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ELLIPTIC CURVES WITH p -SELMER GROWTH FOR ALL p

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ABSTRACT. It is known, that for every elliptic curve over \mathbb{Q} there exists a quadratic extension in which the rank does not go up. For a large class of elliptic curves, the same is known with the rank replaced by the size of the 2-Selmer group. We show, however, that there exists a large supply of semistable elliptic curves E/\mathbb{Q} whose 2-Selmer group grows in size in every bi-quadratic extension, and such that moreover, for any odd prime p , the size of the p -Selmer group grows in every D_{2p} -extension and every elementary abelian p -extension of rank at least 2. We provide a simple criterion for an elliptic curve over an arbitrary number field to exhibit this behaviour. We also discuss generalisations to other Galois groups.

1. INTRODUCTION

In [6] it is shown that there exist elliptic curves over number fields for which in every quadratic extension of the base field either the rank goes up or the Tate-Shafarevich group becomes infinite. Equivalently, every quadratic twist of such a curve has either positive rank or an infinite Tate-Shafarevich group (the latter is of course conjectured to never happen). Over \mathbb{Q} , such curves do not exist by the combined work of Kolyvagin [10] and Bump–Friedberg–Hoffstein–Murty–Murty–Waldspurger [3, 12, 16]. In fact, it is conjectured that half of all quadratic twists of an elliptic curve over \mathbb{Q} have rank 0. Moreover, if E/\mathbb{Q} has no cyclic 4-isogeny, then there exists a quadratic extension F/\mathbb{Q} such that the size of the 2-Selmer group of E over F is the same as over \mathbb{Q} [15, Theorem 1], [11, Theorem 1.5], [9, Theorems 1.1 and 1.3].

As we shall show, however, if we allow only slightly bigger extensions of \mathbb{Q} than quadratic, then there are lots of elliptic curves over \mathbb{Q} whose Selmer groups grow in size in all such extensions. Below, $S^p(E/F)$ will denote the p -Selmer group of E over a number field F , which we recall to be an elementary abelian p -group defined as the kernel of the map $H^1(F, E[p]) \rightarrow \prod_{\mathfrak{p}} H^1(F_{\mathfrak{p}}, E)$. Here, the product runs over all places of F , and each $\text{Gal}(\bar{F}_{\mathfrak{p}}/F_{\mathfrak{p}})$ is identified with a subgroup of $\text{Gal}(\bar{F}/F)$.

Theorem 1.1. *Let E/K be a semistable elliptic curve over a number field with positive rank. Let p and G be one of the following combinations of a prime number and a finite group:*

- (a) $p = 2$, $G \cong C_2 \times C_2$;
- (b) p is odd, $G \cong D_{2p}$, the dihedral group of order $2p$;
- (c) p is odd, and either $G \cong C_p \times C_p$ or $G \cong C_p \rtimes C_q$, where q is an odd prime, and C_q acts faithfully on C_p .

Suppose further that if p and G are as in (b), then the rank of E/K is greater than the number of primes of K at which E has non-split multiplicative reduction, while if p and G are as in (a), then the rank of E/K is greater

than the number of primes v of non-split multiplicative reduction for which $\text{ord}_v(\Delta(E))$ is even. Then, the following conclusions hold.

- (1) If $\text{III}(E/K)[p^\infty]$ is finite, then we have that for every Galois extension F/K with Galois group G , either the order of the p -primary part of the Tate-Shafarevich group changes at some step in F/K , or $E(F) \supsetneq E(K)$.
- (2) If $\text{III}(E/K)[p] = 0$, then for every Galois extension F/K with Galois group G , either $\#\text{III}(E/F)[p^\infty] > 0$, or $E(F) \supsetneq E(K)$.
- (3) If $\text{III}(E/K)[p] = 0$ and $E(K)[p] = 0$, then for every Galois extension F/K with Galois group G , we have $\#S^p(E/F) > \#S^p(E/K)$.

Example 1.2. The first few curves over \mathbb{Q} in Cremona's database that satisfy the hypotheses of Theorem 1.1 (1) and (2) for all the combinations of p and G listed there are 91b1, 91b2, 91b3, 123a1, 123a2, 141a1, 142a1, 155a1, all of rank 1 and with no primes of non-split multiplicative reduction, and with trivial Tate-Shafarevich groups [8]. Out of these, 91b3, 123a2, 141a1, 142a1 have trivial torsion subgroup over \mathbb{Q} and thus also satisfy the stronger hypothesis of Theorem 1.1 (3) for all the pairs p, G .

Also, the huge majority of rank 2 curves over \mathbb{Q} are expected to have trivial Tate-Shafarevich groups [4]. For example the curve with Cremona label 389a1 almost certainly satisfies all the hypotheses of Theorem (1.1) for all p, G . Unfortunately, we do not even know that the Tate-Shafarevich group is finite for a single rank 2 elliptic curve. In principle, triviality of the p -part for any fixed prime p can be checked algorithmically using descent (see [14] and the references therein). Such checks have been performed for thousands of higher rank curves, including 389a1, and thousands of primes in [13] (using methods quite different from descent).

Example 1.3. The following example illustrates that the above results are not a parity phenomenon. Let E be the curve with Cremona label 65a1. It has rank 1 over \mathbb{Q} and square-free discriminant, and $\text{III}(E/\mathbb{Q})$ is trivial, so the conclusion of Theorem 1.1 (2) holds for all combinations of p and G listed in the theorem, and in particular for $p = 2, G \cong C_2 \times C_2$. Let F be the bi-quadratic field $\mathbb{Q}(\sqrt{3}, \sqrt{5})$. Then we have $E(\mathbb{Q}) = E(F)$. However, $\text{III}(E/F)[2] \cong C_2 \times C_2$, as predicted by Theorem 1.1. This cannot be detected by root numbers.

We will collect the necessary ingredients of the proof in great generality, with no assumption on the Galois group of the extension F/K , although we will simplify the exposition by assuming early on that E is semistable. Then we will perform the necessary calculations in the case of dihedral and bi-quadratic extensions, thereby proving Theorem 1.1 for (a) and (b).

In the last section, we will discuss generalisations to other Galois groups, such as those of Theorem 1.1 (c). We will also explain why our approach cannot be pushed any further than that. This will rely on a certain representation theoretic classification [2, Corollary 9.2].

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2. TAMAGAWA NUMBERS AND REGULATORS

We begin with a simplifying reduction to a seemingly special case:

Lemma 2.1. *Let E/K be a semistable elliptic curve over a number field. Let the prime p and the finite group G be one of the combinations of Theorem 1.1. Theorem 1.1 holds for all extensions F/K with Galois group G if and only if it holds for all F/K with Galois group G that satisfy the following additional conditions:*

- (1) $\text{III}(E/F)[p^\infty]$ is finite (and consequently, by the inflation-restriction exact sequence, so are the p -primary parts of the Tate-Shafarevich groups over all subfields);
- (2) $E(K) \otimes \mathbb{Z}_p = E(F) \otimes \mathbb{Z}_p$ under the natural inclusion map.

Proof. If $\text{III}(E/K)[p^\infty]$ is infinite, then Theorem 1.1 is empty. Also, if $\text{III}(E/K)[p^\infty]$ is finite, then clearly the conclusions of Theorem 1.1 automatically hold for all F/K with Galois group G for which $\text{III}(E/F)[p^\infty]$ is infinite. This proves (1).

Finally, if $E(K) \otimes \mathbb{Z}_p \neq E(F) \otimes \mathbb{Z}_p$, then either $\text{rk}(E/F) > \text{rk}(E/K)$, or a point of infinite order on $E(K)$ becomes p -divisible over F , or $\#E(F)[p^\infty] > \#E(K)[p^\infty]$. In all three cases, the conclusions (1) and (2) of Theorem 1.1 obviously hold. As for (3), certainly if $\text{rk}(E/F) > \text{rk}(E/K)$ or if $\#E(F)[p^\infty] > \#E(K)[p^\infty]$, then the assumption $\text{III}(E/K)[p] = 0$ forces $\#S^p(E/F) > \#S^p(E/K)$. If on the other hand a point of infinite order on $E(K)$ becomes p -divisible, then by Kummer theory, $E(F)[p]$ must be non-trivial, so the assumption $E(K)[p] = 0$ again forces $\#S^p(E/F) > \#S^p(E/K)$. \square

We will therefore henceforth restrict our attention to semistable elliptic curves E/K and to Galois extensions F/K such that E/F satisfies the additional conditions of Lemma 2.1.

Definition 2.2. Let G be a finite group. A formal \mathbb{Z} -linear combination of representatives of conjugacy classes of subgroups $\Theta = \sum_H n_H H$ is called a *Brauer relation* if the virtual permutation representation $\bigoplus \mathbb{C}[G/H]^{\oplus n_H}$ is zero.

For a detailed discussion of the concept of Brauer relations, see the introduction to [2].

Let E/K be an elliptic curve over a number field, F/K a Galois extension with Galois group G , and $\Theta = \sum_H n_H H$ a Brauer relation. There is a corresponding relation between L -functions of E over the intermediate fields:

$$\prod_H L(E/F^H, s)^{n_H} = 1.$$

If E is semistable and has finite Tate-Shafarevich groups over all intermediate extensions of F/K , then a combination of various compatibility results on the Birch and Swinnerton-Dyer conjecture yields a relation¹ between

¹see [7, Theorem 2.3] and the remarks at the beginning of [7, §2.2], as well as [1, Remark 2.3]

arithmetic invariants of E over the intermediate fields. We shall recall the necessary notation shortly:

$$(2.3) \quad \prod_H \left(\frac{c(E/F^H) \# \text{III}(E/F^H) \text{Reg}(E/F^H)}{|E(F^H)_{\text{tors}}|^2} \right)^{n_H} = 1.$$

Moreover, if only the p -primary parts of the Tate-Shafarevich groups are assumed to be finite for some prime p , then the p -part of equation (2.3) holds.

Here, c denotes the product of Tamagawa numbers² of E over the finite places of the respective field.

Recall that the regulator of an elliptic curve is the determinant of the Néron-Tate height pairing evaluated on any basis of the free part of the Mordell-Weil group. Note that since each of the regulators is some real number, in general transcendental, it does not make any sense to speak of its p -part. However, since the quotient $\prod_H \text{Reg}(E/F^H)^{n_H}$ is a rational number (this is an immediate consequence of [5, Theorem 2.17]), it does make sense to speak of the p -parts of the regulator quotient and of the remaining terms.

The precise normalisation of the Néron-Tate height pairing that enters the Birch and Swinnerton-Dyer conjecture will be crucial for us. If M/K is a finite extension of fields, and if $\langle \cdot, \cdot \rangle_K$, respectively $\langle \cdot, \cdot \rangle_M$ denotes the Néron-Tate height pairing on $E(K)$, respectively on $E(M)$, then for any $P, Q \in E(K)$, $\langle P, Q \rangle_M = [M : K] \langle P, Q \rangle_K$. In particular, if E/K does not acquire any new points of infinite order over F , then the regulator quotient in (2.3) does not vanish in general, but rather

$$(2.4) \quad \prod_H \text{Reg}(E/F^H)^{n_H} = \prod_H \frac{1}{|H|^{n_H \text{rk} E(K)}}.$$

More generally, if $E(K) \otimes \mathbb{Z}_p = E(F) \otimes \mathbb{Z}_p$, then the p -part of equation (2.4) holds.

In summary, if E/K is a semistable elliptic curve with $\text{III}(E/F)[p^\infty]$ finite and if $E(K) \otimes \mathbb{Z}_p = E(F) \otimes \mathbb{Z}_p$, then

$$(2.5) \quad \prod_H \# \text{III}(E/F^H)^{n_H} c(E/F^H)^{n_H} =_{p'} \prod_H |H|^{n_H \text{rk} E(K)}.$$

Here, $=_{p'}$ means that both sides have the same p -adic valuation.

3. DIHEDRAL AND BI-QUADRATIC EXTENSIONS

3.1. Dihedral extensions. Suppose that $G \cong D_{2p}$, where p is an odd prime. There is a Brauer relation in G of the form $\Theta = 1 - 2C_2 - C_p + 2G$. For this relation, we have

$$\prod_H |H|^{n_H} = \frac{(2p)^2}{4p} = p,$$

so that the right hand side of equation (2.5) is $p^{\text{rk} E(K)}$. If v is a place of K , write $c_v(E/K)$ for the Tamagawa number at v , and $c_v(E/F^H)$ for the

²It is here that we use the assumption that E is semistable. Otherwise, the Tamagawa numbers have to be re-normalised. See [7, Theorem 2.3] for the general case.

product of Tamagawa numbers at places of F^H above v . Write $c_v(E/\Theta)$ for the quotient $\frac{c_v(E/K)^2 c_v(E/F)}{c_v(E/F^{C_2})^2 c_v(E/F^{C_p})}$. Similarly, write $\#\text{III}(E/\Theta)[p^\infty]$ for the p -primary part of the corresponding quotient of sizes of Tate-Shafarevich groups. Finally, let M denote the intermediate quadratic extension $M = F^{C_p}$. The following table gives the possible values of the p -part of $c_v(E/\Theta)$, depending on the reduction type of E at v (horizontal axis) and on the splitting behaviour of v in F/K (vertical axis):

redn. type of E splitting of v	split mult. over K	non-split mult. over F	non-split mult. over K , split mult. over M
splits into more than one prime	1	1	1
inert in M/K , ramified in F/M	$1/p$	—	p
totally ramified in F/K	$1/p$	1	—

It follows immediately from this table and from equation (2.5) that if E/K is semistable, if $\text{III}(E/F)[p^\infty]$ is finite, if $E(K) \otimes \mathbb{Z}_p = E(F) \otimes \mathbb{Z}_p$, and if the rank of E/K is greater than the number of primes of non-split multiplicative reduction, then $\#\text{III}(E/\Theta)[p^\infty] > 1$, and thus at least one of $\#\text{III}(E/K)[p^\infty]$, $\#\text{III}(E/F)[p^\infty]$ is strictly larger than at least one of $\#\text{III}(E/M)[p^\infty]$, $\#\text{III}(E/F^{C_2})[p^\infty]$. This, together with Lemma 2.1 concludes the proof of Theorem 1.1 for $G \cong D_{2p}$.

3.2. Bi-quadratic extensions. We now apply the same reasoning to $G \cong C_2 \times C_2$, with $p = 2$. Denote the three distinct subgroups of order 2 by C_2^a, C_2^b, C_2^c . The space of Brauer relations in G is generated by the relation $\Theta = 1 - C_2^a - C_2^b - C_2^c + 2G$, for which we have

$$\prod |H|^{n_H} = \frac{16}{8} = 2.$$

Again writing $c_v(E/\Theta)$ for the corresponding quotient of Tamagawa numbers of E at places above v over the corresponding intermediate fields of F/K , the following table gives the possible values of $c_v(E/\Theta)$:

redn. type of E splitting of v	split mult. over K	non-split mult. over F	non-split mult. over K , split mult. over some F^{C_2}
splits in some F^{C_2}	1	1	1
inert in some F^{C_2}/K , ramified in F/F^{C_2}	$1/2$	—	2: $\text{ord}_v(\Delta(E))$ is even 1/2: otherwise
totally ramified in F/K	$1/2$	1: $\text{ord}_v \Delta(E)$ even 1/4: otherwise	—

As above, this table together with equation (2.5) and with Lemma 2.1 proves Theorem 1.1 for $G \cong C_2 \times C_2$.

4. GENERALISATION TO OTHER GALOIS GROUPS

If G is a subgroup of a group \tilde{G} , then by transitivity of induction, a Brauer relation Θ in G automatically gives a Brauer relation $\text{Ind}_{\tilde{G}/G} \Theta$ in

\tilde{G} . Also, if G is a quotient of a group Γ , $G = \Gamma/N$, then a Brauer relation $\Theta = \sum_H n_H H$ in G gives rise to a Brauer relation $\text{Inf}_{\Gamma/N} \Theta = \sum_H n_H N H$ in Γ .

In general, in order to prove by the same technique as above that the size of the p -Selmer group of some elliptic curve grows in all Galois extensions with Galois group G , we need to have a Brauer relation $\Theta = \sum_H n_H H$ in G such that $\text{ord}_p(\prod |H|^{n_H}) \neq 0$. This quantity is clearly invariant under inductions and lifts of Brauer relations.

Proposition 4.1 ([2], Corollary 9.2). *Let p be a prime number. A finite group \tilde{G} has a Brauer relation $\Theta = \sum_H n_H H$ with $\text{ord}_p(\mathcal{C}_\Theta(\mathbf{1})) \neq 0$ if and only if \tilde{G} has a subquotient G isomorphic either to $C_p \times C_p$ or to $C_p \rtimes C_q$ with C_q cyclic of prime order acting faithfully on C_p . Moreover, in the former case Θ can be taken to be induced and/or lifted from the relation $1 - \sum_{U \leq_p G} U + pG$; while in the latter case Θ can be taken to be induced and/or lifted from the relation $1 - qC_q - C_p + qG$.*

Suppose that \tilde{G} is a finite group that has a quotient isomorphic to G . If some inequalities between the rank of an elliptic curve and the number of places of certain reduction types ensure that sizes of Tate-Shafarevich groups or Mordell-Weil groups grow in all G -extensions of number fields, as in Theorem 1.1, then clearly the same inequalities imply the same conclusions for \tilde{G} -extensions.

On the other hand, suppose that G is a subgroup of a finite group \tilde{G} . Suppose that some condition on the rank of E and on the number of places of given reduction types implies the conclusions of Theorem 1.1 (1), say, for all Galois extensions with Galois group G . Some care is then needed to deduce the same result for \tilde{G} -extensions. Indeed, if F/K is Galois with Galois group \tilde{G} , then one would like to deduce the conclusion of Theorem 1.1 (1) for F/K by applying the theorem to the extension F/F^G . But in order to satisfy the required inequalities for the number of places of F^G , one might need to impose stronger conditions on E/K , since the number of places of given reduction type might grow in F^G/K (by at most a factor of $[\tilde{G} : G]$). This is of course a straightforward modification.

Thus, we may restrict attention to the groups $C_p \times C_p$ and $C_p \rtimes C_q$ as in Proposition 4.1.

When $G \cong C_p \times C_p$ and $\Theta = 1 - \sum_{U \leq_p G} U + pG$, we have

$$\prod |H|^{n_H} = \frac{(p)^{2p}}{p^{p+1}} = p^{p-1}.$$

When $G \cong C_p \rtimes C_q$ and $\Theta = 1 - qC_q - C_p + qG$, we have

$$\prod |H|^{n_H} = \frac{(pq)^q}{pq^q} = p^{q-1}.$$

Having already dealt with such groups of even order, we may now restrict our attention to groups of odd order, so that only the primes of K at which E has split multiplicative reduction contribute to the p -part of the corresponding Tamagawa number quotients.

Here are the possible values of $c_v(E/\Theta)$ when E/K has split multiplicative reduction at v , first for $G \cong C_p \times C_p$ and $\Theta = 1 - \sum_{U \leq_p G} U + pG$, and then

for $G \cong C_p \times C_q$ and $\Theta = 1 - qC_q - C_p + qG$, where p and q are odd primes:

v splits	v is inert in some F^{C_p}/K and ramified in F/F^{C_p}	v is totally ramified
1	p^{1-p}	p^{1-p}

v splits	v is inert in F^{C_p}/K and ramified in F/F^{C_p}	v is totally ramified
1	p^{1-q}	p^{1-q}

These tables together with equation (2.5) finish the proof of Theorem 1.1 for $G \cong C_p \times C_p$ and $G \cong C_p \times C_q$.

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