The Numbers Lead a Dance

Mathematics of the Sestina

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Abstract.

Sestinas are poems of 39 lines comprising six verses of six lines each, and a three line final verse or 'envoi'. The structure of the sestina is built around word repetition rather than strict rhyme. Each verse uses the same set line ending words, but in a permuted order. The form of the permutation is highly specific, and is equivalent to iteration of the tent map. This paper considers for which number N of verses, other than 6, can a sestina-like poem be formed. That is, which N will the prescribed permutation lead to a poem of N verses where no two verses have the same order of their end words. In so doing, a link is found between permutation groups, chaotic dynamics, and Cunningham numbers.

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1. Introduction

Sestinas are a form of highly complex poems designed around a particular pattern, see e.g. [6, 13]. Each verse of the poem has six lines and there are 6 verses in total. In addition there is a coda, called an *envoi* that contains just three lines. For the main poem, the final word of each line is crucial. The collection of six such end-words is invariant from verse to verse, yet the word order is permuted. The permutation from one verse to the next takes a specific form. The idea is similar to that of a rifle shuffle of a pack of cards. The list of words is split in two and the words from the second half are meshed with the words from the first half, but in reverse order. Thus, what was the last word is now first, what was the penultimate is now third, etc. This mixing is is sometimes represented in a spiral pattern, as illustrated in Figure 1. Schimel[11] describes the sestina as

> "like a *dance* [12], with each stanza representing a reel. Each stanza is based on the stanza directly preceding it. The order of the stanza peels off

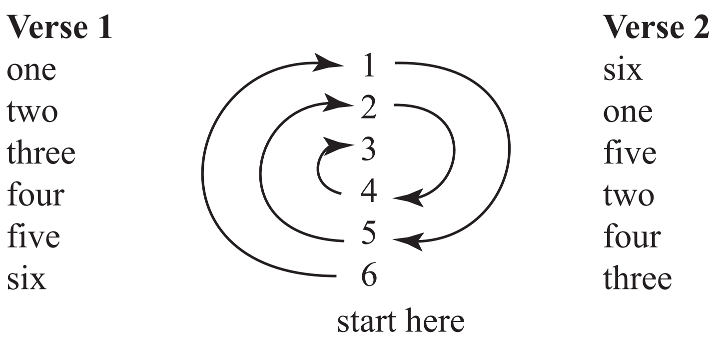

Figure 1. Illustrating the permutation of the order of the end words when passing from the first to the second verse. Here "one" to "six" represent the end words used in the first verse, and the numbers 1 to 6 represent the position within a verse. This spiral illustration, found in several poetry text books e.g. [6, 13], appears somewhat confusing as it is does not represent the actual permutation map. Instead, the arrangement of the second verse is found by following the spiral, beginning with 6.

> the lines of the prior stanza, moving ever inwards towards the core: last, first, penultimate, second, antepenultimate, third."

And what of the envoi? It contains all six end words, two per line, with half of the end-words being placed somewhere within the body of the line, and half at the end. It is like a closing passage of the dance in double time. Some versions of the sestina demand a strict order for the placement of the six words on these lines,

$$\begin{array}{ll} \ldots 5 & \ldots 2 \\ \ldots 3 & \ldots 4 \\ \ldots 1 & \ldots 6 \end{array}$$

some poets chose to vary the form by using synonyms, or otherwise shake up the form, but the reason behind these alterations has to be implicit in either the imagery or the language e.g., the numbers of a rocket countdown. Such numbers might be fudged if the poem is about someone setting off fireworks in their garden and one firework goes off unexpectedly early.

The envoi is a crucial part of each sestina, but because it is not involved directly in the permutation of words from verse to verse, we shall ignore the envoi in the mathematical arguments that follow.

Scholars continue to debate on the sestina's precise origins because of the volume of poems thought to be comprised of sextains (six line verses) between the 11th and 13th centuries. However, the form's invention is commonly attributed to early 13th century literary giant Arnaut Daniel with his poem (English translation: *The Firm Desire That Entered My Heart*)[1] At the time his poetic abilities were incontestable and it set a deliberate challenge to those around him to make the poetry of courtship as tough as possible by forcing the poet to utilise a pattern of repeated words while still keeping their bride to be entertained. It also was incidentally quite pioneering, exhibiting a kind of free verse, years ahead of other exponents of free verse like Christopher Smart or Walt Whitman, with attention largely focused on content over rhyme. While any challengers would be able to express themselves in a manner that was strictly regimented it also allowed allowed poets to dispense with strict rhyme and meter. This is probably one of the main reasons why in the 20th Century he was championed by the likes of T.S. Eliot and Ezra Pound. Both Eliot and Pound were exponents of an imagistic poetry which was sparse, and that lacked the romance, nostalgia and high rhetoric that had so dominated the poetry of the previous century.

Think of poems such as Tennyson's *The Charge of the Light Brigade* or Wordsworth's *Daffodils* and the line, 'I wandered lonely as a cloud'. Poetry's horizons opened up under the modernists as they fractured the syntax and rhyme and formal conventions of poetry, and now a poem could be made from found text, and fragments of dialogue,

[1] We shall not give precise references to the poems or poets mentioned in this introduction; this is, after all, primarily an article on *mathematics* and the text of all the poems can easily be found online.

and musical hall songs, or it could be made to look like rain trickling down a page, and most importantly of all, the poem was no longer a piece of text reflecting on an event, but it was the event itself. This shifted the spotlight to both the form a poem could take, and how this interacted directly with its meaning. It was as if someone in an art gallery pointed at a portrait and suddenly made the pronouncement that the frame the portrait was sitting in was just as integral to the artwork's meaning as the painting itself. This shift in approach is still felt today, and particularly in the workings of the sestina. The sestina structure itself is what frames the poem, and its structure, therefore is subject to the same scrutiny as the poem's literal meaning.

Well composed sestinas can either make the word repetition seem utterly necessary to the unfolding narrative, which is the conventional view, or they can emphasize each endword deliberately to bring the structure more to the fore. Great modern examples of the form include John Ashbery's *The Painter* and Paul Muldoon's extraordinary work *Yarrow*. There are several modern literary journals such as McSweeneys in San Francisco that have in the past, purposefully asked for sestinas to ward off amateur poets and mention the word ‘sestina’ to any poetry workshop now and you can expect a sharp intake of breath from around the room because of its infamous complexity.

Mathematically, we can describe the sestina permutation as follows. Let m be the number of m-line verses and let n represent the word that is at the end of the nth line of verse p. Then the position in the $(p+1)st$ verse is given by the rule

$$n \mapsto \begin{cases} 2n & \text{if } n \leq \left[\frac{m}{2}\right] \\ 2m+1-2n & \text{if } \left[\frac{m}{2}\right] < n \leq m \end{cases}, \tag{1}$$

where m is the number of lines in a verse and $[\cdot]$ represents the integer part of an expression. Thus, for $m = 6$ we have

$$1 \mapsto 2, \quad 2 \mapsto 4, \quad 3 \mapsto 6, \quad 4 \mapsto 5, \quad 5 \mapsto 3, \quad 6 \mapsto 1, \tag{2}$$

as constructed in Figure 1.

For a sestina to work properly, each of the end-words, should have a turn at the end of nth line of a verse, for each $n = 1, 2, \ldots m$. This indeed occurs if $m = 6$ as indicated in Table 1 and illustrated in the poems embedded in this article. Here, each word has a turn in each position, and each line within a verse sees each end-word precisely once during the poem. If one were to construct a seventh verse according to the rule (1), then the order of the end-words would be identical to that of the first verse. Thus we find that the permutation forms a cycle of length six.

The question we wish to address in this article is for which other verse lengths m does this symmetric, egalitarian distribution of line order among the end-words occur if we use the same basic rule (1) from verse to verse? A simple test shows, for example, that something goes wrong if $m = 7$, or $m = 8$, see Tables 2 and 3 respectively.

verse	one	two	three	four	five	six
1st line	1	6	3	5	4	2
2nd line	2	1	6	3	5	4
3rd line	3	5	4	2	1	6
4th line	4	2	1	6	3	5
5th line	5	4	2	1	6	3
6th line	6	3	5	4	2	1

Table 1. Position of the end-words of each line among the six verses of a six-line sestina. Here, the number 1 represents the word that ends the first line of the first verse, 2 represents the word that ends the second line of the first verse, etc. The word in the first line moves to the second line, the word in the second line moves to the fourth line and so on.

verse	one	two	three	four	five	six	seven
1st line	1	7	4	2	1	7	4
2nd line	2	1	7	4	2	1	7
3rd line	3	6	3	3	3	6	3
4th line	4	2	1	7	4	2	1
5th line	5	5	5	5	5	5	5
6th line	6	3	6	3	6	3	6
7th line	7	4	2	1	7	4	2

Table 2. Similar to Table 1 but for $m = 7$. Note the fifth line of each verse always ends with the same word. That is, the number 5 is a fixed point of the rule (1). Also the sixth and third lines share the same two words repeatedly; (6 3) is a period-two cycle of (1).

verse	one	two	three	four	five	six	seven	eight
1st line	1	8	4	2	1	8	4	2
2nd line	2	1	8	4	2	1	8	4
3rd line	3	7	5	6	3	7	5	6
4th line	4	2	1	8	4	2	1	8
5th line	5	6	3	7	5	6	3	7
6th line	6	3	7	5	6	3	7	5
7th line	7	5	6	3	7	5	6	3
8th line	8	4	2	1	8	4	2	1

Table 3. Similar to Table 1 but for $m = 8$. Note that the pattern repeats at the fifth verse, so that the word that end the first line of the first verse only ever ends the first, second, fourth and eighth lines of any subsequent verse, never the third, fifth, sixth or seventh. In fact, $(1, 8, 4, 2)$ is a period-four cycle of the rule (1), and so is $(3, 7, 5, 6)$.

There are also poems known as double sestinas, which have $m = 12$. For example, one of the first known sestina in English, *The Complaint of Lisa* by Algernon Charles Swinburne, is actually a double, although the permutation of the end-words does not follow the rule (1) but appears somewhat random. We shall see shortly that there is also a problem applying rule (1) when $m = 12$.

Strictly speaking, we should always choose m to be even in order for their to be an envoi of length $m/2$ with each line containing two end words. But, for purposes of the mathematics, we shall ignore this restriction. Poets who engage on highly regular forms like sestinas tend to do it in part for the challenge, and the requirement for a non-integer number of lines in the envoi could provide an opportunity to subvert the form creatively. Of course, the phrase "poetic license" springs to mind.

2. Recent history

Before proceeding, it might be interesting to point out the happenstance that led to this article being written. It started with a chance meeting more than 10 years ago between the first and last author on their regular daily commute from the same bus stop on the outskirts of Bath to Bristol. Harry at the time was on a placement with a publishing house following a successful MA in Creative Writing at Bath Spa University. Alan had recently become the youngest ever head of Department of Engineering Mathematics at Bristol. Harry had written the first two stanzas of a sestina, his opus magnus, with $m = 78$.

Harry initially decided upon the number 78 for the simple fact that it was twice the length (39) of the total number of lines of a conventional sestina including its envoi. A familiar construction technique to most poets in this form is write down on a blank piece of paper a guide to illustrate which words are going to arrive in what position allowing the poem to be grafted onto this template and adjusted. Aware that the total number of lines was now 6,123 it became necessary for Harry to plot out the new larger sestina using a fractionally more adept system than pen and paper! Microsoft Excel provided him with a chart illustrating the word positions for each verse. To his horror, he noticed that already at verse 26, the entire sestina collapsed to the original order and renewed its cycle once more, something that should only happen in a hypothetical extra verse prior to the final envoi. The pleasure in the reading of a sestina for most literary critics is precisely its strategic avoidance of this outcome!

With all the carefully chosen words in play already, and all the source material under his belt and the idea firmly in his mind, there was little that could be done to remedy the problem except for the increasingly large possibility of it all ending up in the wastepaper basket. Not to be outdone by this setback he began to try and establish a means by which to accurately predict the relationship between the number of lines in a verse and the potentially disastrous outcome of word positions prematurely coming back to their original order and thus to restructure the two original verses while causing minimum damage to the sense of the writing. So should the sestina be grown or shrunk?

Unknown to us, a solution was actually available in French, starting with the work in the 1960s of the French Poet Raymond Queneau and his colleague the mathematician Jacques Roubaud [8, 9]. Those results were recently summarised in the excellent article by Michael Saclolo in *Notices of the American Mathematical Society* [10]. Queneau asked exactly the same question. For what numbers m is an m-ina possible? In French a sestina is called a *sestine* and so Roubard coined the phrase q-ines or *quenines* in honour of Queneau for the admissible q-verse poem. This was later formalised by Monique Bringer [2], a student of Roubard, who coined the phase *Queneau-Daniell group* for the cyclic subgroup of the general linear group of order m generated by the quinine permutation. She was able to provide a partial set of necessary and sufficient conditions for admissible numbers m. The complete characterisation of which numbers m are admissible was not actually solved until 2008, in the work of Dumas [5], whose results are reproduced in English in [10]. Dumas' proof is not however constructive, in the following sense. It is easy to state necessary and sufficient conditions for a number p to be prime, but there is no simple checkable formula that generates the m'th prime number. So it seems to be also with sestina numbers.

In what follows we describe an investigation of the generalisation of a sestina, which was arrived at independently of the French group theorists. In so doing so, we uncover an alternative view, establishing a connection with a different branch of mathematics, namely chaotic dynamical systems.

3. Permutation groups

We call m a **sestina number** if the permutation represented by (1) on the set of m integers has minimal period m. Let us recall some basic facts from permutation group theory.

The group of all permutations of m symbols is denoted by S_n. Basic theorems [7] tell us that any element of the group has a unique minimal representation in terms of disjoint cycles. Take the permutation (1) with $m = 6$ as described in (2) and Table 1. A far more compact way of writing this is to look at the orbit of the position of the first end-word after each successive application of the permutation: That is, following the arrows around the circle of (2), we see that the first end-word of the first verse, becomes the second end-word of the second verse, the fourth end-word of the third verse, the 5th end-word of the fourth verse and so on. The circular representation also allows us to find the orbit of any other end-word. For example, to see what happens to the third end-word of the first verse, we start at the number 3 on the clock face and

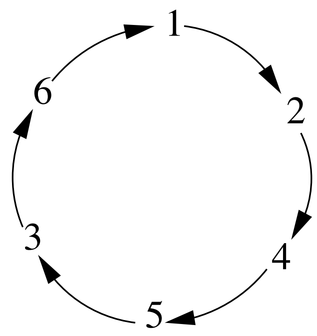

Figure 2. The orbit of the end-words for a sestina of length $m = 6$

follow the arrows six times. So, in the second verse, this word ends the sixth line; it ends the first line of the third verse and so on. This then leads to the more compact notation

$$(1, 2, 4, 5, 3, 6),$$

where the round brackets mean "and repeat". Now $m = 6$ is a sestina number because there is representation of the effect of the transformation (1) in terms of a single cycle.

If we try the same for $m = 7$, based on the information in table 2 we see that the permutation is now written

$$(1, 2, 4, 7)(3, 6)(5)$$

which has three separate disjoint cycles. The end-words of lines one, two, four and seven cycle, whereas lines three and six swap end-words between each successive verse, while the fifth line ends with the same word each time.

Similarly for $m = 8$, we have

$$(1, 2, 4, 8)(3, 6, 5, 7)$$

two four-cycles, and for $m = 12$ we have

$$(1, 2, 4, 8, 9, 7, 11, 3, 6, 12)(5, 10)$$

a 10-cycle and a 2-cycle. This latter case shows the difficulty of trying to construct a "double sestina" using the same transformation (1) as for the standard six-verse transformation.

Thus, we have established a criterion for m to be a sestina number; namely that the permutation (1) can be expressed as a single cycle of length m. Table 4 lists the disjoint cycle representation for the first few m. Note that there is no obvious pattern governing

m	cycle representation	sestina number ?
1	(1)	yes
2	(12)	yes
3	(123)	yes
4	(124)(3)	no
5	(12435)	yes
6	(124536)	**yes**
7	(1247)(36)(5)	no
8	(1248)(3657)	no

Table 4. Cycle structure of the sestina permutation for the first few values of m.

which m's lead to a single m-cycle and which do not. It is precisely this pattern that we aim to uncover in the rest of this article.

4. A connection with chaotic dynamics

The equation (1) can be represented as discrete-time dynamical system acting on the first m integers. A simple re-scaling, letting $y = 2n/(2m+1)$, shows that repeated iteration of (1) is equivalent to the dynamics of the **tent map** for $y \in [0, 1]$:

$$y \mapsto \begin{cases} 2y & \text{if } y \le 1/2, \\ 2 - 2y & \text{if } 1/2 < y \le 1. \end{cases} \tag{3}$$

Instead of the integers from 1 to m we now have the points $2j/(2m+1), j = 1, \cdots m$ distributed between 0 and 1. For any value of m we will call these points **sestina points**.

The dynamics of the map is represented graphically in Fig. 4. To be more precise, this is the tent map with slope 2, which is part of the general family of tent maps

$$y \mapsto \begin{cases} \mu y & \text{if } y \le 1/2, \\ \mu(1-y) & \text{if } 1/2 < y \le 1, \end{cases} \tag{4}$$

with slope $\mu > 0$ [4]. Straightforward analysis shows that if $\mu < 1$, then the fixed point $x = 0$ is the unique attractor of the system. That is, all initial conditions will eventually converge towards $x = 0$ under repeated iteration of (4). If $\mu = 1$, then all points with $y \le 1/2$ are fixed points of this dynamical system.

It is when $\mu > 1$ that things get interesting. See Figure 4. In fact, among chaotic maps, the tent map is rather special because of the sharp point at $y = 1/2$. So as μ increases through 1, rather than a Feigenbaum period-doubling cascade that is familiar

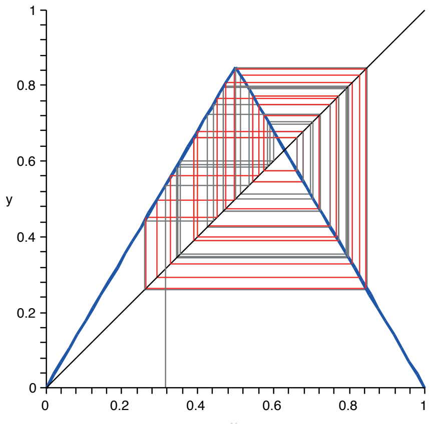

Figure 3. Constructing the dynamics of the tent map via the so-called cobwebbing process. Here y is replaced at the next unit of time by its value given by the formula (3). This value is then fed back as the next value of y into the same formula, and so-on. This feedback process is represented as the reflection of the value of the image of a given y-value in the 45° line.

to all who have studied smooth chaotic dynamical systems (see e.g. [4]) the dynamics immediately becomes chaotic. There are still two fixed points, $y = 0$ and $y = \mu/(\mu + 1)$, but both are unstable. That is, if you choose an initial condition arbitrarily close to one of these points, it moves away under iteration. For $1 < \mu < \sqrt{2}$ then the attractor of the map splits into two non-overlapping sub-intervals of $(0, 1)$, Arbitrary initial conditions are attracted to these two sub-intervals within which there is a chaotic cycling of points. For $\sqrt{2} \leq \mu \leq 2$ the seperate intervals start to overlap.

For $\mu = 2$ the map is fully chaotic. That is, almost all initial conditions are part of the chaotic set and each region of the chaotic set are visited with equal probability. Starting from some arbitrary y-value in the interval $(0, 1)$ and repeatedly iterating the formula (3), we reach a infinite sequence of further y-values that never repeat. Nevertheless, this sequence eventually visits arbitrary close to *every* y-value in the interval $[0, 1]$. Moreover, no points that start in this interval ever escape. That is, the interval $[0, 1]$ is the unique chaotic attractor of the dynamics.

However, embedded within the chaos are a (countable) infinity of unstable periodic orbits with all possible periods. In particular, all rational initial conditions of (3) lie on periodic orbits. To see this, note that if an initial condition $y = p/q$ for integers p and q then all forward images of this point must be expressible as a fraction r/q for some integer r. Moreover, the map takes the unit interval to itself, hence $0 \leq r \leq q$. Since there are only $q + 1$ such fractions, this must be a periodic orbit of period at most $q + 1$. In particular we are interested in the case that $q = N$ for odd $N = 2m + 1$ and $p = 2n$ for some $n \leq m$.

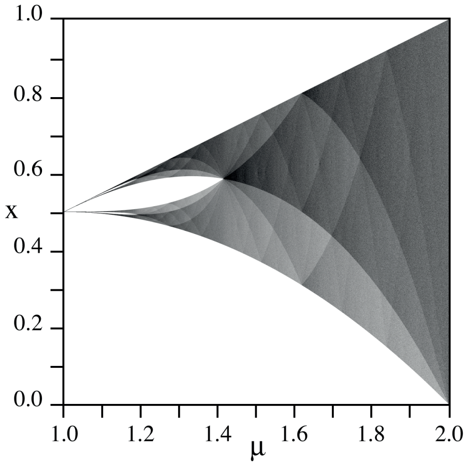

Figure 4. Bifurcation diagram showing points on the attractor of the tent map (4) for $0 \le \mu \le 2$. At $\mu = 1$

The question we seek to address then is: what is the image under repeated iteration of (3) of the specific initial condition $y = 2/(2m+1)$, for each odd integer $2m+1$? If this orbit has minimum period m then we say that m is a sestina number. The only other possibility is that this initial condition lies on a periodic orbit with a lower period q. So, it seems we must look at conditions for the existence of periodic orbits of (3) (and hence of (1)) of arbitrary period $q \le m$.

5. Conditions for cycles

The example in Table 2 above shows that $m = 7$ fails to be a sestina number because there exists a fixed point (a 1-cycle) and a 2-cycle. Also, from Table 3, $m = 8$ fails to be a sestina number because the permutation is decomposed into two disjoint 4-cycles. So in order to characterise which numbers are *not* sestina numbers, we need to consider conditions for a position j $(0 < j \le m)$ to be part of a period-q cycle for $q \le m$.

Consider first the case of a fixed point. The fixed point for the tent map is at the intersection between the map and the line $x = y$, and (disregarding the trivial fixed point $x = 0$ which will not be relevant here) occurs at $x = 2/3$. If one of the m sestina points, $x_j = 2j/(2m+1), j = 1, \cdots, m$ happens to coincide with the value $x = 2/3$, then a 1-cycle will occur, and the number m (if different from 1) will fail as a sestina number. This will happen for all numbers m such that

$$\frac{2j}{2m+1} = \frac{2}{3}$$

or

$$3|(2m+1)$$

and is obviously the case for $m = 7$.

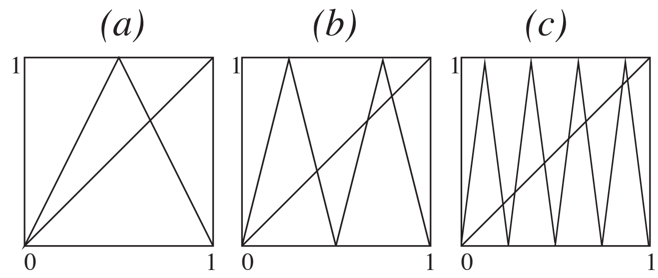

Figure 5. Location of (a) period-1 (fixpoint), (b) period-2 and (c) period-3 points for the tent map is the abcissa for the intersection between repeated tents and the line $y = x$, see e.g., ([4]).

If we study the condition for 2-cycles, we must find the loci for period-2 points of the tent map. These points are located where the *twice repeated* tent map intersects the line $x = y$, i.e., at $x = 2/5, 2/3, 4/5$, see figure 5. For sestina points to coincide with these values, we find that in addition to $3|(2m+1)$ that

$$\frac{2j}{2m+1} = \frac{2}{5} \quad \text{or} \quad \frac{2j}{2m+1} = \frac{2}{3} \quad \text{or} \quad \frac{2j}{2m+1} = \frac{4}{5}$$

The middle condition gives us $3|(2m+1)$ (because a period 1 orbit is also a period 2 orbit) but we now also have to exclude

$$5|(2m+1)$$

to avoid period-2 orbits, so this condition prevents m (if different from 2) from being a sestina number. For the value $m = 7$ we have both a 1-cycle and a 2-cycle present, since both 3 and 5 are factors of $(2m+1)$.

3-cycles occur see figure 5 at the $2^3 - 1$ values $x = 2/9, 2/7, 4/9, 4/7, 6/9, 6/7, 8/9$, and they will coincide with sestina values if $7|(2m+1)$ or $9|(2m+1)$.

Continuing in this manner, one finds:

Proposition 1. *The q-cycle points are located at*

$$x = \frac{2}{2^q+1}, \frac{2}{2^q-1}, \frac{4}{2^q+1}, \frac{4}{2^q-1}, \cdots, \frac{2k}{2^q-1}, \cdots, \frac{2^q-2}{2^q+1}, \frac{2^q-2}{2^q-1}, \frac{2^q}{2^q+1}$$

□

All in all there are $2^q - 1$ such points.

If there is a j such that one of the sestina points $x = 2j/(2m+1)$ coincides with a q-cycle point, then the sestina permutation will contain a q-cycle.

This happens when $\exists j, k \in \mathbb{N}$, $k = 1, ..m$ and $k = 1, ..2^{m-1}$ such that

$$\frac{2j}{2m+1} = \frac{2k}{2^q \pm 1}$$

or

$$k(2m+1) = j(2^q \pm 1)$$

Here, $\pm$ is taken as “plus or minus“.

The necessary existence of at least one q-cycle ($q \leq m$) for a sestina permutation over m can be noted in the following

Proposition 2. *For any odd number* $2m+1$ *there must be a number* $q \leq m$ *such that* $(2m+1)|(2^q \pm 1)$.

□

We are now in position to give a necessary and sufficient conditions for a number m to be a sestina number. The first sestina point ($j = 1$) must be part of a m-cycle which takes it to all the other positions, i.e., the m-cycle is not caused by successive q-cycles where q is a factor of m:

Theorem 1. *A number m is a sestina number if and only if* $(2m+1)|(2^m \pm 1)$ *and* $(2m+1) \nmid (2^q \pm 1)$ *for any q which is a factor of m.* □

Unfortunately Theorem 1 is not constructive, since in order to check whether an arbitrary m is a sestina number, we have to factorise several potentially very large numbers. In particular, it is not clear from the theorem how many sestina numbers there are, even if there are infinitely many or not.

The following corollaries establish some more information.

Corollary 1. *For m to be a sestina number,* $2m+1$ *must be prime.*

Proof. Suppose that $2m + 1$ is composite, let $2m + 1 = rs$. where r and s are both odd and smaller than $2m + 1$ By the above corollary applied to r, there must exist a $q \leq (r-1)/2$ such that r divides $2^q \pm 1$. Let $(2^q \pm 1) = rk$. Now we have $k(2m+1) = (2^q \pm 1)s$ where, by construction, $k < 2^{q-1}$ and $s < m$. This is precisely the condition, according to Proposition 1 for the existence of a q-cycle. But $q < m$ by construction, and hence m cannot be a sestina number. □

Corollary 2. *Let* $2m+1$ *be a prime number that divides* $2^m \pm 1$. *If m is also a prime, then m is a sestina number.*

Proof. This follows immediately from Theorem 1, since if m is prime its only factors q are 1 and m itself. □

Remarks:

1. Note that Corollary 2 is not a necessary condition for a sestina number. For example, $m = 6$ and $m = 9$ are non-prime sestina numbers.
2. The Corollary does not establish how many sestina numbers there are, but at least we have a simple algorithm for finding sestinas of large length. Take a prime m such that $2m + 1$ is also prime. Test whether $(2m + 1)$ is a factor of $2^m \pm 1$. If it is, then m is a sestina number. Perhaps this could be the point of departure for a study of the cardinality of sestina numbers.
3. When m is itself a prime number, the numbers $2^m - 1$ are the so-called *Mersenne primes*. More generally Primes of the form $2^m \pm 1$ are examples of what are known as *Cunningham Primes* [1]. Such numbers are named after the British number theorist who in 1925 [3] started what has become known as the Cunningham project of finding factors of numbers of the form $b^n \pm 1$, for $b = 2, 3, 5, 6, 7, 10, 11, 12$ and large n.

6. Discussion

The above description of sestina numbers is in some way less than satisfactory. It relies on the factorisation of large primes of the form $2^m \pm 1$. As is well known such factorisation is a complex computational task. In fact the brute force approach of simply letting the numbers lead a dance, i.e., iterating the tent map m times, provides a far quicker (order m) method of deciding whether m is a sestina number (in fact this is the essence of Dumas' theorem [5, 10]). Using this method it is a straightforward computational task to construct all the sestina numbers less than a certain positive integer. Here, for example, is a list of all sestina numbers up to $m = 200$:

1, 2, 3, 5, 6, 9, 11, 14, 18, 23, 26, 29, 30, 33, 35, 39, 41, 50, 51, 53, 65, 69, 74, 81, 83, 86, 89, 90, 95, 98, 99, 105, 113, 119, 131, 134, 135, 146, 155, 158, 173, 174, 179, 183 ,186, 189, 191, 194

Also, the characterisation given here does not immediately tell us whether there are infinitely many sestina numbers or not. This question is still open.

Finally, we return to the original motivation to this article. How to construct a sestina with $m = 78$. Here $2m + 1$ is prime but m isn't. The permutation (1) splits into three 26-cycles:

(1,2,4,8,16,32,64,29,58,41,75,7,14,28,56,45,67,23,46,65,27,54,49,59,39,78)
(3,6,12,24,48,61,35,70,17,34,68,21,42,73,11,22,44,69,19,38,76,5,10,20,40,77)
(9,18,36,72,13,26,52,53,51,55,47,63,31,62,33,66,25,50,57,43,71,15,30,60,37,74)

So $m = 78$ is not a sestina number as defined here; the usual method of making a sestina of this length will not work. Instead, an alternative strategy might be to use the basic sestina permutation (1) 25 times to generate the first 26 verses, then apply something else to perturb the situation so that we do not get locked into a 26 cycle. One example of such a perturbation can be found by noticing that each successive cycle in the above 26-cycles is the image of the previous one under

$$n \mapsto \begin{cases} 2n & \text{if } n \leq \left[\frac{m}{3}\right] \\ 2m + 1 - 3n & \text{if } \left[\frac{m}{3}\right] < n \leq \left[\frac{2m}{3}\right] \\ 3n - (2m + 1) & \text{if } \left[\frac{2m}{3}\right] < n \leq m \end{cases} \tag{5}$$

Hence we can apply this transformation to create the 27th verse, followed by 25 more applications of the basic permutation (1) to create verses 28 to 52, one more application of (5) to create verse 53, finishing off with a final 25 iterations of (1) to complete the sestina.

However, this is a mathematical solution. From the point of view of a contemporary poet it might be better for the variation to occur within the poem itself; creatively intentional rather than merely because of the sestina number's mathematical torsion. Instead, the poet could take the opportunity to to build a relationship between the number 26 and the poem's content. The obvious example being the number of letters in the alphabet. Harry is still writing his *magnum opus*.

7. Afterword

This article itself has, in fact, undergone a merry dance. The original chance encounter mentioned in section 2 happened some ten years or so ago. Some of the theory was worked out at the time and presented at the British Applied Mathematics Colloquium which in the year 2007 was held in Bristol. Harry provided an impromptu performance sestina at the event. Then both authors went back to their day jobs and no article was written. Harry is now fulfilling his then dream ambition to be a published poet and Alan, having completed his stint as Head of Department and other managerial roles, continues to slug it out as a regular engineering mathematics professor.

And so it would have remained had it not been for another chance encounter some fifteen years or so before that, coincidentally also in Bristol. Alan, then a finishing PhD at Oxford, got chatting with Poul, then a recently appointed Assistant Prof at the Technical University of Denmark, at a conference on the *Dynamics of Numerics and the Numerics of Dynamics*. It was quickly established that, in addition to scientific interests in common, each has the same quirky sense of humour and attitude to life. An invitation to Lyngby for the following year ensued and together they studied the dynamics of chaos amid the beautiful deer park there. A lifelong friendship has become established, but no joint publication has ever result from their collaboration. Until now. A few years ago, chatting over a pint of British real ale, Poul expressed the desire to pick up the pathetic half-finished manuscript that had resulted from Alan and Harry's original collaboration. He soon discovered Saclolo's article and the preceding work of the French poets and group theorists. We had been scooped. So, once again our desire to publish seems to have been thwarted.

Then Poul was invited to speak at the birthday symposium for another longstanding scientific friend, Helge Holden. The story of the sestina mathematics and its links to chaos was once again resurrected, and Poul was even inspired to compose a sestina for Helge (see elsewhere in this Volume). An invitation to write a paper based on his talk has put new impetus into the collaboration; Alan and Harry are now back in touch after their paths were separated. And so the number dance continues.

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