AN INFINITE C*-ALGEBRA WITH A DENSE, STABLY FINITE *-SUBALGEBRA

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Abstract. We construct a unital pre-C*-algebra A_0 which is stably finite, in the sense that every left invertible square matrix over A_0 is right invertible, while the C*-completion of A_0 contains a non-unitary isometry, and so it is infinite.

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1. Introduction

Let A be a unital algebra. We say that A is *finite* (also called directly finite or Dedekind finite) if every left invertible element of A is right invertible, and we say that A is *infinite* otherwise. This notion originates in the seminal studies of projections in von Neumann algebras carried out by Murray and von Neumann in the 1930s. At the 22^{nd} International Conference on Banach Algebras and Applications, held at the Fields Institute in Toronto in 2015, Yemon Choi raised the following questions:

(1) Let A be a unital, finite normed algebra. Must its completion be finite?
(2) Let A be a unital, finite pre-C*-algebra. Must its completion be finite?

Choi also stated Question (1) in [7, Section 6].

A unital algebra A is said to be *stably finite* if the matrix algebra $M_n(A)$ is finite for each $n \in \mathbb{N}$. This stronger form of finiteness is particularly useful in the context of K-theory, and so it has become a household item in the Elliott classification programme for C*-algebras. The notions of finiteness and stable finiteness differ even for C*-algebras, as was shown independently by Clarke [8] and Blackadar [4] (or see [5, Exercise 6.10.1]). A much deeper result is due to Rørdam [9, Corollary 7.2], who constructed a unital, simple C*-algebra which is finite (and separable and nuclear), but not stably finite.

We shall answer Question (2), and hence Question (1), in the negative by proving the following result:

Theorem 1.1. *There exists a unital, infinite C*-algebra which contains a dense, unital, stably finite *-subalgebra.*

Let A be a unital *-algebra. Then there is a natural variant of finiteness in this setting, namely we say that A is **-finite* if whenever we have $u \in A$ satisfying $u^*u = 1$, then $uu^* = 1$. However, it is known (see, *e.g.*, [10, Lemma 5.1.2]) that a C*-algebra is finite if and only if it is *-finite, so in this article we shall not need to refer to *-finiteness again.

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2. Preliminaries

Our approach is based on semigroup algebras. Let S be a monoid, that is, a semigroup with an identity, which we shall usually denote by e. By an *involution* on S we mean a map from S to S, always denoted by $s \mapsto s^*$, satisfying $(st)^* = t^*s^*$ and $s^{**} = s$ $(s, t \in S)$. By a **-monoid* we shall mean a pair $(S, *)$, where S is a monoid, and $*$ is an involution on S. Given a *-monoid S, the semigroup algebra $\mathbb{C}S$ becomes a unital *-algebra simply by defining $\delta_s^* = \delta_{s^*}$ $(s \in S)$, and extending conjugate-linearly.

Next we shall recall some basic facts about free products of *-monoids, unital *-algebras, and their C*-representations.

Let S and T be monoids, and let A and B be unital algebras. Then we denote the free product (*i.e.* the coproduct) of S and T in the category of monoids by $S * T$, and similarly we denote the free product of the unital algebras A and B by $A * B$. It follows from the universal property satisfied by free products that, for monoids S and T, we have $\mathbb{C}(S*T) \cong (\mathbb{C}S)*(\mathbb{C}T)$.

Given *-monoids S and T, we can define an involution on $S * T$ by

$$(s_1 t_1 \cdots s_n t_n)^* = t_n^* s_n^* \cdots t_1^* s_1^*$$

for $n \in \mathbb{N}, s_1 \in S, s_2, \ldots, s_n \in S \setminus \{e\}, t_1, \ldots, t_{n-1} \in T \setminus \{e\}$, and $t_n \in T$. The resulting *-monoid, which we continue to denote by $S*T$, is the free product in the category of *-monoids. We can analogously define an involution on the free product of two unital *-algebras, and again the result is the free product in the category of unital *-algebras. We then find that $\mathbb{C}(S * T) \cong (\mathbb{C}S) * (\mathbb{C}T)$ as unital *-algebras.

Let A be a *-algebra. If there exists an injective *-homomorphism from A into some C*-algebra, then we say that A admits a *faithful C*-representation.* In this case, A admits a norm such that the completion of A in this norm is a C*-algebra, and we say that A admits a *C*-completion.* Our construction will be based on C*-completions of *-algebras of the form $\mathbb{C}S$, for S a *-monoid.

We shall denote by S_∞ the free *-monoid on countably many generators; that is, as a monoid S_∞ is free on some countably-infinite generating set $\{t_n, s_n : n \in \mathbb{N}\}$, and the involution is determined by $t_n^* = s_n$ $(n \in \mathbb{N})$. For the rest of the text we shall simply write t_n^* in place of s_n. We define BC to be the bicyclic monoid $\langle p, q : pq = e\rangle$. This becomes a *-monoid when an involution is defined by $p^* = q$, and the corresponding $*$-algebra $\mathbb{C}BC$ is infinite because $\delta_p \delta_q = \delta_e$, but $\delta_q \delta_p = \delta_{qp} \neq \delta_e$.

Lemma 2.1. *The following unital *-algebras admit faithful C*-representations:*

(i) $\mathbb{C}(BC)$,
(ii) $\mathbb{C}(S_\infty)$.

Proof. (i) Since BC is an inverse semigroup, this follows from [2, Theorem 2.3].

(ii) By [3, Theorem 3.4] $\mathbb{C}S_2$ admits a faithful C*-representation, where S_2 denotes the free monoid on two generators $S_2 = \langle a, b\rangle$, endowed with the involution determined by $a^* = b$. There is a *-monomorphism $S_\infty \hookrightarrow S_2$ defined by $t_n \mapsto a(a^*)^n a$ $(n \in \mathbb{N})$ and this induces a *-monomorphism $\mathbb{C}S_\infty \hookrightarrow \mathbb{C}S_2$. The result follows. ☐

By a *state* on a unital *-algebra A we mean a linear functional $\mu \colon A \to \mathbb{C}$ satisfying $\langle a^*a, \mu\rangle \geq 0$ $(a \in A)$ and $\langle 1, \mu\rangle = 1$. We say that a state μ is *faithful* if $\langle a^*a, \mu\rangle > 0$ $(a \in A \setminus \{0\})$. A unital *-algebra with a faithful state admits a faithful C*-representation via the GNS representation associated with the state.

The following theorem appears to be folklore in the theory of free products of C^*-algebras; it can be traced back at least to the seminal work of Avitzour [1, Proposition 2.3] (see also [6, Section 4] for a more general result).

Theorem 2.2. *Let A and B be unital $*$-algebras which admit faithful states. Then their free product $A * B$ also admits a faithful state, and hence it has a faithful C^*-representation.*

We make use of this result in our next lemma.

Lemma 2.3. *The unital *-algebra $\mathbb{C}(BC * S_\infty)$ admits a faithful C^*-representation.*

Proof. We first remark that a separable C*-algebra A always admits a faithful state. To see this, note that the unit ball of A^* with the weak*-topology is a compact metric space, and hence also separable. It follows that the set of states $S(A)$ is weak*-separable. Taking $\{\rho_n : n \in \mathbb{N}\}$ to be a dense subset of $S(A)$, we then define $\rho = \sum_{n=1}^\infty 2^{-n}\rho_n$, which is easily seen to be a faithful state on A.

By Lemma 2.1, both $\mathbb{C}(BC)$ and $\mathbb{C}(S_\infty)$ admit C*-completions. Since both of these algebras have countable dimension, their C*-completions are separable, and, as such, each admits a faithful state, which we may then restrict to obtain faithful states on $\mathbb{C}BC$ and $\mathbb{C}S_\infty$. By Theorem 2.2, $(\mathbb{C}BC) * (\mathbb{C}S_\infty) \cong \mathbb{C}(BC * S_\infty)$ admits a faithful C*-representation. □

3. Proof of Theorem 1.1

The main idea of the proof is to embed $\mathbb{C}S_\infty$, which is finite, as a dense *-subalgebra of some C*-completion of $\mathbb{C}(BC * S_\infty)$, which will necessarily be infinite. In fact we have the following:

Lemma 3.1. *The *-algebra $\mathbb{C}S_\infty$ is stably finite.*

Proof. As we remarked in the proof of Lemma 2.1, $\mathbb{C}S_\infty$ embeds into $\mathbb{C}S_2$. It is also clear that, as an algebra, $\mathbb{C}S_2$ embeds into $\mathbb{C}F_2$, where F_2 denotes the free group on two generators. Hence $\mathbb{C}S_\infty$ embeds into $\mathrm{vN}(F_2)$, the group von Neumann algebra of F_2, which is stably finite since it is a C*-algebra with a faithful tracial state. It follows that $\mathbb{C}S_\infty$ is stably finite as well. □

We shall next define a notion of length for elements of $BC * S_\infty$. Indeed, each $u \in (BC * S_\infty) \setminus \{e\}$ has a unique expression of the form $w_1 w_2 \cdots w_n$, for some $n \in \mathbb{N}$ and some $w_1, \ldots, w_n \in (BC \setminus \{e\}) \cup \{t_j, t_j^* : j \in \mathbb{N}\}$, satisfying $w_{i+1} \in \{t_j, t_j^* : j \in \mathbb{N}\}$ whenever $w_i \in BC \setminus \{e\}$ $(i = 1, \ldots, n-1)$. We then define $\operatorname{len} u = n$ for this value of n, and set $\operatorname{len} e = 0$. This also gives a definition of length for elements of S_∞ by considering S_∞ as a submonoid of $BC * S_\infty$ in the natural way. For $m \in \mathbb{N}_0$ we set

$$L_m(BC * S_\infty) = \{u \in BC * S_\infty : \operatorname{len} u \leq m\}, \quad L_m(S_\infty) = \{u \in S_\infty : \operatorname{len} u \leq m\}.$$

We now describe our embedding of $\mathbb{C}S_\infty$ into $\mathbb{C}(BC * S_\infty)$. By Lemma 2.3, $\mathbb{C}(BC * S_\infty)$ has a C*-completion $(A, \|\cdot\|)$. Let $\gamma_n = (n\|\delta_{t_n}\|)^{-1}$ $(n \in \mathbb{N})$ and define elements a_n in $\mathbb{C}(BC * S_\infty)$ by $a_n = \delta_p + \gamma_n \delta_{t_n}$ $(n \in \mathbb{N})$, so that $a_n \to \delta_p$ as $n \to \infty$. Using the universal property of S_∞ we may define a unital *-homomorphism $\varphi \colon \mathbb{C}S_\infty \to \mathbb{C}(BC * S_\infty)$ by setting $\varphi(\delta_{t_n}) = a_n$ $(n \in \mathbb{N})$ and extending to $\mathbb{C}S_\infty$. In what follows, given a monoid S and $s \in S$, δ'_s will denote the linear functional on $\mathbb{C}S$ defined by $\langle \delta_t, \delta'_s \rangle = \mathbb{1}_{s,t}$ $\quad (t \in S)$, where $\mathbb{1}_{s,t} = 1$ if $s = t$ and $\mathbb{1}_{s,t} = 0$ otherwise.

Lemma 3.2. *Let $w \in S_\infty$ with $\operatorname{len} w = m$. Then*

(i) $\varphi(\delta_w) \in \operatorname{span}\{\delta_u : u \in L_m(BC * S_\infty)\}$;

(ii) *for each $y \in L_m(S_\infty)$ we have*

$$\langle \varphi(\delta_y), \delta'_w \rangle \neq 0 \Leftrightarrow y = w.$$

Proof. We proceed by induction on m. When $m = 0$, w is forced to be e and hence, as φ is unital, $\varphi(\delta_e) = \delta_e$, so that (i) is satisfied. In (ii), y is also equal to e, so that (ii) is trivially satisfied as well.

Assume $m \geq 1$ and that (i) and (ii) hold for all elements of $L_{m-1}(S_\infty)$. We can write w as $w = vx$ for some $v \in S_\infty$ with $\operatorname{len} v = m - 1$ and some $x \in \{t_j, t_j^* : j \in \mathbb{N}\}$.

First consider (i). By the induction hypothesis, we can write $\varphi(\delta_v) = \sum_{u \in E} \alpha_u \delta_u$, for some finite set $E \subset L_{m-1}(BC * S_\infty)$ and some scalars $\alpha_u \in \mathbb{C}$ $(u \in E)$. Suppose that $x = t_j$ for some $j \in \mathbb{N}$. Then

$$\varphi(\delta_w) = \varphi(\delta_v)\varphi(\delta_{t_j}) = \left(\sum_{u \in E} \alpha_u \delta_u \right) (\delta_p + \gamma_j \delta_{t_j}) = \sum_{u \in E} \alpha_u \delta_{up} + \alpha_u \gamma_j \delta_{ut_j},$$

which belongs to $\operatorname{span} \{\delta_u : u \in L_m(BC * S_\infty)\}$ because

$$\operatorname{len}(up) \leq \operatorname{len}(u) + 1 \leq m \quad \text{and} \quad \operatorname{len}(ut_j) = \operatorname{len}(u) + 1 \leq m$$

for each $u \in L_{m-1}(BC * S_\infty)$. The case $x = t_j^*$ is established analogously.

Next consider (ii). Let $y \in L_m(S_\infty)$. If $\operatorname{len} y \leq m - 1$ then, by (i), we know that $\varphi(\delta_y) \in \operatorname{span} \{\delta_u : u \in L_{m-1}(BC * S_\infty)\} \subset \ker \delta'_w$. Hence in this case $y \neq w$ and $\langle \varphi(\delta_y), \delta'_w \rangle = 0$.

Now suppose instead that $\operatorname{len} y = m$, and write $y = uz$ for some $u \in L_{m-1}(S_\infty)$ and $z \in \{t_j, t_j^* : j \in \mathbb{N}\}$. By (i) we may write $\varphi(\delta_u) = \sum_{s \in F} \beta_s \delta_s$ for some finite subset $F \subset L_{m-1}(BC * S_\infty)$ and some scalars $\beta_s \in \mathbb{C}$ $(s \in F)$, and we may assume that $v \in F$ (possibly with $\beta_v = 0$). We prove the result in the case that $z = t_j$ for some $j \in \mathbb{N}$, with the argument for the case $z = t_j^*$ being almost identical. We have $\varphi(\delta_z) = \delta_p + \gamma_j \delta_{t_j}$ and it follows that

$$\varphi(\delta_y) = \varphi(\delta_u)\varphi(\delta_z) = \sum_{s \in F} \beta_s \delta_{sp} + \beta_s \gamma_j \delta_{st_j}.$$

Observe that $sp \neq w$ for each $s \in F$. This is because we either have $\operatorname{len}(sp) < m = \operatorname{len}(w)$, or else sp ends in p when considered as a word over the alphabet $\{p, p^*\} \cup \{t_j, t_j^* : j \in \mathbb{N}\}$, whereas $w \in S_\infty$. Moreover, given $s \in F$, $st_j = w = vx$ if and only if $s = v$ and $t_j = x$. Hence

$$\langle \varphi(\delta_y), \delta'_w \rangle = \beta_v \gamma_j \mathbb{1}_{t_j, x} = \langle \varphi(\delta_u), \delta'_v \rangle \gamma_j \mathbb{1}_{t_j, x}.$$

As $\gamma_j > 0$, this implies that $\langle \varphi(\delta_y), \delta'_w \rangle \neq 0$ if and only if $\langle \varphi(\delta_u), \delta'_v \rangle \neq 0$ and $t_j = x$, which, by the induction hypothesis, occurs if and only if $u = v$ and $t_j = x$. This final statement is equivalent to $y = w$. □

Corollary 3.3. *The map φ is injective.*

Proof. Assume towards a contradiction that $\sum_{u \in F} \alpha_u \delta_u \in \ker \varphi$ for some non-empty finite set $F \subset S_\infty$ and $\alpha_u \in \mathbb{C} \setminus \{0\}$ $(u \in F)$. Take $w \in F$ of maximal length. Then

$$0 = \left\langle \varphi \left(\sum_{u \in F} \alpha_u \delta_u \right), \delta'_w \right\rangle = \sum_{u \in F} \alpha_u \langle \varphi(\delta_u), \delta'_w \rangle = \alpha_w \langle \varphi(\delta_w), \delta'_w \rangle,$$

where the final equality follows from Lemma 3.2(ii). That lemma also tells us that $\langle \varphi(\delta_w), \delta'_w \rangle \neq 0$, forcing $\alpha_w = 0$, a contradiction. □

We can now prove our main theorem.

Proof of Theorem 1.1. Recall that $(A, \|\cdot\|)$ denotes a C*-completion of $\mathbb{C}(BC * S_\infty)$, which exists by Lemma 2.3, and A is infinite since $\delta_p, \delta_q \in A$. Let $A_0 \subset A$ be the image of φ. Corollary 3.3 implies that $A_0 \cong \mathbb{C}S_\infty$, which is stably finite by Lemma 3.1. Moreover, $\varphi(\delta_{t_n}) = a_n \to \delta_p$ as $n \to \infty$, so that $\delta_p \in \overline{A_0}$, and we see also that $\delta_{t_n} = \frac{1}{\gamma_n}(a_n - \delta_p) \in \overline{A_0}$ $(n \in \mathbb{N})$. The elements δ_p and δ_{t_n} $(n \in \mathbb{N})$ generate A as a C*-algebra, and since $\overline{A_0}$ is a C*-subalgebra containing them, we must have $A = \overline{A_0}$, which completes the proof. □

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