1	-1
6R15B		−	±	+	+	+	−	4	2	2	1	1	1	1	1
6R16A		+	±	+	+	+	+	4	4	2	1	1	-1	1	-1
6R16B		−	±	+	+	+	+	4	4	2	1	1	-1	1	-1
6R17A		+	±	+	+	−	+	4	2	2	1	1	1	1	1
6R17B		−	±	+	+	−	+	4	2	1	1	1	-1	1	-1
6R18A		+	±	+	+	+	+	4	4	4	1	1	1	1	1
6R18B		−	±	+	+	+	+	4	4	1	1	1	1	1	1
6R19A		+	±	+	+	+	−	4	2	1	1	1	-1	1	-1
6R19B		−	±	+	+	+	−	4	2	2	1	1	1	1	1
6R20A		+	±	+	+	−	−	4	1	1	1	1	1	1	1
6R20B		−	±	+	+	−	−	4	1	1	1	1	1	1	1
6R21A		+	±	+	+	+	−	4	2	2	1	1	1	1	1
6R21B		−	±	+	+	+	−	4	2	1	1	1	-1	1	-1
6R22A		+	±	+	+	+	+	4	1	1	1	1	1	1	1
6R22B		−	±	+	+	+	+	4	1	1	1	1	1	1	1
6R23aA		+	+	+	+	−	+	4	1	1	1	1	1	1	1
6R23aB		−	+	+	+	−	+	4	1	1	1	1	1	1	1
6R23bA		+	−	+	+	−	+	4	1	1	1	2	-1	1	-1
6R23bB		−	−	+	+	−	+	4	1	1	1	2	-1	1	-1
6R24A		+	±	+	+	+	+	4	4	2	1	1	-1	1	-1
6R24B		−	±	+	+	+	+	4	4	2	1	1	-1	1	-1
6R25A		+	±	+	+	+	+	4	4	1	1	1	1	1	1
6R25B		−	±	+	+	+	+	4	4	4	1	1	1	1	1
6R26A		+	±	+	+	−	+	4	2	1	1	1	-1	1	-1
6R26B		−	±	+	+	−	+	4	2	2	1	1	1	1	1
6R27A		+	±	+	+	+	+	4	4	2	1	1	-1	1	-1
6R27B		−	±	+	+	+	+	4	4	2	1	1	-1	1	-1

4.6 $C_2 \times D_4$ curves at finite places $v \mid 2$

In order to use the results of Section 3.5, we impose the condition that our $C_2 \times D_4$ curves belong to the family $\mathcal{C}$ given in Proposition 3.5.34 at 2-adic places. Theorem 3.5.36 yields $\lambda_v = 1$ in this case. Moreover, E_v and ω_v are as follows.

Lemma 4.6.17. *$E_v = 1$ for all curves $C \in \mathcal{C}$.*

Proof. See Section 6.5. □

Lemma 4.6.18. *$\omega_v = 1$ for all curves $C \in \mathcal{C}$.*

Proof. By Corollary 3.5.33, J has split totally toric reduction. The result follows from [10][Proposition 3.23]. □

Theorem 4.6.19. *Conjecture 4.4.10 holds for places $v \mid 2$ such that $C \in \mathcal{C}$.*

Proof. Clear from Theorem 3.5.36 and Lemmata 4.6.17, 4.6.18. □

4.7 $C_2 \times D_4$ curves at finite places $v \nmid 2$

Notation

In this section, we fix a finite place $v \in M_{\mathcal{K}}$ such that $v \nmid 2$ and denote by K the completion $\mathcal{K}_v$ of $\mathcal{K}$ at v. We let $\mathcal{O}_K$ be the ring of integers of K, choose a uniformizer π and denote by v the corresponding normalized valuation. Let Gal_K denote the Galois group of $\overline{K}$ over K, $Frob$ the Frobenius automorphism and I_K the inertia subgroup.

For a given $C_2 \times D_4$ curve C, we wish to compute λ_v, E_v and ω_v. By Theorem 3.2.16, this means computing Tamagawa numbers at v for J and $\hat{J}$, deficiency at v for C and $\hat{C}$, E_v and ω_v. As proven in Chapters 5 and 6, this can be done once we know the cluster picture of C at v and its $C_2 \times D_4$ factorization together with some specific local data. We therefore consider all the possible semistable reduction types of C over the local field K together with each possible $C_2 \times D_4$ factorization and compute λ_v, E_v and ω_v in each case. The results are tabulated below.

Table convention

Each table corresponds to a specific reduction type with a specific $C_2 \times D_4$ factorization. In a given table, each row corresponds to a particular Galois action and values of extra invariants. The first column names the case considered, the second column gives the corresponding cluster picture for C (using without loss of generality the symmetry between the roots of $G_2(x)$ and $G_3(x)$, blue diamonds and purple stars), the Tamagawa number of J and the deficiency for C. The third, fourth and fifth columns specify the valuations of the leading terms of $G(x)$ and $L(x)$ as well as the valuation of Δ_G. The last column before the vertical double lines gives conditions on invariants of C that determine the Galois action on the cluster pictures of C and $\hat{C}$. We use the dash symbol "-" to mean that the value of a given invariant is not determined by the case considered. The vertical double lines represent the application of the Richelot isogeny. The next column after the vertical double line therefore gives the balanced representative of the cluster picture of $\hat{C}$, the Tamagawa number of $\hat{J}$ and the deficiency of $\hat{C}$. Finally the three last columns compute λ_v, E_v and ω_v.

Tables are indexed by reduction types for C, where for a given type, each $C_2 \times D_4$ factorization for $G(x)$ is given a particular number, followed by a letter which indicates a specific Galois action. Type 2 cases are called GR for Good Reduction, type 1_{2a} cases are called ON for One Node, type $I_{2a,2b}$ cases TN for Two Nodes, type $U_{2a,2b,2n}$ are called U for Ubereven and types 1×1 are called TC for Two Cusps. The following example illustrates this construction.

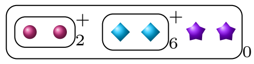
Example 4.7.20.
Let C/K be a $C_2 \times D_4$ curve with factorization $G(x) = G_1(x)G_2(x)G_3(x)$ such that $G(x)$ has 2 double roots $\mod \pi$. Following notation in Section 3.4, possible types for C are A) $I^{+,+}_{2a,2b}$, B) $I^{+,-}_{2a,2b}$, C) $I^{-,+}_{2a,2b}$, D) $I^{-,-}_{2a,2b}$, E) $I^{+}_{2a\tilde{}2a}$ and F) $I^{-}_{2a\tilde{}2a}$ for some $a, b \in \frac{1}{2}\mathbb{Z}$ (all curves considered are assumed to be semistable; $2a, 2b \in \mathbb{Z}$ follows directly from the semistability criterion 3.4.29). First note that Tamagawa numbers for J, $\hat{J}$ and deficiency for C and $\hat{C}$ depend on the exact type of C. We therefore have to consider all types and give the conditions on C to differentiate them. In types A, B, C, D, Gal_K acts trivially on clusters, while in types E and F, $Frob$ permutes both clusters (see Table 3.1). To distinguish these types one needs to know the field of definition of the roots of $G(x)$. The signs in the types are given by the field of definition of the slopes of the tangents at the nodes on the reduced curve, which can be computed using Proposition 3.4.30.

Now the double roots of $G(x)$ mod π can be distributed among the 6 roots of its 3 quadratic factors. With the restriction that $G_1(x) \in K[x]$ by construction, this yields several cases to consider, each of them giving a different reduction type (cluster picture) for $\hat{C}$ and therefore different values for λ_v, E_v and ω_v. All possibilities in that specific case are listed, defined and computed in Tables 4.9 to 4.22.

As an example, consider the curve $C/\mathbb{Q}_{17} : y^2 = G_1(x)G_2(x)G_3(x)$, with

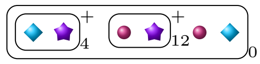
$$G_1(x) = (x - 17^2)(x + 17^2), G_2(x) = (x - 3 + 17^6)(x - 3 - 17^6), G_3(x) = (x - 1)(x - 4).$$

Here, $G(x)$ has two double roots when reduced modulo 17. These concern the roots $\alpha_1, -\alpha_1$ of $G_1(x)$, say $\mathfrak{t}_1$ is the twin cluster around the two red circles, and the roots α_2, β_2 of $G_2(x)$, say $\mathfrak{t}_2$ is the twin cluster around the blue diamonds. By Proposition

3.4.30 the sign of $\mathfrak{t}_1$ is $+$ since $T_{\alpha_1} \equiv 15 \bmod 17 \in \mathbb{F}_{17}^{\times 2}$, and the sign of $\mathfrak{t}_2$ is $+$ since $T_{\alpha_2} \equiv 16 \bmod 17 \in \mathbb{F}_{17}^{\times 2}$. The cluster picture of C is  , so that C is of type $I_{4,12}^{+,+}$ (case TN1A in Table 4.9 below). It follows from the Richelot construction that

$$L_1(x) = \frac{1}{2913111186232305}x^2 + 1165244474459512x - 2913111186148784,$$

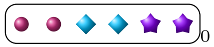
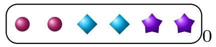
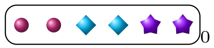
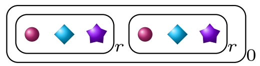
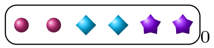
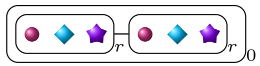
$$L_2(x) = 5x^2 - 167050x + 417605, \quad L_3(x) = -6x^2 - 1165244474292462x - 501126.$$

Computing the roots of $L(x)$, we find that the cluster picture of $\hat{C}$ is  , so that $\hat{C}$ is of type $I_{8,24}^{+,+}$. Using Table 3.1 it is now possible to compute λ_v and ω_v. Moreover, from the cluster picture of C and the local data associated to this case, i.e. $v(\Delta_G) = 0, T_{\alpha_1}, T_{\alpha_2} \in K^{\times 2}$, one can compute E_v (see Case TN1A in Chapter 6 for the computation).

Theorem 4.7.21. *Let C be a $C_2 \times D_4$ curve and v a place of $\mathcal{K}$ of odd residue characteristic. If the cluster picture of C is one of Table 3.1, then Conjecture 4.4.10 holds for C at v whenever the $C_2 \times D_4$ invariants involved in E_v are non-zero.*

Proof. Follows since Tables 4.4 to 4.33 list all possible cases for $C_2 \times D_4$ curves with the required cluster pictures and since in all cases we find $\lambda_v = E_v\omega_v$. $\square$

Table 4.4: C is of type 2, GR1

Isogeny	C	$v(c)$	$\begin{cases} v(\ell) \\ v(\ell_1) \end{cases}$	$v(\Delta_G^2)$	$(A_1 - B_1)^2$	$\hat{C}$	λ_v	w_v	E_v
GR1A	 $c_v = 1, \quad m_v = 1$	0	$\begin{cases} \equiv 0\ (2) \\ - \end{cases}$	0	–	 $\hat{c}_v = 1, \quad \hat{m}_v = 1$	1	1	1
GR1B	 $c_v = 1, \quad m_v = 1$	0	$\begin{cases} \equiv r\ (2) \\ \equiv r\ (2) \end{cases}$	$2r$	$\in K^{\times 2}$	 $\hat{c}_v = 1, \quad \hat{m}_v = 1$	1	1	1
GR1C	 $c_v = 1, \quad m_v = 1$	0	$\begin{cases} \equiv r\ (2) \\ \equiv r\ (2) \end{cases}$	$2r$	$\notin K^{\times 2}$	 $\hat{c}_v = 1, \quad \hat{m}_v = \widetilde{r+1}$	(-1)^r	1	(-1)^r

Notation: $\tilde{x} = 2$ if $2|x$ and $\tilde{x} = 1$ if $2 \nmid x$.

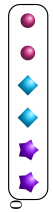
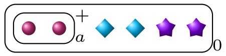
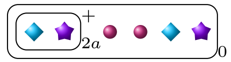
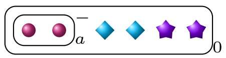
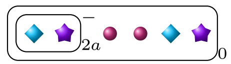
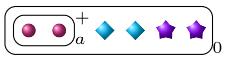
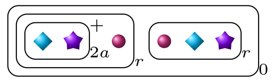
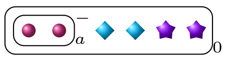
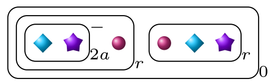
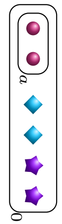


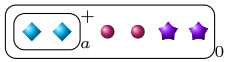
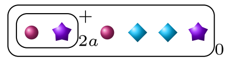
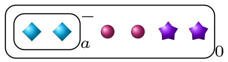
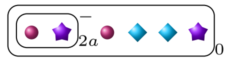
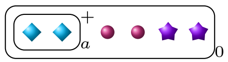
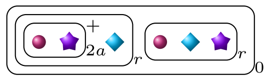
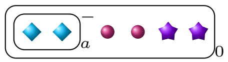
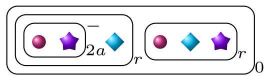
Table 4.5: C is of type 1_{2a}, ON1

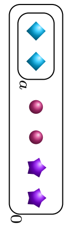
Isogeny	C	$v(c)$	$\begin{cases} v(\ell) \\ v(\ell_1) \end{cases}$	$v(\Delta_G^2)$	T_{α_1}	$\hat{C}$	λ_v	w_v	E_v
ON1A	 $c_v = 2a, \quad m_v = 1$	0	$\begin{cases} \equiv 0\ (2) \\ \equiv 0\ (2) \end{cases}$	0	$\in K^{\times 2}$	 $\hat{c}_v = 4a, \quad \hat{m}_v = 1$	-1	-1	1
ON1B	 $c_v = \widetilde{2a}, \quad m_v = 1$	0	$\begin{cases} \equiv 0\ (2) \\ \equiv 0\ (2) \end{cases}$	0	$\notin K^{\times 2}$	 $\hat{c}_v = 2, \quad \hat{m}_v = 1$	(-1)^{2a}	1	(-1)^{2a}
ON1C	 $c_v = 2a, \quad m_v = 1$	0	$\begin{cases} \equiv r\ (2) \\ \equiv r\ (2) \end{cases}$	$2r$	$\in K^{\times 2}$	 $\hat{c}_v = 4a, \quad \hat{m}_v = 1$	-1	-1	1
ON1D	 $c_v = \widetilde{2a}, \quad m_v = 1$	0	$\begin{cases} \equiv r\ (2) \\ \equiv r\ (2) \end{cases}$	$2r$	$\notin K^{\times 2}$	 $\hat{c}_v = 2, \quad \hat{m}_v = 1$	(-1)^{2a}	1	(-1)^{2a}



Notation: $T_{\alpha_1} = c(\alpha_1 - \alpha_2)(\alpha_1 - \beta_2)(\alpha_1 - \alpha_3)(\alpha_1 - \beta_3)$,
$\tilde{x} = 2$ if $2|x$ and $\tilde{x} = 1$ if $2 \nmid x$.

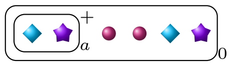
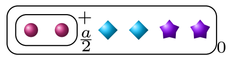
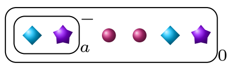
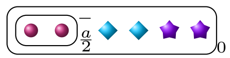
Table 4.6: C is of type 1_{2a}, ON2

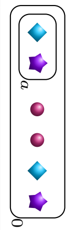
Isogeny	C	$v(c)$	$\begin{cases} v(\ell) \\ v(\ell_1) \end{cases}$	$v(\Delta_G^2)$	T_{α_2}	$\hat{C}$	λ_v	w_v	E_v
ON2A	 $c_v = 2a, \quad m_v = 1$	0	$\begin{cases} \equiv 0\ (2) \\ - \end{cases}$	0	$\in K^{\times 2}$	 $\hat{c}_v = 4a, \quad \hat{m}_v = 1$	-1	-1	1
ON2B	 $c_v = \widetilde{2a}, \quad m_v = 1$	0	$\begin{cases} \equiv 0\ (2) \\ - \end{cases}$	0	$\notin K^{\times 2}$	 $\hat{c}_v = 2, \quad \hat{m}_v = 1$	(-1)^{2a}	1	(-1)^{2a}
ON2C	 $c_v = 2a, \quad m_v = 1$	0	$\begin{cases} \equiv r\ (2) \\ - \end{cases}$	$2r$	$\in K^{\times 2}$	 $\hat{c}_v = 4a, \quad \hat{m}_v = 1$	-1	-1	1
ON2D	 $c_v = \widetilde{2a}, \quad m_v = 1$	0	$\begin{cases} \equiv r\ (2) \\ - \end{cases}$	$2r$	$\notin K^{\times 2}$	 $\hat{c}_v = 2, \quad \hat{m}_v = 1$	(-1)^{2a}	1	(-1)^{2a}



Notation: $T_{\alpha_2} = c(\alpha_2 - \alpha_1)(\alpha_2 + \alpha_1)(\alpha_2 - \alpha_3)(\alpha_2 - \beta_3)$,
$\tilde{x} = 2$ if $2|x$ and $\tilde{x} = 1$ if $2 \nmid x$.

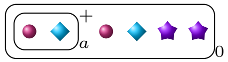
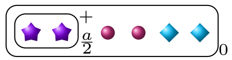
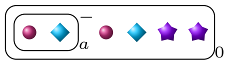
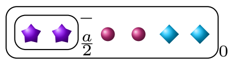
Table 4.7: C is of type 1_{2a}, ON3

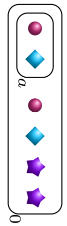
Isogeny	C	$v(c)$	$\begin{cases} v(\ell) \\ v(\ell_1) \end{cases}$	$v(\Delta_G^2)$	T_{α_2}	$\hat{C}$	λ_v	w_v	E_v
ON3A	 $c_v = 2a, \quad m_v = 1$	0	$\begin{cases} \equiv 0 \ (2) \\ - \end{cases}$	0	$\in K^{\times 2}$	 $\hat{c}_v = a, \quad \hat{m}_v = 1$	-1	-1	1
ON3B	 $c_v = 2, \quad m_v = 1$	0	$\begin{cases} \equiv 0 \ (2) \\ - \end{cases}$	0	$\notin K^{\times 2}$	 $\hat{c}_v = \widetilde{a}, \quad \hat{m}_v = 1$	(-1)^a	1	(-1)^a



Notation: $T_{\alpha_2} = c(\alpha_2 - \alpha_1)(\alpha_2 + \alpha_1)(\alpha_2 - \beta_2)(\alpha_2 - \beta_3)$,
$\tilde{x} = 2$ if $2|x$ and $\tilde{x} = 1$ if $2 \nmid x$.

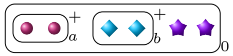
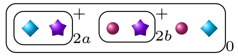
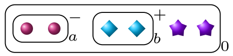
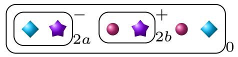
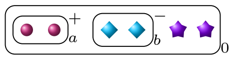
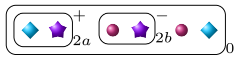
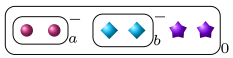
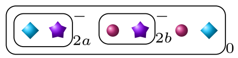
Table 4.8: C is of type 1_{2a}, ON4

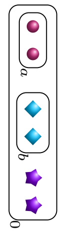
Isogeny	C	$v(c)$	$\begin{cases} v(\ell) \\ v(\ell_1) \end{cases}$	$v(\Delta_G^2)$	T_{α_1}	$\hat{C}$	λ_v	w_v	E_v
ON4A	 $c_v = 2a, \quad m_v = 1$	0	$\begin{cases} \equiv 0\ (2) \\ - \end{cases}$	0	$\in K^{\times 2}$	 $\hat{c}_v = a, \quad \hat{m}_v = 1$	-1	-1	1
ON4B	 $c_v = 2, \quad m_v = 1$	0	$\begin{cases} \equiv 0\ (2) \\ - \end{cases}$	0	$\notin K^{\times 2}$	 $\hat{c}_v = \widetilde{a}, \quad \hat{m}_v = 1$	(-1)^a	1	(-1)^a



Notation: $T_{\alpha_1} = 2c\alpha_1(\alpha_1 - \beta_2)(\alpha_1 - \alpha_3)(\alpha_1 - \beta_3)$,
$\tilde{x} = 2$ if $2|x$ and $\tilde{x} = 1$ if $2 \nmid x$.

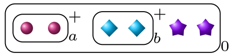
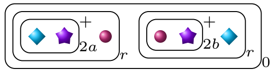
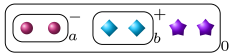
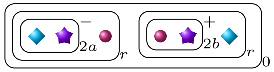
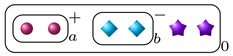
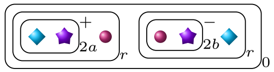
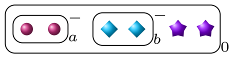
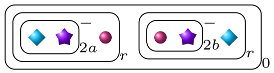
Table 4.9: C is of type $I_{2a,2b}$, TN1

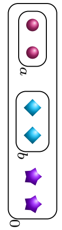
Isogeny	C	$v(c)$	$\begin{cases} v(\ell) \\ v(\ell_1) \end{cases}$	$v(\Delta_G^2)$	$T_{\alpha_1}, T_{\alpha_2}$	$\hat{C}$	λ_v	w_v	E_v
TN1A	 $c_v = 4ab, \quad m_v = 1$	0	$\begin{cases} \equiv 0\ (2) \\ - \end{cases}$	0	$\in K^{\times 2}, \in K^{\times 2}$	 $\hat{c}_v = 16ab, \quad \hat{m}_v = 1$	1	1	1
TN1B	 $c_v = \widetilde{2a}2b, \quad m_v = 1$	0	$\begin{cases} \equiv 0\ (2) \\ - \end{cases}$	0	$\notin K^{\times 2}, \in K^{\times 2}$	 $\hat{c}_v = 8b, \quad \hat{m}_v = 1$	(-1)$^{2a+1}$	-1	(-1)^{2a}
TN1C	 $c_v = 2a\widetilde{2b}, \quad m_v = 1$	0	$\begin{cases} \equiv 0\ (2) \\ - \end{cases}$	0	$\in K^{\times 2}, \notin K^{\times 2}$	 $\hat{c}_v = 8a, \quad \hat{m}_v = 1$	(-1)$^{2b+1}$	-1	(-1)^{2b}
TN1D	 $c_v = \widetilde{2a}\widetilde{2b}, \quad m_v = 1$	0	$\begin{cases} \equiv 0\ (2) \\ - \end{cases}$	0	$\notin K^{\times 2}, \notin K^{\times 2}$	 $\hat{c}_v = 4, \quad \hat{m}_v = 1$	(-1)$^{2a+2b}$	1	(-1)$^{2a+2b}$



Notation: $T_{\alpha_1} = c(\alpha_1 - \alpha_2)(\alpha_1 - \beta_2)(\alpha_1 - \alpha_3)(\alpha_1 - \beta_3)$, $T_{\alpha_2} = c(\alpha_2 - \alpha_1)(\alpha_2 + \alpha_1)(\alpha_2 - \alpha_3)(\alpha_2 - \beta_3)$, $\tilde{x} = 2$ if $2|x$ and $\tilde{x} = 1$ if $2 \nmid x$.

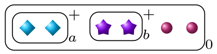
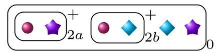
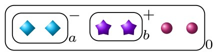
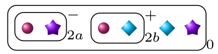
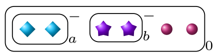
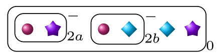
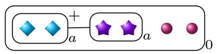
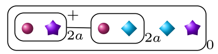
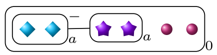
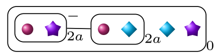
Table 4.10: C is of type $I_{2a,2b}$, TN1

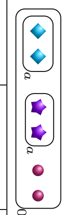
Isogeny	C	$v(c)$	$\begin{cases} v(\ell) \\ v(\ell_1) \end{cases}$	$v(\Delta_G^2)$	$T_{\alpha_1}, T_{\alpha_2}$	$\hat{C}$	λ_v	w_v	E_v
TN1E	 $c_v = 4ab, \quad m_v = 1$	0	$\begin{cases} \equiv r\ (2) \\ -r \end{cases}$	$2r > 0$	$\in K^{\times 2}, \in K^{\times 2}$	 $\hat{c}_v = 16ab, \quad \hat{m}_v = 1$	1	1	1
TN1F	 $c_v = \widetilde{2a}2b, \quad m_v = 1$	0	$\begin{cases} \equiv r\ (2) \\ -r \end{cases}$	$2r > 0$	$\notin K^{\times 2}, \in K^{\times 2}$	 $\hat{c}_v = 8b, \quad \hat{m}_v = 1$	(-1)$^{2a+1}$	-1	(-1)^{2a}
TN1G	 $c_v = 2a\widetilde{2b}, \quad m_v = 1$	0	$\begin{cases} \equiv r\ (2) \\ -r \end{cases}$	$2r > 0$	$\in K^{\times 2}, \notin K^{\times 2}$	 $\hat{c}_v = 8a, \quad \hat{m}_v = 1$	(-1)$^{2b+1}$	-1	(-1)^{2b}
TN1H	 $c_v = \widetilde{2a}\widetilde{2b}, \quad m_v = 1$	0	$\begin{cases} \equiv r\ (2) \\ -r \end{cases}$	$2r > 0$	$\notin K^{\times 2}, \notin K^{\times 2}$	 $\hat{c}_v = 4, \quad \hat{m}_v = 1$	(-1)$^{2a+2b}$	1	(-1)$^{2a+2b}$



Notation: $T_{\alpha_1} = c(\alpha_1 - \alpha_2)(\alpha_1 - \beta_2)(\alpha_1 - \alpha_3)(\alpha_1 - \beta_3)$, $T_{\alpha_2} = c(\alpha_2 - \alpha_1)(\alpha_2 + \alpha_1)(\alpha_2 - \alpha_3)(\alpha_2 - \beta_3)$, $\tilde{x} = 2$ if $2|x$ and $\tilde{x} = 1$ if $2 \nmid x$.

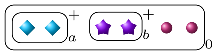
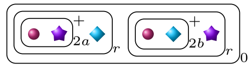
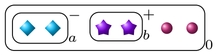
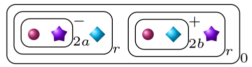
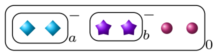
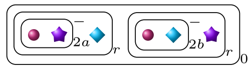
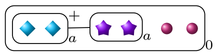
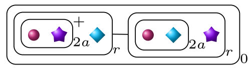
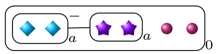
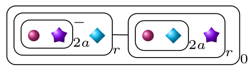
Table 4.11: C is of type $I_{2a,2b}$, TN2

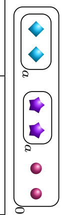
Isogeny	C	$v(c)$	$\begin{cases} v(\ell) \\ v(\ell_1) \end{cases}$	$v(\Delta_G^2)$	$T_{\alpha_2}, T_{\alpha_3}$	$(A_1 - B_1)^2$	$\hat{C}$	λ_v	w_v	E_v
TN2A	 $c_v = 4ab, \quad m_v = 1$	0	$\begin{cases} \equiv 0 \ (2) \\ 0 \end{cases}$	0	$\in K^{\times 2}, \in K^{\times 2}$	$\in K^{\times 2}$	 $\hat{c}_v = 16ab, \quad \hat{m}_v = 1$	1	1	1
TN2B	 $c_v = \widetilde{2a}2b, \quad m_v = 1$	0	$\begin{cases} \equiv 0 \ (2) \\ 0 \end{cases}$	0	$\notin K^{\times 2}, \in K^{\times 2}$	$\in K^{\times 2}$	 $\hat{c}_v = 8b, \quad \hat{m}_v = 1$	(-1)$^{2a+1}$	-1	(-1)^{2a}
TN2C	 $c_v = \widetilde{2a}\widetilde{2b}, \quad m_v = 1$	0	$\begin{cases} \equiv 0 \ (2) \\ 0 \end{cases}$	0	$\notin K^{\times 2}, \notin K^{\times 2}$	$\in K^{\times 2}$	 $\hat{c}_v = 4, \quad \hat{m}_v = 1$	(-1)$^{2a+2b}$	1	(-1)$^{2a+2b}$
TN2D	 $c_v = 2a, \quad m_v = 1$	0	$\begin{cases} \equiv 0 \ (2) \\ 0 \end{cases}$	0	$-, -$	$\notin K^{\times 2}$	 $\hat{c}_v = 4a, \quad \hat{m}_v = 1$	-1	-1	1
TN2E	 $c_v = \tilde{2}a, \quad m_v = 1$	0	$\begin{cases} \equiv 0 \ (2) \\ 0 \end{cases}$	0	$-, -$	$\notin K^{\times 2}$	 $\hat{c}_v = 2, \quad \hat{m}_v = 1$	(-1)^{2a}	1	(-1)^{2a}



Notation: $T_{\alpha_2} = c(\alpha_2 - \alpha_1)(\alpha_2 + \alpha_1)(\alpha_2 - \alpha_3)(\alpha_2 - \beta_3)$, $T_{\alpha_3} = c(\alpha_3 - \alpha_1)(\alpha_3 + \alpha_1)(\alpha_3 - \alpha_2)(\alpha_3 - \beta_2)$, $\tilde{x} = 2$ if $2|x$ and $\tilde{x} = 1$ if $2 \nmid x$.

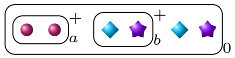
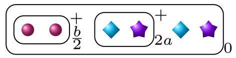
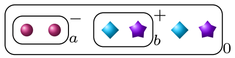
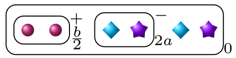
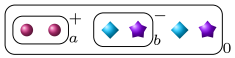
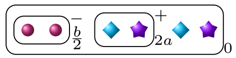
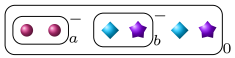
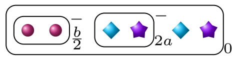
Table 4.12: C is of type $I_{2a,2b}$, TN2

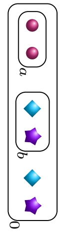
Isogeny	C	$v(c)$	$\begin{cases} v(\ell) \\ v(\ell_1) \end{cases}$	$v(\Delta_G^2)$	$T_{\alpha_2}, T_{\alpha_3}$	$(A_1-B_1)^2$	$\hat{C}$	λ_v	w_v	E_v
TN2F	 $c_v = 4ab, \quad m_v = 1$	0	$\begin{cases} \equiv r \ (2) \\ -r \end{cases}$	$2r$	$\in K^{\times 2}, \in K^{\times 2}$	$\in K^{\times 2}$	 $\hat{c}_v = 16ab, \quad \hat{m}_v = 1$	1	1	1
TN2G	 $c_v = \widetilde{2a}2b, \quad m_v = 1$	0	$\begin{cases} \equiv r \ (2) \\ -r \end{cases}$	$2r$	$\notin K^{\times 2}, \in K^{\times 2}$	$\in K^{\times 2}$	 $\hat{c}_v = 8b, \quad \hat{m}_v = 1$	$(-1)^{2a+1}$	-1	$(-1)^{2a}$
TN2H	 $c_v = \widetilde{2a}\widetilde{2b}, \quad m_v = 1$	0	$\begin{cases} \equiv r \ (2) \\ -r \end{cases}$	$2r$	$\notin K^{\times 2}, \notin K^{\times 2}$	$\in K^{\times 2}$	 $\hat{c}_v = 4, \quad \hat{m}_v = 1$	$(-1)^{2a+2b}$	1	$(-1)^{2a+2b}$
TN2I	 $c_v = 2a, \quad m_v = 1$	0	$\begin{cases} \equiv r \ (2 \\ -r \end{cases}$	$2r$	$-, -$	$\notin K^{\times 2}$	 $\hat{c}_v = 4a, \quad \hat{m}_v = \widetilde{r+1}$	$(-1)^{r+1}$	-1	$(-1)^{r}$
TN2J	 $c_v = \tilde{2}a, \quad m_v = 1$	0	$\begin{cases} \equiv r \ (2) \\ -r \end{cases}$	$2r$	$-, -$	$\notin K^{\times 2}$	 $\hat{c}_v = 2, \quad \hat{m}_v = \widetilde{r+1}$	$(-1)^{2a+r}$	1	$(-1)^{2a+r}$



Notation: $T_{\alpha_2} = c(\alpha_2-\alpha_1)(\alpha_2+\alpha_1)(\alpha_2-\alpha_3)(\alpha_2-\beta_3)$, $T_{\alpha_3} = c(\alpha_3-\alpha_1)(\alpha_3+\alpha_1)(\alpha_3-\alpha_2)(\alpha_3-\beta_2)$, $\tilde{x} = 2$ if $2|x$ and $\tilde{x} = 1$ if $2 \nmid x$.

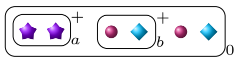
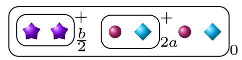
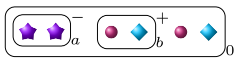
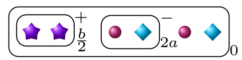
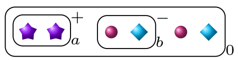
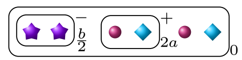
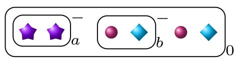
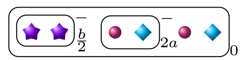
Table 4.13: C is of type $I_{2a,2b}$, TN3

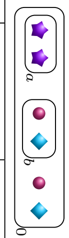
Isogeny	C	$v(c)$	$\begin{cases} v(\ell) \\ v(\ell_1) \end{cases}$	$v(\Delta_G^2)$	$T_{\alpha_1}, T_{\alpha_2}$	$\hat{C}$	λ_v	w_v	E_v
TN3A	 $c_v = 4ab, \quad m_v = 1$	0	$\begin{cases} \equiv 0 \ (2) \\ 0 \end{cases}$	0	$\in K^{\times 2}, \in K^{\times 2}$	 $\hat{c}_v = 4ab, \quad \hat{m}_v = 1$	1	1	1
TN3B	 $c_v = \widetilde{2a}2b, \quad m_v = 1$	0	$\begin{cases} \equiv 0 \ (2) \\ 0 \end{cases}$	0	$\notin K^{\times 2}, \in K^{\times 2}$	 $\hat{c}_v = 2b, \quad \hat{m}_v = 1$	(-1)$^{2a+1}$	-1	(-1)^{2a}
TN3C	 $c_v = 4a, \quad m_v = 1$	0	$\begin{cases} \equiv 0 \ (2) \\ 0 \end{cases}$	0	$\in K^{\times 2}, \notin K^{\times 2}$	 $\hat{c}_v = 4a\tilde{b}, \quad \hat{m}_v = 1$	(-1)$^{b+1}$	-1	(-1)^{b}
TN3D	 $c_v = \widetilde{2a}\widetilde{2b}, \quad m_v = 1$	0	$\begin{cases} \equiv 0 \ (2) \\ 0 \end{cases}$	0	$\notin K^{\times 2}, \notin K^{\times 2}$	 $\hat{c}_v = 2\tilde{b}, \quad \hat{m}_v = 1$	(-1)$^{2a+b}$	1	(-1)$^{2a+b}$



Notation: $T_{\alpha_1} = c(\alpha_1 - \alpha_2)(\alpha_1 - \beta_2)(\alpha_1 - \alpha_3)(\alpha_1 - \beta_3)$, $T_{\alpha_2} = c(\alpha_2 - \alpha_1)(\alpha_2 + \alpha_1)(\alpha_2 - \alpha_3)(\alpha_2 - \beta_3)$, $\tilde{x} = 2$ if $2|x$ and $\tilde{x} = 1$ if $2 \nmid x$.

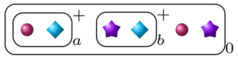
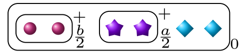
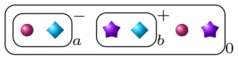
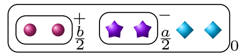
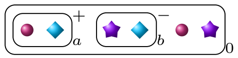
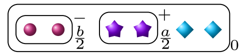
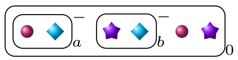
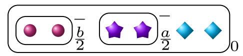
Table 4.14: C is of type $I_{2a,2b}$, TN4

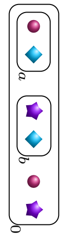
Isogeny	C	$v(c)$	$\begin{cases} v(\ell) \\ v(\ell_1) \end{cases}$	$v(\Delta_G^2)$	$T_{\alpha_3}, T_{\alpha_1}$	$\hat{C}$	λ_v	w_v	E_v
TN4A	 $c_v = 4ab, \quad m_v = 1$	0	$\begin{cases} \equiv 0\ (2) \\ - \end{cases}$	0	$\in K^{\times 2}, \in K^{\times 2}$	 $\hat{c}_v = 4ab, \quad \hat{m}_v = 1$	1	1	1
TN4B	 $c_v = \widetilde{2a}2b, \quad m_v = 1$	0	$\begin{cases} \equiv 0\ (2) \\ - \end{cases}$	0	$\notin K^{\times 2}, \in K^{\times 2}$	 $\hat{c}_v = 2b, \quad \hat{m}_v = 1$	(-1)$^{2a+1}$	-1	(-1)^{2a}
TN4C	 $c_v = 4a, \quad m_v = 1$	0	$\begin{cases} \equiv 0\ (2) \\ - \end{cases}$	0	$\in K^{\times 2}, \notin K^{\times 2}$	 $\hat{c}_v = 4a\tilde{b}, \quad \hat{m}_v = 1$	(-1)$^{b+1}$	-1	(-1)^{b}
TN4D	 $c_v = 2\widetilde{2a}, \quad m_v = 1$	0	$\begin{cases} \equiv 0\ (2) \\ - \end{cases}$	0	$\notin K^{\times 2}, \notin K^{\times 2}$	 $\hat{c}_v = 2\tilde{b}, \quad \hat{m}_v = 1$	(-1)$^{2a+b}$	1	(-1)$^{2a+b}$



Notation: $T_{\alpha_3} = c(\alpha_3 - \alpha_1)(\alpha_3 + \alpha_1)(\alpha_3 - \alpha_2)(\alpha_3 - \beta_2)$, $T_{\alpha_1} = 2c\alpha_1(\alpha_1 - \beta_2)(\alpha_1 - \alpha_3)(\alpha_1 - \beta_3)$, $\tilde{x} = 2$ if $2|x$ and $\tilde{x} = 1$ if $2 \nmid x$.

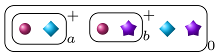
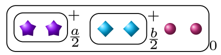
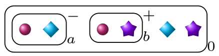
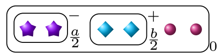
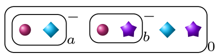
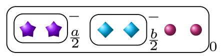
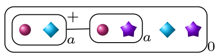
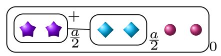
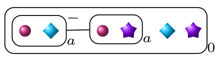
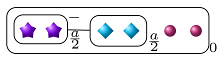
Table 4.15: C is of type $I_{2a,2b}$, TN5

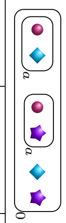
Isogeny	C	$v(c)$	$\begin{cases} v(\ell) \\ v(\ell_1) \end{cases}$	$v(\Delta_G^2)$	$T_{\alpha_1}, T_{\alpha_3}$	$\hat{C}$	λ_v	w_v	E_v
TN5A	 $c_v = 4ab, \quad m_v = 1$	0	$\begin{cases} \equiv 0\ (2) \\ - \end{cases}$	0	$\in K^{\times 2}, \in K^{\times 2}$	 $\hat{c}_v = ab, \quad \hat{m}_v = 1$	1	1	1
TN5B	 $c_v = 4b, \quad m_v = 1$	0	$\begin{cases} \equiv 0\ (2) \\ - \end{cases}$	0	$\notin K^{\times 2}, \in K^{\times 2}$	 $\hat{c}_v = \tilde{a}b, \quad \hat{m}_v = 1$	(-1)$^{a+1}$	-1	(-1)^{a}
TN5C	 $c_v = 4a, \quad m_v = 1$	0	$\begin{cases} \equiv 0\ (2) \\ - \end{cases}$	0	$\in K^{\times 2}, \notin K^{\times 2}$	 $\hat{c}_v = a\tilde{b}, \quad \hat{m}_v = 1$	(-1)$^{b+1}$	-1	(-1)^{b}
TN5D	 $c_v = 4, \quad m_v = 1$	0	$\begin{cases} \equiv 0\ (2) \\ - \end{cases}$	0	$\notin K^{\times 2}, \notin K^{\times 2}$	 $\hat{c}_v = \tilde{a}\tilde{b}, \quad \hat{m}_v = 1$	(-1)$^{a+b}$	1	(-1)$^{a+b}$



Notation: $T_{\alpha_1} = c2\alpha_1(\alpha_1 - \beta_2)(\alpha_1 - \alpha_3)(\alpha_1 - \beta_3)$, $T_{\alpha_3} = c(\alpha_3 - \alpha_1)(\alpha_3 + \alpha_1)(\alpha_3 - \alpha_2)(\alpha_3 - \beta_3)$,
$\tilde{x} = 2$ if $2|x$ and $\tilde{x} = 1$ if $2 \nmid x$.

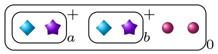
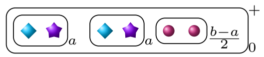
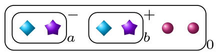
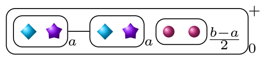
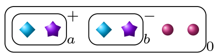
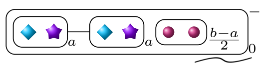
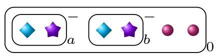
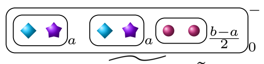
Table 4.16: C is of type $I_{2a,2b}$, TN6

Isogeny	C	$v(c)$	$\begin{cases} v(\ell) \\ v(\ell_1) \end{cases}$	$v(\Delta_G^2)$	$T_{\alpha_1}, T_{-\alpha_1}$	$(A_2 - A_3)^2$	$\hat{C}$	λ_v	w_v	E_v
TN6A	 $c_v = 4ab, \quad m_v = 1$	0	$\begin{cases} \equiv 0\ (2) \\ - \end{cases}$	0	$\in K^{\times 2}, \in K^{\times 2}$	$\in K^{\times 2}$	 $\hat{c}_v = ab, \quad \hat{m}_v = 1$	1	1	1
TN6B	 $c_v = 4b, \quad m_v = 1$	0	$\begin{cases} \equiv 0\ (2) \\ - \end{cases}$	0	$\notin K^{\times 2}, \in K^{\times 2}$	$\in K^{\times 2}$	 $\hat{c}_v = \tilde{a}b, \quad \hat{m}_v = 1$	(-1)$^{a+1}$	-1	(-1)^{a}
TN6C	 $c_v = 4, \quad m_v = 1$	0	$\begin{cases} \equiv 0\ (2) \\ - \end{cases}$	0	$\notin K^{\times 2}, \notin K^{\times 2}$	$\in K^{\times 2}$	 $\hat{c}_v = \tilde{a}\tilde{b}, \quad \hat{m}_v = 1$	(-1)$^{a+b}$	1	(-1)$^{a+b}$
TN6D	 $c_v = 2a, \quad m_v = 1$	0	$\begin{cases} \equiv 0\ (2) \\ - \end{cases}$	0	$-, -$	$\notin K^{\times 2}$	 $\hat{c}_v = a, \quad \hat{m}_v = 1$	-1	-1	1
TN6E	 $c_v = 2, \quad m_v = 1$	0	$\begin{cases} \equiv 0\ (2) \\ - \end{cases}$	0	$-, -$	$\notin K^{\times 2}$	 $\hat{c}_v = \tilde{a}, \quad \hat{m}_v = 1$	(-1)^{a}	1	(-1)^{a}



Notation: $T_{\alpha_1} = 2c\alpha_1(\alpha_1 - \alpha_3)(\alpha_1 - \beta_2)(\alpha_1 - \beta_3)$, $T_{-\alpha_1} = -2c\alpha_1(-\alpha_1 - \alpha_2)(-\alpha_1 - \beta_2)(-\alpha_1 - \beta_3)$,
$\tilde{x} = 2$ if $2|x$ and $\tilde{x} = 1$ if $2 \nmid x$.

Table 4.17: C is of type $I_{2a,2b}$, TN7

Isogeny	C	$v(c)$	$\begin{cases} v(\ell) \\ v(\ell_1) \end{cases}$	$v(\Delta_G^2)$	$T_{\alpha_2}, T_{\beta_2}$	$(A_2-B_2)^2$	$\hat{C}$	λ_v	w_v	E_v
TN7A $a<b$	 $c_v = 4ab, \quad m_v = 1$	0	$\begin{cases} \equiv 0 \ (2) \\ 0 \end{cases}$	$2a$	$\in K^{\times 2}, \in K^{\times 2}$	$\in K^{\times 2}$	 $\hat{c}_v = 4ab, \quad \hat{m}_v = 1$	1	1	1
TN7B $a<b$	 $c_v = 2\tilde{a}b, \quad m_v = 1$	0	$\begin{cases} \equiv 0 \ (2) \\ 0 \end{cases}$	$2a$	$\notin K^{\times 2}, \in K^{\times 2}$	$\notin K^{\times 2}$	 $\hat{c}_v = 2b, \quad \hat{m}_v = 1$	(-1)$^{2a+1}$	-1	(-1)^{2a}
TN7C $a<b$	 $c_v = 2a\tilde{b}, \quad m_v = 1$	0	$\begin{cases} \equiv 0 \ (2) \\ 0 \end{cases}$	$2a$	$\in K^{\times 2}, \notin K^{\times 2}$	$\notin K^{\times 2}$	 $\hat{c}_v = 2a, \quad \hat{m}_v = \widetilde{b+a+1}$	(-1)$^{a+b+1}$	-1	(-1)$^{a+b}$
TN7D $a<b$	 $c_v = \widetilde{2a}\widetilde{2b}, \quad m_v = 1$	0	$\begin{cases} \equiv 0 \ (2) \\ 0 \end{cases}$	$2a$	$\notin K^{\times 2}, \notin K^{\times 2}$	$\in K^{\times 2}$	 $\hat{c}_v = \widetilde{4ab/D} \cdot \tilde{D}$, $\hat{m}_v = \widetilde{b+a+1}$	(-1)$^{a+b}$	1	(-1)$^{a+b}$

Notation: $T_{\alpha_2} = c(\alpha_2-\alpha_1)(\alpha_2+\alpha_1)(\alpha_2-\beta_2)(\alpha_2-\beta_3)$, $T_{\beta_2} = c(\beta_2-\alpha_1)(\beta_2+\alpha_1)(\beta_2-\alpha_2)(\beta_2-\alpha_3)$, $\tilde{x} = 2$ if $2|x$ and $\tilde{x} = 1$ if $2 \nmid x$, $\quad D = gcd(2a, b-a)$.

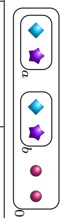
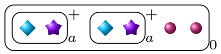
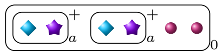
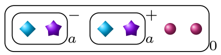
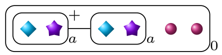
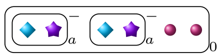
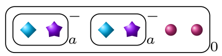
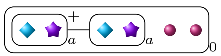
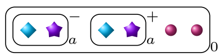
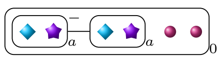
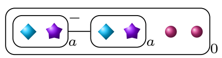


Table 4.18: C is of type $I_{2a,2a}$, TN7

Isogeny	C	$v(c)$	$\begin{cases} v(\ell) \\ v(\ell_1) \end{cases}$	$v(\Delta_G^2)$	$T_{\alpha_2}, T_{\beta_2}$	$(A_2{-}B_2)^2$	$\hat{C}$	λ_v	w_v	E_v
TN7E	 $c_v = 4a^2, \quad m_v = 1$	0	$\begin{cases} \equiv 0\ (2) \\ 0\ (2) \end{cases}$	$2a$	$\in K^{\times 2}, \in K^{\times 2}$	$\in K^{\times 2}$	 $\hat{c}_v = 4a^2, \quad \hat{m}_v = 1$	1	1	1
TN7F	 $c_v = 2\tilde{a}a, \quad m_v = 1$	0	$\begin{cases} \equiv 0\ (2) \\ 0\ (2) \end{cases}$	$2a$	$\notin K^{\times 2}, \in K^{\times 2}$	$\notin K^{\times 2}$	 $\hat{c}_v = 2a, \quad \hat{m}_v = 1$	(-1)$^{2a+1}$	-1	(-1)^{2a}
TN7G	 $c_v = \widetilde{2a}\widetilde{2a}, \quad m_v = 1$	0	$\begin{cases} \equiv 0\ (2) \\ 0\ (2) \end{cases}$	$2a$	$\notin K^{\times 2}, \notin K^{\times 2}$	$\in K^{\times 2}$	 $\hat{c}_v = c_v, \quad \hat{m}_v = 1$	1	1	1
TN7H	 $c_v = 2a, \quad m_v = 1$	0	$\begin{cases} \equiv 0\ (2) \\ 0\ (2) \end{cases}$	$2a$	$-, -$	$\in K^{\times 2}$	 $\hat{c}_v = 2a\tilde{a}, \quad \hat{m}_v = 1$	(-1)$^{2a+1}$	-1	(-1)^{2a}
TN7I	 $c_v = \tilde{2}a, \quad m_v = 1$	0	$\begin{cases} \equiv 0\ (2) \\ 0\ (2) \end{cases}$	$2a$	$-, -$	$\notin K^{\times 2}$	 $\hat{c}_v = \tilde{2}a, \quad \hat{m}_v = 1$	1	1	1

Notation: $T_{\alpha_2} = c(\alpha_2 - \alpha_1)(\alpha_2 + \alpha_1)(\alpha_2 - \beta_2)(\alpha_2 - \beta_3)$, $T_{\beta_2} = c(\beta_2 - \alpha_1)(\beta_2 + \alpha_1)(\beta_2 - \alpha_2)(\beta_2 - \alpha_3)$, $\tilde{x} = 2$ if $2|x$ and $\tilde{x} = 1$ if $2 \nmid x$.

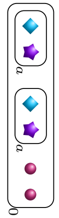
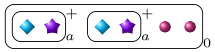
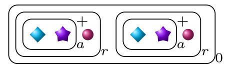
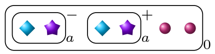
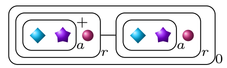
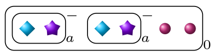
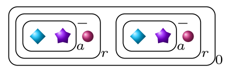
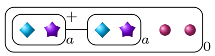
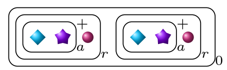
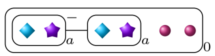
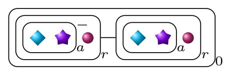
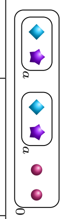


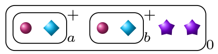
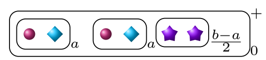
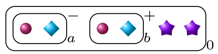
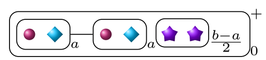
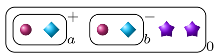
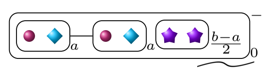
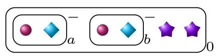
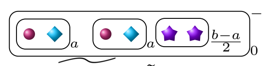
Table 4.19: C is of type $I_{2a,2a}$, TN7

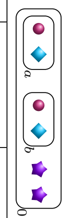
Isogeny	C	$v(c)$	$\begin{cases} v(\ell) \\ v(\ell_1) \end{cases}$	$v(\Delta_G^2)$	$T_{\alpha_2}, T_{\beta_2}$	$(A_2-B_2)^2$	$\hat{C}$	λ_v	w_v	E_v
TN7J	 $c_v = 4a^2, \quad m_v = 1$	0	$\begin{cases} \equiv r\ (2) \\ -r \end{cases}$	$2r+2a$ $r>0$	$\in K^{\times 2}, \in K^{\times 2}$	$\in K^{\times 2}$	 $\hat{c}_v = 4a^2, \quad \hat{m}_v = 1$	1	1	1
TN7K	 $c_v = 2\tilde{a}a, \quad m_v = 1$	0	$\begin{cases} \equiv r\ (2) \\ -r \end{cases}$	$2r+2a$ $r>0$	$\notin K^{\times 2}, \in K^{\times 2}$	$\notin K^{\times 2}$	 $\hat{c}_v = 2a, \quad \hat{m}_v = \widetilde{r+1}$	$(-1)^{2a+r+1}$	-1	$(-1)^{2a+r}$
TN7L	 $c_v = \widetilde{2a2a}, \quad m_v = 1$	0	$\begin{cases} \equiv r\ (2) \\ -r \end{cases}$	$2r+2a$ $r>0$	$\notin K^{\times 2}, \notin K^{\times 2}$	$\in K^{\times 2}$	 $\hat{c}_v = c_v, \quad \hat{m}_v = 1$	1	1	1
TN7M	 $c_v = 2a, \quad m_v = 1$	0	$\begin{cases} \equiv r\ (2) \\ -r \end{cases}$	$2r+2a$ $r>0$	$-,-$	$\in K^{\times 2}$	 $\hat{c}_v = 2a\tilde{a}, \quad \hat{m}_v = 1$	$(-1)^{2a+1}$	-1	$(-1)^{2a}$
TN7N	 $c_v = \tilde{2a}, \quad m_v = 1$	0	$\begin{cases} \equiv r\ (2) \\ -r \end{cases}$	$2r+2a$ $r>0$	$-,-$	$\notin K^{\times 2}$	 $\hat{c}_v = \tilde{2a}, \quad \hat{m}_v = \widetilde{r+1}$	$(-1)^r$	1	$(-1)^r$



Notation: $T_{\alpha_2} = c(\alpha_2-\alpha_1)(\alpha_2+\alpha_1)(\alpha_2-\beta_2)(\alpha_2-\beta_3)$, $T_{\beta_2} = c(\beta_2-\alpha_1)(\beta_2+\alpha_1)(\beta_2-\alpha_2)(\beta_2-\alpha_3)$,
$\tilde{x} = 2$ if $2|x$ and $\tilde{x} = 1$ if $2\nmid x$.

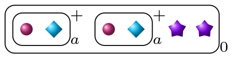
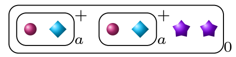
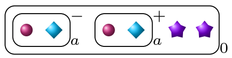
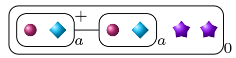
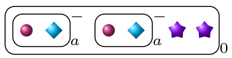
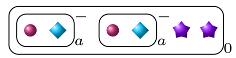
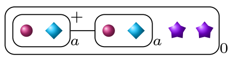
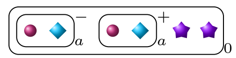
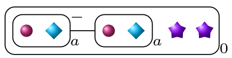
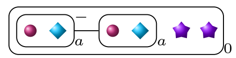
Table 4.20: C is of type $I_{2a,2b}$, TN8

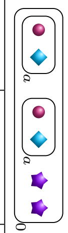
Isogeny	C	$v(c)$	$\begin{cases} v(\ell) \\ v(\ell_1) \end{cases}$	$v(\Delta_G^2)$	$T_{\alpha_1}, T_{-\alpha_1}$	$(A_1-B_1)^2$	$\hat{C}$	λ_v	w_v	E_v
TN8A $a<b$	 $c_v = 4ab, \quad m_v = 1$	0	$\begin{cases} \equiv 0 \ (2) \\ - \end{cases}$	$2a$	$\in K^{\times 2}, \in K^{\times 2}$	$\in K^{\times 2}$	 $\hat{c}_v = 4ab, \quad \hat{m}_v = 1$	1	1	1
TN8B $a<b$	 $c_v = 4b, \quad m_v = 1$	0	$\begin{cases} \equiv 0 \ (2) \\ - \end{cases}$	$2a$	$\notin K^{\times 2}, \in K^{\times 2}$	$\notin K^{\times 2}$	 $\hat{c}_v = 2b, \quad \hat{m}_v = 1$	-1	-1	1
TN8C $a<b$	 $c_v = 4a, \quad m_v = 1$	0	$\begin{cases} \equiv 0 \ (2) \\ - \end{cases}$	$2a$	$\in K^{\times 2}, \notin K^{\times 2}$	$\notin K^{\times 2}$	 $\hat{c}_v = 2a, \quad \hat{m}_v = b+a+1$	(-1)$^{a+b+1}$	-1	(-1)$^{a+b}$
TN8D $a<b$	 $c_v = 4, \quad m_v = 1$	0	$\begin{cases} \equiv 0 \ (2) \\ - \end{cases}$	$2a$	$\notin K^{\times 2}, \notin K^{\times 2}$	$\in K^{\times 2}$	 $\hat{c}_v = 4a^2/D \cdot \tilde{D}, \quad \hat{m}_v = 1$	(-1)$^{a+b}$	1	(-1)$^{a+b}$



Notation: $T_{\alpha_1} = 2c\alpha_1(\alpha_1-\beta_2)(\alpha_1-\alpha_3)(\alpha_1-\beta_3)$, $T_{-\alpha_1} = -2c\alpha_1(-\alpha_1-\alpha_2)(-\alpha_1-\alpha_3)(-\alpha_1-\beta_3)$,
$\tilde{x} = 2$ if $2|x$ and $\tilde{x} = 1$ if $2 \nmid x$, $\quad D = gcd(2a, b-a)$.

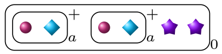
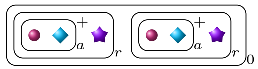
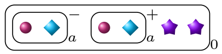
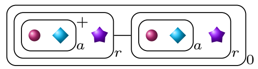
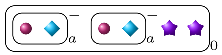
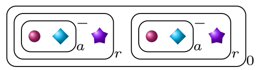
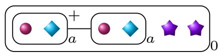
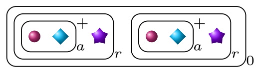
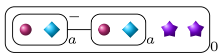
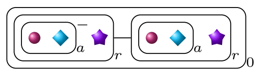
Table 4.21: C is of type $I_{2a,2a}$, TN8

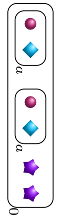
Isogeny	C	$v(c)$	$\begin{cases} v(\ell) \\ v(\ell_1) \end{cases}$	$v(\Delta_G^2)$	$T_{\alpha_1}, T_{-\alpha_1}$	$(A_1 - B_1)^2$	$\hat{C}$	λ_v	w_v	E_v
TN8E	 $c_v = 4a^2, \quad m_v = 1$	0	$\begin{cases} \equiv 0\ (2) \\ - \end{cases}$	$2a$	$\in K^{\times 2}, \in K^{\times 2}$	$\in K^{\times 2}$	 $\hat{c}_v = 4a^2, \quad \hat{m}_v = 1$	1	1	1
TN8F	 $c_v = 4a, \quad m_v = 1$	0	$\begin{cases} \equiv 0\ (2) \\ - \end{cases}$	$2a$	$\notin K^{\times 2}, \in K^{\times 2}$	$\notin K^{\times 2}$	 $\hat{c}_v = 2a, \quad \hat{m}_v = 1$	-1	-1	1
TN8G	 $c_v = 4, \quad m_v = 1$	0	$\begin{cases} \equiv 0\ (2) \\ - \end{cases}$	$2a$	$\notin K^{\times 2}, \notin K^{\times 2}$	$\in K^{\times 2}$	 $\hat{c}_v = c_v, \quad \hat{m}_v = 1$	1	1	1
TN8H	 $c_v = 2a, \quad m_v = 1$	0	$\begin{cases} \equiv 0\ (2) \\ - \end{cases}$	$2a$	$-, -$	$\in K^{\times 2}$	 $\hat{c}_v = 4a, \quad \hat{m}_v = 1$	-1	-1	1
TN8I	 $c_v = 2, \quad m_v = 1$	0	$\begin{cases} \equiv 0\ (2) \\ - \end{cases}$	$2a$	$-, -$	$\notin K^{\times 2}$	 $\hat{c}_v = 2, \quad \hat{m}_v = 1$	1	1	1



Notation: $T_{\alpha_1} = 2c\alpha_1(\alpha_1 - \beta_2)(\alpha_1 - \alpha_3)(\alpha_1 - \beta_3)$, $T_{-\alpha_1} = -2c\alpha_1(-\alpha_1 - \alpha_2)(-\alpha_1 - \alpha_3)(-\alpha_1 - \beta_3)$,
$\tilde{x} = 2$ if $2|x$ and $\tilde{x} = 1$ if $2 \nmid x$, $\quad D = gcd(2a, b - a)$.

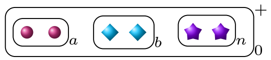
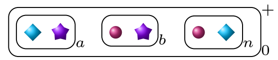
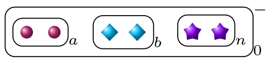
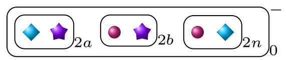
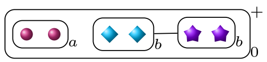
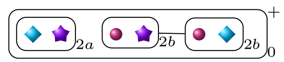
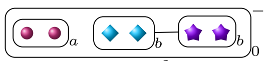
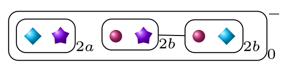
Table 4.22: C is of type $I_{2a,2a}$, TN8

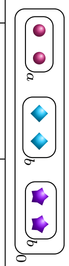
Isogeny	C	$v(c)$	$\begin{cases} v(\ell) \\ v(\ell_1) \end{cases}$	$v(\Delta_G^2)$	$T_{\alpha_1}, T_{-\alpha_1}$	$(A_1-B_1)^2$	$\hat{C}$	λ_v	w_v	E_v
TN8J	 $c_v = 4a^2, \quad m_v = 1$	0	$\begin{cases} \equiv r\ (2) \\ -r \end{cases}$	$2r+2a$ $r>0$	$\in K^{\times 2}, \in K^{\times 2}$	$\in K^{\times 2}$	 $\hat{c}_v = 4a^2, \quad \hat{m}_v = 1$	1	1	1
TN8K	 $c_v = 4a, \quad m_v = 1$	0	$\begin{cases} \equiv r\ (2) \\ -r \end{cases}$	$2r+2a$ $r>0$	$\notin K^{\times 2}, \in K^{\times 2}$	$\notin K^{\times 2}$	 $\hat{c}_v = 2a, \quad \hat{m}_v = \widetilde{r+1}$	(-1)$^{r+1}$	-1	(-1)^r
TN8L	 $c_v = 4, \quad m_v = 1$	0	$\begin{cases} \equiv r\ (2) \\ -r \end{cases}$	$2r+2a$ $r>0$	$\notin K^{\times 2}, \notin K^{\times 2}$	$\in K^{\times 2}$	 $\hat{c}_v = c_v, \quad \hat{m}_v = 1$	1	1	1
TN8M	 $c_v = 2a, \quad m_v = 1$	0	$\begin{cases} \equiv r\ (2) \\ -r \end{cases}$	$2r+2a$ $r>0$	$-,-$	$\in K^{\times 2}$	 $\hat{c}_v = 4a, \quad \hat{m}_v = 1$	-1	-1	1
TN8N	 $c_v = 2, \quad m_v = 1$	0	$\begin{cases} \equiv r\ (2) \\ -r \end{cases}$	$2r+2a$ $r>0$	$-,-$	$\notin K^{\times 2}$	 $\hat{c}_v = 2, \quad \hat{m}_v = \widetilde{r+1}$	(-1)^r	1	(-1)^r



Notation: $T_{\alpha_1} = 2c\alpha_1(\alpha_1-\beta_2)(\alpha_1-\alpha_3)(\alpha_1-\beta_3)$, $T_{-\alpha_1} = -2c\alpha_1(-\alpha_1-\alpha_2)(-\alpha_1-\alpha_3)(-\alpha_1-\beta_3)$,
$\tilde{x} = 2$ if $2|x$ and $\tilde{x} = 1$ if $2 \nmid x$, $\quad D = gcd(2a, b-a)$.

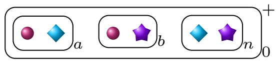
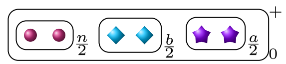
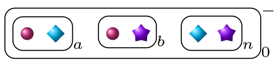
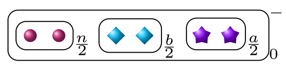
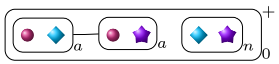
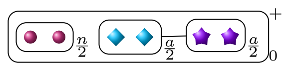
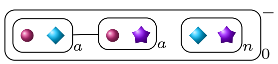
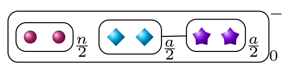
Table 4.23: C is of type $U_{2a,2b,2n}$, U1

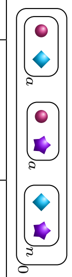
Isogeny	C	$v(c)$	$\begin{cases} v(\ell) \\ v(\ell_1) \end{cases}$	$v(\Delta_G^2)$	c	$(A_1 - B_1)^2$	$\hat{C}$	λ_v	w_v	E_v
U1A	 $c_v = N, \quad m_v = 1$	0	$\begin{cases} 0 \\ 0 \end{cases}$	0	$\in K^{\times 2}$	$\in K^{\times 2}$	 $\hat{c}_v = 4N, \quad \hat{m}_v = 1$	1	1	1
U1B	 $c_v = \widetilde{N/D} \cdot \widetilde{D}, \quad m_v = d$	0	$\begin{cases} 0 \\ 0 \end{cases}$	0	$\notin K^{\times 2}$	$\in K^{\times 2}$	 $\hat{c}_v = 4, \quad \hat{m}_v = 1$	(-1)^N	1	(-1)^N
U1C	 $c_v = 2b + 4a, \quad m_v = 1$	0	$\begin{cases} 0 \\ 0 \end{cases}$	0	$\in K^{\times 2}$	$\notin K^{\times 2}$	 $\hat{c}_v = 4b + 8a, \quad \hat{m}_v = 1$	-1	-1	1
U1D	 $c_v = 2b,$ $m_v = \widetilde{2a+1}$	0	$\begin{cases} 0 \\ 0 \end{cases}$	0	$\notin K^{\times 2}$	$\notin K^{\times 2}$	 $\hat{c}_v = 4b, \quad \hat{m}_v = 1$	(-1)$^{2a+1}$	-1	(-1)^{2a}



Notation: $\tilde{x} = 2$ if $2|x$ and $\tilde{x} = 1$ if $2 \nmid x$,
$D = gcd(2a, 2b, 2n)$, $N = 4ab + 4an + 4bn$, $d = 2$ if $2a, 2b, 2n$ are odd, $d = 1$ otherwise.

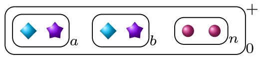
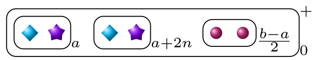
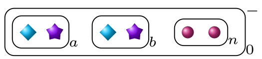
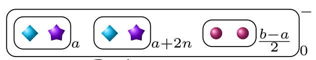
Table 4.24: C is of type $U_{2a,2b,2n}$, U2

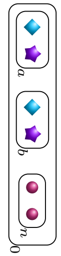
Isogeny	C	$v(c)$	$\begin{cases} v(\ell) \\ v(\ell_1) \end{cases}$	$v(\Delta_G^2)$	c	δ_2, δ_3	$\hat{C}$	λ_v	w_v	E_v
U2A	 $c_v = 4N, \quad m_v = 1$	0	$\begin{cases} 0 \\ 0 \end{cases}$	0	$\in K^{\times 2}$	$\in K^{\times 2}$	 $\hat{c}_v = N, \quad \hat{m}_v = 1$	1	1	1
U2B	 $c_v = 4, \quad m_v = 1$	0	$\begin{cases} 0 \\ 0 \end{cases}$	0	$\notin K^{\times 2}$	$\in K^{\times 2}$	 $\hat{c}_v = \widetilde{N/D} \cdot \widetilde{D}, \quad \hat{m}_v = d$	(-1)^N	1	(-1)^N
U2C	 $c_v = 2a + 4n, \quad m_v = 1$	0	$\begin{cases} 0 \\ 0 \end{cases}$	0	$\in K^{\times 2}$	$\notin K^{\times 2}$	 $\hat{c}_v = a + 2n, \quad \hat{m}_v = 1$	-1	-1	1
U2D	 $c_v = 2a, \quad m_v = 1$	0	$\begin{cases} 0 \\ 0 \end{cases}$	0	$\notin K^{\times 2}$	$\notin K^{\times 2}$	 $\hat{c}_v = a, \quad \hat{m}_v = \widetilde{n+1}$	(-1)$^{n+1}$	-1	(-1)^n



Notation: $\tilde{x} = 2$ if $2|x$ and $\tilde{x} = 1$ if $2 \nmid x$,
$D = gcd(a,b,n)$, $N = ab + an + bn$, $d = 2$ if a, b, n are odd, $d = 1$ otherwise.

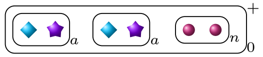
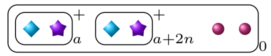
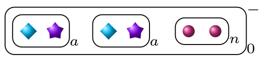
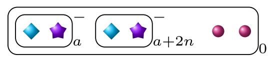
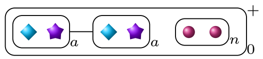
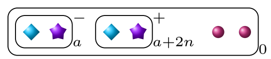
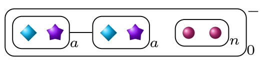
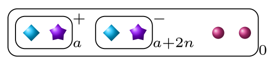
Table 4.25: C is of type $U_{2a,2b,2n}$, U3

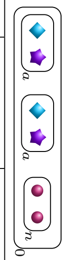
Isogeny	C	$v(c)$	$\begin{cases} v(\ell) \\ v(\ell_1) \end{cases}$	$v(\Delta_G^2)$	c	δ_2, δ_3	$\hat{C}$	λ_v	w_v	E_v
U3A $a < b$	 $c_v = N, \quad m_v = 1$	0	$\begin{cases} \equiv 0 \ (2) \\ 0 \end{cases}$	$2a$	$\in K^{\times 2}$	$\in K^{\times 2}$	 $\hat{c}_v = N, \quad \hat{m}_v = 1$	1	1	1
U3B $a < b$	 $c_v = \widetilde{N/D} \cdot \widetilde{D}, \quad m_v = d$	0	$\begin{cases} \equiv 0 \ (2) \\ 0 \end{cases}$	$2a$	$\notin K^{\times 2}$	$\in K^{\times 2}$	 $\hat{c}_v = \widetilde{N/\hat{D}} \cdot \widetilde{\hat{D}}, \quad \hat{m}_v = \hat{d}$	(-1)$^{2n+a+b}$	1	(-1)$^{2n+a+b}$



Notation: $T = c$, $\tilde{x} = 2$ if $2|x$ and $\tilde{x} = 1$ if $2 \nmid x$,
$D = gcd(2a, 2b, 2n)$, $N = 4ab + 4an + 4bn$, $d = 2$ if $2a, 2b, 2n$ are odd, $d = 1$ otherwise,
$\hat{D} = gcd(2a, b - a)$, $\hat{d} = 2$ if $2a, b - a$ are odd, $\hat{d} = 1$ otherwise .

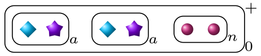
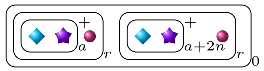
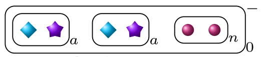
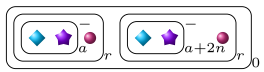
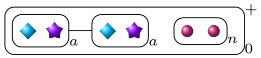
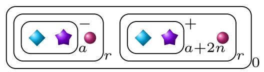
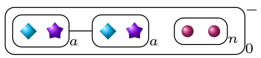
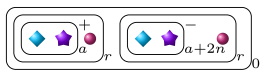
Table 4.26: C is of type $U_{2a,2b,2n}$, U3

Isogeny	C	$v(c)$	$\begin{cases} v(\ell) \\ v(\ell_1) \end{cases}$	$v(\Delta_G^2)$	c	δ_2, δ_3	$\hat{C}$	λ_v	w_v	E_v
U3C	 $c_v = N, \quad m_v = 1$	0	$\begin{cases} \equiv 0\ (2) \\ \equiv 0\ (2) \end{cases}$	$2a$	$\in K^{\times 2}$	$\in K^{\times 2}$	 $\hat{c}_v = N, \quad \hat{m}_v = 1$	1	1	1
U3D	 $c_v = \widetilde{N/D} \cdot \widetilde{D}, \quad m_v = d$	0	$\begin{cases} \equiv 0\ (2) \\ \equiv 0\ (2) \end{cases}$	$2a$	$\notin K^{\times 2}$	$\in K^{\times 2}$	 $\hat{c}_v = \widetilde{2a}\widetilde{(2a+4n)}, \hat{m}_v = 1$	(-1)^{2n}	1	(-1)^{2n}
U3E	 $c_v = 2a + 4n, \quad m_v = 1$	0	$\begin{cases} \equiv 0\ (2) \\ \equiv 0\ (2) \end{cases}$	$2a$	$\in K^{\times 2}$	$\notin K^{\times 2}$	 $\hat{c}_v = \tilde{2a}(2a+4n), \quad \hat{m}_v = 1$	(-1)$^{2a+1}$	-1	(-1)^{2a}
U3F	 $c_v = 2a, \quad m_v = d$	0	$\begin{cases} \equiv 0\ (2) \\ \equiv 0\ (2) \end{cases}$	$2a$	$\notin K^{\times 2}$	$\notin K^{\times 2}$	 $\hat{c}_v = \widetilde{(2a+4n)}2a, \hat{m}_v = 1$	(-1)$^{2n+2a+1}$	-1	(-1)$^{2n+2a}$



Notation: $\tilde{x} = 2$ if $2|x$ and $\tilde{x} = 1$ if $2 \nmid x$,
$D = gcd(2a, 2n)$, $N = 4a^2 + 8an$, $d = 2$ if $2a, 2n$ are odd, $d = 1$ otherwise.

Table 4.27: C is of type $U_{2a,2b,2n}$, U3

Isogeny	C	$v(c)$	$\begin{cases} v(\ell) \\ v(\ell_1) \end{cases}$	$v(\Delta_G^2)$	c	δ_2, δ_3	$\hat{C}$	λ_v	w_v	E_v
U3G	 $c_v = N, \quad m_v = 1$	0	$\begin{cases} \equiv r\ (2) \\ -r \end{cases}$	$2a + 2r$ $r > 0$	$\in K^{\times 2}$	$\in K^{\times 2}$	 $\hat{c}_v = N, \quad \hat{m}_v = 1$	1	1	1
U3H	 $c_v = \widetilde{N/D} \cdot \widetilde{D}, \quad m_v = d$	0	$\begin{cases} \equiv r\ (2) \\ -r \end{cases}$	$2a + 2r$ $r > 0$	$\notin K^{\times 2}$	$\in K^{\times 2}$	 $\hat{c}_v = \widetilde{2a}\widetilde{(2a+4n)}, \hat{m}_v = 1$	(-1)^{2n}	1	(-1)^{2n}
U3I	 $c_v = 2a + 4n, \quad m_v = 1$	0	$\begin{cases} \equiv r\ (2) \\ -r \end{cases}$	$2a + 2r$ $r > 0$	$\in K^{\times 2}$	$\notin K^{\times 2}$	 $\hat{c}_v = \tilde{2a}(2a+4n), \quad \hat{m}_v = 1$	(-1)$^{2a+1}$	-1	(-1)^{2a}
U3J	 $c_v = 2a, \quad m_v = d$	0	$\begin{cases} \equiv r\ (2) \\ -r \end{cases}$	$2a + 2r$ $r > 0$	$\notin K^{\times 2}$	$\notin K^{\times 2}$	 $\hat{c}_v = 2a\widetilde{(2a+4n)}, \hat{m}_v = 1$	(-1)$^{2n+2a+1}$	-1	(-1)$^{2n+2a}$

Notation: $\tilde{x} = 2$ if $2|x$ and $\tilde{x} = 1$ if $2 \nmid x$,
$D = gcd(2a, 2n)$, $N = 4a^2 + 8an$, $d = 2$ if $2a, 2n$ are odd, $d = 1$ otherwise.

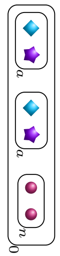
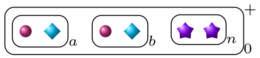
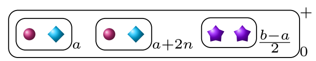
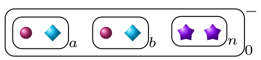
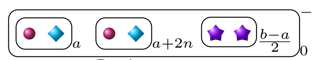
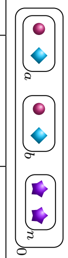


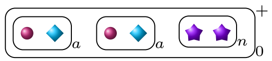
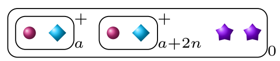
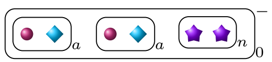
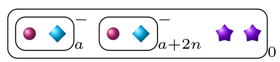
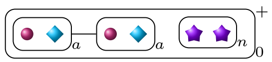
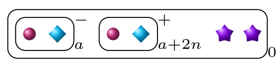
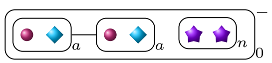
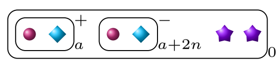
Table 4.28: C is of type $U_{2a,2b,2n}$, U4

Isogeny	C	$v(c)$	$\begin{cases} v(\ell) \\ v(\ell_1) \end{cases}$	$v(\Delta_G^2)$	c	δ_1, δ_2	$\hat{C}$	λ_v	w_v	E_v
U4A $a < b$	 $c_v = N, \quad m_v = 1$	0	$\begin{cases} \equiv 0 \ (2) \\ - \end{cases}$	$2a$	$\in K^{\times 2}$	$\in K^{\times 2}$	 $\hat{c}_v = N, \quad \hat{m}_v = 1$	1	1	1
U4B $a < b$	 $c_v = \widetilde{N/D} \cdot \widetilde{D}, \quad m_v = 1$	0	$\begin{cases} \equiv 0 \ (2) \\ - \end{cases}$	$2a$	$\notin K^{\times 2}$	$\in K^{\times 2}$	 $\hat{c}_v = \widetilde{N/\hat{D}} \cdot \widetilde{\hat{D}}, \quad \hat{m}_v = 1$	(-1)$^{2n+a+b}$	1	(-1)$^{2n+a+b}$



Notation: $\tilde{x} = 2$ if $2|x$ and $\tilde{x} = 1$ if $2 \nmid x$,
$D = gcd(2a, 2b, 2n)$, $N = 4ab + 4an + 4bn$, $\hat{D} = gcd(2a, b - a)$.

Table 4.29: C is of type $U_{2a,2b,2n}$, U4

Isogeny	C	$v(c)$	$\begin{cases} v(\ell) \\ v(\ell_1) \end{cases}$	$v(\Delta_G^2)$	c	δ_1, δ_2	$\hat{C}$	λ_v	w_v	E_v
U4C	 $c_v = N, \quad m_v = 1$	0	$\begin{cases} \equiv 0 \ (2) \\ - \end{cases}$	$2a$	$\in K^{\times 2}$	$\in K^{\times 2}$	 $\hat{c}_v = N, \quad \hat{m}_v = 1$	1	1	1
U4D	 $c_v = \widetilde{N/D} \cdot \widetilde{D}, \quad m_v = 1$	0	$\begin{cases} \equiv 0 \ (2) \\ - \end{cases}$	$2a$	$\notin K^{\times 2}$	$\in K^{\times 2}$	 $\hat{c}_v = 4, \quad \hat{m}_v = 1$	$(-1)^{2n}$	1	$(-1)^{2n}$
U4E	 $c_v = 2a + 4n, \quad m_v = 1$	0	$\begin{cases} \equiv 0 \ (2) \\ - \end{cases}$	$2a$	$\in K^{\times 2}$	$\notin K^{\times 2}$	 $\hat{c}_v = 2(2a + 4n), \quad \hat{m}_v = 1$	-1	-1	1
U4F	 $c_v = 2a, \quad m_v = \widetilde{2n+1}$	0	$\begin{cases} \equiv 0 \ (2) \\ - \end{cases}$	$2a$	$\notin K^{\times 2}$	$\notin K^{\times 2}$	 $\hat{c}_v = 2a, \quad \hat{m}_v = 1$	$(-1)^{2n+1}$	-1	$(-1)^{2n}$

Notation: $\tilde{x} = 2$ if $2|x$ and $\tilde{x} = 1$ if $2 \nmid x$,
$D = gcd(2a, 2n)$, $N = 4a^2 + 8an$.

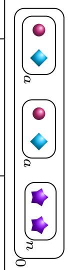
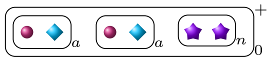
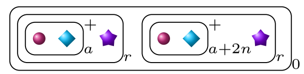
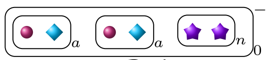
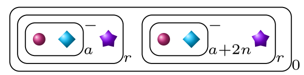
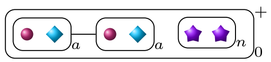
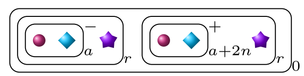
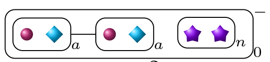
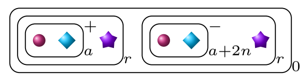
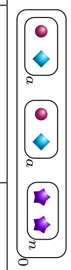


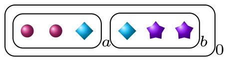
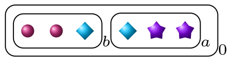
Table 4.30: C is of type $U_{2a,2b,2n}$, U4

Isogeny	C	$v(c)$	$\begin{cases} v(\ell) \\ v(\ell_1) \end{cases}$	$v(\Delta_G^2)$	c	δ_1, δ_2	$\hat{C}$	λ_v	w_v	E_v
U4G	 $c_v = N, \quad m_v = 1$	0	$\begin{cases} \equiv r \ (2) \\ -r \end{cases}$	$2a+2r$ $r>0$	$\in K^{\times 2}$	$\in K^{\times 2}$	 $\hat{c}_v = N, \quad \hat{m}_v = 1$	1	1	1
U4H	 $c_v = \widetilde{N/D} \cdot \widetilde{D}$, $m_v = \widetilde{2n+1}$	0	$\begin{cases} \equiv r \ (2) \\ -r \end{cases}$	$2a+2r$ $r>0$	$\notin K^{\times 2}$	$\in K^{\times 2}$	 $\hat{c}_v = 4, \quad \hat{m}_v = 1$	$(-1)^{2n}$	1	$(-1)^{2n}$
U4I	 $c_v = 2a+4n, \quad m_v = 1$	0	$\begin{cases} \equiv r \ (2) \\ -r \end{cases}$	$2a+2r$ $r>0$	$\in K^{\times 2}$	$\notin K^{\times 2}$	 $\hat{c}_v = 2(2a+4n), \quad \hat{m}_v = 1$	-1	-1	1
U4J	 $c_v = 2a$, $m_v = \widetilde{2n+1}$	0	$\begin{cases} \equiv r \ (2) \\ -r \end{cases}$	$2a+2r$ $r>0$	$\notin K^{\times 2}$	$\notin K^{\times 2}$	 $\hat{c}_v = 2a, \quad \hat{m}_v = 1$	$(-1)^{2n+1}$	-1	$(-1)^{2n}$



Notation: $\tilde{x} = 2$ if $2|x$ and $\tilde{x} = 1$ if $2 \nmid x$,
$D = gcd(2a, 2n)$, $N = 4a^2 + 8an$.

Table 4.31: C is of type 1×1, TC1

Isogeny	C	$v(c)$	δ_1	$\begin{cases} v(\ell) \\ v(\ell_1) \end{cases}$	$v(\Delta_G^2)$	$(A_1 - B_1)^2$	$\hat{C}$	λ_v	w_v	E_v
TC1	 $c_v = 1, \quad m_v = 1$	$\equiv a \equiv b\ (2)$	–	$\begin{cases} v(c) \\ -v(c) \end{cases}$	$2v(c)$	–	 $\hat{c}_v = 1, \quad \hat{m}_v = 1$	1	1	1

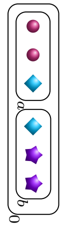
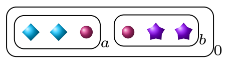
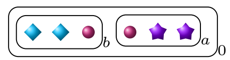
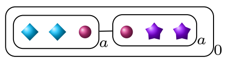
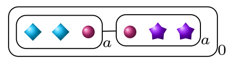


Table 4.32: C is of type 1×1, TC2

Isogeny	C	$v(c)$	δ_1	$\begin{cases} v(\ell) \\ v(\ell_1) \end{cases}$	$v(\Delta_G^2)$	$(A_1 - B_1)^2$	$\hat{C}$	λ_v	w_v	E_v
TC2A	 $c_v = 1, \quad m_v = 1$	$\equiv a \equiv b\ (2)$	$\in K^{\times 2}$	$\begin{cases} v(c) \\ -v(c) \end{cases}$	$2v(c)$	$\in K^{\times 2}$	 $\hat{c}_v = 1, \quad \hat{m}_v = 1$	1	1	1
TC2B	 $c_v = 1, \quad m_v = \widetilde{a+1}$	$\equiv a\ (2)$	$\notin K^{\times 2}$	$\begin{cases} v(c) \\ -v(c) \end{cases}$	$2v(c)$	$\notin K^{\times 2}$	 $\hat{c}_v = 1, \quad \hat{m}_v = \widetilde{a+1}$	1	1	1

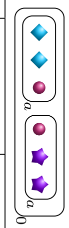
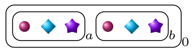
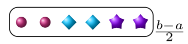
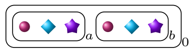
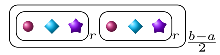
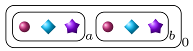
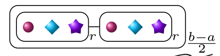
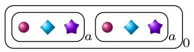
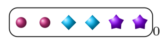
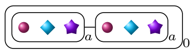
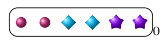


Table 4.33: C is of type 1×1, TC3

Isogeny	C	$v(c)$	δ_1	$\begin{cases} v(\ell) \\ v(\ell_1) \end{cases}$	$v(\Delta_G^2)$	$(A_1 - B_1)^2$	$\hat{C}$	λ_v	w_v	E_v
TC3A $a < b$	 $c_v = 1, \quad m_v = 1$	$\equiv a \equiv b\ (2)$	$\in K^{\times 2}$	$\begin{cases} v(c)+2a-b \\ -b-v(c) \end{cases}$	$2(a+b+v(c))$	–	 $\hat{c}_v = 1, \quad \hat{m}_v = 1$	1	1	1
TC3B $a < b$	 $c_v = 1, \quad m_v = 1$	$\equiv a \equiv b\ (2)$	$\in K^{\times 2}$	$\begin{cases} v(c)+2a-b-r \\ -b-v(c)-r \end{cases}$	$2(a+b+v(c)+r)$ $r > 0$	$\in K^{\times 2}$	 $\hat{c}_v = 1, \quad \hat{m}_v = 1$	1	1	1
TC3C $a < b$	 $c_v = 1, \quad m_v = 1$	$\equiv a \equiv b\ (2)$	$\in K^{\times 2}$	$\begin{cases} v(c)+2a-b-r \\ -b-v(c)-r \end{cases}$	$2(a+b+v(c)+r)$ $r > 0$	$\notin K^{\times 2}$	 $\hat{c}_v = 1, \quad \hat{m}_v = \widetilde{r+1}$	(-1)^r	1	(-1)^r
TC3D	 $c_v = 1, \quad m_v = 1$	$\equiv a\ (2)$	$\in K^{\times 2}$	$\begin{cases} \equiv 0\ (2) \\ v(u_1)-2a-v(c) \end{cases}$	$2(2a+v(c))$	–	 $\hat{c}_v = 1, \quad \hat{m}_v = 1$	1	1	1
TC3E	 $c_v = 1, \quad m_v = \widetilde{a+1}$	$\equiv a\ (2)$	$\notin K^{\times 2}$	$\begin{cases} \equiv 0\ (2) \\ v(u_1)-2a-v(c) \end{cases}$	$2(2a+v(c))$	–	 $\hat{c}_v = 1, \quad \hat{m}_v = 1$	(-1)^a	1	(-1)^a

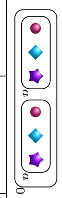
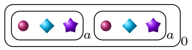
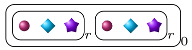
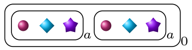
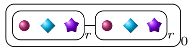
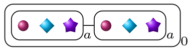
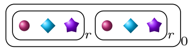
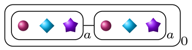
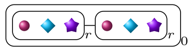
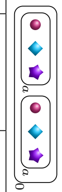


Table 4.34: C is of type 1×1, TC3

Isogeny	C	$v(c)$	δ_1	$\begin{cases} v(\ell) \\ v(\ell_1) \end{cases}$	$v(\Delta_G^2)$	$(A_1{-}B_1)^2$	$\hat{C}$	λ_v	w_v	E_v
TC3F	 $c_v = 1, \quad m_v = 1$	$\equiv a\ (2)$	$\in K^{\times 2}$	$\begin{cases} \equiv r\ (2) \\ v(u_1){-}2a{-}v(c){-}r \end{cases}$	$2(2a{+}v(c){+}r)$ $r > 0$	$\in K^{\times 2}$	 $\hat{c}_v = 1, \quad \hat{m}_v = 1$	1	1	1
TC3G	 $c_v = 1, \quad m_v = 1$	$\equiv a\ (2)$	$\in K^{\times 2}$	$\begin{cases} \equiv r\ (2) \\ v(u_1){-}2a{-}v(c){-}r \end{cases}$	$2(2a{+}v(c){+}r)$ $r > 0$	$\notin K^{\times 2}$	 $\hat{c}_v = 1, \quad \hat{m}_v = \widetilde{r+1}$	$(-1)^r$	1	$(-1)^r$
TC3H	 $c_v = 1, \quad m_v = \widetilde{a+1}$	$\equiv a\ (2)$	$\notin K^{\times 2}$	$\begin{cases} \equiv r\ (2) \\ v(u_1){-}2a{-}v(c){-}r \end{cases}$	$2(2a{+}v(c){+}r)$ $r > 0$	$\in K^{\times 2}$	 $\hat{c}_v = 1, \quad \hat{m}_v = 1$	$(-1)^a$	1	$(-1)^a$
TC3I	 $c_v = 1, \quad m_v = \widetilde{a+1}$	$\equiv a\ (2)$	$\notin K^{\times 2}$	$\begin{cases} \equiv r\ (2) \\ v(u_1){-}2a{-}v(c){-}r \end{cases}$	$2(2a{+}v(c){+}r)$ $r > 0$	$\notin K^{\times 2}$	 $\hat{c}_v = 1, \quad \hat{m}_v = \widetilde{r+1}$	$(-1)^{a+r}$	1	$(-1)^{a+r}$



Chapter 5

Richelot isogeny in odd residue characteristic

5.1 Introduction

In this chapter, we prove the results presented in Tables 4.4 to 4.34. Each of these tables takes a cluster picture for C with extra local data as an input and displays the cluster picture of $\hat{C}$ together with the Tamagawa numbers of J and $\hat{J}$, the deficiency of C and $\hat{C}$ as well as λ_v and ω_v as an output (it also displays E_v which is treated in Chapter 6). Table 3.1 in Section 3.4 computes Tamagawa numbers, deficiency and root numbers for semistable curves (Jacobians) of genus 2 from their cluster pictures. Therefore it remains to compute the cluster picture of $\hat{C}$ to obtain λ_v in each case. As introduced in Section 2.2, the Richelot construction is entirely explicit. In particular, we can compute the valuations of differences of the roots of $\hat{C}$, and hence the cluster picture of $\hat{C}$, from the valuations of the differences of the roots of $G(x)$ (that is from the cluster picture of C). We first present the properties of the Richelot construction that allow us to do such computations before presenting all computations case by case.

List of notation for this chapter

Henceforth addition of indices is performed modulo 3.

K	local field with odd residue characteristic
π	uniformiser of K
v	normalized valuation of K
$G_1(x)$	$c(x-\alpha_1)(x+\alpha_1)$ with $\alpha_1^2 \in K$
$G_i(x), i=2,3$	$(x-\alpha_i)(x-\beta_i)$
$G(x)$	$G_1(x)G_2(x)G_3(x)$
L	Splitting field of $G(x)$
c	leading term of $G(x)$
C	$C_2 \times D_4$ curve defined over K with the given factorization $y^2 = G(x) = G_1(x)G_2(x)G_3(x)$
J	Jacobian of C
ϕ	Richelot isogeny on J given by the factorization of $G(x)$
$L_1(x)$	$L_1(x) = \frac{1}{\Delta_G}[G_2(x), G_3(x)] = \ell_1(x-A_1)(x-B_1)$
$\ell_1 = \frac{u_1}{\Delta_G}$	leading term of $L_1(x)$ with $u_1 = \alpha_2 + \beta_2 - \alpha_3 - \beta_3$
$L_i(x), i=2,3$	$L_i(x) = [G_{i+1}(x), G_{i+2}(x)] = \ell_i(x-A_i)(x-B_i)$
$L(x)$	$L_1(x)L_2(x)L_3(x)$
$\ell = \ell_1\ell_2\ell_3$	leading term of $L(x)$
$\hat{C}$	$C_2 \times D_4$ dual curve of C defined over K by $y^2 = L(x) = L_1(x)L_2(x)L_3(x)$
$\hat{J}$	Jacobian of $\hat{C}$
$\delta_i,\ i=1,2,3$	discriminant of $G_i(x)$, $\delta_1 = 4c^2\alpha_1^2, \delta_2 = (\alpha_2-\beta_2)^2, \delta_3 = (\alpha_3 - b_3)^2$
$\hat{\delta}_i,\ i=1,2,3$	discriminant of $L_i(x)$, $\hat{\delta}_i = \ell_i^2(A_i - B_i)^2$
$\alpha_i, \beta_i,\ i=1,2,3$	roots of $G_i(x)$
$A_i, B_i,\ i=1,2,3$	roots of $L_i(x)$
$x \equiv y$	$x \equiv y \bmod \pi$
$x \equiv_\square y$	$x \equiv yz$ where z is a square element in $K^\times$
$x =_\square y$	$x = yz$ where z is a square element in $K^\times$
$Frob$	Frobenius automorphism in $Gal(\bar{K}/K)$
I_K	inertia subgroup of $Gal(\overline{K}/K)$
● ●	α_1, β_1 in the cluster picture of C (A_1, B_1 is that of $\hat{C}$)
◆ ◆	α_2, β_2 in the cluster picture of C (A_2, B_2 is that of $\hat{C}$)
★ ★	α_3, β_3 in the cluster picture of C (A_3, B_3 is that of $\hat{C}$)

5.2 Algebraic identities of a $C_2 \times D_4$ Richelot curve

Proposition 5.2.1. *Let $G(x)$ be a $C_2 \times D_4$ polynomial and $L(x)$ the defining polynomial of its associated $C_2 \times D_4$ dual curve as in Definition 4.2.5. Then for $i = 2, 3$,*

i) $\Delta_L = -2\Delta_G$,

ii) $\hat{\delta_1} = \ell_1^2(A_1 - B_1)^2 = \dfrac{4}{\Delta_G^2}(\alpha_2 - \beta_3)(\alpha_2 - \alpha_3)(\beta_2 - \alpha_3)(\beta_2 - \beta_3)$,

ii) $\hat{\delta_i} = \ell_i^2(A_i - B_i)^2 = 4c^2(\alpha_{i+1} - \beta_{i+2})(\alpha_{i+1} - \alpha_{i+2})(\beta_{i+1} - \alpha_{i+2})(\beta_{i+1} - \beta_{i+2})$,

and the discriminants of $G(x)$ and $L(x)$ are given by

$$iii)\ Disc(G(x)) = \frac{1}{2^{12}}\delta_1\delta_2\delta_3\hat{\delta_1}^2\hat{\delta_2}^2\hat{\delta_3}^2,$$
$$iv)\ Disc(L(x)) = \Delta_G^{12}\delta_1^2\delta_2^2\delta_3^2\hat{\delta_1}\hat{\delta_2}\hat{\delta_3}.$$

Proof. Follow from direct computations. □

Proposition 5.2.2. *Let $G(x), L(x) \in \mathcal{K}[x]$ be as in Proposition 5.2.1 above. Then applying the $C_2 \times D_4$ construction to $L(x)$ as in Definition 4.2.5 gives*

$$H(x) = \prod_{i=1}^{3} H_i(x), \qquad H_i(x) = -2\Delta_G G_i(x), \quad i{=}1{,}2{,}3.$$

In particular $H(x) = -8\Delta_G^3 G(x)$.

Proof. Clear from computation. □

Corollary 5.2.3. *The discriminants $\delta_1, \delta_2, \delta_3$ associated to the Richelot factorization of $G(x)$ satisfy for $i = 2, 3$*

$$\begin{aligned}
\Delta_G^2\delta_1 &= \Delta_G^2 c^2(\alpha_1 - \beta_1)^2 \\
&= \ell_2^2\ell_3^2(A_2 - B_3)(A_2 - A_3)(B_2 - B_3)(B_2 - B_3) \\
\Delta_G^2\delta_i &= \Delta_G^2(\alpha_i - \beta_i)^2 \\
&= \ell_{i+1}^2\ell_{i+2}^2(A_{i+1} - B_{i+2})(A_{i+1} - A_{i+2})(B_{i+1} - B_{i+2})(B_{i+1} - B_{i+2}).
\end{aligned}$$

Proof. It follows from Proposition 5.2.2 that for $i = 1, 2, 3$, the discriminants of $H_i(X)$ are equal to $4\Delta_G^2\delta_i$. Now the result follows from applying Proposition 5.2.1 to $L(x)$ and $H(x)$. □

Proposition 5.2.4. *The $C_2 \times D_4$ dual curve of $\hat{C}$ is isomorphic to C and is given by*

$$\hat{\hat{C}}\ :\ y^2 = 4G(x).$$

Proof. We have that $\hat{C} : y^2 = L(x)$. It suffices to apply the Richelot construction to $L(x) = L_1(x)L_2(x)L_3(x)$ to get
$[L_1(x), L_2(x)] = -2G_3(x), \quad [L_1(x), L_3(x)] = -2G_2(x), \quad [L_2(x), L_3(x)] = -2\Delta_G G_1(x)$, and $\Delta_L = -2\Delta_G$. Now, by definition $\hat{\hat{C}} : \Delta_L\, y^2 = -8\Delta_G G(x)$, so that $\hat{\hat{C}} : y^2 = 4G(x)$, as required. □

The following results follow either by definition, direct computations or properties given above.

Proposition 5.2.5. *Keeping notation for the roots of $G(x)$ and $L(x)$ as in Section 2.2, we have*

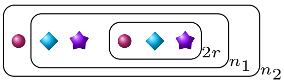
1. $\delta_1 = 4c^2\alpha_1^2, \quad \delta_2 = (\alpha_2 - \beta_2)^2, \quad \delta_3 = (\alpha_3 - \beta_3)^2$,
2. $\Delta_G = -c(\alpha_1^2(\alpha_2 + \beta_2 - \alpha_3 - \beta_3) + \alpha_2\beta_2(\alpha_3 + \beta_3)$
   $+\alpha_3\beta_3(-\alpha_2 - \beta_2))$,
3. $\ell_1 = \frac{u_1}{\Delta_G} = \frac{\alpha_2+\beta_2-\alpha_3-\beta_3}{\Delta_G}, \quad \ell_2 = c(\alpha_3 + \beta_3), \quad \ell_3 = c(-\alpha_2 - \beta_2)$,
4. $\hat{\delta_1} = \ell_1^2(A_1 - B_1)^2 = \frac{4}{\Delta_G^2}(\alpha_2 - \alpha_3)(\alpha_2 - \beta_3)(\beta_2 - \alpha_3)(\beta_2 - \beta_3)$,
5. $\hat{\delta_2} = \ell_2^2(A_2 - B_2)^2 = 4c^2(\alpha_3 - \alpha_1)(\alpha_3 + \alpha_1)(\beta_3 - \alpha_1)(\beta_3 + \alpha_1)$,
6. $\hat{\delta_3} = \ell_3^2(A_3 - B_3)^2 = 4c^2(\alpha_1 - \alpha_2)(\alpha_1 - \beta_2)(-\alpha_1 - \alpha_2)(-\alpha_1 - \beta_2)$,
7. $\Delta_G^2\delta_1 = \ell_2^2\ell_3^2(A_2 - A_3)(A_2 - B_3)(B_2 - A_3)(B_2 - B_3)$,
8. $\delta_2 = \ell_3^2\ell_1^2(A_3 - A_1)(A_3 - B_1)(B_3 - A_1)(B_3 - B_1)$,
9. $\delta_3 = \ell_1^2\ell_2^2(A_1 - A_2)(A_1 - B_2)(B_1 - A_2)(B_1 - B_2)$.
10. $Disc(G(x)) = \frac{\Delta_G^4}{2^{12}}\delta_1\delta_2\delta_3\hat{\delta_1}^2\hat{\delta_2}^2\hat{\delta_3}^2$,
11. $Disc(L(x)) = \Delta_G^4\delta_1^2\delta_2^2\delta_3^2\hat{\delta_1}\hat{\delta_2}\hat{\delta_3}$,
12. $cu_1 + \ell_2 + \ell_3 = 0$.

Remark 5.2.6. $\Delta_G \neq 0$ by definition of a Richelot curve (see Definition 2.2.12),

We will use extensively Proposition 5.2.5 in the proofs below. Therefore we will use P.k to refer to Propostion 5.2.5.k.

5.3 Proof of Tables 4.4 to 4.34

Remark 5.3.7. 1) In the following proofs, we do not include formal computations that can be readily verified using a computer algebra. All of these computations have been performed using Maple and can be made available if needed.
2) In the cluster pictures displayed in the following section, we denote the depth of a cluster relatively to that of its parent. For instance, in the cluster picture



, we have $v(\alpha_1-\beta_1) = v(\alpha_1-\alpha_2) = v(\alpha_1-\beta_2) = v(\alpha_1-\alpha_3) = v(\alpha_1-\beta_3) = n_2$, $v(\alpha_2-\beta_1) = v(\alpha_2-\beta_2) = v(\alpha_2-\beta_3) = v(\alpha_3-\beta_1) = v(\alpha_3-\beta_2) = v(\alpha_3-\beta_3) = n_1+n_2$ and $v(\beta_1-\beta_2) = v(\beta_1-\beta_3) = v(\beta_2-\beta_3) = 2r+n_1+n_2$.

Lemma 5.3.8. *If C is of type 2, 1_n, $I_{a,b}$ or $U_{a,b,n}$ as in Table 3.1 then $v(c) \in 2\mathbb{Z}$.*

Proof. Clear from the semistability criterion 3.4.29. □

Corollary 5.3.9. *If C is type 2, 1_n, $I_{a,b}$ or $U_{a,b,n}$ as in Table 3.1 then without loss of generality $v(c) = 0$. If C is of type $1 \times_{\frac{n+m}{2}} 1$ then without loss of generality $v(c) = 0$ or $v(c) = 1$.*

Proof. The first case follows from Lemma 5.3.8. If C is of type $1 \times_{\frac{n+m}{2}} 1$ then by semistability criterion $v(c) \equiv n \equiv m \mod 2$. □

5.3.1 C is of type 2

Lemma 5.3.10. *Suppose that C is of type 2, i.e. its given Weiertrass model has unit discriminant. Then the same hold for $\hat{C}$ if and only if $v(\Delta_G^2) = 0$.*

Proof. By Corollary 5.3.9, we have $v(c) = 0$. Since C is of type 2 we have that $v(Disc(G(x)) = 0$, and by P.10

$$v(\delta_i) = 0, \quad \forall i = 1, 2, 3, \quad v(\Delta_G^2 \hat{\delta_1}) = v(\hat{\delta_2}) = v(\hat{\delta_3}) = 0.$$

It follows from P.11 that $v(Disc(L(x)) = 0$ if and only if $v(\Delta_G^2) = 0$. In particular, $\hat{C}$ has good reduction if and only if $v(\Delta_G^2) = 0$. □

Proof of Table 4.4

By Corollary 5.3.9 we have $v(c) = 0$. We note that since J has good reduction, it follows that $\hat{J}$ also has good reduction. Therefore, if $\hat{C}$ has bad reduction, it is of type $1 \times_n 1$ or $1 \tilde{\times}_n 1$ for some $n \neq 0$.
Case GR1A. Here $v(\Delta_G^2) = 0$, therefore $\hat{C}$ is of type 2 by Lemma 5.3.10. From

P.4, P.5 and P.6 we have $v(A_1 - B_1)^2 = -2v(\Delta_G) - 2v(\ell_1) = -2v(u_1)$,

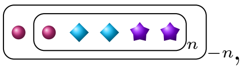
$$v(A_2 - B_2)^2 = -2v(\ell_2), \quad v(A_3 - B_3)^2 = -2v(\ell_3),$$

and P.7, P.8 and P.9 give

$$v((A_2-A_3)(A_2-B_3)(B_2-A_3)(B_2-B_3)) = 2v(\Delta_G)-2v(\ell_2)-2v(\ell_3) = 2v(\ell_2)-2v(\ell_3),$$

$$v((A_3 - A_1)(A_3 - B_1)(B_3 - A_1)(B_3 - B_1)) = -2v(\ell_1) - 2v(\ell_3) = -2v(u_1) - 2v(\ell_3),$$

$$v((A_1 - A_2)(A_1 - B_2)(B_1 - A_2)(B_1 - B_2)) = -2v(\ell_1) - 2v(\ell_2) = -2v(u_1) - 2v(\ell_2).$$

If $v(\ell) = 0$ then the cluster picture of $L(x)$ is the balanced one given in Table 4.4. Otherwise if $v(\ell) \neq 0$ then $v(\ell_i) = 0$ for some $i = 1, 2, 3$, for otherwise P.2 and P.12 would yield $v(\Delta_G^2) > 0$. Assume without loss of generality that $v(\ell_1) = n \neq 0$. Then $n \in 2\mathbb{Z}$ by semistability criterion 3.4.29 and the cluster picture for $L(x)$ is 

, which is equivalent to the claimed balanced cluster picture as shown in Table 3.2.

Cases GR1B/GR1C. Since $v(\Delta_G^2) = 2r > 0$, $\hat{C}$ has bad reduction by Lemma 5.3.10. From P.4, P.5 and P.6 we have $v(A_1 - B_1)^2 = -2v(\Delta_G) - 2v(\ell_1) = -2v(u_1)$,

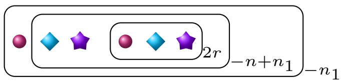
$$v(A_2 - B_2)^2 = -2v(\ell_2), \quad v(A_3 - B_3)^2 = -2v(\ell_3),$$

and P.7, P.8 and P.9 give

$$v((A_2-A_3)(A_2-B_3)(B_2-A_3)(B_2-B_3)) = 2v(\Delta_G)-2v(\ell_2)-2v(\ell_3) = 2r-2v(\ell_2)-2v(\ell_3),$$

$$v((A_3 - A_1)(A_3 - B_1)(B_3 - A_1)(B_3 - B_1)) = -2v(\ell_1) - 2v(\ell_3) = 2r - 2v(u_1) - 2v(\ell_3),$$

$$v((A_1 - A_2)(A_1 - B_2)(B_1 - A_2)(B_1 - B_2)) = -2v(\ell_1) - 2v(\ell_2) = 2r - 2v(u_1) - 2v(\ell_2).$$

If $v(u_1) = v(\ell_2) = v(\ell_3) = 0$ then the cluster picture for $\hat{C}$ is that of Table 4.4. Otherwise, if $v(u_1) = n_1 > 0$ then by P.2 and P.12, we have $v(\ell_2) = v(\ell_3)$, say $v(\ell_2) = n$. We obtain the following cluster picture for $\hat{C}$ 

, which is equivalent to the claimed balanced cluster picture as shown in Table 3.2. Similarly if $v(\ell_2), v(\ell_3) > 0$.

Frobenius action. If $(A_1 - B_1)^2 \notin K^{\times 2}$, then $Frob$ permutes A_1 and B_1 and hence permutes both clusters yielding the required automorphism.

5.3.2 C is of type 1_{2a}

Proof of Table 4.5

Case ON1A/B. Here $v(\Delta_G^2) = 0$, and

$$v(\delta_1) = 2a, v(\delta_2) = 0, v(\delta_3) = 0, v(\hat{\delta_1}) = 0, v(\hat{\delta_2}) = 0, v(\hat{\delta_3}) = 0,$$

so that P.4, P.5 and P.6 yield

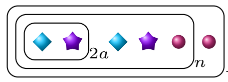
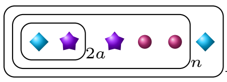
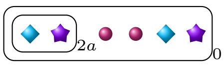
$$v(A_1 - B_1)^2 = -2v(\ell_1), \quad v(A_2 - B_2)^2 = -2v(\ell_2), \quad v(A_3 - B_3)^2 = -2v(\ell_3),$$

and P.7, P.8 and P.9 give

$$v((A_2 - A_3)(A_2 - B_3)(B_2 - A_3)(B_2 - B_3)) = 2a - 2v(\ell_2) - 2v(\ell_3),$$

$$v((A_3 - A_1)(A_3 - B_1)(B_3 - A_1)(B_3 - B_1)) = -2v(\ell_1) - 2v(\ell_3),$$

$$v((A_1 - A_2)(A_1 - B_2)(B_1 - A_2)(B_1 - B_2)) = -2v(\ell_1) - 2v(\ell_2).$$

If $v(\ell_1) = v(\ell_2) = v(\ell_3) = 0$ then without loss of generality, let $v(A_2 - A_3)$=2a so that the cluster picture of $\hat{C}$ is that of Table 4.5 for ON1A/B. Otherwise, since $v(\Delta_G^2) = 0$, either $v(\ell_1) = n > 0$ and $v(\ell_2) = v(\ell_3) = 0$ or without loss of generality, $v(\ell_2) = n > 0$ and $v(\ell_1) = v(\ell_3) = 0$. Then in the first case, one readily verifies using the valuations above that the cluster picture for $\hat{C}$ is $_{-n}$. And similarly, in the second case if $v(\ell_2) = n > 0$ then the cluster picture for $\hat{C}$ is $_{-n}$. In both cases, these cluster pictures are in the equivalence class of $_0$ as required.

Case ON1C/D. Here $v(\Delta_G^2) = 2r > 0$ so that $v(\ell_1) = v(u_1) - r$, and

$$v(\delta_1) = 2a, v(\delta_2) = 0, v(\delta_3) = 0, v(\hat{\delta_1}) = 0, v(\hat{\delta_2}) = 0, v(\hat{\delta_3}) = 0,$$

so that P.4, P.5 and P.6 yield

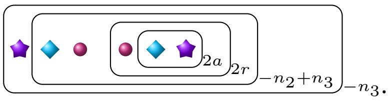
$$v(A_1 - B_1)^2 = -2v(u_1), \quad v(A_2 - B_2)^2 = -2v(\ell_2), \quad v(A_3 - B_3)^2 = -2v(\ell_3),$$

P.7, P.8 and P.9 give

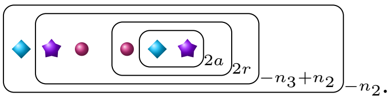
$$v((A_2 - A_3)(A_2 - B_3)(B_2 - A_3)(B_2 - B_3)) = 2a + 2r - 2v(\ell_2) - 2v(\ell_3),$$

$$v((A_3 - A_1)(A_3 - B_1)(B_3 - A_1)(B_3 - B_1)) = 2r - 2v(u_1) - 2v(\ell_3),$$

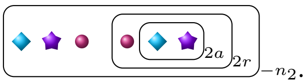
$$v((A_1 - A_2)(A_1 - B_2)(B_1 - A_2)(B_1 - B_2)) = 2r - 2v(u_1) - 2v(\ell_2).$$

If $v(u_1) = v(\ell_2) = v(\ell_3) = 0$ then without loss of generality, let $v(A_2 - A_3)$=2a so that the cluster picture of $\hat{C}$ is that of Table 4.5 for ON1C/D. Otherwise, note that since $v(\Delta_G^2) > 0$, we have $v(u_1), v(\ell_2), v(\ell_3) > 0$ (for otherwise, this forces others double roots $\mod \pi$ a contradiction to the cluster picture of C).
Write $v(u_1) = n_1, v(\ell_2) = n_2, v(\ell_3) = n_3$. Assume that $n_2 < n_3$, then $r = n_1 = n_2$ and the cluster picture for $\hat{C}$ is 

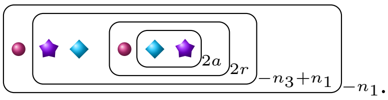
Assume that $n_3 < n_2$, then $r = n_1 = n_3$ and the cluster picture for $\hat{C}$ is

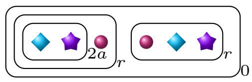


Assume that $n_3 = n_2$ and $r = n_1 = n_3$. Then the cluster picture for $\hat{C}$ is



Assume that $n_3 = n_2$ and $n_1 > n_3 = r$. Then the cluster picture for $\hat{C}$ is



In all cases, the cluster pictures obtained are in the equivalence class of 

as required.
Frobenius action. By Proposition 3.4.30, the reduction at the node at α_1 is split if and only if $T_{\alpha_1} = c(\alpha_1 - \alpha_2)(\alpha_1 - \beta_2)(\alpha_1 - \alpha_3)(\alpha_1 - \beta_3) \equiv_\square 1$.

Proof of Table 4.6

Case ON2A/B. Here $v(\Delta_G^2) = 0$, and

$$v(\delta_1) = 0, v(\delta_2) = 2a, v(\delta_3) = 0, v(\hat{\delta_1}) = 0, v(\hat{\delta_2}) = 0, v(\hat{\delta_3}) = 0,$$

so that P.4, P.5 and P.6 yield

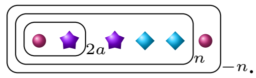
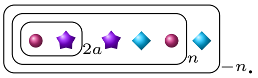
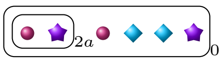
$$v(A_1 - B_1)^2 = -2v(\ell_1), \quad v(A_2 - B_2)^2 = -2v(\ell_2), \quad v(A_3 - B_3)^2 = -2v(\ell_3),$$

and P.7, P.8 and P.9 give

$$v((A_2 - A_3)(A_2 - B_3)(B_2 - A_3)(B_2 - B_3)) = -2v(\ell_2) - 2v(\ell_3),$$

$$v((A_3 - A_1)(A_3 - B_1)(B_3 - A_1)(B_3 - B_1)) = 2a - 2v(\ell_1) - 2v(\ell_3),$$

$$v((A_1 - A_2)(A_1 - B_2)(B_1 - A_2)(B_1 - B_2)) = -2v(\ell_1) - 2v(\ell_2).$$

If $v(\ell_1) = v(\ell_2) = v(\ell_3) = 0$ then without loss of generality, let $v(A_3 - A_1)$=2a so that the cluster picture of $\hat{C}$ is that of Table 4.6 for ON2A/B. Otherwise, since $v(\Delta_G^2) = 0$, either $v(\ell_1) = n > 0$ and $v(\ell_2) = v(\ell_3) = 0$ or without loss of generality, $v(\ell_2) = n > 0$ and $v(\ell_1) = v(\ell_3) = 0$. Then in the first case, one readily verifies using the valuations above that the cluster picture for $\hat{C}$ is  And similarly, in the second case if $v(\ell_2) = n > 0$ then the cluster picture for $\hat{C}$ is  In both cases, these cluster pictures are in the equivalence class of  as required.

<u>**Case ON2C/D**</u>. Here $v(\Delta_G^2) = 2r > 0$ so that $v(\ell_1) = v(u_1) - r$, and

$$v(\delta_1) = 0, v(\delta_2) = 2a, v(\delta_3) = 0, v(\hat{\delta_1}) = 0, v(\hat{\delta_2}) = 0, v(\hat{\delta_3}) = 0,$$

so that P.4, P.5 and P.6 yield

$$v(A_1 - B_1)^2 = -2v(u_1), \quad v(A_2 - B_2)^2 = -2v(\ell_2), \quad v(A_3 - B_3)^2 = -2v(\ell_3),$$

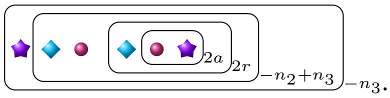
P.7, P.8 and P.9 give

$$v((A_2 - A_3)(A_2 - B_3)(B_2 - A_3)(B_2 - B_3)) = 2r - 2v(\ell_2) - 2v(\ell_3),$$

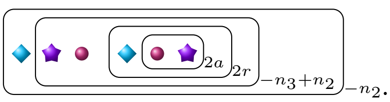
$$v((A_3 - A_1)(A_3 - B_1)(B_3 - A_1)(B_3 - B_1)) = 2a + 2r - 2v(u_1) - 2v(\ell_3),$$

$$v((A_1 - A_2)(A_1 - B_2)(B_1 - A_2)(B_1 - B_2)) = 2r - 2v(u_1) - 2v(\ell_2).$$

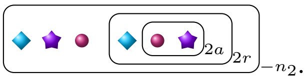
If $v(u_1) = v(\ell_2) = v(\ell_3) = 0$ then without loss of generality, let $v(A_3 - A_1)$=2a so that the cluster picture of $\hat{C}$ is that of Table 4.6 for ON2C/D. Otherwise, note that since $v(\Delta_G^2) > 0$, we have $v(u_1), v(\ell_2), v(\ell_3) > 0$ (for otherwise, this forces others double roots $\mod \pi$ a contradiction to the cluster picture of C).

Write $v(u_1) = n_1, v(\ell_2) = n_2, v(\ell_3) = n_3$. Assume that $n_2 < n_3$, then $r = n_1 = n_2$ and the cluster picture for $\hat{C}$ is 

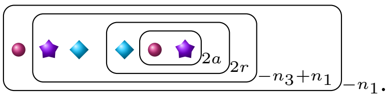
Assume that $n_3 < n_2$, then $r = n_1 = n_3$ and the cluster picture for $\hat{C}$ is



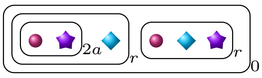
Assume that $n_3 = n_2$ and $r = n_1 = n_3$. Then the cluster picture for $\hat{C}$ is



Assume that $n_3 = n_2$ and $n_1 > n_3 = r$. Then the cluster picture for $\hat{C}$ is



In all cases the cluster pictures obtained are in the equivalence class of



as required.

<u>**Frobenius action**</u>. By Proposition 3.4.30, the reduction at the node at α_2 is split if and only if $T_{\alpha_2} = c(\alpha_2 - \alpha_1)(\alpha_2 + \alpha_1)(\alpha_2 - \alpha_3)(\alpha_2 - \beta_3) \equiv_{\square} 1$.

Proof of Table 4.7

<u>**Case ON3A/B**</u>. Computing ℓ_1 we find that $v(\ell_1) = 0$ with $v(\Delta_G^2) = 0$. From the definition of the isogeny, either $\alpha_2, \beta_2, \alpha_3, \beta_3 \in K$ or $Frob$ permutes δ_2 and δ_3 so that $Frob(\alpha_2) = \alpha_3$, $Frob(\alpha_3) = \alpha_2$ and similarly for β_2, β_3

$$v(\delta_1) = 0, v(\delta_2) = 0, v(\delta_3) = 0, v(\hat{\delta_1}) = a, v(\hat{\delta_2}) = 0, v(\hat{\delta_3}) = 0,$$

so that P.4, P.5 and P.6 yield

$$v(A_1 - B_1)^2 = a, \quad v(A_2 - B_2)^2 = -2v(\ell_2), \quad v(A_3 - B_3)^2 = -2v(\ell_3),$$

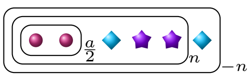
and P.7, P.8 and P.9 give

$$v((A_2 - A_3)(A_2 - B_3)(B_2 - A_3)(B_2 - B_3)) = -2v(\ell_2) - 2v(\ell_3),$$

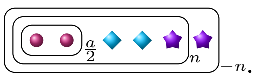
$$v((A_3 - A_1)(A_3 - B_1)(B_3 - A_1)(B_3 - B_1)) = -2v(\ell_3),$$

$$v((A_1 - A_2)(A_1 - B_2)(B_1 - A_2)(B_1 - B_2)) = -2v(\ell_2).$$

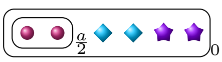
It follows that $v(A_1 - B_1) = \frac{a}{2}$ and if $v(\ell_2) = v(\ell_3) = 0$ then the cluster picture of $\hat{C}$ is that of Table 4.7 for ON3A/B. Otherwise, if $v(\ell_2) = n_2 > 0$ then $v(\ell_3) = 0$. Indeed, by definition of ℓ_2 we have $\beta_3 \equiv -\alpha_2$. By definition of ℓ_3, if $v(\ell_3) > 0$ then $\beta_2 \equiv -\alpha_2$, a contradiction since $\beta_3 \not\equiv \beta_2$. Therefore we obtain the following cluster picture for $\hat{C}$



or



In both cases, these cluster pictures are in the equivalence class of



as required.

<u>**Frobenius action**</u>. By Proposition 3.4.30, the reduction at the node at α_2 is split

if and only if $T_{\alpha_2} = c(\alpha_2 - \alpha_1)(\alpha_2 + \alpha_1)(\alpha_2 - \beta_2)(\alpha_2 - \beta_3) \equiv_{\square} 1$.

Proof of Table 4.8

Case ON4A/B. Computing ℓ_1 we find that $v(\Delta_G^2) = 0$. From the definition of the isogeny, $\alpha_1, \alpha_2, \beta_2 \in K$ $v(\alpha_1 - \alpha_2) = a \in \mathbb{Z}$ and

$$v(\delta_1) = 0, v(\delta_2) = 0, v(\delta_3) = 0, v(\hat{\delta_1}) = 0, v(\hat{\delta_2}) = 0, v(\hat{\delta_3}) = a,$$

so that P.4, P.5 and P.6 yield

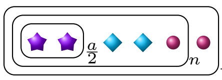
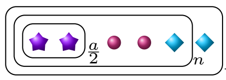
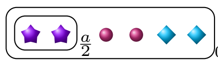
$$v(A_1 - B_1)^2 = -2v(\ell_1), \quad v(A_2 - B_2)^2 = -2v(\ell_2), \quad v(A_3 - B_3)^2 = a - 2v(\ell_3),$$

and P.7, P.8 and P.9 give

$$v((A_2 - A_3)(A_2 - B_3)(B_2 - A_3)(B_2 - B_3)) = -2v(\ell_2) - 2v(\ell_3),$$

$$v((A_3 - A_1)(A_3 - B_1)(B_3 - A_1)(B_3 - B_1)) = -2v(\ell_1) - 2v(\ell_3),$$

$$v((A_1 - A_2)(A_1 - B_2)(B_1 - A_2)(B_1 - B_2)) = -2v(\ell_1) - 2v(\ell_2).$$

It follows that $v(A_3 - B_3) = \frac{a}{2}$ and if $v(\ell_1) = v(\ell_2) = v(\ell_3) = 0$ then the cluster picture of $\hat{C}$ is that of Table 4.8 for ON4A/B. Otherwise, since $v(\Delta_G) = 0$ it follows from Lemma 6.2.2.5, that either $v(\ell_1) = n > 0$ and $v(\ell_2) = v(\ell_3) = 0$ or without loss of generality, $v(\ell_2) = n > 0$ and $v(\ell_1) = v(\ell_3) = 0$. Therefore we obtain the following cluster picture for $\hat{C}$: $_{-n}$ or $_{-n}$. In both cases, these cluster pictures are in the equivalence class of $_0$ as required.

Frobenius action. By Proposition 3.4.30, the reduction at the node at α_1 is split if and only if $T_{\alpha_1} = 2c\alpha_1(\alpha_1 - \beta_2)(\alpha_1 - \alpha_3)(\alpha_1 - \beta_3) \equiv_{\square} 1$.

5.3.3 C is of type $I_{2a,2b}$

Proof of Tables 4.9 and 4.10

By definition of the isogeny we have

$$v(\delta_1) = 2a, v(\delta_2) = 2b, v(\delta_3) = 0, v(\hat{\delta_1}) = 0, v(\hat{\delta_2}) = 0, v(\hat{\delta_3}) = 0,$$

$\alpha_1 \equiv 0 \not\equiv \alpha_2, \alpha_3, \beta_3$. This yields $v(\ell_3) = 0$ since $\ell_3 \equiv -2c\alpha_2$.

Case TN1A/B/C/D. Here $v(\Delta_G) = 0$.

From P.4, P.5 and P.6 we have

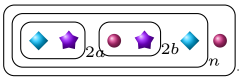
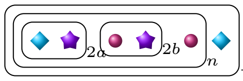
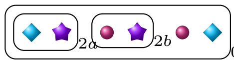
$$v(A_1 - B_1)^2 = -2v(\ell_1), \quad v(A_2 - B_2)^2 = -2v(\ell_2), \quad v(A_3 - B_3)^2 = 0,$$

and P.7, P.8 and P.9 give

$$v((A_2 - A_3)(A_2 - B_3)(B_2 - A_3)(B_2 - B_3)) = 2a - 2v(\ell_2),$$

$$v((A_3 - A_1)(A_3 - B_1)(B_3 - A_1)(B_3 - B_1)) = 2b - 2v(\ell_1),$$

$$v((A_1 - A_2)(A_1 - B_2)(B_1 - A_2)(B_1 - B_2)) = -2v(\ell_1) - 2v(\ell_2).$$

If $v(\ell_1) = v(\ell_2) = 0$ then without loss of generality, let $v(A_2 - A_3) = 2a$ and $(A_1 - B_3) = 2b$ so that the cluster picture of $\hat{C}$ is that of Table 4.9 for TN1A/B/C/D. Otherwise, since $v(\Delta_G^2) = 0$, either $v(\ell_1) = n > 0$ and $v(\ell_2) = 0$ or $v(\ell_2) = n > 0$ and $v(\ell_1) = 0$. Then in the first case, one readily verifies using the valuations above that the cluster picture for $\hat{C}$ is $_{-n}$. And similarly, in the second case if $v(\ell_2) = n > 0$ then the cluster picture for $\hat{C}$ is $_{-n}$. In both cases, these cluster pictures are in the equivalence class of $_0$ as required.

Case TN1E/F/G/H. Here $v(\Delta_G) = r > 0$ so that $v(\ell_1) = v(u_1) - r$. Note that here $\alpha_1 \equiv 0 \not\equiv \alpha_2 \equiv \beta_2 \bmod \pi$. In particular, computing ℓ_2, ℓ_3, u_1, one finds that $\ell_3 \equiv -2c\beta_2$ and hence $v(\ell_3) = 0$. Since $v(\Delta_G^2) > 0$ we find that if $v(\ell_2) > 0$ or equivalently $v(u_1) > 0$ then $\alpha_3 \equiv 0 \bmod \pi$ a contradiction to the cluster picture of C. Hence $v(u_1) = v(\ell_2) = 0$.
From P.4, P.5 and P.6 we have

$$v(A_1 - B_1)^2 = 0, \quad v(A_2 - B_2)^2 = 0, \quad v(A_3 - B_3)^2 = 0,$$

and P.7, P.8 and P.9 give

$$v((A_2 - A_3)(A_2 - B_3)(B_2 - A_3)(B_2 - B_3)) = 2r + 2a,$$

$$v((A_3 - A_1)(A_3 - B_1)(B_3 - A_1)(B_3 - B_1)) = 2r + 2b,$$

$$v((A_1 - A_2)(A_1 - B_2)(B_1 - A_2)(B_1 - B_2)) = 2r.$$

Thus without loss of generality, let $v(A_2 - A_3) = 2a + r$ and $(A_1 - B_3) = 2b + r$ so that the cluster picture of $\hat{C}$ is that of Table 4.9 for TN1E/F/G/H.

Frobenius action. By Proposition 3.4.30, the reduction at the node at α_1 is split if and only if $T_{\alpha_1} = c(\alpha_1-\alpha_2)(\alpha_1-\beta_2)(\alpha_1-\alpha_3)(\alpha_1-\beta_3) \equiv_\square 1$ and the reduction at the node at α_2 is split if and only if $T_{\alpha_2} = c(\alpha_2-\alpha_1)(\alpha_2+\alpha_1)(\alpha_2-\alpha_3)(\alpha_2-\beta_3) \equiv_\square 1$. It follows that C is of type $I_{a,b}^{+,+}$, $I_{a,b}^{-,+}$,$I_{a,b}^{+,-}$,$I_{a,b}^{-,-}$ in TN1A/B/C/D respectively. Similarly for $\hat{C}$, using Proposition 3.4.30 at the nodes A_2 and A_1, one finds that $\hat{C}$ is of type $I_{2a,2b}^{+,+}$, $I_{2a,2b}^{-,+}$,$I_{2a,2b}^{+,-}$,$I_{2a,2b}^{-,-}$ in TN1A/B/C/D respectively. Finally, since $G(x)$ is a $C_2 \times D_4$ polynomial, we have that $\delta_1 \in K$ so that $Frob$ does not permute any clusters in the cluster picture of C and similarly for $\hat{C}$.

Proof of Tables 4.11 and 4.12

By definition of the isogeny

$$v(\delta_1) = 0, v(\delta_2) = 2a, v(\delta_3) = 2b, v(\hat{\delta_1}) = 0, v(\hat{\delta_2}) = 0, v(\hat{\delta_3}) = 0.$$

Reducing invariants, we find that $u_1 \equiv_\square (\alpha_3 - \beta_2)^2$ so that $v(u_1) = 0$.
Case TN2A/B/C/D/E. Here $v(\Delta_G) = 0$ hence $v(\ell_1) = 0$.

From P.4, P.5 and P.6 we have

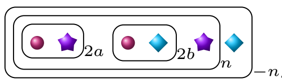
$$v(A_1 - B_1)^2 = 0, \quad v(A_2 - B_2)^2 = -2v(\ell_2), \quad v(A_3 - B_3)^2 = -2v(\ell_3),$$

and P.7, P.8 and P.9 give

$$v((A_2 - A_3)(A_2 - B_3)(B_2 - A_3)(B_2 - B_3)) = -2v(\ell_2) - 2v(\ell_3),$$

$$v((A_3 - A_1)(A_3 - B_1)(B_3 - A_1)(B_3 - B_1)) = 2a - 2v(\ell_3),$$

$$v((A_1 - A_2)(A_1 - B_2)(B_1 - A_2)(B_1 - B_2)) = 2b - 2v(\ell_2).$$

If $v(\ell_2) = v(\ell_3) = 0$ then without loss of generality, let $v(A_1 - A_3) = 2a$ and $(A_1 - A_2) = 2b$ so that the cluster picture of $\hat{C}$ is that of Table 4.11 for TN2A/B/C/D/E. Otherwise, computing ℓ_2 and ℓ_3 and reducing mod π we find that $\ell_2 \equiv 2c\alpha_3$ and $\ell_3 \equiv -2c\beta_2$. It follows from the isogeny that $\alpha_3 \not\equiv \beta_2$ therefore either $v(\ell_2) > 0$ and $v(\ell_3) = 0$ or conversely $v(\ell_3) > 0$ and $v(\ell_2) = 0$. In the first case, one readily verifies using the valuations above that the cluster picture for $\hat{C}$ is 

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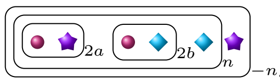
And similarly, in the second case, the cluster picture for $\hat{C}$ is 

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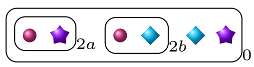
In both cases, these cluster pictures are in the equivalence class of 

as required.
Case TN2E/F/G/H. Here $v(\Delta_G) = r > 0$ hence $v(\ell_1) = -r$. Computing ℓ_2

and ℓ_3 and reducing mod π we find that $\ell_2 \equiv 2c\alpha_3$ and $\ell_3 \equiv -2c\beta_2$. But since $v(\Delta_G) = r > 0$, we also have $\alpha_1^2 \equiv \alpha_3\beta_2$. It follows from the cluster picture of C that $\alpha_1 \not\equiv 0$ so that $v(\ell_2) = v(\ell_3) = 0$.
From P.4, P.5 and P.6 we have

$$v(A_1 - B_1)^2 = 0, \quad v(A_2 - B_2)^2 = 0, \quad v(A_3 - B_3)^2 = 0,$$

and P.7, P.8 and P.9 give

$$v((A_2 - A_3)(A_2 - B_3)(B_2 - A_3)(B_2 - B_3)) = 2r,$$

$$v((A_3 - A_1)(A_3 - B_1)(B_3 - A_1)(B_3 - B_1)) = 2r + 2a,$$

$$v((A_1 - A_2)(A_1 - B_2)(B_1 - A_2)(B_1 - B_2)) = 2r + 2b.$$

Thus without loss of generality, let $v(A_1 - A_3) = 2a + r$ and $(A_1 - A_2) = 2b + r$ so that the cluster picture of $\hat{C}$ is that of Table 4.12 for TN2F/G/H/I/J.
Frobenius action. By Proposition 3.4.30, the reduction at the node at α_2 is split if and only if $T_{\alpha_2} = c(\alpha_2 - \alpha_1)(\alpha_2 + \alpha_1)(\alpha_2 - \alpha_3)(\alpha_2 - \beta_3) \equiv_\square 1$ and the reduction at the node at α_3 is split if and only if $T_{\alpha_3} = c(\alpha_3 - \alpha_1)(\alpha_3 + \alpha_1)(\alpha_3 - \alpha_2)(\alpha_3 - \beta_2) \equiv_\square 1$. It follows that C is of type $I^{+,+}_{2a,2b}$, $I^{-,+}_{2a,2b}$,$I^{-,-}_{2a,2b}$ in TN1A/B/C respectively. Similarly for $\hat{C}$, using Proposition 3.4.30 at the nodes A_1 and B_1, one finds that $\hat{C}$ is of type $I^{+,+}_{4a,4b}$, $I^{-,+}_{4a,4b}$,$I^{-,-}_{4a,4b}$ in TN1A/B/C respectively. Computing $(A_1 - B_1)^2$ and reducing mod π one finds that $(A_1 - B_1)^2 \equiv (\alpha_3 - \beta_2)^2$ so that $(A_1 - B_1)^2 \in K^{\times 2}$ in these 3 cases.

In TN2D/E, $Frob$ permutes δ_2 and δ_3 (by semistability criterion, I_K acts trivially on clusters of size > 1). It follows that $T_{\alpha_2}, T_{\alpha_3} \in K(\delta_2) \subseteq K^{nr}$. For TN2D, we let $T_{\alpha_2}, T_{\alpha_3} \in K(\delta_2)^{\times 2}$ so that C is of type $I^{+}_{2a\tilde{\ }2a}$. For TN2E, we let $T_{\alpha_2}, T_{\alpha_3} \notin K(\delta_2)^{\times 2}$ so that C is of type $I^{-}_{2a\tilde{\ }2a}$. Since $(A_1 - B_1)^2 \equiv (\alpha_3 - \beta_2)^2$, it follows that $(A_1 - B_1)^2 \notin K^{\times 2}$ in both cases. Finally, using Proposition 3.4.30 at the nodes A_1 and B_1 one finds that T_{A_1} and T_{B_1} are congruent to $T_{\alpha_2}, T_{\alpha_3}$ mod π. Therefore $\hat{C}$ is of types $I^{+}_{4a\tilde{\ }4a}$ and $I^{-}_{4a\tilde{\ }4a}$ respectively.

Cases TN2F/G/H are similar to TN2A/B/C. However, cases TN2I/J are different since $\hat{C}$ is deficient for v if r is odd.

Proof of Table 4.13

Case TN3A/B/C/D. By definition of the isogeny

$$v(\delta_1) = 2a, v(\delta_2) = 0, v(\delta_3) = 0, v(\hat{\delta_1}) = b, v(\hat{\delta_2}) = 0, v(\hat{\delta_3}) = 0,$$

$\alpha_1 \equiv 0 \not\equiv \alpha_2, \beta_2, \alpha_3$ and $\alpha_2, \beta_2, \alpha_3, \beta_3 \in K^{nr}$ so that $b \in \mathbb{Z}$. Reducing invariants, we find that

$$\Delta_G \equiv -\alpha_3^2(\beta_2 - \beta_3), \quad \ell_1 \equiv \frac{-1}{c\alpha_3^2}, \quad \ell_2 \equiv c(\alpha_3 + \beta_3), \quad \ell_3 \equiv -c(\alpha_3 + \beta_2),$$

so that $v(\Delta_G) = v(\ell_1) = 0$ and if $v(\ell_2) > 0$ ($v(\ell_3) > 0$ respectively) then $v(\ell_3) = 0$ ($v(\ell_2) = 0$ respectively).
From P.4, P.5 and P.6 we have

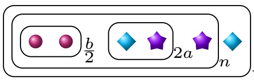
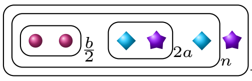
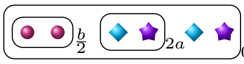
$$v(A_1 - B_1)^2 = b, \quad v(A_2 - B_2)^2 = -2v(\ell_2), \quad v(A_3 - B_3)^2 = -2v(\ell_3),$$

and P.7, P.8 and P.9 give

$$v((A_2 - A_3)(A_2 - B_3)(B_2 - A_3)(B_2 - B_3)) = 2a - 2v(\ell_2) - 2v(\ell_3),$$

$$v((A_3 - A_1)(A_3 - B_1)(B_3 - A_1)(B_3 - B_1)) = 0 - 2v(\ell_3),$$

$$v((A_1 - A_2)(A_1 - B_2)(B_1 - A_2)(B_1 - B_2)) = 0 - 2v(\ell_2).$$

If $v(\ell_2) = v(\ell_3) = 0$ then without loss of generality, let $v(A_2 - A_3) = 2a$ so that the cluster picture of $\hat{C}$ is that of Table 4.13 for TN3A/B/C/D. Otherwise, if $v(\ell_2) = n > 0$ and $v(\ell_3) = 0$ or conversely $v(\ell_3) = n > 0$ and $v(\ell_2) = 0$ one readily verifies using the valuations above that the cluster picture for $\hat{C}$ is  $_{-n}$. And similarly, in the second case, the cluster picture for $\hat{C}$ is  $_{-n}$. In both cases, these cluster pictures are in the equivalence class of  $_0$ as required.

Frobenius action. By Proposition 3.4.30, the reduction at the node at α_1 is split if and only if $T_{\alpha_1} = c(\alpha_1 - \alpha_2)(\alpha_1 - \beta_2)(\alpha_1 - \alpha_3)(\alpha_1 - \beta_3) \equiv_\square 1$ and the reduction at the node at α_2 is split if and only if $T_{\alpha_2} = c(\alpha_2 - \alpha_1)(\alpha_2 + \alpha_1)(\alpha_2 - \alpha_3)(\alpha_2 - \beta_3) \equiv_\square 1$. It follows that C is of type $I_{2a,2b}^{+,+}$, $I_{2a,2b}^{-,+}$, $I_{2a,2b}^{+,-}$, $I_{2a,2b}^{-,-}$ in TN3A/B/C/D respectively. Similarly for $\hat{C}$, using Proposition 3.4.30 at the nodes A_1 and A_2, one finds that the reduction at the node at A_1 is split if and only if $\ell(A_1 - A_2)(A_1 - B_2)(A_1 - A_3)(A_1 - B_3) \equiv T_{\alpha_2} \equiv_\square 1$. Finally, the reduction at the node at A_2 is split if and only if $\ell(A_2 - A_1)(A_2 - B_1)(A_2 - A_3)(A_2 - B_3) \equiv T_{\alpha_1} \equiv_\square 1$. Therefore $\hat{C}$ is of type $I_{b,4a}^{+,+}$, $I_{b,4a}^{-,+}$, $I_{b,4a}^{+,-}$, $I_{b,4a}^{-,-}$ in TN3A/B/C/D respectively. Finally, since $G(x)$ is a $C_2 \times D_4$ polynomial, we have that $\delta_1 \in K$ so that $Frob$ does not permute any clusters in the cluster picture of C and similarly for $\hat{C}$.

Proof of Table 4.14

<u>**Case TN4A/B/C/D**</u>. By definition of the isogeny

$$v(\delta_1)=0, v(\delta_2)=0, v(\delta_3)=2a, v(\hat{\delta_1})=0, v(\hat{\delta_2})=0, v(\hat{\delta_3})=b,$$

$\alpha_1,\alpha_2,\beta_2\in K$ so that $b\in\mathbb{Z}$. Reducing invariants, we find that

$$\Delta_G\equiv c(\alpha_1-\alpha_3)^2(\alpha_1+\beta_2),\quad u_1\equiv\alpha_1+\beta_2-2\alpha_3,\quad \ell_2\equiv 2c\alpha_3,\quad \ell_3\equiv -c(\alpha_1+\beta_2),$$

so that $v(\Delta_G)=v(\ell_3)=0$ and $v(\ell_1)=v(u_1)$. Moreover, by Lemma 6.2.2.5., if $v(\ell_1)>0$ ($v(\ell_2)>0$ respectively) then $v(\ell_2)=0$ ($v(\ell_1)=0$ respectively).
From P.4, P.5 and P.6 we have

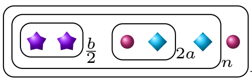
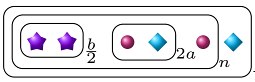
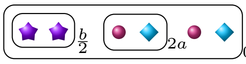
$$v(A_1-B_1)^2=-2v(\ell_1),\quad v(A_2-B_2)^2=-2v(\ell_2),\quad v(A_3-B_3)^2=b,$$

and P.7, P.8 and P.9 give

$$v((A_2-A_3)(A_2-B_3)(B_2-A_3)(B_2-B_3))=-2v(\ell_2),$$

$$v((A_3-A_1)(A_3-B_1)(B_3-A_1)(B_3-B_1))=-2v(\ell_1),$$

$$v((A_1-A_2)(A_1-B_2)(B_1-A_2)(B_1-B_2))=2a-2v(\ell_1)-2v(\ell_2).$$

If $v(\ell_1)=v(\ell_2)=0$ then without loss of generality, let $v(A_1-A_2)=2a$ so that the cluster picture of $\hat{C}$ is that of Table 4.14 for TN4A/B/C/D. Otherwise, if $v(\ell_1)=n>0$ and $v(\ell_2)=0$ or conversely $v(\ell_2=n)>0$ and $v(\ell_1)=0$ one readily verifies using the valuations above that the cluster picture for $\hat{C}$ is  $_{-n}$. And similarly, in the second case, the cluster picture for $\hat{C}$ is  $_{-n}$. In both cases, these cluster pictures are in the equivalence class of  $_0$ as required.
<u>**Frobenius action**</u>. By Proposition 3.4.30, the reduction at the node at α_3 is split if and only if $T_{\alpha_3}=c(\alpha_3-\alpha_1)(\alpha_3+\alpha_1)(\alpha_3-\alpha_2)(\alpha_3-\beta_2)\equiv_\square 1$ and the reduction at the node at α_1 is split if and only if $T_{\alpha_1}=2c\alpha_1(\alpha_1-\beta_2)(\alpha_1-\alpha_3)(\alpha_1-\beta_3)\equiv_\square 1$. It follows that C is of type $I^{+,+}_{2a,2b}$, $I^{-,+}_{2a,2b}$, $I^{+,-}_{2a,2b}$, $I^{-,-}_{2a,2b}$ in TN4A/B/C/D respectively. Similarly for $\hat{C}$, using Proposition 3.4.30 at the nodes A_3 and A_1, one finds that the reduction at the node at A_1 is split if and only if $\ell(A_3-A_1)(A_3-B_1)(A_3-A_2)(A_3-B_2)\equiv T_{\alpha_1}\equiv_\square 1$. Finally, the reduction at the node at A_1 is split if and only if $\ell(A_1-B_1)(A_1-B_2)(A_1-A_3)(A_1-B_3)\equiv T_{\alpha_3}\equiv_\square 1$. Therefore $\hat{C}$ is of type

$I^{+,+}_{b,4a}$, $I^{-,+}_{b,4a}$, $I^{+,-}_{b,4a}$, $I^{-,-}_{b,4a}$ in TN4A/B/C/D respectively.

Proof of Table 4.15

Case TN5A/B/C/D. By definition of the isogeny

$$v(\delta_1) = 0, v(\delta_2) = 0, v(\delta_3) = 0, v(\hat{\delta_1}) = b, v(\hat{\delta_2}) = 0, v(\hat{\delta_3}) = a,$$

$\alpha_1, \alpha_2, \beta_2, \alpha_3, \beta_3 \in K$ so that $a, b \in \mathbb{Z}$. Reducing invariants, we find that

$$v(\Delta_G) = 0, \quad \ell_2 \equiv c(\alpha_3 + \beta_3), \quad \ell_3 \equiv -c(\alpha_1 + \alpha_3),$$

In particular, $v(\ell_3) = 0$.
From P.4, P.5 and P.6 we have

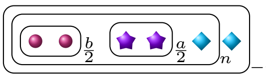
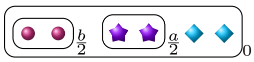
$$v(A_1 - B_1)^2 = b, \quad v(A_2 - B_2)^2 = -2v(\ell_2), \quad v(A_3 - B_3)^2 = a - 2v(\ell_3),$$

and P.7, P.8 and P.9 give

$$v((A_2 - A_3)(A_2 - B_3)(B_2 - A_3)(B_2 - B_3)) = 2v(\ell_2) - 2v(\ell_3),$$

$$v((A_3 - A_1)(A_3 - B_1)(B_3 - A_1)(B_3 - B_1)) = -2v(\ell_3),$$

$$v((A_1 - A_2)(A_1 - B_2)(B_1 - A_2)(B_1 - B_2)) = 2v(\ell_2).$$

If $v(\ell_2) = 0$ then $v(A_1 - B_1) = \frac{b}{2}$ and $v(A_3 - B_3) = \frac{a}{2}$ so that the cluster picture of $\hat{C}$ is that of Table 4.15 for TN5A/B/C/D. Otherwise, if $v(\ell_2) = n > 0$ then one readily verifies using the valuations above that the cluster picture for $\hat{C}$ is , which is in the equivalence class of  as required.

Frobenius action. By Proposition 3.4.30, the reduction at the node at α_1 is split if and only if $T_{\alpha_1} = c2\alpha_1(\alpha_1 - \beta_2)(\alpha_1 - \alpha_3)(\alpha_1 - \beta_3) \equiv_\square 1$ and the reduction at the node at α_3 is split if and only if $T_{\alpha_3} = c(\alpha_3 - \alpha_1)(\alpha_3 + \alpha_1)(\alpha_3 - \alpha_2)(\alpha_3 - \beta_3) \equiv_\square 1$. It follows that C is of type $I^{+,+}_{a,b}$, $I^{-,+}_{a,b}$, $I^{+,-}_{a,b}$, $I^{-,-}_{a,b}$ in TN5A/B/C/D respectively. Similarly for $\hat{C}$, using Proposition 3.4.30 at the nodes A_1 and A_3, one finds that the reduction at the node at A_1 is split if and only if $\ell(A_1 - A_2)(A_1 - B_2)(A_1 - A_3)(A_1 - B_3) \equiv T_{\alpha_3} \equiv_\square 1$. Finally, the reduction at the node at A_3 is split if and only if $\ell(A_3 - A_1)(A_3 - B_1)(A_3 - A_2)(A_3 - B_2) \equiv T_{\alpha_1} \equiv_\square 1$. Therefore $\hat{C}$ is of type $I^{+,+}_{b,a}$, $I^{-,+}_{b,a}$, $I^{+,-}_{b,a}$, $I^{-,-}_{b,a}$ in TN5A/B/C/D respectively.

Proof of Table 4.16

Case TN6A/B/C/D/E. By definition of the isogeny

$$v(\delta_1)=0, v(\delta_2)=0, v(\delta_3)=0, v(\hat{\delta_1})=0, v(\hat{\delta_2})=b, v(\hat{\delta_3})=a,$$

$\alpha_1,\alpha_2,\beta_2,\alpha_3,\beta_3\in K^{nr}$ for otherwise I_K would permute both clusters, a contradiction to the semistability criterion 3.4.29. Hence $a,b\in\mathbb{Z}$. Reducing invariants, we find that

$$v(\Delta_G)=v(\ell_2)=v(\ell_3)=0.$$

In particular, we have $v(\ell_1)=v(u_1)$.
From P.4, P.5 and P.6 we have

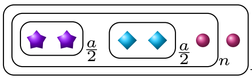
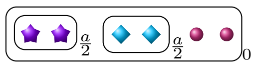
$$v(A_1-B_1)^2=-2v(u_1), \quad v(A_2-B_2)^2=b, \quad v(A_3-B_3)^2=a,$$

and P.7, P.8 and P.9 give

$$v((A_2-A_3)(A_2-B_3)(B_2-A_3)(B_2-B_3))=0,$$

$$v((A_3-A_1)(A_3-B_1)(B_3-A_1)(B_3-B_1))=-2v(u_1),$$

$$v((A_1-A_2)(A_1-B_2)(B_1-A_2)(B_1-B_2))=2v(u_1).$$

If $v(u_1)=0$ then $v(A_2-B_2)=\frac{b}{2}$ and $v(A_3-B_3)=\frac{a}{2}$ so that the cluster picture of $\hat{C}$ is that of Table 4.16 for TN6A/B/C/D/E. Otherwise, if $v(u_1)=n>0$ then one readily verifies using the valuations above that the cluster picture for $\hat{C}$ is  $_{-n}$, which is in the equivalence class of  $_0$ as required.

Frobenius action. Computing roots and invariants and reducing $\mod \pi$ we see that $(A_2-A_3)^2\equiv_\square \alpha_1^2$. It follows that $Frob$ acts trivially on clusters for TN6A/B/C, while permuting them for TN6D/E. By Proposition 3.4.30, the reduction at the node at α_1 is split if and only if $T_{\alpha_1}=2c\alpha_1(\alpha_1-\alpha_3)(\alpha_1-\beta_2)(\alpha_1-\beta_3)\equiv_\square 1$ and the reduction at the node at $-\alpha_1$ is split if and only if $T_{-\alpha_1}=-2c\alpha_1(-\alpha_1-\alpha_2)(-\alpha_1-\beta_2)(-\alpha_1-\beta_3)\equiv_\square 1$. It follows that C is of type $I_{2a,2b}^{+,+}$, $I_{2a,2b}^{-,+}$, $I_{2a,2b}^{-,-}$ in TN6A/B/C respectively. Similarly for $\hat{C}$, using Proposition 3.4.30 at the nodes A_3 and A_2, one finds that the reduction at the node at A_3 is split if and only if $T_{A_3}=\ell(A_3-A_1)(A_3-B_1)(A_3-A_2)(A_3-B_2)\equiv T_{\alpha_1}\equiv_\square 1$. Finally, the reduction at the node at A_2 is split if and only if $T_{A_2}\equiv \ell(A_2-A_1)(A_2-B_1)(A_2-A_3)(A_2-B_3)\equiv T_{-\alpha_1}\equiv_\square 1$. Therefore $\hat{C}$ is of type $I_{a,b}^{+,+}$, $I_{a,b}^{-,+}$, $I_{a,b}^{-,-}$ in TN6A/B/C respectively.
For TN6D/E we have $\alpha_1^2\notin K^{\times 2}$ and hence since $(A_2-A_3)^2\equiv_\square \alpha_1^2$, it follows

that $Frob$ permutes both clusters in the cluster picture of $\hat{C}$. For TN6D, we let $T_{\alpha_1}, T_{-\alpha_1} \in K(\delta_2)^{\times 2} = K(\delta_3)^{\times 2}$ so that C is of type $I^+_{2a\tilde{}2a}$. Moreover as noted above, since $T_{A_3} \equiv_\square T_{\alpha_1}$ and $T_{A_2} \equiv_\square T_{-\alpha_1}$, it follows that $\hat{C}$ is of type $I^+_{a\tilde{}a}$. Similarly, for TN6E, so that C is of type $I^-_{2a\tilde{}2a}$ and $I^-_{a\tilde{}a}$.

Proof of Tables 4.17, 4.18 and 4.19

By definition of the isogeny

$$v(\delta_1) = 0, v(\delta_2) = 0, v(\delta_3) = 0, v(\hat{\delta_1}) = a + b, v(\hat{\delta_2}) = 0, v(\hat{\delta_3}) = 0,$$

Without loss of generality, assume $a \leq b$. Write

$$\alpha_3 = \alpha_2 + a_3\pi^a, \quad \beta_3 = \beta_2 + b_3\pi^b, \quad a_3, b_3 \in \mathcal{O}_K^\times.$$

By definition of $\Delta_G, \ell_1, \ell_2, \ell_3$ we have

$$\Delta_G = c\pi^a \Big(a_3(\beta_2 - \alpha_1)(\beta_2 + \alpha_1) + b_3\pi^{b-a}(\alpha_2 - \alpha_1)(\alpha_2 + \alpha_1) + a_3b_3\pi^b(\alpha_2 + \beta_2)\Big),$$

$$u_1 = -a_3\pi^a - b_3\pi^b, \quad \ell_2 = c(\alpha_2 + \beta_2 + a_3\pi^a + b_3\pi^b), \quad \ell_3 = -c(\alpha_2 + \beta_2).$$

It follows that if $a < b$ then $v(\Delta_G) = a, \quad v(u_1) = a$. In particular, $v(\ell_1) = 0$ and $v(\ell) \in 2\mathbb{Z}$.

If $a = b$ then

$$\Delta_G = c\pi^a \left(a_3(\beta_2 - \alpha_1)(\beta_2 + \alpha_1) + b_3(\alpha_2 - \alpha_1)(\alpha_2 + \alpha_1) + a_3b_3\pi^a(\alpha_2 + \beta_2)\right),$$

$$u_1 = -\pi^a(a_3 + b_3), \quad \ell_2 = c(\alpha_2 + \beta_2 + \pi^a(a_3 + b_3)), \quad \ell_3 = -c(\alpha_2 + \beta_2).$$

Therefore

$$v(\Delta_G) = a + r, \quad r = v\left(a_3(\beta_2 - \alpha_1)(\beta_2 + \alpha_1) + b_3(\alpha_2 - \alpha_1)(\alpha_2 + \alpha_1) + a_3b_3\pi^a(\alpha_2 + \beta_2)\right),$$

$$v(u_1) = a + n_1, \quad n_1 = v(a_3 + b_3),$$

$$v(\ell_2) = v(\alpha_2 + \beta_2 + \pi^a(a_3 + b_3)), \quad v(\ell_3) = v(\alpha_2 + \beta_2).$$

In particular, $v(\ell_1) = n_1 - r$.

<u>**Case TN7A/B/C/D**</u> : Here we let $a < b$ so that $v(\Delta_G) = a$, $v(u_1) = a$ and $v(\ell_1) = 0$. $v(\ell_2) = 0$ if $v(\ell_3) = 0$ or $v(\ell_2) \geq \min\{a, v(\ell_3)\}$ and by semistability criterion 3.4.29, we have $v(\ell) \in 2\mathbb{Z}$.

From P.4, P.5 and P.6 we have

$$v(A_1 - B_1)^2 = a + b - 2v(\Delta_G) - 2v(\ell_1) = a + b - 2a = b - a,$$

$$v(A_2-B_2)^2 = -2v(\ell_2), \quad v(A_3-B_3)^2 = -2v(\ell_3),$$

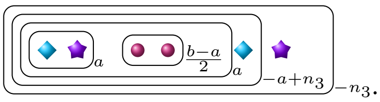
and P.7, P.8 and P.9 give

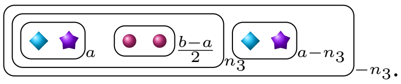
$$v((A_2-A_3)(A_2-B_3)(B_2-A_3)(B_2-B_3)) = 2v(\Delta_G)-2v(\ell_2)-2v(\ell_3) = 2a-2v(\ell_2)-2v(\ell_3),$$

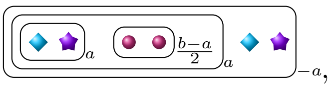
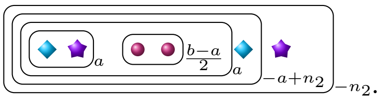
$$v((A_3-A_1)(A_3-B_1)(B_3-A_1)(B_3-B_1)) = -2v(\ell_1)-2v(\ell_3) = -2v(\ell_3),$$

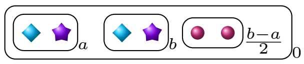
$$v((A_1-A_2)(A_1-B_2)(B_1-A_2)(B_1-B_2)) = -2v(\ell_1)-2v(\ell_2) = -2v(\ell_2).$$

If $v(\ell_2) = v(\ell_3) = 0$ then we have, without loss of generality, let $v(A_2-A_3) = v(B_2-B_3) = a$ and $v(A_1-B_1) = \frac{b-a}{2}$ so that the cluster picture of $\hat{C}$ is that of Table 4.17 for TN7A/B/C/D. Otherwise,

if $v(\ell_3) = n_3 > 0$ with $n_3 > a$, then $v(\ell_2) = a$ and the cluster picture for $\hat{C}$ is .

If $v(\ell_3) = n_3 > 0$ with $n_3 < a$, then $v(\ell_2) = v(\ell_3)$ and the cluster picture for $\hat{C}$ is .

If $v(\ell_3) = n_3 > 0$ with $n_3 = a$, then if $v(\ell_2) = v(\ell_3) = a$, the cluster picture for $\hat{C}$ is , and if $v(\ell_2) > v(\ell_3) = a$, the cluster picture for $\hat{C}$ is .

All are in the equivalence class of  as required.

Frobenius action. By Proposition 3.4.30, the reduction at the node at α_2 is split if and only if $T_{\alpha_2} = c(\alpha_2-\alpha_1)(\alpha_2+\alpha_1)(\alpha_2-\beta_2)(\alpha_2-\beta_3) \equiv_\square 1$ and the reduction at the node at β_2 is split if and only if $T_{\beta_2} = c(\beta_2-\alpha_1)(\beta_2+\alpha_1)(\beta_2-\alpha_2)(\beta_2-\alpha_3) \equiv_\square 1$. It follows that C is of type $I^{+,+}_{2a,2b}$, $I^{-,+}_{2a,2b}$, $I^{+,-}_{2a,2b}$ $I^{-,-}_{2a,2b}$ in TN7A/B/C/D respectively.

Now computing invariants, one finds that $(A_2-B_2)^2 \equiv_\square (\alpha_2+\beta_2)^2 T_{\alpha_2} T_{\beta_2}$. Here $(\alpha_2+\beta_2)^2 \in K^{\times 2}$, it follows that $(A_2-B_2)^2 \in K^{\times 2}$ for TN7A/D and $(A_2-B_2)^2 \notin K^{\times 2}$ for TN7B/C. Also, using Proposition 3.4.30, one finds that $\hat{C}$ is of type U^+ (U^- respectively) if $\ell \in K^{\times 2}$ ($\ell \notin K^{\times 2}$ respectively). Computing ℓ yields $\ell \equiv_\square c(\beta_2-\alpha_1)(\beta_2+\alpha_1)(\alpha_2+\beta_2)^2 \equiv T_{\beta_2}$. Therefore $\ell \in K^{\times 2}$ for TN7A/B and $\ell \notin K^{\times 2}$ for TN7C/D. This yields that $\hat{C}$ is of type $U^+_{2a,2a,b-a}$ for TN7A, $U^+_{2a\tilde{\ }2a,b-a}$ for TN7B, $U^-_{2a\tilde{\ }2a,b-a}$ for TN7C, $U^-_{2a,2a,b-a}$ for TN7D.

Remark 5.3.11. Since $\alpha_2 \equiv \alpha_3$ and $\beta_2 \equiv \beta_3$, it follows that $2a$ odd $\Leftrightarrow$ $2b$ odd.

Tamagawa numbers and deficiency : λ_v is clear for computations of Tamagawa numbers and deficiency given in Table 3.1 except for TN7C/D.Therefore, for TN7C we have

$$(-1)^{ord_2(\frac{c_v}{\hat{c}_v})} = 1 \Leftrightarrow b \equiv 1 \bmod 2, \quad (-1)^{ord_2(\frac{m_v}{\hat{m}_v})} = 1 \Leftrightarrow b - a \equiv 1 \bmod 2,$$

which yields that $\lambda_v = -1$ if and only if $b \equiv a \bmod 2$ as required.

For TN7D we use the following lemma.

Lemma 5.3.12.

1. *For $a, b \in \mathbb{Z}$. If $b - a$ is odd then $d = gcd(2a, b - a)$ is odd and $\frac{4ab}{d}$ is even. If $b - a$ is even then d is even and $\frac{4ab}{d}$ is even.*

2. *For $a, b \in \frac{1}{2}\mathbb{Z}$ with $2a, 2b$ odd. If $b - a$ odd then $d = gcd(2a, b - a)$ is odd and $\frac{4ab}{d}$ is odd. If $b - a$ even then d is odd and $\frac{4ab}{d}$ is odd.*

Proof. 1)The first claim is clear. If $b-a$ is even with a, b odd then 2 divides exactly once $2a$, and hence d. On the other hand, 4 divides exactly once $4ab$ so that $\frac{4ab}{d}$ is even. Finally if $b-a$ is even with a, b even write $a = 2^n a'$ and $b = 2^m b'$ with $n \leq m$. Then 2^n divides exactly d and $4ab = 2^{n+m+2}a'b'$ so that $\frac{4ab}{d}$ is even.

2) This is clear since $4ab$ is odd and d is odd. □

Hence, if $a, b \in \mathbb{Z}$ then $c_v = 4$, m_v=1 and
if $b - a$ is even then $\hat{c}_v = 4$ and $\hat{m}_v = 1$,
if $b - a$ is odd, then , $\hat{c}_v = 2$ and $\hat{m}_v = 2$. Therefore in this case

$$(-1)^{ord_2(\frac{c_v}{\hat{c}_v})} = 1 \Leftrightarrow b - a \equiv 0 \bmod 2, \quad (-1)^{ord_2(\frac{m_v}{\hat{m}_v})} = 1 \Leftrightarrow b - a \equiv 0 \bmod 2.$$

If $a, b \in \frac{1}{2}\mathbb{Z}$ then $c_v = 1$, m_v=1 and
if $b - a$ is even then $\hat{c}_v = 1$ and $\hat{m}_v = 1$,
if $b - a$ is odd, then , $\hat{c}_v = 1$ and $\hat{m}_v = 2$. Therefore in this case

$$(-1)^{ord_2(\frac{c_v}{\hat{c}_v})} = 1, \quad (-1)^{ord_2(\frac{m_v}{\hat{m}_v})} = 1 \Leftrightarrow b - a \equiv 0 \bmod 2.$$

Therefore, $\lambda_v = 1$ if and only if $2a \equiv b - a \bmod 2$, equivalently if $b - a \equiv 0 \bmod 2$ as required.

Case TN7E/F/G/H/I : Here $a = b$ but we let $v(\Delta_G) = 2a$ so that $v(\ell_1) = n_1$. Also, $v(\ell_2) = 0$ if $v(\ell_3) = 0$ or $v(\ell_2) \geq \min\{a, v(\ell_3)\}$ and by semistability criterion 3.4.29, we have $v(\ell) \in 2\mathbb{Z}$. Recall that in this case

$$\Delta_G = c\pi^a \left(a_3(\beta_2 - \alpha_1)(\beta_2 + \alpha_1) + b_3(\alpha_2 - \alpha_1)(\alpha_2 + \alpha_1) + a_3 b_3 \pi^a(\alpha_2 + \beta_2)\right),$$

$$u_1 = -\pi^a(a_3 + b_3), \quad \ell_2 = c(\alpha_2 + \beta_2 + \pi^a(a_3 + b_3)), \quad \ell_3 = -c(\alpha_2 + \beta_2).$$

Therefore

$$v(\Delta_G) = a \quad v\left(a_3(\beta_2 - \alpha_1)(\beta_2 + \alpha_1) + b_3(\alpha_2 - \alpha_1)(\alpha_2 + \alpha_1) + a_3 b_3 \pi^a(\alpha_2 + \beta_2)\right) = 0,$$

$$v(u_1) = a + n_1, \quad n_1 = v(a_3 + b_3),$$

$$v(\ell_2) = v(\alpha_2 + \beta_2 + \pi^a(a_3 + b_3)), \quad v(\ell_3) = v(\alpha_2 + \beta_2).$$

In particular, $v(\ell_1) = n_1$.
From P.4, P.5 and P.6 we have

$$v(A_1 - B_1)^2 = 2a - 2v(\Delta_G) - 2v(\ell_1) = 2a - 2a - 2n_1 = -2n_1,$$

$$v(A_2 - B_2)^2 = -2v(\ell_2), \quad v(A_3 - B_3)^2 = -2v(\ell_3),$$

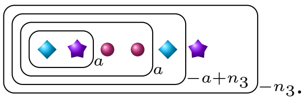
and P.7, P.8 and P.9 give

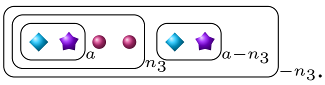
$$v((A_2 - A_3)(A_2 - B_3)(B_2 - A_3)(B_2 - B_3)) = 2v(\Delta_G) - 2v(\ell_2) - 2v(\ell_3) = 2a - 2v(\ell_2) - 2v(\ell_3),$$

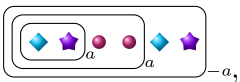
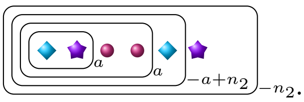
$$v((A_3 - A_1)(A_3 - B_1)(B_3 - A_1)(B_3 - B_1)) = -2v(\ell_1) - 2v(\ell_3) = -2n_1 - 2v(\ell_3),$$

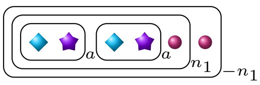
$$v((A_1 - A_2)(A_1 - B_2)(B_1 - A_2)(B_1 - B_2)) = -2v(\ell_1) - 2v(\ell_2) = -2n_1 - 2v(\ell_2).$$

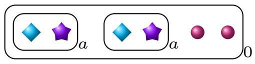
If $n_1 = v(\ell_2) = v(\ell_3) = 0$ then, without loss of generality, let $v(A_2 - A_3) = v(B_2 - B_3) = a$ so that the cluster picture of $\hat{C}$ is that of Table 4.18 for TN7E/F/G/H/I.
Otherwise, assume that n_1=0 then

if $v(\ell_3) = n_3 > 0$ with $n_3 > a$, then $v(\ell_2) = a$ and the cluster picture for $\hat{C}$ is 
.

If $v(\ell_3) = n_3 > 0$ with $n_3 < a$, then $v(\ell_2) = v(\ell_3)$ and the cluster picture for $\hat{C}$ is 
.

If $v(\ell_3) = n_3 > 0$ with $n_3 = a$, then if $v(\ell_2) = v(\ell_3) = a$, the cluster picture for $\hat{C}$ is 
, and if $v(\ell_2) > v(\ell_3) = a$, the cluster picture for $\hat{C}$ is 
.

Finally, if $n_1 > 0$ then since $v(\Delta_G) = a$ we have $v(\ell_2) = v(\ell_3) = 0$ and the cluster picture of $\hat{C}$ is 
.

All are in the equivalence class of  as required.

Frobenius action. By Proposition 3.4.30, the reduction at the node at α_2 is split if and only if $T_{\alpha_2} = c(\alpha_2-\alpha_1)(\alpha_2+\alpha_1)(\alpha_2-\beta_2)(\alpha_2-\beta_3) \equiv_\square 1$ and the reduction at the node at β_2 is split if and only if $T_{\beta_2} = c(\beta_2-\alpha_1)(\beta_2+\alpha_1)(\beta_2-\alpha_2)(\beta_2-\alpha_3) \equiv_\square 1$. It follows that C is of type $I^{+,+}_{2a,2a}$, $I^{-,+}_{2a,2a}$, $I^{-,-}_{2a,2a}$ in TN7E/F/G respectively.

Now computing invariants, one finds that $\hat{\delta_2} \equiv_\square \hat{\delta_3} \equiv_\square T_{\alpha_2}T_{\beta_2}$. It follows that $\hat{\delta_2}, \hat{\delta_3} \in K^{\times 2}$ for TN7F and $\hat{\delta_2}, \hat{\delta_3} \notin K^{\times 2}$ for TN7E/G. Similarly for $\hat{C}$, using Proposition 3.4.30 at the nodes A_2 and A_B, one finds that the reduction at the node at A_2 is split multiplicative if and only if $T_{A_2} = \ell(A_2-A_1)(A_2-B_1)(A_2-B_2)(A_2-B_3) \equiv_\square 1$. Finally, the reduction at the node at B_2 is split multiplicative if and only if $T_{B_2} \equiv \ell(B_2-A_1)(B_2-B_1)(B_2-A_2)(B_2-A_3) \equiv_\square 1$. However, computing T_{A_2}, T_{B_2} we have that $T_{A_2}T_{B_2} \equiv_\square \delta_2 \equiv \delta_3$. Therefore $\hat{C}$ is of type $I^{+,+}_{2a,2a}$, $I^{+,+}_{2a\tilde{2}a}$, $I^{-,-}_{2a,2a}$ in TN7E/F/G respectively.

Remark 5.3.13. It could be argued that for TN7G, since $T_{A_2}T_{B_2} \equiv_\square \delta_2 \equiv \delta_3 \equiv 1$, we could have $T_{A_2}, T_{B_2} \in K^{\times 2}$. However, this is not possible since the order of $Frob$ on the components of both special fibers of C and $\hat{C}$ is preserved (e.g. because it can be read of the local factor of the L-function). That is, for TN7E, $Frob$ acts trivially on the components of the special fiber of C, therefore its action is also trivial on that of $\hat{C}$. Similarly, for TN7G, $Frob$ has order 2 on both $2a$-gone of the special fiber of C. It follows that $Frob$ has order 2 on both $2a$-gone of the special fiber of $\hat{C}$ and hence that $\hat{C}$ is of type $I^{-,-}_{2a,2a}$.

Finally, for TN7H : Let $t^+_{\alpha_2}, t^-_{\alpha_2}$, (respectively $t^+_{\beta_2}, t^-_{\beta_2}$) denote the square roots of T_{α_2}, (resp. T_{β_2}); i.e. $(t^+_{\alpha_2})^2 = (t^-_{\alpha_2})^2 = T_{\alpha_2}$. Since C is of type $I^{+,+}_{2a\tilde{2}a}$, it follows that $Frob$ has order 2 on $t^+_{\alpha_2}, t^+_{\beta_2}$. Hence, without loss of generality, let $Frob(t^+_{\alpha_2}) = t^+_{\beta_2}$, $Frob(t^+_{\beta_2}) = (t^+_{\alpha_2})$ and $Frob(t^-_{\alpha_2}) = t^-_{\beta_2}$, $Frob(t^-_{\beta_2}) = (t^-_{\alpha_2})$. Since, as above

$$d\ell_2 \equiv d\ell_3 \equiv_\square T_{\alpha_2}T_{\beta_2},$$

and

$$T_{\alpha_2}T_{\beta_2} = (t^+_{\alpha_2})^2(t^+_{\beta_2})^2 = (t^+_{\alpha_2}t^+_{\beta_2})^2,$$

with $Frob(t^+_{\alpha_2}t^+_{\beta_2}) = t^+_{\alpha_2}t^+_{\beta_2}$, it follows that $t^+_{\alpha_2}t^+_{\beta_2} \in K^\times$ and $d\ell_2, d\ell_3 \in K^{\times 2}$.

Also, using P.8 we have that $\delta_2 = 4\ell_1^2\ell_3^2(A_3-A_1)(A_3-B_1)(B_3-A_1)(B_3-B_1)$. Write $\alpha_3 = a_3\pi^a + \alpha_2$ and $\beta_3 = b_3\pi^a + \beta_2$, then either $Frob(\alpha_2) = \beta_2$ and $Frob(\alpha_3) = \beta_3$ in which case $Frob(a_3) = b_3, Frob(b_3) = a_3$ or $Frob(\alpha_2) = \beta_3$ and $Frob(\alpha_3) = \beta_2$ in which case $Frob(a_3) = -b_3, Frob(b_3) = -a_3$. In both cases, $\delta_2 \equiv_\square T_{A_2}T_{B_2}$, with $T_{A_2}, T_{B_2} \in K$. Since $\delta_2 \equiv \delta_3 \notin K^{\times 2}$ it follows that, without loss of generality, $T_{A_2} \in K^{\times 2}$ and $T_{B_2} \notin K^{\times 2}$. Therefore, $\hat{C}$ is of type $I^{-,+}_{2a,2a}$ as required.

For TN7F : Here $Frob$ has order 4 on $t^+_{\alpha_2}$, hence we have $Frob(t^+_{\alpha_2}) = t^+_{\beta_2}$, $Frob(t^+_{\beta_2}) = t^-_{\alpha_2}$, $Frob(t^-_{\alpha_2}) = t^-_{\beta_2}$, $Frob(t^-_{\beta_2}) = t^+_{\alpha_2}$. It follows that $Frob(t^+_{\alpha_2}t^+_{\beta_2}) = t^+_{\beta_2}t^-_{\alpha_2}$ so that $t^+_{\alpha_2}t^+_{\beta_2} \notin K$ and $\hat{\delta_2} \equiv \hat{\delta_3} \equiv T_{\alpha_2}T_{\beta_2} = (t^+_{\alpha_2}t^+_{\beta_2})^2 \notin K^{\times 2}$. It follows that $Frob(T_{A_2}) = T_{B_2}$ and using P.8 as above we have $T_{A_2}T_{B_2} \equiv_\square \delta_2 \equiv \delta_3 \notin K^{\times 2}$. Writing $T_{A_2}T_{B_2} = (t_{A_2})^2(t_{B_2})^2 = (t_{A_2}t_{B_2})^2$, it follows that $t_{A_2}t_{B_2} \notin K$ and therefore $\hat{C}$ is of type $I^{-,+}_{2a\tilde{}2a}$.

Case TN7J/K/L/M/N : Here $a = b$ but we let $v(\Delta_G) = a + r > a$. Recall that

$$\Delta_G = c\pi^a\left(a_3(\beta_2 - \alpha_1)(\beta_2 + \alpha_1) + b_3(\alpha_2 - \alpha_1)(\alpha_2 + \alpha_1) + a_3b_3\pi^a(\alpha_2 + \beta_2)\right),$$

$$u_1 = -\pi^a(a_3 + b_3), \quad \ell_2 = c(\alpha_2 + \beta_2 + \pi^a(a_3 + b_3)), \quad \ell_3 = -c(\alpha_2 + \beta_2).$$

In particular, $v(\ell_1) = a + n_1 - r$ where $v(a_3 + b_3) = n_1$. Note that since $v(\Delta_G) > a$ we have $a_3(\beta_2 - \alpha_1)(\beta_2 + \alpha_1) \equiv -b_3(\alpha_2 - \alpha_1)(\alpha_2 + \alpha_1)$. Now if $v(\ell_3) > 0$ or $v(u_1) > a$ this yields another congruence between roots, contradicting the cluster picture of C. Therefore $v(\ell_3) = v(\ell_2) = n_1 = 0$.

From P.4, P.5 and P.6 we have

$$v(A_1 - B_1)^2 = 2a - 2v(\Delta_G) - 2v(\ell_1) = 2a - 2(a + r) - 2(-r) = 0,$$

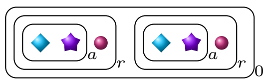
$$v(A_2 - B_2)^2 = 0, \quad v(A_3 - B_3)^2 = 0,$$

and P.7, P.8 and P.9 give

$$v((A_2 - A_3)(A_2 - B_3)(B_2 - A_3)(B_2 - B_3)) = 2v(\Delta_G) - 2v(\ell_2) - 2v(\ell_3) = 2(a + r),$$

$$v((A_3 - A_1)(A_3 - B_1)(B_3 - A_1)(B_3 - B_1)) = -2v(\ell_1) - 2v(\ell_3) = 2r,$$

$$v((A_1 - A_2)(A_1 - B_2)(B_1 - A_2)(B_1 - B_2)) = -2v(\ell_1) - 2v(\ell_2) = 2r.$$

Thus we have the following cluster picture 

as required.

Frobenius action, Tamagawa numbers and deficiency. Follows from the previous case, except for TN7K/N where deficiency is different but readily computable from Table 3.1.

Proof of Tables 4.20, 4.21 and 4.22

By definition of the isogeny

$$v(\delta_1) = 0, v(\delta_2) = 0, v(\delta_3) = 0, v(\hat{\delta_1}) = 0, v(\hat{\delta_2}) = 0, v(\hat{\delta_3}) = a + b,$$

Since $G(x)$ is a $C_2 \times D_4$ polynomial, we have $\delta_1 \in K$, therefore $\alpha_1, \alpha_2, \beta_2 \in K^{nr}$ for otherwise I_K would permute both clusters, a contradiction to the semistability criterion 3.4.29. Hence $a, b \in \mathbb{Z}$. Without loss of generality, assume $a \leq b$. Write

$$\alpha_2 = \alpha_1 + a_2\pi^a, \quad \beta_2 = -\alpha_1 + b_2\pi^b, \quad a_2, b_2 \in \mathcal{O}_K^\times.$$

By definition of $\Delta_G, \ell_1, \ell_2, \ell_3$ we have

$$\Delta_G = c\pi^a(a_2(\alpha_1+\alpha_3)(\alpha_1+\beta_3) + b_2\pi^{b-a}(\alpha_1-\alpha_3)(\alpha_1-\beta_3) - a_2b_2\pi^b(\alpha_3+\beta_3)),$$

$$u_1 = a_2\pi^a + b_2\pi^b - \alpha_3 - \beta_3, \quad \ell_2 = c(\alpha_3+\beta_3), \quad \ell_3 = -c(a_2\pi^a + b_2\pi^b).$$

It follows that if $a < b$ then $v(\Delta_G) = a, \quad u_1 = a_2\pi^a - \alpha_3 - \beta_3, \quad v(\ell_3) = a.$

If $a = b$ then

$$\Delta_G = c\pi^a(a_2(\alpha_1+\alpha_3)(\alpha_1+\beta_3) + b_2(\alpha_1-\alpha_3)(\alpha_1-\beta_3) - a_2b_2\pi^a(\alpha_3+\beta_3)),$$

$$u_1 = \pi^a(a_2+b_2) - \alpha_3 - \beta_3, \quad \ell_2 = c(\alpha_3+\beta_3), \quad \ell_3 = -c\pi^a(a_2+b_2).$$

Therefore

$$v(\Delta_G) = a+r, \quad r = v\left(a_2(\alpha_1+\alpha_3)(\alpha_1+\beta_3) + b_2(\alpha_1-\alpha_3)(\alpha_1-\beta_3) - a_2b_2\pi^a(\alpha_3+\beta_3)\right),$$

$$v(\ell_3) = a + r_3, \quad r_3 = v(a_2+b_2),$$

$$v(u_1) = v(\pi^a(a_2+b_2) + \alpha_3 + \beta_3), \quad v(\ell_2) = v(\alpha_3+\beta_3).$$

In particular, $v(\ell_1) = v(u_1) - a - r$.

Case TN8A/B/C/D : Here we let $a < b$ so that $v(\Delta_G) = a$, $v(\ell_3) = a$ and $v(\ell_1) = v(u_1) - a$. $v(u_1) = 0$ if $v(\ell_2) = 0$ or $v(u_1) \geq \min\{a, v(\ell_2)\}$ and by semistability criterion 3.4.29, we have $v(\ell) \in 2\mathbb{Z}$. From P.4, P.5 and P.6 we have

$$v(A_1 - B_1)^2 = -2v(\Delta_G) - 2v(\ell_1) = -2a - 2v(u_1) + 2a = -2v(u_1),$$

$$v(A_2 - B_2)^2 = -2v(\ell_2), \quad v(A_3 - B_3)^2 = a + b - 2v(\ell_3) = a + b - 2a = b - a,$$

and P.7, P.8 and P.9 give

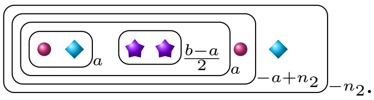
$$v((A_2-A_3)(A_2-B_3)(B_2-A_3)(B_2-B_3)) = 2v(\Delta_G) - 2v(\ell_2) - 2v(\ell_3)$$

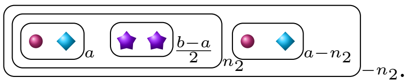
$$= 2a - 2v(\ell_2) - 2a = -2v(\ell_2),$$

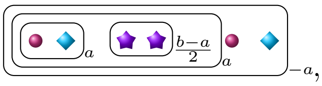
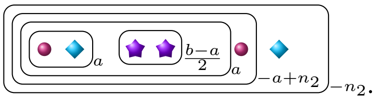
$$v((A_3-A_1)(A_3-B_1)(B_3-A_1)(B_3-B_1)) = -2v(\ell_1) - 2v(\ell_3) = -2v(u_1) + 2a - 2a = -2v(u_1),$$

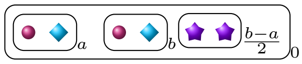
$$v((A_1-A_2)(A_1-B_2)(B_1-A_2)(B_1-B_2)) = -2v(\ell_1) - 2v(\ell_2) = -2v(u_1) + 2a - 2v(\ell_2).$$

If $v(\ell_2)=0$ then $v(u_1)=0$ and we have, without loss of generality, let $v(A_1-A_2)=v(B_1-B_2)=a$ and $v(A_3-B_3)=\frac{b-a}{2}$ so that the cluster picture of $\hat{C}$ is that of Table 4.20 for TN8A/B/C/D. Otherwise,

if $v(\ell_2)=n_2>0$ with $n_2>a$, then $v(u_1)=a$ and the cluster picture for $\hat{C}$ is 

If $v(\ell_2)=n_2>0$ with $n_2<a$, then $v(u_1)=v(\ell_2)$ and the cluster picture for $\hat{C}$ is 

If $v(\ell_2)=n_2>0$ with $n_2=a$, then if $v(u_1)=v(\ell_2)=a$, the cluster picture for $\hat{C}$ is , and if $v(u_1)>v(\ell_2)=a$, the cluster picture for $\hat{C}$ is 

All are in the equivalence class of  as required.

<u>Frobenius action</u>. By Proposition 3.4.30, the reduction at the node at α_1 is split if and only if $T_{\alpha_1}=2c\alpha_1(\alpha_1-\beta_2)(\alpha_1-\alpha_3)(\alpha_1-\beta_3)\equiv_{\square}1$ and the reduction at the node at $-\alpha_1$ is split if and only if $T_{-\alpha_1}=-2c\alpha_1(-\alpha_1-\alpha_2)(-\alpha_1-\alpha_3)(-\alpha_1-\beta_3)\equiv_{\square}1$. It follows that C is of type $I^{+,+}_{2a,2b}$, $I^{-,+}_{2a,2b}$, $I^{+,-}_{2a,2b}$ $I^{-,-}_{2a,2b}$ in TN8A/B/C/D respectively.

Now computing invariants, one finds that $(A_1-B_1)^2\equiv_{\square}(\alpha_3+\beta_3)^2T_{\alpha_1}T_{-\alpha_1}$. Here $(\alpha_3+\beta_3)^2\in K^{\times 2}$, it follows that $(A_1-B_1)^2\in K^{\times 2}$ for TN8A/D and $(A_1-B_1)^2\notin K^{\times 2}$ for TN8B/C. Also, using Proposition 3.4.30, one finds that $\hat{C}$ is of type U^+ (U^- respectively) if $\ell\in K^{\times 2}$ ($\ell\notin K^{\times 2}$ respectively). Computing ℓ yields $\ell\equiv_{\square}c(\alpha_1+\beta_3)(\alpha_1+\alpha_3)\equiv T_{-\alpha_1}$. Therefore $\ell\in K^{\times 2}$ for TN8A/B and $\ell\notin K^{\times 2}$ for TN8C/D. This yields that $\hat{C}$ is of type $U^{+}_{2a,2a,b-a}$ for TN8A, $U^{+}_{2a\tilde{}2a,b-a}$ for TN8B, $U^{-}_{2a\tilde{}2a,b-a}$ for TN8C, $U^{-}_{2a,2a,b-a}$ for TN8D.

<u>Tamagawa numbers and deficiency</u>:

λ_v is clear for computations of Tamagawa numbers and deficiency given in Table 3.1 except for TN8D. Here $D=gcd(2a,b-a)$ with $2a\in 2\mathbb{Z}$. Hence $D\equiv b-a \bmod 2$. It follows that if $b-a$ is odd, then $\frac{4a^2}{D}$ is even so that $\hat{c}_v=2$, Otherwise, $\frac{4a^2}{D}$ is even and D is even, hence $\hat{c}_v=4$. Therefore $\lambda_v=(-1)^{b-a}$.

<u>Case TN8E/F/G/H/I</u> : Here $a=b$ but we let $v(\Delta_G)=2a$. Recall that if $a=b$

$$\Delta_G=c\pi^a(a_2(\alpha_1+\alpha_3)(\alpha_1+\beta_3)+b_2(\alpha_1-\alpha_3)(\alpha_1-\beta_3)-a_2b_2\pi^a(\alpha_3+\beta_3)),$$

$$u_1=\pi^a(a_2+b_2)-\alpha_3-\beta_3,\quad \ell_2=c(\alpha_3+\beta_3),\quad \ell_3=-c\pi^a(a_2+b_2).$$

Therefore

$$v(\Delta_G) = a+r, \quad r = v\left(a_2(\alpha_1+\alpha_3)(\alpha_1+\beta_3) + b_2(\alpha_1-\alpha_3)(\alpha_1-\beta_3) - a_2b_2\pi^a(\alpha_3+\beta_3)\right),$$

$$v(\ell_3) = a + n_3, \quad r_3 = v(a_2+b_2),$$

$$v(u_1) = v(\pi^a(a_2+b_2)+\alpha_3+\beta_3), \quad v(\ell_2) = v(\alpha_3+\beta_3).$$

In particular, we set $r = 0$ so that $v(\ell_1) = v(u_1) - a$ and $v(u_1) = 0$ if $v(\ell_2) = 0$, otherwise if $v(\ell_2) = n_2 > 0$, then $v(u_1) \geq \min\{n_2, a\}$. By semistability criterion 3.4.29, we have $v(\ell) \in 2\mathbb{Z}$. From P.4, P.5 and P.6 we have

$$v(A_1-B_1)^2 = -2v(\Delta_G) - 2v(\ell_1) = -2a - 2v(u_1) + 2a + 2r = -2v(u_1),$$

$$v(A_2-B_2)^2 = -2v(\ell_2), \quad v(A_3-B_3)^2 = 2a - 2a - 2n_3 = -2n_3,$$

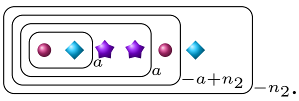
and P.7, P.8 and P.9 give

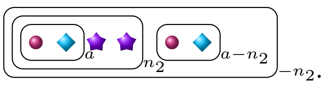
$$v((A_2-A_3)(A_2-B_3)(B_2-A_3)(B_2-B_3)) = 2v(\Delta_G) - 2v(\ell_2) - 2v(\ell_3) = -2v(\ell_2) - 2n_3,$$

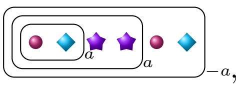
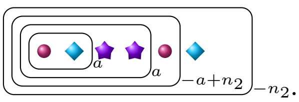
$$v((A_3-A_1)(A_3-B_1)(B_3-A_1)(B_3-B_1)) = -2v(\ell_1) - 2v(\ell_3) = -2v(u_1) - 2n_3,$$

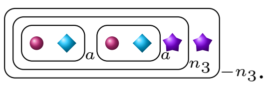
$$v((A_1-A_2)(A_1-B_2)(B_1-A_2)(B_1-B_2)) = -2v(\ell_1) - 2v(\ell_2) = -2v(u_1) + 2a - 2v(\ell_2).$$

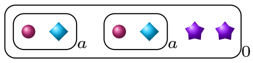
If $n_3 = v(\ell_2) = v(u_1) = 0$ then we have, without loss of generality, let $v(A_1 - A_2) = v(B_1 - B_2) = a$ so that the cluster picture of $\hat{C}$ is that of Table 4.21 for TN8E/F/G/H/I. Otherwise, assume that $n_3 = 0$, then

if $v(\ell_2) = n_2 > 0$ with $n_2 > a$, then $v(u_1) = a$ and the cluster picture for $\hat{C}$ is .

If $v(\ell_2) = n_2 > 0$ with $n_2 < a$, then $v(u_1) = v(\ell_2)$ and the cluster picture for $\hat{C}$ is .

If $v(\ell_2) = n_2 > 0$ with $n_2 = a$, then if $v(u_1) = v(\ell_2) = a$, the cluster picture for $\hat{C}$ is , and if $v(u_1) > v(\ell_2) = a$, the cluster picture for $\hat{C}$ is .

Finally, if $n_3 > 0$ then since $v(\Delta_G) = a$ we have $v(u_1) = v(\ell_2) = 0$ and the cluster picture of $\hat{C}$ is .

All are in the equivalence class of  as required.

Frobenius action. By Proposition 3.4.30, the reduction at the node at α_1 is split if and only if $T_{\alpha_1} = 2c\alpha_1(\alpha_1-\beta_2)(\alpha_1-\alpha_3)(\alpha_1-\beta_3) \equiv_\square 1$ and the reduction at the node at $-\alpha_1$ is split if and only if $T_{-\alpha_1} = -2c\alpha_1(-\alpha_1-\alpha_2)(-\alpha_1-\alpha_3)(-\alpha_1-\beta_3) \equiv_\square 1$. It follows that C is of type $I^{+,+}_{2a,2a}$, $I^{-,+}_{2a,2a}$, $I^{-,-}_{2a,2a}$ in TN8E/F/G respectively.

Now computing invariants, one finds that $(A_1-B_1)^2 \equiv_\square T_{\alpha_1}T_{-\alpha_1}$. It follows that $(A_1-B_1)^2 \in K^{\times 2}$ for TN8F and $(A_1-B_1)^2 \notin K^{\times 2}$ for TN8E/G. Similarly for $\hat{C}$, using Proposition 3.4.30 at the nodes A_1 and B_1, one finds that the reduction at the node at A_1 is split if and only if $T_{A_1} = \ell(A_1-B_1)(A_1-B_2)(A_1-A_3)(A_1-B_3) \equiv_\square 1$. Finally, the reduction at the node at B_1 is split if and only if $T_{B_1} \equiv \ell(B_1-A_1)(B_1-A_2)(B_1-A_3)(B_1-B_3) \equiv_\square 1$. However, computing T_{A_1}, T_{B_1} we have that $T_{A_1}T_{B_1} \equiv_\square \delta_2 \equiv \delta_2$. Therefore $\hat{C}$ is of type $I^{+,+}_{2a,2a}$, $I^{+,+}_{2a,2a}$, $I^{-,-}_{2a,2a}$ in TN8E/F/G respectively. We note that the same remark as in Remark 5.3.13 can be done about TN8G. Finally, for TN8H/I, the proof follows directly from that of TN7H/I.

Case TN8J/K/L/M/N : Here $a = b$ but we let $v(\Delta_G) = a + r > a$. Recall that

$$\Delta_G = c\pi^a(a_2(\alpha_1+\alpha_3)(\alpha_1+\beta_3) + b_2(\alpha_1-\alpha_3)(\alpha_1-\beta_3) - a_2b_2\pi^a(\alpha_3+\beta_3)),$$

$$u_1 = \pi^a(a_2+b_2) - \alpha_3 - \beta_3, \quad \ell_2 = c(\alpha_3+\beta_3), \quad \ell_3 = -c\pi^a(a_2+b_2).$$

In particular, $v(\ell_1) = v(u_1) - a - r$, $v(\ell_3) = a + r_3$, where $v(a_3+b_3) = r_3$. Note that since $v(\Delta_G) > a$ we have $a_2(\alpha_1+\alpha_3)(\alpha_1+\beta_3) \equiv -b_2(\alpha_1-\alpha_3)(\alpha_1-\beta_3)$. Now if $v(\ell_2) > 0$ or $r_3 > 0$ this yields another congruence between roots, contradicting the cluster picture of C. Therefore $r_3 = v(\ell_2) = v(u_1) = 0$.

From P.4, P.5 and P.6 we have

$$v(A_1-B_1)^2 = -2v(\Delta_G) - 2v(\ell_1) = -2a - 2r - 2v(u_1) + 2a + 2r = 0,$$

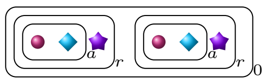
$$v(A_2-B_2)^2 = -0, \quad v(A_3-B_3)^2 = 2a - 2a - 2r_3 = 0,$$

and P.7, P.8 and P.9 give

$$v((A_2-A_3)(A_2-B_3)(B_2-A_3)(B_2-B_3)) = 2v(\Delta_G) - 2v(\ell_2) - 2v(\ell_3) = 2r,$$

$$v((A_3-A_1)(A_3-B_1)(B_3-A_1)(B_3-B_1)) = -2v(\ell_1) - 2v(\ell_3) = 2r,$$

$$v((A_1-A_2)(A_1-B_2)(B_1-A_2)(B_1-B_2)) = -2v(\ell_1) - 2v(\ell_2) = 2r.$$

Thus we have the following cluster picture  as required.

The Frobenius action, Tamagawa numbers and deficiency follows from the previous case, except for TN8K/N where deficiency is different but readily computable from Table 3.1.

5.3.4 C is of type $U_{2a,2b,2n}$

Proof of Tables 4.23

Case U1A/B/C/D: By definition of the isogeny

$$v(\delta_1) = 2a, v(\delta_2) = 2b, v(\delta_3) = 2n, v(\hat{\delta_1}) = 0, v(\hat{\delta_2}) = 0, v(\hat{\delta_3}) = 0,$$

By definition of $\Delta_G, \ell_1, \ell_2, \ell_3$ we have

$$\Delta_G \equiv -2c\alpha_2\alpha_3(\alpha_2 - \alpha_3), \quad u_1 \equiv 2(\alpha_2 - \alpha_3), \quad \ell_2 \equiv 2c\alpha_3, \quad \ell_3 \equiv -2c\alpha_2,$$

and $\ell \equiv_\square c$. Since $G(x)$ is a $C_2 \times D_4$ polynomial, we have $\alpha_1 \equiv 0 \not\equiv \alpha_2 \not\equiv \alpha_3$ so that $v(\Delta_G) = v(u_1) = v(\ell_2) = v(\ell_3) = 0$. From P.4, P.5 and P.6 we have

$$v(A_1 - B_1)^2 = -2v(\Delta_G) - 2v(\ell_1) = 0, \quad v(A_2 - B_2)^2 = -2v(\ell_2) = 0,$$

$$v(A_3 - B_3)^2 = -2v(\ell_3) = 0,$$

and P.7, P.8 and P.9 give

$$v((A_2 - A_3)(A_2 - B_3)(B_2 - A_3)(B_2 - B_3)) = 2a + 2v(\Delta_G) - 2v(\ell_2) - 2v(\ell_3) = 2a,$$

$$v((A_3 - A_1)(A_3 - B_1)(B_3 - A_1)(B_3 - B_1)) = 2b - 2v(\ell_1) - 2v(\ell_3) = 2b,$$

$$v((A_1 - A_2)(A_1 - B_2)(B_1 - A_2)(B_1 - B_2)) = 2n - 2v(\ell_1) - 2v(\ell_2) = 2n.$$

It follows that, without loss of generality, $v(A_2 - A_3) = 2a$, $v(A_1 - B_3) = 2b$ and $v(B_1 - A_2) = 2n$ so that the cluster picture of $\hat{C}$ is that of Table 4.23 for U1A/B/C/D.

Frobenius action. Since $G(x)$ is a $C_2 \times D_4$ polynomial, we have $\delta_1 \in K$, therefore either $\delta_2, \delta_3 \in K$ or $Frob$ swaps δ_2, δ_3. In particular in that case, $b = n \in \mathbb{Z}$ (otherwise I_K swaps the two clusters in the cluster picture of C which contradicts the semistability criterion 3.4.29).

By Proposition 3.4.30, C is of type U^+ if and only if $c \in K^{\times 2}$. Since $\ell \equiv_\square c$, it follows that $\hat{C}$ is of type U^+ whenever C is. Moreover, computing A_1, B_1, one finds that $(A_1 - B_1)^2 \equiv (\alpha_2 - \alpha_3)^2$. It follows that C is of type $U^+_{2a,2b,2n}$, $U^-_{2a,2b,2n}$, $U^+_{2a,2b\tilde{}2b}$, $U^-_{2a,2b\tilde{}2b}$ for U1A/B/C/D respectively and that $\hat{C}$ is of type $U^+_{4a,4b,4n}$, $U^-_{4a,4b,4n}$, $U^+_{4a,4b\tilde{}4b}$, $U^-_{4a,4b\tilde{}4b}$ for U1A/B/C/D respectively.

Tamagawa numbers and deficiency: Tamagawa numbers, deficiency and λ_v

are clear from Table 3.1, except for U1B. Here $N = 4ab + 4an + 4bn$ and $D = gcd(2a, 2b, 2n)$. We have the following :
if $2a, 2b, 2n$ are odd, then N is odd, D is odd and $\frac{N}{D}$ is odd so that $c_v = 1$ and $m_v = 2$,
if $2a, 2b, 2n$ are even, then N is even, D is even and $\frac{N}{D}$ is even so that $c_v = 4$ and $m_v = 1$,
if one of $2a, 2b, 2n$ is odd, then N is even, D is odd and $\frac{N}{D}$ is even so that $c_v = 2$ and $m_v = 1$,
if two of $2a, 2b, 2n$ are odd, then N is odd, D is odd and $\frac{N}{D}$ is odd so that $c_v = 1$ and $m_v = 1$. It then follows that $\lambda_v = -1$ if and only if one or three of $2a, 2b, 2n$ is odd, which is the same as N is odd.

Proof of Tables 4.24

Case U2A/B/C/D: By definition of the isogeny

$$v(\delta_1) = 0, v(\delta_2) = 0, v(\delta_3) = 0, v(\hat{\delta_1}) = n, v(\hat{\delta_2}) = b, v(\hat{\delta_3}) = a,$$

By definition of $\Delta_G, \ell_1, \ell_2, \ell_3$ we have

$$\Delta_G \equiv 2\alpha_1 c(\alpha_1 - \beta_2)(\alpha_1 + \beta_2), \quad u_1 \equiv 2\alpha_1, \quad \ell_2 \equiv -c(\alpha_1 - \beta_2),$$

and $\ell_3 \equiv -c(\alpha_1 + \beta_2), \quad \ell \equiv c$. Since $\alpha_1 \not\equiv 0$, we have $v(\Delta_G) = v(u_1) = v(\ell_2) = v(\ell_3) = 0$. From P.4, P.5 and P.6 we have

$$v(A_1 - B_1)^2 = n - 2v(\Delta_G) - 2v(\ell_1) = n, \quad v(A_2 - B_2)^2 = b - 2v(\ell_2) = b,$$

$$v(A_3 - B_3)^2 = 2a - 2v(\ell_3) = a,$$

and P.7, P.8 and P.9 give

$$v((A_2 - A_3)(A_2 - B_3)(B_2 - A_3)(B_2 - B_3)) = -2v(\Delta_G) - 2v(\ell_2) - 2v(\ell_3) = 0,$$

$$v((A_3 - A_1)(A_3 - B_1)(B_3 - A_1)(B_3 - B_1)) = -2v(\ell_1) - 2v(\ell_3) = 0,$$

$$v((A_1 - A_2)(A_1 - B_2)(B_1 - A_2)(B_1 - B_2)) = -2v(\ell_1) - 2v(\ell_2) = 0.$$

It follows that $v(A_1 - B_1) = \frac{n}{2}$, $v(A_2 - B_2) = \frac{b}{2}$ and $v(A_3 - B_3) = \frac{a}{2}$ so that the cluster picture of $\hat{C}$ is that of Table 4.24 for U2A/B/C/D.
Frobenius action. Since $G(x)$ is a $C_2 \times D_4$ polynomial, we have either $\delta_1 \in K^{\times 2}$ and $\delta_2, \delta_3 \in K$ or $\delta_1 \notin K^{\times 2}$ and $\delta_2, \delta_3 \notin K$. In particular in both case, $a, b, n \in \mathbb{Z}$ (otherwise I_K swaps the two clusters in the cluster picture of C which contradicts the semistability criterion 3.4.29).

By Proposition 3.4.30, C is of type U^+ if and only if $c \in K^{\times 2}$. It follows that C is of type $U^+_{2a,2b,2n}$, $U^-_{2a,2b,2n}$, $U^+_{2a,2b\tilde{}2b}$, $U^-_{2a,2b\tilde{}2b}$ for U2A/B/C/D respectively. Moreover, since $\delta_2 \equiv (\alpha_1 - \beta_2)^2$ and $\delta_3 \equiv (\alpha_1 + \beta_2)^2$ and $\ell \equiv_\square c$, it follows that $\hat{C}$ is of type U^+ whenever C is so that $\hat{C}$ is of type $U^+_{a,b,n}$, $U^-_{a,b,n}$, $U^+_{a,b\tilde{}b}$, $U^-_{a,b\tilde{}b}$ for U2A/B/C/D respectively.
Tamagawa numbers and deficiency: Tamagawa numbers, deficiency and λ_v are clear from Table 3.1, except for U2B. Here $N = ab + an + bn$ and $D = gcd(a, b, n)$. We have the following :
if a, b, n are odd, then N is odd, D is odd and $\frac{N}{D}$ is odd so that $c_v = 1$ and $m_v = 2$,
if a, b, n are even, then N is even, D is even and $\frac{N}{D}$ is even so that $c_v = 4$ and $m_v = 1$,
if one of a, b, n is odd, then N is even, D is odd and $\frac{N}{D}$ is even so that $c_v = 2$ and $m_v = 1$,
if two of a, b, n are odd, then N is odd, D is odd and $\frac{N}{D}$ is odd so that $c_v = 1$ and $m_v = 1$. It then follows that $\lambda_v = -1$ if and only if one or three of a, b, n is odd, which is the same as N is odd.

Proof of Tables 4.25, 4.26 and 4.27

By definition of the isogeny

$$v(\delta_1) = 2n, v(\delta_2) = 0, v(\delta_3) = 0, v(\hat{\delta_1}) = a + b, v(\hat{\delta_2}) = 0, v(\hat{\delta_3}) = 0,$$

where $2a, 2b, 2n \in \mathbb{Z}$.
Write

$$\alpha_1 = a_1\pi^n, \quad \alpha_3 = \alpha_2 + a_3\pi^a, \quad \beta_3 = \beta_2 + b_3\pi^b,$$

for some $a_1, a_3, b_3 \in \mathcal{O}_K^\times$. By definition of $\Delta_G, \ell_1, \ell_2, \ell_3$ we have

$$\Delta_G = c\pi^a \Big(a_3(\beta_2 - \alpha_1)(\beta_2 + \alpha_1) + b_3\pi^{b-a}(\alpha_2 - \alpha_1)(\alpha_2 + \alpha_1) + a_3b_3\pi^b(\alpha_2 + \beta_2)\Big),$$

$$u_1 = \pi^a(a_3 + b_3\pi^{b-a}), \quad \ell_2 = c(\pi^a(a_3 + b_3\pi^{b-a}) + \alpha_2 + \beta_2), \quad \ell_3 = -c(\alpha_2 + \beta_2).$$

Case U3A/B: Here $a < b$ hence

$$\Delta_G = c\pi^a\left(a_3(\beta_2 - \alpha_1)(\beta_2 + \alpha_1)\right) + o(\pi^a),$$

$$u_1 = \pi^a a_3 + o(\pi^a), \quad \ell_2 = \alpha_2 + \beta_2 + o(\pi^a), \quad \ell_3 = -c(\alpha_2 + \beta_2).$$

Therefore $v(u_1) = v(\Delta_G) = a$ so that $v(\ell_1) = 0$ and $v(\ell_2) = v(\ell_3)$.
From P.4, P.5 and P.6 we have

$$v(A_1 - B_1)^2 = a + b - 2v(\Delta_G) - 2v(\ell_1) = b - a, \quad v(A_2 - B_2)^2 = -2v(\ell_2),$$

$$v(A_3 - B_3)^2 = -2v(\ell_3),$$

and P.7, P.8 and P.9 give

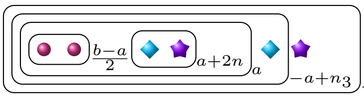
$$v((A_2 - A_3)(A_2 - B_3)(B_2 - A_3)(B_2 - B_3)) = 2n + 2v(\Delta_G) - 2v(\ell_2) - 2v(\ell_3)$$

$$= 2n + 2a - 2v(\ell_2) - 2v(\ell_3),$$

$$v((A_3 - A_1)(A_3 - B_1)(B_3 - A_1)(B_3 - B_1)) = -2v(\ell_1) - 2v(\ell_3) = -2v(\ell_3),$$

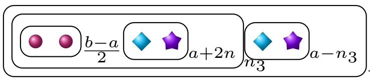
$$v((A_1 - A_2)(A_1 - B_2)(B_1 - A_2)(B_1 - B_2)) = -2v(\ell_1) - 2v(\ell_2) = -2v(\ell_2).$$

Assume that $v(\ell_2) = v(\ell_3) = 0$. It follows that $v(A_1 - B_1) = \frac{b-a}{2}$ and without loss of generality $v(A_2 - A_3) = a$ and $v(B_2 - B_3) = a + 2n$ so that the cluster picture of $\hat{C}$ is that of Table 4.25 for U3A/B. On the other hand,
if $v(\ell_3) = n_3 > a$ then $v(\ell_2) = a$ and the cluster picture for $\hat{C}$ is



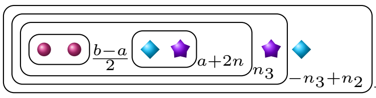
$_{-n_3}$.

If $v(\ell_3) = n_3 \leq a$ and $v(\ell_2) = n_3$ and the cluster picture for $\hat{C}$ is

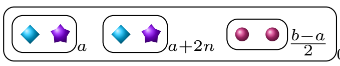


$_{-n_3}$.

If $v(\ell_3) = a$ and $v(\ell_2) = n_2 > n_3$ then the cluster picture for $\hat{C}$ is



$_{-n_2}$.

All are in the equivalence class of 

$_0$, as required.

Frobenius action. First note that since $G(x)$ is a $C_2 \times D_4$ polynomial, if $\delta_2, \delta_3 \notin K$ then $2a$ is odd if and only if $2b$ is odd. By Proposition 3.4.30, C is of type U^+ if and only if $c \in K^{\times 2}$. It follows that C is of type $U^+_{2a,2b,2n}$, $U^-_{2a,2b,2n}$ for U3A/B respectively. Moreover, since $\ell = c(\alpha_2 + \beta_2)^2\beta_2^2 + o(\pi)$, it follows from the cluster picture of C that $\hat{C}$ is of type U^+ whenever C is so that $\hat{C}$ is of type $U^+_{2a,2a+4n,b-a}$, $U^-_{2a,2a+4n,b-a}$ for U3A/B respectively.

Tamagawa numbers and deficiency : Tamagawa numbers, deficiency and λ_v are clear from Table 3.1, except for U4B. Here $N = 4ab + 4an + 4bn$ and $D = gcd(2a, 2b, 2n)$, $\hat{D} = gcd(2a, b - a)$. We have either i) $2a, 2b, 2n$ are odd; or ii) $2a, 2b$ are odd and $2n$ is even; or iii) $2a, 2b$ are even and $2n$ is odd; or iv) $2a, 2b, 2n$ are

even.
i) N is odd, D is odd and $\frac{N}{D}$ is odd so that $c_v = 1$ and $m_v = 1$; $\hat{D}$ is odd so that $\hat{c}_v = 1$ and $\hat{m}_v = 2$ if and only if $b - a$ is odd.
ii) N is odd, D is odd and $\frac{N}{D}$ is odd so that $c_v = 1$ and $m_v = 2$; $\hat{D}$ is odd so that $\hat{c}_v = 1$ and $\hat{m}_v = 2$ if and only if $b - a$ is odd.
iii) N is even, D is odd and $\frac{N}{D}$ is even so that $c_v = 2$ and $m_v = 1$; $\hat{D}$ is even if and only if $b - a$ is even so that $\hat{c}_v = 4$ if and only if $b - a$ is even, and $\hat{m}_v = 1$.
iv) N is even, D is even and $\frac{N}{D}$ is even so that $c_v = 4$ and $m_v = 1$; $\hat{D}$ is even if and only if $b - a$ is even so that $\hat{c}_v = 4$ if and only if $b - a$ is even, and $\hat{m}_v = 1$.
It follows that $\lambda_v = 1$ if and only if $2n \equiv b - a \mod 2$, equivalently, $\lambda_v = (-1)^{2n+b+a}$ as required.
Case U3C/D/E/F: Here $a = b$ but we let $v(\Delta_G) = a$ hence

$$\Delta_G = c\pi^a \Big(a_3(\beta_2 - \alpha_1)(\beta_2 + \alpha_1) + b_3(\alpha_2 - \alpha_1)(\alpha_2 + \alpha_1) + a_3 b_3 \pi^b(\alpha_2 + \beta_2)\Big),$$

$$u_1 = \pi^a(a_3 + b_3), \quad \ell_2 = c(\pi^a(a_3 + b_3)) + \alpha_2 + \beta_2), \quad \ell_3 = -c(\alpha_2 + \beta_2).$$

Write $v(a_3 + b_3) = n_1$ so that $v(\ell_1) = n_1$ and note that since $v(\Delta_G) = a$, we have either $n_1 = 0$ and $v(\ell_2) \geq v(\ell_3)$, or $n_1 > 0$ and $v(\ell_2) = v(\ell_3) = 0$.
From P.4, P.5 and P.6 we have

$$v(A_1 - B_1)^2 = 2a - 2v(\Delta_G) - 2v(\ell_1) = -2n_1, \quad v(A_2 - B_2)^2 = -2v(\ell_2),$$

$$v(A_3 - B_3)^2 = -2v(\ell_3),$$

and P.7, P.8 and P.9 give

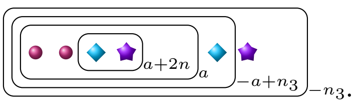
$$v((A_2 - A_3)(A_2 - B_3)(B_2 - A_3)(B_2 - B_3)) = 2n + 2v(\Delta_G) - 2v(\ell_2) - 2v(\ell_3)$$

$$= 2n + 2a - 2v(\ell_2) - 2v(\ell_3),$$

$$v((A_3 - A_1)(A_3 - B_1)(B_3 - A_1)(B_3 - B_1)) = -2v(\ell_1) - 2v(\ell_3) = -2n_1 - 2v(\ell_3),$$

$$v((A_1 - A_2)(A_1 - B_2)(B_1 - A_2)(B_1 - B_2)) = -2v(\ell_1) - 2v(\ell_2) = -2n_1 - 2v(\ell_2).$$

Assume that $n_1 = v(\ell_2) = v(\ell_3) = 0$. It follows that $v(A_1 - B_1) = 0$ and without loss of generality $v(A_2 - A_3) = a$ and $v(B_2 - B_3) = a + 2n$ so that the cluster picture of $\hat{C}$ is that of Table 4.26 for U3C/D/E/F. On the other hand, assume that $n_1 = 0$ then
if $v(\ell_3) = n_3 > a$ then $v(\ell_2) = a$ and the cluster picture for $\hat{C}$ is



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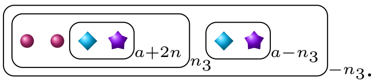
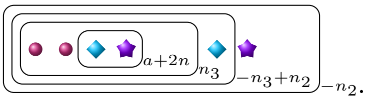
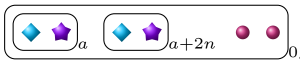
If $v(\ell_3) = n_3 \leq a$ and $v(\ell_2) = n_3$ and the cluster picture for $\hat{C}$ is



If $v(\ell_3) = a$ and $v(\ell_2) = n_2 > n_3$ then the cluster picture for $\hat{C}$ is



All are in the equivalence class of 

, as required.

Frobenius action.

For U3C/D, by Proposition 3.4.30, C is of type U^+ if and only if $c \in K^{\times 2}$. It follows that C is of type $U^+_{2a,2b,2n}$, $U^-_{2a,2b,2n}$, $U^+_{2a\tilde{~}2a,2n}$, $U^-_{2a~\tilde{2}a,2n}$ for U3C/D/E/F respectively. Also, for U3C/D, given the cluster picture of C we have either, $\alpha_2, \beta_2, \alpha_3, \beta_3 \in K$ or I_K permutes α_2 and α_3 as well as β_2 and β_3. In particular $I_K(\delta_2) = \delta_3$ and by Hensel's lemma, $\delta_2, \delta_3, \alpha_2^2, \beta_2^2 \in K^{\times 2}$. Moreover, $(A_2 - B_2)^2 \in K^{\times 2}$ and by Proposition 3.4.30, we find that the reduction of $\hat{C}$ at the node A_2 is split if and only if $T_{A_2} = \ell(A_2 - A_1)(A_2 - B_1)(A_2 - B_2)^2 \in K^{\times 2}$. Similarly, the reduction of $\hat{C}$ at the node B_2 is split if and only if $T_{B_2} = \ell(B_2 - A_1)(B_2 - B_1)(B_2 - A_2)^2 \in K^{\times 2}$. Now, computing T_{A_2}, T_{B_2}, one finds that $T_{A_2} \in K^{\times 2} \Leftrightarrow c\delta_2 \in K^{\times 2}$ and $T_{B_2} \in K^{\times 2} \Leftrightarrow c \in K^{\times 2}$. Therefore $\hat{C}$ is of type $I^{+,+}_{2a,2a+4n}$ and $I^{-,-}_{2a,2a+4n}$ for U3C/D respectively. Finally, for U3E/F, we have that $\delta_2, \delta_3 \notin K^{\times 2}$ so that $\hat{C}$ is of type $I^{-,+}_{2a,2a+4n}$ and $I^{+,-}_{2a,2a+4n}$ respectively.

Tamagawa numbers and deficiency: Tamagawa numbers, deficiency and λ_v are clear from Table 3.1, except for U3D/H. Here $N = 4a^2 + 8an$ so that N is odd if and only if $2a$ is odd, and $D = gcd(2a, 2n)$. We have either i) $2a, 2n$ are odd; or ii) $2a$ is odd and $2n$ is even; or iii) $2a$ is even and $2n$ is odd; or iv) $2a, 2n$ are even.

i) N is odd, D is odd and $\frac{N}{D}$ is odd so that $c_v = 1$ and $m_v = 2$; $\hat{c}_v = 1$ and $\hat{m}_v = 1$.

ii) N is odd, D is odd and $\frac{N}{D}$ is odd so that $c_v = 1$ and $m_v = 1$; $\hat{c}_v = 1$ and $\hat{m}_v = 1$.

iii) N is even, D is odd and $\frac{N}{D}$ is even so that $c_v = 2$ and $m_v = 1$; $\hat{c}_v = 4$ and $\hat{m}_v = 1$.

iv) N is even, D is even and $\frac{N}{D}$ is even so that $c_v = 4$ and $m_v = 1$; $\hat{c}_v = 4$ and $\hat{m}_v = 1$.

It follows that $\lambda_v = 1$ if and only if $2n \equiv 0 \bmod 2$, equivalently, $\lambda_v = (-1)^{2n}$ as required.

Case U3G/H/I/J: Here $a = b$ and we let $v(\Delta_G) = a + r > a$ hence

$$\Delta_G = c\pi^a \Big(a_3(\beta_2 - \alpha_1)(\beta_2 + \alpha_1) + b_3(\alpha_2 - \alpha_1)(\alpha_2 + \alpha_1) + a_3 b_3 \pi^b (\alpha_2 + \beta_2) \Big),$$

$$u_1 = \pi^a(a_3 + b_3), \quad \ell_2 = c(\pi^a(a_3 + b_3) + \alpha_2 + \beta_2), \quad \ell_3 = -c(\alpha_2 + \beta_2).$$

Write $v(a_3+b_3)=n_1$ so that $v(\ell_1)=n_1-r$ and $v(\ell_2)\geq v(\ell_3)$. Note that since $v(\Delta_G)>a$ we have $a_3(\beta_2-\alpha_1)(\beta_2+\alpha_1)\equiv -b_3(\alpha_2-\alpha_1)(\alpha_2+\alpha_1)$. Now if $v(\ell_3)>0$ or $v(u_1)>a$ this yields another congruence between roots, contradicting the cluster picture of C. Therefore $v(\ell_3)=v(\ell_2)=n_1=0$.

From P.4, P.5 and P.6 we have

$$v(A_1-B_1)^2=2a-2v(\Delta_G)-2v(\ell_1)=0,\quad v(A_2-B_2)^2=0,$$

$$v(A_3-B_3)^2=-0,$$

and P.7, P.8 and P.9 give

$$v((A_2-A_3)(A_2-B_3)(B_2-A_3)(B_2-B_3))=2n+2v(\Delta_G)-2v(\ell_2)-2v(\ell_3)=2n+2a+2r,$$

$$v((A_3-A_1)(A_3-B_1)(B_3-A_1)(B_3-B_1))=-2v(\ell_1)-2v(\ell_3)=2r,$$

$$v((A_1-A_2)(A_1-B_2)(B_1-A_2)(B_1-B_2))=-2v(\ell_1)-2v(\ell_2)=2r.$$

Thus the cluster picture of $\hat{C}$ is that of Table 4.27 for U3G/H/I/J.
Frobenius action. Follows directly from cases U3C/D/E/F.

Proof of Tables 4.28, 4.29 and 4.30

By definition of the isogeny

$$v(\delta_1)=0, v(\delta_2)=0, v(\delta_3)=2n, v(\hat{\delta_1})=0, v(\hat{\delta_2})=0, v(\hat{\delta_3})=a+b,$$

where $a,b,2n\in\mathbb{Z}$ (since $G(x)$ is a $C_2\times D_4$ polynomial $\delta_1\in K$, and if $a,b\notin\mathbb{Z}$, I_K permutes both clusters, a contradiction to the semistability criterion 3.4.29).
Write

$$\alpha_2=\alpha_1+a_2\pi^a,\quad \beta_2=-\alpha_1+b_2\pi^b,\quad \beta_3=\alpha_3+b_3\pi^n,$$

for some $a_2,b_2,b_3\in\mathcal{O}_{\overline{K}}^{\times}$. By definition of $\Delta_G,\ell_1,\ell_2,\ell_3$ we have

$$\Delta_G=c\pi^a(a_2(\alpha_1+\alpha_3)^2+a_2b_3\pi^n(\alpha_1+\alpha_3)+b_2\pi^{b-a}(\alpha_1-\alpha_3)^2$$

$$-b_2b_3\pi^{b-a+n}(\alpha_1-\alpha_3)-2a_2b_2\alpha_3\pi^b-a_2b_2b_3\pi^{b+n}),$$

$$u_1=\pi^a(a_2+b_2\pi^{b-a})-2\alpha_3-b_3\pi^n,\quad \ell_2=c(2a_3+b_3\pi^n)),\quad \ell_3=-c\pi^a(a_2+b_2\pi^{b-a}).$$

Case U4A/B: Here $a<b$ hence

$$\Delta_G=c\pi^a(a_2(\alpha_1+\alpha_3)^2)+o(\pi),$$

$$u_1 = -2\alpha_3 - b_3\pi^n + o(\pi^a), \quad \ell_2 = c(2a_3 + b_3\pi^n)), \quad \ell_3 = -c\pi^a(a_2) + o(\pi^a).$$

Therefore $v(\ell_3) = v(\Delta_G) = a$ and $v(u_1) \geq \min\{v(\ell_2), a\}$.
From P.4, P.5 and P.6 we have

$$v(A_1-B_1)^2 = -2v(\Delta_G)-2v(\ell_1) = -2a-2v(u_1)+2a = -2v(u_1), \quad v(A_2-B_2)^2 = -2v(\ell_2),$$

$$v(A_3 - B_3)^2 = a + b - 2v(\ell_3) = b - a,$$

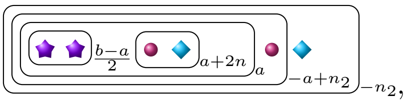
and P.7, P.8 and P.9 give

$$v((A_2 - A_3)(A_2 - B_3)(B_2 - A_3)(B_2 - B_3)) = 2v(\Delta_G) - 2v(\ell_2) - 2v(\ell_3) = -2v(\ell_2),$$

$$v((A_3 - A_1)(A_3 - B_1)(B_3 - A_1)(B_3 - B_1)) = -2v(\ell_1) - 2v(\ell_3) = -2v(u_1),$$

$$v((A_1-A_2)(A_1-B_2)(B_1-A_2)(B_1-B_2)) = 2n-2v(\ell_1)-2v(\ell_2) = 2n+2a-2v(u_1)-2v(\ell_2).$$

Assume that $v(u_1) = v(\ell_2) = 0$. It follows that $v(A_3 - B_3) = \frac{b-a}{2}$ and without loss of generality $v(A_1 - A_2) = a$ and $v(B_1 - B_2) = a + 2n$ so that the cluster picture of $\hat{C}$ is that of Table 4.28 for U4A/B. On the other hand,
if $v(\ell_2) = n_2 > a$ then $v(u_1) = a$ and the cluster picture for $\hat{C}$ is



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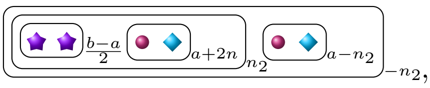
if $v(\ell_2) = n_2 \leq a$ and $v(u_1) = n_2$ and the cluster picture for $\hat{C}$ is



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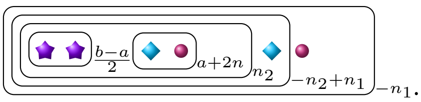
if $v(\ell_2) = a$ and $v(u_1) = n_1 > n_2$ then the cluster picture for $\hat{C}$ is



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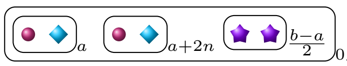
All are in the equivalence class of 

, as required.

Frobenius action. First note that since $G(x)$ is a $C_2 \times D_4$ polynomial, $a, b \in \mathbb{Z}$ for otherwise I_K would permute α_1 and α_2 a contradiction. By Proposition 3.4.30, C is of type U^+ if and only if $c \in K^{\times 2}$. It follows that C is of type $U^+_{2a,2b,2n}$, $U^-_{2a,2b,2n}$ for U4A/B respectively. Moreover, since $\ell =_\square c\alpha_3^2(\alpha_1 + \alpha_3)^2 + o(\pi)$, it follows from the cluster picture of C that $\hat{C}$ is of type U^+ whenever C is so that $\hat{C}$ is of type $U^+_{2a,2a+4n,b-a}$, $U^-_{2a,2a+4n,b-a}$ for U4A/B respectively.

Tamagawa numbers and deficiency : Tamagawa numbers, deficiency and λ_v are clear from Table 3.1, except for U4B. Here $2a, 2b \in 2\mathbb{Z}$, $N = 4ab + 4an + 4bn$ and $D = gcd(2a, 2b, 2n)$. It follows that $m_v = \hat{m}_v = 1$ and that $c_v = 2$ if $2n$ is odd, $c_v = 4$ otherwise. Similarly, $\hat{c}_v = 2$ if $b - a$ is odd, $\hat{c}_v = 4$ otherwise. It follows that

$\lambda_v = 1$ if and only if $2n \equiv b - a \bmod 2$, equivalently, $\lambda_v = (-1)^{2n+b+a}$ as required.
Case U4C/D/E/F: Here $a = b$ but we let $v(\Delta_G) = a$ hence

$$\Delta_G = c\pi^a(a_2(\alpha_1 + \alpha_3)^2 + a_2 b_3 \pi^n(\alpha_1 + \alpha_3) + b_2(\alpha_1 - \alpha_3)^2$$

$$-b_2 b_3 \pi^n(\alpha_1 - \alpha_3) - 2a_2 b_2 \alpha_3 \pi^b - a_2 b_2 b_3 \pi^{a+n}),$$

$$u_1 = \pi^a(a_2 + b_2) - 2\alpha_3 - b_3\pi^n, \quad \ell_2 = c(2a_3 + b_3\pi^n)), \quad \ell_3 = -c\pi^a(a_2 + b_2).$$

Write $v(a_2 + b_2) = n_3$ so that $v(\ell_3) = a + n_3$ and note that since $v(\Delta_G) = a$, we have either $n_3 = 0$ and $v(u_1) \geq v(\ell_3)$, or $n_3 > 0$ and $v(u_1) = v(\ell_2) = 0$.
From P.4, P.5 and P.6 we have

$$v(A_1 - B_1)^2 = -2v(\Delta_G) - 2v(\ell_1) = -2v(u_1), \quad v(A_2 - B_2)^2 = -2v(\ell_2),$$

$$v(A_3 - B_3)^2 = a + b - 2v(\ell_3) = -n_3,$$

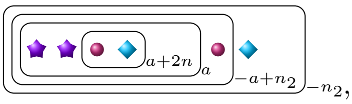
and P.7, P.8 and P.9 give

$$v((A_2 - A_3)(A_2 - B_3)(B_2 - A_3)(B_2 - B_3)) = 2v(\Delta_G) - 2v(\ell_2) - 2v(\ell_3) = -2v(\ell_2) - 2n_3,$$

$$v((A_3 - A_1)(A_3 - B_1)(B_3 - A_1)(B_3 - B_1)) = -2v(\ell_1) - 2v(\ell_3) = -2v(u_1) - 2n_3,$$

$$v((A_1 - A_2)(A_1 - B_2)(B_1 - A_2)(B_1 - B_2)) = -2v(\ell_1) - 2v(\ell_2) = 2n + 2a - 2v(u_1) - 2v(\ell_2).$$

Assume that $v(u_1) = v(\ell_2) = v(\ell_3) = 0$. It follows that $v(A_3 - B_3) = 0$ and without loss of generality $v(A_1 - A_2) = a$ and $v(B_1 - B_2) = a + 2n$ so that the cluster picture of $\hat{C}$ is that of Table 4.29 for U4C/D/E/F. On the other hand, assume that $n_3 = 0$ then
if $v(\ell_2) = n_2 > a$ then $v(u_1) = a$ and the cluster picture for $\hat{C}$ is



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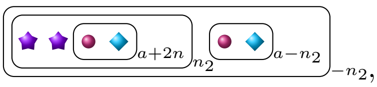
if $v(\ell_2) = n_2 \leq a$ and $v(u_1) = n_2$ and the cluster picture for $\hat{C}$ is



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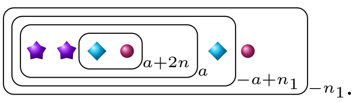
if $v(\ell_2) = a$ and $v(u_1) = n_1 > a$ then the cluster picture for $\hat{C}$ is



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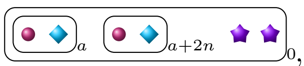
All are in the equivalence class of 

, as required.
Frobenius action.

By Proposition 3.4.30, C is of type U^+ if and only if $c \in K^{\times 2}$. It follows

that C is of type $U^{+}_{2a,2b,2n}$, $U^{-}_{2a,2b,2n}$, $U^{+}_{2a\tilde{\ }2a,2n}$, $U^{-}_{2a\tilde{2}a,2n}$ for U4C/D/E/F respectively. Also, for U4C/D, given the cluster picture of C we have $\alpha_1\alpha_2, \beta_2 \in K$ and $\beta_3 \equiv \alpha_3$, hence $\delta_1, \delta_2, \alpha_3^2 \in K^{\times 2}$. Moreover, by Proposition 3.4.30, we find that the reduction of $\hat{C}$ at the node A_1 is split if and only if $T_{A_1} = \ell(A_1 - B_1)(A_1 - B_2)(A_1 - A_3)(A_1 - B_3) \in K^{\times 2}$. Similarly, the reduction of $\hat{C}$ at the node B_1 is split if and only if $T_{B_1} \equiv \ell(B_1 - A_1)(B_1 - A_2)(B_1 - A_3)(B_1 - B_3) \in K^{\times 2}$. Now, computing T_{A_1}, T_{B_1}, one finds that $T_{A_1} \in K^{\times 2} \Leftrightarrow c\alpha_1^2\delta_2 \in K^{\times 2}$ and $T_{B_1} \in K^{\times 2} \Leftrightarrow c \in K^{\times 2}$. Therefore $\hat{C}$ is of type $I^{+,+}_{2a,2a+4n}$ and $I^{-,-}_{2a,2a+4n}$ for U4C/D respectively. Finally, for U4E/F, we have that $\delta_1, \delta_2 \notin K^{\times 2}$ so that $\hat{C}$ is of type $I^{-,+}_{2a,2a+4n}$ and $I^{+,-}_{2a,2a+4n}$ respectively.
Tamagawa numbers and deficiency: Tamagawa numbers, deficiency and λ_v are clear from Table 3.1 since $2a \in 2\mathbb{Z}$.
Case U4G/H/I/J: Here $a = b$ and we let $v(\Delta_G) = a + r > a$ hence

$$\Delta_G = c\pi^a(a_2(\alpha_1 + \alpha_3)^2 + a_2 b_3 \pi^n(\alpha_1 + \alpha_3) + b_2(\alpha_1 - \alpha_3)^2$$

$$-b_2 b_3 \pi^n(\alpha_1 - \alpha_3) - 2a_2 b_2 \alpha_3 \pi^b - a_2 b_2 b_3 \pi^{a+n}),$$

$$u_1 = \pi^a(a_2 + b_2) - 2\alpha_3 - b_3\pi^n, \quad \ell_2 = c(2a_3 + b_3\pi^n)), \quad \ell_3 = -c\pi^a(a_2 + b_2).$$

Write $v(a_2 + b_2) = r_3$ so that $v(\ell_3) = a + r_3$. Note that since $v(\Delta_G) > a$ we have $a_2(\alpha_1 + \alpha_3)(\alpha_1 + \beta_3) \equiv -b_2(\alpha_1 - \alpha_3)(\alpha_1 - \beta_3)$. Now if $v(\ell_2) > 0$ or $r_3 > 0$ this yields another congruence between roots, contradicting the cluster picture of C. Therefore $r_3 = v(\ell_2) = v(u_1) = 0$.

From P.4, P.5 and P.6 we have

$$v(A_1 - B_1)^2 = 2v(\Delta_G) - 2v(\ell_1) = 0, \quad v(A_2 - B_2)^2 = 0,$$

$$v(A_3 - B_3)^2 = 0,$$

and P.7, P.8 and P.9 give

$$v((A_2 - A_3)(A_2 - B_3)(B_2 - A_3)(B_2 - B_3)) = 2n + 2v(\Delta_G) - 2v(\ell_2) - 2v(\ell_3) = 2r,$$

$$v((A_3 - A_1)(A_3 - B_1)(B_3 - A_1)(B_3 - B_1)) = -2v(\ell_1) - 2v(\ell_3) = 2r,$$

$$v((A_1 - A_2)(A_1 - B_2)(B_1 - A_2)(B_1 - B_2)) = -2v(\ell_1) - 2v(\ell_2) = 2n + 2a + 2r.$$

Thus the cluster picture of $\hat{C}$ is that of Table 4.30 for U4G/H/I/J.
Frobenius action. Follows directly from cases U4C/D/E/F.
Tamagawa numbers and deficiency: Follows directly from cases U4C/D/E/F.

5.3.5 C is of type 1×1

Proof of Tables 4.31, 4.32, 4.33

Since C is if type 1×1 or $1 \tilde{\times} 1$, it follows from the semistability criterion 3.4.29 that $v(c) \equiv a \equiv b \bmod 2$.

Case TC1. Since $\beta_1 = -\alpha_1$, it follows that $\alpha_1 \equiv \beta_1 \equiv \alpha_2 \equiv 0 \not\equiv \beta_2 \equiv \alpha_3 \equiv \beta_3 \bmod \pi$. Reducing invariants we find

$$\ell \equiv 2c \bmod \pi, \quad \ell_1 \equiv \frac{-1}{c\beta_2^2} \bmod \pi, \quad \ell_2 \equiv 2c\beta_2 \bmod \pi, \quad \ell_3 \equiv -c\beta_2 \bmod \pi,$$

$$\Delta_G \equiv c\beta_2^3 \bmod \pi,$$

hence $v(\Delta_G) = v(c) = v(\ell)$. Now

$$v(\delta_1) = 2a + 2c, v(\delta_2) = 0, v(\delta_3) = 2b, v(\hat{\delta_1}) = 2b - 2c, v(\hat{\delta_2}) = 2c, v(\hat{\delta_3}) = 2a + 2c,$$

so that P.4, P.5 and P.6 yield

$$v(A_1 - B_1)^2 = 2b, \quad v(A_2 - B_2)^2 = 0, \quad v(A_3 - B_3)^2 = 2a,$$

and P.7, P.8 and P.9 give

$$v((A_2 - A_3)(A_2 - B_3)(B_2 - A_3)(B_2 - B_3)) = 2a,$$

$$v((A_3 - A_1)(A_3 - B_1)(B_3 - A_1)(B_3 - B_1)) = 0,$$

$$v((A_1 - A_2)(A_1 - B_2)(B_1 - A_2)(B_1 - B_2)) = 2b,$$

from which we obtain the cluster picture for $\hat{C}$.

Frobenius action. If $Frob$ swaps α_2 and β_2 then by the cluster configuration, it must swap δ_1 and δ_3. This is a contradiction since $G(x)$ is a $C_2 \times D_4$ polynomial. Similarly for $L(x)$ so that the action of $Frob$ on the cluster picture follows.

Cases TC2A/B. Since $\beta_1 = -\alpha_1$, it follows that $\alpha_1 \not\equiv 0 \bmod \pi$. Reducing invariants we find

$$\ell \equiv 2c \bmod \pi, \quad \ell_1 \equiv \frac{1}{2c\alpha_1^2} \bmod \pi, \quad \ell_2 \equiv -2c\alpha_1 \bmod \pi, \quad \ell_3 \equiv -2c\alpha_1 \bmod \pi,$$

$$\Delta_G \equiv 8c\alpha_1^3 \bmod \pi, \quad \delta_1 \equiv 4c^2\alpha_1^2 \bmod \pi, \quad \hat{\delta_1} \equiv \frac{1}{c^2\alpha_1^2} \bmod \pi,$$

and the valuations and Frobenius action of clusters follow.

Since

$$v(\delta_1) = 2v(c), \quad v(\delta_2) = 2a, \quad v(\delta_3) = 2b,$$

$$v(\hat{\delta_1}) = -2v(c), \quad v(\hat{\delta_2}) = 2b + 2v(c), \quad v(\hat{\delta_3}) = 2a + 2v(c),$$

P.4, P.5 and P.6 yield

$$v(A_1 - B_1)^2 = 0, \quad v(A_2 - B_2)^2 = 2b, \quad v(A_3 - B_3)^2 = 2a,$$

and P.7, P.8 and P.9 give

$$v((A_2 - A_3)(A_2 - B_3)(B_2 - A_3)(B_2 - B_3)) = 0,$$

$$v((A_3 - A_1)(A_3 - B_1)(B_3 - A_1)(B_3 - B_1)) = 2a,$$

$$v((A_1 - A_2)(A_1 - B_2)(B_1 - A_2)(B_1 - B_2)) = 2b,$$

from which we obtain the cluster picture for $\hat{C}$.

Cases TC3. Since $\beta_1 = -\alpha_1$, it follows that $\alpha_1 \not\equiv 0 \bmod \pi$. Write

$$\alpha_2 = \alpha_1 + a_2\pi^a, \quad \alpha_3 = \alpha_1 + a_3\pi^a, \quad \beta_2 = -\alpha_1 + b_2\pi^b, \quad \beta_3 = -\alpha_1 + b_3\pi^b,$$

where $a_2, a_3, b_2, b_3 \in \mathcal{O}_{\bar{K}}^{\times}$. It follows that $\delta_1 \equiv 4c^2\alpha_1^2 \bmod \pi$, $\quad \delta_2 \equiv \delta_3 \equiv 4\alpha_1^2 \bmod \pi$,so that $v(\delta_1) = 2v(c)$, $\quad v(\delta_2) = v(\delta_3) = 0$ and

$$v(\hat{\delta_1}) = a + b - 2v(\Delta_G), \quad v(\hat{\delta_2}) = 2v(c) + a + b, \quad v(\hat{\delta_3}) = 2v(c) + a + b,$$

and

$$u_1 = \pi^a(a_2 - a_3) + \pi^b(b_2 - b_3), \quad \ell_2 = c(a_3\pi^a + b_3\pi^b), \quad \ell_3 = -c(a_2\pi^a + b_2\pi^b),$$

$$\Delta_G = c\pi^{a+b}\left(2\alpha_1(a_2b_3 - a_3b_2) + a_2a_3\pi^a(b_3 - b_2) + b_2b_3\pi^b(a_3 - a_2)\right),$$

Case TC3A. Here $a < b$ and $v(\Delta_G) = v(c) + a + b$. Therefore

$$v(\ell_1) = -b - v(c), \quad v(\ell_2) = a + v(c), \quad v(\ell_3) = a + v(c),$$

P.4, P.5 and P.6 yield

$$v(A_1 - B_1)^2 = a + b - 2v(\Delta_G) - 2v(\ell_1) = b - a,$$

$$v(A_2 - B_2)^2 = a + b + 2v(c) - 2v(\ell_2) = b - a, \quad v(A_3 - B_3)^2 = a + b + 2v(c) - 2v(\ell_3) = b - a,$$

and P.7, P.8 and P.9 give

$$v((A_2 - A_3)(A_2 - B_3)(B_2 - A_3)(B_2 - B_3)) = 2v(\Delta_G) + 2v(c) - 2v(\ell_2) - 2v(\ell_3) = 2b - 2a,$$

$$v((A_3 - A_1)(A_3 - B_1)(B_3 - A_1)(B_3 - B_1)) = -2v(\ell_1) - 2v(\ell_3) = 2b - 2a,$$

$$v((A_1-A_2)(A_1-B_2)(B_1-A_2)(B_1-B_2)) = -2v(\ell_1) - 2v(\ell_2) = 2b - 2a.$$

Hence the cluster picture for $\hat{C}$ is that of the isogeny TC3A.

Case TC3B/C. Here $a < b$ and $v(\Delta_G) = v(c) + a + b + r$, with $r > 0$. Therefore

$$v(\ell_1) = -b - v(c) - r, \quad v(\ell_2) = a + v(c), \quad v(\ell_3) = a + v(c),$$

P.4, P.5 and P.6 yield

$$v(A_1 - B_1)^2 = a + b - 2v(\Delta_G) - 2v(\ell_1) = b - a,$$

$$v(A_2-B_2)^2 = a+b+2v(c)-2v(\ell_2) = b-a, \quad v(A_3-B_3)^2 = a+b+2v(c)-2v(\ell_3) = b-a,$$

and P.7, P.8 and P.9 give

$$v((A_2-A_3)(A_2-B_3)(B_2-A_3)(B_2-B_3)) = 2v(\Delta_G)+2v(c)-2v(\ell_2)-2v(\ell_3) = 2b-2a+2r,$$

$$v((A_3 - A_1)(A_3 - B_1)(B_3 - A_1)(B_3 - B_1)) = -2v(\ell_1) - 2v(\ell_3) = 2r + 2b - 2a,$$

$$v((A_1 - A_2)(A_1 - B_2)(B_1 - A_2)(B_1 - B_2)) = -2v(\ell_1) - 2v(\ell_2) = 2r + 2b - 2a.$$

Hence the cluster picture for $\hat{C}$ is that of the isogeny TC3B/C.

Frobenius action. Given the cluster picture for $\hat{C}$, it follows that $Frob$ permutes both clusters if and only if $(A_1 - B_1)^2 \notin K^{\times 2}$.

Case TC3D/E/F/G/H/I. Here $a = b$ so that

$$u_1 = \pi^a(a_2 - a_3 + b_2 - b_3), \quad \ell_2 = c\pi^a(a_3 + b_3), \quad \ell_3 = -c\pi^a(a_2 + b_2),$$

$$\Delta_G = c\pi^{2a}\left(2\alpha_1(a_2b_3 - a_3b_2) + \pi^a(a_2a_3(b_3 - b_2) + b_2b_3(a_3 - a_2))\right).$$

Let $n_1 = v(a_2 - a_3 + b_2 - b_3), n_2 = v(a_3 + b_3), n_3 = v(a_2 + b_2)$ so that $v(u_1) = a + n_1, v(\ell_2) = a + n_2, v(\ell_3) = a + n_3$.

Case TC3D/E. Here we set $v(\Delta_G) = v(c) + 2a$. Therefore

$$v(\ell_1) = n_1 - a - v(c), \quad v(\ell_2) = n_2 + a + v(c), \quad v(\ell_3) = n_3 + a + v(c),$$

and since $v(\Delta_G) = v(c) = 2a$, it follows that if $n_i > 0$ then $n_{i+1} = n_{i+2} = 0$ for $i = 1, 2, 3$ and where addition of indices is performed modulo 3.

P.4, P.5 and P.6 yield

$$v(A_1 - B_1)^2 = 2a - 2v(\Delta_G) - 2v(\ell_1) = -2n_1,$$

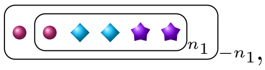
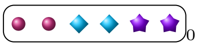
$$v(A_2-B_2)^2 = 2a+2v(c)-2v(\ell_2) = -2n_2, \quad v(A_3-B_3)^2 = 2a+2v(c)-2v(\ell_3) = -2n_3,$$

and P.7, P.8 and P.9 give

$$v((A_2-A_3)(A_2-B_3)(B_2-A_3)(B_2-B_3)) = 2v(\Delta_G)+2v(c)-2v(\ell_2)-2v(\ell_3) = -2n_2-2n_3,$$

$$v((A_3-A_1)(A_3-B_1)(B_3-A_1)(B_3-B_1)) = -2v(\ell_1) - 2v(\ell_3) = -2n_1 - 2n_3,$$

$$v((A_1-A_2)(A_1-B_2)(B_1-A_2)(B_1-B_2)) = -2v(\ell_1) - 2v(\ell_2) = -2n_1 - 2n_2.$$

If $n_1 = n_2 = n_3 = 0$ then the cluster picture for $\hat{C}$ is that of the isogenies TC3D/E. Otherwise, without loss of generality, let $n_1 > 0$. As noted above this implies $n_2 = n_3 = 0$ so that the cluster picture for $\hat{C}$ is  which is in the equivalence class of  as required.

Case TC3F/G/H/I. Here $a = b$ and $v(\Delta_G) = v(c) + a + b + r$, with $r > 0$. Therefore

$$v(\ell_1) = n_1 - a - v(c) - r, \quad v(\ell_2) = n_2 + a + v(c), \quad v(\ell_3) = n_3 + a + v(c),$$

P.4, P.5 and P.6 yield

$$v(A_1 - B_1)^2 = 2a - 2v(\Delta_G) - 2v(\ell_1) = -2n_1,$$

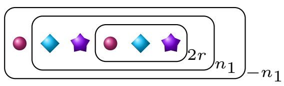
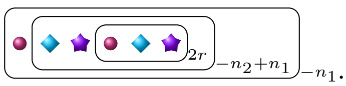
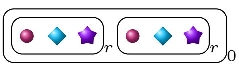
$$v(A_2-B_2)^2 = 2a+2v(c)-2v(\ell_2) = -2n_2, \quad v(A_3-B_3)^2 = 2a+2v(c)-2v(\ell_3) = -2n_3,$$

and P.7, P.8 and P.9 give

$$v((A_2-A_3)(A_2-B_3)(B_2-A_3)(B_2-B_3)) = 2v(\Delta_G)+2v(c)-2v(\ell_2)-2v(\ell_3) = 2r-2n_2-2n_3,$$

$$v((A_3-A_1)(A_3-B_1)(B_3-A_1)(B_3-B_1)) = -2v(\ell_1) - 2v(\ell_3) = 2r - 2n_1 - 2n_3,$$

$$v((A_1-A_2)(A_1-B_2)(B_1-A_2)(B_1-B_2)) = -2v(\ell_1) - 2v(\ell_2) = 2r - 2n_2 - 2n_2.$$

If $n_1 = n_2 = n_3 = 0$ then the cluster picture for $\hat{C}$ is that of the isogenies TC3F/G/H/I.
Otherwise, if $n_i > 0$ then $n_{i+1} = n_{i+2}$ for $i = 1, 2, 3$ and where addition on indices is performed modulo 3. Assume first that $n_{i+1} = n_{i+2} = 0$ and without loss of generality, let $i = 1$. Then the cluster picture of $\hat{C}$ is  Finally if $n_2 = n_3 > 0$ the cluster picture of $\hat{C}$ is  In both cases, the cluster pictures are in the same equivalence class of  as required.

Frobenius action. Given the cluster pictures for C and $\hat{C}$, it follows that $Frob$ permutes clusters if and only if $\delta_1 \notin K^{\times 2}$, $(A_1 - B_1)^2 \notin K^{\times 2}$ respectively.

Chapter 6

Proof of local discrepancy conjecture

6.1 Introduction

In this chapter we compute the term of local discrepancy E_v associated to a $C_2 \times D_4$ curve and prove the last columns of Tables 4.1 to 4.34. Recall from Definition 4.3.9 that, in addition to the leading terms of $G(x)$ and $L(x)$, c and $\ell = \ell_1\ell_2\ell_3$, we associated the following $C_2 \times D_4$ polynomial invariants to $G(x)$

$I_{20} = \frac{1}{2^3}\left(\delta_2 + \delta_3\right),$

$I_{21} = (\alpha_2 + \beta_2)(\alpha_3 + \beta_3),$

$I_{22} = \frac{1}{2}(\Delta_G^2 \ell_1^2 - \delta_2 - \delta_3) = (\alpha_2 - \alpha_3)(\beta_2 - \beta_3) + (\beta_2 - \alpha_3)(\alpha_2 - \beta_3),$

$I_{23} = 4c^2\alpha_1^2,$

$I_{40} = \frac{1}{2^6}(\delta_2 - \delta_3)^2,$

$I_{41} = 16\left(\alpha_2\beta_2\alpha_3\beta_3 + \alpha_1^2\left(\alpha_1^2 + \alpha_2\beta_2 + \alpha_3\beta_3 + (\alpha_2 + \beta_2)(\beta_3 + \alpha_3)\right)\right),$

$I_{42} = 4(2\alpha_1^2 - \alpha_2^2 - \beta_2^2)(2\alpha_1^2 - \alpha_3^2 - \beta_3^2),$

$I_{43} = \delta_2\left(\alpha_2^2 + \beta_2^2 - 2\alpha_1^2\right) + \delta_3\left(\alpha_3^2 + \beta_3^2 - 2\alpha_1^2\right),$

$I_{44} = \delta_2\delta_3 = (\alpha_2 - \beta_2)^2(\alpha_3 - \beta_3)^2,$

$I_{45} = \Delta_G^2\hat{\delta_1} = 4(\beta_3 - \beta_2)(\alpha_3 - \beta_2)(\alpha_2 - \beta_3)(\alpha_2 - \alpha_3),$

$I_{60} = 4\hat{\delta_3}\left(\alpha_2^2 + \beta_2^2 - 2\alpha_1^2\right) + 4\hat{\delta_2}\left(\alpha_3^2 + \beta_3^2 - 2\alpha_1^2\right),$

$I_{80} = \frac{1}{c^4}\hat{\delta_2}\hat{\delta_3},$

and for each place v of $\mathcal{K}$, we defined the following Hilbert symbols at v

$$H_1 = (-1, I_{22}I_{41}I_{43}I_{60}), \quad H_2 = (I_{20}, -I_{40}I_{44}), \quad H_3 = (I_{40}, \ell I_{60}I_{43}),$$

$$H_4 = (c, I_{23}I_{44}I_{80}), \quad H_5 = (I_{23}, I_{41}), \quad H_6 = (I_{45}, -\ell I_{22}I_{21}),$$

$$H_7 = (I_{44}, 2I_{22}I_{42}I_{43}), \quad H_8 = (I_{80}, -2I_{41}I_{42}I_{60}), \quad H_9 = (I_{42}, -I_{60}I_{43}),$$

and formed the following product

$$E_v = \prod_{i=1}^{9} H_i.$$

Throughout the entire chapter, we assume that none of the $I_{i,j}$, c and ℓ are zero. In Sections 4.5–4.7, we claimed that E_v correctly matches the local discrepancy between the local factors λ_v and the local root numbers ω_v. In this chapter, we explicitly compute E_v for each case and prove the claim.

We first derive a few properties of the $C_2 \times D_4$ invariants involved that will prove essential in the computations of E_v. We also include a few properties concerning Hilbert Symbols at finite places.

List of notation for this chapter

K	local field of odd residue characteristic
$\mathcal{O}_K$	ring of integers of K
K^{nr}	maximal unramified extension
k	residue field
π	uniformiser of K
v	normalized valuation on K
$(.,.)$	Hilbert Symbol at v
c	leading term of $G(x)$
$\ell_1 = \frac{u_1}{\Delta_G}$	leading term of $L_1(x)$
	with $u_1 = \alpha_2 + \beta_2 - \alpha_3 - \beta_3$
$\ell = \ell_1\ell_2\ell_3$	leading term of $L(x)$
δ_i, $i = 1,2,3$	discriminant of $G_i(x)$
$\hat{\delta}_1$	discriminant of $\frac{1}{\Delta_G} L_1(x)$
$\hat{\delta}_i$, $i = 2,3$	discriminant of $L_i(x)$
α_i, β_i, $i = 1,2,3$	roots of $G_i(x)$
A_i, B_i, $i = 1,2,3$	roots of $L_i(x)$
$x \equiv y$	$x \equiv y \bmod \pi$
$x \equiv_\square y$	$x \equiv yz$ where z is a square element in K and $x, y, z \in \mathcal{O}_K^\times$
$x =_\square y$	$x = yz$ where z is a square element in K
$Frob$	Frobenius automorphism in $Gal(\bar{K}/K)$
I_K	inertia subgroup of $Gal(\overline{K}/K)$

6.2 Properties of $C_2 \times D_4$ invariants

We derive a series of Lemmata concerning interesting properties of $C_2 \times D_4$ invariants. When readily verifiable by simple computations, the proofs are omitted.

Lemma 6.2.1.

1. $I_{20}^2 = \frac{1}{16} I_{44} + I_{40}$,
2. $I_{21} = -\frac{1}{c^2}\ell_2\ell_3$ *hence* $\ell = -c^2\ell_1 I_{21}$,
3. *Let* $A_{23} = (\alpha_2 - \alpha_3)(\beta_2 - \beta_3)$, $\quad B_{23} = (\beta_2 - \alpha_3)(\alpha_2 - \beta_3)$,
   so that $I_{45} = 4A_{23}B_{23}$ *and* $I_{22} = A_{23} + B_{23}$.
   Then $I_{22}^2 = I_{45} + I_{44}$ *and* $I_{22} = \frac{1}{2}u_1^2 - \frac{1}{2}(\delta_2 + \delta_3)$,
4. $I_{40} = \frac{u_1^2}{64} J_{40}^2$, *for some* $J_{40} \in K$
5. *Let* $A_{41p} = 8(\alpha_2 + \alpha_1)(\beta_2 + \alpha_1)(\alpha_3 + \alpha_1)(\beta_3 + \alpha_1)$,
   $A_{41m} = 8(\alpha_2 - \alpha_1)(\beta_2 - \alpha_1)(\alpha_3 - \alpha_1)(\beta_3 - \alpha_1)$. *Then*
   $I_{41} = A_{41p} + A_{41m}$ *and* $I_{80} = A_{41p}A_{41m}$. *Moreover*
   $I_{41}^2 = 16 I_{80} + 16^2\alpha_1^2 J_{41}^2 = 16 I_{80} + \frac{64}{c^2} I_{23} J_{41}^2$, *for some* $J_{41} \in K$
6. *Let* $A_{21} = (\alpha_2 - \alpha_1)(\alpha_2 + \alpha_1) + (\beta_2 - \alpha_1)(\beta_2 + \alpha_1)$,
   $A_{31} = (\alpha_3 - \alpha_1)(\alpha_3 + \alpha_1) + (\beta_3 - \alpha_1)(\beta_3 + \alpha_1)$. *Then*
   (a) $I_{42} = 4A_{21}A_{31}$,
   (b) $I_{60} = \frac{4}{c^2}(\hat{\delta_2} A_{31} + \hat{\delta_3} A_{21})$,
   (c) $I_{43} = \delta_2 A_{21} + \delta_3 A_{31}$.
   (d) $A_{21}^2 = \frac{\hat{\delta_3}}{c^2} + \delta_2(\alpha_2 + \beta_2)^2$,
   (e) $A_{31}^2 = \frac{\hat{\delta_2}}{c^2} + \delta_3(\alpha_3 + \beta_3)^2$,
7. $I_{43}^2 - I_{42}I_{44} = u_1^2 J_{43}^2$, *for some* $J_{43} \in K$.
8. $I_{60}^2 - 16 I_{42} I_{80} = u_1^2 J_{60}^2$, *for some* $J_{60} \in K$.
9. $4c\Delta_G u_1 = c^2 I_{45} + u_1^2 I_{23} - 4c^2(\alpha_2\beta_2 - \alpha_3\beta_3)^2$.
10. $J_{43} = \frac{\delta_2 A_{21} - \delta_3 A_{31}}{u_1}$.
11. $J_{60} = \frac{4}{c^2}\frac{\hat{\delta_2} A_{31} - \hat{\delta_3} A_{21}}{u_1}$
12. $J_{41} = \frac{A_{41p} - A_{41m}}{4\alpha_1}$

Proof. 1. 2. 3. are clear from computations.
4. $J_{40} = \frac{(\alpha_2 - \beta_2 - \alpha_3 + \beta_3)(\alpha_2 - \beta_2 + \alpha_3 - \beta_3)}{(\alpha_2 + \beta_2 - \alpha_3 - \beta_3)}$.
5. $J_{41} = \alpha_1^2(\alpha_2 + \beta_2 + \alpha_3 + \beta_3) + \alpha_2\beta_2(\alpha_3 + \beta_3) + \alpha_3\beta_3(\alpha_2 + \beta_2)$.

6. is clear from computation.
7. $J_{43} = \frac{-\alpha_2^4-\beta_2^4+\alpha_3^4+\beta_3^4-4\alpha_1^2\alpha_2\beta_2-2\alpha_3^2\alpha_1^2-2\beta_3^2\alpha_1^2+4\alpha_1^2\alpha_3\beta_3+2\alpha_2\beta_2^3-2\alpha_3^3\beta_3-2\alpha_3\beta_3^3+2\alpha_2^3\beta_2+2\alpha_2^2\alpha_1^2+2\beta_2^2\alpha_1^2+2\beta_3^2\alpha_3^2-2\beta_2^2\alpha_2^2}{\alpha_2+\beta_2-\alpha_3-\beta_3}$.
8. $J_{60} = \frac{\alpha_1^2\beta_3^4-3\alpha_1^4\beta_3^2+\alpha_2^2\beta_2^4+\alpha_2^4\beta_2^2-\alpha_3^2\beta_3^4+4\alpha_1^2\alpha_3^2\beta_3^2-3\alpha_1^4\alpha_3^2+3\alpha_1^4\beta_2^2+3\alpha_2^2\alpha_1^4-4\alpha_1^2\alpha_2^2\beta_2^2-\alpha_3^4\beta_3^2+\alpha_1^2\alpha_3^4-\alpha_1^2\alpha_2^4-\alpha_1^2\beta_2^4}{\alpha_2+\beta_2-\alpha_3-\beta_3}$.
Note that all the denominators are u_1, and hence non-zero as otherwise $u_1 = \ell_1 = \ell = 0$. □

Lemma 6.2.2. *All the following expressions are equivalent ways to define Δ_G.*

1. $\Delta_G = c\big((\alpha_1+\beta_2)(\alpha_1-\alpha_2)(\alpha_2+\beta_2-\alpha_3-\beta_3)+(\alpha_2+\beta_2)(\alpha_2-\alpha_3)(\alpha_2-\beta_3)\big)$,
2. $\Delta_G = c\big((\alpha_1+\beta_2)(\alpha_1-\beta_2)(\alpha_2-\alpha_3)+(\beta_3-\beta_2)\left((\beta_2-\alpha_1)(\alpha_3-\alpha_2)+(\alpha_2+\alpha_1)(\alpha_3-\alpha_1)\right)\big)$,
3. $\Delta_G = c\big((\alpha_1+\beta_2)(\alpha_1-\beta_2)(\alpha_2-\alpha_3)+(\beta_3-\beta_2)\left(\alpha_2(\alpha_3-\beta_2)+(\beta_2\alpha_3-\alpha_1^2)\right)\big)$,
4. $\Delta_G = c\big((\alpha_1+\alpha_2)(\alpha_3-\alpha_1)(\beta_3-\alpha_1)+(\beta_2-\alpha_1)\left((\alpha_1-\beta_3)(\alpha_2-\alpha_3)+(\alpha_3+\alpha_1)(\alpha_1-\alpha_2)\right)\big)$.
5. $\Delta_G = c(\alpha_1^2 u_1 - \alpha_2\beta_2\ell_2 - \alpha_3\beta_3\ell_3)$.

6.3 Standard properties of Hilbert Symbols

For convenience we recall some basic properties of Hilbert symbols. First recall that $(A, B) = 1$ if A or B is a square and whenever A, B are both units for odd places.

Lemma 6.3.3. *Let $A, B, C \in K^\times$. Then*

1. $(A+B, -AB) = (A, B)$,
2. $(A, 1-A) = (A, -A) = 1$,
3. $(A, B)(-A, -C)(B, -C) = 1$ *if* $A+B+C = 0$.

Lemma 6.3.4. *Let K be a finite extension of $\mathbb{Q}_p$ for an odd prime p. Let $A, B, C \in K^\times$ such that $A^2 = B + C$. Write $A = u_A\pi^a$, $B = u_B\pi^b$, $C = u_C\pi^c$, where $u_A, u_B, u_C \in \mathcal{O}_K^\times$ and $a, b, c \in \mathbb{Z}$. Then*

1. *if* $v(A^2) > v(B) = v(C)$ *then* $(A, -BC) = 1$,
2. *if* $v(B) > v(A^2) = v(C)$ *then* $(A, -BC) = (A, -B)$,
3. *if* $v(C) > v(A^2) = v(B)$ *then* $(A, -BC) = (A, -C)$,
4. *if* $v(A^2) = v(B) = v(C)$ *then* $(A, -BC) = (A, -u_Bu_C)$.

Proof. 1. Since $v(A^2) > v(B) = v(C)$, we have $b = c$ and $u_B + u_C = u_n\pi^n$ for some $u_n \in \mathcal{O}_K^\times$ and $n \in \mathbb{Z}_{>0}$. Hence $(A, -BC) = (A, -\pi^{2b}u_Bu_C) = (A, -u_Bu_C) = (A, -(u_n\pi^n - u_C)u_C) = (A, u_C^2) = 1$.
2. We have $C = u_A^2\pi^{2a} - u_B\pi^b$ and $2a < b$ since $v(B) > v(A^2) = v(C)$. Hence

$(A,-BC)=(A,-B)(A,u_A^2\pi^{2a}-u_B\pi^b)=(A,-B)(A,\pi^{2a}(u_A^2-u_B\pi^{b-2a}))$
$=(A,-B)(A,u_A^2)=(A,-B)$.
3. Follows from ii).
4. Here $b=c$ as $v(A^2)=v(B)=v(C)$. Hence $(A,-BC)=(A,-u_Bu_C\pi^{2b})=(A,-u_Bu_C)$. □

6.4 Local discrepancy at infinite places

This section computes E_v at real places and proves the last columns of Tables 4.1, 4.2 and 4.3. Since for two real numbers $a,b\in\mathbb{R}$ we have $(a,b)=-1$ if and only if $a,b\in\mathbb{R}_{<0}$, we are primally interested here in the signs of all invariants involved in $E_{\mathbb{R}}$. These signs are sometimes obvious from the definitions of the invariants and the configuration of the real roots considered. When this is the case, the sign appears in Tables 6.1, 6.2 and 6.3. On the other hand, when the signs can vary, its corresponding entry is left blank and we use the properties of the $C_2\times D_4$ invariants in Section 6.2 to prove that $E_{\mathbb{R}}$ gives the required result. This is done case by case in the second part of this section.

Table 6.1: Sign of $C_2 \times D_4$ invariants over $\mathbb{R}$

Isogeny	I_{44}	I_{23}	I_{80}	I_{45}	I_{40}	I_{20}	I_{21}	I_{22}	I_{41}	I_{42}	I_{43}	I_{60}	c	ℓ	$E_{\mathbb{R}}$
6C1A	+	−	+	+	+	−		+					+		1
6C1B	+	−	+	+	+	−		+					−		-1
6C2Aa	+	−	+	−	−		+			+			+		-1
6C2Ab	+	−	+	+	−		+	−		+			+	+	-1
6C2Ac	+	−	+	+	−		+	−		+			+	−	1
6C2Ba	+	−	+	−	−		+			+			−		1
6C2Bb	+	−	+	+	−		+	−		+			−	+	1
6C2Bc	+	−	+	+	−		+	−		+			−	−	-1
4C1A/B	+	+	+	+	+	−		+	+						1
4C2A/B	−	−	+	+	+										1
4C3a	+	+	+	−	−		+		+	+					-1
4C3b	+	+	+	+	−		+		+	+				+	-1
4C3c	+	+	+	+	−		+		+	+				−	1
2C1A	+	−	+	+	+	+		+		+			+		1
2C1B	+	−	+	+	+	+		+		+			−		-1
2C2A/B	+	−	+	−	+	+				+					-1
2C3A	+	−	+	+	+	+		−		+			+		-1
2C3B	+	−	+	+	+	+		−		+			−		1
2C4A	−	+	+	+	+								+		1
2C4B	−	+	+	+	+								−		-1
2C5A/B	−	+	−	+	+										-1
2C6aA	−	+	+	+	+				+				+		-1
2C6bA	−	+	+	+	+				−				+		-1
2C6aB	−	+	+	+	+				+				−		1
2C6bB	−	+	+	+	+				−				−		1

Table 6.2: Sign of $C_2 \times D_4$ invariants over $\mathbb{R}$

Isogeny	I_{44}	I_{23}	I_{80}	I_{45}	I_{40}	I_{20}	I_{21}	I_{22}	I_{41}	I_{42}	I_{43}	I_{60}	c	ℓ	$E_{\mathbb{R}}$
6R1A	+	+	+	+	+	+		+	+	+	+	+	+		1
6R1B	+	+	+	+	+	+		+	+	+	+	+	−		1
6R2A	+	+	+	−	+	+	+		+	+	+	+	+		-1
6R2B	+	+	+	−	+	+	+		+	+	+	+	−		1
6R3A	+	+	+	+	+	+	+	−	+	+	+	+	+		-1
6R3B	+	+	+	+	+	+	+	−	+	+	+	+	−		-1
6R4A	+	+	−	+	+	+		+					+		-1
6R4B	+	+	−	+	+	+		+					−		1
6R5A	+	+	−	−	+	+							+		1
6R5B	+	+	−	−	+	+							−		1
6R6A	+	+	−	+	+	+		−		+	−	−	+		1
6R6B	+	+	−	+	+	+		−		+	−	−	−		-1
6R7A	+	+	+	+	+	+		+	+	−			+		-1
6R7B	+	+	+	+	+	+		+	+	−			−		-1
6R8aA	+	+	+	−	+	+	+		+				+	+	1
6R8aB	+	+	+	−	+	+	+		+				−	+	1
6R8bA	+	+	+	−	+	+	+		+				+	−	-1
6R8bB	+	+	+	−	+	+	+		+				−	−	-1
6R9A	+	+	+	+	+	+		−	+				+		1
6R9B	+	+	+	+	+	+		−	+				−		1
6R10A	+	+	−	+	+	+		+					+		1
6R10B	+	+	−	+	+	+		+					−		-1
6R11A	+	+	−	−	+	+							+		1
6R11B	+	+	−	−	+	+							−		1
6R12A	+	+	−	+	+	+		−					+		-1
6R12B	+	+	−	+	+	+		−					−		1

Table 6.3: Sign of $C_2 \times D_4$ invariants over $\mathbb{R}$

Isogeny	I_{44}	I_{23}	I_{80}	I_{45}	I_{40}	I_{20}	I_{21}	I_{22}	I_{41}	I_{42}	I_{43}	I_{60}	c	ℓ	$E_{\mathbb{R}}$
6R13A	+	+	+	+	+	+		+	+	+	−	−	+		1
6R13B	+	+	+	+	+	+		+	+	+	−	−	−		1
6R14A	+	+	+	−	+	+			+	+	−	−	+		1
6R14B	+	+	+	−	+	+			+	+	−	−	−		-1
6R15A	+	+	−	+	+	+		−		+	+	+	+		-1
6R15B	+	+	−	+	+	+		−		+	+	+	−		1
6R16A	+	+	+	+	+	+		+	−	+	+	+	+		-1
6R16B	+	+	+	+	+	+		+	−	+	+	+	−		-1
6R17A	+	+	+	−	+	+			−	+	+	+	+		1
6R17B	+	+	+	−	+	+			−	+	+	+	−		-1
6R18A	+	+	+	+	+	+		−	−	+	+	+	+		1
6R18B	+	+	+	+	+	+		−	−	+	+	+	−		1
6R19A	+	+	−	+	+	+		+					+		-1
6R19B	+	+	−	+	+	+		+					−		1
6R20A	+	+	−	−	+	+							+		1
6R20B	+	+	−	−	+	+							−		1
6R21A	+	+	−	+	+	+		−		+	−	−	+		1
6R21B	+	+	−	+	+	+		−		+	−	−	−		-1
6R22A	+	+	+	+	+	+		+	−				+		1
6R22B	+	+	+	+	+	+		+	−				−		1
6R23aA	+	+	+	−	+	+	−		−				+	+	1
6R23aB	+	+	+	−	+	+	−		−				−	+	1
6R23bA	+	+	+	−	+	+	−		−				+	−	-1
6R23cB	+	+	+	−	+	+	−		−				−	−	-1
6R24A	+	+	+	+	+	+		−	−	−			+		-1
6R24B	+	+	+	+	+	+		−	−	−			−		-1
6R25A	+	+	+	+	+	+		+	+	+	+	+	+		1
6R25B	+	+	+	+	+	+		+	+	+	+	+	−		1
6R26A	+	+	+	−	+	+	+		+	+	+	+	+		-1
6R26B	+	+	+	−	+	+	+		+	+	+	+	−		1
6R27A	+	+	+	+	+	+	+	−	+	+	+	+	+		-1
6R27B	+	+	+	+	+	+	+	−	+	+	+	+	−		-1

6.4.1 C has 0, 1 or 2 real connected components

Proof of Table 4.1

Cases 6C1A/B. $I_{22} > 0$ follows from the expression of I_{22} given in Lemma 6.2.1.3. Hence

$$E_{\mathbb{R}} = (-1, I_{41}I_{43}I_{60})(-1)(c, -1)(-1, I_{41})(I_{42}, -I_{60}I_{43})$$

$$= -(c, -1)(-I_{42}, I_{60})(-I_{42}, I_{43})(I_{42}, -1).$$

Hence if $I_{42} < 0$ then $E_{\mathbb{R}} = (c, -1) = \text{sign}(c)$ and we are done. Otherwise from Lemma 6.2.1.6.(a), we have $A_{21}, A_{31} > 0$ or $A_{21}, A_{31} < 0$. Assume the former, then

$I_{60} = \hat{\delta_2}A_{31} + \hat{\delta_3}A_{21} > 0$ and $I_{43} = \delta_2 A_{21} + \delta_3 A_{31} < 0$ which yields $E_{\mathbb{R}} = (c, -1)$. Finally if $A_{21}, A_{31} < 0$ then $I_{60} = \hat{\delta_2}A_{31} + \hat{\delta_3}A_{21} < 0$ and $I_{43} = \delta_2 A_{21} + \delta_3 A_{31} > 0$ thus $E_{\mathbb{R}} = (c, -1)$ as required.
Cases 6C2Aa/b/c and 6C2Ba/b/c. $I_{21} = (\alpha_2 + \beta_2)(\alpha_3 + \beta_3) = (\alpha_2 + \beta_2)(\bar{\alpha_2} + \bar{\beta_2}) = |(\alpha_2 + \beta_2)|^2 > 0$ and $I_{42} = A_{21}\overline{A_{21}} = |A_{21}|^2 > 0$. It follows that

$$E_{\mathbb{R}} = (-1, I_{22})(-1, \ell)(-1, c)(I_{45}, -I_{22}\ell).$$

We wish to prove that if $I_{45} > 0$ then $E_{\mathbb{R}}$ =-sign(ℓc), and if $I_{45} < 0$ then $E_{\mathbb{R}}$ =-sign(c). Assume the latter, then $E_{\mathbb{R}} = -(c, -1)$ = -sign(c) proving the result. Finally assume that $I_{45} > 0$. By Lemma 6.2.1.3, it follows that $I_{22} < 0$. Therefore in this case, $E_{\mathbb{R}} = -(-1, \ell)(c, -1) = -\text{sign}(\ell c)$ as required.
Cases 4C1A/B. $I_{22} > 0$ and $I_{41} > 0$ follow from $\alpha_2 = \bar{\beta_2}$, $\alpha_3 = \bar{\beta_3}$ and Lemmata 6.2.1.3 and 6.2.1.5. This yields $E_{\mathbb{R}} = -(I_{43}, -I_{42})(I_{60}, -I_{42})(I_{42}, -1)$. If $I_{42} < 0$ then $E_{\mathbb{R}} = 1$ and we are done. Otherwise from Lemma 6.2.1.6.(a), we have $A_{21}, A_{31} > 0$ or $A_{21}, A_{31} < 0$. Assume the former, then $I_{60} = \hat{\delta_2}A_{31} + \hat{\delta_3}A_{21} > 0$ and $I_{43} = \delta_2 A_{21} + \delta_3 A_{31} < 0$ which yields $E_{\mathbb{R}} = 1$. Finally if $A_{21}, A_{31} < 0$ then $I_{60} = \hat{\delta_2}A_{31} + \hat{\delta_3}A_{21} < 0$ and $I_{43} = \delta_2 A_{21} + \delta_3 A_{31} > 0$ so that $E_{\mathbb{R}} = 1$ as required.
Cases 4C2A/B. Here

$$E_{\mathbb{R}} = (-1, I_{22}I_{41}I_{43}I_{60})(I_{41}, -1)(-1, I_{22}I_{42}I_{43})(I_{42}, -I_{60}I_{43})$$

$$= (I_{42}, I_{60}I_{43})(I_{60}, -1)$$

and $\delta_3 < 0$ and $\delta_2 > 0$. From Lemma 6.2.1.6 we have $A_{21} > 0$. Hence from Lemma 6.2.1.6.(a) and (b), if $I_{42} > 0$ then $A_{31} > 0$ and $I_{60} > 0$ so that $E_{\mathbb{R}} = 1$ as required. On the other hand, if $I_{42} < 0$ then $A_{31} < 0$ and $I_{43} > 0$ from Lemma 6.2.1.6.(c). Therefore $E_{\mathbb{R}} = 1$ proving the result.
Cases 4C3a/b/c. $I_{21} > 0$, $I_{42} > 0$ and $I_{41} > 0$ follow from $\alpha_2 = \bar{\alpha_3}$, $\beta_2 = \bar{\beta_3}$ and Lemmata 6.2.1.5 and 6.2.1.6. This yields

$$E_{\mathbb{R}} = (-1, I_{22})(-1, \ell)(I_{45}, -\ell I_{22}).$$

We wish to prove that if $I_{45} < 0$ then $E_{\mathbb{R}} = -1$ and if $I_{45} > 0$ then $E_{\mathbb{R}} = -\text{sign}(\ell)$. This is clear if If $I_{45} < 0$. On the other hand, if $I_{45} > 0$ then by lemma 6.2.1.3 we have $I_{22} < 0$, therefore $E_{\mathbb{R}} = -(-1, \ell)$ proving the result.
Cases 2C1/2/3. By definition of I_{42} we have $I_{42} > 0$ since $\alpha_1^2 < 0$. Hence

$$E_{\mathbb{R}} = (-1, c)(I_{22}, -I_{45})(I_{43}, -1)(I_{60}, -1)(I_{45}, -\ell I_{21}).$$

Now it follows from Lemma 6.2.1.6.(a),(b) and (c) that $(I_{43}, -1)(I_{60}, -1) = 1$. Therefore it remains to compute $(-1,c)(I_{22},-I_{45})(I_{45},-\ell I_{21})$.

Cases 2C1A/B. Here $I_{45}>0$ and $I_{22}>0$ hence $E_{\mathbb{R}}=$sign(c), as required.

Cases 2C2A/B. Here $I_{45}<0$ hence $E_{\mathbb{R}}=-(-1,c)(-1,\ell I_{21})$. Lemma 6.2.1.2 yields $E_{\mathbb{R}}=-(-1,c)(-1,-\frac{u_1}{\Delta_G})=(cu_1\Delta_G,-1)$ since $u_1,\Delta_G\in\mathbb{R}$. It follows from Lemma 6.2.1.9 that $cu_1\Delta_G<0$ since $I_{23},I_{45}<0$ and $\alpha_2,\beta_2,\alpha_3,\beta_3\in\mathbb{R}$. Therefore $E_{\mathbb{R}}=-1$ as required.

Case 2C3A/B. Here $I_{45}>0$ and $I_{22}<0$ hence $E_{\mathbb{R}}=$-sign(c), as required.

Cases 2C4/5/6. In this cases we have

$$E_{\mathbb{R}}=(I_{42},-I_{43}I_{60})(I_{80},-I_{41}I_{42}I_{60})(c,-I_{80})(-1,I_{41}I_{60})(-1,I_{42}).$$

Cases 2C4A/B. Here $I_{80}>0$ therefore $\hat{\delta_3}>0$ and

$$E_{\mathbb{R}}=(I_{42},-I_{43}I_{60})(c,-1)(-1,I_{41}I_{60})(-1,I_{42}).$$

Using Lemma 6.2.1.6.(a), we have that $A_{21}>0$ hence if $I_{42}>0$ then $I_{60}>0$. On the other hand, if $I_{42}<0$ then $I_{43}>0$; both cases yield $E_{\mathbb{R}}=(I_{41},-1)(c,-1)$. Now from the expression given in Lemma 6.2.1.5 we have that $I_{41p},I_{41m}>0$ hence $E_{\mathbb{R}}=(-1,c)$ as required.

Cases 2C5A/B. Here $I_{80}<0$ hence $\hat{\delta_3}<0$ and

$$E_{\mathbb{R}}=-(I_{42},-I_{43}I_{60}).$$

If $I_{42}>0$ then $E_{\mathbb{R}}=-1$ and we are done. Otherwise, from Lemma 6.2.1.6.(a), either $A_{21}>0,A_{31}<0$ or $A_{21}<0,A_{31}>0$. In the former case we have $I_{43}<0$ and $I_{60}>0$, while in the latter case we have $I_{43}>0$ and $I_{60}<0$. In both cases $-(I_{42},-I_{43}I_{60})=-1$ as required.

Cases 2C6aA/B. Here $I_{80}>0$ therefore $\hat{\delta_3}>0$ and

$$E_{\mathbb{R}}=(I_{42},-I_{43}I_{60})(c,-1)(-1,I_{41}I_{60})(-1,I_{42}).$$

From Lemma 6.2.1.5 we have that $A_{41p},A_{41m}>0$ for 2C6aA and 2C6aB hence $I_{41}>0$. Using Lemma 6.2.1.6.(a), we have that $A_{21}<0$ hence if $I_{42}>0$ then $A_{31}<0$ and $I_{60}<0$. On the other hand, if $I_{42}<0$ then $A_{31}>0$ and $I_{43}<0$. In both cases this yields $E_{\mathbb{R}}=-(-1,c)$ as required.

Cases 2C6bA/B. This is similar as for 2C6aA/B except that from Lemma 6.2.1.5 we have that $A_{41p},A_{41m}<0$ and hence $I_{41}<0$. Using Lemma 6.2.1.6.(a), we have that $A_{21}>0$ hence if $I_{42}>0$ then $A_{31}>0$ and $I_{60}>0$. On the other hand, if $I_{42}<0$ then $A_{31}<0$ and $I_{43}>0$. In both cases this yields $E_{\mathbb{R}}=-(-1,c)$ as required.

6.4.2 C has 3 real connected components

Proof of Tables 4.2 and 4.3

From now on, all roots of $G(x)$ are real therefore $I_{44}, I_{23}, I_{20}, I_{40} > 0$ and

$$E_v = (-1, I_{22}I_{41}I_{43}I_{60})(c, I_{80})(I_{45}, -\ell I_{22}I_{21})(I_{80}, -2I_{41}I_{42}I_{60})(I_{42}, -I_{60}I_{43}).$$

Cases 6R1A/B. $E_{\mathbb{R}} = 1$. Clear.
Cases 6R2A/B. $E_{\mathbb{R}} = (-1, -\ell) = (-1, c)(-1, \frac{\Delta_G}{c}u_1)$. The result follows since $\frac{\Delta_G}{c}u_1 < 0$ from Lemma 6.2.2.
Cases 6R3A/B. $E_{\mathbb{R}} = -1$. Clear.
Case 6R4A/B. $E_{\mathbb{R}} = -(c, -1)(I_{42}, I_{43}I_{60})(-1, I_{43})$. From Lemma 6.2.1.6.(a) and by definition of 6R4, we have that $A_{31} > 0$ hence if $I_{42} > 0$ then $I_{21} > 0$ and $I_{43} > 0$; whereas if $I_{42} < 0$ then $I_{21} < 0$ and $I_{60} > 0$. Both case yield $(I_{42}, I_{43}I_{60})(-1, I_{43}) = 1$ so that $E_{\mathbb{R}} = -c(, -1)$ as required.
Cases 6R5A/B. $E_{\mathbb{R}} = -(-1, -\ell I_{21})(I_{42}, -I_{43}I_{60})(-1, I_{42})(-1, I_{43})(c, -1)$. By definition of 6R5, we have $\hat{\delta_3} < 0$ and by Lemma 6.2.1.6.(a), we have $A_{31} > 0$. Hence if $I_{42} > 0$ then $I_{43} > 0$. On the other hand, if $I_{42} < 0$ then $I_{60} < 0$. Both cases yield $E_{\mathbb{R}} = -(-1, -\ell I_{21})(c, -1) = -(-1, \frac{\Delta_G}{c}u_1)$. The result follows since $\frac{\Delta_G}{c}u_1 < 0$ from Lemma 6.2.2.
Cases 6R6A/B. $E_{\mathbb{R}} = (I_{42}, -I_{43}I_{60})(-1, I_{42})(-1, I_{43})(c, -1)$. By definition of 6R6, we have $\hat{\delta_3} < 0$ and by Lemma 6.2.1.6.(a), we have $A_{31} > 0$. Hence if $I_{42} > 0$ then $I_{43} > 0$. On the other hand, if $I_{42} < 0$ then $I_{60} > 0$. Both cases yield $E_{\mathbb{R}} = (c, -1)$ as required.
Cases 6R7A/B. $E_{\mathbb{R}} = -1$. Clear.
Cases 6R8aA/B and 6R8bA/B. $I_{21} > 0$ follows from $0 < \beta_2 - \alpha_1 < \alpha_2 + \beta_2$ and $0 < \beta_3 - a_1 < \alpha_3 + \beta_3$. Therefore

$$E_{\mathbb{R}} = (-1, -\ell I_{21})(I_{42}, -I_{43}I_{60})(-1, I_{43}I_{60}).$$

By definition of 6R8, we have $\hat{\delta_2}, \hat{\delta_3} < 0$ hence by Lemma 6.2.1.6, if $I_{42} < 0$ then $E_{\mathbb{R}} = (-1, \ell)$ and we are done. Otherwise, if $I_{42} > 0$ with $A_{21}, A_{31} > 0$ then $I_{43} > 0$ and $I_{60} < 0$. On the other hand, if $I_{42} > 0$ with $A_{21}, A_{31} < 0$ then $I_{43} < 0$ and $I_{60} > 0$. This yields $E_{\mathbb{R}} = -(-1, -\ell I_{21}) = (-1, \ell)$ as required.
Cases 6R9A/B. $E_{\mathbb{R}} = -(I_{42}, -I_{43}I_{60})(-1, I_{43}I_{60})$. By definition of 6R9B, we have $\hat{\delta_2}, \hat{\delta_3} < 0$ hence by Lemma 6.2.1.6, if $I_{42} < 0$ then $E_{\mathbb{R}} = 1$ and we are done. Otherwise, if $I_{42} > 0$ with $A_{21}, A_{31} > 0$ then $I_{43} > 0$ and $I_{60} < 0$. On the other hand, if $I_{42} > 0$ with $A_{21}, A_{31} < 0$ then $I_{43} < 0$ and $I_{60} > 0$. Both cases yield $E_{\mathbb{R}} = 1$ as required.
Cases 6R10A/B. $E_{\mathbb{R}} = -(-1, I_{43})(I_{42}, I_{43}I_{60})(c, -1)$. By definition of 6R10, we

have $\hat{\delta_2} < 0, \hat{\delta_3} > 0$ and $A_{21} < 0$. Therefore, by Lemma 6.2.1.6, if $I_{42} > 0$ then $I_{43} < 0$ and $E_{\mathbb{R}} = (c, -1)$ and we are done. Otherwise, if $I_{42} < 0$ then $A_{31} > 0$ and $I_{60} < 0$ so that $E_{\mathbb{R}} = (c, -1)$ as required.

Cases 6R11A/B. $E_{\mathbb{R}} = -(-1, -\ell I_{21})(I_{42}, -I_{43}I_{60})(-1, I_{42})(-1, I_{43})(c, -1)$. By definition of 6R11, we have $\hat{\delta_3} < 0$ and by Lemma 6.2.1.6.(a), we have $A_{31} < 0$. Hence if $I_{42} > 0$ then $I_{43} < 0$. On the other hand, if $I_{42} < 0$ then $I_{60} < 0$. Both cases yield $E_{\mathbb{R}} = (-1, -\ell I_{21})(c, -1) = (-1, \frac{\Delta_G}{c} u_1)$. The result follows since $\frac{\Delta_G}{c} u_1 > 0$ from Lemma 6.2.2.

Cases 6R12A/B. $E_{\mathbb{R}} = (-1, I_{43})(I_{42}, I_{43}I_{60})(c, -1)$. By definition of 6R12, we have $\hat{\delta_2} > 0, \hat{\delta_3} < 0$ and $A_{31} < 0$. Therefore, by Lemma 6.2.1.6, if $I_{42} > 0$ then $I_{43} < 0$ and $E_{\mathbb{R}} = -(c, -1)$ and we are done. Otherwise, if $I_{42} < 0$ then $A_{21} > 0$ and $I_{60} < 0$ so that $E_{\mathbb{R}} = -(c, -1)$ as required.

Cases 6R13A/B . $E_{\mathbb{R}} = 1$. Clear.

Cases 6R14A/B. $E_{\mathbb{R}} = (-1, -\ell I_{21}) = (-1, c)(-1, \frac{\Delta_G}{c} u_1)$. Since $\frac{\Delta_G}{c} u_1 > 0$ from Lemma 6.2.2 we have $E_{\mathbb{R}} = (c, -1)$ as required.

Cases 6R15A/B. $E_{\mathbb{R}} = (I_{42}, -I_{43}I_{60})(-1, I_{42})(-1, I_{43})(c, -1)$. By definition of 6R15, we have $\hat{\delta_3} < 0$ and by Lemma 6.2.1.6.(a), we have $A_{31} < 0$. Hence if $I_{42} > 0$ then $I_{43} < 0$. On the other hand, if $I_{42} < 0$ then $I_{60} < 0$. Both cases yield $E_{\mathbb{R}} = -(c, -1)$ as required.

Cases 6R16A/B. $E_{\mathbb{R}} = -(-1, I_{43}I_{60})(I_{42}, I_{43}I_{60})$. By definition of 6R16, we have $I_{42}, I_{43}, I_{60} > 0$. Therefore $E_{\mathbb{R}} = -1$ as required.

Cases 6R17A/B. $E_{\mathbb{R}} = -(-1, -\ell I_{21}) = -(-1, c)(-1, \frac{\Delta_G}{c} u_1)$. Since $\frac{\Delta_G}{c} u_1 < 0$ from Lemma 6.2.2 we have $E_{\mathbb{R}} = (c, -1)$ as required.

Cases 6R18A/B. $E_{\mathbb{R}} = 1$. Clear.

Cases 6R19A/B. $E_{\mathbb{R}} = -(-1, I_{43})(I_{42}, I_{43}I_{60})(c, -1)$. By definition of 6R19, we have $\hat{\delta_2} > 0, \hat{\delta_3} < 0$ and $A_{31} > 0$. Therefore, by Lemma 6.2.1.6, if $I_{42} > 0$ then $I_{43} > 0$ and $E_{\mathbb{R}} = -(c, -1)$ and we are done. Otherwise, if $I_{42} < 0$ then $A_{21} < 0$ and $I_{60} > 0$ so that $E_{\mathbb{R}} = -(c, -1)$ as required.

Cases 6R20A/B. $E_{\mathbb{R}} = -(-1, I_{43})(I_{42}, I_{43}I_{60})(c, -1)(-1, -\ell I_{21})$. By definition of 6R20, we have $\hat{\delta_2} < 0, \hat{\delta_3} > 0$ and $A_{21} > 0$. Therefore, by Lemma 6.2.1.6, if $I_{42} > 0$ then $I_{43} > 0$ and $E_{\mathbb{R}} = -(c, -1)(-1, -\ell I_{21})$. Otherwise, if $I_{42} < 0$ then $A_{31} < 0$ and $I_{60} > 0$ so that $E_{\mathbb{R}} = -(c, -1)(-1, -\ell I_{21})$. Using that $\ell = -c^2 \ell_1 I_{21}$, this yields $E_{\mathbb{R}} = -(-1, c\ell_1)$. Moreover, $\ell_1 = \frac{u_1}{\Delta_G}$ with $u_1 = \alpha_2 + \beta_2 - \alpha_3 - \beta_3$. By definition of 6R20 we have that $\alpha_2 - \alpha_3 < 0$ since $|\alpha_2| > |\alpha_3|$. Also, $\beta_2 - \beta_3 < 0$, hence $u_1 < 0$. Using Lemma 6.2.2.2, we have that $c\Delta_G > 0$ if

$$|(\beta_2 - \alpha_1)(\alpha_3 - \alpha_2)| > |(\alpha_2 + \alpha_1)(\alpha_3 - \alpha_1)|.$$

But $|\beta_2 - \alpha_1| > |-2\alpha_1|$, hence it suffices to prove that

$$|-2\alpha_1||\alpha_3 - \alpha_2| > |\alpha_2 + \alpha_1||\alpha_3 - \alpha_1|$$

with $\alpha_2 < \alpha_1 < \alpha_3 < -\alpha_1 = \beta_1$. Let $\alpha_1 = \alpha_2 + b, \alpha_3 = \alpha_2 + b + e, \beta_1 = \alpha_2 + b + e + d$, with $b, e, d > 0$. Then $\beta_1 - \alpha_1 = e + d, \alpha_3 - \alpha_2 = b + e, \alpha_2 - \beta_1 = -b - e - d, \alpha_3 - \alpha_1 = e$. Hence $|-2\alpha_1||\alpha_3 - \alpha_2| = e^2 + be + ed + bd$ and $|\alpha_2 + \alpha_1||\alpha_3 - \alpha_1| = eb + e^2 + ed$. The result follows since $bd > 0$.

Thus $c\Delta_G > 0$ and $c\ell_1 < 0$ so that $E_{\mathbb{R}} = 1$.

Cases 6R21A/B. $E_{\mathbb{R}} = (I_{42}, -I_{43}I_{60})(-1, I_{42})(-1, I_{43})(c, -1)$. By definition of 6R21, we have $\hat{\delta_3} < 0$ and by Lemma 6.2.1.6.(a), we have $A_{31} > 0$. Hence if $I_{42} > 0$ then $I_{43} > 0$. On the other hand, if $I_{42} < 0$ then $I_{60} > 0$. Both cases yield $E_{\mathbb{R}} = (c, -1)$ as required.

Cases 6R22A/B. $E_{\mathbb{R}} = -(I_{42}, -I_{43}I_{60})(-1, I_{43}I_{60})$. By definition of 6R22, we have $\hat{\delta_2}, \hat{\delta_3} < 0$ hence by Lemma 6.2.1.6, if $I_{42} < 0$ then $E_{\mathbb{R}} = 1$ and we are done. Otherwise, if $I_{42} > 0$ with $A_{21}, A_{31} > 0$ then $I_{43} > 0$ and $I_{60} < 0$. On the other hand, if $I_{42} > 0$ with $A_{21}, A_{31} < 0$ then $I_{43} < 0$ and $I_{60} > 0$. Both cases yield $E_{\mathbb{R}} = 1$ as required.

Cases 6R23aA/B and 6R23bA/B. $I_{21} > 0$ follows from $\alpha_2 + \beta_2 < \beta_2 - \alpha_1 < 0$ and $0 < \beta_3 - a_1 < \alpha_3 + \beta_3$. $E_{\mathbb{R}} = -(-1, -\ell I_{21})(I_{42}, -I_{43}I_{60})(-1, I_{43}I_{60})$. By definition of 6R23, we have $\hat{\delta_2}, \hat{\delta_3} < 0$ hence by Lemma 6.2.1.6, if $I_{42} < 0$ then $E_{\mathbb{R}} = (-1, -\ell I_{21}) = (-1, \ell)$ and we are done. Otherwise, if $I_{42} > 0$ with $A_{21}, A_{31} > 0$ then $I_{43} > 0$ and $I_{60} < 0$. On the other hand, if $I_{42} > 0$ with $A_{21}, A_{31} < 0$ then $I_{43} < 0$ and $I_{60} > 0$. Both cases yield $E_{\mathbb{R}} = (-1, -\ell I_{21}) = (-1, \ell)$ as required.

Cases 6R24A/B . $E_{\mathbb{R}} = -1$. Clear.

Cases 6R25A/B. $E_{\mathbb{R}} = (-1, I_{43}I_{60})(I_{42}, I_{43}I_{60})$. By definition of 6R25, we have $I_{42}, I_{43}, I_{60} > 0$ hence $E_{\mathbb{R}} = 1$ as required.

Cases 6R26A/B. $E_{\mathbb{R}} = (-1, -\ell I_{21}) = (-1, c)(-1, \frac{\Delta_G}{c}u_1)$. Since $\frac{\Delta_G}{c}u_1 < 0$ from Lemma 6.2.2 we have $E_{\mathbb{R}} = -(c, -1)$ as required.

Cases 6R27A/B . $E_{\mathbb{R}} = -1$. Clear.

6.5 Local discrepancy at finite places $v \mid 2$

Fix a 2-adic place v of $\mathcal{K}$. Recall from Section 4.6 that we required our $C_2 \times D_4$ curve to belong to the family $\mathcal{C}$ given by

$$\mathcal{C} : y^2 = f(x) = G_1(x)G_2(x)G_3(x),$$

$$G_1(x) = (x^2 - (8 + 4n)^2),$$

$$G_2(x) = (x^2 + x(-2m - 23) + \frac{441}{4} - 2d + 14m),$$

$$G_3(x) = (x^2 + x(-8k - 18) + 105 + 8r + 56k),$$

for $n, m, d, k, r \in \mathcal{O}_K$.
We wish to prove that $E_v = 1$ for curves $C \in \mathcal{C}$ as claimed in Lemma 4.6.17. Recall the following results on Hilbert Symbols (see [9][Lemma 15]).

Lemma 6.5.5. *Let $F/\mathbb{Q}_p$ be a finite extension. Then*
(1) $(1 + 4x, y) = 1$ if $v(x) > 0$ and $y \in F^\times$,
(2) $(1 + 4x, y) = 1$ if $p = 2, v(x) = 0$ and $y \in \mathcal{O}_F^\times$,

Let C be a $C_2 \times D_4$ curve such that $C \in \mathcal{C}$ and consider the model for such curves given above. Computing its corresponding $C_2 \times D_4$ invariants one finds that

$$I_{45} \equiv_\square 1 \bmod 16, \quad I_{40} =_\square 1,$$

$$I_{80} = 1 + 4t, \quad I_{41} =_\square (1 + 4t'), \quad I_{42} =_\square (1 + 4t''),$$

with $v(t), v(t'), v(t'') > 0$, so that by Lemma 6.5.5 we have

$$E_v = (-1, I_{22}I_{43}I_{60})(I_{20}, -I_{44})(I_{44}, 2I_{22}I_{43}).$$

Moreover

$$I_{22} \equiv \frac{1}{2} + 4d \bmod 8 =_\square (2 + 16d) \bmod 32 =_\square 2(1 + 8d),$$

$$I_{60} \equiv \frac{1}{2} + 4(d + m^2) \bmod 8 =_\square (2 + 16(d + m^2)) \bmod 32 =_\square 2(1 + 8(d + m^2)),$$

so that

$$(-1, I_{22}I_{60}) = (I_{44}, 2I_{22}) = 1.$$

We therefore have that

$$E_v = (I_{20}, -I_{44})(I_{43}, -I_{44})$$

and we show that $E_v = 1$.

By definition we have $I_{44} = \delta_2\delta_3$, $I_{20} = \frac{1}{2^3}(\delta_2 + \delta_3)$ and $I_{43} = -\delta_2 A_{21} - \delta_3 A_{31}$ with $A_{21}A_{31} = I_{42} = \square$ and $A_{21}, A_{31} \in K$ since $G_2(x), G_3(x) \in K[x]$. Replacing invariants by their expression above and using twice Lemma 6.3.3 we have

$$E_v = (\frac{1}{2^3}(\delta_2 + \delta_3), -\delta_2\delta_3)(-\delta_2 A_{21} - \delta_3 A_{31}, -\delta_2\delta_3),$$

$$= (\frac{1}{2^3}, -\delta_2\delta_3)(\delta_2, \delta_3)(-\delta_2 A_{21}, -\delta_3 A_{31}).$$

But $A_{21} =_\square A_{31}$, hence we can replace occurrences of A_{31} by A_{21} and obtain after simplification

$$E_v = (-\delta_2\delta_3, -\frac{A_{21}}{2^3}) = 1$$

since $-2A_{21} = 1 \bmod 8$.

6.6 Local discrepancy at finite places $v \nmid 2$

Recall that we consider a $C_2 \times D_4$ curve $C : y^2 = G_1(x)G_2(x)G_3(x)$ such that its cluster picture at places $v \nmid 2$ of $\mathcal{K}$ is one of Table 3.1. In particular, the roots of $G_1(x), G_2(x), G_3(x)$ are integral and hence all the $I_{i,j}$, $\delta_i, \hat{\delta_i}$, Δ_G, u_1, ℓ_2, ℓ_3 are integral (the only potentially non-integral invariant is $\ell_1 = \frac{u_1}{\Delta_G}$). Also, it follows from the definition of the invariants that without loss of generality, we may assume $v(c) = 0$ or $v(c) = 1$. In this section, since $v \nmid 2$, we extensively use Hensel's Lemma to claim that $x \in K^{\times 2}$ if and only if x reduces to a non zero square element in k.

Lemma 6.6.6. *If $v(A_{21}), v(A_{31}) > 0$ and $v(I_{80}) = v(I_{44}) = 0$ then $I_{80} \in K^{\times 2} \Leftrightarrow I_{44} \in K^{\times 2}$.*

Proof. Since $v(I_{80}) = v(I_{44}) = 0$ and $I_{80} =_\square \hat{\delta_2}\hat{\delta_3}$ and $I_{44} =_\square \delta_2\delta_3$, it follows that $v(\delta_2) = v(\delta_3) = v(\hat{\delta_2}) = v(\hat{\delta_3}) = 0$. From Lemma 6.2.1.6 (c),(d), we have $\hat{\delta_2} \equiv_\square -\delta_3(\alpha_3 + \beta_3)^2$ and $\hat{\delta_3} \equiv_\square -\delta_2(\alpha_2 + \beta_2)^2$. Hence $I_{80} =_\square \hat{\delta_2}\hat{\delta_3} \equiv_\square \delta_2\delta_3 I_{21}^2 \equiv_\square \delta_2\delta_3 =_\square I_{44}$ and the result follows. □

Lemma 6.6.7. *If $v(A_{21}) \neq v(A_{31})$ then $A_{21}, A_{31} \in K$ and moreover $\delta_2, \delta_3, \hat{\delta_2}, \hat{\delta_3}, u_1 \in K$.*

Proof. Since the action of $\sigma \in Gal(\overline{K}/K)$ preserves distances between the roots, it follows that $\sigma(A_{21}) \neq A_{31}$ in this case. Therefore $\sigma(A_{21}) = A_{21}$ and $\sigma(A_{31}) = A_{31}$ which implies that $\sigma(\alpha_i) = \alpha_i$ or $\sigma(\alpha_i) = \beta_i$ for i=2,3. In particular, it follows that σ fixes $\delta_2, \delta_3, \hat{\delta_2}, \hat{\delta_3}, u_1$. □

From this point onwards, our results concern computations of Hilbert Symbols. We give an extra detailed proof for the first Lemma. For the remaining of this section, the reader might find helpful to keep the list of notation for this chapter, the list of invariants $I_{i,j}$, Lemma 6.2.1 and the properties of Hilbert Symbols of Section 6.3 at hand.

Lemma 6.6.8. *If $v(I_{44}) = v(I_{80}) = v(I_{42}) = 0$ then $(I_{43}, -I_{40}I_{42}I_{44}) = (I_{60}, -I_{40}I_{42}I_{80}) = 1$.*

Proof. Write $S_{43}^2 = I_{43}^2 - I_{42}I_{44}$, $S_{60}^2 = I_{60}^2 - 16I_{42}I_{80}$ as given in Lemmata 6.2.1.7 and 6.2.1.8. If $v(I_{43}^2) > v(I_{42}I_{44}) = v(S_{43}^2)$ then $(I_{43}, -I_{40}I_{42}I_{44}) = 1$ by Lemma

6.3.4.1. If $v(I_{43}^2) = v(I_{42}I_{44}) = v(S_{43}^2) = 0$ then $(I_{43}, -I_{40}I_{42}I_{44}) = 1$ trivially. Else, if $v(S_{43}^2) > v(I_{43}^2) = v(I_{42}I_{44}) = 0$ then $(I_{43}, -S_{43}^2 I_{42}I_{44}) = (I_{43}, -S_{43}^2)$ by Lemma 6.3.4.3. In particular since $S_{43}^2 = u_1^2 S^2 =_{\square} I_{40}$ where $S \in K$ and since $v(I_{43}) = 0$, we have $(I_{43}, -S_{43}^2) = (I_{43}, -u_1^2)$. Now, here $v(I_{44}) = v(I_{80}) = 0$ so that $\alpha_2 \not\equiv \beta_2 \not\equiv \alpha_3 \not\equiv \beta_3$ and hence inertia acts trivially on these roots. In particular, $v(u_1) \in \mathbb{Z}$ so that $v(u_1^2) \in 2\mathbb{Z}$. Hence $(I_{43}, -u_1^2) = 1$. The proof is similar for $(I_{60}, -I_{40}I_{42}I_{80})$.

$\square$

Lemma 6.6.9. *If $v(I_{44}) = v(I_{80}) = 0$ then*

$$H = (I_{43}, -I_{40}I_{42}I_{44})(I_{60}, -I_{40}I_{42}I_{80})(I_{42}, -I_{80}I_{44}) = 1.$$

In particular, this holds when C is of type 2.

Proof. Note that since $v(I_{44}) = v(I_{80}) = 0$, $\alpha_2 \not\equiv \beta_2 \not\equiv \alpha_3 \not\equiv \beta_3$ and hence inertia acts trivially on these roots so that $v(u_1^2), v(I_{40}) \in 2\mathbb{Z}$ and $v(A_{21}), v(A_{31}) \in \mathbb{Z}$.

1) First assume that $v(I_{42}) = 0$. Then $(I_{42}, -I_{80}I_{44}) = 1$ trivially since $v(I_{42}) = v(I_{44}) = v(I_{80}) = 0$ and the result follows from Lemma 6.6.8.

2) Now let $v(I_{42}) > 0$ with $v(A_{21}) > v(A_{31}) \geq 0$. Then by Lemma 6.6.7, we have $\delta_2, \delta_3 \in K^\times$ and $I_{40} \in K^2$ by Lemma 6.2.1.4. Also, from Lemma 6.2.1.6.(b) and (c) we have $I_{60} =_{\square} \hat{\delta_2} A_{31} + \hat{\delta_3} A_{21}$ and $I_{43} = \delta_2 A_{21} + \delta_3 A_{31}$. Hence $(I_{43}, -I_{40}I_{42}I_{44})(I_{60}, -I_{40}I_{42}I_{80}) = (\delta_3 A_{31}, -I_{42}I_{44})(\hat{\delta_2} A_{31}, -I_{42}I_{80})$ in this case. Using Lemma 6.2.1.6.(a) and replacing in H yields

$$H = (\delta_3 A_{31}, -A_{21}A_{31}\delta_2\delta_3)(\hat{\delta_2} A_{31}, -A_{21}A_{31}\hat{\delta_2}\hat{\delta_3})(A_{21}A_{31}, -\hat{\delta_2}\hat{\delta_3}\delta_2\delta_3),$$

which, once simplified using the multiplicativity property of Hilbert Symbols yields $H = (A_{21}, -\delta_2\hat{\delta_3})(A_{31}, -\hat{\delta_2}\delta_3)$. Since $v(A_{21}) > 0$, it follows from Lemma 6.2.1.6.(d) that $\hat{\delta_3} \equiv_{\square} -\delta_2$ so that $-\delta_2\hat{\delta_3} \in K^{\times 2}$ and $(A_{21}, -\delta_2\hat{\delta_3}) = 1$. Lastly, If $v(A_{31}) = 0$ then $(A_{31}, -\hat{\delta_2}\delta_3) = 1$ trivially and we are done, otherwise by Lemma 6.2.1.6.(e), it follows that $\hat{\delta_2} \equiv_{\square} -\delta_3$ so that $-\hat{\delta_2}\delta_3 \in K^{\times 2}$ and $(A_{31}, -\hat{\delta_2}\delta_3) = 1$.

3) Lastly, let $v(I_{42}) > 0$ with $v(A_{21}) = v(A_{31}) > 0$. Then $v(I_{42}) \in 2\mathbb{Z}$ and by Lemma 6.6.6, we have $(I_{42}, -I_{80}I_{44}) = 1$ so that

$$H = (I_{43}, -I_{40}I_{42}I_{44})(I_{60}, -I_{40}I_{42}I_{80}).$$

It follows from Lemma 6.2.1.6.(b) and (c) that $v(I_{43}), v(I_{60}) \geq v(A_{21}) = v(A_{31})$, and hence by Lemma 6.2.1.6.(a) we have $v(I_{42}) \leq v(I_{60}^2), v(I_{43}^2)$.

i) If $v(I_{42}) = v(I_{43}^2) = v(I_{60}^2)$. Write $I_{42} = \pi^{2n} U_{42}$ for some $n \in \mathbb{Z}_{>0}$ and $U_{42} \in K^\times$. By Lemmata 6.2.1.7 and 6.2.1.8, if $v(u_1^2 J_{43}^2) = v(u_1^2 J_{60}^2) = v(I_{43}^2)$ then using Lemma 6.3.4.4 it follows that $(I_{43}, -I_{40}I_{42}I_{44}) = (I_{43}, -u_1^2U_{42}I_{44})$ and that $(I_{60}, -I_{40}I_{42}I_{80}) = (I_{60}, -u_1^2U_{42}I_{80})$. Since $v(I_{43}) = v(I_{60})$, this yields

$$H = (I_{43}, I_{44})(I_{60}, I_{80}),$$

and the result follows from Lemma 6.6.6. On the other hand, if $v(u_1^2J_{43}^2) > v(I_{43}^2)$ then by Lemma 6.3.4.2 we have that

$$(I_{43}, -I_{40}I_{42}I_{44}) = (I_{43}, -u_1^2).$$

If in addition, $v(u_1^2J_{60}^2) > v(I_{60}^2)$ then by the same Lemma we have

$$(I_{60}, -I_{40}I_{42}I_{80}) = (I_{60}, -u_1^2),$$

so that

$$(I_{43}, -I_{40}I_{42}I_{44})(I_{60}, -I_{40}I_{42}I_{80}) = 1,$$

since $v(I_{43}) = v(I_{60})$.

Finally, if $v(u_1^2J_{43}^2) > v(I_{43}^2)$ and $v(u_1^2J_{60}^2) = v(I_{60}^2)$ then writing $u_1^2 = \pi^{2b}U_1^2$ for some $b \in \mathbb{Z}$ and $U_1 \in \mathcal{O}_{\overline{K}}^{\times}$, and using Lemma 6.3.4.2 and 4, this yields

$$(I_{43}, -I_{40}I_{42}I_{44}) = (I_{43}, -U_1^2), \quad (I_{60}, -I_{40}I_{42}I_{80}) = (I_{60}, -U_1^2U_{42}I_{80}).$$

Hence

$$H = (I_{43}, -U_1^2)(I_{60}, -U_1^2U_{42}I_{80}) = (I_{60}, U_{42}I_{80}).$$

Write $A_{21} = \pi^aU_{21}$ and $A_{31} = \pi^aU_{31}$ for some $a \in \mathbb{Z}$ and $U_{21}, U_{31} \in \mathcal{O}_{\overline{K}}^{\times}$. Using Lemma 6.2.1.10, we have that $u_1^2J_{43}^2 = (\delta_2A_{21} - \delta_3A_{31})^2$ and since $v((\delta_2A_{21} - \delta_3A_{31})^2) > v(I_{43}^2)$, it follows that $v(\delta_2U_{21} - \delta_3U_{31}) > 0$. In particular $\delta_2U_{21} \equiv \delta_3U_{31}$. By Lemma 6.2.1.6.1 we can write $U_{42} = U_{21}U_{31}$ so that

$$U_{42}I_{80} = U_{21}U_{31}\hat{\delta_2}\hat{\delta_3}.$$

Also, by Lemma 6.2.1.6.(d) and (e) we have

$$\frac{\hat{\delta_3}}{c^2} \equiv -\delta_2(\alpha_2 + \beta_2)^2, \quad \frac{\hat{\delta_2}}{c^2} \equiv -\delta_3(\alpha_3 + \beta_3)^2.$$

It follows that

$$U_{42}I_{80} = U_{21}U_{31}\hat{\delta_2}\hat{\delta_3} \equiv U_{21}U_{31}\delta_2\delta_3((\alpha_2 + \beta_2)(\alpha_3 + \beta_3))^2$$

$$\equiv_{\square} U_{21}U_{31}\delta_2\delta_3 \equiv U_{21}^2\delta_2^2 \equiv_{\square} 1.$$

The last congruence follows from the fact that $U_{21}\delta_2 \in k$, therefore $U_{42}I_{80} \equiv_{\square} 1$ and $(I_{60}, U_{42}I_{80}) = 1$. The proof is similar if $v(u_1^2 J_{60}^2) > v(I_{60}^2)$ and $v(u_1^2 J_{43}^2) = v(I_{43}^2)$.

ii) If $v(I_{42}) = v(I_{43}^2) < v(I_{60}^2)$ then by Lemmata 6.2.1.8 and 6.3.4.1 we have $(I_{60}, -I_{40}I_{42}I_{80}) = 1$ so that $H = (I_{43}, -I_{40}I_{42}I_{44})$. Using Lemma 6.3.4.3, we have

$$H = (\delta_2 A_{21} + \delta_3 A_{31}, -I_{40}A_{21}A_{31}\delta_2\delta_3) = (\pi^a(\delta_2 U_{21} + \delta_3 U_{31}), -U_1^2 U_{21}U_{31}\delta_2\delta_3).$$

Also, since $v(I_{43}^2) = v(I_{42})$ it follows that $v(\delta_2 U_{21} + \delta_3 U_{31}) = 0$ so that

$$H = (\pi^a, -U_1^2 U_{21}U_{31}\delta_2\delta_3),$$

Now, by Lemma 6.2.1.6.(d) and (e) we have $\hat{\delta_2} =_{\square} -\delta_3(\alpha_3+\beta_3)^2$ and $\hat{\delta_3} =_{\square} -\delta_2(\alpha_2+\beta_2)^2$. Also, since $v(I_{43}^2) < v(I_{60}^2)$, using Lemma 6.2.1.6.(b) we have

$$I_{60} =_{\square} \hat{\delta_2}A_{31} + \hat{\delta_3}A_{21} = \pi^a(\hat{\delta_2}U_{31} + \hat{\delta_3}U_{21}), \quad \text{and } v(\hat{\delta_2}U_{31} + \hat{\delta_3}U_{21}) > 0,$$

so that $\hat{\delta_2}U_{31} \equiv -\hat{\delta_3}U_{21}$. Therefore $U_{21}U_{31}\hat{\delta_2}\hat{\delta_3} =_{\square} -U_{21}^2\hat{\delta_3}^2 =_{\square} -U_{21}^2\delta_2^2(\alpha_2+\beta_2)^4$. Hence

$$H = (\pi^a, U_1^2 U_{21}^2 \delta_2^2(\alpha_2+\beta_2)^4).$$

If $\alpha_2, \beta_2, \alpha_3, \beta_3 \in K$ then $U_1^2 U_{21}^2\delta_2^2(\alpha_2+\beta_2)^4 \in K^{\times 2}$ and we are done. Otherwise $U_1^2, U_{21}^2 \notin K^{\times 2}$ but their product is a square element in K hence $H = 1$. The proof is similar if $v(I_{42}) = v(I_{60}^2) < v(I_{43}^2)$.

iii) If $v(I_{42}) < v(I_{60}^2), v(I_{43}^2)$ then since by Lemmata 6.2.1.7 and 6.2.1.8 we have $I_{43}^2 = I_{42}I_{44} + I_{40}I$ for some $I \in K^{\times 2}$ and $I_{60}^2 = 16I_{42}I_{80} + I_{40}I'$ for some $I' \in K^{\times 2}$, it follows from Lemma 6.3.4.1. that $H = 1$.

□

Proposition 6.6.10. *Suppose that $\alpha_2, \beta_2, \alpha_3, \beta_3 \in K^{nr}$ and*
i) $v(\Delta_G^2) = v(I_{23}) = v(c) = v(I_{45}) = 0$ and $v(u_1^2) > 0$; or
ii) $v(u_1), v(I_{23}) \neq 0$ and $v(c) = v(I_{45}) = 0$. Then

$$c^2 I_{45} \equiv 4c^2(\alpha_2\beta_2 - \alpha_3\beta_3)^2 \text{ and } (\ell_1, u_1^2 I_{45}) = 1.$$

Proof. It follows from Lemma 6.2.1.9 that $c^2 I_{45} \equiv 4c^2(\alpha_2\beta_2 - \alpha_3\beta_3)^2$ in this case.

i) Recall that $u_1 = \alpha_2 + \beta_2 - \alpha_3 - \beta_3$ and $\ell_1 = \frac{u_1}{\Delta_G}$. Write $u_1 = \pi^a c_1$ with $c_1 \in \mathcal{O}_K^{\times}, a \in \mathbb{Z}$ since we assumed $\alpha_2, \beta_2, \alpha_3, \beta_3 \in K^{nr}$. We have

$$(\ell_1, u_1^2 I_{45}) = (\frac{\pi^a c_1}{\Delta_G}, \pi^{2a}c_1^2 I_{45}) = (\pi^a, c_1^2 I_{45}),$$

since $v(\Delta_G) = v(I_{45}) = 0$. If $u_1 \notin K$ then $(\alpha_2\beta_2 - \alpha_3\beta_3)^2, c_1^2 \notin K^{\times 2}$ so that $c_1^2(\alpha_2\beta_2 - \alpha_3\beta_3)^2 \in K^{\times 2}$ and hence $c_1^2 I_{45} \in K^{\times 2}$. Conversely, if $u_1 \in K$ then

$(\alpha_2\beta_2 - \alpha_3\beta_3)^2 \in K^{\times 2}$ and $c_1^2 I_{45} \in K^{\times 2}$.

ii) Write $u_1 = \pi^a c_1, \Delta_G = \pi^b u_d$ with $c_1, u_d \in \mathcal{O}_K^\times$. We have $a \in \mathbb{Z}$ since we assumed $\alpha_2, \beta_2, \alpha_3, \beta_3 \in K^{nr}$, from which it follows that $b \in \mathbb{Z}$ since $\ell_1 = \frac{u_1}{\Delta_G} \in K$. If $c_1 \notin K$(equivalently $u_d \notin K$, as $\ell_1 \in K$), then $(\alpha_2\beta_2 - \alpha_3\beta_3) \notin K^\times$ hence $I_{45} \notin K^{\times 2}$ and $c_1^2 I_{45} \in K^{\times 2}$. Conversely, if $c_1 \in K^\times$ then $(\alpha_2\beta_2 - \alpha_3\beta_3)^2 \in K^{\times 2}$ and $I_{45}, c_1^2, c_1^2 I_{45} \in K^{\times 2}$.

Now $(\frac{u_1}{\Delta_G}, u_1^2 I_{45}) = (\pi^{a-b}\frac{c_1}{u_d}, \pi^{2a} c_1^2 I_{45}) = (\pi^{a-b}, c_1^2 I_{45}) = 1$ as required.

□

6.6.1 C is of type 2

Lemma 6.6.11. *If C has good reduction then I_K acts trivially on $J[2]$ which implies $\alpha_1, \alpha_2, \beta_2, \alpha_3, \beta_3 \in K^{nr}$.*

Proof. This follows from Neron-Ogg-Shafarevich Theorem and the characterization of $J[2]$ in terms of $\alpha_1, \alpha_2, \beta_2, \alpha_3, \beta_3$ of Lemma 2.1.9. □

Proof of Table 4.4

Since C is of type 2, it follows that $v(I_{23}) = v(I_{45}) = v(I_{44}) = v(I_{80}) = 0$. Recall that either $v(c) = 0$ or $v(c) = 1$ so by semistability criterion 3.4.29, $v(c) = 0$. Moreover $\alpha_1, \alpha_2, \beta_2, \alpha_3, \beta_3 \in K^{nr}$ so that valuations of invariants are integers.

Using Lemmata 6.2.1.5, 6.2.1.3 and 6.2.1.1, and since $v(I_{40}) \in 2\mathbb{Z}$ by Lemma 6.2.1.4, we have that $(I_{41}, -I_{23}I_{80}) = (I_{22}, -I_{45}I_{44}) = (I_{20}, -I_{40}I_{44}) = 1$ by Lemma 6.3.4.1. Also, $(2, I_{44}) = (-2, I_{80}) = 1$ since $v \nmid 2$ and

$$(I_{43}, -I_{40}I_{42}I_{44})(I_{60}, -I_{40}I_{42}I_{80})(I_{42}, -I_{80}I_{44}) = 1$$

by Lemma 6.6.9. Therefore using definitions of invariants and simplifying gives

$$E_v = (\ell, I_{40})(I_{45}, -\ell I_{21}) = (\ell_1, (A_1 - B_1)^2)(u_1^2, \ell_2\ell_3).$$

<u>Case GR1A</u>. If $v(\Delta_G \ell_1) = 0$ then $E_v = (\ell_2\ell_3, u_1^2)$. If $v(\ell_2\ell_3) = 0$ we are done. Otherwise, if $v(\ell_2) \neq v(\ell_3)$ then $\alpha_2, \beta_2, \alpha_3, \beta_3 \in K$ and $u_1^2 \in K^{\times 2}$ so that $E_v = 1$. On the other hand, if $v(\ell_2) = v(\ell_3)$ then $v(\ell_2\ell_3) \in 2\mathbb{Z}$ and $E_v = 1$.

If $v(\Delta_G \ell_1) > 0$ then since $v(\Delta_G \ell_1) = u_1 = \alpha_2 + \beta_2 - \alpha_3 - \beta_3$ we have that $v(u_1^2) \in 2\mathbb{Z}$. Also recall that $v(\ell) = v(\ell_1) + v(\ell_2) + v(\ell_3)$ so that $v(\ell_1) \in 2\mathbb{Z}$ since by semistability criterion 3.4.29, we have $v(\ell) \in 2\mathbb{Z}$. Therefore $(u_1^2, \ell_2\ell_3) = 1$ and $E_v = (\ell_1, (A_1 - B_1)^2) = 1$.

If $v(\ell_2\ell_3) > 0$ then by definition of Δ_G and since $v(\Delta_G) = 0$, it follows that $v(\Delta_G \ell_1) = 0$. If $v(\ell_2) = v(\ell_3)$ then $v(\ell_2\ell_3) \in 2\mathbb{Z}$ and $E_v = 1$. Lastly, if $v(\ell_2) \neq v(\ell_3)$ then $u_1 = \Delta_G \ell_1 \in K$ so that $u_1^2 \in K^{\times 2}$ and $E_v = 1$.

Cases GR1B/GR1C. Assume first that $v(\Delta_G \ell_1) = 0$. If $v(\ell_2\ell_3) = 0$ then $E_v = (\ell_1, (A_1 - B_1)^2)$. If $v(\ell_2\ell_3) > 0$ with $v(\ell_2) \neq v(\ell_3)$ then $\Delta_G \ell_1 \in K$. On the other hand, if $v(\ell_2) = v(\ell_3)$ then $v(\ell_2\ell_3) \in 2\mathbb{Z}$. In both cases, $E_v = (\ell_1, (A_1 - B_1)^2)$.

Assume now that $v(\Delta_G \ell_1) > 0$. Then since $v(\Delta_G \ell_1) = u_1$, as above we have $v(u_1^2) \in 2\mathbb{Z}$. Now, if $v(\ell_2) = v(\ell_3)$ then $v(\ell_2\ell_3) \in 2\mathbb{Z}$ and $(u_1^2, \ell_2\ell_3) = 1$. Otherwise $u_1 \in K$ and $u_1^2 \in K^2$ so that $(u_1^2, \ell_2\ell_3) = 1$. In both cases we have $E_v = (\ell_1, (A_1 - B_1)^2)$. The result follows since in this case $v(\ell_1) \equiv r \bmod 2$.

6.6.2 C is of type 1_{2a}

Proof of Tables 4.5, 4.6, 4.7 and 4.8

Cases ON1. From the definition of the isogeny, we have

$$v(I_{23}) = 2a, \quad v(c) = v(I_{45}) = v(I_{80}) = v(I_{44}) = 0.$$

Also, $\alpha_2 \not\equiv \beta_2 \not\equiv \alpha_3 \not\equiv \beta_3 \not\equiv 0$ so that inertia acts trivially on these roots and hence $\alpha_2, \beta_2, \alpha_3, \beta_3 \in K^{\times nr}$. We have $(c, I_{80}I_{44})$=1. Making repetitive use of Lemma 6.3.4.1, it follows from Lemma 6.2.1.1 that $v(I_{20}) = 0$ since $v(I_{44}) = 0$ so that $(I_{20}, -I_{44}I_{40}) = (I_{20}, -I_{40})$, since $v(I_{45}) = v(I_{44}) = 0$, it follows from Lemma 6.2.1.3 that $(I_{22}, -I_{45}I_{44}) = 1$, since $v(\delta_1) > v(I_{80}) = 0$, it follows from Lemma 6.2.1.5 that $v(I_{41}) = 0$ and hence $(I_{41}, -I_{23}I_{80}) = (I_{41}, -I_{23})$. Therefore $E_v = (I_{23}, cI_{41})H_1H_2$, with

$$H_1 = (I_{40}, I_{20}\ell)(I_{45}, \ell I_{21}), \quad H_2 = (I_{42}, -I_{44}I_{80})(I_{43}, -I_{40}I_{42}I_{44})(I_{60}, -I_{40}I_{42}I_{80}).$$

From Lemma 6.6.9 we have that $H_2 = 1$. We show that $H_1 = 1$. By Lemma 6.2.1.4, we have $I_{40} =_\square u_1^2$, where $u_1 = \alpha_2 + \beta_2 - \alpha_3 - \beta_3$ so that $\ell_1 = \frac{u_1}{\Delta_G} =_\square \Delta_G u_1$. Using this notation, we have $H_1 = (\Delta_G u_1, u_1^2 I_{45})(\ell_2\ell_3, u_1^2)$.

Cases ON1A/B. Here $v(\Delta_G) = 0$ hence $H_1 = (\Delta_G u_1, u_1^2 I_{45})(\ell_2\ell_3, u_1^2)$.

If $v(u_1) = 0$ then $H_1 = (\ell_2\ell_3, u_1^2)$. If $v(\ell_2) \neq v(\ell_3)$ then $Frob(\alpha_2) = \alpha_2$ or $Frob(\alpha_2) = \beta_2$ and similarly for α_3. In particular $u_1 \in K$ and $u_1^2 \in \mathcal{O}_K^{\times 2}$; if $v(\ell_2) = v(\ell_3)$ then $v(\ell_2\ell_3) \in 2\mathbb{Z}$

If $v(u_1) > 0$, since $v(\Delta_G) = 0$, $v(\ell_2) = 0$ or $v(\ell_3) = 0$ or both from Lemma 6.2.2.5. But also since $u_1 = \ell_2 + \ell_3$, we must have $v(\ell_2) = v(\ell_3) = 0$. Moreover $v(u_1^2) \in 2\mathbb{Z}$ since $\alpha_2, \beta_2, \alpha_3, \beta_3 \in K^{\times nr}$. Hence $H_1 = (\Delta_G u_1, u_1^2 I_{45})$. The result follows from Proposition 6.6.10.2.

Cases ON1C/D. Here $v(\Delta_G) = r > 0$. Since $\ell_1 = \frac{u_1}{\Delta_G} \in K$, it follows that $r \in \mathbb{Z}$ and as above we have $H_1 = (\Delta_G u_1, u_1^2 I_{45})$.

The result follows from Proposition 6.6.10.2.

Therefore $E_v = (I_{23}, cI_{41})$. By definition of I_{41} we have that $cI_{41} \equiv_\square T_{\alpha_1}$ so

that $E_v = 1$ for ON1A/C and $E_v = (-1)^{2a}$ for ON1B/D as claimed.
Cases ON2. From the definition of the isogeny, we have

$$v(I_{44}) = 2a, \quad v(c) = v(I_{23}) = v(I_{80}) = v(I_{45}) = 0,$$

Therefore $(c, I_{23}I_{80})$=1. Reducing invariants $\bmod \pi$ yields

$$I_{20} \equiv_\square 2(\alpha_3 - \beta_3)^2, \quad I_{40} \equiv_\square I_{20}^2 \equiv_\square 1, \quad I_{45} \equiv_\square ((\alpha_2 - \alpha_3)(\alpha_2 - \beta_3))^2 \equiv_\square 1$$

and $v(I_{40}) = 0$. It follows that $(I_{45}, -\ell I_{21}) = 1$ and from Lemma 6.2.1.1 that $(I_{20}, -I_{44}I_{40}) = (I_{20}, -I_{44})$, with $v(I_{20}) = 0$. Making repetitive use of Lemma 6.3.4, since $v(I_{45}) = 0$, it follows from Lemma 6.2.1.3 that $(I_{22}, -I_{45}I_{44}) = (I_{22}, -I_{44})$ with $v(I_{22}) = 0$; since $v(I_{23}) = v(I_{80}) = 0$, it follows from Lemma 6.2.1.5 that $(I_{41}, -I_{23}I_{80}) = 1$. From the definition of the isogeny we have that $\delta_2, \delta_3 \in K$ so that using the definition of I_{42} in Lemma 6.2.1.6.(a) we have $I_{42} = 4A_{21}A_{31}$ with $A_{21}, A_{31} \in K$ and $v(A_{21}) = 0$. Using this notation, we have $E_v = (I_{44}, -cI_{22}A_{21})H$, with

$$H = (I_{44}, -2I_{20}A_{31}I_{43})(-I_{42}, -I_{43}I_{60})(I_{80}, I_{42}I_{60}).$$

We show that $H = 1$. Recall from Lemma 6.2.1.6.(b) and (c) that $I_{43} = \delta_2 A_{21} + \delta_3 A_{31}$ and $I_{60} = \hat{\delta_2} A_{31} + \hat{\delta_3} A_{21}$.

If $v(I_{31}) = 0$ then $v(I_{43}) = 0$ and $(I_{60}, -I_{42}I_{80}) = 1$ by Lemmata 6.3.4 and 6.2.1.8. It follows that

$$H = (I_{44}, -2I_{20}A_{31}I_{43}) = (I_{44}, 4(\alpha_3 - \beta_3)^4(-\alpha_3^2 - \beta_3^2 + 2\alpha_1^2)^2) = 1.$$

If $v(I_{31}) > 0$ then using definitions of invariants we obtain

$$H = (\delta_2\delta_3, \delta_3 A_{31}I_{43})(A_{21}A_{31}, -\hat{\delta_2}\hat{\delta_3})(I_{43}, -A_{21}A_{31})(\hat{\delta_3}A_{21}, -A_{21}A_{31}\hat{\delta_2}\hat{\delta_3}),$$

after reorganizing and simplifying we have

$$= (\delta_2, \delta_3 A_{31}I_{43})(A_{31}, A_{21}I_{43})(I_{43}, -\delta_3 A_{21}).$$

Now since $I_{43} - \delta_2 A_{21} - \delta_3 A_{31} = 0$ it follows from Lemma 6.3.3.3 that

$$(I_{43}, -\delta_2 A_{21})(-I_{43}, \delta_3 A_{31})(-\delta_2 A_{21}, \delta_3 A_{31}) = 1.$$

Hence

$$(I_{43}, -A_{21}\delta_3)(\delta_2, \delta_3)(A_{31}, A_{21})(I_{43}, \delta_2)(I_{43}, A_{31})(\delta_2, A_{31})$$
$$= (\delta_2, \delta_3 A_{31}I_{43})(A_{31}, A_{21}I_{43})(I_{43}, -\delta_3 A_{21}) = 1$$

as required. Therefore $E_v = (I_{44}, -cI_{22}A_{21})$ and since $-cI_{22}A_{21} \equiv_\square T_{a_2}$ it follows that $E_v = 1$ for ON2A/C and $E_v = (-1)^{2a}$ for ON2B/D.

<u>Cases ON3</u>. From the definition of the isogeny, we have

$$v(I_{45}) = a, \quad v(c) = v(I_{23}) = v(I_{80}) = v(I_{44}) = 0,$$

Therefore $(c, I_{23}I_{80}I_{44})$=1. Reducing invariants $\mod \pi$ yields

$$\ell_1 \equiv_\square c(\alpha_1 - \alpha_2)(\alpha_1 + \alpha_2) \quad I_{40} \equiv_\square (\beta_2 - \beta_3)^2,$$

so that $v(\ell_1) = 0$ and $v(I_{40}) \in 2\mathbb{Z}$. Also from Lemma 6.2.1.3 we have that $v(I_{22}) = 0$ and using Lemma 6.3.4, since $v(I_{23}) = v(I_{80}) = 0$, it follows from Lemma 6.2.1.5 that $(I_{41}, -I_{23}I_{80}) = 1$, similarly since $v(I_{44}) = 0$ and $v(I_{40}) \in 2\mathbb{Z}$ we have $(I_{20}, -I_{40}I_{44}) = 1$. Moreover since $v(I_{40}) \in 2\mathbb{Z}$,

$$(I_{43}, -I_{40}I_{42}I_{44})(I_{60}, -I_{40}I_{42}I_{80})(I_{42}, -I_{80}I_{44}) = 1$$

by Lemma 6.6.9. This yields $E_v = (I_{40}, \ell)(I_{45}, \ell I_{21}I_{22})$. We show that $(I_{40}, \ell) = 1$. Clearly, if $v(\ell_2\ell_3) = 0$ we are done. Otherwise, if $v(\ell_2) > 0$ then $v(\ell_3) = 0$. Indeed, by definition of ℓ_2 that yields $\beta_3 \equiv -\alpha_2$. By definition of ℓ_3, if $v(\ell_3) > 0$ then $\beta_2 \equiv -\alpha_2$, a contradiction since $\beta_3 \not\equiv \beta_2$. Now either $\alpha_2, \beta_2, \alpha_3, \beta_3 \in K$ and $I_{40} \in K^{\times 2}$ and $(I_{40}, \ell) = 1$, or from the definition of the isogeny, the roots α_2 and α_3 are permuted, similarly for β_2 and β_3. In particular ℓ_2 and $-\ell_3$ are permuted, a contradiction since their valuation is different. Therefore $E_v = (I_{45}, \ell I_{21}I_{22})$ and since $\ell I_{21}I_{22} =_\square T_{\alpha_2\alpha_3}$, it follows that $E_v = 1$ for ON3A and $E_v = (-1)^a$ for ON3B as required.

<u>Cases ON4</u>. From the definition of the isogeny, we have

$$v(I_{80}) = a, \quad v(c) = v(I_{23}) = v(I_{45}) = v(I_{44}) = 0,$$

and $\alpha_1, \alpha_2, \beta_2 \in K$. Therefore $(c, I_{23}I_{44})$=1 and since $I_{40} =_\square (\alpha_2 + \alpha_2 - \alpha_3 - \beta_3)^2$ it follows that $v(I_{40}) \in K^{\times 2}$ and $(I_{40}, \ell I_{60}I_{43}) = 1$. Computing Δ_G we find that $v(\Delta_G) = 0$. Also from Lemma 6.2.1.1 we have that $v(I_{20}) = 0$ so that and using Lemma 6.3.4, we have $(I_{20}, -I_{44}I_{40}) = 1$. Similarly, from Lemma 6.2.1.3, we have that $(I_{22}, -I_{44}I_{45}) = 1$; and from Lemma 6.2.1.5 we have that $v(I_{41}) = 0$ and $(I_{41}, -I_{23}I_{80}) = (I_{41}, I_{80})$. Moreover, by definitions of the invariants we have $(I_{45}, -\ell I_{21}) = (I_{45}, \ell_1)$. Recall that $u_1 = \alpha_2 + \beta_2 - \alpha_3 - \beta_3$ so that $\ell_1 = \frac{u_1}{\Delta_G}$. Then, since $v(\Delta_G) = 0$, $(I_{45}, \ell_1) = 1$ if $v(u_1) = 0$. Otherwise by Lemma 9.1 we have that

$(I_{45}, \ell_1) = 1$. This yields

$$E_v = (I_{80}, -2cI_{41}I_{42}I_{60})(I_{43}, -I_{42}I_{44})(I_{60}, -I_{42})(I_{42}, -I_{44}).$$

From the definition of the isogeny we have that $\delta_2, \delta_3, \hat{\delta_2}, \hat{\delta_3} \in K$ so that using the definition of I_{42} in Lemma 6.2.1.6.(a) we have $I_{42} = 4A_{21}A_{31}$ with $A_{21}, A_{31} \in K$, $I_{60} = \hat{\delta_2}A_{31} + \hat{\delta_3}A_{21}$ and $I_{43} = \delta_2 A_{21} + \delta_3 A_{31}$. Note that $v(\delta_2) = v(\delta_3) = v(\hat{\delta_2}) = v(A_{21}) = 0$.

If $v(A_{31}) = 0$ then $v(I_{42}) = 0$ so that $(I_{42}, -I_{44}) = 1$ and by Lemmata 6.3.4 and 6.2.1.7 we have $(I_{43}, -I_{42}I_{44}) = 1$. Also, in this case $I_{60} \equiv \hat{\delta_2}A_{31}$ and $v(I_{60}) = 0$, hence $(I_{60}, -I_{42}) = 1$. This yields

$$E_v = (I_{80}, -2cI_{41}A_{21}A_{31}\hat{\delta_2}A_{31}) = (I_{80}, -2cI_{41}A_{21}\hat{\delta_2}).$$

On the other hand, if $v(A_{31}) > 0$ then $I_{43} \equiv \delta_2 A_{21}$ and using this notation and the definitions of invariants and simplifying, we have

$$E_v = (\hat{\delta_2}\hat{\delta_3}, I_{60})(\hat{\delta_2}, A_{31})(\hat{\delta_3}, -2cI_{41}A_{21}A_{31})$$

$$(\delta_2 A_{21}, -A_{21}A_{31}\delta_2\delta_3)(I_{60}, -A_{21}A_{31})(A_{31}, -\delta_2\delta_3).$$

Since $I_{60} - \hat{\delta_2}A_{31} - \hat{\delta_3}A_{21} = 0$ it follows from Lemma 6.3.3.3 that

$$(I_{60}, -\hat{\delta_2}A_{31})(-I_{60}, \hat{\delta_3}A_{21})(-\hat{\delta_2}A_{31}, \hat{\delta_3}A_{21}) = 1,$$

which yields

$$(I_{60}, \hat{\delta_2}\hat{\delta_3}) = (I_{60}, -A_{21}A_{31})(\hat{\delta_3}A_{21}, \hat{\delta_2}A_{31}).$$

Using this expression for $(I_{60}, \hat{\delta_2}\hat{\delta_3})$ in E_v and simplifying again gives

$$E_v = (\hat{\delta_3}, -2cI_{41}A_{21}\hat{\delta_2})(A_{31}, -\hat{\delta_2}\delta_3).$$

By Lemma 6.2.1.6.(e), since if $v(A_{31}) > 0$ then we have $\hat{\delta_2} \equiv -\delta_3$. Hence $E_v = (\hat{\delta_3}, -2cI_{41}A_{21}\hat{\delta_2})$ as in the previous case.

Since $-2cI_{41}A_{21}\hat{\delta_2} \equiv_\square T_{\alpha_1\alpha_2}$, it follows that $E_v = 1$ for ON4A and $E_v = (-1)^a$ for ON4B as required.

6.6.3 C is of type $I_{2a,2b}$

Proof of Tables 4.9 to 4.22

Cases TN1. From the definition of the isogeny we see that $v(I_{23}) = 2a$ and $v(I_{44}) = 2b$ with $v(\delta_2) = 2b \neq v(\delta_3) = 0$. Hence $\delta_2, \delta_3 \in K$ and $\hat{\delta_2}, \hat{\delta_3}, A_{21}, A_{31} \in K$. Reducing invariants yields

$$I_{45} \equiv_\square \hat{\delta_2} \equiv_\square \hat{\delta_3} \equiv_\square I_{40} \equiv_\square 1, \quad I_{20} \equiv_\square 2\delta_3, \quad I_{22} \equiv_\square 2(\alpha_2 - \beta_3)(\alpha_2 - \alpha_3),$$

$$A_{21} \equiv_\square 2\alpha_2^2, \quad A_{31} \equiv_\square \alpha_3^2 + \beta_3^2.$$

It follows that $E_v = (I_{23}, cI_{41})(I_{44}, cI_{22}A_{21})H$, where $H = (I_{44}, 2I_{20}I_{42}I_{43})(-I_{42}, -I_{43}I_{60})$. We show that $H = 1$.

If $v(A_{31}) = 0$ then by Lemma 6.2.1.7, $v(I_{43}) = 0$ and $I_{43} \equiv_\square \delta_3 A_{31}$ so that

$$H = (I_{44}, \delta_3^2 A_{31}^2)(I_{60}, -A_{21}A_{31}) = (\hat{\delta_2} A_{31} + \hat{\delta_3} A_{21}, -A_{21}A_{31}).$$

Using Lemma 6.3.3.1 and since $\hat{\delta_2}, \hat{\delta_3} \in K^{\times 2}$, this gives

$$= (\hat{\delta_2} A_{31} + \hat{\delta_3} A_{21}, -A_{21}A_{31}\hat{\delta_2}\hat{\delta_3}) = 1.$$

If $v(A_{31}) > 0$ then $I_{60} \equiv_\square \hat{\delta_3} A_{21}$ and $\delta_3 \equiv -\hat{\delta_2} \equiv_\square -1$ from Lemma 6.2.1.6.(e), so that

$$H = (-\delta_2, -A_{31}I_{43})(A_{21}A_{31}, -I_{43}A_{21})(I_{43}A_{21}, -1).$$

But since $I_{43} = \delta_2 A_{21} + \delta_3 A_{31}$, it follows from Lemma 6.3.3.3 that

$$1 = (I_{43}, -\delta_2 A_{21})(-I_{43}, \delta_3 A_{31})(-\delta_2 A_{21}, \delta_3 A_{31})$$

$$= (I_{43}, -\delta_2 A_{21})(-I_{43}, -A_{31})(-\delta_2 A_{21}, -A_{31}) = H,$$

proving the result.

Therefore $E_v = (I_{23}, cI_{41})(I_{44}, cI_{22}A_{21})$. Now $cI_{41} \equiv_\square T_{\alpha_1}$ and $cI_{22}A_{21} \equiv_\square T_{\alpha_2}$. It follows that $E_v = 1$ for TN1A/E, $E_v = (-1)^{2a}$ for TN1B/F, $E_v = (-1)^{2b}$ for TN1C/G and $E_v = (-1)^{2a+2b}$ for TN1D/H as required.

Cases TN2. From the definition of the isogeny we see that $v(I_{23}) = v(I_{45}) = v(I_{80}) = 0$ and $v(I_{44}) = 2a + 2b$. Write $\beta_2 = \alpha_2 + a_2\pi^a$ and $\beta_3 = \alpha_3 + a_3\pi^b$, $a_2, a_3 \in \mathcal{O}_K^\times$. We have

$$I_{44} \equiv a_2^2 a_3^2 \pi^{2a+2b}, \quad I_{45} \equiv_\square 1 \quad I_{40} \equiv_\square u_1^2 \equiv_\square (\alpha_2 - \alpha_3)^2,$$

$$I_{20} \equiv_\square 2(a_2^2\pi^{2a} + a_3^2\pi^{2b}), \quad I_{22} \equiv_\square 2(\alpha_2 - \alpha_3)^2,$$

$$I_{42} = 4A_{21}A_{31} \equiv_\square (\alpha_2 - \alpha_1)(\alpha_2 + \alpha_1)(\alpha_3 - \alpha_1)(\alpha_3 + \alpha_1), \quad I_{43} \equiv a_2^2\pi^{2a}A_{21} + a_3^2\pi^{2b}A_{31}.$$

It follows that $v(I_{22}) = v(I_{42}) = 0$. Also by Lemma 6.2.1.5 and using Lemma 6.3.4 we have that $(I_{41}, -I_{23}I_{80}) = 1$, and by Lemma 6.2.1.8, we have $(I_{60}, -I_{40}I_{42}I_{80}) =$

1. This yields $E_v = (I_{20}, -I_{40}I_{44})(I_{43}, -I_{40}I_{42}I_{44})(I_{44}, 2cI_{22}I_{42})(\ell, I_{40})$.
Cases TN2A/B/C/F/G/H: Here $\alpha_2, \alpha_3 \in K_v$ therefore

$$I_{44} \equiv \pi^{2a+2b}, \quad I_{45} \equiv I_{40} \equiv_{\square} 1, \quad I_{22} \equiv_{\square} 2, \quad I_{20} \equiv_{\square} 2(a_2^2\pi^{2a} + a_3^2\pi^{2b}),$$

$$A_{21} \equiv 2(\alpha_2-\alpha_1)(\alpha_2+\alpha_1), \quad A_{31} \equiv 2(\alpha_3-\alpha_1)(\alpha_3+\alpha_1), \quad I_{43} \equiv a_2^2\pi^{2a}A_{21}+a_3^2\pi^{2b}A_{31}.$$

Hence $E_v = (I_{20}, -I_{44})(I_{43}, -I_{42}I_{44})(I_{44}, 2cI_{22}I_{42})$.
If $a < b$ then $I_{20} \equiv_{\square} 2\pi^{2a}$ and $I_{43} \equiv_{\square} \pi^{2a}A_{21}$ so that

$$E_v = (2\pi^{2a}, -\pi^{2a+2b})(A_{21}\pi^{2a}, -A_{21}A_{31}\pi^{2a+2b})(\pi^{2a+2b}, cA_{21}A_{31}).$$

Simplifying yields $E_v = (\pi^{2a}, 2cA_{21})(\pi^{2b}, 2cA_{31})$. Now the results follows since we have $T_{\alpha_2} \equiv_{\square} 2cA_{21}$ and $T_{\alpha_3} \equiv_{\square} 2cA_{31}$. Therefore $E_v = 1$ for TN2A/F, $E_v = (-1)^{2a}$ for TN2B/G and $E_v = (-1)^{2a+2b}$ for TN2C/H as required.

If $a = b$ then $I_{44} \in K_v^2$ so that $E_v = (I_{20}, -1)(I_{43}, -I_{42})$ with

$$I_{20} \equiv \pi^{2a}(a_2^2 + a_3^2) \text{ and } I_{43} \equiv \pi^{2a}(a_2^2A_{21} + a_3^2A_{31})$$

hence

$$E_v = (\pi^{2a}, A_{21}A_{31})(a_2^2 + a_3^2, -1)(a_2^2A_{21} + a_3^2A_{31}, -A_{21}A_{31}).$$

Now $(a_2^2+a_3^2, -1) = (a_2^2+a_3^2, -a_2^2a_3^2) = 1$ and $(a_2^2A_{21}+a_3^2A_{31}, -A_{21}A_{31}) = (a_2^2A_{21}+a_3^2A_{31}, -a_2^2a_3^2A_{21}A_{31}) = 1$ by Lemma 6.3.3, therefore $E_v = (\pi^{2a}, A_{21}A_{31})$.

For TN2A (respectively TN2C), we have $T_{\alpha_2} \equiv_{\square} c(\alpha_2-\alpha_1)(\alpha_2+\alpha_1), T_{\alpha_3} \equiv_{\square} c(\alpha_3-\alpha_1)(\alpha_3+\alpha_1) \in K^{\times 2}(\notin K^{\times 2}$ respectively) so that (in both cases) $(\alpha_2-\alpha_1)(\alpha_2+\alpha_1) \in K^{\times 2} \Leftrightarrow (\alpha_3-\alpha_1)(\alpha_3+\alpha_1) \in K^{\times 2}$. It follows that $A_{21}A_{31} \in K^{\times 2}$ and hence $E_v = 1$ which proves the result.

For TN2B, we have $T_{\alpha_2} \equiv_{\square} c(\alpha_2-\alpha_1)(\alpha_2+\alpha_1) \notin K^{\times 2}, T_{\alpha_3} \equiv_{\square} v(\alpha_3-\alpha_1)(\alpha_3+\alpha_1) \in K^{\times 2}$ so that $A_{21}A_{31} \notin K^{\times 2}$ and $E_v = (-1)^{2a}$ as required.
Cases TN2D/E/I/J: Here $a = b$ and $Frob$ swaps δ_2 and δ_3 and hence $\alpha_2, \beta_2, \alpha_3, \beta_3 \notin K$. Without loss of generality, let $Frob(\alpha_2) = \alpha_3$ so that $Frob(\beta_2) = \beta_3$. By Lemma 6.2.1.4, we have $v(I_{40}) \in 2\mathbb{Z}$ since $v(u_1) = 0$. Also, since $a = b$ we have $v(I_{44}) \in 2\mathbb{Z}$ and as above

$$I_{20} \equiv \pi^{2a}(a_2^2 + a_3^2), \quad I_{43} \equiv \pi^{2a}(a_2^2A_{21} + a_3^2A_{31}), \quad I_{44} \equiv_{\square} a_2^2a_3^2$$

This yields $E_v = (I_{20}, -I_{40}I_{44})(I_{43}, -I_{40}I_{42}I_{44})(\ell, I_{40})$.

If $2a \in 2\mathbb{Z}$, since $v(I_{42}) = 0$, it follows from Lemma 6.2.1.7 that $(I_{43}, -I_{40}I_{42}I_{44}) = 1$. Hence $E_v = (\pi^{2a}(a_2^2+a_3^2), -I_{40}a_2^2a_3^2)(\ell, I_{40})$. Simplifying using Lemma 6.3.3 which gives $(a_2^2+a_3^2, -a_2^2a_3^2) = (a_2^2, a_3^2)$ as above, yields $E_v = (\ell, I_{40})$. Now, $\ell = \ell_1\ell_2\ell_3$ with $\ell_1 = \frac{u_1}{\Delta_G}, \ell_2 \equiv_{\square} 2c\alpha_3, \ell_3 \equiv_{\square} -2c\alpha_2$. It follows that $v(\ell_2) = v(\ell_3) = 0$ and that

$v(\ell_1) = 0$ for TND/E while $v(\ell_1) = -r$ for TNI/J.
Therefore $E_v = (\ell_1, u_1^2)$. For TN2D/E, $v(\ell_1) = 0$ and $E_v = 1$. For TN2I/J, $v(\ell_1) = -r$ so that $E_v = (-1)^r$ since $u_1^2 \notin K^{\times 2}$ as required.

If $2a$ is odd then using the above expression for the invariants yields

$$E_v = (\pi^{2a}(a_2^2 + a_3^2), -I_{40}I_{44})(\pi^{2a}(a_2^2 A_{21} + a_3^2 A_{31}), -I_{40}I_{42}I_{44})(\ell, I_{40})$$

$$= (\ell, I_{40})(\pi^{2a}, I_{40}^2 I_{44}^2 I_{42})H_1 H_2,$$

where

$$H_1 = (a_2^2 + a_3^2, -I_{40}I_{44}), \quad H_2 = (a_2^2 A_{21} + a_3^2 A_{31}, -I_{40}I_{42}I_{44}).$$

If $v(a_2^2 + a_3^2) > 0$ then $v(I_{20}) > v(I_{44}) = v(I_{40})$ by Lemma 6.2.1.1. Therefore $H_1 = 1$ by Lemma 6.3.4. Similarly for H_2 since if $v(a_2^2 A_{21} + a_3^2 A_{31}) > 0$ then $v(I_{43}) > v(I_{42}I_{44})$ so that $H_2 = 1$ by Lemma 6.2.1.7.

Therefore $E_v = (\pi^{2a}, I_{42})(\ell, I_{40})$.

Recall from Proposition 3.4.30 that $T_{\alpha_2} = c(\alpha_2 - \alpha_3)(\alpha_2 - \beta_3)(\alpha_2 - \alpha_1)(\alpha_2 + \alpha_1)$ and $T_{\alpha_3} = c(\alpha_3 - \alpha_2)(\alpha_3 - \beta_2)(\alpha_3 - \alpha_1)(\alpha_3 + \alpha_1)$. In particular in these cases, $Frob(T_{\alpha_2}) = T_{\alpha_3}$ and $Frob(T_{\alpha_3}) = T_{\alpha_2}$. Moreover,

$$T_{\alpha_2}T_{\alpha_3} \equiv c^2(\alpha_2 - \alpha_3)^4(\alpha_2 - \alpha_1)(\alpha_2 + \alpha_1)(\alpha_3 - \alpha_1)(\alpha_3 + \alpha_1)$$

so that $T_{\alpha_2}T_{\alpha_3} \equiv_\square I_{42}$.

Now, let $t_{\alpha_2}^{\pm}$ and $t_{\alpha_3}^{\pm}$ denote the square roots of $T_{\alpha_2}, T_{\alpha_3}$ respectively. By definition of TN2D/I, we have without loss of generality $Frob(t_{\alpha_2}^{+}) = t_{\alpha_3}^{+}$ and $Frob(t_{\alpha_3}^{+}) = t_{\alpha_2}^{+}$. Therefore $T_{\alpha_2}T_{\alpha_3} = (t_{\alpha_2}^{+}t_{\alpha_3}^{+})^2 \in \mathcal{O}_K^{\times 2}$, and hence $I_{42} \in \mathcal{O}_K^{\times 2}$. On the other hand, by definition of TN2E/J, we have $Frob(t_{\alpha_2}^{+}) = t_{\alpha_3}^{+}$, $Frob(t_{\alpha_3}^{+}) = t_{\alpha_2}^{-}$, $Frob(t_{\alpha_2}^{-}) = t_{\alpha_3}^{-}$ and $Frob(t_{\alpha_3}^{-}) = t_{\alpha_2}^{+}$. It follows that $T_{\alpha_2}T_{\alpha_3} = (t_{\alpha_2}^{+}t_{\alpha_3}^{+})^2 \notin \mathcal{O}_K^{\times 2}$, and hence $I_{42} \notin \mathcal{O}_K^{\times 2}$.

This yields $E_v = (\pi^{2a}, I_{42}) = 1$ for TN2D, $E_v = (\pi^{2a}, I_{42}) = (-1)^{2a}$ for TN2E, $E_v = (\pi^{2a}, I_{42})(\ell, I_{40}) = (-1)^r$ for TN2I, and $E_v = (\pi^{2a}, I_{42})(\ell, I_{40}) = (-1)^{2a+r}$ for TN2J as required.

Cases TN3. From the definition of the isogeny we see that $v(I_{23}) = 2a, v(I_{45}) = b$ and $v(I_{44}) = v(I_{80}) = 0$. Also since $v(\alpha_2 - \alpha_3) \neq v(\beta_2 - \beta_3)$ and $\beta_2 \not\equiv \beta_3$, it follows that $\alpha_2, \beta_2, \alpha_3, \beta_3 \in K$ (since otherwise I_K would permutes α_2 and α_3 while $Frob$ would permute β_2 and β_3, a contradiction). In particular $b \in \mathbb{Z}$, $A_{21}, A_{31} \in K$ and $\delta_2, \delta_3, \hat{\delta_2}, \hat{\delta_3} \in K^{\times 2}$. Reducing invariants yields

$$I_{44} \equiv_\square I_{40} \equiv_\square I_{80} \equiv_\square 1,$$

$$\ell_1 \equiv_\square \frac{-1}{c}, \quad I_{22} \equiv -(\alpha_2 - \beta_3)(\alpha_3 - \beta_2), \quad I_{41} \equiv_\square \beta_2\beta_3,$$

so that $v(\ell_1) = v(I_{22}) = v(I_{41}) = 0$. Now from Lemma 6.2.1.1 and using Lemma 6.3.4, we have that $(I_{20}, -I_{40}I_{44}) = 1$. Therefore $E_v = (I_{23}, cI_{41})(I_{45}, -\ell I_{22}I_{21})H$, where $H = (-I_{42}, -I_{43})(-I_{42}, I_{60})$. We show that $H = 1$. Using definitions of invariants and since $A_{21}, A_{31} \in K$ and $\delta_2, \delta_3, \hat{\delta_2}, \hat{\delta_3} \in K^{\times 2}$, we can rewrite H as

$$= (-A_{21}A_{31}, -A_{21}\delta_2 - A_{31}\delta_3)(-A_{21}A_{31}, A_{21}\hat{\delta_3} + A_{31}\hat{\delta_2})$$

$$= (-A_{21}A_{31}\delta_2\delta_3, -A_{21}\delta_2 - A_{31}\delta_3)(-A_{21}A_{31}\hat{\delta_2}\hat{\delta_3}, A_{21}\hat{\delta_3} + A_{31}\hat{\delta_2}).$$

Using Lemma 6.3.3.1 we have $H = (A_{21}\delta_2, A_{31}\delta_3)(A_{21}\hat{\delta_3}, A_{31}\hat{\delta_2}) = 1$.
Therefore $E_v = (I_{23}, cI_{41})(I_{45}, -\ell I_{22}I_{21})$. Since $\ell = \ell_1\ell_2\ell_3$ and $I_{21} = -\ell_2\ell_3$, we have $E_v = (I_{23}, cI_{41})(I_{45}, \ell_1 I_{22})$. Noting that $cI_{41} \equiv_\square T_{\alpha_1}$ and $\ell_1 I_{22} \equiv_\square T_{\alpha_2}$, we obtain that $E_v = 1$ for TN3A, $E_v = (-1)^{2a}$ for TN3B, $E_v = (-1)^b$ for TN3C and $E_v = (-1)^{2a+b}$ for TN3D as required.
Cases TN4. From the definition of the isogeny we see that $v(I_{44}) = v(\delta_3) = 2a$, $v(I_{80}) = v(\hat{\delta_3}) = b$ and $v(I_{23}) = v(I_{45}) = 0$. Also $\alpha_1, \alpha_2, \beta_2 \in K$. In particular $b \in \mathbb{Z}$ and $\delta_2, \delta_3, \hat{\delta_2}, \hat{\delta_3} \in K^{\times 2}$. Reducing invariants yields

$$I_{23} \equiv_\square I_{45} \equiv_\square I_{40} \equiv_\square 1, \quad I_{20} \equiv_\square 2, \quad I_{22} \equiv_\square 2(\alpha_3 - \alpha_1)(\alpha_3 - \beta_2),$$

$$I_{41} \equiv_\square \alpha_1(\alpha_1 + \beta_2), \quad I_{42} \equiv_\square 2(\alpha_1 - \beta_2)(\alpha_1 + \beta_2)(\alpha_1 - \alpha_3)(\alpha_1 + \beta_3),$$

$$I_{43} \equiv_\square -(\alpha_1 + \beta_2)(\alpha_1 - \beta_2), \quad I_{60} \equiv_\square -2(\alpha_1 - \alpha_3)(\alpha_1 + \alpha_3).$$

Therefore $E_v = (I_{44}, 2cI_{20}I_{22}I_{42}I_{43})(I_{80}, -2cI_{41}I_{42}I_{60})$. Replacing invariants with their values above and clearing squares in K yields

$$E_v = (I_{44}, c(\alpha_3 - \beta_2)(\alpha_1 + \beta_3))(I_{80}, 2c\alpha_1(\alpha_1 - \beta_2)).$$

Noting that $c(\alpha_3 - \beta_2)(\alpha_1 + \beta_3) \equiv_\square T_{\alpha_3}$ and $2c\alpha_1(\alpha_1 - \beta_2) \equiv_\square T_{\alpha_1}$, we obtain that $E_v = 1$ for TN4A, $E_v = (-1)^{2a}$ for TN4B, $E_v = (-1)^b$ for TN4C and $E_v = (-1)^{2a+b}$ for TN4D as required.
Cases TN5. From the definition of the isogeny we see that $v(I_{45}) = b$, $v(I_{80}) = v(\hat{\delta_3}) = a$ and $v(I_{23}) = v(I_{44}) = 0$. Also $\alpha_1, \alpha_2, \beta_2, \alpha_3, \beta_3 \in K$. In particular $a, b \in \mathbb{Z}$, $A_{21}, A_{31} \in K$ and $I_{40}, \delta_2, \delta_3, \hat{\delta_2}, \hat{\delta_3} \in K^{\times 2}$. Reducing invariants yields

$$I_{23} \equiv_\square I_{44} \equiv_\square I_{40} \equiv_\square 1, \quad I_{41} \equiv_\square \alpha_1(\alpha_1 + \beta_3), \quad A_{21} \equiv_\square (\alpha_3 + \alpha_1)(\alpha_3 - \alpha_1).$$

Using Lemma 6.3.4, it follows from Lemma 6.2.1.1 that $(I_{20}, -I_{40}I_{44}) = 1$. Therefore

$$E_v = (I_{45}, -\ell I_{21}I_{22})(I_{80}, -2cI_{41}I_{42}I_{60})(-I_{42}, -I_{43}I_{60}).$$

If $v(I_{31}) = 0$ then by Lemma 6.2.1.6.(a), we have $v(I_{42}) = 0$ and Lemmata

6.2.1.7 and 6.3.4 yields $(I_{43}, -I_{42}I_{44}) = 0$. Also, by Lemma 6.2.1.6.(b), we have $I_{60} \equiv \hat{\delta_2} A_{31}$ so that

$$E_v = (I_{45}, -\ell I_{21} I_{22})(I_{80}, -2cI_{41}A_{21}A_{31}^2\hat{\delta_2})(-A_{21}A_{31}, \hat{\delta_2}A_{31})$$

$$E_v = (I_{45}, -\ell I_{21} I_{22})(I_{80}, -2cI_{41}A_{21}\hat{\delta_2}).$$

If $v(A_{31}) > 0$ then by Lemma 6.2.1.6.(c) we have $I_{43} \equiv \delta_2 A_{21} \equiv_\square A_{21}$ and $\hat{\delta_2} \equiv_\square -1$. Therefore

$$E_v = (I_{45}, -\ell I_{21} I_{22})(I_{80}, -2cI_{41}A_{21}\hat{\delta_2})H, \quad H = (I_{80}, \hat{\delta_2}A_{31}I_{60})(-A_{21}A_{31}, -A_{21}I_{60}).$$

We show that $H = 1$. From Lemma 6.2.1.6.(b) we have that $I_{60} - \hat{\delta_2}A_{31} - \hat{\delta_3}A_{21} = 0$, therefore using Lemma 6.3.3.3 we have

$$(I_{60}, -\hat{\delta_2}A_{31})(-I_{60}, \hat{\delta_3}A_{21})(-\hat{\delta_2}A_{31}, \hat{\delta_3}A_{21}) = (I_{60}, A_{31})(-I_{60}, \hat{\delta_3}A_{21})(A_{31}, \hat{\delta_3}A_{21}) = 1$$

Now H can be re written as follows $H = (\hat{\delta_3}, \hat{\delta_2}A_{31}I_{60})(-A_{21}A_{31}, -A_{21}I_{60})$, and we see that $H = (I_{60}, A_{31})(-I_{60}, \hat{\delta_3}A_{21})(A_{31}, \hat{\delta_3}A_{21}) = 1$.

Therefore in both case we have $E_v = (I_{45}, -\ell I_{21} I_{22})(I_{80}, -2cI_{41}A_{21}\hat{\delta_2})$. Noting that $-\ell I_{21}I_{22} \equiv_\square T_{\alpha_1}$ and $-2cI_{41}A_{21}\hat{\delta_2} \equiv_\square T_{\alpha_3}$, we obtain that $E_v = 1$ for TN5A, $E_v = (-1)^a$ for TN5B, $E_v = (-1)^b$ for TN5C and $E_v = (-1)^{a+b}$ for TN5D as required.

<u>Cases TN6</u>. From the definition of the isogeny we see that $v(I_{80}) = a + b$ and $v(I_{45}) = v(I_{23}) = v(I_{44}) = 0$.

<u>Cases TN6A/B/C</u>. Here $\alpha_1, \alpha_2, \beta_2, \alpha_3, \beta_3 \in K$. In particular $a, b \in \mathbb{Z}$, $A_{21}, A_{31} \in K$ and $I_{40}, I_{23}, \delta_2, \delta_3 \in K^{\times 2}$. By Lemma 6.2.1.1 we have that $v(I_{20}) = 0$ and Lemma 6.3.4 yields $(I_{20}, -I_{40}I_{44}) = 1$. Also, Lemma 6.2.1.3 yields $v(I_{22}) = 0$ so that $(I_{22}, -I_{44}I_{45}) = 1$. Similarly, Lemma 6.2.1.7 gives $v(I_{43}) = 0$ so that $(I_{43}, -I_{40}I_{42}I_{44}) = 1$. Finally, we have $v(\ell_2) = v(\ell_3) = 0$. Reducing invariants yields

$$A_{21} \equiv_\square -(\alpha_1 - \beta_2)(\alpha_1 + \beta_2), \quad A_{31} \equiv_\square -(\alpha_1 - \beta_3)(\alpha_1 + \beta_3),$$

$$I_{45} \equiv 2\alpha_1(\beta_2 - \beta_3)(\alpha_1 + \beta_2)(\alpha_1 - \beta_3)$$

so that $v(I_{42}) = 0$ by Lemma 6.2.1.6.(a). Therefore

$$E_v = (I_{45}, -\ell I_{21})(I_{80}, -2cI_{41}I_{42}I_{60})(-I_{42}, I_{60})(-1, I_{41}).$$

Recall that $\ell = \ell_1\ell_2\ell_3$ and $I_{21} = -\ell_2\ell_3$. It follows that $(I_{45}, -\ell I_{21}) = (I_{45}, \ell_1)$. Recall that $u_1 = \alpha_2 + \beta_2 - \alpha_3 - \beta_3$ so that $\ell = \frac{u_1}{\Delta_G}$. In particular, we have $v(\Delta_G) = 0$ and $u_1 \equiv 2\alpha_1 + \beta_2 - \beta_3$. If $v(u_1) = 0$ then $(I_{45}, \ell_1) = 1$. On the other hand, if $v(u_1) > 0$

then by Lemma 9 we have $I_{45} \equiv_\square 1$. Hence $(I_{45}, \ell_1) = 1$ and

$$E_v = (I_{80}, -2cI_{41}I_{42}I_{60})(-I_{42}, I_{60})(-1, I_{41}).$$

If $a < b$, by Lemma 6.2.1.6.(b) we have $I_{60} \equiv \hat{\delta_3} A_{21}$. Moreover, using Lemma 6.2.1.5 we can write $I_{41} \equiv 2u_a\pi^a(\alpha_1 - \beta_2)(\alpha_1 - \alpha_3)(\alpha_1 - \beta_3)$ for some $u_a \in \mathcal{O}_K^\times$. Replacing invariants by their value and simplifying yields

$$E_v = (\hat{\delta_2}\hat{\delta_3}, -u_a\pi^a(\alpha_1-\beta_2)(\alpha_1-\alpha_3)(\alpha_1-\beta_3)A_{21}A_{31}\hat{\delta_3}A_{21})(A_{21}A_{31}, \hat{\delta_3}A_{21})(\pi^{2a}, -1)$$

$$= (\pi^a, c(\alpha_1 - \beta_2)(\alpha_1 - \beta_3))(\pi^b, c(\alpha_1 + \beta_2)(\alpha_1 + \beta_3)).$$

Noting that $T_{\alpha_1} \equiv_\square c(\alpha_1-\beta_2)(\alpha_1-\beta_3)$ and $T_{-\alpha_1} \equiv_\square c(\alpha_1+\beta_2)(\alpha_1+\beta_3)$, we obtain that $E_v = 1$ for TN6A, $E_v = (-1)^a$ for TN6B and $E_v = (-1)^{a+b}$ for TN6C as required.

If $a = b$ then $v(I_{80}) \in 2\mathbb{Z}$ and $E_v = (I_{80}, I_{41}I_{60})(I_{60}, -I_{42})(I_{41}, -1)$. Using the definitions of invariants in Lemmata 6.2.1.6.(a), 6.2.1.5 and since $I_{80} = \hat{\delta_2}\hat{\delta_3}$, we may write

$$I_{80} = u_a u_b \pi^{2a}\alpha_1^2(\alpha_1 - \beta_3)(\alpha_1 + \beta_3)(\alpha_1 - \beta_2)(\alpha_1 + \beta_2)$$

$$= u_a u_b \pi^{2a}\alpha_1^2 A_{21}A_{31},$$

$$I_{41} = \alpha_1\pi^a((u_b(\alpha_1 + \beta_2)(\alpha_1 + \beta_3) + u_a(\alpha_1 - \beta_2)(\alpha_1 - \beta_3)),$$

$$I_{60} = -2\alpha_1\pi^a(u_a(\alpha_1 - \beta_2)^2(\alpha + \beta_2)^2 + u_b(\alpha_1 - \beta_3)^2(\alpha_1 - \beta_2)^2),$$

for some $u_a, u_b \in \mathcal{O}_K^\times$. Hence replacing invariants by their values and simplifying using Lemma 6.3.3 gives

$$(\pi^a, A_{21}A_{31})(u_a(\alpha_1 - \beta_2)^2(\alpha + \beta_2)^2 + u_b(\alpha_1 - \beta_3)^2(\alpha_1 - \beta_2)^2, -u_a u_b)$$

$$((u_b(\alpha_1+\beta_2)(\alpha_1+\beta_3)+u_a(\alpha_1-\beta_2)(\alpha_1-\beta_3)), -u_a u_b(\alpha_1-\beta_3)(\alpha_1+\beta_3)(\alpha_1-\beta_2)(\alpha_1+\beta_2))$$

$$= (\pi^a, A_{21}A_{31}).$$

Noting that $A_{21}A_{31} \equiv_\square T_{\alpha_1}T_{-\alpha_1}$, it follows that $E_v = 1$ for TN6A, $E_v = (-1)^a$ for TN6B and $E_v = (-1)^{2a} = 1$ for TN6C as required.

Cases TN6D/E. Here $\alpha_1\alpha_2, \beta_2, \alpha_3, \beta_3 \notin K$ and $a = b$. However, by semistability criterion 3.4.29, we have that $\alpha_1, \alpha_2, \beta_2, \alpha_3, \beta_3 \in K^{nr}$ so that $v(I_{40}) \in 2\mathbb{Z}$. Therefore, repeating the same arguments as above we have $(I_{20}, -I_{40}I_{44}) = 1$, $(I_{22}, -I_{44}I_{45}) = 1$ and $(I_{43}, -I_{40}I_{42}I_{44}) = 1$. This yields

$$E_v = (I_{40}, \ell)(I_{45}, -\ell I_{21})(I_{41}, -I_{23}I_{80})(I_{60}, -I_{40}I_{42}I_{80}).$$

Using notation for ℓ_1 as above, and since $v(\ell_2) = v(\ell_3) = 0$, we can simplify $(I_{40}, \ell)(I_{45}, -\ell I_{21})$ into $(u_1^2, u_1)(I_{45}, u_1) = (u_1^2 I_{45}, u_1)$. Now, if $v(u_1) = 0$ then $(u_1^2 I_{45}, u_1) = 1$. Otherwise, by Lemma 6.2.1.9 we have $I_{45} \notin K^{\times 2}$. But $u_1^2 \notin K^{\times 2}$, hence $u_1^2 I_{45} \in K^{\times 2}$ and $(u_1^2 I_{45}, u_1) = 1$. It follows that $E_v = (I_{41}, -I_{23}I_{80})(I_{60}, -I_{40}I_{42}I_{80})$. Using the notation for I_{80}, I_{41}, I_{60} set up above, using Lemma 6.3.3 and simplifying we have

$$E_v = (\pi^a(u_b(\alpha_1+\beta_2)(\alpha_1+\beta_3) + u_a(\alpha_1-\beta_2)(\alpha_1-\beta_3)), -\alpha_1^2 u_a u_b \alpha_1^2 A_{21}A_{31})$$

$$(\pi^a(u_a(\alpha_1-\beta_2)^2(\alpha+\beta_2)^2 + u_b(\alpha_1-\beta_3)^2(\alpha_1-\beta_2)^2), -I_{40}A_{21}A_{31}u_a u_b \alpha_1^2 A_{21}A_{31})$$

$$= (\pi^a, A_{21}A_{31})(\pi^a, \alpha_1^2 I_{40}) = (\pi^a, A_{21}A_{31})(\pi^a, \alpha_1^2 u_1^2) = 1,$$

since $u_1, \alpha_1^2 \notin K^{\times 2}$. Therefore $E_v = (\pi^a, A_{21}A_{31})$.

Recall that $A_{21}A_{31} \equiv_\square T_{\alpha_1}T_{-\alpha_1}$. In particular in these cases, $Frob(T_{\alpha_1}) = T_{-\alpha_1}$ and $Frob(T_{-\alpha_1}) = T_{\alpha_1}$. Now, let $t_{\alpha_1}^\pm$ and $t_{-\alpha_1}^\pm$ denote the square roots of $T_{\alpha_1}, T_{-\alpha_1}$ respectively. By definition of TN6D, we have without loss of generality $Frob(t_{\alpha_1}^+) = t_{-\alpha_1}^+$ and $Frob(t_{-\alpha_1}^+) = t_{\alpha_1}^+$. Therefore $T_{\alpha_1}T_{-\alpha_1} = (t_{\alpha_1}^+ t_{-\alpha_1}^+)^2 \in \mathcal{O}_K^{\times 2}$, and hence $I_{42} \in \mathcal{O}_K^{\times 2}$. On the other hand, by definition of TN6E, we have $Frob(t_{\alpha_1}^+) = t_{-\alpha_1}^+$, $Frob(t_{-\alpha_1}^+) = t_{\alpha_1}^-$, $Frob(t_{\alpha_1}^-) = t_{-\alpha_1}^-$ and $Frob(t_{-\alpha_1}^-) = t_{\alpha_1}^+$. It follows that $T_{\alpha_1}T_{-\alpha_1} = (t_{\alpha_1}^+ t_{-\alpha_1}^+)^2 \notin \mathcal{O}_K^{\times 2}$, and hence $I_{42} \notin \mathcal{O}_K^{\times 2}$.

This yields $E_v = (\pi^a, I_{42}) = 1$ for TN6D, $E_v = (\pi^a, I_{42}) = (-1)^a$ for TN2E as required.

<u>Cases TN7</u>. From the definition of the isogeny we see that $v(I_{45}) = a + b$ and $v(I_{80}) = v(I_{23}) = v(I_{44}) = 0$.
Write $\alpha_3 = a_3\pi^a + \alpha_2$ and $\beta_3 = b_3\pi^b + \beta_2$, with $2a, 2b \in \mathbb{Z}$ and $u_1 = \alpha_2 + \beta_2 - \alpha_3 - \beta_3$ so that $\ell_1 = \frac{u_1}{\Delta_G}$. We have

$$\Delta_G = c(a_3\pi^a(\beta_2-\alpha_1)(\beta_2+\alpha_1) + b_3\pi^b(\alpha_2-\alpha_1)(\alpha_2+\alpha_1) + a_3b_3\pi^{a+b}(\alpha_2+\beta_2)),$$

$$\ell_1 = -(a_3\pi^a + b_3\pi^b), \quad I_{21} \equiv (\alpha_2+\beta_2)^2, \quad \delta_2 \equiv \delta_3, \quad I_{44} \equiv \delta_2^2, \quad \hat{\delta_2} \equiv \hat{\delta_3}, \quad I_{80} \equiv \hat{\delta_2}^2,$$

$$I_{45} = 4a_3b_3\pi^{a+b}(\alpha_2-\beta_2-b_3\pi^b)(\beta_2-\alpha_2-a_3\pi^a), \quad I_{20} \equiv_\square \delta_2, \quad I_{22} \equiv -\delta_2, \quad I_{40} =_\square u_1^2,$$

$$I_{41} \equiv 2((\alpha_1+\alpha_2)^2(\alpha_1+\beta_2)^2 + (\alpha_2-\alpha_1)^2(\beta_2-\alpha_1)^2), \quad I_{43} \equiv 2\delta_2 A_{21}, \quad I_{60} \equiv 2\hat{\delta_2}A_{21},$$

$$A_{21} \equiv A_{31} \equiv (\alpha_2-\alpha_1)(\alpha_2+\alpha_1) + (\beta_2-\alpha_1)(\beta_2+\alpha_1), \quad I_{42} \equiv_\square A_{21}^2,$$

so that

$$E_v = (I_{40}, \ell I_{20})(I_{45}, -I_{22}\ell I_{21})(I_{41}, -I_{23}I_{80})(I_{42}, -I_{44}I_{80})(I_{43}, -I_{40}I_{42}I_{44})(I_{60}, -I_{40}I_{42}I_{80})$$

and $(I_{41}, -I_{23}I_{80}) = 1$ by Lemmata 6.3.4 and 6.2.1.5 since $v(I_{41}) \geq 0 = v(I_{80})$.

Therefore

$$E_v = (I_{40}, \ell I_{20})(I_{45}, -I_{22}\ell I_{21})(I_{42}, -I_{44}I_{80})(I_{43}, -I_{40}I_{42}I_{44})(I_{60}, -I_{40}I_{42}I_{80})$$

Replacing invariants by their reduced expression above gives

$$E_v = (u_1^2, \ell\delta_2)(I_{45}, \delta_2\ell(\alpha_2+\beta_2)^2)(A_{21}^2, -\delta_2^2\hat{\delta_2}^2)(2\delta_2 A_{21}, -u_1^2 A_{21}^2\delta_2^2)(2\hat{\delta_2}A_{21}, -u_1^2 A_{21}^2\hat{\delta_2}^2).$$

By definition of the isogeny, either $\alpha_2, \beta_2, \alpha_3, \beta_3 \in K$ and $(\alpha_2+\beta_2)^2, A_{21}^2 \in K^{\times 2}$, or $\alpha_2, \beta_2, \alpha_3, \beta_3 \notin K$ but $Frob(\alpha_2+\beta_2) = (\alpha_2+\beta_2)$ or $Frob(\alpha_2+\beta_2) = (\alpha_3+\beta_3)$. Either way, $(\alpha_2+\beta_2)^2, A_{21}^2 \in k^{\times 2}$ and hence $(\alpha_2+\beta_2)^2, A_{21}^2 \in K^{\times 2}$. Therefore E_v simplifies to

$$(u_1^2, \ell\delta_2)(I_{45}, \delta_2\ell)(2\delta_2 A_{21}, -u_1^2)(2\hat{\delta_2}A_{21}, -u_1^2) = (u_1^2, \ell\delta_2)(I_{45}, \delta_2\ell)(\delta_2, u_1^2)(\hat{\delta_2}, u_1^2).$$

Now since $I_{45} = u_1^2(A_1 - B_1)^2$ it follows that $E_v =$

$$(u_1^2, \ell\delta_2)(u_1^2(A_1-B_1)^2, \delta_2\ell)(\delta_2, u_1^2)(\hat{\delta_2}, u_1^2) = ((A_1-B_1)^2, \delta_2\ell)(\delta_2, u_1^2)(\hat{\delta_2}, u_1^2).$$

Cases TN7A/B/C/D. Here $v(u_1^2) = 2a$, $v(A_1 - B_1)^2 = b - a$ and α_2, β_2 are fixed by Frobenius so that $\delta_2 \in K^{\times 2}$. Therefore $E_v = ((A_1-B_1)^2, \delta_2\ell)(\hat{\delta_2}, u_1^2)$. Note that by the semistability criterion 3.4.29, $v(\ell) \in 2\mathbb{Z}$.

TN7A. Since $\hat{\delta_2} \equiv_\square T_{\alpha_2}T_{\beta_2} \equiv_\square 1$, we have that $(\hat{\delta_2}, u_1^2) = 1$.
Now $\ell\delta_2 \equiv \frac{(\alpha_2+\beta_2)^2(\alpha_2-\beta_2)^2}{c(\beta_2-\alpha_1)(\beta_2+\alpha_1)} \equiv_\square T_{\beta_2} \equiv_\square 1$. Hence $((A_1-B_1)^2, \delta_2\ell) = 1$ and $E_v = 1$.

TN7B. Since $\hat{\delta_2} \equiv T_{\alpha_2}T_{\beta_2} \notin K^{\times 2}$, we have that $(\hat{\delta_2}, u_1^2) = (-1)^{2a}$.
Now $\ell\delta_2 \equiv \frac{(\alpha_2+\beta_2)^2(\alpha_2-\beta_2)^2}{c(\beta_2-\alpha_1)(\beta_2+\alpha_1)} \equiv T_{\beta_2} \equiv \square$. Hence $((A_1-B_1)^2, \delta_2\ell) = 1$. Hence $E_v = (-1)^{2a}$.

TN7C. Since $\hat{\delta_2} \equiv T_{\alpha_2}T_{\beta_2} \notin K^{\times 2}$, we have that $(\hat{\delta_2}, u_1^2) = (-1)^{2a}$.
Now $\ell\delta_2 \equiv \frac{(\alpha_2+\beta_2)^2(\alpha_2-\beta_2)^2}{c(\beta_2-\alpha_1)(\beta_2+\alpha_1)} \equiv T_{\beta_2} \notin K^{\times 2}$. Hence $((A_1-B_1)^2, \delta_2\ell) = 1 \Leftrightarrow b - a$ is even. Therefore $E_v = 1 \Leftrightarrow 2a \equiv b - a \bmod 2$, equivalently $E_v = 1 \Leftrightarrow a + b \equiv 0 \bmod 2$.

TN7D. Since $\hat{\delta_2} \equiv T_{\alpha_2}T_{\beta_2} \equiv_\square 1$, we have that $(\hat{\delta_2}, u_1^2) = 1$.
Now $\ell\delta_2 \equiv \frac{(\alpha_2+\beta_2)^2(\alpha_2-\beta_2)^2}{c(\beta_2-\alpha_1)(\beta_2+\alpha_1)} \equiv T_{\beta_2} \notin K^{\times 2}$. Hence $((A_1-B_1)^2, \delta_2\ell) = 1 \Leftrightarrow b - a$ is even. Hence $E_v = 1 \Leftrightarrow a + b \equiv 0 \bmod 2$.

Cases TN7E/F/G. Here $v(A_1 - B_1)^2 = 0$ and α_2, β_2 are fixed by Frobenius so that $\delta_2 \in K^{\times 2}$. Let $v(u_1) = a + r_1 \geq a$ as in the proof of Table 4.18 in Section 5.3.3. By definition, we have $v(\ell) = v(\ell_1) + v(\ell_2) + v(\ell_3)$ and by the semistability criterion 3.4.29, $v(\ell) \in 2\mathbb{Z}$. Here $v(\ell_1) = r_1$. Either $r_1 = 0$ and $v(\ell) = v(\ell_2) + v(\ell_3) \in 2\mathbb{Z}$ or $r_1 > 0$ and $v(\ell_2) = v(\ell_3) = 0$ so that $r_1 \in 2\mathbb{Z}$. Therefore $E_v = (\hat{\delta_2}, u_1^2)$ and $v(u_1) = a + r_1 \equiv a \bmod 2$.

TN7E. Since $\hat{\delta_2} \equiv_\square T_{\alpha_2}T_{\beta_2} \equiv_\square 1$, we have that $E_v = (\hat{\delta_2}, u_1^2) = 1$.

TN7F. Since $\hat{\delta_2} \equiv T_{\alpha_2}T_{\beta_2} \notin K^{\times 2}$, we have that $E_v = (\hat{\delta_2}, u_1^2) = 1 \Leftrightarrow 2a \equiv 0 \bmod 2$.

TN7G. Since $\hat{\delta_2} \equiv T_{\alpha_2}T_{\beta_2} \in K^{\times 2}$, we have that $E_v = (\hat{\delta_2}, u_1^2) = 1$.

Cases TN7H/I. Here $v(A_1 - B_1)^2 = 0$ and α_2, β_2 are not fixed by Frobenius, hence $\delta_2 \notin K^{\times 2}$. Let $v(u_1) = a + r_1 \geq a$, as for the cases TN7E/F/G, we have $v(u_1) = a + r_1 \equiv a \bmod 2$. It follows that $E_v = (\delta_2, u_1^2)(\hat{\delta_2}, u_1^2)$ with $(\delta_2, u_1^2) = 1 \Leftrightarrow 2a \equiv 0 \bmod 2$.

TN7H. Following the proof of Table 4.18 in Section 5.3.3, we see that $\hat{\delta_2} \in K^{\times 2}$ and hence $E_v = (\delta_2, u_1^2)(\hat{\delta_2}, u_1^2) = (\delta_2, u_1^2) = 1 \Leftrightarrow 2a \equiv 0 \bmod 2$.

TN7I. Following the proof of Table 4.18 in Section 5.3.3, we see that $\hat{\delta_2} \notin K^{\times 2}$ and hence $E_v = (\delta_2, u_1^2)(\hat{\delta_2}, u_1^2) = (\delta_2, u_1^2)^2 = 1$.

Cases TN7J/K/L/M/N: Here $v(A_1 - B_1)^2 = 0$ and $v(\Delta_G) = 2r > 2a$, $v(u_1) = a$ so that $v(\ell_1) = -r$ and $v(\ell_2) = v(\ell_3) = 0$ as in Section 5.3.3. Therefore $E_v = ((A_1 - B_1)^2, \ell)(\delta_2, u_1^2)(\hat{\delta_2}, u_1^2)$. Note that from the cluster picture of $\hat{C}$ we see that $(A_1 - B_1)^2 \in K^{\times 2} \Leftrightarrow (A_2 - B_2)^2 \in K^{\times 2} \Leftrightarrow \hat{\delta_2} \in K^{\times 2}$.

TN7J/L. Here $(A_1 - B_1)^2, \delta_2, \hat{\delta_2} \in K^{\times 2}$. Hence $E_v = 1$.

TN7K. Here $\delta_2 \in K^{\times 2}$ and $(A_1 - B_1)^2, \hat{\delta_2} \notin K^{\times 2}$ hence $E_v = ((A_1 - B_1)^2, \ell)(\hat{\delta_2}, u_1^2)$, with $((A_1 - B_1)^2, \ell) = 1 \Leftrightarrow r \equiv 0 \bmod 2$ and $(\hat{\delta_2}, u_1^2) = 1 \Leftrightarrow 2a \equiv 0 \bmod 2$. Therefore $E_v = 1 \Leftrightarrow 2a + r \equiv 0 \bmod 2$.

TN7M. Here $\delta_2 \notin K^{\times 2}$ and $(A_1 - B_1)^2, \hat{\delta_2} \in K^{\times 2}$ hence $E_v = (\delta_2, u_1^2) = 1 \Leftrightarrow 2a \equiv 0 \bmod 2$.

TN7N. Here $(A_1 - B_1)^2, \delta_2, \hat{\delta_2} \notin K^{\times 2}$ hence $E_v = ((A_1 - B_1)^2, \ell)(\delta_2, u_1^2)(\hat{\delta_2}, u_1^2) = ((A_1 - B_1)^2, \ell)(\delta_2, u_1^2)^2 = 1 \Leftrightarrow r \equiv 0 \bmod 2$.

Cases TN8. From the definition of the isogeny we see that $v(I_{80}) = a + b$ and $v(I_{45}) = v(I_{23}) = v(I_{44}) = 0$.

Write $\alpha_2 = \alpha_1 + a_2\pi^a, \quad \beta_2 = -\alpha_1 + b_2\pi^b, \quad a_2, b_2 \in \mathcal{O}_K^\times$, with $a, b \in \mathbb{Z}$ (since otherwise I_K would permute both clusters in the cluster picture of C, which contradicts the semistability criterion 3.4.29). In particular, $v(u_1) \in \mathbb{Z}$ and $A_{21}, A_{31}, A_{41p}, A_{41m} \in K$. Let $u_1 = \alpha_2 + \beta_2 - \alpha_3 - \beta_3$ so that $\ell_1 = \frac{u_1}{\Delta_G}$. We have

$$\Delta_G = c\pi^a(a_2(\alpha_1 + \alpha_3)(\alpha_1 + \beta_3) + b_2\pi^{b-a}(\alpha_1 - \alpha_3)(\alpha_1 - \beta_3) - a_2b_2\pi^b(\alpha_3 + \beta_3)),$$

$$u_1 = a_2\pi^a + b_2\pi^b - \alpha_3 - \beta_3, \quad \ell_2 = c(\alpha_3 + \beta_3), \quad \ell_3 = -c(a_2\pi^a + b_2\pi^b),$$

$$I_{23} \equiv_\square \delta_2 \equiv_\square \alpha_1^2, \quad I_{44} \equiv_\square \alpha_1^2\delta_3, \quad I_{80} \equiv_\square -\alpha_1^2 a_2 b_2 \pi^{a+b} + o(\pi^{a+b}),$$

$$I_{21} = (a_2\pi^a + b_2\pi^b)(\alpha_3 + \beta_3), \quad A_{21} = 2\alpha_1(a_2\pi^a - b_2\pi^b) + o(\pi^a), \quad I_{40} =_\square u_1^2,$$

$$I_{41p} = \alpha_1 b_2\pi^b(\alpha_1 + \alpha_3)(\alpha_1 + \beta_3), \quad I_{41m} = -\alpha_1 a_2\pi^a(\alpha_1 - \alpha_3)(\alpha_1 - \beta_3),$$

$$T_{\alpha_1} \equiv c(2\alpha_1)^2(\alpha_1 - \alpha_3)(\alpha_1 - \beta_3), \quad T_{-\alpha_1} = c(2\alpha_1)^2(-\alpha_1 - \alpha_3)(-\alpha_1 - \beta_3),$$

$$I_{45} \equiv \hat{\delta_2} \equiv_\square T_{\alpha_1}T_{-\alpha_1}.$$

Using Lemma 6.3.4 and Lemmata 6.2.1.3 and 6.2.1.1, we have $(I_{20}, -I_{44}) = (I_{22}, -I_{44}I_{45}) = 1$. After simplification, this yields

$$E_v = (-1, I_{41}I_{43}I_{60})(I_{23}, I_{41})(I_{45}, -\ell I_{21})(I_{44}, I_{42}I_{43})(I_{80}, -2cI_{41}I_{42}I_{60})(I_{42}, -I_{60}I_{43})$$

$= (-1, I_{41}A_{21})(I_{23}, I_{41})(I_{45}, -\ell I_{21})(I_{80}, -2cI_{41}A_{21})H$, where

$$H = (-1, I_{43}I_{60})(\delta_2\delta_3, A_{21}A_{31}I_{43})(\hat{\delta_3}\hat{\delta_2}, A_{31}I_{60})(A_{21}, I_{60}I_{43})(A_{31}, -I_{60}I_{43}) = 1,$$

We show that $H = 1$. Since $I_{43} = \delta_2 A_{21} + \delta_3 A_{31}$ and $I_{60} = \hat{\delta_2}A_{31} + \hat{\delta_3}A_{21}$, regrouping and using Lemma 6.3.3, we obtain

$$E_v = (I_{43}, -\delta_2\delta_3A_{21}A_{31})(I_{60}, -\hat{\delta_2}\hat{\delta_3}A_{21}A_{31})(\delta_2\delta_3, A_{21}A_{31})(A_{31}, \hat{\delta_2}\hat{\delta_3})$$

$$= (\delta_2A_{21}, \delta_3A_{31})(\hat{\delta_2}A_{31}, \hat{\delta_3}A_{21})(\delta_2\delta_3, A_{21}A_{31})(A_{31}, \hat{\delta_2}\hat{\delta_3}),$$

which gives $E_v = (A_{31}, -\delta_3)$ after simplification. We are done by noting that if $v(A_{31}) > 0$ then $\hat{\delta_2} \equiv_\square -\delta_3$ by Lemma 6.2.1.6.(e), so that in this case, $-\delta_3$ is a square. Therefore $H = 1$.

We also show that if $a = b$ then $v(I_{80}) \in 2\mathbb{Z}$ and $(-1, I_{41}A_{21})(I_{80}, -2cI_{41}A_{21}) = 1$. We have $(-1, I_{41}A_{21})(I_{80}, -2cI_{41}A_{21}) = (-1, I_{41}A_{21})(I_{80}, I_{41}A_{21})$, with

$$I_{41} = \alpha_1\pi^a(b_2(\alpha_1 + \alpha_3)(\alpha_1 + \beta_3) - a_2(\alpha_1 - \alpha_3)(\alpha_1 - \beta_3)), \quad A_{21} = 2\alpha_1\pi^a(a_2 - b_2).$$

Now using Lemma 6.3.3

$$(I_{41}, -I_{80}) = (\alpha_1\pi^a, a_2b_2)(b_2(\alpha_1 + \alpha_3)(\alpha_1 + \beta_3) - a_2(\alpha_1 - \alpha_3)(\alpha_1 - \beta_3), a_2b_2\hat{\delta_2})$$

$$= (\alpha_1\pi^a, a_2b_2)(b_2(\alpha_1 + \alpha_3)(\alpha_1 + \beta_3), a_2(\alpha_1 - \alpha_3)(\alpha_1 - \beta_3)) = (\alpha_1\pi^a, a_2b_2).$$

Similarly,

$$(A_{21}, -I_{80}) = (2\alpha_1\pi^a(a_2 - b_2), a_2b_2) = (2\alpha_1\pi^a, a_2b_2).$$

Therefore $(I_{41}A_{21}, -I_{80}) = 1$.

Cases TN8A/D/E/F/G/J/L. Here $\alpha_1^2 \equiv I_{23} \equiv \delta_2 \equiv I_{45} \equiv \hat{\delta_2} \equiv_\square 1$, hence $E_v = (-1, I_{41}A_{21})(I_{80}, -2cI_{41}A_{21})$.

Cases TN8A/D. Here $a < b$, hence

$$A_{21} = 2\alpha_1a_2\pi^a, \quad I_{41} = -\alpha_1a_2\pi^a(\alpha_1 - \alpha_3)(\alpha_1 - \beta_3),$$

so that

$$E_v = (-1, I_{41}A_{21})(I_{80}, -2cI_{41}A_{21}) = (I_{80}, -2cI_{41}A_{21}).$$

We have $-2gI_{41}A_{21} \equiv_\square T_{\alpha_1}$ so that $E_v = 1$ for TN8A, and $E_v = (-1)^{a+b}$ for TN8D.

Cases TN8E/F/G/J/L. Here $a=b$ hence $E_v=1$ for TN8E/F/G/J/L.
Cases TN8B/C/F/K. Here $\alpha_1^2 \equiv_\square I_{23} \equiv \delta_2 \equiv_\square 1$ but $T_{-\alpha_1}T_{\alpha_1}, I_{45}, \hat{\delta_2} \not\equiv_\square 1$. Hence

$$E_v = (-1, I_{41}A_{21})(I_{45}, -\ell I_{21})(I_{80}, -2cI_{41}A_{21}).$$

Recall that $\ell = \ell_1\ell_2\ell_3$, $I_{21} = -\ell_2\ell_3$ and $\ell_1 = \frac{u_1}{\Delta_G}$ so that hence $(I_{45}, -\ell I_{21}) = (I_{45}, \frac{u_1}{\Delta_G})$.
Cases TN8B/C. Here $a < b$, $u_1 = -\alpha_3 - \beta_3 + o(\pi^a), v(\Delta_G) = a$ so that $(I_{45}, -\ell I_{21}) = (I_{45}, u_1\pi^a) = (I_{45}, u_1)(I_{45}, \pi^a)$, since $u_1 \in K$. Also note that if $v(u_1) > 0$ then $I_{45} \in K^{\times 2}$ hence $(I_{45}, -\ell I_{21}) = (I_{45}, \pi^a)$. It follows that

$$E_v = (-1, I_{41}A_{21})(I_{80}, T_{\alpha_1})(I_{45}, \pi^a) = (I_{80}, T_{\alpha_1})(T_{\alpha_1}T_{-\alpha_1}, \pi^a) = (\pi^{a+b}, T_{-\alpha_1}).$$

Therefore, $E_v = 1$ for TN8B and $E_v = (-1)^{a+b}$ for TN8C as required.
Cases TN8F. Here $a=b$ but $v(\Delta_G) = 2a$. It follows directly from the TN8B/C cases that $E_v = 1$.
Cases TN8K. Here $a=b$ and $v(\Delta_G) = 2a+2r$, with $r>0$. We have $E_v = (I_{45}, \frac{u_1}{\Delta_G}) = (I_{45}, \pi^{a+r})(I_{45}, u_1)$. As in Section 5.3.3, we have $v(u_1) = a$. Therefore, $E_v = (I_{45}, \frac{u_1}{\Delta_G}) = (I_{45}, \pi^{a+r})(I_{45}, \pi^a) = (I_{45}, \pi^r)$, and $E_v = (-1)^r$ for TN8K.
Cases TN8H/M. Here $I_{45} \equiv_\square \hat{\delta_2} \equiv_\square 1$ and $\alpha_1^2 \equiv_\square I_{23}, \not\equiv_\square 1$ so that

$$E_v = (I_{23}, I_{41}).$$

Using Lemma 6.2.1.5 and Lemma 6.3.4, we have that $E_v = 1$ since $v(I_{80}) \in 2\mathbb{Z}$.
Cases TN8I/N. Here $\alpha_1^2 \equiv_\square I_{23}, I_{45}, \not\equiv_\square 1$ and $v(\Delta_G) = a$ so that

$$E_v = (I_{23}, I_{41})(I_{45}, -\ell I_{21}).$$

as in the cases of TN8B/C and TN8K, it follows that $E_v = (I_{45}, -\ell I_{21}) = (I_{45}, \frac{u_1}{\Delta_G})$ and $E_v = 1$ for TN8I but $E_v = (-1)^r$ for TN8N as required.

6.6.4 C is of type $U_{2a,2b,2n}$

Proof of Tables 4.23 to 4.30

Cases U1. From the definition of the isogeny we see that $v(I_{23}) = 2a$, $v(I_{44}) = 2b+2n$ and $v(I_{45}) = v(I_{80}) = 0$. By semistability criterion 3.4.29, we have $v(c) = 0$. Let $u_1 = \alpha_2 + \beta_2 - \alpha_3 - \beta_3$ so that $\ell_1 = \frac{u_1}{\Delta_G}$.
Computing invariants and reducing mod π we find that

$$u_1^2 \equiv_\square (\alpha_2 - \alpha_3)^2, \quad \ell_1 \equiv_\square -c\alpha_2\alpha_3, \quad I_{21} \equiv_\square \alpha_2\alpha_3,$$

$$I_{80} \equiv_\square I_{45} \equiv_\square I_{41} \equiv_\square I_{42} \equiv_\square 1, \quad I_{22} \equiv_\square 2(\alpha_2 - \alpha_3)^2,$$

$$A_{21} \equiv_\square 2\alpha_2^2, \quad A_{31} \equiv_\square 2\alpha_3^2.$$

Using Lemma 6.3.4 and Lemma 6.2.1.8, we have $(I_{60}, -I_{40}I_{42}I_{80}) = 1$. After simplification, this yields

$$E_v = (-1, I_{43})(I_{20}, -I_{40}I_{44})(I_{40}, cI_{43})(c, I_{23}I_{44})(I_{44}, 2I_{22}I_{43}).$$

Cases U1A/B. Here $\delta_2, \delta_3 \in K$ and $2a, 2b, 2n \in \mathbb{Z}$. In particular, $I_{40} \in K^{\times 2}$, $I_{22} \equiv_\square A_{21} \equiv_\square A_{31} \equiv_\square 2$. Therefore $E_v = (I_{43}, -1)(I_{20}, -I_{44})(I_{44}, I_{43})(c, I_{23}I_{44})$. Write $\beta_2 = \alpha_2 + a_2\pi^b$, $\beta_3 = \alpha_3 + a_3\pi^n$. If $b < n$ we have

$$I_{20} =_\square 2a_2^2\pi^{2b} + o(\pi^{2b}), \quad I_{43} =_\square a_2^2\pi^{2b} + o(\pi^{2b}),$$

so that $(I_{43}, -1)(I_{20}, -I_{44})(I_{44}, I_{43}) = (\pi^{2b}, -1)(2\pi^{2b}, -\pi^{2b+2n})(\pi^{2b+2n}, \pi^{2b}) = 1$. On the other hand, if $b = n$ then $v(I_{44}) \in 2\mathbb{Z}$ and it follows from Lemmata 6.2.1.1 and 6.2.1.7, using Lemma 6.3.4 that $(I_{43}, -I_{44}) = (I_{20}, -I_{44}) = 1$. Therefore $E_v = (c, I_{23}I_{44})$.

For U1A, we have $c \in K^{\times 2}$ hence $E_v = 1$. For U1B, we have $c \notin K^{\times 2}$ hence $E_v = -1$ if and only if one or three of $2a, 2b, 2n$ are odd. Equivalently, $E_v = -1$ if and only if $4ab + 4ac + 4bn$ is odd as required.

Cases U1C/D. Here $\delta_2, \delta_3 \notin K$. In particular, $I_{22}, I_{40} \notin K^{\times 2}$ but $v(I_{44}), v(I_{40}) \in 2\mathbb{Z}$. Therefore $(I_{22}, I_{44}) = 1$ and by Lemmata 6.2.1.1 and 6.2.1.7, using Lemma 6.3.4 we have that $(I_{43}, -I_{40}I_{44}) = (I_{20}, -I_{40}I_{44}) = 1$. Therefore $E_v = (c, I_{23})$ and $E_v = 1$ for U1C, while $E_v = -1$ if $2a$ is odd, $E_v = 1$ otherwise for U1D as required.

Cases U2. From the definition of the isogeny we see that $v(I_{45}) = n$, $v(I_{80}) = a + b$ and $v(I_{23}) = v(I_{44}) = 0$. By semistability criterion 3.4.29, we may assume that $v(c) = 0$. Let $u_1 = \alpha_2 + \beta_2 - \alpha_3 - \beta_3$ so that $\ell_1 = \frac{u_1}{\Delta_G}$.

Computing invariants and reducing $\bmod \pi$ we find that

$$u_1^2 \equiv 2\alpha_1, \quad \ell_1 \equiv_\square c(\alpha_1 - \beta_2)(\alpha_1 + \beta_2), \quad I_{21} \equiv_\square -(\alpha_1 - \beta_2)(\alpha_1 + \beta_2),$$

$$I_{22} \equiv_\square -I_{21}, \quad I_{40} \equiv_\square \delta_1, \quad A_{21} \equiv_\square A_{31}, \quad I_{42} \equiv_\square I_{44} \equiv_\square 1.$$

Write $\alpha_2 = \alpha_1 + a_2\pi^a, \quad \alpha_3 = -\alpha_1 + a_3\pi^b, \quad \beta_3 = \beta_2 + b_3\pi^n$. Then

$$\hat{\delta_2} =_\square -2a_3\pi^b\alpha_1 A_{21}, \quad \hat{\delta_3} =_\square 2a_2\pi^a\alpha_1 A_{21}, \quad I_{80} =_\square -a_2a_3\alpha_1^2\pi^{a+b},$$

$$I_{41} \equiv_\square \alpha_1 a_2\pi^b(\alpha_1 + \beta_2)^2 - \alpha_1 a_2\pi^a(\beta_2 - \alpha_1)^2.$$

Using Lemma 6.3.4 and Lemmata 6.2.1.1, 6.2.1.7, we have

$$(I_{20}, -I_{40}I_{44}) = (I_{43}, -I_{40}I_{44}I_{42}) = 1.$$

Also since $I_{22} \equiv_\square -I_{21}$ and $\ell \equiv c$, we have $(I_{45}, -\ell I_{21}I_{22}) = (I_{45}, \ell) = (I_{45}, c)$. After simplification, this yields $E_v = (c, I_{80}I_{45})(I_{80}, -2I_{41}I_{60})(I_{41}, -I_{23})(I_{60}, -I_{40})$.
Cases U2A/B. Here $I_{23} \in K^{\times 2}$ and $\delta_2, \delta_3 \in K$. In particular, $I_{40} \in K^{\times 2}$ and $E_v = (c, I_{80}I_{45})(I_{80}, -2I_{41}I_{60})(I_{41}I_{60}, -1)$.
If $a < b$ we have $I_{60} =_\square 2a_2^2\alpha_1\pi^a$, $\quad I_{41} =_\square -\alpha_1 a_2\pi^a(\beta_2-\alpha_1)^2$, so that $-2I_{41}I_{60} \equiv_\square 1$ and $(I_{80}, -2I_{41}I_{60})(I_{41}I_{60}, -1) = 1$.
On the other hand, if $a = b$ then $v(I_{80}) \in 2\mathbb{Z}$ and it follows from Lemmata 6.2.1.8 and 6.2.1.5, using Lemma 6.3.4 that $(I_{41}, -I_{80}) = (I_{60}, -I_{80}) = 1$. Therefore $E_v = (c, I_{45}I_{80})$.
For U2A, we have $c \in K^{\times 2}$ hence $E_v = 1$. For U2B, we have $c \notin K^{\times 2}$, hence $E_v = -1$ if and only if one or three of a, b, n are odd. Equivalently, $E_v = -1$ if and only if $ab + an + bn$ is odd as required.
Cases U2C/D. Here $I_{23} \notin K^{\times 2}$, $\delta_2, \delta_3 \notin K$ and $a = b$ so that $v(I_{80}) \in 2\mathbb{Z}$. We have $E_v = (c, I_{80}I_{45})(I_{80}, -2I_{41}I_{60})(I_{41}, -I_{23})(I_{60}, -I_{40})$. By Lemmata 6.2.1.8 and 6.2.1.5, using Lemma 6.3.4 we have that $(I_{41}, -I_{40}I_{80}) = (I_{60}, -I_{40}I_{80}) = 1$. Therefore $E_v = (c, I_{45})$ and $E_v = 1$ for U2C, while $E_v = -1$ if n is odd, $E_v = 1$ otherwise for U2D as required.
Cases U3. From the definition of the isogeny we see that $v(I_{45}) = a+b$, $v(I_{23}) = 2n$ and $v(I_{80}) = v(I_{44}) = 0$. By semistability criterion 3.4.29, we may assume that $v(c) = 0$. Let $u_1 = \alpha_2 + \beta_2 - \alpha_3 - \beta_3$ so that $\ell_1 = \frac{u_1}{\Delta_G}$. Write

$$\alpha_1 = a_1\pi^n, \quad \alpha_3 = \alpha_2 + a_3\pi^a, \quad \beta_3 = \beta_2 + b_3\pi^b, \quad u_1 = \alpha_2 + \beta_2 - \alpha_3 - \beta_3,$$

for some $a_1, a_3, b_3 \in \mathcal{O}_K^\times$, so that $\ell_1 = \frac{u_1}{\Delta_G}$. By definition of $\Delta_G, \ell_1, \ell_2, \ell_3$ we have

$$\Delta_G = c\pi^a\left(a_3(\beta_2-\alpha_1)(\beta_2+\alpha_1) + b_3\pi^{b-a}(\alpha_2-\alpha_1)(\alpha_2+\alpha_1) + a_3b_3\pi^b(\alpha_2+\beta_2)\right),$$

$$u_1 = \pi^a(a_3 + b_3\pi^{b-a}), \quad \ell_2 = c(\pi^a(a_3+b_3\pi^{b-a}) + \alpha_2 + \beta_2), \quad \ell_3 = -c(\alpha_2+\beta_2).$$

Reducing invariants $\bmod \pi$,

$$I_{21} \equiv_\square 1, \quad \delta_2 \equiv \delta_3 \equiv_\square I_{20}, \quad I_{44} \equiv_\square 1, \quad I_{22} \equiv -\delta_2,$$

$$\hat{\delta_2} \equiv \hat{\delta_3} \equiv_\square 1, \quad I_{80} \equiv_\square 1, \quad I_{41} \equiv \hat{\delta_2} \bmod \pi,$$

$$I_{43} \equiv 2(\alpha_2^2+\beta_2^2)\delta_2, \quad I_{60} \equiv_\square 2(\alpha_2^2+\beta_2^2), \quad I_{42} \equiv_\square 1,$$

so that $I_{43}I_{60} \equiv \delta_2$. Therefore $E_v = (I_{20}, I_{40})(c, I_{23})(I_{40}, \ell I_{60}I_{43})(I_{45}, -\ell I_{22})$. Replacing invariants by their reduction $\bmod \pi$ and since $I_{40} =_\square u_1^2$, $d\ell_1 = \ell_1^2(A_1 - B_1)^2$, this yields

$$E_v = (u_1^2, \delta_2)(c, I_{23})(u_1^2, \delta_2)(u_1^2, \ell)(u_1^2(A_1 - B_1)^2, \ell\delta_2)$$

$$= (c, I_{23})(u_1^2, \delta_2)((A_1 - B_1)^2, \ell)((A_1 - B_1)^2, \delta_2).$$

We have

$$I_{45} =_\square -a_3 b_3 \pi^{a+b} + o(\pi^{a+b}), \quad I_{23} \equiv a_1^2 \pi^{2n}.$$

Now if $a < b$ then $\ell \equiv_\square c$. Otherwise, if $a = b$ then $v(A_1 - B_1) = 0$ and $(A_1 - B_1)^2 \in K^{\times 2}$ so that $E_v = (c, I_{23})(u_1^2, \delta_2)$.

Cases U3A. Here $a < b$, $c \in K^{\times 2}$ and $\delta_2 \in K^{\times 2}$ hence $\ell \in K^{\times 2}$ and $E_v = 1$.

Cases U3B. Here $a < b$, $c \notin K^{\times 2}$ and $\delta_2 \in K^{\times 2}$ hence $\ell \notin K^{\times 2}$ and $E_v = (c, I_{23})((A_1 - B_1)^2, \ell) = (-1)^{a+b+2n}$.

Cases U3C/G. Here $a = b$, $c \in K^{\times 2}$ and $\delta_2 \in K^{\times 2}$ and $E_v = (c, I_{23})(u_1^2, \delta_2) = 1$.

Cases U3D/H. Here $a = b$, $c \notin K^{\times 2}$ and $\delta_2 \in K^{\times 2}$ and $E_v = (c, I_{23}) = (-1)^{2n}$.

Cases U3E/I. Here $a = b$, $c \in K^{\times 2}$ and $\delta_2 \notin K^{\times 2}$ and $E_v = (u_1^2, \delta_2) = (-1)^{2a}$.

Cases U3F/J. Here $a = b$, $c \notin K^{\times 2}$ and $\delta_2 \notin K^{\times 2}$ and $E_v = (c, I_{23})(u_1^2, \delta_2) = (-1)^{2n+2a}$.

Cases U4. From the definition of the isogeny we see that $v(I_{80}) = a+b$, $v(I_{44}) = 2n$ and $v(I_{45}) = v(I_{23}) = 0$. By semistability criterion 3.4.29, we have $v(c) = 0$. Let $u_1 = \alpha_2 + \beta_2 - \alpha_3 - \beta_3$ so that $\ell_1 = \frac{u_1}{\Delta_G}$. Write

$$\alpha_2 = \alpha_1 + a_2\pi^a, \quad \beta_2 = -\alpha_1 + b_2\pi^b, \quad \beta_3 = \alpha_3 + b_3\pi^n, \quad u_1 = \alpha_2 + \beta_2 - \alpha_3 - \beta_3,$$

for some $a_2, b_2, b_3 \in \mathcal{O}_K^\times$, so that $\ell_1 = \frac{u_1}{\Delta_G}$. By definition of $\Delta_G, \ell_1, \ell_2, \ell_3$ we have

$$\Delta_G = c\pi^a(a_2(\alpha_1 + \alpha_3)^2 + a_2 b_3 \pi^n(\alpha_1 + \alpha_3) + b_2\pi^{b-a}(\alpha_1 - \alpha_3)^2$$

$$-b_2 b_3 \pi^{b-a+n}(\alpha_1 - \alpha_3) - 2a_2 b_2 \alpha_3 \pi^b - a_2 b_2 b_3 \pi^{b+n}),$$

$$u_1 = \pi^a(a_2 + b_2\pi^{b-a}) - 2\alpha_3 - b_3\pi^n, \quad \ell_2 = c(2a_3 + b_3\pi^n)), \quad \ell_3 = -c\pi^a(a_2 + b_2\pi^{b-a}).$$

Reducing invariants $\bmod \pi$,

$$\delta_2 \equiv I_{23} \equiv_\square \alpha_1^2, \quad I_{20} \equiv_\square 2\alpha_1^2, \quad \hat{\delta_2} \equiv (\alpha_1 - \alpha_3)^2(\alpha_1 + \alpha_3)^2 \equiv I_{45} \equiv_\square 1,$$

$$I_{22} \equiv I_{60} \equiv A_{31} \equiv -2(\alpha_1 - \alpha_3)(\alpha_1 + \alpha_3), \quad I_{40} \equiv_\square 1,$$

$$A_{21} = a_2\pi^a(2\alpha_1 + a_2\pi^a) + b_2\pi^b(-2\alpha_1 + b_2\pi^b),$$

$$I_{41} = 8b_2\pi^b(2\alpha_1 + a_2\pi^a)(\alpha_3 + \alpha_1)(\beta_3 + \alpha_1) + 8a_2\pi^a(-2\alpha_1 + b_2\pi^b)(\alpha_3 - \alpha_1)(\beta_3 - \alpha_1).$$

It follows that

$$E_v = (-1, I_{41}I_{43})(I_{20}, I_{44})(g, I_{44}I_{80})(I_{41}, dg_1)(I_{44}, 2I_{22}I_{42}I_{43})(I_{80}, -2I_{41}I_{42}I_{60})(I_{42}, -I_{60}I_{43}).$$

Cases U4A/B: Here $a < b$, $I_{23} \equiv \delta_2 \equiv_\square \alpha_1^2 \in K^{\times 2}$ therefore $I_{20} \equiv_\square 2$ and

$$E_v = (c, I_{44}I_{80})H, \quad H = (-1, I_{41}I_{43})(I_{44}, I_{22}I_{42}I_{43})(I_{80}, -2I_{41}I_{42}I_{60})(I_{42}, -I_{60}I_{43}).$$

We show that $H = 1$. Since $a < b$ we have $A_{21} \equiv 2\alpha_1 a_2 \pi^a$ and $I_{41} \equiv_\square -a_2\alpha_1\pi^a$. It follows that

$$E = (-1, I_{41}I_{43})(\delta_3, A_{31}^2 A_{21} I_{43})(I_{80}, A_{21}^2 A_{31}^2)(A_{21}A_{31}, -A_{31}I_{43})$$

$$= (-1, I_{41}I_{43})(\delta_3, A_{21}I_{43})(A_{21}A_{31}, -A_{31}I_{43})$$

$$= (-1, A_{21}\delta_2 I_{43})(\delta_3, A_{21}\delta_2 I_{43})(A_{21}\delta_2, -A_{31}I_{43})(I_{43}, A_{31})$$

$$= (I_{43}, -A_{31}\delta_3)(A_{21}\delta_2, \delta_3 A_{31})(A_{21}\delta_2, I_{43}) = 1,$$

by Lemma 6.3.3 since $I_{43} = \delta_2 A_{21} + \delta_3 A_{31}$. Hence $E_v = (c, I_{44}I_{80})$. As required, we obtain that $E_v = 1$ for U4A, since $c \in K^{\times 2}$ and $E_v = (-1)^{a+b+2n}$ for U4B since $c \notin K^{\times 2}$.

Cases U4C/D/G/H. Here $a = b$, $\delta_2 \equiv I_{23} \equiv_\square \alpha_1^2 \in K^{\times 2}$ so that

$$E = (-I_{80}, I_{41}I_{42})(I_{43}, -1)(I_{44}, I_{22}I_{42}I_{43})(I_{42}, I_{60}I_{43}).$$

Now $v(I_{80}) \in 2\mathbb{Z}$ so that if we let $I_{80} = -\pi^{2a} a_2 b_2 (\alpha_1 - \alpha_3)^2 (\alpha_1 + \alpha_3)^2$, we have

$$(-I_{80}, I_{41}I_{42}) = (a_2 b_2, a_2 - b_2)(a_2 b_2, b_2(\alpha_1 + \alpha_3)^2 - a_2(\alpha_3 - \alpha_1)^2) = 1$$

by Lemma 6.3.3 since $(\alpha_3 + \alpha_1)^2, (\alpha_3 - \alpha_1)^2 \in K^{\times 2}$. Therefore

$$E_v = (c, I_{44}I_{80})(-1, I_{43})(I_{44}, I_{22}I_{42}I_{43})(I_{42}, -I_{60}I_{43}),$$

with

$$(I_{43}, -\delta_3 A_{31})(I_{43}, A_{21}\delta_2)(\delta_2 A_{21}, \delta_3 A_{31}) = 1,$$

by Lemma 6.3.3 since $I_{43} = \delta_2 A_{21} + \delta_3 A_{31}$ as above. It follows that $E_v = (c, I_{44})$ and $E_v = 1$ for U4C/G as $c \in K^{\times 2}$ and $E_v = (-1)^{2n}$ for U4D/H since $c \notin K^{\times 2}$ as required.

Cases U4E/F/I/J. Here $a = b$, $\delta_2 \equiv I_{23} \equiv_\square \alpha_1^2 \notin K^{\times 2}$. Hence $E_v = (c, I_{44})H_1 H_2$, with

$$H_1 = (-I_{80}, I_{41}A_{21})(\delta_2, A_{21})(I_{41}, I_{23}),$$

$$H_2 = (I_{43}, -\delta_2 A_{21})(I_{43}, \delta_3 A_{31})(\delta_3, A_{21}\delta_2)(A_{21}, A_{31}).$$

On one hand, we can write H_2 as $H_2 = (I_{43}, -\delta_2 A_{21})(I_{43}, \delta_3 A_{31})(\delta_3 A_{31}, A_{21}\delta_2)$, so that $H_2 = 1$ by Lemma 6.3.3 since $I_{43} - \delta_2 A_{21} - \delta_3 A_{31} = 0$. On the other hand, using the expressions for I_{80}, I_{41} and A_{21} given above, we have

$$H_1 = (a_2 b_2 \alpha_1^2, 2\alpha_1^2(a_2 - b_2)(b_2(\alpha_3 + \alpha_1)^2 - a_2(\alpha_3 - \alpha_1)^2))$$

$$= (a_2 b_2 \alpha_1^2, \alpha_1^2(a_2 - b_2))(a_2 b_2 \alpha_1^2, \alpha_1^2(b_2(\alpha_3 + \alpha_1)^2 - a_2(\alpha_3 - \alpha_1)^2)) = 1$$

by Lemma 6.3.3. It follows that $E_v = (c, I_{44})$ so that $E_v = 1$ for U4E/I as $c \in K^{\times 2}$ and $E_v = (-1)^{2n}$ for U4F/J, as $c \notin K^{\times 2}$.

6.6.5 C is of type 1×1

Proof of Tables 4.31 to 4.34

Case TC1. By definition of the isogeny and since J has good reduction, we have $\alpha_2, \beta_2 \in K$ and $\alpha_1, \alpha_3, \beta_3 \in K^{nr}$ by Lemma 6.6.11. In particular $a, b \in \mathbb{Z}$ and $a \equiv b \equiv v(c) \bmod 2$ by semistability criterion 3.4.29. Also, since $\beta_1 = -\alpha_1$ it follows that $\alpha_1 \equiv \alpha_2 \equiv 0 \not\equiv \beta_2 \equiv \alpha_3 \equiv \beta_3$. Reducing invariants $\bmod \pi$ yields

$$I_{20} \equiv I_{21} \equiv_\square I_{42} \equiv_\square I_{60} \equiv_\square 2, \quad I_{40} \equiv_\square I_{43} \equiv_\square 1, \ell \equiv_\square 2c,$$

so that

$$E_v = (-1, I_{22}I_{41})(I_{44}, 2cI_{22})(I_{23}, cI_{41})(I_{45}, -cI_{22})(I_{80}, -2cI_{41}).$$

Since the following valuations are even,

$$v(I_{23}) = 2a, \quad v(I_{45}) = 2b, \quad v(I_{44}) = 2b, \quad v(I_{80}) = 2a,$$

$$E_v = (I_{22}, -I_{44}I_{45})(I_{41}, -I_{23}I_{80})(c, I_{44}I_{45}I_{23}I_{80}).$$

Write $\alpha_1 = a_1\pi^a$, $\alpha_2 = \alpha_1 + a_2\pi^a$, $\alpha_3 = \beta_2 + a_3\pi^b$, $\beta_3 = \beta_2 + b_3\pi^b$, with $a_1, a_2, a_3, b_3 \in \mathcal{O}_{K^{nr}}^\times$. We have

$$I_{22} = \beta_2\pi^b(a_3 + b_3) + o(\pi^b), \quad I_{41} = \pi^a\beta_2(a_1 + a_2) + o(\pi^a),$$

$$I_{44} =_\square (a_3 - b_3)^2 + o(\pi^b), \quad I_{45} =_\square -a_3b_3 + o(1) \equiv_\square -a_3b_3,$$

$$I_{80} =_\square -a_2(2a_1 + a_2) + o(\pi^b), \quad I_{23} =_\square a_1^2.$$

Now if $v(I_{22}) > b$, then by Lemma 6.2.1.3 and using Lemma 6.3.4, we have that $(I_{22}, -I_{44}I_{45}) = 1$. Also in this case $a_3 \equiv -b_3$ so that $I_{44}I_{45} =_\square 1$ and $(c, I_{44}I_{45}) = 1$.

Similarly if $v(I_{41}) > a$ then by Lemma 6.2.1.5 and using Lemma 6.3.4, we have that $(I_{41}, -I_{23}I_{80}) = 1$. Also, in this case $a_1 \equiv -a_2$ so that $I_{23}I_{80} =_\square 1$ and $(c, I_{23}I_{80}) = 1$. Finally assume that $v(I_{22}) = b, v(I_{41}) = a$. If $a \equiv 0 \bmod 2$ then since $a \equiv b \equiv v(c) \bmod 2$, we are done. Otherwise assume $a \equiv b \equiv v(c) \equiv 1 \bmod 2$. Then rewriting

$$E_v = (cI_{22}, I_{44}I_{45})(cI_{41}, I_{23}I_{80})(I_{22}I_{41}, -1),$$

it is clear that $E_v = 1$.

Case TC2. By definition of the isogeny and since J has good reduction, we have $\alpha_1, \alpha_2, \beta_2, \alpha_3, \beta_3 \in K^{nr}$ by Lemma 6.6.11. In particular $a, b \in \mathbb{Z}$ and $a \equiv b \equiv v(c) \bmod 2$ by semistability criterion 3.4.29, and $v(I_{44}) = v(I_{80}) = 2(a+b) \in 2\mathbb{Z}$. Reducing invariants $\bmod \pi$ yields

$$\ell \equiv 2c, \quad I_{23} =_\square \alpha_1^2, \quad (A_1 - B_1)^2 =_\square \alpha_1^2, \quad I_{21} \equiv_\square -\alpha_1^2, \quad I_{22} \equiv_\square 2\alpha_1^2, I_{45} \equiv_\square 1.$$

Also by Lemma 6.2.1.4, $I_{40} =_\square u_1^2$ so that $I_{40} \equiv_\square \alpha_1^2$. Therefore

$$E_v = (I_{41}, -\alpha_1^2 I_{80})(I_{43}, -\alpha_1^2 I_{44}I_{42})(I_{60}, -\alpha_1^2 I_{80}I_{42})(I_{20}, -\alpha_1^2 I_{44})(c, I_{44}I_{80})(I_{42}, -I_{44}I_{80}).$$

Let $a_2, b_2, a_3, b_3 \in \mathcal{O}_{K^{nr}}^\times$ and write

$$\alpha_2 = \alpha_1 + a_2\pi^a, \quad \beta_2 = \alpha_1 + b_2\pi^a, \quad \alpha_3 = -\alpha_1 + a_3\pi^b, \quad \beta_3 = -\alpha_1 + \alpha_3\pi^b,$$

so that

$$I_{42} =_\square -\alpha_1^2\pi^{a+b}U_{42} \text{ with } U_{42} = (a_3+b_3)(a_2+b_2),$$

$$I_{44} = \pi^{2a+2b}U_{44} \text{ with } U_{44} = (a_2-b_2)^2(a_3-b_3)^2,$$

$$I_{80} =_\square \pi^{2a+2b}U_{80} \text{ with } U_{80} = a_2b_2a_3b_3.$$

If $a < b$ then $\alpha_1 \in K^\times$ since otherwise $Frob(\alpha_1) = -\alpha_1$, a contradiction to the definition of the isogeny since $a \neq b$. Therefore we have

$$I_{20} =_\square 2(a_2-b_2)^2 + o(\pi^{2a}), \quad I_{41} =_\square 2a_2b_2 + o(\pi^{2a}),$$

$$I_{43} =_\square 2\pi^a\alpha_1(a_2+b_2)(a_2-b_2)^2 + o(\pi^{3a}), \quad I_{60} =_\square 2\pi^a\alpha_1 a_2b_2(a_2+b_2) + o(\pi^{3a}).$$

It follows that $(I_{41}, -\alpha_1^2 I_{80}) = (I_{20}, -\alpha_1^2 I_{44}) = 1$ and since $v(c) \equiv a \bmod 2$ we can rewrite

$$E_v = (\pi^a, I_{42}^2 I_{44}^2 I_{80}^2)(2\alpha_1(a_2+b_2)(a_2-b_2)^2, (a_2-b_2)^2(a_3-b_3)^2(a_2+b_2)(a_3+b_3))$$

$$(2\alpha_1 a_2b_2(a_2+b_2), a_2b_2a_3b_3(a_2+b_2)(a_3+b_3))(-(a_2+b_2)(a_3+b_3), -a_2b_2a_3b_3(a_2-b_2)^2(a_3-b_3)^2).$$

Since $v((a_2-b_2)^2(a_3-b_3)^2)=1$, simplifying yields

$$E_v=(a_2+b_2,-a_2b_2(a_2-b_2)^2)(a_3+b_3,-a_3b_3(a_3-b_3)^2).$$

Finally, assume that $v(a_2+b_2)>0$, then $a_2\equiv -b_2$ so that $-a_2b_2(a_2-b_2)^2\equiv a_2^4\equiv_\square 1$ and similarly for (a_3+b_3). Therefore $E_v=1$.
If $a=b$ then

$$I_{20}=_\square 2(a_2-b_2)^2+2(a_3-b_3)^2,\quad I_{41}=_\square 2\alpha_1^2U_{41}+o(\pi^{2a}) \text{ with } U_{41}=(a_2b_2+a_3b_3),$$

$$I_{43}=_\square 2\alpha_1\pi^aU_{43}+o(\pi^{3a}) \text{ with } U_{43}=(a_2+b_2)(a_2-b_2)^2-(a_3+b_3)(a_3-b_3)^2,$$

$$I_{60}=_\square 2\alpha_1^3\pi^aU_{60}+o(\pi^{3a}) \text{ with } U_{60}=(a_2b_2(a_2+b_2)-a_3b_3(a_3+b_3)).$$

If $v(2(a_2-b_2)^2+2(a_3-b_3)^2)>0$ then $v(I_{20})>2a$ and $v(I_{20}^2)>v(I_{44})$ hence by Lemmata 6.2.1.1 and 6.3.4 we have $(I_{20},-\alpha_1^2U_{44})=1$. Similarly, if $v(U_{41})>0$ then $v(I_{41})>2a$ and $v(I_{41}^2)>v(I_{80})$, therefore by Lemmata 6.2.1.5 and 6.3.4 we have $(I_{41},-\alpha_1^2I_{80})=1$. Hence replacing invariants and simplifying yields

$$E_v=(2\alpha_1\pi^aU_{43},U_{44}U_{42})(2\alpha_1^3\pi^aU_{60},U_{80}U_{42})(\pi^a,U_{44}U_{80})(-\alpha_1^2U_{42},-U_{44}U_{80}).$$

Factoring out π^a we have $(\pi^a,U_{44}^2U_{42}^2U_{80}^2)=1$, hence simplifying gives

$$E_v=(U_{43},U_{44}U_{42})(U_{60},U_{80}U_{42})(U_{42},-U_{44}U_{80}).$$

If $\alpha_1\in K$ then by definition of the isogeny $\delta_2,\delta_3\in K$ and using Lemma 6.3.3.1 we have

$$(U_{43},U_{44}U_{42})=((a_2+b_2)(a_2-b_2)^2,(a_3+b_3)(a_3-b_3)^2),$$

$$(U_{60},U_{80}U_{42})=(a_2b_2(a_2+b_2),a_3b_3(a_3+b_3))$$

$$(U_{42},-U_{44}U_{80})=(a_2+b_2,-U_{44}U_{80})(a_3+b_3,-U_{44}U_{80}).$$

Factoring out and simplifying gives

$$E_v=(a_2+b_2,-a_2b_2(a_2-b_2)^2)(a_3+b_3,-a_3b_3(a_3-b_3)^2).$$

Now if $v(a_2+b_2)>0$ then $a_2\equiv -b_2$ and $-a_2b_2(a_2-b_2)^2\equiv 4a_2^4\equiv_\square 1$, similarly if $v(a_3+b_3)>0$. Hence E_v=1.
If $\alpha_1\notin K$ then by definition of the isogeny $\delta_2,\delta_3\in K$ and $Frob((a_2+b_2))=(a_3+b_3)$ it follows that $v(U_{42})\in 2\mathbb{Z}$ hence

$$E_v=(U_{43},U_{42}U_{44})(U_{60},U_{42}U_8).$$

If $v(U_{43}) > 0$ then $(a_2 - b_2)^2(a_2 + b_2) \equiv (a_3 - b_3)^2(a_3 + b_3)$, therefore $U_{42}U_{44} \equiv (a_2 - b_2)^4(a_2 + b_2)^2$ so that $U_{42}U_{44} \in K^{\times 2}$.

If $v(U_{60}) > 0$ then $a_2b_2(a_2+b_2) \equiv a_3b_3(a_3+b_3)$, therefore $U_{42}U_8 \equiv (a_2b_2)^2(a_2+b_2)^2$ so that $U_{42}U_8 \in K^{\times 2}$. Both cases yield $E_v = 1$.

Case TC3. By definition of the isogeny and since J has good reduction, we have $\alpha_1, \alpha_2, \beta_2, \alpha_3, \beta_3 \in K^{nr}$ from Lemma 6.6.11, and $\alpha_1 \not\equiv 0$. In particular $a, b \in \mathbb{Z}$ and $a \equiv b \equiv v(c) \mod 2$ by semistability criterion 3.4.29. Here $v(I_{23}) = v(I_{44}) = 0$ and $v(I_{45}) = a + b \in 2\mathbb{Z}, v(I_{80}) = 2a + 2b \in 2\mathbb{Z}$. Reducing invariants $\mod \pi$ yields

$$I_{23} \equiv_{\square} \alpha_1^2, \quad I_{44} \equiv_{\square} 1, \quad I_{20} \equiv \alpha_1^2, \quad I_{22} \equiv_{\square} -\alpha_1^2.$$

Hence

$$E_v = (-1, I_{41}I_{43}I_{60})(I_{40}, \ell I_{20}I_{43}I_{60})(I_{45}, -\ell I_{21}I_{22})(cI_{41}, I_{23})(I_{80}, cI_{41}I_{42}I_{60})(I_{42}, -I_{43}I_{60}).$$

Let $a_2, b_2, a_3, b_3 \in \mathcal{O}_K^{\times}$ and write

$$\alpha_2 = \alpha_1 + a_2\pi^a, \quad \beta_2 = -\alpha_1 + b_2\pi^b, \quad \alpha_3 = \alpha_1 + a_3\pi^a, \quad \beta_3 = -\alpha_1 + b_3\pi^b.$$

This yields

$$I_{45} =_{\square} -\alpha_1^2\pi^{a+b}U_{45}, \text{ with } U_{45} = (a_2 - a_3)(b_2 - b_3),$$

$$I_{80} =_{\square} \pi^{2a+2b}U_{80} \text{ with } U_{80} = a_2b_2a_3b_3.$$

Cases TC3A/B/C. Here $a < b$, $\alpha_1 \in K$ hence

$$E_v = (I_{45}, \ell I_{21})(I_{40}, \ell)H, \quad H = (-1, I_{41}I_{43}I_{60})(I_{40}, I_{43}I_{60})(I_{80}, cI_{41}I_{42}I_{60})(I_{42}, -I_{43}I_{60}).$$

We show that $H = 1$. Computing invariants we find

$$I_{21} = a_2a_3\pi^{2a} + o(\pi^{2a}), \quad I_{40} =_{\square} \pi^{2a}(a_2 - a_3)^2 + o(\pi^{2a}),$$

$$I_{41} =_{\square} 2\pi^{2a}a_2a_3, \quad I_{42} =_{\square} \pi^{2a}a_2a_3,$$

$$I_{43} =_{\square} 2\alpha_1\pi^a(a_2 + a_3) + o(\pi^a), \quad I_{60} =_{\square} \pi^{a+b}(-2\alpha_1\pi^a(a_2^2b_2 + a_3^2b_3)) + o(\pi^{2a+b}).$$

Hence $(I_{43}I_{60}, -I_{40}I_{42}) = (-(a_2+a_3)(a_2^2b_2+a_3^2b_3), -(a_2-a_3)^2a_2a_3)$, and $(I_{41}, -I_{80}) = (-I_{80}, I_{42}) = 1$ since $v(I_{41}), v(I_{42}), v(I_{80}) \in 2\mathbb{Z}$. Similarly $(I_{80}, cI_{60}) = (a_2^2b_2 + a_3^2b_3, a_2b_2a_3b_3)$. Simplifying yields

$$H = (a_2^2b_2 + a_3^2b_3, -(a_2 - a_3)^2a_2^2a_3^2b_2b_3)(a_2 + a_3, -(a_2 - a_3)^2a_2a_3).$$

i) If $a_2, a_3, b_2, b_3 \in K$ then $(a_2 - a_3)^2 \in K^{\times 2}$ and by Lemma 6.3.3 we have

$$H = (a_2^2 b_2, a_3^2 b_3)(a_2, a_3) = 1.$$

ii) If $Frob$ permutes a_2 and a_3 as well as b_2 and b_3, then $(a_2 - a_3)^2 \notin K^{\times 2}$. If $a_2 + a_3$ and $a_2^2 b_2 + a_3^2 b_3$ are units then $H = 1$. Otherwise, if $v(a_2 + a_3) > 0$ then $a_2 \equiv -a_3$ so that $-a_2 a_3 (a_2 - a_3)^2 \equiv 4a_2^4$ hence $-a_2 a_3 (a_2 - a_3)^2 \in K_v^2$ and $H = 1$. Finally, if $v(a_2^2 b_2 + a_3^2 b_3) > 0$, then $a_2^2 b_2 \equiv -a_3^2 b_3$ so that $-a_2^2 b_2 a_3^2 b_3 \equiv_\square b_2^2$. In particular $-a_2^2 b_2 a_3^2 b_3 \notin K_v^2$ since $Frob(b_2) = b_3$, thus $-a_2^2 b_2 a_3^2 b_3 (a_2 - a_3)^2 \in K_v^2$ (as $(a_2 - a_3)^2 \notin K_v^2$) and $H = 1$. Hence $E_v = (I_{40}, \ell)(I_{45}, \ell I_{21})$.
Now, $\ell = -c^2 \ell_1 I_{21} =_\square -\ell_1 I_{21}$ so that $\ell I_{21} =_\square -\ell_1$. Therefore

$$E_v = ((a_2 - a_3)^2, -\ell_1 I_{21})(-(a_2 - a_3)(b_2 - b_3), -\ell_1) = (\ell_1, -(a_2 - a_3)^3 (b_2 - b_3)),$$

since I_{21} and $(a_2 - a_3)$ are units in K. For isogeny TC3A, we have $v(\ell_1) \in 2\mathbb{Z}$ hence $E_v = 1$ as required.
Finally, computing $(A_1 - B_1)^2$ in this case, one finds

$$(A_1 - B_1)^2 =_\square \frac{-(a_2 - a_3)(b_2 - b_3)}{(a_2 - a_3)^2} + o(\pi^{a+b}),$$

hence $(A_1 - B_1)^2 =_\square -(a_2 - a_3)^3 (b_2 - b_3)$. It follows that since $(A_1 - B_1)^2 \in K^2$ in isogeny TC3B, $E_v = 1$. Finally, $(A_1 - B_1)^2 \notin K^{\times 2}$ in isogeny TC3C, and $v(\ell_1) \equiv r \bmod 2$, therefore $E_v = (-1)^r$ as required.
Cases TC3D/E/F/G/H/I. Here $a = b$ and $E_v = (I_{45}, \ell I_{21})(I_{40}, -\ell I_{20} I_{22})(c I_{41}, I_{23}) H$, with

$$H = (-1, I_{41} I_{43} I_{60})(I_{40}, I_{43} I_{60})(I_{80}, c I_{41} I_{42} I_{60})(I_{42}, -I_{43} I_{60}).$$

We show that $H = 1$. Computing invariants, we find

$$I_{21} =_\square (a_2 + b_2)(a_3 + b_3), \quad I_{40} =_\square (a_2 + b_2 - a_3 - b_3)^2,$$

$$I_{41} =_\square 2\alpha_1^2 (a_2 a_3 + b_2 b_3) + o(\pi^{2a}), \quad I_{42} =_\square \alpha_1^2 (a_2 - b_2)(a_3 - b_3) + o(\pi^{2a}),$$

$$I_{43} =_\square 2\alpha_1^3 \pi^a U_{43} + o(\pi^a) \text{ with } U_{43} = (a_2 - b_2 + a_3 - b_3),$$

$$I_{60} =_\square -2\alpha_1 \pi^a U_{60} + o(\pi^{3a}) \text{ with } U_{60} = a_2 b_2 (a_2 - b_2) + a_3 b_3 (a_3 - b_3).$$

Also note that $v(I_{40}) \in 2\mathbb{Z}$ hence $(\alpha_1^2, -I_{40}) = 1$.
Note that since either $\alpha_2, \beta_2, \alpha_3, \beta_3 \in K$ or $Frob(a_2 - b_2) = (a_3 - b_3)$, we have that $v(I_{42}) \in 2\mathbb{Z}$. Using Lemma 6.3.3 and simplifying gives $(I_{41}, -I_{80}) = (a_2 a_3 + b_2 b_3, -a_2 b_2 a_3 b_3) = 1$, since if $v(I_{41}) > 0$ then $a_2 a_3 \equiv -b_2 b_3$ so that $-a_2 b_2 a_3 b_3 \equiv (a_2 a_3)^2 \equiv_\square 1$. Also $(I_{43}, -I_{40} I_{42}) = (U_{43}, -U_{40} U_{42}) = 1$ since if $v(U_{43}) > 0$ then

$v(I_{43}) > a$ thus $v(I_{43}^2) > v(I_{42}I_{44})$ and the result follows from Lemma 6.3.4. Otherwise if $v(U_{43}) = 0$ then the result follows since $v(U_{42}), v(U_{40}) \in 2\mathbb{Z}$. Similarly, $(I_{42}, I_{80}) = 1$. Finally, $(I_{60}, -I_{40}I_{42}I_{80}) = (U_{60}, -U_{40}U_{42}U_{80}) = 1$ since if $v(U_{60}) > 0$ then $v(I_{60}) > 3a$ thus $v(I_{60}^2) > 6a = v(I_{42}I_{80})$ hence $(U_{60}, -U_{40}U_{42}U_{80}) = 1$ by Lemma 6.3.4. Otherwise, if $v(U_{60}) = 0$ then the result follows since $v(U_{42}), v(U_{40}) \in 2\mathbb{Z}$.

Therefore when $a = b$, $H = 1$ and $E_v = (I_{45}, \ell I_{21})(I_{40}, -\ell I_{20}I_{22})(cI_{41}, I_{23})$.

Cases TC3D/F/G. Here $\alpha_1 \in K$ so that $E_v = (I_{45}, \ell I_{21})(I_{40}, \ell)$. As above, we have, $\ell = -c^2\ell_1 I_{21} =_\square -\ell_1 I_{21}$ so that $\ell I_{21} =_\square -\ell_1$. Recall that $I_{21} = -\ell_2\ell_3$. Hence $E_v = (-(a_2 - a_3)(b_2 - b_3), -\ell_1)((a_2 + b_2 - a_3 - b_3)^2, \ell_1\ell_2\ell_3)$. Moreover, computing the roots of $L_1(x)$, we find

$$(A_1 - B_1)^2 =_\square \frac{-(a_2 - a_3)(b_2 - b_3)}{(a_2 + b_2 - a_3 - b_3)^2} =_\square -(a_2 - a_3)(b_2 - b_3)(a_2 + b_2 - a_3 - b_3)^2,$$

so that $E_v = ((A_1 - B_1)^2, \ell_1)((a_2 + b_2 - a_3 - b_3)^2, -(a_2 + b_2)(a_3 + b_3))$.

For TC3D, we have $v(\ell_1) = v(u_1) - 2a - v(c)$. If $v(u_1) > 0$ then $a_2 + b_2 \equiv a_3 + b_3$ and hence $v(\ell_2) = v(\ell_3) = 0$ (see Proof of Table 4.33 in Section 5.3.5). Moreover, in this case, $2v(u_1) \in 2\mathbb{Z}$ and $a_2 - a_3 \equiv -b_2 + b_3$. It follows that $E_v = ((b_2 - b_3)^4, u_1) = 1$. Finally, if $v(u_1) = 0$ then $((A_1 - B_1)^2, \ell_1) = 1$. If $\alpha_2, \beta_2, \alpha_3, \beta_3 \in K$ then $u_1^2 \in K^{\times 2}$ and $E_v = 1$. Otherwise $Frob(\alpha_2) = \alpha_3$ and $Frob(\alpha_3) = \alpha_2$ and similarly for β_2, β_3, so that $Frob(\ell_2) = -\ell_3$. Therefore $v(I_{21}) \in 2\mathbb{Z}$ and $E_v = (u_1^2, -I_{21}) = 1$.

For TC3F, we have $(A_1 - B_1)^2 \in K^{\times 2}$ hence $E_v = ((a_2 + b_2 - a_3 - b_3)^2, -(a_2 + b_2)(a_3 + b_3))$. If $v(a_2 + b_2) = v(a_3 + b_3) = 0$ then $E_v = 1$ as $v(u_1)^2 \in 2\mathbb{Z}$ and we are done. Otherwise either $\alpha_2, \beta_2, \alpha_3, \beta_3 \in K$ and $u_1^2 \in K^{\times 2}$ and we are done; or $Frob(\ell_2) = -\ell_3$ and$v(a_2 + b_2) = v(a_3 + b_3)$ so that $v(a_2 + b_2)(a_3 + b_3)) \in 2\mathbb{Z}$ and $E_v = 1$ as required.

For TC3G, we have $(A_1 - B_1)^2 \notin K^{\times 2}$. Now by definition of $\Delta_G, u_1, \ell_2, \ell_3$ (see Proof of Table 4.33 in Section 5.3.5), we have $v(\ell_2) = v_(\ell_3)$ when $v(\Delta_G) > 2a$. Therefore, since $v(u_1)^2 \in 2\mathbb{Z}$ we have $E_v = ((A_1 - B_1)^2, \ell_1)$. Moreover, by semistabilty criterion 3.4.29, we have that $v(\ell) \equiv r \bmod 2$. It follows that $v(u_1) \equiv v(c) \bmod 2$ and hence that $v(\ell_1) \equiv r \bmod 2$. Thus $E_v = (-1)^r$ as required.

Cases TC3E/H/I. Here $\alpha_1 \in K$ so that $E_v = (I_{45}, -\ell I_{21}I_{22})(I_{40}, \ell)(cI_{41}, I_{23})$.

Using Lemmata 6.3.4, 6.2.1.1 and 6.2.1.3 we have that $(I_{41}, I_{23}) = (I_{40}, I_{20}) = 1$. Moreover replacing invariants by their reduced values as above yields

$E_v = (I_{40}, \ell)(c, I_{23})(I_{45}, \ell\alpha_1^2 I_{21})$ Since $I_{23} \equiv_\square \alpha_1^2$ it follows that $E_v = (I_{45}, \ell I_{21})(I_{40}, \ell)(\alpha_1^2, c)$.

The behaviour of the first two Hilbert Symbols is given in the cases of TC3D/F/G. Hence since $v(c) \equiv a \bmod 2$, if follows that $E_v = (-1)^a$ for TC3E/H, and $E_v = (-1)^{a+r}$ for TC3I as required.

Chapter 7

Conclusion

7.1 Forthcoming result

The result on the 2-parity conjecture presented in Theorem 4.4.11 is used in a joint work with Vladimir Dokchitser in [17]. Combined with the theory of regulator constants of [10], [11], it yields the following result on the parity conjecture in this setting.

Theorem 7.1.1. *Let $C/\mathcal{K}$ be a hyperelliptic curve of genus 2 over a number field given by*

$$C : y^2 = f(x),$$

and satisfying the conditions of Theorem 4.4.11. Let J denote its Jacobian, $\mathcal{K}_f$ the splitting field of $f(x)$ and assume that $Ш(J/\mathcal{K}_f)[p^\infty]$ is finite for $p = 2, 3, 5$. Then the parity conjecture holds for J/K.

Remark 7.1.2. If $\mathcal{K} \subseteq \mathcal{L} \subseteq \mathcal{K}_f$ is an intermediate field with $[\mathcal{K}_f : \mathcal{L}]$ a power of 2 then $Gal(\mathcal{K}_f/\mathcal{L}) \subseteq C_2 \times D_4$, as the latter is the Sylow 2-subgroup of S_6. In particular $C/\mathcal{L}$ is a $C_2 \times D_4$ curve. Thus Theorem 4.4.11 shows that the parity conjecture holds over all such fields $\mathcal{L}$. Combined with the theory of regulator constants, this is sufficient to deduce that the parity conjecture holds over $\mathcal{K}$.

7.2 Work in progress

We are currently working on improving Theorem 4.4.11, and hence Theorem 7.1.1, by removing some extra conditions. Namely, the following are work in progress:
i) proving the local discrepancy conjecture 4.4.10 when the reduction at finite odd places of the polynomial defining C has double roots inside two triple roots,
ii) showing that the term of the local discrepancy E_v is stable under the change of variables performed to balance a cluster picture as in [15][Definition 3.41],

iii) controlling the term of the local discrepancy E_v when some $C_2 \times D_4$ invariants vanish,
iv) weakening the conditions at 2-adic places.

By proving i), we would prove the 2-parity conjecture for curves C with semistable balanced cluster picture. Adding ii) would then remove the balanced condition so that all semistable cluster pictures at odd finite places could be considered. Finally showing iii) would release the extra condition on non-vanishing of $C_2 \times D_4$ invariants.

7.3 Obstructions to generalization

As mentioned in Section 3.2.1, the parity of the 2^∞ Selmer rank of a principally polarized abelian variety admitting an isogeny through which multiplication by 2 factors, is given by Lemma 3.2.5. However, in order to express this parity as a sum of local terms, one needs to control the order of their Shafarevic-Tate group (up to squares). This is achievable using a result of Poonen and Stoll in [32], whenever both the variety and the codomain of the isogeny are Jacobians. This was true in our case thanks to the property of a Richelot isogeny. In higher dimension, it is not clear how to get a hold of the order of the Shafarevich-Tate group up to squares since the codomain of the isogeny may be a principally polarized abelian variety that is not a Jacobian. Furthermore, even if it were the case, curves of genus $g > 2$ are not necessarily hyperelliptic so that we cannot explicitly control their local invariants as we have done here. And finally, even if they were hyperelliptic, finding the term of local discrepancy in order to prove the 2-parity conjecture in this case seems rather optimistic, unless a better conceptual understanding of the invariants involved is achieved.

Appendix A

Isogenies between abelian varieties with split totally toric reduction (by Adam Morgan)

Preliminaries

Definition A.0.1. Let A and B be abelian groups and $f : A \to B$ a homomorphism with finite kernel and cokernel. Then we define

$$z(f) := |\text{coker}(f)|/|ker(f)|.$$

Lemma A.0.2. *We have the following properties of z:*

(i) Let A be a finite abelian group and $f : A \to A$ a homomorphism. Then $z(f) = 1$.

(ii) Suppose we have a commutative diagram of abelian groups

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \\ 0 & \longrightarrow & B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & 0 \end{array}$$

whose rows are exact. Suppose further that f_1, f_2 and f_3 all have finite kernel and cokernel. Then $z(f_2) = z(f_1)z(f_3)$.

(iii) Let A be an abelian group, let $A = A_0 \supseteq A_1 \supseteq A_2 \supseteq \ldots$ be a filtration on A with finite quotients and let $f : A \to A$ be a homomorphism having finite kernel and cokernel. Suppose further that f respects the filtration and that for each n the induced maps $f_n : A_n \to A_n$ have finite kernel and cokernel. Then we have $z(f) = z(f_n)$ for all $n \geq 0$.

Proof. (i). The first isomorphism theorem gives $A/ker(A) \cong \mathrm{im}(A)$, with each of A, $ker(A)$ and $\mathrm{im}(A)$ finite by assumption. In particular, we have

$$|A| = |\mathrm{im}(A)||ker(A)|$$

from which the result follows immediately.

(ii). Apply the snake lemma to the commutative diagram in the statement.

(iii). For each $n \geq 0$, apply parts (i) and (ii) to the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_{n+1} & \longrightarrow & A_n & \longrightarrow & A_n/A_{n+1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle \bar{f_n}} & & \\ 0 & \longrightarrow & A_{n+1} & \longrightarrow & A_n & \longrightarrow & A_n/A_{n+1} & \longrightarrow & 0 \end{array}$$

(the top and bottom rows induced by the natural inclusion $A_{n+1} \subseteq A_n$ and the map $\bar{f_n}$ being induced by f_n). □

The multiplicative group

Let K be a finite extension of $\mathbb{Q}_p$ for some prime p, let $g \geq 1$ and let $M = (m_{i,j}) \in \mathrm{Mat}_g(\mathbb{Z})$ be a full rank matrix. Write $f_M : \bar{K}^{\times g} \to \bar{K}^{\times g}$ for the map

$$\mathbf{x} = (x_i) \longmapsto \mathbf{x}^M := \left(\prod_{j=1}^{g} x_j^{m_{i,j}} \right).$$

Lemma A.0.3. *Write f_K for the restriction of f_M to a map $K^{\times g} \to K^{\times g}$. Then we have*

$$z(f_K) = \frac{|\det(M)|}{|\det(M)|_K},$$

where here $|\cdot|_K$ denotes the usual normalised abolute value on K (sending a uniformiser π_K for K to $|k|^{-1}$ where k is the residue field of K) and $|\cdot|$ denotes the usual archimedean absolute value on $\mathbb{Z}$.

Proof. Write f_0 for the map $\mathcal{O}_K^{\times g} \to \mathcal{O}_K^{\times g}$ induced by f. Let v_K denote the normalised valuation on K. We have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_K^{\times g} & \longrightarrow & K^{\times g} & \xrightarrow{v_K} & \mathbb{Z}^g & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_K} & & \downarrow{\scriptstyle \bar{f_K}} & & \\ 0 & \longrightarrow & \mathcal{O}_K^{\times g} & \longrightarrow & K^{\times g} & \xrightarrow{v_K} & \mathbb{Z}^g & \longrightarrow & 0, \end{array}$$

where the map $\bar{f_K} : \mathbb{Z}^g \to \mathbb{Z}^g$ induced by f is just left multiplication by the matrix M. Considering the Smith Normal Form of M over $\mathbb{Z}$ gives $|\mathrm{coker}(\bar{f_K})| =$

$|\det(M)|$ and since M has full rank, $ker(\bar{f_K}) = 0$. By A.0.2(ii) we now have $z(f_K) = |\det(M)|z(f_0)$. Consider the filtration

$$\mathcal{O}_K^{\times g} \supseteq (1 + \pi_K \mathcal{O}_K)^g \supseteq (1 + \pi_K^2 \mathcal{O}_K)^g \supseteq ...$$

which is preserved by f_K and has each successive quotient finite. Write f_n for the induced map on $(1 + \pi_K^n \mathcal{O}_K)^g$. Then A.0.2(iii) gives $z(f_0) = z(f_n)$ for each $n \geq 1$. Taking $n \geq 1$ sufficiently large, the formal logarithm gives an isomorpism

$$(1 + \pi_K^n \mathcal{O}_K)^g \xrightarrow{\sim} \mathcal{O}_K^g.$$

Since f_n is induced by the matrix $M \in \mathrm{Mat}_g(\mathbb{Z})$, the map f_n gets transported under this isomorphism to left multiplication by the matrix M on $\mathcal{O}_K^g$. Considering the Smith Normal Form of M over $\mathcal{O}_K$ and again using the fact that M has full rank over $\mathbb{Z}$ this gives $z(f_n) = |\det(M)|_K^{-1}$ which completes the proof. □

Abelian varieties

Let K be a finite extension of $\mathbb{Q}_p$ and A/K an abelian variety. Let $\phi : A \to B$ be an isogeny.

Suppose that A/K has split totally toric reduction. Then the same is true also of B. We have an isomorphism of $G_K := \mathrm{Gal}(\bar{K}/K)$-modules

$$A(\bar{K}) \cong \bar{K}^{\times g}/\Lambda_A$$

for some lattice $\Lambda_A \subseteq K^{\times g}$ (see [31, Section 5.3] for a review of the uniformisation of abelian varieties with split totally toric reduction and the definition of a lattice inside $K^{\times g}$). The same is true for B with some lattice $\Lambda_B \subseteq K^{\times g}$. Note that the induced Galois action on Λ_A and Λ_B are trivial.

By [20, Theorem 3], there is a matrix $M(\phi) \in \mathrm{Mat}_g(\mathbb{Z})$ such that the isogeny $\phi : A \to B$ is induced by the map $\mathbf{x} \mapsto \mathbf{x}^{M(\phi)}$ on $\bar{K}^{\times g}$, and $M(\phi)$ sends Λ_A into Λ_B. We write ϕ_Λ for the induced map $\Lambda_A \to \Lambda_B$ of free $\mathbb{Z}$-modules of rank g. In summary, we have a G_K-equivariant commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Lambda_A & \longrightarrow & \bar{K}^{\times g} & \longrightarrow & A(\bar{K}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi_\Lambda} & & \downarrow{\scriptstyle f_{M(\phi)}} & & \downarrow{\scriptstyle \phi} & & \\ 0 & \longrightarrow & \Lambda_B & \longrightarrow & \bar{K}^{\times g} & \longrightarrow & B(\bar{K}) & \longrightarrow & 0 \end{array} \tag{A.0.4}$$

with exact rows. Since $H^1(K, \Lambda_A) = 0 = H^1(K, \Lambda_B)$ (G_K is profinite whilst both Λ_A and Λ_B are torsion free with trivial action) we have the same diagram with $A(\bar{K})$ (resp. $B(\bar{K})$) replaced by $A(K)$ (resp. $B(K)$) and $\bar{K}^{\times g}$ replaced by $K^{\times g}$.

We note that the isogeny ϕ determines the matrix $M(\phi)$. Indeed, since Λ_A and Λ_B are lattices, $\mathcal{O}_{\bar{K}}^{\times g} \subseteq \bar{K}^{\times g}$ injects into $\bar{K}^{\times g}/\Lambda_A$ (resp. $\bar{K}^{\times g}/\Lambda_B$). Now since the maps $x \mapsto x^n$ for $n \in \mathbb{Z}$ are all distinct as endomorphisms of $\mathcal{O}_{\bar{K}}^{\times g}$ (consider their kernels), the same is true for that maps $\mathbf{x} \mapsto \mathbf{x}^M$ for all $M \in \mathrm{Mat}_g(\mathbb{Z})$ as endomorphisms of $\mathcal{O}_{\bar{K}}^{\times g}$, from which the claim follows.

Lemma A.0.5. *Write ϕ_K for the map $A(K) \to B(K)$ on K-points induced by ϕ. The we have*

$$z(\phi_K) = \frac{|A[\phi] \cap A(\bar{K})_0||\det(M)|}{\deg(\phi)|\det(M)|_K}.$$

(Here $A(\bar{K})_0$ denotes the points in $A(\bar{K})$ reducing to the identity component of the Neron model of A/K).

Proof. Consider first the diagram A.0.4. Applying A.0.2 we obtain

$$z(f_{M(\phi)}) = z(\phi_{\bar{K}})z(\phi_\Lambda) \tag{A.0.6}$$

where here $\phi_{\bar{K}}$ denotes the map $A(\bar{K}) \to B(\bar{K})$ on $\bar{K}$-points induced by ϕ. Now ϕ is surjective on $\bar{K}$-points and $|ker(\phi_{\bar{K}})| = \deg(\phi)$ by definition. In particular, $z(\phi_{\bar{K}}) = \deg(\phi)^{-1}$. Next, let $M(\phi)^a$ denote the adjugate matrix of $M(\phi)$, so that $M(\phi)M(\phi)^a = \det(M(\phi)) = M(\phi)^a M(\phi)$. Now for any $0 \neq n \in \mathbb{Z}$, multiplication by n is surjective on $\bar{K}^{\times g}$. In particular, since $M(\phi)$ has non-zero determinant we see that $f_{M(\phi)}$ is surjective. Thus $z(f_{M(\phi)}) = |ker(f_{M(\phi)})|^{-1}$. Note also that with $n = \det(M(\phi))$ we have $ker(f_{M(\phi)}) \subseteq \boldsymbol{\mu}_n^g \subseteq \bar{K}^{\times g}$.

We now have from A.0.6 that

$$z(\phi_\Lambda) = \frac{\deg(\phi)}{|ker(f_{M(\phi)})|}.$$

Let $\mathcal{A}/\mathcal{O}_K$ denote the Neron model of A/K and let $\mathcal{A}^0/\mathcal{O}_K$ denote the identity component of $\mathcal{A}$. Since A is assumed to have semistable reduction over K, the formation of $\mathcal{A}^0$ commutes with base-change to any finite exctension L/K (though the same need not be true of the full Neron model). We have a commutative square

$$\begin{array}{ccc} \mathcal{O}_{\bar{K}}^{\times g} & \longrightarrow & \bar{K}^{\times g}/\Lambda \\ \downarrow & & \downarrow \\ \mathcal{A}^0(\mathcal{O}_{\bar{K}}) & \longrightarrow & A(\bar{K}). \end{array} \tag{A.0.7}$$

where both vertical arrows are isomorphisms and the horizontal arrows are injections (see [31, Figure 1] and the surrounding discussion).

The observation that $ker(f_{M(\phi)}) \subseteq \boldsymbol{\mu}_n^g$ shows that $ker(f_{M(\phi)}) = ker(f_{M(\phi)}|_{\mathcal{O}_{\bar{K}}^{\times g}})$ which by the above diagram is equal to $A(\bar{K})_0[\phi] = A[\phi] \cap A(\bar{K})_0$.

We conclude that

$$z(\phi_\Lambda) = \frac{\deg(\phi)}{|A[\phi] \cap A(\bar{K})_0|}. \tag{A.0.8}$$

We now turn to the commutative diagram A.0.4 over K rather than $\bar{K}$. A.0.2 (ii) gives

$$z(\phi_K) = \frac{z(f_{M(\phi)}|_K)}{z(\phi_\Lambda)}.$$

Combining this with A.0.8 and A.0.3 gives the result. □

The main result

Suppose now that A is principally polarised with fixed principal polarisation λ defined over K, and suppose that the isogeny $\phi : A \to B$ is such that the kernel of ϕ is a maximal isotropic subspace of $A[2]$ with respect to the Weil pairing associated to λ. Note that in particular this forces $\deg(\phi) = 2^g$. In particular, B is principally polarised also and the dual isogeny $\phi^\vee : B \to A$ is such that $\phi\phi^\vee = [2] = \phi^\vee\phi$ (see, for example, [24, Proposition 16.8]). Let $M(\phi^\vee)$ be the matrix associated to $\phi^\vee$. Then we have $M(\phi)M(\phi^\vee) = 2 = M(\phi^\vee)M(\phi)$. In particular, the determinant of $M(\phi)$ is $\pm$ a power of 2. As before, $A(\bar{K})_0$ denotes the points in $A(\bar{K})$ reducing to the identity component of the Neron model of A/K

Proposition A.0.9. *Write ϕ_K for the map $A(K) \to B(K)$ on K-points induced by ϕ. Then we have*

$$z(\phi_K) = \begin{cases} \frac{|A[\phi] \cap A(\bar{K})_0|^2}{2^g} & p > 2 \\ \frac{|A[\phi] \cap A(\bar{K})_0|^{[K:\mathbb{Q}_2]+2}}{2^g} & p = 2. \end{cases}$$

In particular,

$$\mathrm{ord}_2 z(\phi_K) \equiv \begin{cases} g \pmod 2 & p > 2 \\ g + [K : \mathbb{Q}_2]\mathrm{ord}_2|A[\phi] \cap A(\bar{K})_0| \pmod 2 & p = 2. \end{cases}$$

Proof. As observed previously, the assumption on ϕ mean that the determinant of $M(\phi)$ is a power of 2. In particular, we have

$$|\det(M)|_K = \begin{cases} 1 & p > 2 \\ \frac{1}{|\det(M(\phi))|^{[K:\mathbb{Q}_2]}} & p = 2. \end{cases}$$

Thus in light of A.0.5 (and the fact that, as remarked previously, $\deg(\phi) = 2^g$), it suffices to show that we have

$$|\det(M(\phi))| = |A[\phi] \cap A(\bar{K})_0|.$$

Now $A[\phi] \cap A(\bar{K})_0$ is the kernel of the map $\mathbf{x} \mapsto \mathbf{x}^{M(\phi)}$ on $\mathcal{O}_{\bar{K}}^{\times}$. Since $M(\phi^\vee)M(\phi) = 2$, this is contained in $\boldsymbol{\mu}_2^g$. Writing $\boldsymbol{\mu}_2^g$ additively, the map $\mathbf{x} \mapsto \mathbf{x}^{M(\phi)}$ on $\boldsymbol{\mu}_2^g$ is just left multiplication by the reduction modulo 2 of the matrix $M(\phi)$. Denote this matrix by $\bar{M}$. Then $|A[\phi] \cap A(\bar{K})_0|$ is just the size of the kernel of $\bar{M} : \mathbb{F}_2^g \to \mathbb{F}_2^g$.

Write $M(\phi) = UDV$ where $U, V \in GL_g(\mathbb{Z})$ and $D \in \mathrm{Mat}_g(\mathbb{Z})$ is diagonal (i.e. write $M(\phi)$ in Smith Normal form). Now $M(\phi)M(\phi^\vee)$ is twice the identity matrix. Thus

$$2V^{-1} = M(\phi^\vee)UD.$$

In particular, each coefficient of $M(\phi^\vee)UD$ is divisible by 2, yet $\frac{1}{2}M(\phi^\vee)UD$ has determinant 1. If one of the entries of D were divisible by 4 then 2 would divide each entry of some row of the integral matrix $\frac{1}{2}M(\phi^\vee)UD$, and hence its determinant, a contradiction. We deduce that each entry of D is divisible by 2 at most once. On the other hand, the determinant of $M(\phi)$ is a power of 2 (it divides 2^g) so we deduce that each entry on the diagonal of D is in the set $\{\pm 1, \pm 2\}$. Noting that U and V are both invertible modulo 2, we see that the number of ± 2's appearing on the diagonal of D is equal to the $\mathbb{F}_2$-dimension of the kernel of $\bar{M}$. In particular, we deduce that

$$|A[\phi] \cap A(\bar{K})_0| = 2^{\dim_{\mathbb{F}_2} ker(\bar{M})} = |\det(M(\phi))|$$

as desired. □

Remark A.0.10. Under the isomorphism $A(\bar{K})_0 \cong \mathcal{O}_{\bar{K}}^{\times g}$, the subgroup $A(\bar{K})_1$ of points reducing to the identity corresponds to $(1 + \pi_K \mathcal{O}_K)^g$. In particular, if $p = 2$ then $\boldsymbol{\mu}_2^g$ lies in $(1 + \pi_K \mathcal{O}_K)^g$ and so, when $p = 2$, we may replace $|A[\phi] \cap A(\bar{K})_0|$ with the quantity $|A[\phi] \cap A(\bar{K})_1|$.

Remark A.0.11. Note that in the setup above, $A(\bar{K})_0[2]$ corresponds to $\{\pm 1\}^g$ sitting inside $\mathcal{O}_{\bar{K}}^{\times g} \subseteq \bar{K}^{\times g}$ and is fixed by the action of $\mathrm{Gal}(\bar{K}/K)$. In particular, $A(\bar{K})_0[2] = A(K)_0[2]$. Suppose we have $|A(K)[2]| = 2^g$. Then we must necessarily have $A(K)[2] = A(\bar{K})_0[2]$. In particular, under this assumption, we have $A[\phi] \cap A(\bar{K})_0 = A(K)[\phi]$.

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