Wary of the worst: Maximizing award guarantees when new claimants may arrive

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Abstract

When rationing a resource or adjudicating conflicting claims, the arrival of new agents necessitates revision. Adopting a worst-case perspective, we introduce guarantee structures to measure the protection a rule provides to either individuals or groups in these circumstances. With the goal of maximizing guarantees for those in the original group, we characterize the constrained equal awards rule. Requiring that a rule provide protection for both the original and arriving agents, so that both gains and losses are shared, we characterize the Talmud rule.

Keywords: Claims problem, guarantee structures, worst-case analysis, Talmud rule, constrained equal awards rule.

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1 Introduction

Facing pervasive scarcity, we often find several agents with incompatible claims to a resource and must resort to rationing.[1] Whether dividing an estate, liquidating a firm, or allocating funds among researchers,[2] we seek methods with desirable properties. The problem of adjudication, already challenging when all relevant information is known, is compounded in a fluid and dynamic environment. Changes in the environment – whether the amount to divide, demands of the claimants, or even the population of claimants themselves – can be expected. Whether a division method responds favorably to such changes becomes an important consideration. Pursuing this line of inquiry, we investigate how rules respond to the arrival of new agents with valid claims.

To evaluate rules, we measure the extent to which they may revise original awards to accommodate new agents. Rules which limit this revision are able to guarantee that agents retain at least some portion of their original award, thereby providing a valuable sense of protection to the original agents. In more detail, fixing the numbers of original and arriving agents, we compare the awards proposed for the original agents before and after the arrival of the new agents. A *guarantee* measures the smallest fraction that a single agent may receive of her tentative award in the augmented problem and a *group guarantee* identifies the smallest average of the individual award ratios.[3] For example, a guarantee of one-third ensures that each original agent retains at least one-third of her original award, regardless of the claims of the arriving agents.

Just as agents may legitimately object to the expropriation of their entire award to compensate arriving agents, so too may they complain when forced to bear all of the losses when new agents arrive. Guarantees respond to the first objection; to respond to the second, we introduce *dual guarantees*. Dual guarantees apply the ideas underpinning guarantees to losses rather than awards, thereby ensuring that arriving agents share the required sacrifice.

So far, we have defined guarantees for fixed group sizes, whereas we seek rules that perform well for all populations. Taking this into account, the *guarantee structure* of a rule parameterizes guarantees (or group guarantees) by the numbers of original and arriving agents. Rather than summarize the performance of a rule by a single number, the typical approach to worst case analysis, guarantee structures offer a comprehensive description of performance across settings. Consequently, guarantee structures permit both nuanced comparisons between rules and strong recommendations when these comparisons are uniformly favorable or unfavorable to a rule.

Given their richness, we might suspect that guarantee structures preclude unambiguous com-

[1] The literature on claims (or bankruptcy) problems begins with O'Neill (1982). For a thorough analysis of the existing literature, see Thomson (2003), Thomson (2014), and Thomson (2015a).

[2] Rationing is also required in the commonly studied school assignment problem (e.g., Abdulkadiroğlu and Sönmez (2003), Abdulkadiroğlu et al. (2006), and Abdulkadiroğlu et al. (2009)). When lotteries are allowed, seats are effectively divisible and techniques developed for adjudicating conflicting claims can be applied.

[3] If the original award is zero, the fraction is undefined. For concreteness, we set $\frac{0}{0} = 1$ so that an agent who receives a zero award in both cases has a full guarantee, although our results do not depend on this choice.

parisons. In fact, we find that guarantee structures not only permit comparisons, but identify some rules as uniformly best. As a preliminary result, we establish bounds on guarantees (Lemma 2) and compute the guarantees of rules in a one-parameter family[4] among which these bounds are attained (Proposition 1). Our main results characterize two well-known rules. First, among *endowment continuous* and *consistent* rules, the constrained equal awards rule uniquely maximizes group guarantees (Theorem 1).[5] Thus, maximizing group guarantees essentially requires a rule to assign awards as equally as possible among claimants. Our second result characterizes the Talmud rule,[6] which aims to equal awards for small endowments but to equalize losses for large endowments. Among *consistent* rules, the Talmud rule is the only rule to simultaneously maximize guarantees and dual guarantees (Theorem 2). The Talmud rule thus arises as the most robust to revision when new agents arrive. Taken together, these results help to relate guarantee structures to equity and robustness properties appearing in other characterizations of these rules.[7]

Testing the robustness of our approach, we also consider collective guarantees, which measure group protection by the total award received by the group. This leads to a measure generally equivalent to guarantees (Lemma 1). Furthermore, we show that maximizing guarantees and group guarantees is generally equivalent to meeting focal lower bounds (Proposition 2).[8]

Deferring further discussion of related literature, we formalize the model and introduce guarantee structures in Section 2. Section 3 presents our characterizations and Section 4 discusses extensions and related literature. Proofs appear in the appendix.

2 Model

A claims problem consists of a finite set of agents with conflicting claims over an amount to divide. Formally, there is a countably infinite set of potential agents indexed by $\mathbb{N}$.[9] For each $N \in \mathcal{N}$, a **claims problem for $\boldsymbol{N}$** is a pair $\boldsymbol{(c, E)} \in \mathbb{R}^N_+ \times \mathbb{R}_+$ such that $E \leq \sum_{i \in N} c_i$ where c_i is agent i's **claim** and E is the **endowment**.[10] The set of all claims problems for N is $\boldsymbol{\mathcal{C}^N} \equiv \{(c, E) \in \mathbb{R}^N_+ \times \mathbb{R}_+ : E \leq \sum_{i \in N} c_i\}$. A **rule** is a mapping $\boldsymbol{\varphi}$ defined on $\bigcup_{N \in \mathcal{N}} \mathcal{C}^N$ such that for each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$, $0 \leq \varphi(c, E) \leq c$ and $\sum_{i \in N} \varphi_i(c, E) = E$. Finally, by convention, let $\frac{0}{0} = 1$

[4] This is the "TAL-family" introduced by Moreno-Ternero and Villar (2006b).

[5] *Endowment continuity* requires that small changes in the endowment lead to small changes in awards; *consistency* requires that awards remain the same when some agents depart with their awards and the situation is reassessed.

[6] The Talmud rule first appears in Aumann and Maschler (1985). Dagan (1996) provides several characterizations of the Talmud and constrained equal awards rules by standard properties.

[7] For example, see Dagan (1996), Aumann and Maschler (1985), and Moreno-Ternero and Villar (2006b).

[8] Our formal results adapt the properties introduced by Herrero and Villar (2001) and Moreno-Ternero and Villar (2004).

[9] By $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}_+$, and $\mathbb{R}_{++}$, we denote respectively the natural (excluding zero), real, non-negative real, and positive real numbers. Also, $\mathcal{N}$ denotes the finite subsets of $\mathbb{N}$.

[10] For vector $x \in \mathbb{R}^N_+$, $x_{N'} \equiv (x_i)_{i \in N'}$ denotes the components of x corresponding to N'. With slight abuse of notation, we write x_i for $x_{\{i\}}$ and x_{-i} for $x_{N \setminus \{i\}}$. For vectors $x, y \in \mathbb{R}^N_+$, we write $x \ll y$, $x < y$, and $x \leq y$ for standard inequalities.

and for each $\delta \in \mathbb{R}_{++}$, let $\frac{\delta}{0} = \infty$.

2.1 Properties and rules

Several standard properties play a role in our analysis. First, *equal treatment of equals* says that agents with the same claim receive the same award; *endowment continuity* requires that small changes in the endowment lead to at most small revisions in awards[11]; and *endowment monotonicity* requires that when the endowment decreases, no agent's award increases. Finally, *consistency* says that when one agent departs with her award, then the awards of the remaining agents should be unchanged when the rule is reapplied to distribute the remaining endowment among those agents.[12] Given φ, the formal requirements, stated for each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$, are:

Equal treatment of equals: For each pair $i, j \in N$, if $c_i = c_j$, then $\varphi_i(c, E) = \varphi_j(c, E)$.

Endowment continuity: For each sequence $\{E^\nu\}_{\nu \in \mathbb{N}} \subseteq \mathbb{R}_+$ such that for each $\nu \in \mathbb{N}$, $E^\nu \in [0, \sum_{i \in N} c_i]$, if $E^\nu \to E$,[13] then $\varphi(c, E^\nu) \to \varphi(c, E)$.

Endowment monotonicity: For each $E' \in \mathbb{R}_+$, if $E' \leq E$, then $\varphi(c, E') \leq \varphi(c, E)$.

Consistency: For each $M \subseteq N$, $\varphi_M(c, E) = \varphi\big(c_M, E - \sum_{i \in N \setminus M} \varphi_i(c, E)\big)$.

We now turn to rules. Those arising from our analysis are members of a one-parameter family. Each rule in the family begins by equally dividing awards until a pre-determined fraction of each claim is filled and then switches to equally dividing losses.[14]

TAL-family rule with respect to $\boldsymbol{\theta \in [0, 1]}$, $\boldsymbol{T^\theta}$: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $i \in N$,

$$T_i^\theta(c, E) \equiv \begin{cases} \min\{\theta c_i, \delta\} & \text{if } E \leq \theta \cdot \sum_{i \in N} c_i \\ \max\{\theta c_i, c_i - \delta\} & \text{otherwise} \end{cases}$$

where $\delta \in \mathbb{R}$ is chosen so that $\sum_{i \in N} T_i^\theta(c, E) = E$.

All rules in the family satisfy *equal treatment of equals*, *endowment continuity*, *endowment monotonicity*, and *consistency*. Within this family, we distinguish three members: T^1 is the **constrained**

[11] This property is sometimes called *endowment endowment continuity* to distinguish it from a stronger property which requires the same conclusion for joint changes in the claims and endowment. As we only consider this version, we adopt the shorter name.

[12] See Thomson (2012) for a thorough normative analysis of the consistency principle and Thomson (2015b) for a survey on its applications.

[13] We write $\varphi(c, E^\nu) \to \varphi(c, E)$ fi for each $\epsilon \in \mathbb{R}_{++}$, there is $\nu_0 \in \mathbb{N}$ such that for all $\nu \in \mathbb{N}$ such that $\nu \geq \nu_0$, $\sum_{i \in N} |\varphi_i(c, E^\nu) - \varphi_i(c, E)| < \epsilon$.

[14] This family, studied by Moreno-Ternero and Villar (2006b), generalizes a rule originating from the Talmud (Aumann and Maschler, 1985). It comprises the *consistent* and *homogeneous* members of a further generalized family introduced by Thomson (2008). This in turn is a subfamily of the class of (fully) *continuous* and *consistent* rules that satisfy *equal treatment of equals* (Young, 1987).

equal awards (CEA) rule, $T^{1/2}$ is the **Talmud (T) rule**, and T^0 is the **constrained equal losses (CEL) rule**.

Each rule may also be applied to distribute losses instead of awards, which defines its dual. Formally, for each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$, the **dual of $\boldsymbol{\varphi}$**, φ^d, allocates losses according to φ: $\varphi^d(c, E) \equiv c - \varphi\big(c, \sum_{i \in N} c_i - E\big)$.[15] Within the TAL-family, for each $\theta \in [0, 1]$, T^θ and $T^{1-\theta}$ are dual (Proposition 1, Moreno-Ternero and Villar (2006b)). Thus, CEA and CEL are dual rules and T is its own dual, or **self dual**.

2.2 Guarantee structures

In general, adding agents to a problem without increasing the available resources decreases the awards of original agents. Our goal is to bound the "worst case" revision that may be required for original agents considered individually or as a group. To fix the context, let φ be a rule and $n, m \in \mathbb{N}$ represent the numbers of original and arriving agents. Comparing awards before and after including the new agents, we compute the ratio of awards for each original agent.[16] The **guarantee of $\boldsymbol{\varphi}$ for $\boldsymbol{(n, m)}$** is then the smallest (infimum) ratio over all problems involving the specified numbers of agents. Guarantees measure the extent to which a single agent's award may be revised.

In addition to this individualistic perspective, we are interested in the revisions required of the original agents considered as a group. For our primary measure, we compute the ratio of awards for each original agent in a pair of problems as before and then average these ratios. Again searching over all problems, the **group guarantee of $\boldsymbol{\varphi}$ for $\boldsymbol{(n, m)}$** is the smallest (infimum) average ratio. This perspective gives each agent's proportional sacrifice equal weight, judging a given absolute sacrifice to be more significant when it represents a larger share of an agent's award. An alternative measure instead equally weights the absolute sacrifices, summing the awards of the original agents in each problem and then computing the ratio of these amounts. The **collective guarantee of $\boldsymbol{\varphi}$ for $\boldsymbol{(n, m)}$** is the smallest (infimum) average ratio.[17] Formally,[18] we define

[15] Aumann and Maschler (1985) introduce duality to the study of claims problems.

[16] Absolute differences also measure the extent of revision required. For an agent with a particular award, absolute and proportional measures provide the same information. In general, however, absolute revisions increase with the size of claims, so worst-case analysis requires some form of normalization. To the extent that absolute differences are deemed the appropriate measures of revision, proportional revisions can be viewed as their normalizations.

[17] Both guarantees and group guarantees have been applied to bargaining solutions (Thomson and Lensberg, 1983; Thomson, 1983a) while collective guarantees have not. While adding awards is natural in our environment, the corresponding summation is would be questionable in bargaining problems where it would involve addition of utilities across agents and be susceptible to re-scaling.

[18] Some agents may receive zero awards. To interpret these cases, recall our convention that $\frac{0}{0} = 1$ and for each $a \in \mathbb{R}_{++}$, $\frac{a}{0} = \infty$.

Guarantee of φ for (n,m), $\mathcal{G}(\varphi,n,m)$:

$$\mathcal{G}(\varphi,n,m) \equiv \inf \left\{ \frac{\varphi_i((c,\hat{c}),E)}{\varphi_i(c,E)} : \begin{array}{c} N,M \in \mathcal{N},\ N \cap M = \emptyset,\ |N| = n,\ |M| = m, \\ i \in N,\ (c,E) \in \mathcal{C}^N,\ \hat{c} \in \mathbb{R}_+^M \end{array} \right\}.$$

Group guarantee of φ for (n,m), $\bar{\mathcal{G}}(\varphi,n,m)$:

$$\bar{\mathcal{G}}(\varphi,n,m) \equiv \inf \left\{ \frac{1}{n} \sum_{i \in N} \frac{\varphi_i((c,\hat{c}),E)}{\varphi_i(c,E)} : \begin{array}{c} N,M \in \mathcal{N},\ N \cap M = \emptyset,\ |N| = n,\ |M| = m, \\ (c,E) \in \mathcal{C}^N,\ \hat{c} \in \mathbb{R}_+^M \end{array} \right\}.$$

Collective guarantee of φ for (n,m), $\hat{\mathcal{G}}(\varphi,n,m)$:

$$\hat{\mathcal{G}}(\varphi,n,m) \equiv \inf \left\{ \frac{1}{E} \sum_{N} \varphi_i(c_{N\cup M},E) : \begin{array}{c} N,M \in \mathcal{N},\ N \cap M = \emptyset,\ |N| = n,\ |M| = m, \\ (c_N,E) \in \mathcal{C}^N,\ c_M \in \mathbb{R}_+^M \end{array} \right\}.$$

Gathering values for all population sizes defines a guarantee structure. We call $\boldsymbol{\mathcal{G}^\varphi} \equiv (\mathcal{G}(\varphi,n,m))_{n,m\in\mathbb{N}}$, $\boldsymbol{\bar{\mathcal{G}}^\varphi} \equiv (\bar{\mathcal{G}}(\varphi,n,m))_{n,m\in\mathbb{N}}$, and $\boldsymbol{\hat{\mathcal{G}}^\varphi} \equiv (\hat{\mathcal{G}}(\varphi,n,m))_{n,m\in\mathbb{N}}$ respectively the **guarantee structure**, **group guarantee structure**, and **collective guarantee structure** of φ. Of course, when the original group consists of a single agent, these measures coincide. More interestingly, collective guarantees are generally equivalent to guarantees.

Lemma 1. *Let φ be an* endowment monotonic *and* consistent *rule. Then for each pair $n,m \in \mathbb{N}$, $\mathcal{G}(\varphi,n,m) = \hat{\mathcal{G}}(\varphi,n,m)$.*

Intuitively, these measures coincide because the "worst cases"occur when a single agent in the original group has a non-trivial claim.

Further methods of comparison might consider the profiles of individual ratios directly. Proceeding lexicographically beginning with the smallest ratios, the goal of maximizing the minimum ratios would refine the comparisons made according to our guarantees. Beginning instead with the largest ratios defines an approach which deems worse situations in which even the most protected agent's ratio is small. Comparing rules in this way refines comparisons according to our group guarantees. Thus, while our focus remains on guarantees and group guarantees, our conclusions also inform reasonable alternatives.

Finally, we introduce the **dual guarantees** and **dual group guarantees** of φ, the guarantees and group guarantees of φ^d, which shift the focus from awards to losses. To interpret, we rewrite the ratio in the definition of a guarantee for φ^d in terms of φ:

$$\frac{\varphi_i^d((c,\hat{c}),E)}{\varphi_i^d(c,E)} = \frac{c_i - \varphi_i((c,\hat{c}), \sum_{i\in N\cup M} c_i - E)}{c_i - \varphi_i(c, \sum_{i\in N} c_i - E)}.$$

Moving from $(c, \sum_{i \in N} c_i - E)$ to $\big((c, \hat{c}), \sum_{i \in N \cup M} c_i - E\big)$ postulates the arrival of additional agents together with sufficient additional funds to cover their entire claims. If the ratio is positive, then the original agent incurs at least some of the loss in the augmented problem; a positive dual guarantee says that the original agents always share in the sacrifice in this case. Normatively, the dual guarantee ensures that the arriving agents are protected. Focusing on the case where $E = 0$ and reversing the roles of the original arriving agents, the situation now matches the hypothetical considered by a guarantee.[19] Summarizing, dual guarantees protect agents against incurring all of the additional losses created when new agents arrive.

3 Results

3.1 Bounds on guarantees and the TAL-family

Our goal is to identify those rules which provide the largest guarantees. Our first result establishes general upper bounds on guarantees and group guarantees. From the definitions, the bounds apply to the dual notions as well.

Lemma 2. *For rule φ and each pair $n, m \in \mathbb{N}$, $\mathcal{G}(\varphi, n, m) \leq \frac{1}{1+m}$ and $\bar{\mathcal{G}}(\varphi, n, m) \leq \frac{n}{n+m}$.*

Proof. Let φ be a rule, $n, m \in \mathbb{N}$, $E \in \mathbb{R}_{++}$, and $N, M \in \mathcal{N}$ with $N \cap M = \emptyset$, $|N| = n$, and $|M| = m$. Let $c \in \mathbb{R}^N_+$ and $\hat{c} \in \mathbb{R}^M_+$ denote claims for the two groups to be specified subsequently. We consider two cases according to whether φ treats agents symmetrically.

Case 1: φ satisfies equal treatment of equals. To establish the first bound, let $c \equiv (0, \ldots, 0)$ and $\hat{c} \equiv (E, \ldots, E)$. By feasibility and *equal treatment of equals*,

$$\begin{aligned} \varphi(c, E) &= (0, \ldots, 0, E) \\ \varphi((c, \hat{c}), E) &= \left(0, \ldots, 0, \tfrac{E}{1+m}, \tfrac{E}{1+m}, \ldots, \tfrac{E}{1+m}\right). \end{aligned}$$

Moving from the smaller to larger economy, the ratio of awards for agent n is

$$\frac{E/(1+m)}{E} = \frac{1}{1+m}.$$

Therefore, $\mathcal{G}(\varphi, n, m) \leq \frac{1}{1+m}$.

To establish the second bound, let $c \equiv (E, \ldots, E)$ and $\hat{c} \equiv (E, \ldots, E)$. By *equal treatment of equals*, $\varphi(c, E) = \left(\frac{E}{n}, \ldots, \frac{E}{n}\right)$ and $\varphi((c, \hat{c}), E) = \left(\frac{E}{n+m}, \ldots, \frac{E}{n+m}\right)$. For each $i \in N$,

$$\frac{\varphi_i((c, \hat{c}), E)}{\varphi_i(c, E)} = \frac{E/(n+m)}{E/n} = \frac{n}{n+m}.$$

[19] Thus, for each pair $n, m \in \mathbb{N}$, it is appropriate to pair $\mathcal{G}(\varphi, n, m)$ with $\mathcal{G}(\varphi^d, m, n)$. Since our goal will be to maximize all guarantees, the interchange of group sizes does not bear on the interpretation of our results.

Therefore, $\bar{\mathcal{G}}(\varphi, n, m) \leq \frac{n}{n+m}$.

Case 2: General rules. To establish the first bound, let $i \in N$ and

$$\mathcal{G}_i^* \equiv \inf \left\{ \frac{\varphi_i((c,\hat{c}),E)}{\varphi_i(c,E)} : (c,E) \in \mathcal{C}^N,\ \hat{c} \in \mathbb{R}_+^M \right\}.$$

Suppose that $\mathcal{G}_i^* > \frac{1}{1+m}$ and $c_i \equiv E$, $c_{N\setminus\{i\}} \equiv (0,\ldots,0)$, and $\hat{c} \equiv (E,\ldots,E)$.

By feasibility, $\varphi_i(c,E) = E$, so $\varphi_i((c,\hat{c}),E) \geq \mathcal{G}_i^* \cdot \varphi_i(c,E) > \frac{E}{1+m}$. Then there is $j \in M$ such that $\varphi_j((c,\hat{c}),E) < \frac{E}{1+m}$. Let $N' \equiv (N\setminus\{i\}) \cup \{j\}$, $M' \equiv (M\setminus\{j\}) \cup \{i\}$, $c' \equiv (c_j)_{j\in N'}$, and $\hat{c}' \equiv (c_j)_{j \in M'}$. Then $|N'| = n$ and $|M'| = m$. By feasibility, $\varphi_j(c_{N'},E) = E$ and $\varphi_j(c_{N'\cup M'},E) = \varphi_j((c,\hat{c}),E)$. Then

$$\mathcal{G}(\varphi,n,m) \leq \frac{\varphi_j((c',\hat{c}'),E)}{\varphi_j(c',E)} = \frac{\varphi_j((c,\hat{c}),E)}{E} < \frac{E/(1+m)}{E} = \frac{1}{1+m}.$$

To establish the second bound, for each $i \in N \cup M$, let $c_i \equiv \frac{E}{n}$, and $x \equiv \varphi((c,\hat{c}),E)$. Label the agents in increasing order according to x, breaking ties arbitrarily. Let $N' \subseteq N \cup M$ consist of n agents with the smallest awards, $M' \equiv (N \cup M)\setminus N'$, $c' \equiv (c_j)_{j\in N'}$, and $\hat{c}' \equiv (c_j)_{j\in M'}$. If $x_1 = x_{n+m}$, then the arguments from Case 1 apply, so suppose instead that $x_1 < x_{n+m}$.

By feasibility, for each $i \in N'$, $\varphi_i(c',E) = \frac{E}{n}$, so

$$\bar{\mathcal{G}}(\varphi,n,m) \leq \frac{1}{n}\sum_{i\in N'} \frac{\varphi_i((c',\hat{c}'),E)}{\varphi_i(c',E)} = \frac{1}{n}\sum_{i\in N'}\frac{x_i}{E/n} = \frac{1}{E}\sum_{i\in N'} x_i < \frac{nE/(n+m)}{E} = \frac{n}{n+m}.$$

□

In the proof, we first identify problems in which the bounds bind for rules that satisfy *equal treatment of equals*. The second part of the argument illustrates that asymmetric treatment entails a trade-off: Improving some of the ratios comes at the expense of other ratios, ultimately decreasing the smallest ratios and the guarantees of the rule.

We now search for rules which attain these bounds. To this end, we compute the guarantees and group guarantees of rules in the TAL-family and thereby show that the bounds are attained.

Proposition 1. *For each pair $n, m \in \mathbb{N}$ with $n \geq 2$, and each $\theta \in [0,1]$, $\mathcal{G}(T^\theta,n,m) = \min\{\theta, \frac{1}{1+m}\}$ and $\bar{\mathcal{G}}(T^\theta,n,m) = \min\{\theta, \frac{n}{n+m}\}$.*

Proof. Let $\theta \in [0,1]$, $n,m \in \mathbb{N}$, and $N, M \in \mathcal{N}$ with $N \cap M = \emptyset$, $|N| = n$, and $|M| = m$.

Upper bounds: $\mathcal{G}(T^\theta,n,m) \leq \min\{\theta, \frac{1}{1+m}\}$ and $\bar{\mathcal{G}}(T^\theta,n,m) = \min\{\theta,\frac{n}{n+m}\}$. By Lemma 2, $\mathcal{G}(T^\theta,n,m) \leq \frac{1}{1+m}$ and $\bar{\mathcal{G}}(T^\theta,n,m) \leq \frac{n}{n+m}$. To establish the remaining bounds, let $d \in \mathbb{R}_{++}$, $c \equiv (\delta,\ldots,\delta)$, $\hat{c} \equiv ((n+1)\delta, 0, \ldots, 0)$, and $E \equiv n\delta$. Then $T^\theta(c,E) = (\delta,\ldots,\delta)$. Because $\delta - \theta\delta \leq (n+1)\delta - n\delta - n\theta\delta$, each original agent will receive at most $\theta\delta$ in the augmented economy.

First consider guarantees and suppose $\theta \leq \frac{1}{1+m}$. Since $\theta \leq \frac{1}{1+m} \leq \frac{1}{2}$, $E = n\delta \geq (n+1)\theta\delta$ and so $E - (n+1)\theta\delta \geq \theta\delta$. Therefore,

$$T^{\theta}(C_{N\cup M}, E) = (\theta\delta, \ldots, \theta\delta, E - n\theta\delta, 0, \ldots, 0).$$

For each $i \in N$, the ratio of awards is $\frac{\theta\delta}{\delta} = \theta$. Thus, $\mathcal{G}(T^{\theta}, n, m) \leq \theta$.

Next consider group guarantees. We have

$$T^{\theta}(C_{N\cup M}, E) = \left(\theta\delta, \ldots, \theta\delta, \tfrac{E-n\theta\delta}{m}, \ldots, \tfrac{E-n\theta\delta}{m}\right).$$

For each $i \in N$, the ratio of awards is $\frac{\theta\delta}{\delta} = \theta$. Thus, $\bar{\mathcal{G}}(T^{\theta}, n, m) \leq \theta$.

Lower bounds: $\boldsymbol{\mathcal{G}(T^{\theta}, n, m) \geq \min\{\theta, \frac{1}{1+m}\}}$ and $\boldsymbol{\bar{\mathcal{G}}(T^{\theta}, n, m) \geq \min\{\theta, \frac{n}{n+m}\}}$. Let $(c, E) \in \mathcal{C}^N$, $\hat{c} \in \mathbb{R}^M_+$, $x \equiv T^{\theta}(c, E)$, and $x' \equiv T^{\theta}((c, \hat{c}), E)$. Without loss of generality, label the agents so that $c_1 \leq \cdots \leq c_n$. We distinguish three cases.

Case 1: $E \leq \sum_{i\in N} \theta c_i$. Then $x = CEA(\theta c, E)$, $x' = CEA(\theta(c, \hat{c}), E)$, and the bounds for the constrained equal awards rule apply: For each $i \in N$, $\frac{x'_i}{x_i} \geq \frac{1}{1+m}$.

Case 2: $E \geq \sum_{i\in N\cup M} \theta c_i$. Then for each $i \in N$, $x'_i \geq \theta c_i$. Since $x_i \leq c_i$, this implies $\frac{x'_i}{x_i} \geq \frac{\theta c_i}{c_i} = \theta$.

Case 3: $\sum_{i\in N} \theta c_i < E < \sum_{i\in N\cup M} \theta c_i$. Then $x' = CEA(\theta(c, \hat{c}), E)$ and there are $\delta \in \mathbb{R}_{++}$ and $k \in \mathbb{N}$ such that

$$x_M = (\theta c_1, \ldots, \theta c_k, \delta, \ldots, \delta).$$

Moreover, $\delta = \max\{x'_i : i \in N \cup M\}$. Let $i \in N$. If $i \leq k$, then $\frac{x'_i}{x_i} \geq \frac{\theta c_i}{c_i} = \theta$. If $i > k$, then $x_i - x'_i \leq \sum_{j\in M} x'_j \leq m\delta$, so $x_i \leq m\theta + x'_i = (1+m)\theta$. Then $\frac{x'_i}{x_i} \geq \frac{\delta}{(1+m)\delta} = \frac{1}{1+m}$. Altogether, $\mathcal{G}(T^{\theta}, n, m) \geq \min\{\theta, \frac{1}{1+m}\}$.

For each $i \in \{1, \ldots, k\}$, $\frac{x'_i}{x_i} \geq \frac{\theta c_i}{c_i} = \theta$. The average ratio among the agents $\{k+1, \ldots, n\}$ is smallest when the losses are spread equally and $k = 0$. In this case, for each $i \in \{k+1, \ldots, n\}$, $x_i \leq \frac{E}{n}$ and $x'_i \geq \frac{E}{n+m}$ so

$$\frac{x'_i}{x_i} \geq \frac{E/(n+m)}{E/n} = \frac{n}{n+m}.$$

The group guarantee is at least $\frac{k}{n} \cdot \theta + \frac{n-k}{n} \cdot \frac{n}{n+m} \geq \min\left\{\theta, \frac{n}{n+m}\right\}$. Therefore, $\bar{\mathcal{G}}(T^{\theta}, n, m) \geq \min\left\{\theta, \frac{n}{n+m}\right\}$. □

Since for each $\theta \in [0,1]$, T^{θ} and $T^{1-\theta}$ are dual, the results also describe the dual guarantees and dual group guarantees: For each pair $n, m \in \mathbb{N}$, $\mathcal{G}((T^{\theta})^d, n, m) = \min\{1-\theta, \frac{1}{1+m}\}$ and $\bar{\mathcal{G}}((T^{\theta})^d, n, m) = \min\{1-\theta, \frac{n}{n+m}\}$.

Comparing with the bounds in Lemma 2, we see that several rules in the family maximize guarantees. This is true of all members for which $\theta \geq \frac{1}{2}$. On the other hand, since n and m may be chosen so that $\frac{n}{n+m}$ is arbitrarily close to 1, maximizing group guarantees requires $\theta = 1$ so that T^θ is the constrained equal awards rule. Simultaneously maximizing guarantees and dual guarantees is possible, though again achieved by a single member of the family, the Talmud rule itself. The constrained equal losses rule maximizes dual group guarantees and does so uniquely within the family; consequently, no rule in the family simultaneously maximizes group guarantees and dual group guarantees.

3.2 Characterizations

As our observations about the TAL-family suggest, the goal of maximizing notions of guarantees leads quickly to specific recommendations of rules. Beginning with a group perspective, we identify the constrained equal awards as the unique *endowment continuous* and *consistent* rule to maximize group guarantees.

Theorem 1. *The constrained equal awards rule is the unique* endowment continuous *and* consistent *rule to maximize group guarantees.*

Proof. The constrained equal awards rule maximizes group guarantees (Proposition 1) and is well known to satisfy the other properties (e.g., Young (1987)). To prove the converse, let φ be a *endowment continuous* and *consistent* rule that maximizes group guarantees.

Step 1: φ satisfies *equal treatment of equals.* Let $k, n \in \mathbb{N}$ with $2 \leq k < n$, $N \in \mathcal{N}$ with $|N| = n$, and $\delta \in \mathbb{R}_+$. Also let $i \in N$, $E \equiv k\delta$, $c^\delta \equiv (\delta)_{j \in N}$, and $x \equiv \varphi(c^\delta, E)$. For each $S \subseteq N$ with $i \in S$ and $|S| = k$, $\varphi(c^\delta_S, E) = (\delta)_{j \in S}$, so

$$\frac{k}{n} = \bar{\mathcal{G}}(\varphi, k, n-k) \leq \frac{1}{k} \sum_{j \in S} \frac{x_j}{\delta} = \frac{1}{E} \sum_{j \in S} x_j.$$

That is, $\sum_{j \in S} x_j \geq \frac{kE}{n}$. Now agent i is a member of $\binom{n-1}{k-1}$ such subgroups and each other agent is a member of $\binom{n-2}{k-1}$ such subgroups. Summing awards over these subsets,

$$\begin{aligned}
\binom{n-1}{k-1} \frac{kE}{n} &= \binom{n-1}{k-1} x_i + \binom{n-2}{k-2} \sum_{j \in N \setminus \{i\}} x_j \\
&= \left[\binom{n-1}{k-1} - \binom{n-2}{k-2} \right] x_i + \binom{n-2}{k-2} \sum_{j \in N} x_j \\
&= \frac{n-k}{k-1} \binom{n-2}{k-2} x_i + \binom{n-2}{k-2} E.
\end{aligned}$$

Rearranging,

$$\frac{n-k}{k-1} \binom{n-2}{k-2} x_i = \left[\frac{k}{n} \binom{n-1}{k-1} - \binom{n-2}{k-2} \right] E = \frac{n-k}{n(k-1)} \binom{n-2}{k-2} E.$$

Therefore, $x_i = \frac{E}{n} = \frac{k\delta}{n}$. Since this is true for each $i \in N$, $\varphi(c^\delta, E) = \left(\frac{k\delta}{n}, \dots, \frac{k\delta}{n}\right) = CEA(c^\delta, E)$.

By *consistency*, for each pair $i, j \in N$, $\varphi\big((c_i^\delta, c_j^\delta), \frac{2k\delta}{n}\big) = \left(\frac{k\delta}{n}, \frac{k\delta}{n}\right)$. Moreover, the argument shows that this is true for each pair $k, n \in \mathbb{N}$ with $2 \leq k < n$. For each $E \in [\delta, 2\delta]$ and each $\epsilon \in \mathbb{R}_{++}$, there is a pair $k, n \in \mathbb{N}$ with $2 \leq k < n$ such that $\left|\frac{2k\delta}{n} - E\right| < \epsilon$. Then by *endowment continuity*, $\varphi(c^\delta, E) = CEA(c^\delta, E)$. Since $\delta \in \mathbb{R}_+$ was arbitrary, φ satisfies *equal treatment of equals* on the domain of two-claimant problems. By *consistency*, φ satisfies *equal treatment of equals* generally. Since φ satisfies *equal treatment of equals*, *endowment continuity*, and *consistency*, it also satisfies *endowment monotonicity* (Young, 1987).[20]

Step 2: $\boldsymbol{\varphi = CEA}$ on the domain of two-claimant problems. Let $N \in \mathcal{N}$ with $|N| = 2$, $(c, E) \in \mathcal{C}^N$ with $c_1 \leq c_2$, and $x \equiv \varphi(c, E)$. We consider three cases according to the size of the endowment relative to the smallest claim.

Case 1: $E \in [0, c_1]$. For each $i \in N$, $(c_i, E) \in \mathcal{C}^{\{i\}}$ and $\varphi_i(c_i, E) = E$. Since $\bar{\mathcal{G}}(\varphi, 1, 1) = \frac{1}{2}$, $x_i \geq \frac{E}{2}$. This is true for both agents, so $x = \left(\frac{E}{2}, \frac{E}{2}\right) = CEA(c, E)$.

Case 2: $E \in [c_1, 2c_1]$. Suppose by way of contradiction that $x \neq \left(\frac{E}{2}, \frac{E}{2}\right)$ and label the agents so that $x_i < \frac{E}{2} < x_j$. Then there is $k \in \mathbb{N}$ such that $kx_i + E < (k+1)c_1$. Let $M \in \mathcal{N}$ with $N \cap M = \emptyset$ and $|M| = k$, $\hat{c} \equiv (c_i)_{h \in M}$. Then $((c_i, \hat{c}), kx_i + E) \in \mathcal{C}^{\{i\} \cup M}$. By *equal treatment of equals*, $\varphi(c_{\{i\} \cup M}, kx_i + E) = \left(\frac{kx_i + E}{k+1}, \dots, \frac{kx_i + E}{k+1}\right)$. Also, by *consistency*, for each $h \in \{i\} \cup M$, $\varphi_h((c, \hat{c}), kx_i + E) = x_i$. Now comparing group guarantees,

$$\frac{k+1}{k+2} = \bar{\mathcal{G}}(\varphi, k+1, 1) \leq \frac{1}{k+1} \sum_{h \in \{i\} \cup M} \frac{\varphi_h((c, \hat{c}), kx_i + E)}{\varphi_h((c_i, \hat{c}), kx_i + E)} = \frac{x_i}{(kx_i + E)/(k+1)}.$$

But then $E \leq 2x_i$, which contradicts $E = x_i + x_j > 2x_i$. Instead, $x_i = x_j = \frac{E}{2}$ and $x = CEA(c, E)$.

Case 3: $E \in [2c_1, c_1 + c_2]$. By Case 2, $\varphi(c, 2c_1) = (c_1, c_1)$. By *endowment monotonicity*, for each $E \in [2c_1, c_1 + c_2]$, $\varphi(c, E) = (c_1, E - c_1) = CEA(c, E)$. Therefore, $\varphi = CEA$ on the domain of two-claimant problems.

Step 3: $\boldsymbol{\varphi = CEA}$. By Step 2, $\varphi = CEA$ for all two-claimant problems. As CEA is its own unique *consistent* extension from the two-claimant case, $\varphi = CEA$ generally.[21] □

Adopting the perspective of an individual, we find that a wide range of rules maximize either guarantees or dual guarantees (see Example 1). In fact, it is possible to combine these objectives to ensure both protection of awards and sharing of losses. Combining these goals with *consistency* characterizes the Talmud rule.

[20] Although Young (1987) invokes *continuity* with respect to both claims and endowment, the argument for this implication (Lemma 1) relies only on *endowment continuity*.

[21] Because CEA is also "conversely consistent," this step is an application of the "Elevator Lemma" (Thomson, 2015b).

Theorem 2. *The Talmud rule is the unique* consistent *rule to maximize guarantees and maximum dual guarantees.*

Proof. The Talmud rule maximizes guarantees (Proposition 1) and is well known to be *consistent* (e.g., Aumann and Maschler (1985)). To prove the converse, let φ be a *consistent* rule such that for each pair $n, m \in \mathbb{N}$, $\mathcal{G}(\varphi, n, m) = \mathcal{G}(\varphi^d, n, m) = \frac{1}{1+m}$. We show that φ coincides with T on the domain of two-claimant problems and conclude by invoking *consistency*. Let $N \in \mathcal{N}$ with $|N| = 2$, $(c, E) \in \mathcal{C}^N$, and label the agents so that $c_1 \leq c_2$. We consider three cases according to the size of the endowment relative to the claims.

Case 1: $\boldsymbol{E \in [0, c_1]}$. If $\varphi(c, E) \neq T(c, E) = (\frac{E}{2}, \frac{E}{2})$, then there there is a labeling of the agents $i, j \in N$ such that $\varphi_i(c, E) < \frac{E}{2} < \varphi_j(c, E)$. Let $i \in N$. By feasibility, $\varphi_i(c_i, E) = E$, so

$$\frac{1}{2} = \mathcal{G}(\varphi, 1, 1) \leq \frac{\varphi_i(c, E)}{\varphi_i(c_i, E)} = \frac{\varphi_i(c, E)}{E}.$$

Then $\varphi_i(c, E) \geq \frac{E}{2}$. Since this is true for both agents and $\varphi_1(c, E) + \varphi_2(c, E) = E$, the statements hold with equality and $\varphi_i(c, E) = \left(\frac{E}{2}, \frac{E}{2}\right) = T(c, E)$.

Case 2: $\boldsymbol{E \in [c_2, c_1 + c_2]}$. Then $\sum_{i \in N} c_i - E \in [0, c_1]$. Now $\varphi(c, E) = c - \varphi^d(c, \sum_{i \in N} c_i - E)$ and $\mathcal{G}(\varphi^d, 1, 1) = \frac{1}{1+1} = \frac{1}{2}$, so the argument from Case 1 applies to φ^d. Thus, $\varphi^d(c, \sum_{i \in N} c_i - E) = (\frac{c_1+c_2-E}{2}, \frac{c_1+c_2-E}{2})$ and $\varphi(c, E) = c - \varphi^d(c, \sum_{i \in N} c_i - E) = (c_1 - \frac{c_1+c_2-E}{2}, c_2 - \frac{c_1+c_2-E}{2}) = T(c, E)$.

Case 3: $\boldsymbol{E \in [c_1, c_2]}$. Suppose by way of contradiction that $\varphi(c, E) \neq T(c, E) = (\frac{c_1}{2}, E - \frac{c_1}{2})$. Then $c_1 < c_2$ and either $\varphi_1(c, E) < \frac{c_1}{2}$ or $\varphi_1(c, E) > \frac{c_1}{2}$.

Subcase 3.1: $\varphi_1(c, E) < \frac{c_1}{2}$. Then $\varphi_2(c, E) > E - \frac{c_1}{2}$ and there is $k \in \mathbb{N}$ such that $\frac{E}{k} < \frac{c_1}{2} - \varphi_1(c, E) = \varphi_2(c, E) + \frac{c_1}{2} - E$. Let $M \in \mathcal{N}$ with $N \cap M = \emptyset$ and $|M| = k$ and for each $i \in M$, let $c_i \equiv c_2$. Since $\frac{E}{k+2} < \frac{E}{k} < \frac{c_1}{2} < \frac{c_2}{2}$, for each $i \in N \cup M$, $\varphi((c, \hat{c}), E) = \frac{E}{k+2}$. Now the ratio for agent 2 is

$$\frac{\varphi_2((c, \hat{c}), E)}{\varphi_2(c, E)} < \frac{E/(k+2)}{E/k + E - c_1/2} < \frac{E/(k+2)}{E/k + E} = \frac{k}{(k+1)(k+2)} < \frac{1}{k+1}.$$

But then $\mathcal{G}(\varphi, 2, k) < \frac{1}{k+1}$, a contradiction.

Subcase 3.2: $\varphi_1(c, E) > \frac{c_1}{2}$. Then $\varphi_2(c, E) < E - \frac{c_1}{2}$ and $T_2^d(c, c_1 + c_2 - E) = c_2 - (E - \frac{c_1}{2}) < c_2 - \varphi_2(c, E) = \varphi_2^d(c, c_1 + c_2 - E)$. Repeating the argument from subcase 3.1, $\mathcal{G}(\varphi^d, 2, k) < \frac{1}{k+1}$, again a contradiction. Instead, $\varphi(c, E) = T(c, E)$.

Altogether, φ coincides with the Talmud rule on the domain of two-claimant problems. By *consistency*, $\varphi = T$. □

Steps 1 and 2 in the proof establish that a rule satisfying the properties must coincide with the Talmud rule for large and small endowments. The conclusion of Step 3 would be implied directly by either *endowment continuity* or *endowment monotonicity*, but in fact neither is necessary. Instead, the argument relies on *consistency* to compare awards across populations of different sizes, effectively establishing *endowment continuity*.

By duality, Theorem 1 can be adapted to characterize the constrained equal losses rule. Emphasizing duality also provides a way to distinguish the Talmud rule among the many *consistent* rules which maximize guarantees.

Corollary 1. *(i) The constrained equal losses rule is the unique* endowment continuous *and* consistent *rule to maximize dual group guarantees.*
(ii) The Talmud rule is the unique consistent *and* self dual *rule to maximize guarantees.*

Unfortunately, as the characterizations make clear, no rule simultaneously maximizes group guarantees and dual group guarantees.

To conclude, we illustrate the diversity of rules which maximize guarantees. Further examples establishing independence of axioms appear in the appendix.

Example 1. A family of endowment continuous and consistent rules that maximize guarantees. Let $\tilde{\varphi}$ be *endowment continuous* and *consistent* and $\theta \in [\frac{1}{2}, 1]$. For each $(c, E) \in \mathcal{C}^N$, let

$$\varphi(c,E) \equiv \begin{cases} T^\theta\big(\theta c, E\big) & \text{if } E \leq \sum_{i \in N} \frac{c_i}{2} \\ \frac{c}{2} + \tilde{\varphi}\big(\frac{c}{2}, E\big) & \text{if } E > \sum_{i \in N} \frac{c_i}{2}. \end{cases}$$

Then φ is *endowment continuous* and *consistent*. Moreover, based on our previous results, φ maximizes guarantees: For each $n, m \in \mathbb{N}$, $\mathcal{G}(\varphi, n, m) = \frac{1}{1+m}$. The family is large because, aside from the two basic properties, $\tilde{\varphi}$ is unrestricted. For example, $\tilde{\varphi}$ may give priority to large claims over small claims or vice versa and thus diverge widely from both the constrained equal awards and Talmud rules. It may even give priority to some agents over others, thereby violating *equal treatment of equals*.

The rules defined in Example 1 coincide with the constrained equal awards rule for small endowments. Once small awards are "locked in," however, only the implications of *endowment continuity* and *consistency* restrict the behavior of the rule. Thus, maximizing guarantees is consistent with highly asymmetric treatment among agents.

3.3 Lower Bounds

Like guarantees, lower bounds[22] provide protection or insurance for the awards agents may expect. In contrast with guarantees, lower bounds apply to individual claims problems. They are therefore parameterized by the data in a problem, namely the sizes of the population, the agents' claims,

[22]See Thomson (2015a) for a thorough discussion of lower bounds in the literature.

and the endowment. Reflecting the similar normative motivation for these ideas, we find a close relation between lower bounds and the guarantee structures we introduce.

We consider two leading lower bounds applied to *endowment monotonic* and *consistent* rules. First, the $\frac{1}{n}$*-truncated-claims lower bound* requires that each agent receive at least an equal share of the smaller between the endowment and the agent's claim. Second, the *conditional equal division lower bound* requires that each agent who is not fully compensated receive at least an equal share of the endowment. Given φ, the formal requirements, stated for each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$, are:

$\boldsymbol{\frac{1}{n}}$**-truncated-claims lower bound:** For each $i \in N$, $\varphi_i(c, E) \geq \frac{1}{n}\min\{c_i, E\}$.

Conditional equal division lower bound: For each $i \in N$, $\varphi_i(c, E) \geq \min\{c_i, \frac{E}{n}\}$.

Whereas the $\frac{1}{n}$*-truncated-claims lower bound* imposes a bound satisfied by many rules, the *conditional equal division lower bound* is much more restrictive; not only does it directly imply the $\frac{1}{n}$*-truncated-claims lower bound*, but together with *consistency*, characterizes CEA.[23] This characterization parallels our Theorem 1, thereby connecting the bound to maximal group guarantees. The $\frac{1}{n}$*-truncated-claims lower bound* also carries implications for guarantees. In fact, rules satisfying this bound maximize guarantees.

Proposition 2. *Let φ be an* endowment monotonic *and* consistent *rule. Then φ (i) maximizes guarantees if and only if it satisfies the* $\frac{1}{n}$-truncated-claims lower-bound *and (ii) maximizes group guarantees if and only if it satisfies the* conditional equal division lower-bound.

To prove Proposition 2, we show that a rule satisfies the $\frac{1}{n}$*-truncated-claims lower-bound* if and only if it maximizes guarantee when the original population consists of a single agent. This is straightforward because the agent's award is always bounded by sizes of the endowment and her claim. The generalization to all guarantees is a consequence of a more subtle argument in the proof of Lemma 1.

4 Discussion and related literature

Formally, we study the familiar model of claims problems and characterize rules known to satisfy many desirable properties. The goal of maximizing guarantees provides a new perspective on claims problems while confirming the centrality of two leading rules.

As with the model, our normative motivation is familiar. Worst-case analysis has become a common standard of comparison with recent applications to cost sharing, probabilistic assignment of

[23] It is straightforward and well-known that only CEA satisfies the bound with two agents; *consistency* extends this result to populations of all sizes (see, for example, Thomson (2014)).

objects, and network routing.[24] However, in contrast with guarantee structures, these applications propose measures which summarize the performance of a rule by a single number. By parameterizing by population sizes, guarantee structures offer a more nuanced description of performance and allow more definitive comparisons among rules.

We have shown that guarantees relate to important lower bounds which are motivated by similar normative goals (Herrero and Villar, 2001; Moreno-Ternero and Villar, 2004).[25] Guarantees also reflect solidarity principles, notably *population monotonicity*.[26] Applied to claims problems, this property requires that when new agents arrive, no agent's award increases. It therefore codifies the intuition that the arrival of new agents will require downward revision of awards. In contrast with its application in other settings, this version of *population monotonicity* is very mild, satisfied by all seriously considered rules and in fact implied by *consistency* together with *endowment monotonicity* (Chun, 1999).

Although our application to claims problems is new, the guarantee structures we study parallel related notions first defined to compare anonymous and (weakly) Pareto-efficient bargaining solutions (Thomson and Lensberg, 1983; Thomson, 1983a). In this context, the Kalai-Smorodinsky solution maximizes guarantees (Thomson and Lensberg, 1983) whereas the Nash solution maximizes group guarantees (Thomson, 1983a). A related measure asks how the original agents share the sacrifice required to include the arriving agents. Seeking to minimize the maximum disparity of sacrifice among these agents leads to characterizations of the Kalai-Smorodinsky and Egalitarian solutions. Recognizing that the arrival of new agents may make some original agents better off, solutions may be further compared according to the opportunities they create (Thomson, 1987). Our analysis of dual guarantees, although formally distinct, follows the spirit of this approach.

To formally compare these results to ours, we may view claims problems as a subclass of bargaining problems with "rectangular" structure. Here, both the Nash and lexicographic extension of the egalitarian solution correspond to the constrained equal awards rules (Dagan and Volij, 1993). Similarly, weighted Nash solution with weights determined by the claims and the Kalai-Smorodinsky solution correspond respectively to the proportional and truncated proportional rules (Dagan and Volij, 1993). Like the Nash solution, the constrained equal awards rule maximizes group guarantees. However, group guarantees no longer distinguish the Nash and egalitarian solutions as these rules

[24] Some of the focal and recent applications include: Moulin and Shenker (2001), Hashimoto and Saitoh (2015), Moulin (2008), Juarez (2008), and Massó et al. (2015) for cost sharing; Bhalgat et al. (2011) and Bogomolnaia and Moulin (2014) for probabilistic assignment; and Koutsoupias and Papadimitriou (1999), Roughgarden and Tardos (2002), Roughgarden (2002), and Anshelevich et al. (2008) for network routing.

[25] The $\frac{1}{n}$*-truncated-claims lower bound* is introduced as "securement" by Moreno-Ternero and Villar (2004), called the "secured lower bound" by (Yeh, 2008), and decomposed into sub-properties in Moreno-Ternero and Villar (2006a). The *conditional equal division lower bound* first appears in Moulin (2002) under the name "lower bound." It implies "exemption" (Herrero and Villar, 2001; Yeh, 2006) and is closely related to "sustainability" (Moreno-Ternero and Villar, 2004; Yeh, 2006). Extending lower bounds to baselines leads to related "baseline first" composition properties (Hougaard et al., 2012, 2013). For a complete discussion of lower bounds and generalizations, see Thomson (2015a).

[26] This axiom is first proposed for axiomatic bargaining where it is a key property in characterizations of the Kalai-Smorodinsky and egalitiarian solutions (Thomson, 1983b,c).

coincide on the smaller domain. Like its bargaining counterpart, the truncated proportional rule maximizes guarantees. This is because it coincide with the constrained equal awards rule when the endowment is no larger than the smallest claim. As many rules even in the TAL-family also maximize guarantees, the truncated proportional rule is far from uniquely identified on the smaller domain.

Conceptual differences between the claims and bargaining model are also important to the study of guarantees. For example, whereas adding monetary awards of agents in claims problems is natural and meaningful, bargaining problems lack a common similar common scale rendering summation of utilities across agents suspect. Similarly, while the relation between the zero and claims points immediately suggests a notion of duality in claims problems, general bargaining problems lack an obvious upper reference point; different choices are possible, none with compelling conceptual interpretation. Consequently, there is no analog of our Theorem 2 nor of the measure $\hat{\mathcal{G}}$.

While our goal has been to maximize guarantees, our guarantee structures also facilitate comparisons among rules which do not achieve the maximum guarantees. An important avenue for further research would identify and compare the guarantee structures of rules in families beyond the TAL-family. Another approach would derive the consequences for guarantee structures of other desirable properties, identifying those guarantees compatible with or implied by various monotonicity and independence properties, lower bounds, and operators on rules. Similarly, while our guarantees consider the possibility of new agents arriving, other hypotheticals are possible. For example, our measures could be adapted to define the worst-case revision to awards following a reduction in the the endowment or an increase in the claims of other agents. Given the adaptability of the underlying principles, as well as the numerous fruitful applications of worst-case analysis generally, we further expect that analogues of guarantee structures will provide rich information about allocation methods in a broad range of settings from surplus sharing to marriage problems and school assignment. Similarly, revisiting classical results on worst case analysis, the richer information of guarantee structures may improve our understanding of these familiar problems.

A Appendix

A.1 Proof of Lemma 1

Proof. Let φ be an *endowment monotonic* and *consistent* rule, $n, m \in \mathbb{N}$, $N, M \in \mathcal{N}$ with $N \cap M = \emptyset$, $|N| = n$, and $|M| = m$, $i \in N$, and $\hat{c} \in \mathbb{R}^M_+$.

Step 1: $\boldsymbol{\mathcal{G}(\varphi, 1, m) = \mathcal{G}(\varphi, n, m)}$. First let $(c, E) \in \mathcal{C}^{\{i\}}$, $c' \equiv (c, \hat{c})$, $x \equiv \varphi(c, E)$, and $y \equiv \varphi(c', E)$. Now define $\bar{c} \in \mathbb{R}^N_+$ such that $\bar{c}_i = c_i$ and for each $j \in N \backslash \{i\}$, $\bar{c}_j = 0$, $\bar{c}' \equiv (\bar{c}, \hat{c})$, $\bar{x} \equiv \varphi(\bar{c}, E)$, and $\bar{y} \equiv \varphi(\bar{c}', E)$. Then for each $j \in N \backslash \{i\}$, $\varphi_j(\bar{c}, E) = \varphi_j(\bar{c}', E) = 0$. By *consistency*, $\bar{x}_i = x_i$ and $\bar{y}_i = y_i$. Therefore, $\mathcal{G}(\varphi, n, m) \leq \frac{\bar{y}_i}{\bar{x}_i} = \frac{y_i}{x_i}$. Since this is true for $(c, E) \in \mathcal{C}^{\{i\}}$,

$\mathcal{G}(\varphi, n, m) \leq \mathcal{G}(\varphi, 1, m)$.

Next let $(c, E) \in \mathcal{C}^N$, $c' \equiv (c, \hat{c})$, $x \equiv \varphi(c, E)$, and $y \equiv \varphi(c', E)$. Now define $\bar{c} \equiv (c_i, \hat{c})$, $\bar{E} \equiv \min\left\{c_i, \sum_{j \in M \cup \{i\}} y_j\right\}$, $\bar{x} \equiv \varphi(c_i, E)$, and $\bar{y} \equiv \varphi(\bar{c}, \bar{E})$. By *endowment monotonicity*, $\bar{y}_i \leq y_i$. First suppose that $\bar{E} = c_i$. Then $x_i \leq c_i = \bar{E} = \bar{x}_i$ and $\frac{\bar{y}_i}{\bar{x}_i} \leq \frac{y_i}{x_i}$. Suppose instead that $\bar{E} = E$. Then by *consistency*, $\bar{x}_i = x_i$ and $\bar{y}_i = y_i$, so $\frac{\bar{y}_i}{\bar{x}_i} = \frac{y_i}{x_i}$. Since this is true for $(c, E) \in \mathcal{C}^N$, $\mathcal{G}(\varphi, 1, m) \leq \mathcal{G}(\varphi, n, m)$. Altogether, $\mathcal{G}(\varphi, 1, m) = \mathcal{G}(\varphi, n, m)$.

Step 2: $\boldsymbol{\mathcal{G}(\varphi, n, m) = \hat{\mathcal{G}}(\varphi, n, m)}$. First, the ratio of total awards is at least as large as the smallest individual ratio: $\min\left\{\frac{y_i}{x_i}\right\} \leq \frac{\sum_{i \in N} y_i}{\sum_{i \in N} x_i}$. Therefore, $\mathcal{G}(\varphi, n, m) \leq \hat{\mathcal{G}}(\varphi, n, m)$. To show the reverse inequality, by Step 1, it suffices to show that $\hat{\mathcal{G}}(\varphi, n, m) \leq \mathcal{G}(\varphi, 1, m)$.

Let $(c, E) \in \mathcal{C}^{\{i\}}$, $c' \equiv (c, \hat{c})$, $x \equiv \varphi(c, E)$, and $y \equiv \varphi(c', E)$. Now define $\bar{c} \in \mathbb{R}^N_+$ such that $\bar{c}_i = c_i$ and for each $j \in N \backslash \{i\}$, $\bar{c}_j = 0$, $\bar{c}' \equiv (\bar{c}, \hat{c})$, $\bar{x} \equiv \varphi(\bar{c}, E)$, and $\bar{y} \equiv \varphi(\bar{c}', E)$. Then for each $j \in N \backslash \{i\}$, $\varphi_j(\bar{c}, E) = \varphi_j(\bar{c}', E) = 0$. Therefore,

$$\hat{\mathcal{G}}(\varphi, n, m) \leq \frac{\sum_{j \in N} \bar{y}_j}{\sum_{j \in N} \bar{x}_j} = \frac{\bar{y}_i}{\bar{x}_i} = \frac{y_i}{x_i}.$$

Since this is true for $(c, E) \in \mathcal{C}^{\{i\}}$, $\hat{\mathcal{G}}(\varphi, n, m) \leq \mathcal{G}(\varphi, 1, m)$. Altogether, $\mathcal{G}(\varphi, n, m) = \hat{\mathcal{G}}(\varphi, n, m)$. □

A.2 Proof of Proposition 2

Proof. Let φ be an *endowment monotonic* and *consistent* rule.

(i). By Step 1 in the proof of Lemma 1, for each pair $n, m \in \mathbb{N}$, $\mathcal{G}(\varphi, 1, m) = \mathcal{G}(\varphi, n, m)$, so it suffices to consider the case $1 = n < m$. By Lemma 2, the maximal guarantee in this case is $\mathcal{G}(\varphi, n, m) = \mathcal{G}(\varphi, 1, m) = \frac{1}{1+m}$.

First suppose that φ satisfies the $\frac{1}{n}$*-truncated-claims lower-bound*. Let $m \in \mathbb{N}$, $N \in \mathcal{N}$ with $|N| = 1 + m$, $(c, E) \in \mathcal{C}^N$, and $i \in N$. If $c_i \geq E$, then $\varphi_i(c, E) \geq \frac{E}{1+m}$. Since $\varphi_i(c_i, E) = E$, $\frac{\varphi_i(c,E)}{\varphi_i(c_i,E)} \geq \frac{E/(1+m)}{E} = \frac{1}{1+m}$. Suppose instead that $c_i \leq E$. For each $E' \in \mathbb{R}_+$ with $E' \leq c_i$, by $\varphi_i(c, E') \leq \varphi_i(c, E)$. By the $\frac{1}{n}$*-truncated-claims lower-bound*, $\varphi_i(c, E') \geq \frac{E'}{1+m}$. Since $\varphi_i(c_i, E') = E'$, $\frac{\varphi_i(c,E')}{\varphi_i(c_i,E')} \geq \frac{E'/(1+m)}{E'} = \frac{1}{1+m}$. Together, these inequalities show that $\mathcal{G}(1, m) \geq \frac{1}{1+m}$.

Conversely, suppose that φ violates the $\frac{1}{n}$*-truncated-claims lower-bound*. Since a violation requires at least two agents, there are $m \in \mathbb{N}$, $N \in \mathcal{N}$ with $|N| = 1 + m$, $(c, E) \in \mathcal{C}^N$, and $i \in N$ such that $\varphi_i(c, E) < \frac{1}{1+m} \min\{c_i, E\}$. Let $E' \equiv c_i$. By *endowment monotonic*, $\varphi_i(c, E') \leq \varphi_i(c, E) < \frac{1}{1+m} \min\{c_i, E\} \leq \frac{E'}{1+m}$. Since $\varphi_i(c_i, E') = E'$, $\frac{\varphi_i(c,E')}{\varphi_i(c_i,E')} < \frac{E'/(1+m)}{E'} = \frac{1}{1+m}$. Therefore, $\mathcal{G}(1, m) < \frac{1}{1+m}$ and φ fails to maximize guarantees.

(ii). By extension of Theorem 2 in Herrero and Villar (2001), if φ satisfies the *conditional equal division lower-bound*, then $\varphi = CEA$. Conversely, by Theorem 1, if φ maximizes average guarantees, then $\varphi = CEA$. □

A.3 Independence of axioms

Example 2 shows that *consistency* is required in Theorem 1. To maximize group guarantee, the rule "switches" from the path of the less egalitarian Talmud rule to the more egalitarian constrained equal awards rule as the size of the population increases. Although this violates *consistency*, the rule continues to maximize group guarantees. Whether *endowment continuity* is independent in Theorem 1 is an open question.

Example 2. A endowment continuous and endowment monotonic rule that satisfies equal treatment of equals and maximizes group guarantees. For each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$, let

$$\varphi(c, E) \equiv \begin{cases} T(c, E) & \text{if } |N| = 2 \\ CEA(c, E) & \text{otherwise} \end{cases}.$$

It is immediate that φ is *endowment continuous* and *endowment monotonic* and satisfies *equal treatment of equals*. We verify that φ maximizes group guarantees. Let $n, m \in \mathbb{N}$. First, if $n \neq 2$ and $n+m \neq 2$, then φ applies CEA for all populations of the relevant sizes and $\bar{\mathcal{G}}(\varphi, n, m) = \bar{\mathcal{G}}^{CEA}(n, m) = \frac{n}{n+m}$. Also, since all rules coincide for one-claimant problems, $\bar{\mathcal{G}}(\varphi, 1, 1) = \bar{\mathcal{G}}^T(1, 1) = \frac{1}{2} = \bar{\mathcal{G}}^{CEA}(1, 1)$.

Now consider $n = 2$ and let $N, M \in \mathcal{N}$ with $N \cap M = \emptyset$, $|N| = n$, and $|M| = m$. Let $(c, E) \in \mathcal{C}^N$ with $c_1 \leq c_2$ and $\hat{c} \in \mathbb{R}^M_+$. Then $\varphi(c, E) = T(c, E)$ and $\varphi((c, \hat{c}), E) = CEA((c, \hat{c}), E)$. First, if either $c_1 = 0$ or $E \leq c_1$, then $\varphi(c, E) = CEA(c, E)$ and the group guarantee for CEA applies. Suppose instead that $0 < c_1 < E$. We distinguish two cases.

Case 1: $\boldsymbol{E \leq (2+m)c_1}$**.** Then $\varphi(c, E) = \left(\frac{c_1}{2}, E - \frac{c_1}{2}\right)$ and for each $i \in N$, $\varphi_i((c, \hat{c}), E) \geq \frac{E}{n+m} = \frac{E}{2+m}$. The average ratio is at least

$$\frac{1}{2}\left(\frac{E/(2+m)}{c_1/2} + \frac{E/(2+m)}{E - c_1/2}\right) = \frac{E}{2(2+m)} \cdot \frac{E}{c_1/2 \cdot (E - c_1/2)} = \frac{E^2}{(2+m)c_1(E - c_1/2)}.$$

As E varies, $\frac{E^2}{c_1(E-c_1/2)}$ attains a minimum at $E = c_1$ in which case $\frac{E^2}{c_1(E-c_1/2)} = 2$. Therefore, $\frac{E^2}{(2+m)c_1(E-c_1/2)} \geq \frac{2}{2+m}$ and the maximum group guarantee is achieved.

Case 2: $\boldsymbol{E > (2+m)c_1}$**.** Then $\varphi_1((c, \hat{c}), E) = c_1$. Also, $\varphi(c, E) \leq (c_1, c_2)$ and $\varphi(c, E) \leq (E, E)$. If $m \geq 2$, then the average ratio is at least

$$\frac{1}{2}\left(\frac{c_1}{c_1} + \frac{c_1}{c_2}\right) > \frac{1}{2}(1 + 0) = \frac{1}{2} \geq \frac{2}{2+m}.$$

If $m = 1$, then $\varphi_2((c, \hat{c}), E) \geq \frac{E}{3}$ and the average ratio is at least

$$\frac{1}{2}\left(\frac{c_1}{c_1} + \frac{E/3}{E}\right) > \frac{1}{2}\left(1 + \frac{1}{3}\right) = \frac{2}{3} = \frac{2}{2+m}.$$

Therefore, φ maximizes group guarantee.

Among *consistent* rules, many maximize guarantees. For example, the constrained equal awards rule is *consistent* and maximizes guarantees, as do all rules constructed in Example 1. Similarly, the constrained equal losses rule is *consistent* and maximizes dual guarantees, as do as the duals of all rules constructed in Example 1. Example 3 show the independence of *consistency* in Theorem 2. The rule modifies the Talmud rule by carefully altering the "drop out" points so that guarantees and dual guarantees continue to be maximized. Of course, this rule is not *consistent*.

Example 3. A endowment continuous, endowment monotonic, and self dual rule that maximizes guarantees and maximum dual guarantees. For each $N \in \mathcal{N}$ and each $(c,E) \in \mathcal{C}^N$ with $|N| = 3$ and $c_1 \leq c_2 \leq c_3$, let

$$\varphi(c,E) \equiv \begin{cases} \left(\frac{E}{3}, \frac{E}{3}, \frac{E}{3}\right) & \text{if } E \leq c_1 \\ \left(\frac{c_1}{3}, \frac{c_1}{3} + \frac{E-c_1}{2}, \frac{c_1}{3} + \frac{E-c_1}{2}\right) & \text{if } c_1 < E \leq c_2 \\ \left(\frac{c_1}{3} + \frac{E-c_2}{3}, \frac{c_1}{3} + \frac{c_2-c_1}{2} + \frac{E-c_2}{3}, \frac{c_1}{3} + \frac{c_2-c_1}{2} + \frac{E-c_2}{3}\right) & \text{if } c_2 < E \leq c_2 + \frac{c_1}{2} \\ \left(\frac{c_1}{2}, \frac{c_2}{2}, \frac{c_2}{2} + (E - c_2 - \frac{c_1}{2})\right) & \text{if } c_2 + \frac{c_1}{2} < E \leq \frac{c_1+c_2+c_3}{2} \\ c - \varphi(c, c_1+c_2+c_3-E) & \text{if } E < \frac{c_1+c_2+c_3}{2} \end{cases}$$

and $\varphi(c,E) = T(c,E)$ otherwise. That is, φ modifies T so that agent 1 "drops out" earlier: Under T, all agents share equally the incremental endowment between c_1 and $\frac{3}{2}c_1$ and then agents 2 and 3 share the incremental endowment between $\frac{3}{2}c_1$ and $c_2 - c_1$; φ reverses these increments.

From the definition, φ is *endowment continuous*, *endowment monotonic*, and *self dual*. By *self dual*, it suffices to show that φ maximizes guarantees. Let $N, M \in \mathcal{N}$ with $N \cap M = \emptyset$, $(c,E) \in \mathcal{C}^N$, and $\hat{c} \in \mathbb{R}^M_+$. Let $n \equiv |N|$ and $m \equiv |M|$. Since φ differs from T only for three-claimant problems, there are three cases to consider.

Case 1: $\boldsymbol{n = 1}$ and $\boldsymbol{m = 2}$. Then $E \leq c_i$ and $(c_i, E) \in \mathcal{C}^{\{i\}}$ so that $\varphi_i((c,\hat{c}),E) \geq \frac{E}{3}$. Since $\varphi_i(c_i,E) = E$, the ratio for agent i is at least $\frac{E/3}{E} = \frac{1}{3}$. Therefore, $\mathcal{G}(\varphi, 1, 2) \geq \frac{1}{3}$.

Case 2: $\boldsymbol{n = 2}$ and $\boldsymbol{m = 1}$. Then $N = \{i,j\}$ and $E \leq c_i + c_j$ so $(c_{\{i,j\}}, E) \in \mathcal{C}^{\{i,j\}}$. If $\varphi_i((c,\hat{c}),E) \geq \frac{c_i}{2}$, then the ratio for agent i is at least $\frac{1}{2}$, so suppose instead that $\varphi_i((c,\hat{c}),E) < \frac{c_i}{2}$ so $E \leq \frac{c_1+c_2+c_3}{2}$. Then $\varphi_2((c,\hat{c}),E) \geq T_2((c,\hat{c}),E)$ and $\varphi_3((c,\hat{c}),E) \geq T_3((c,\hat{c}),E)$, so suppose $i = 1$. Then $E \leq c_2 + \frac{c_1}{2}$. If $E \leq c_1$, then $\varphi_1(c,E) = \frac{E}{2}$ and $\varphi_1((c,\hat{c}),E) = \frac{E}{3}$ so the ratio for agent 1 is $\frac{E/3}{E/2} = \frac{2}{3}$. If $E > c_1$, then $\varphi_1(c,E) = \frac{c_1}{2}$ and $\varphi_((c,\hat{c}),E) = \frac{c_1}{3}$ so the ratio for agent 1 is $\frac{c_1/3}{c_1/2} = \frac{2}{3}$. Altogether, $\mathcal{G}(\varphi, 2, 1) \geq \frac{1}{2}$.

Case 3: $\boldsymbol{n = 3}$. Let $i \in N$. Since $\varphi((c,\hat{c}),E) = T((c,\hat{c}),E)$, if $\varphi_i(c,E) \leq T_i(c,E)$, then the ratio for agent i is at least $\mathcal{G}^T(n,m)$, so suppose instead that $\varphi_i(c,E) > T_i(c,E)$. There are two subcases.

Subcase 3.1: $i \neq 1$ and $c_1 \leq E \leq c_2 + \frac{c_1}{2}$. If $\varphi_i((c,\hat{c}),E) \geq \frac{c_i}{2}$, then the ratio for agent i is at least $\frac{1}{2}$, so suppose $\varphi_i((c,\hat{c}),E) < \frac{c_i}{2}$. Then $\varphi_i((c,\hat{c}),E) \geq \frac{E}{n+m}$. If $c_2 \leq E$, then $\varphi_i(c,E) \leq \frac{c_2}{2}$ and the ratio for agent i is at least

$$\frac{\varphi_i((c,\hat{c}),E)}{\varphi_i(c,E)} \geq \frac{E/(n+m)}{c_2/2} = \frac{2}{n+m} \cdot \frac{E}{c_2/2} > \frac{4}{n+m} = \frac{4}{m+3} \geq \frac{1}{1+m}.$$

If $E < c_2$, then $\varphi_i(c,E) = \frac{c_1}{3} + \frac{E-c_1}{2}$ and the ratio for agent i is at least

$$\frac{\varphi_i((c,\hat{c}),E)}{\varphi_i(c,E)} \geq \frac{E/(n+m)}{E/2 - c_1/6} \geq \frac{1}{n+m} \cdot \frac{E}{E/2} > \frac{2}{n+m} = \frac{2}{m+3} \geq \frac{1}{1+m}.$$

Subcase 3.2: $i = 1$ and $c_3 + \frac{c_1}{2} \leq E \leq c_2 + c_3$. Then $\varphi_1(c,E) \leq \frac{2c_1}{3}$ and $\varphi_1((c,\hat{c}),E) = T_1((c,\hat{c}),E)$. If $\varphi_1((c,\hat{c}),E) \geq \frac{c_1}{3}$, then the ratio for agent 1 is at least $\frac{c_1/3}{2c_1/3} = \frac{1}{2}$. If instead $\varphi_1((c,\hat{c}),E) < \frac{c_1}{3}$, then $\varphi_1((c,\hat{c}),E) \geq \frac{E}{n+m}$ and the ratio for agent 1 is at least

$$\frac{\varphi_1((c,\hat{c}),E)}{\varphi_1(c,E)} \geq \frac{E/(n+m)}{2c_1/3} \geq \frac{1}{(n+m)} \cdot \frac{c_3 + c_1/2}{2c_1/3} \geq \frac{1}{n+m} \cdot \frac{9}{4} = \frac{9}{4(m+3)} \geq \frac{1}{1+m}.$$

Altogether, $\mathcal{G}(\varphi, n, m) = \frac{1}{1+m}$. By *self duality*, φ also maximizes dual guarantees.

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