Barnett Replies In their comment [1] on my letter [2], Iwo Bialynicki-Birula and Zofia Bialynicka-Birula identify the circulation, a measure derived from fluid mechanics, as their method of choice to search for vortices. They require, in particular, that a vortex has a nonvanishing circulation for any path circling the vortex core even arbitrarily close to the axis, where this would require a singular vorticity and, with it, a divergent particle velocity. They justify the statement that there is no vortex for a relativistic electron by the absence of such a divergent velocity. As we show, there are no places in Nature where a singular vorticity is to be found and conclude that their preferred criterion for the existence of a vortex is too restrictive. We illustrate this point with examples from fluid mechanics, atmospheric physics, superfluidity, and optics.

Let us begin by summarizing the argument presented in the Comment [1]. The circulation is the line integral of the particle velocity around a closed loop:

$$\Gamma = \oint \mathbf{v} \cdot d\mathbf{l} \tag{1}$$

and may be expected to take a nonzero value if the path encloses a vortex. For Γ to take a constant value for *any* curve enclosing the vortex core, however short in length, we require a *divergent* azimuthal particle velocity near the core, $v_\phi(r) = \Gamma/(2\pi r)$, corresponding to a singular vorticity. This idealization is unphysical, although it does have its mathematical uses [3].

The problem of a divergent particle velocity in the field of fluid mechanics was addressed by Rankine by introducing his combined vortex [4] in which the azimuthal particle velocity near the core is replaced by $v_\phi(r) = \Gamma r/(2\pi R^2)$ for $r \leq R$. The Rankine vortex and variants of it have been widely employed, notably in atmospheric physics [5] and optics [6]. For distances from the axis that are less that R, the circulation tends to zero as r^2 so that by the criterion required in [1] we would have to conclude that the Rankine vortex is not a vortex.

In the quantum domain the study of vortices has long been a key component in the field of superfluidity [7–9]. In a superfluid we introduce a local velocity density that is proportional to the gradient of the phase of the wave function [10,11] and for wave function with azimuthal phase dependence, $e^{i\phi}$, this suggests the existence of a singular vorticity. This simple description breaks down for small distances from the axis, at least on the atomic scale [7]. A superfluid will have a vanishing circulation for loops with a radius less than this distance. As with classical fluids, adopting the criterion required in the Comment [1] would mean that superfluid vortices are not vortices.

As a final example, we turn to optics. It is here, I suggest, that the closest analogy to the relativistic-electron vortex is to be found. Phase singularities are ubiquitous in optics and, indeed, in other wave phenomena [12]. Laguerre-Gaussian laser modes, for example, have an azimuthal phase-dependence $e^{i\ell\phi}$ [13–15]. The associated phase singularity on the beam axis goes by many names including, perhaps most regularly, an optical vortex [6,16,17]. There is no suggestion of photons localized near this vortex orbiting it at unbounded speeds and the use of the vorticity and circulation from classical fluid mechanics is clearly inappropriate. Optical vortices do not have a singular vorticity and so, once again, by the criterion advocated in the Comment [1] are not vortices.

The term "vortex" has been used in a wide variety of ways in physics and even in fluid mechanics there are many distinct phenomena, including a vortex, a vortex line, and a vortex tube [18]. Of these the most demanding and also the most unphysical is the line vortex which is "a vortex filament of finite strength and zero cross section; it is a singular distribution of vorticity" [18]. This is the criterion by which Bialynicki-Birula and Bialynicka-Birula have chosen to define a vortex. Yet regions of divergent vorticity exist nowhere in Nature and if we are to accept the proposed criterion then we have no choice but to abandon the term "vortex" altogether and leave vortices as purely unphysical barren objects. I venture to suggest that few physicists would be prepared to accept this.

Relativistic electrons do indeed exhibit vortices but these are vortices analogous to those already familiar from optics rather than from fluid mechanics.

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[1] I. Bialynicki-Birula and Z. Bialynicka-Birula, Comment on "Relativistic Electron Vortices", preceding Comment, Phys. Rev. Lett. **119**, 029501 (2017).
[2] S. M. Barnett, Relativistic Electron Vortices, Phys. Rev. Lett. **118**, 114802 (2017). I take this opportunity to correct a transcription error in this paper. There is a missing factor of

2 in both lines of Eq. (11) where the second term should be $\pm i\beta\alpha \times \mathbf{p}/(2\sqrt{m^2 + p^2})$, respectively
[3] A. J. Majda and A. L. Bertozzi, *Vorticity and Incompressible Fluid Flow* (Cambridge University Press, Cambridge, England, 2002).
[4] W. J. M. Rankine, *A Manual of Applied Mechanics* (Richard Griffin, London, 1858).
[5] V. Wood and L. W. White, A parametric wind-pressure relationship for Rankine versus non-Rankine cyclostrophic vortices, J. Atmos. Ocean. Technol. **30**, 2850 (2013).
[6] G. A. Swartzlander and R. I. Hernandez-Aranda, Optical Rankine Vortex and Anomalous Circulation of Light, Phys. Rev. Lett. **99**, 163901 (2007).
[7] R. P. Feynman, Application of quantum mechanics to liquid helium, Prog. Low. Temp. Phys. **1**, 17 (1954).
[8] D. R. Tilley and J. Tilley, *Superfluidity and Superdonductivity*, 3rd ed. (Institute of Physics, Bristol, 1990).
[9] R. J. Donnelly, *Quantized Vortices in Helium II* (Cambridge University Press, Cambridge, England, 1991).
[10] L. Landau, The theory of superfluidity of helium II, J. Phys. **5**, 71 (1941). Reprinted in translation in [11]. L. Landau, The theory of superfluidity of helium II, Phys. Rev. **60**, 356 (1941).
[11] I. M. Khalatnikov, *An Introduction to the Theory of Superfluidity* (Perseus, Cambridge, MA, 2000).
[12] J. F. Nye, *Natural Focusing and Fine Structure of Light* (Institute of Physics, Bristol, 1999).
[13] A. E. Siegman, *Lasers* (University Science Books, Saulsalito, CA, 1986).
[14] L. Allen, M. W. Beijersbergen, R. J. C. Spreeuw, and J. P. Woerdman, Orbital angular momentum of light and the transformation of Laguerre-Gaussian laser modes, Phys. Rev. A **45**, 8185 (1992).
[15] A. M. Yao and M. J. Padgett, Orbital angular momentum: Origins, behavior and applications, Adv. Opt. Photonics **3**, 161 (2011).
[16] M. Harris, C. A. Hill, and J. M. Vaughan, Optical helices and spiral interference fringes, Opt. Commun. **106**, 161 (1994).
[17] A. Bekshaev, M. Soskin, and M. Vasnetsov, *Paraxial Light Beams with Angular Momentum* (Nova, New York, 2008).
[18] P. G. Saffman and G. R. Baker, Vortex interactions, Annu. Rev. Fluid Mech. **11**, 95 (1979).