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Online Appendix for Peer Effects in Endogenous Networks: One-sided Link Formation

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Abstract

We solve for the case of one-sided network formation and show that all Nash equilibrium networks are again nested split graphs.

1 Model Description

The one-sided specification differs from the two-sided model in that, in order for a pair of agents to benefit from each other's effort level, only one agent needs to extend a link (and bear the cost). This allows us to use Nash equilibrium as equilibrium concept. Note that under pairwise Nash equilibrium pairs of agents can create only *one* link at a time and *both* agents may adjust their effort levels. Under Nash equilibrium we consider deviations where an agent may extend *multiple* links (and simultaneously delete any subset of existing ones), but only the (single) deviating agent may adjust effort levels. Note that, since the respective other agent in a deviation now does not alter his effort level, we now need strict convexity of the value function for our characterization to go through.

Let again $N = \{1, 2, ..., n\}$ be the set of players with $n \geq 3$. As before, each player i chooses a personal effort level $x_i \in X$ and a set of links, which are represented as a row vector $\mathbf{g}_i = (g_{i,1}, ..., g_{ii-1}, g_{ii+1}, ..., g_{in})$, where $g_{ij} \in \{0, 1\}$ for each $j \in N \backslash \{i\}$. Assume $X = [0, +\infty)$ and $\mathbf{g}_i \in G_i = \{0, 1\}^{n-1}$. The set of strategies of i is denoted by $S_i = X \times G_i$ and the set of strategies of all players by $S = S_1 \times S_2 \times ... \times S_n$. A strategy profile $\mathbf{s} = (\mathbf{x}, \mathbf{g}) \in S$ again specifies the individual effort level of each player, $\mathbf{x} = (x_1, x_2, ..., x_n)$, and a set of links $\mathbf{g} = (\mathbf{g}_1, \mathbf{g}_2, ..., \mathbf{g}_n)$. Agent i is said to sustain or extend a link to j if $g_{i,j} = 1$ and to receive a link from j if $g_{j,i} = 1$. The network of relations $\mathbf{g}$ is a directed graph, i.e. it is possible that $g_{i,j} \neq g_{j,i}$. Let $N_i(\mathbf{g}) = \{j \in N : g_{i,j} = 1\}$ be the set of agents i has extended a link to and define $\eta_i(\mathbf{g}) = |N_i(\mathbf{g})|$. Call again the closure of $\mathbf{g}$ an undirected network, which we denote by $\bar{\mathbf{g}} = cl(\mathbf{g})$, where $\bar{g}_{i,j} = \max\{g_{i,j}, g_{j,i}\}$ for each i and j in N. Denote by $N_i(\bar{\mathbf{g}}) = \{j \in N : \bar{g}_{i,j} = 1\}$ the set of players that are directly connected to i in $\bar{\mathbf{g}}$. The effort level of i's direct neighbors can then be written as $y_i(\bar{\mathbf{g}}) = \sum_{j \in N_i(\bar{\mathbf{g}})} x_j$. We will drop the subscript of y_i when it is clear from the context. The network is said to be empty and denoted by $\bar{\mathbf{g}}^e$ if $\bar{g}_{i,j} = 0 \; \forall i, j \in N$ and complete and denoted by $\bar{\mathbf{g}}^c$ if $\bar{g}_{i,j} = 1 \; \forall i, j \in N$.

Payoffs of player i under strategy profile $\mathbf{s} = (\mathbf{x}, \mathbf{g})$ are given by

$$\Pi_i(\mathbf{s}) = \pi_i(\mathbf{x}, \mathbf{g}) - \eta_i(\mathbf{g})\kappa,$$

where κ denotes the cost of extending a link. The assumptions on the payoff function are as in the one-sided specification.

A Nash equilibrium is a strategy profile $\mathbf{s}^* = (\mathbf{x}^*, \mathbf{g}^*)$ such that

$$\Pi_i(\mathbf{s}_i^*, \mathbf{s}_{-i}^*) \geq \Pi_i(\mathbf{s}_i, \mathbf{s}_{-i}^*), \; \forall \mathbf{s}_i \in S_i, \forall i \in N.$$

2 Equilibrium Characterization

We only present the corresponding result for Theorem 1 of the main part of the paper, namely that all Nash networks are nested split graphs. This model is solved in a previous working paper version (Hiller, 2013). Lemma 1 (OA) shows that there can be only one directed link between any two agents.

Lemma 1 (OA): *In any NE, $(\mathbf{x}, \mathbf{g})$, there is at most one directed link between any pair of agents $i, j \in N$.*

Proof. Assume that $(\mathbf{x}, \mathbf{g})$ is a Nash equilibrium and that $g_{i,j} = g_{j,i} = 1$. But then i can profitably deviate by cutting the link to j such that $g_{i,j} = 0$. Gross payoffs remain unchanged, while i's linking total cost decrease by κ. *Q.E.D.*

In Lemma 2 (OA) we show that in any Nash equilibrium if i extends a link to l, then i must also be connected to agent k for any k such that $x_k \geq x_l$. Note that we do not require that i extends a link to k, but only that i and k are connected. That is, it may be agent k extending the link to agent i.

Lemma 2 (OA): *In any NE, $(\mathbf{x}, \mathbf{g})$, if $g_{i,l} = 1$ then $\bar{g}_{i,k} = 1 \; \forall k : x_k \geq x_l$.*

Proof. For $g_{i,j} = 1$ to be part of a NE it must be that $v(y_i(\bar{\mathbf{g}})) - v(y_i(\bar{\mathbf{g}}) - x_l) \geq \kappa$. Assume contrary to the above statement that $\bar{g}_{i,k} = 0$ for some k with $x_k \geq x_l$. This, however, cannot be a NE since i then finds it profitable to extend a link to agent k. To see this, note that $v(y_i(\bar{\mathbf{g}}) + x_k) - v(y_i(\bar{\mathbf{g}})) > v(y_i(\bar{\mathbf{g}})) - v(y_i(\bar{\mathbf{g}}) - x_l) \geq \kappa$, where the inequalities follow from the convexity of the value function. We have reached a contradiction and therefore $\bar{g}_{i,k} = 1$ for all agents k with $x_k \geq x_l$. *Q.E.D.*

The following lemma shows that if i extends a link to l, then any agent k with a higher or equal effort level than i must also be connected to l. Again this follows from the convexity of the value function.

Lemma 3 (OA): *In any NE,* $(\mathbf{x}, \mathbf{g})$*, if* $g_{i,l} = 1$ *then* $\bar{g}_{k,l} = 1$ $\forall k : x_k \geq x_i$*.*

Proof. For $g_{i,j} = 1$ to be part of a NE, it must be that $v(y_i(\bar{\mathbf{g}})) - v(y_i(\bar{\mathbf{g}}) - x_l) \geq \kappa$. Assume contrary to the above statement that $\bar{g}_{k,l} = 0$ for some k with $x_k \geq x_i$. Note next that for $x_k \geq x_i$ to hold we must have $y_k(\bar{\mathbf{g}}) \geq y_i(\bar{\mathbf{g}})$, which follows directly from strict strategic complementarities. Therefore, $v(y_k(\bar{\mathbf{g}}) + x_l) - v(y_k(\bar{\mathbf{g}})) > v(y_i(\bar{\mathbf{g}})) - v(y_i(\bar{\mathbf{g}}) - x_l) \geq \kappa$, where the inequalities follow from the convexity of the value function. We have reached a contradiction. *Q.E.D.*

The following Lemma shows that in any Nash equilibrium if a pair of agents exert same effort levels, then they must share the same neighborhoods. The proof is a direct consequence of the convexity of the value function.

Lemma 4 (OA): *In any NE,* $(\mathbf{x}, \mathbf{g})$*,* $x_i = x_k \Leftrightarrow N_i(\bar{\mathbf{g}}) \setminus \{k\} = N_k(\bar{\mathbf{g}}) \setminus \{i\}$*.*

Proof. First we show that $N_i(\bar{\mathbf{g}}) \setminus \{k\} = N_k(\bar{\mathbf{g}}) \setminus \{i\} \Rightarrow x_i = x_k$. If $\bar{g}_{i,k} = 0$, then $y_i(\bar{\mathbf{g}}) = y_k(\bar{\mathbf{g}})$ and therefore $x_i = x_k$. Assume next that $\bar{g}_{i,k} = 1$ and without loss of generality that $x_i > x_k$. But then $y_i(\bar{\mathbf{g}}) < y_k(\bar{\mathbf{g}})$ and we have reached a contradiction. Next we show that $x_i = x_k \Rightarrow N_i(\bar{\mathbf{g}}) \setminus \{k\} = N_k(\bar{\mathbf{g}}) \setminus \{i\}$. Assume to the contrary that $x_i = x_k$ and $N_i(\bar{\mathbf{g}}) \setminus \{k\} \neq N_k(\bar{\mathbf{g}}) \setminus \{i\}$. Note that for $x_i = x_k$ to hold it must be that $y_i(\bar{\mathbf{g}}) = y_k(\bar{\mathbf{g}})$. For $N_i(\bar{\mathbf{g}}) \setminus \{k\} \neq N_k(\bar{\mathbf{g}}) \setminus \{i\}$ to hold, there must exist an agent l such that $l \in N_k(\bar{\mathbf{g}})$ and $l \notin N_i(\bar{\mathbf{g}})$. For the link $\bar{g}_{k,l} = 1$ to be in place in $\bar{\mathbf{g}}$ we must have that $v(y_k(\bar{\mathbf{g}}) - v(y_k(\bar{\mathbf{g}}) - x_l) \geq \kappa$. But from $y_i(\bar{\mathbf{g}}) = y_k(\bar{\mathbf{g}})$ and the convexity of the value function $v(y_i(\bar{\mathbf{g}}) + x_l) - v(y_k(\bar{\mathbf{g}})) > v(y_k(\bar{\mathbf{g}})) - v(y_k(\bar{\mathbf{g}}) - x_l) \geq \kappa$ holds and we reach a contradiction. *Q.E.D.*

Lemma 5 (OA) shows that in any Nash equilibrium if an agent i exerts a weakly lower effort level than another agent k, then agent i's neighborhood is contained in k's neighborhood.

Lemma 5 (OA): *In any NE,* $(\mathbf{x}, \mathbf{g})$*,* $x_i \leq x_k \Leftrightarrow N_i(\bar{\mathbf{g}}) \setminus \{k\} \subseteq N_k(\bar{\mathbf{g}}) \setminus \{i\}$*.*

Proof. We first show that $N_i(\bar{\mathbf{g}}) \setminus \{k\} \subseteq N_k(\bar{\mathbf{g}}) \setminus \{i\} \Rightarrow x_i \leq x_k$. If $\bar{g}_{i,k} = 0$, then $y_i(\bar{\mathbf{g}}) \leq y_k(\bar{\mathbf{g}})$ and therefore $x_i \leq x_k$. Assume next that $\bar{g}_{i,k} = 1$ and $x_i > x_k$ holds. But then $y_i(\bar{\mathbf{g}}) < y_k(\bar{\mathbf{g}})$ and we have reached a contradiction. Next we show that $x_i \leq x_k \Rightarrow N_i(\bar{\mathbf{g}}) \setminus \{k\} \subseteq N_k(\bar{\mathbf{g}}) \setminus \{i\}$. Assume to the contrary that $x_i \leq x_k$ and there exists an agent l such that $l \in N_i(\bar{\mathbf{g}})$ and $l \notin N_k(\bar{\mathbf{g}})$. For the link $\bar{g}_{i,l} = 1$ to be in place in $\bar{\mathbf{g}}$ either $g_{i,l} = 1$ or $g_{l,i} = 1$. If $g_{i,l} = 1$, then $v(y_i(\bar{\mathbf{g}})) - v(y_i(\bar{\mathbf{g}}) - x_l) \geq \kappa$ must hold. But from $y_i \leq y_k$ and the convexity of the value function we can write $v(y_k(\bar{\mathbf{g}}) + x_l) - v(y_k(\bar{\mathbf{g}})) > v(y_i(\bar{\mathbf{g}})) - v(y_i(\bar{\mathbf{g}}) - x_l) \geq \kappa$ and we have reached a contradiction. We can apply an analogous argument for $g_{l,i} = 1$. *Q.E.D.*

In Theorem 1 (OA) we show that all Nash equilibria are nested split graphs.

Theorem 1 (OA): *In any NE, the network* $\mathbf{g}$ *is a nested split graph.*

Proof. Note first that the complete and the empty network are nested split graphs. We start by showing that in any NE if $\eta_k(\bar{\mathbf{g}}) \geq \eta_l(\bar{\mathbf{g}})$, then $x_k \geq x_l$. Assume to the contrary that $x_l > x_k$. We distinguish two cases. Consider first the case that $\eta_k(\bar{\mathbf{g}}) > \eta_l(\bar{\mathbf{g}})$ holds. Then there exists an agent $m \in N_k(\bar{\mathbf{g}})$ and $m \notin N_l(\bar{\mathbf{g}})$. We consider two subcases. Assume first that agent m extends a link to k and $g_{m,k} = 1$. But then, by $x_l > x_k$ and Lemma 2 (OA), $\bar{g}_{m,l} = 1$ and we have reached a contradiction. Next assume that agent k extends a link to m and $g_{k,m} = 1$. But then, by Lemma 3 (OA), $\bar{g}_{l,m} = 1$ must hold and we have reached a contradiction. Assume next that $\eta_k(\bar{\mathbf{g}}) = \eta(\bar{\mathbf{g}})$. We distinguish two cases. If $N_k(\bar{\mathbf{g}}) \setminus \{l\} = N_l(\bar{\mathbf{g}}) \setminus \{k\}$, then $x_k = x_l$ by Lemma 4 (OA) and we have reached a contradiction. If $N_k(\bar{\mathbf{g}}) \setminus \{l\} \neq N_l(\bar{\mathbf{g}}) \setminus \{k\}$, then there exists an agent $m \in N_k(\bar{\mathbf{g}}) \setminus \{l\}$ and $m \notin N_l(\bar{\mathbf{g}}) \setminus \{k\}$. The argument that $m \in N_l(\bar{\mathbf{g}}) \setminus \{k\}$ must hold is analogous to the previous case. We have established that in any NE if $\eta_k(\bar{\mathbf{g}}) \geq \eta_l(\bar{\mathbf{g}})$ holds, then $x_k \geq x_l$ also holds. Next we show that in any NE if $\bar{g}_{i,l} = 1$ and $\eta_k(\bar{\mathbf{g}}) \geq \eta_l(\bar{\mathbf{g}})$ (and therefore $x_k \geq x_l$), then $\bar{g}_{i,k} = 1$. We distinguish two cases. We first assume $g_{i,l} = 1$. Then by Lemma 2 (OA) $\bar{g}_{i,k} = 1$ holds. Assume next that $g_{l,i} = 1$. Then by Lemma 3 (OA) $\bar{g}_{i,k} = 1$. That is, $\bar{\mathbf{g}}$ is a nested split graph. $Q.E.D.$

References

[1] Hiller, T., (2013), Peer Effects in Endogenous Networks, http://eprints.lse.ac.uk/58176/