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A simultaneous iterative scheme for the Craig-Bampton reduction based substructuring

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ABSTRACT

A simultaneous iterative procedure for the fixed-interface component modal synthesis (CMS) method is developed in this paper toward fast calculating the modal parameters and ROM of a large-scale and/or complicated structure. Different from the existing iterative fixed-interface CMS methods, in the proposed iterative scheme, an eigenvalue independent matrix, whose column projections in the exact reduced space are the interested global eigenvectors, is chosen as the iterative term and then used as a Ritz basis to generate the reduced system matrices. Consequently, all the interested modes can be solved simultaneously and a ROM can be derived after one round of iterations. For reference, an implementation is given together with some computational considerations. Compared with other methods for solving modal parameters and/or model order reduction, the proposed method has such merits as high computational efficiency, especially for reanalysis tasks and parallel programming. A numerical example is provided to illustrate and validate the proposed method.

KEY WORDS: component modal synthesis; substructure; simultaneous iterative procedure; model order reduction; Craig-Bampton method

1. Introduction

Efficiently calculating high precision modal parameters and/or reduce-order models (ROMs) of large-scale and/or complicated structures, for instance the civil and aerospace engineering structures [1–4], are receiving more attentions in the optimal design, model modification and updating tasks [5–7]. Among the existing solution techniques, component modal synthesis (CMS) [8,9] is a well-known method addressing such problems. By analyzing a global structure at its component level, the CMS method can significantly reduce the computational cost of an analysis process, especially for reanalysis problems and parallel computing, and thus can be attractive to the engineers.

Historically, the CMS method was firstly developed by Hurty in the 1960s [10, 11], where the dynamic properties of components are approximated via their lower modes. Then, extra mode bases, specifically the static constrained modes and residual flexible modes, corresponding to the well-known "fixed-interface" [12] and "free-interface" [13] CMS methods respectively, were added to the original low-order modal basis for truncation compensation. In practice, the fixed-interface CMS method, or the Craig-Bampton (C-B) method, is more popular and widely used due to its simplicity and robustness, especially for obtaining the ROM, and thus will be focused in this paper. Developments in decades for improving the performance of C-B type methods can be summarized with reference to the method for truncation compensation, specifically the approximation based [14] and iterated [15] method. Besides, the C-B method can be generalized to the automated multilevel substructuring (AMLS) method [16] by applying modal truncation on the junction parts between components and generalizing the concept of component partitioning. Notice that the C-B method has a close relationship with the dynamic condensation (DC) based substructuring method [17, 18], since both of them assemble the components in the primal form. Based on this relationship, ideas of these two kinds of methods can be compared and unitized reciprocally. For example, both Qiu's iterative C-B (ICB) method [15] and Friswell's iterated improved reduced system (IIRS) method [19] solve the nonlinear reduced eigen-equaiton iteratively; Kim and Lee [20] employed the O'Callahan's idea [21] to improve the classic

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C-B method.

Nevertheless, for the case that high precision modal parameters and ROMs are required, all the existing C-B type methods have their own shortcomings and thus may not be the most appropriate choice. For the approximated methods, truncation errors exist in the final results and thus more component modes should be kept for highly accurate results. However, the computational cost of solving component level eigen-problems, which usually cost the most of the CPU time in a CMS method, will be heavily increased correspondingly and the order of the ROM will be higher as well [22, 23]. For the iterative methods, the precision of modes has to be improved one by one and a linear ROM is not available as a result of the eigenvalue dependent reduced mass matrix, which can largely increase the computational burden when the global dynamic properties are of interest in a relatively wider frequency band. In general, it may still be necessary to improve the C-B type CMS method for enhancing the efficiency.

In this paper, a new iteration scheme with complete theoretical frameworks is developed for the C-B type CMS method, which employs the idea of Friswell for improving the IIRS method [24] and can be treated as an extension of the previous simultaneous iterative procedure for the free-interface CMS approach [23]. In this iteration scheme, an eigenvalue independent matrix, whose column projections in the exact reduced space are the interested global eigenvectors, is chosen as the iterative term and then used as a Ritz basis to generate the reduced system matrices. Therefore, all the interested modes can be solved simultaneously and a linear ROM can be derived after one round of iterations. Furthermore, an expansion formula of the iterative term is developed based on the series expansion of the component receptance matrices. Consequently, by decomposing the components' constrained stiffness matrices during the initializing, only the forward and back substitution processes are needed to update the iterative term in each iteration step. Then, an implementation is given for reference together with some computational considerations. Finally, a simple numerical example is presented to illustrate the method and validate its precision and convergence.

The remainder of this paper is organized as follows: Section 2 defines the problem of C-B reduction, briefly reviews the conventional iterated C-B method and presents the proposed method together with implementation issues. Section 3 presents a numerical example to briefly illustrate and validate the proposed method. Finally, conclusions and future works are summarized in Section 4.

2. The simultaneous iterated C-B substructuring

2.1. Primal assembly and C-B reduction

In this paper, the scope is limited to the eigen-problem of a $\bar{N}$-DoFs global structure consisting of n components, where the over-bar indicates that the term below is associated with the global structure. The mass and stiffness matrices of the jth ($j = 1, 2, ..., n$) component with $N^{(j)}$ DoFs are denoted by $\mathbf{M}^{(j)}$ and $\mathbf{K}^{(j)}$, respectively. In addition, the system matrices of the components can be partitioned with respect to the interior and boundary DoFs as[‡]

$$\mathbf{K}^{(j)} \triangleq \begin{bmatrix} \mathbf{K}_{\mathrm{i}}^{(j)} & \mathbf{K}_{\mathrm{c}}^{(j)} \\ \left(\mathbf{K}_{\mathrm{c}}^{(j)}\right)^{\mathrm{T}} & \mathbf{K}_{\mathrm{b}}^{(j)} \end{bmatrix}, \mathbf{M}^{(j)} \triangleq \begin{bmatrix} \mathbf{M}_{\mathrm{i}}^{(j)} & \mathbf{M}_{\mathrm{c}}^{(j)} \\ \left(\mathbf{M}_{\mathrm{c}}^{(j)}\right)^{\mathrm{T}} & \mathbf{M}_{\mathrm{b}}^{(j)} \end{bmatrix} \tag{1}$$

where the subscripts "i","c" and "d" indicate the interior, coupling and boundary DoFs, respectively. Then, the primly assembled eigen-equations of the global structure can be expressed as

$$\left(\bar{\mathbf{K}} - \bar{\lambda}_k \bar{\mathbf{M}}\right) \bar{\boldsymbol{\phi}}_k = \boldsymbol{0} \tag{2}$$

for $k = 1, 2, ..., \bar{N}$, where

$$\begin{aligned}
\bar{\mathbf{K}} &\triangleq \begin{bmatrix} \bar{\mathbf{K}}_{\mathrm{i}} & \bar{\mathbf{K}}_{\mathrm{c}} \\ \left(\bar{\mathbf{K}}_{\mathrm{c}}\right)^{\mathrm{T}} & \bar{\mathbf{K}}_{\mathrm{b}} \end{bmatrix}, \bar{\mathbf{M}} \triangleq \begin{bmatrix} \bar{\mathbf{M}}_{\mathrm{i}} & \bar{\mathbf{M}}_{\mathrm{c}} \\ \left(\bar{\mathbf{M}}_{\mathrm{c}}\right)^{\mathrm{T}} & \bar{\mathbf{M}}_{\mathrm{b}} \end{bmatrix} \\
\bar{\mathbf{K}}_{\mathrm{i}} &\triangleq \mathrm{diag}\left(\mathbf{K}_{\mathrm{i}}^{(1)}, \mathbf{K}_{\mathrm{i}}^{(2)}, \cdots, \mathbf{K}_{\mathrm{i}}^{(\mathrm{n})}\right), \bar{\mathbf{M}}_{\mathrm{i}} \triangleq \mathrm{diag}\left(\mathbf{M}_{\mathrm{i}}^{(1)}, \mathbf{M}_{\mathrm{i}}^{(2)}, \cdots, \mathbf{M}_{\mathrm{i}}^{(\mathrm{n})}\right) \\
\bar{\mathbf{K}}_{\mathrm{c}} &\triangleq \left[(\mathbf{K}_{\mathrm{c}}^{(1)}\mathbf{L}_{\mathrm{b}}^{(1)})^{\mathrm{T}}, (\mathbf{K}_{\mathrm{c}}^{(2)}\mathbf{L}_{\mathrm{b}}^{(2)})^{\mathrm{T}}, \cdots, (\mathbf{K}_{\mathrm{c}}^{(n)}\mathbf{L}_{\mathrm{b}}^{(n)})^{\mathrm{T}}\right]^{\mathrm{T}}, \bar{\mathbf{M}}_{\mathrm{c}} \triangleq \left[(\mathbf{M}_{\mathrm{c}}^{(1)}\mathbf{L}_{\mathrm{b}}^{(1)})^{\mathrm{T}}, (\mathbf{M}_{\mathrm{c}}^{(2)}\mathbf{L}_{\mathrm{b}}^{(2)})^{\mathrm{T}}, \cdots, (\mathbf{M}_{\mathrm{c}}^{(n)}\mathbf{L}_{\mathrm{b}}^{(n)})^{\mathrm{T}}\right]^{\mathrm{T}} \\
\bar{\mathbf{K}}_{\mathrm{b}} &\triangleq \sum_{j=1}^{n} (\mathbf{L}_{\mathrm{b}}^{(j)})^{\mathrm{T}} \mathbf{K}_{\mathrm{b}}^{(j)} \mathbf{L}_{\mathrm{b}}^{(j)}, \bar{\mathbf{M}}_{\mathrm{b}} \triangleq \sum_{j=1}^{n} (\mathbf{L}_{\mathrm{b}}^{(j)})^{\mathrm{T}} \mathbf{M}_{\mathrm{b}}^{(j)} \mathbf{L}_{\mathrm{b}}^{(j)} \\
\bar{\boldsymbol{\phi}}_k &\triangleq \left[\bar{\boldsymbol{\phi}}_{\mathrm{i},k}^{\mathrm{T}}, \bar{\boldsymbol{\phi}}_{\mathrm{b},k}^{\mathrm{T}}\right]^{\mathrm{T}}, \bar{\boldsymbol{\phi}}_{\mathrm{i},k} \triangleq \left[(\bar{\boldsymbol{\phi}}_{\mathrm{i},k}^{(1)})^{\mathrm{T}}, (\bar{\boldsymbol{\phi}}_{\mathrm{i},k}^{(2)})^{\mathrm{T}}, \cdots, (\bar{\boldsymbol{\phi}}_{\mathrm{i},k}^{(n)})^{\mathrm{T}}\right]^{\mathrm{T}}
\end{aligned} \tag{3}$$

[‡]Throughout this paper, matrices, column vectors, variables and functions and scripts are denoted by bold, bold and italic, italic and roman letters, respectively.

Here, $\mathbf{L}_{\mathrm{b}}^{(j)}$ is a Boolean localization matrix relating assembled boundary DoFs to the jth component's boundary DoFs [9].

To reduce the eigen-problem defined by Equation (2), one can express $\bar{\boldsymbol{\phi}}_{\mathrm{i},k}$ in terms of $\bar{\boldsymbol{\phi}}_{\mathrm{b},k}$ by

$$\bar{\boldsymbol{\phi}}_{\mathrm{i},k} = \underline{\mathbf{T}}_{\mathrm{ib},k}\bar{\boldsymbol{\phi}}_{\mathrm{b},k} \tag{4}$$

where

$$\underline{\mathbf{T}}_{\mathrm{ib},k} \triangleq \left(\bar{\mathbf{K}}_{\mathrm{i}} - \bar{\lambda}_k\bar{\mathbf{M}}_{\mathrm{i}}\right)^{-1}\left(\bar{\mathbf{K}}_{\mathrm{c}} - \bar{\lambda}_k\bar{\mathbf{M}}_{\mathrm{c}}\right) = \bar{\mathbf{K}}_{\mathrm{i}}^{-1}\bar{\mathbf{K}}_{\mathrm{c}} + \bar{\lambda}_k\bar{\mathbf{K}}_{\mathrm{i}}^{-1}\left(\bar{\mathbf{M}}_{\mathrm{i}}\underline{\mathbf{T}}_{\mathrm{ib},k} - \bar{\mathbf{M}}_{\mathrm{c}}\right) \tag{5}$$

represents the transmissibility of interior DoFs to boundary DoFs at $\bar{\lambda}_k$ and the underline together with the index k indicates the term depends on the global eigenvalue $\bar{\lambda}_k$. A ROM and lower global eigen-pairs can be obtained by solving $\underline{\mathbf{T}}_{\mathrm{ib},k}$ exactly via a DC based method, in which an iterative solution scheme is usually required as $\underline{\mathbf{T}}_{\mathrm{ib},k}$ is eigenvalue dependent.

Notice that the transformation in Equation (4) can be expressed in an alternative way as

$$\bar{\boldsymbol{\phi}}_{\mathrm{i},k} = \underline{\mathbf{T}}_{\mathrm{ib},k}\bar{\boldsymbol{\phi}}_{\mathrm{b},k} = \bar{\mathbf{K}}_{\mathrm{i}}^{-1}\bar{\mathbf{K}}_{\mathrm{c}}\bar{\boldsymbol{\phi}}_{\mathrm{b},k} + \bar{\lambda}_k\bar{\mathbf{K}}_{\mathrm{i}}^{-1}\left(\bar{\mathbf{M}}_{\mathrm{i}}\underline{\mathbf{T}}_{\mathrm{ib},k} - \bar{\mathbf{M}}_{\mathrm{c}}\right)\bar{\boldsymbol{\phi}}_{\mathrm{b},k} \triangleq \bar{\mathbf{K}}_{\mathrm{i}}^{-1}\bar{\mathbf{K}}_{\mathrm{c}}\bar{\boldsymbol{\phi}}_{\mathrm{b},k} + \boldsymbol{q}_k \tag{6}$$

The corresponding reduced eigen-equation is

$$\begin{bmatrix} \mathbf{k}_{\mathrm{i}} - \bar{\lambda}_k\mathbf{m}_{\mathrm{i}} & -\bar{\lambda}_k\mathbf{m}_{\mathrm{c}} \\ -\bar{\lambda}_k\mathbf{m}_{\mathrm{c}}^{\mathrm{T}} & \mathbf{k}_{\mathrm{b}} - \bar{\lambda}_k\mathbf{m}_{\mathrm{b}} \end{bmatrix}\begin{Bmatrix} \boldsymbol{q}_k \\ \bar{\boldsymbol{\phi}}_{\mathrm{b},k} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \end{Bmatrix} \tag{7}$$

with

$$\begin{gathered} \mathbf{k}_{\mathrm{i}} \triangleq \bar{\mathbf{K}}_{\mathrm{i}}, \mathbf{m}_{\mathrm{i}} \triangleq \bar{\mathbf{M}}_{\mathrm{i}} \\ \mathbf{k}_{\mathrm{b}} \triangleq \bar{\mathbf{K}}_{\mathrm{b}} - \bar{\mathbf{K}}_{\mathrm{c}}^{\mathrm{T}}\bar{\mathbf{K}}_{\mathrm{i}}^{-1}\bar{\mathbf{K}}_{\mathrm{c}}, \mathbf{m}_{\mathrm{c}} \triangleq \bar{\mathbf{M}}_{\mathrm{c}} - \bar{\mathbf{M}}_{\mathrm{i}}^{\mathrm{T}}\bar{\mathbf{K}}_{\mathrm{i}}^{-1}\bar{\mathbf{K}}_{\mathrm{c}} \\ \mathbf{m}_{\mathrm{b}} \triangleq \bar{\mathbf{M}}_{\mathrm{b}} - \bar{\mathbf{M}}_{\mathrm{c}}^{\mathrm{T}}\bar{\mathbf{K}}_{\mathrm{i}}^{-1}\bar{\mathbf{K}}_{\mathrm{c}} - \bar{\mathbf{K}}_{\mathrm{c}}^{\mathrm{T}}\bar{\mathbf{K}}_{\mathrm{i}}^{-1}\bar{\mathbf{M}}_{\mathrm{c}} + \bar{\mathbf{K}}_{\mathrm{c}}^{\mathrm{T}}\bar{\mathbf{K}}_{\mathrm{i}}^{-1}\bar{\mathbf{M}}_{\mathrm{c}}^{\mathrm{T}}\bar{\mathbf{K}}_{\mathrm{i}}^{-1}\bar{\mathbf{K}}_{\mathrm{c}} \end{gathered} \tag{8}$$

From Equations (7) and (3), we know that this transformation make $\bar{\mathbf{K}}$ block diagonal via Gaussian elimination and thus the reduced model in Equation (7) is exact. Nevertheless, compared with the ROM from the transformation of Equation (4), the size is largely increased, although the reduced eigen-equation is linear.

To solve this problem, the contributions of the high-order term, i.e. $\boldsymbol{q}_k$, can be considered in a certain frequency band. In the C-B method, this certain frequency band is usually the lower interested frequency band, i.e, assuming

$$\boldsymbol{q}_k = \boldsymbol{\Phi}_{\mathrm{L}}\boldsymbol{\psi}_k + \boldsymbol{\Phi}_{\mathrm{H}}\boldsymbol{\eta}_k \approx \boldsymbol{\Phi}_{\mathrm{L}}\boldsymbol{\psi}_k \tag{9}$$

where $\boldsymbol{\psi}_k$ and $\boldsymbol{\eta}_k$ are the coordinate vectors of $\boldsymbol{q}_k$ in the space spanned by $\boldsymbol{\Phi}_{\mathrm{L}}$ and $\boldsymbol{\Phi}_{\mathrm{H}}$, respectively; $(\boldsymbol{\Phi}_{\mathrm{L}}, \boldsymbol{\Lambda}_{\mathrm{L}})$ and $(\boldsymbol{\Phi}_{\mathrm{H}}, \boldsymbol{\Lambda}_{\mathrm{H}})$ are the lower and higher eigen-pair matrices of the matrix pencil $\left(\bar{\mathbf{K}}_{\mathrm{i}}, \bar{\mathbf{M}}_{\mathrm{i}}\right)$ with mass-orthogonality; the subscripts "L" and "H" indicate those terms belong to the low-order (kept) and high-order (omitted) modes group, respectively. With the consideration of Equations (6) and (9), the reduced system matrices of the C-B method are

$$\mathbf{K}_{\mathrm{CB}} \triangleq \mathbf{T}_{\mathrm{CB}}^{\mathrm{T}}\bar{\mathbf{K}}\mathbf{T}_{\mathrm{CB}}, \mathbf{M}_{\mathrm{CB}} \triangleq \mathbf{T}_{\mathrm{CB}}^{\mathrm{T}}\bar{\mathbf{M}}\mathbf{T}_{\mathrm{CB}} \tag{10}$$

with

$$\begin{Bmatrix} \bar{\boldsymbol{\phi}}_{\mathrm{i},k} \\ \bar{\boldsymbol{\phi}}_{\mathrm{b},k} \end{Bmatrix} = \begin{bmatrix} \boldsymbol{\Phi}_{\mathrm{L}} & \bar{\mathbf{K}}_{\mathrm{i}}^{-1}\bar{\mathbf{K}}_{\mathrm{c}} \\ \mathbf{0} & \mathbf{I}_{\mathrm{b}} \end{bmatrix}\begin{Bmatrix} \boldsymbol{\psi}_k \\ \bar{\boldsymbol{\phi}}_{\mathrm{b},k} \end{Bmatrix} \triangleq \mathbf{T}_{\mathrm{CB}}\boldsymbol{p}_{\mathrm{CB},k} \tag{11}$$

Specifically, in Equation (11), $k = 1, 2, ..., n_{\mathrm{CB}}$, where n_{CB} is the dimension of system $(\mathbf{K}_{\mathrm{CB}}, \mathbf{M}_{\mathrm{CB}})$ (or the number of elements of the generalized coordinate vector $\boldsymbol{p}_{\mathrm{CB},k}$) and the subscript "CB" denotes the C-B method. For simplicity, k varies from 1 to n_{CB}, and j varies from 1 to n in the followings of this paper without particular specifications.

In general, the C-B method has a close relationship with the DC method. Instead of compensating the truncation effect $\bar{\lambda}_k\bar{\mathbf{K}}_{\mathrm{i}}^{-1}\left(\bar{\mathbf{M}}_{\mathrm{i}}\underline{\mathbf{T}}_{\mathrm{ib},k} - \bar{\mathbf{M}}_{\mathrm{c}}\right)\bar{\boldsymbol{\phi}}_{\mathrm{b},k}$ approximately or iteratively, an additional basis, the components' constrained modes, is employed for compensation. This can lead to a simpler but more reliable criterion for the selection of generalized coordinates since all the approximations are included via the modal truncation in Equation (9), which can be controlled by adjusting the truncation frequency of the component constrained modes. This criterion can be attractive to the engineers, since an appropriate choice of master DoFs, which largely decides both the precision and convergence of DC methods, may not be easy to find for a large-scale and/or complicated structure. The trade-offs of the C-B method mainly lie in the increment of the size of the final ROM, which will increase the computational burden and may become a serious problem as the precision requirement become stringent, since much more constrained component modes have to be included in such case. The iterative method developed by Qiu could be a solution, which will be reviewed in the next subsection.

2.2. Exact reduced eigen-equation of C-B methods

The exact reduced eigen-equation of the C-B method can be started by omitting the approximation in Equation (9), i.e. let $\boldsymbol{q}_k = \boldsymbol{\Phi}_{\mathrm{L}}\boldsymbol{\psi}_k + \boldsymbol{\Phi}_{\mathrm{H}}\boldsymbol{\eta}_k$. Substituting this transformation into Equation (7) yields

$$\begin{bmatrix} \boldsymbol{\Lambda}_{\mathrm{L}} - \bar{\lambda}_k\mathbf{I}_{\mathrm{L}} & \mathbf{0} & -\bar{\lambda}_k\boldsymbol{\Phi}_{\mathrm{L}}^{\mathrm{T}}\mathbf{m}_{\mathrm{c}} \\ \mathbf{0} & \boldsymbol{\Lambda}_{\mathrm{H}} - \bar{\lambda}_k\mathbf{I}_{\mathrm{H}} & -\bar{\lambda}_k\boldsymbol{\Phi}_{\mathrm{H}}^{\mathrm{T}}\mathbf{m}_{\mathrm{c}} \\ -\bar{\lambda}_k\mathbf{m}_{\mathrm{c}}^{\mathrm{T}}\boldsymbol{\Phi}_{\mathrm{L}} & -\bar{\lambda}_k\mathbf{m}_{\mathrm{c}}^{\mathrm{T}}\boldsymbol{\Phi}_{\mathrm{H}} & \mathbf{k}_{\mathrm{b}} - \bar{\lambda}_k\mathbf{m}_{\mathrm{b}} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\psi}_k \\ \boldsymbol{\eta}_k \\ \bar{\boldsymbol{\phi}}_{\mathrm{b},k} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{Bmatrix} \tag{12}$$

From the second line of Equation (12), one can have

$$\boldsymbol{\eta}_k = \left(\boldsymbol{\Lambda}_{\mathrm{H}} - \bar{\lambda}_k\mathbf{I}_{\mathrm{H}}\right)^{-1}\boldsymbol{\Phi}_{\mathrm{H}}^{\mathrm{T}}\mathbf{m}_c\bar{\boldsymbol{\phi}}_{\mathrm{b},k} \tag{13}$$

Substituting Equation (13) back into Equation (12) gives

$$\begin{bmatrix} \boldsymbol{\Lambda}_{\mathrm{L}} - \bar{\lambda}_k\mathbf{I}_{\mathrm{L}} & -\bar{\lambda}_k\boldsymbol{\Phi}_{\mathrm{L}}^{\mathrm{T}}\mathbf{m}_{\mathrm{c}} \\ -\bar{\lambda}_k\mathbf{m}_{\mathrm{c}}^{\mathrm{T}}\boldsymbol{\Phi}_{\mathrm{L}} & \mathbf{k}_{\mathrm{b}} - \bar{\lambda}_k\mathbf{m}_{\mathrm{b}} - \bar{\lambda}_k^2\mathbf{m}_{\mathrm{c}}^{\mathrm{T}}\underline{\mathbf{X}}_k\mathbf{m}_{\mathrm{c}} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\psi}_k \\ \bar{\boldsymbol{\phi}}_{\mathrm{b},k} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \end{Bmatrix} \tag{14}$$

where

$$\underline{\mathbf{X}}_k \triangleq \boldsymbol{\Phi}_{\mathrm{H}}\left(\boldsymbol{\Lambda}_{\mathrm{H}} - \bar{\lambda}_k\mathbf{I}_{\mathrm{H}}\right)^{-1}\boldsymbol{\Phi}_{\mathrm{H}}^{\mathrm{T}} \tag{15}$$

represents the transmissibility of truncated constrained component modes at $\bar{\lambda}_k$. Besides, the exact transformation matrix of Equation (14) is

$$\underline{\mathbf{T}}_k \triangleq \mathbf{T}_{\mathrm{CB}} + \bar{\lambda}_k\begin{bmatrix} \mathbf{0} & \underline{\mathbf{X}}_k\mathbf{m}_{\mathrm{c}} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \triangleq \mathbf{T}_{\mathrm{CB}} + \underline{\mathbf{t}}_k \tag{16}$$

from Equations (6), (9) and (13). Therefore, the global eigenvectors $\bar{\boldsymbol{\phi}}_k$ can be recovered from the reduced eigen-pairs $\left(\bar{\lambda}_k, \boldsymbol{\psi}_k\right)$ by

$$\bar{\boldsymbol{\phi}}_k = \underline{\mathbf{T}}_k\begin{Bmatrix} \boldsymbol{\psi}_k \\ \bar{\boldsymbol{\phi}}_{\mathrm{b},k} \end{Bmatrix} \triangleq \underline{\mathbf{T}}_k\boldsymbol{p}_k \tag{17}$$

Equation (14) is an exact reduced eigen-equation of the C-B substructuring method. Notice that Equation (14) is nonlinear as it contains an eigenvalue dependent term $\underline{\mathbf{X}}_k$, and thus an iterative procedure is required if the exact solutions are required for a general large-scale and/or complicated structure. Here, for reference and comparison, we briefly list the solution procedure derived by Qiu.

Firstly, rewrite Equation (14) as

$$\left[\mathbf{K}_{\mathrm{CB}} - \bar{\lambda}_k\left(\mathbf{M}_{\mathrm{CB}} + \bar{\lambda}_k\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{m}_{\mathrm{c}}^{\mathrm{T}}\underline{\mathbf{X}}_k\mathbf{m}_{\mathrm{c}} \end{bmatrix}\right)\right]\boldsymbol{p}_k = \boldsymbol{0} \tag{18}$$

Then, the reduced eigen-pairs can be solved iteratively by evaluating

$$\underline{\mathbf{X}}_k^{[i]} \triangleq \boldsymbol{\Phi}_{\mathrm{H}}\left(\boldsymbol{\Lambda}_{\mathrm{H}} - \bar{\lambda}_k^{[i]}\mathbf{I}_{\mathrm{H}}\right)^{-1}\boldsymbol{\Phi}_{\mathrm{H}}^{\mathrm{T}} \tag{19}$$

and solving the eigen-problem

$$\mathbf{K}_{\mathrm{CB}}\boldsymbol{p}_k^{[i]} = \bar{\lambda}_k^{[i]}\left(\mathbf{M}_{\mathrm{CB}} + \bar{\lambda}_k^{[i-1]}\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{m}_{\mathrm{c}}^{\mathrm{T}}\underline{\mathbf{X}}_k^{[i-1]}\mathbf{m}_{\mathrm{c}} \end{bmatrix}\right)\boldsymbol{p}_k^{[i]} \tag{20}$$

for $i = 1, 2, \ldots$ with $\bar{\lambda}_k^{[0]} \triangleq 0$. Assuming that the kth reduced eigen-pair converges after the ith iteration, the kth global eigenvector can be recovered by

$$\bar{\boldsymbol{\phi}}_k^{[i]} = \left(\mathbf{T}_{\mathrm{CB}} + \bar{\lambda}_k^{[i]}\begin{bmatrix} \mathbf{0} & \underline{\mathbf{X}}_k^{[i]}\mathbf{m}_{\mathrm{c}} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\right)\boldsymbol{p}_k^{[i]} \tag{21}$$

In this approach, the iterative process also has to be performed mode by mode as $\underline{\mathbf{X}}_k^{[i]}$ varies from mode to mode. Therefore, the computational efficiency could be raised if the precision of all the interested modes can be simultaneously improved in a single round of iterations. From this point of view, a new iterative procedure will be developed in the following subsection.

2.3. Procedure of simultaneous iterative method

The simultaneous iterative procedure, developed in Reference [23] for solving exact reduced eigen-equation Equation of the free-interface CMS method, will be applied to the C-B method, i.e. the solution of Equation (14) in this subsection. In general, the simultaneous iterative procedure is based on the fact that the exact reduced eigen-equation Equation (14) is eigenvalue dependent as a result of its corresponding eigenvalue dependent transformation matrix $\underline{\mathbf{T}}_k$, defined in Equation (16), and only "$\underline{\mathbf{T}}_k\boldsymbol{p}_k$", a n_{CB}-dimensional vector, instead of the $\bar{N}\times n_{\mathrm{CB}}$ matrix "$\underline{\mathbf{T}}_k$", is indispensable to recover the kth

global eigenvector. Therefore, the new iterative scheme can be started by using an eigenvalue independent matrix, denoted by $\mathbf{T}_{\mathrm{S}}$, as a transformation matrix (reduction basis) to construct the space spanned by all the reduced eigen-pairs $\left(\bar{\lambda}_k, \boldsymbol{p}_k\right)$. Here, the subscript "S" denotes the simultaneous iterative procedure. Specifically, $\mathbf{T}_{\mathrm{S}}$ can be defined as follows.

Lemma 1 *There exists an eigenvalue independent matrix* $\mathbf{T}_{\mathrm{S}}$ *of size* $\bar{N}$*-by-*n_{CB} *such that*

$$\mathbf{T}_{\mathrm{S}}\boldsymbol{p}_k = \underline{\mathbf{T}}_k\boldsymbol{p}_k \tag{22}$$

Specifically, $\mathbf{T}_{\mathrm{S}}$ *can be defined by*

$$\mathbf{T}_{\mathrm{S}} \triangleq \left[\underline{\mathbf{T}}_1\boldsymbol{p}_1 \quad \underline{\mathbf{T}}_2\boldsymbol{p}_2 \quad \dots \quad \underline{\mathbf{T}}_{n_{\mathrm{CB}}}\boldsymbol{P}_{n_{\mathrm{CB}}}\right]\mathbf{P}_{\mathrm{S}}^{-1} \tag{23}$$

In Equation (23), $\mathbf{P}_{\mathrm{S}} \triangleq \left[\boldsymbol{p}_1 \quad \boldsymbol{p}_2 \quad \dots \quad \boldsymbol{p}_{n_{\mathrm{CB}}}\right]$ *is a matrix form of the exact reduced eigenvectors and the corresponding eigenvalue matrix is* $\mathbf{\Lambda}_{\mathrm{S}} \triangleq \mathrm{diag}\left(\bar{\lambda}_1, \bar{\lambda}_2, ..., \bar{\lambda}_{n_{\mathrm{CB}}}\right)$.

Proof Notice that all the exact reduced eigenvectors can construct a basis for the $\mathbb{R}^{n_{\mathrm{CB}}}$ space in a fixed-interface CMS method, i.e. $\mathbf{P}$ is invertible and thus $\mathbf{T}_{\mathrm{S}}$ exists. □

Lemma 2 $\mathbf{P}_{\mathrm{S}}$ *and* $\mathbf{\Lambda}_{\mathrm{S}}$ *can satisfy*

$$\mathbf{K}_{\mathrm{S}}\mathbf{P}_{\mathrm{S}} = \mathbf{M}_{\mathrm{S}}\mathbf{P}_{\mathrm{S}}\mathbf{\Lambda}_{\mathrm{S}} \tag{24}$$

where

$$\mathbf{K}_{\mathrm{S}} \triangleq \mathbf{T}_{\mathrm{S}}^{\mathrm{T}}\bar{\mathbf{K}}\mathbf{T}_{\mathrm{S}},\ \mathbf{M}_{\mathrm{S}} \triangleq \mathbf{T}_{\mathrm{S}}^{\mathrm{T}}\bar{\mathbf{M}}\mathbf{T}_{\mathrm{S}} \tag{25}$$

are the reduced stiffness and mass matrices corresponding to $\mathbf{T}_{\mathrm{S}}$.

Proof Firstly, one can have the followings according to Equations (8), (11), (15) and (16).

$$\begin{aligned}
&\mathbf{T}_{\mathrm{CB}}^{\mathrm{T}}\bar{\mathbf{K}}\underline{\mathbf{t}}_k = \bar{\lambda}_k\begin{bmatrix}\mathbf{0} & \mathbf{\Phi}_{\mathrm{L}}\bar{\mathbf{K}}_{\mathrm{i}}\underline{\mathbf{X}}_k\mathbf{m}_{\mathrm{c}}\\ \mathbf{0} & \mathbf{0}\end{bmatrix} = \mathbf{0}, \mathbf{T}_{\mathrm{CB}}^{\mathrm{T}}\bar{\mathbf{M}}\underline{\mathbf{t}}_k = \bar{\lambda}_k\begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{m}_{\mathrm{c}}^{\mathrm{T}}\underline{\mathbf{X}}_k\mathbf{m}_{\mathrm{c}}\end{bmatrix}\\
&\underline{\mathbf{t}}_r^{\mathrm{T}}\bar{\mathbf{K}}\underline{\mathbf{t}}_k = \bar{\lambda}_k^2\begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{m}_{\mathrm{c}}^{\mathrm{T}}\underline{\mathbf{X}}_r\bar{\mathbf{K}}_{\mathrm{i}}\underline{\mathbf{X}}_k\mathbf{m}_{\mathrm{c}}\end{bmatrix}, \underline{\mathbf{t}}_r^{\mathrm{T}}\bar{\mathbf{M}}\underline{\mathbf{t}}_k = \bar{\lambda}_k^2\begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{m}_{\mathrm{c}}^{\mathrm{T}}\underline{\mathbf{X}}_r\bar{\mathbf{M}}_{\mathrm{i}}\underline{\mathbf{X}}_k\mathbf{m}_{\mathrm{c}}\end{bmatrix}
\end{aligned} \tag{26}$$

for $r = 1, 2, \dots, n_{\mathrm{CB}}$. Then, one can have

$$\begin{aligned}
&\underline{\mathbf{T}}_r^{\mathrm{T}}\bar{\mathbf{K}}\underline{\mathbf{T}}_k - \bar{\lambda}_k\underline{\mathbf{T}}_r^{\mathrm{T}}\bar{\mathbf{M}}\underline{\mathbf{T}}_k\\
=&\mathbf{K}_{\mathrm{CB}} - \bar{\lambda}_k\mathbf{M}_{\mathrm{CB}} - \bar{\lambda}_k^2\begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & 2\mathbf{m}_{\mathrm{c}}^{\mathrm{T}}\underline{\mathbf{X}}_k\mathbf{m}_{\mathrm{c}} - \mathbf{m}_{\mathrm{c}}^{\mathrm{T}}\underline{\mathbf{X}}_r\left(\bar{\mathbf{K}}_{\mathrm{i}} - \bar{\lambda}_k\bar{\mathbf{M}}_{\mathrm{i}}\right)\underline{\mathbf{X}}_k\mathbf{m}_{\mathrm{c}}\end{bmatrix}\\
=&\mathbf{K}_{\mathrm{CB}} - \bar{\lambda}_k\left(\mathbf{M}_{\mathrm{CB}} + \bar{\lambda}_k\begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{m}_{\mathrm{c}}^{\mathrm{T}}\underline{\mathbf{X}}_k\mathbf{m}_{\mathrm{c}}\end{bmatrix}\right)
\end{aligned} \tag{27}$$

for $r = 1, 2, \dots, n_{\mathrm{CB}}$. Therefore, with Equations (18), (22), (25) and (27), one can have

$$\begin{aligned}
\boldsymbol{p}_r^{\mathrm{T}}\left(\mathbf{K}_{\mathrm{S}} - \bar{\lambda}_k\mathbf{M}_{\mathrm{S}}\right)\boldsymbol{p}_k &= \boldsymbol{p}_r^{\mathrm{T}}\mathbf{T}_{\mathrm{S}}^{\mathrm{T}}\left(\bar{\mathbf{K}} - \bar{\lambda}_k\bar{\mathbf{M}}\right)\mathbf{T}_{\mathrm{S}}\boldsymbol{p}_k\\
&= \boldsymbol{p}_r^{\mathrm{T}}\underline{\mathbf{T}}_r^{\mathrm{T}}\left(\bar{\mathbf{K}} - \bar{\lambda}_k\bar{\mathbf{M}}\right)\underline{\mathbf{T}}_k\boldsymbol{p}_k\\
&= \boldsymbol{p}_r^{\mathrm{T}}\left[\mathbf{K}_{\mathrm{CB}} - \bar{\lambda}_k\left(\mathbf{M}_{\mathrm{CB}} + \bar{\lambda}_k\begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{m}_{\mathrm{c}}^{\mathrm{T}}\underline{\mathbf{X}}_k\mathbf{m}_{\mathrm{c}}\end{bmatrix}\right)\right]\boldsymbol{p}_k\\
&= 0
\end{aligned} \tag{28}$$

for $r = 1, 2, ..., n_{\mathrm{KC}}$, i.e. $\mathbf{P}_{\mathrm{S}}^{\mathrm{T}}\left(\mathbf{K}_{\mathrm{S}} - \bar{\lambda}_k\mathbf{M}_{\mathrm{S}}\right)\boldsymbol{p}_k = \boldsymbol{0}$. This can lead to $\left(\mathbf{K}_{\mathrm{S}} - \bar{\lambda}_k\mathbf{M}_{\mathrm{S}}\right)\boldsymbol{p}_k = \boldsymbol{0}$ as $\mathbf{P}_{\mathrm{S}}$ is a basis. □

Remark 1

In Qiu's iterative method, solving Equation (20) leads to n_{CB} vectors in total for a certain mode order k, but $(n_{\mathrm{CB}} - 1)$ of them are non-eigenvectors except for the one associated with $\bar{\lambda}_k$. In contrast, the proposed method only uses the eigenvectors to construct the reduced system, as shown by **Lemma 2**.

Nevertheless, Equation (23) does not provide an efficient way of calculating $\mathbf{T}_{\mathrm{S}}$ in practice. Therefore, an iteration formula of $\mathbf{T}_{\mathrm{S}}$ will be developed, started by the identity equation of $\mathbf{T}_{\mathrm{S}}$ as follow.

Lemma 3 *The term* $\mathbf{T}_{\mathrm{S}}$ *can satisfy*

$$\mathbf{T}_{\mathrm{S}} = \mathbf{T}_{\mathrm{CB}} + \mathbf{S}_0\bar{\mathbf{M}}\mathbf{T}_{\mathrm{S}}\mathbf{M}_{\mathrm{S}}^{-1}\mathbf{K}_{\mathrm{S}} \tag{29}$$

where

$$\begin{aligned} \mathbf{F} &\triangleq \boldsymbol{\Phi}_{\mathrm{H}}\boldsymbol{\Lambda}_{\mathrm{H}}^{-1}\boldsymbol{\Phi}_{\mathrm{H}}^{\mathrm{T}} \\ \mathbf{S}_0 &\triangleq \begin{bmatrix} \mathbf{F} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \end{aligned} \tag{30}$$

are the residual flexibility of the constraint components and projector of the simultaneous iterative procedure, respectively.

Proof Notice that $\mathbf{F}$ can be expanded as

$$\underline{\mathbf{X}}_k = \mathbf{F} + \bar{\lambda}_k\mathbf{F}\bar{\mathbf{M}}\underline{\mathbf{X}}_k \tag{31}$$

and thus one has

$$\begin{aligned} \underline{\mathbf{T}}_k &= \mathbf{T}_{\mathrm{CB}} + \bar{\lambda}_k \begin{bmatrix} \mathbf{0} & \left(\mathbf{F} + \bar{\lambda}_k\mathbf{F}\bar{\mathbf{M}}\underline{\mathbf{X}}_k\right)\mathbf{m}_{\mathrm{c}} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \\ &= \mathbf{T}_{\mathrm{CB}} + \bar{\lambda}_k \begin{bmatrix} \mathbf{0} & \mathbf{F}\mathbf{m}_{\mathrm{c}} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \bar{\lambda}_k\mathbf{S}_0\bar{\mathbf{M}}\underline{\mathbf{t}}_k \\ &= \mathbf{T}_{\mathrm{CB}} + \bar{\lambda}_k\mathbf{S}_0\bar{\mathbf{M}}\left(\mathbf{T}_{\mathrm{CB}} + \underline{\mathbf{t}}_k\right) \\ &= \mathbf{T}_{\mathrm{CB}} + \bar{\lambda}_k\mathbf{S}_0\bar{\mathbf{M}}\underline{\mathbf{T}}_k \end{aligned} \tag{32}$$

with Equations (16), (30) and (31) and the following relationship

$$\mathbf{S}_0\bar{\mathbf{M}}\mathbf{T}_{\mathrm{CB}} = \begin{bmatrix} \mathbf{F}\bar{\mathbf{M}}_{\mathrm{i}} & \mathbf{F}\bar{\mathbf{M}}_{\mathrm{c}} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Phi}_{\mathrm{L}} & \bar{\mathbf{K}}_{\mathrm{i}}^{-1}\bar{\mathbf{K}}_{\mathrm{c}} \\ \mathbf{0} & \mathbf{I}_{\mathrm{b}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{F}\mathbf{m}_{\mathrm{c}} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{33}$$

Then, with Equation (23), post-multiplying both sides of Equation (32) by $\boldsymbol{p}_k$ yields

$$\mathbf{T}_{\mathrm{S}}\boldsymbol{p}_k = \underline{\mathbf{T}}_k\boldsymbol{p}_k = \mathbf{T}_{\mathrm{CB}}\boldsymbol{p}_k + \bar{\lambda}_k\mathbf{S}_0\bar{\mathbf{M}}\underline{\mathbf{T}}_k\boldsymbol{p}_k = \mathbf{T}_{\mathrm{CB}}\boldsymbol{p}_k + \bar{\lambda}_k\mathbf{S}_0\bar{\mathbf{M}}\mathbf{T}_{\mathrm{S}}\boldsymbol{p}_k \tag{34}$$

The matrix form of Equation (34) is "$\mathbf{T}_{\mathrm{S}}\mathbf{P}_{\mathrm{S}} = \mathbf{T}_{\mathrm{CB}}\mathbf{P}_{\mathrm{S}} + \mathbf{S}_0\bar{\mathbf{M}}\mathbf{T}_{\mathrm{S}}\mathbf{P}_{\mathrm{S}}\boldsymbol{\Lambda}_{\mathrm{S}}$", which is equivalent to

$$\mathbf{T}_{\mathrm{S}}\mathbf{P}_{\mathrm{S}} = \mathbf{T}_{\mathrm{CB}}\mathbf{P}_{\mathrm{S}} + \mathbf{S}_0\bar{\mathbf{M}}\mathbf{T}_{\mathrm{S}}\mathbf{M}_{\mathrm{S}}^{-1}\mathbf{K}_{\mathrm{S}}\mathbf{P}_{\mathrm{S}} \tag{35}$$

with the relationship "$\mathbf{P}_{\mathrm{S}}\boldsymbol{\Lambda}_{\mathrm{S}} = \mathbf{M}_{\mathrm{S}}^{-1}\mathbf{K}_{\mathrm{S}}\mathbf{P}_{\mathrm{S}}$" from Equation (24). As $\mathbf{P}_{\mathrm{S}}$ is a basis of the reduced modal space, Equation (35) is equivalent to Equation (29). □

Finally, an algorithm of the simultaneous iterative procedure based C-B (SCB) method can be developed according to **Lemmas 1-3**, as shown in Algorithm 1. Specifically, the maximum interested global eigenvalue is $\bar{\lambda}_{\max}$ and the precision requirement is Tol.

Algorithm 1. Simultaneous iterative C-B method

———*Preparing*———

(1) (a) *Partition* $\left(\bar{\mathbf{K}}, \bar{\mathbf{M}}\right)$ into components $\left(\mathbf{K}^{(j)}, \mathbf{M}^{(j)}\right)$ according to the problem and then (b) *solve* the components' lower eigen-pairs under the condition "$\max\{\mathrm{diag}(\boldsymbol{\Lambda}_{\mathrm{L}})\} = \theta_{\mathrm{ev}} \times \bar{\lambda}_{\max}$".

(2) *Calculate* (a) $\mathbf{F} = \boldsymbol{\Phi}_{\mathrm{H}}\boldsymbol{\Lambda}_{\mathrm{H}}^{-1}\boldsymbol{\Phi}_{\mathrm{H}}^{\mathrm{T}}$ and (b) $\left(\bar{\mathbf{K}}_{\mathrm{i}}\right)^{-1}\bar{\mathbf{K}}_{\mathrm{c}}^{(j)}$ to construct $\mathbf{S}_0$ and $\mathbf{T}_{\mathrm{CB}}$, respectively.

———*Initializing*———

(3) (a) *Start* with $\mathbf{T}_{\mathrm{S}}^{[1]} = \mathbf{T}_{\mathrm{CB}}$ and $\left(\mathbf{K}_{\mathrm{S}}^{[1]}, \mathbf{M}_{\mathrm{S}}^{[1]}\right) = (\mathbf{K}_{\mathrm{CB}}, \mathbf{M}_{\mathrm{CB}})$, then (b) *solve* $\left(\mathbf{K}_{\mathrm{S}}^{[1]}, \mathbf{M}_{\mathrm{S}}^{[1]}\right)$ for $\bar{\lambda}_k^{[1]}$, consequently, a truncation order l_{S} can be decided from $\left\{\bar{\lambda}_1^{[1]}, \bar{\lambda}_2^{[1]}, ..., \bar{\lambda}_{n_{\mathrm{CB}}}^{[1]}\right\}$ and $\bar{\lambda}_{\max}$.

———*Iterating*———

do while $i \geq 2$

(4) *Calculate* (a) $\mathbf{T}_{\mathrm{S}}^{[i]} = \mathbf{T}_{\mathrm{CB}} + \mathbf{S}_0\bar{\mathbf{M}}\mathbf{T}_{\mathrm{S}}^{[i-1]}\left(\mathbf{M}_{\mathrm{S}}^{[i-1]}\right)^{-1}\mathbf{K}_{\mathrm{S}}^{[i-1]}$ and (b) $\mathbf{K}_{\mathrm{S}}^{[i]} = \left(\mathbf{T}_{\mathrm{S}}^{[i]}\right)^{\mathrm{T}}\bar{\mathbf{K}}\mathbf{T}_{\mathrm{S}}^{[i]}$, $\mathbf{M}_{\mathrm{S}}^{[i]} = \left(\mathbf{T}_{\mathrm{S}}^{[i]}\right)^{\mathrm{T}}\bar{\mathbf{M}}\mathbf{T}_{\mathrm{S}}^{[i]}$.

(5) *Solve* $\left(\mathbf{K}_{\mathrm{S}}^{[i]}, \mathbf{M}_{\mathrm{S}}^{[i]}\right)$ for $\lambda_k^{[i]}$.

(6) **if** $\left|\lambda_k^{[i]} - \lambda_k^{[i-1]}\right| / \left|\lambda_k^{[i]}\right| \leq Tol$ for $k = 1, 2, ..., l_{\mathrm{S}}$, **then** set $\bar{\lambda}_k = \lambda_k^{[i]}$ and **stop do**.
else set $i = i + 1$.
end if

end do

———*Solving eigenvectors*———

(7) *Solve* $\boldsymbol{p}_k^{[i]}$ from $\mathbf{K}_{\mathrm{S}}^{[i]}\boldsymbol{p}_k^{[i]} = \lambda_k^{[i]}\mathbf{M}_{\mathrm{S}}^{[i]}\boldsymbol{p}_k^{[i]}$ for $k = 1, 2, ..., l_{\mathrm{S}}$.

(8) *Calculate* $\bar{\boldsymbol{\phi}}_k = \mathbf{T}_{\mathrm{S}}^{[i]}\boldsymbol{p}_k^{[i]}$ for $k = 1, 2, ..., l_{\mathrm{S}}$.

For Algorithm 1, some lines should be commented as follow.

(1a) The partitioning concept is not limited to the conventional substructuring method, that in other methods such as the (a) AMLS or (b) balanced domain decomposition methods can also considered and are clearly compatible with the proposed method since both of them belong to the primal domain decomposition method [25].

(1b) For a general large-scale structure, an algebraic generalized eigenvalue problem solver [26], especially the iterative projection based method such as the subspace iteration method [27] or Lanczos [28] method, is preferred to solve $\left(\mathbf{K}^{(j)}, \mathbf{M}^{(j)}\right)$ for the component modal parameters. Besides, the value of factor θ_{ev} can be selected from 1 to 1.5 according to the experience of the previous CMS approaches [23].

(2a) As the matrices $\mathbf{F}$, $\bar{\mathbf{K}}_{\text{i}}$, $\bar{\mathbf{M}}$ and $\mathbf{\Phi}_{\text{L}}$ are block diagonal with respect to the component modal coordinates, some matrices in the algorithm should be derived by calculating their blocks independently at the component level first and then assembling the blocks for the whole matrices. Specifically, they are $\mathbf{S}_0$, $\mathbf{T}_{\text{CB}}$ in Line (2) and $\mathbf{T}_{\text{S}}^{[i]}$, $\mathbf{K}_{\text{S}}^{[i]}$, $\mathbf{M}_{\text{S}}^{[i]}$ in Line (4).

(2a) The matrix $\mathbf{F}$ should be calculated alternatively by

$$\begin{aligned}\mathbf{F} &\triangleq \operatorname{diag}\left(\mathbf{F}^{(1)}, \mathbf{F}^{(2)}, \cdots, \mathbf{F}^{(n)}\right)\\ &= \operatorname{diag}\left(\mathbf{\Phi}_{\text{H}}^{(1)}(\mathbf{\Lambda}_{\text{H}}^{(1)})^{-1}(\mathbf{\Phi}_{\text{H}}^{(1)})^{\text{T}}, \mathbf{\Phi}_{\text{H}}^{(2)}(\mathbf{\Lambda}_{\text{H}}^{(2)})^{-1}(\mathbf{\Phi}_{\text{H}}^{(2)})^{\text{T}}, \cdots, \mathbf{\Phi}_{\text{H}}^{(n)}(\mathbf{\Lambda}_{\text{H}}^{(n)})^{-1}(\mathbf{\Phi}_{\text{H}}^{(n)})^{\text{T}}\right)\\ &= \operatorname{diag}\left((\mathbf{K}_{\text{i}}^{(1)})^{-1} - \mathbf{\Phi}_{\text{L}}^{(1)}(\mathbf{\Lambda}_{\text{L}}^{(1)})^{-1}(\mathbf{\Phi}_{\text{L}}^{(1)})^{\text{T}}, (\mathbf{K}_{\text{i}}^{(2)})^{-1} - \mathbf{\Phi}_{\text{L}}^{(2)}(\mathbf{\Lambda}_{\text{L}}^{(2)})^{-1}(\mathbf{\Phi}_{\text{L}}^{(2)})^{\text{T}}, \cdots, (\mathbf{K}_{\text{i}}^{(n)})^{-1} - \mathbf{\Phi}_{\text{L}}^{(n)}(\mathbf{\Lambda}_{\text{L}}^{(n)})^{-1}(\mathbf{\Phi}_{\text{L}}^{(n)})^{\text{T}}\right)\end{aligned} \tag{36}$$

instead of by definition, which needs to obtain all the higher component constraint modes. Here, $\left(\mathbf{\Phi}_{\text{L}}^{(j)}, \mathbf{\Lambda}_{\text{L}}^{(j)}\right)$ and $\left(\mathbf{\Phi}_{\text{H}}^{(j)}, \mathbf{\Lambda}_{\text{H}}^{(j)}\right)$ are the lower and higher constraint modes of the jth component, respectively.

(2a) The inverse of $\mathbf{K}_{\text{i}}^{(j)}$ is not explicitly required, instead, $\mathbf{K}_{\text{i}}^{(j)}$ should be factorized first before Line (2) such that its sparseness can be made use of. Consequently, the following processes such as Lines (2b) and (4a) can be realized by performing a forward and backward substitution only. For real modes, as $\mathbf{K}_{\text{i}}^{(j)}$ is symmetric, an LDLT decomposition is preferred to the LU decomposition. Specifically, the supernodal Cholesky decomposition based packages can be considered, which are proved to be effective for large matrices [29]. Furthermore, re-ordering techniques such as those provided by METIS [30] should be used before the decomposition to minimize the half-bandwidths of the factor matrices, which are usually much larger than those of $\mathbf{K}_{\text{i}}^{(j)}$ due to the non-zero fill-ins.

(6) For convergence check, the criterion may not be unique. For instance, the relative mode error or an residual vector based error estimator can be considered as well [27]. Studies on the most appropriate stopping criterion are not contained in this paper and will be conducted in future works.

For the proposed iterative procedure, based on the above derivations, the following issues should be raised.

Issue 1: Comparasions with Qiu's conventional iterative C-B method

As all the terms being iterated, i.e. $\mathbf{T}_{\text{S}}^{[i]}$, $\mathbf{K}_{\text{S}}^{[i]}$ and $\mathbf{M}_{\text{S}}^{[i]}$ do not vary from mode to mode, all the interested modes can be obtained in one round of iterations following the proposed method. Apparently, this is different from Qiu's iterative procedure [15] and can be mathematically explained as the ICB method is intended to find each diagonal term of $\mathbf{\Lambda}_{\text{S}}$ while the proposed method is intended to find $(\mathbf{K}_{\text{S}}, \mathbf{M}_{\text{S}})$ instead, which is $(\mathbf{\Lambda}_{\text{S}}, \operatorname{diag}(\mathbf{I}_{\text{L}}, \mathbf{I}_{\text{b}}))$ in the basis $\mathbf{P}_{\text{S}}$.

Besides, in the proposed method, the reduced system matrices are consistent with the modal transformation matrix. This is also different from the ICB method. Actually, the modal transformation of the proposed method, i.e. Equation (25) performs as a Ritz reduction, and thus the convergence rate of the proposed method can be increased.

Notice that the derivations for development from ICB to SCB, i.e. Lemma 1-3, are very similar to those for the development [24] of the iterated reduced system method, which are usually called the "p-mode" approach in DC methods [24, 31, 32], due to the relationship between the C-B method and the DC method. Their differences lie in the choice of generalized coordinates. More specifically, the proposed method chooses the modal coordinates of retained component constraint modes in addition to the physical coordinates of boundary DoFs. In this way, the initial error can be controlled via the choice of retained modes instead of the selection of master physical DoFs.

Issue 2: Comparisons with approximated fixed-interface CMS methods

Clearly, the first iteration of the proposed method is the C-B method, a widely applied fixed-interface CMS method. This means the precision of the initial guess, which is critical for an iterative procedure from the numerical point of view, can be promised by choosing the retained component constraint modes appropriately under a frequency domain criterion, e.g. Line (1b) of Algorithm 1. Besides, for the second iteration, one can have

$$\mathbf{T}_{\text{S}}^{[2]} = \mathbf{T}_{\text{CB}} + \mathbf{S}_0\bar{\mathbf{M}}\mathbf{T}_{\text{S}}^{[1]}\left(\mathbf{M}_{\text{S}}^{[1]}\right)^{-1}\mathbf{K}_{\text{S}}^{[1]} = \mathbf{T}_{\text{CB}} + \begin{bmatrix}\mathbf{0} & \mathbf{F}\mathbf{m}_{\text{c}}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\mathbf{M}_{\text{CB}}^{-1}\mathbf{K}_{\text{CB}} \triangleq \mathbf{T}_{\text{r}} \tag{37}$$

with Equation (33) and Lines (3a), (4a) of Algorithm 1. Notice that $\mathbf{T}_{\text{r}}$ is the modal transformation of the enhanced C-B method [20], where the approaches of O'Callahan for the improvement of Guyan method are applied to the C-B method.

Compared with these approximation based methods, the SCB method could be a better choice when high precision eigen-solutions are of interest, since the truncation effects are compensated via solving $\underline{\mathbf{X}}_k$ iteratively instead of being reduced via synthesising more lower component modes, which may largely increase the size of the ROM.

Issue 3: Consistency and convergence of the proposed method

The consistency can be promised by the equivalence between the reduced eigen-equation of the proposed method and the exact reduced eigen-equation of the C-B method, i.e. **Lemma 2**. Besides, it can be proved in a similar way to that of the simultaneous iterative free-interface CMS method [23].

The convergence of the simultaneous iterative procedure can be explained by the convergence of the series in Equation (31) and the properties of the Rayleigh-Ritz quotient. A more rigorous explanation can be refer to the convergence verification of the simultaneous iterative free-interface CMS method [23] or the iterated DC methods [24,31,32], since the iterative updating formulas of all these model reduction methods, e.g. Line (4a) of Algorithm 1, have the same mathematical form as that of the improved IIRS method developed by Friswell et al. [24]. A mathematical proof of the convergence for this kind of iteration methods is not available at hand and deserves further studies.

Issue 4: A shortcoming of the proposed method

From Equations (16) and (23), all the boundary DoFs should be retained in $\boldsymbol{p}_k$, and thus the column size of the iterative term and the size of ROM may be much larger than the number of interested global modes, which may include extra computational burdens. This is an important problem for the primal substructuring based reduction techniques. A solution may lie in the reduction of boundary DoFs through modal truncation, as has been done in the AMLS method, where the boundaries shared by components are also treated as substructures. The trade-offs lie in the sacrificing of precision of initial guess. Nevertheless, this error can be compensated by evaluating it precisely and then reducing it iteratively, as has been done in Subsections 2.2 and 2.3 of this paper. Details of this improvement will be conducted in further works.

3. Numerical examples

In this section, a free-free beam model is employed as a numerical example for illustrating the method. As shown in Figure 1, the beam is made of aluminum, with a length of 1.8m, a cross-sectional area of $1.0\times10^{-6}\mathrm{m}^2$, and a moment of inertia of $4.17\times10^{-10}\mathrm{m}^4$. It is equally discretized into 40 beam elements in total and then evenly divided into four components. Each component has 22 DoFs and its maximum eigenvalue is $2.99\times10^{11}\mathrm{rad}^2/\mathrm{s}^2$. For simplicity, the operations in this section are conducted in MATLAB2014a [33].

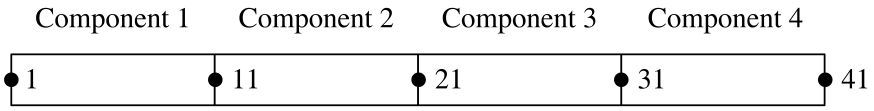

Figure 1: The free-free beam and its components

To examine the precision of the proposed approach, the interested frequency range of the beam is set to (0, 680) Hz (corresponding to the first 11 modes and $\bar{\lambda}_{11} = 1.82\times10^7\mathrm{rad}^2/\mathrm{s}^2$). For reference, the global modal parameters solved by the built function "eig" [34] are employed as the exact values. The configurations of the proposed method are set as: (a) the precision requirement is 1×10^{-6}; (b) the first 3 constrained modes of components 1 & 4, and the first 2 constrained modes of components 2 & 3 are kept, where $\lambda_3^{(1)} = \lambda_3^{(4)} = 2.16\times10^7\mathrm{rad}^2/\mathrm{s}^2$ and $\lambda_2^{(2)} = \lambda_2^{(3)} = 2.23\times10^7\mathrm{rad}^2/\mathrm{s}^2$. The relative errors of the first 9 elastic eigen-pairs solved by the proposed method are listed in Table 1. Here, the error of the kth global eigenvector is calculated with the Modal Assurance Criterion (MAC) by Equation (38).

$$\text{Error of } \bar{\boldsymbol{\phi}}_{\mathrm{CMS},k} \overset{\text{def}}{=} 1 - \frac{\left(\bar{\boldsymbol{\phi}}_k^{\mathrm{T}}\bar{\boldsymbol{\phi}}_{\mathrm{CMS},k}\right)^2}{\left(\bar{\boldsymbol{\phi}}_k^{\mathrm{T}}\bar{\boldsymbol{\phi}}_k\right)\left(\bar{\boldsymbol{\phi}}_{\mathrm{CMS},k}^{\mathrm{T}}\bar{\boldsymbol{\phi}}_{\mathrm{CMS},k}\right)} \tag{38}$$

where the subscript "CMS" indicates the results given by a CMS method. In particular, results in the columns"Init" and "1st" are the modal parameters from the C-B and enhance C-B method, respectively.

It can be known from Table 1 that the eigenvalues of the first 9 elastic modes approach the exact values as the number of iterations increases. The "lower" modes converge faster than the "higher" modes because the truncation errors of the

"higher" modes are larger in the initial step and the approximate convergence rate $\bar{\lambda}_k/\bar{\lambda}_{11}$ approaches unity as k increases. In the meantime, the errors of the eigenvectors are reduced to a negligible level after convergence. Compared with the (enhanced) C-B method, the precision of all the modal parameters is significantly improved. The above results indicate that high precision modal parameters can be obtained with the proposed method.

Table 1: Relative errors of the proposed method for the beam model

Mode order	Exact value (rad^2/s^2)	Eigenvalues				Eigenvectors	
		Init	1st	2nd	3rd	Init	Final
3	1.03×10^{4}	5.52×10^{-6}	7.57×10^{-11}	7.57×10^{-11}	7.34×10^{-11}	1.78×10^{-8}	0
4	7.86×10^{4}	6.33×10^{-5}	2.22×10^{-11}	2.85×10^{-11}	2.87×10^{-11}	7.92×10^{-7}	4.44×10^{-16}
5	3.04×10^{5}	2.36×10^{-4}	2.68×10^{-10}	7.91×10^{-12}	2.01×10^{-12}	6.20×10^{-6}	2.44×10^{-14}
6	8.39×10^{5}	1.79×10^{-4}	1.86×10^{-9}	9.83×10^{-11}	4.15×10^{-11}	1.09×10^{-5}	6.73×10^{-13}
7	1.90×10^{6}	5.23×10^{-4}	8.96×10^{-9}	1.06×10^{-10}	3.33×10^{-11}	2.52×10^{-5}	7.26×10^{-13}
8	3.76×10^{6}	2.75×10^{-3}	6.73×10^{-7}	2.55×10^{-8}	1.06×10^{-8}	3.25×10^{-4}	4.31×10^{-10}
9	6.78×10^{6}	1.50×10^{-2}	1.88×10^{-5}	1.64×10^{-7}	1.86×10^{-8}	2.92×10^{-3}	7.84×10^{-10}
10	1.14×10^{7}	6.01×10^{-3}	1.57×10^{-5}	9.98×10^{-7}	4.14×10^{-7}	2.13×10^{-3}	3.10×10^{-8}
11	1.82×10^{7}	4.62×10^{-2}	2.17×10^{-4}	5.03×10^{-6}	5.67×10^{-7}	1.23×10^{-2}	3.58×10^{-8}

To further study the convergence, the relative errors of the eigenvalues whose modal frequencies are larger than the truncation frequency are plotted in Figure 2 with respect to the number of iterations. As shown in Figure 2, for all the five modes, the proposed method still converges, although the convergence rate is much slower than the lower modes. Here, $\bar{\lambda}_{16} = 1.14\times10^{8}\mathrm{rad}^2/\mathrm{s}^2$, which is five times of $\bar{\lambda}_{\max}$. This again validates the consistency and convergence of the proposed method.

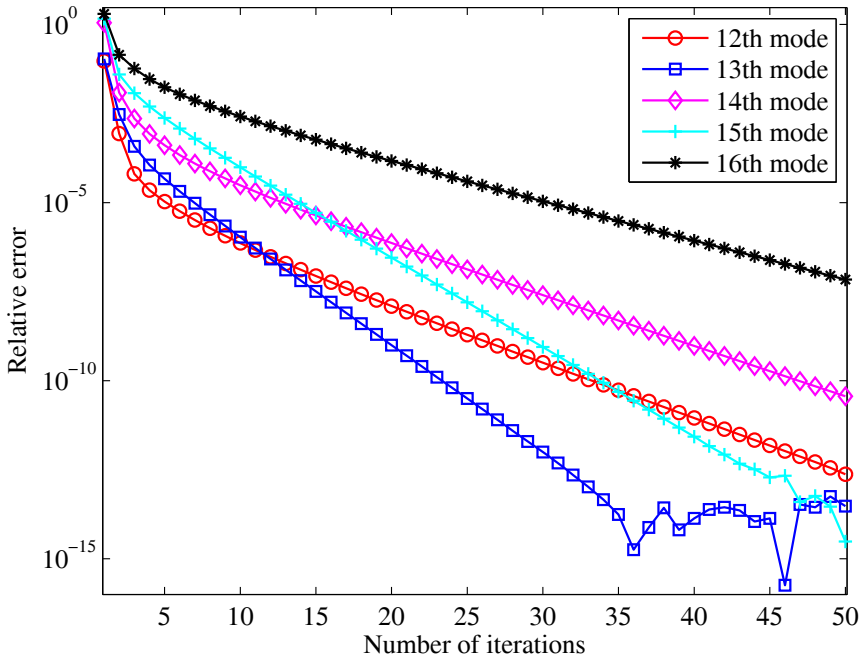

Figure 2: Convergence of the proposed iterative method for modes over the truncation frequency

4. Conclusions

A simultaneous iterative procedure for the fixed-interface CMS method is developed in this paper toward fast calculating the modal parameters and ROM of a large-scale and/or complicated structure. In the proposed iteration scheme, an eigenvalue independent matrix, whose column projections in the exact reduced space are the interested global eigenvectors, is chosen as the iterative term and then used as a Ritz basis to generate the reduced system matrices. Consequently, all the interested modes can be solved simultaneously and a linear ROM can be derived after one round of iterations. Complete theoretical frameworks of the proposed method, including the existence of iterative term, consistency of the ROM and an

identity formula of the iterative term, are presented with mathematical proof. For reference, an implementation is given together with some computational considerations. The numerical example shows that the proposed method can be converged and effective.

In future works, the convergence of the proposed iterative scheme and its generalization for the C-B type methods with interface reduction will be studied. Besides, the upper bound of precision of the ROM of the proposed method will be verified.

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