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Information aggregation in a large multi-stage market game*

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Abstract

A three-stage market-game mechanism is devised that is simple (actions are quantities and outcomes are determined by arithmetic operations that do not depend on details of the economy) and achieves efficiency in a two-divisible-good, pure-exchange setting with potential information-aggregation. After an entry stage, agents make offers which are provisional for all but a small, randomly selected group. Then, those offers are announced, and everyone else makes new offers with payoffs determined by a Shapley-Shubik market game. For a finite and large number of players, there exists an almost ex post efficient equilibrium. Conditions for uniqueness are also provided. (98 words)

Key words: mechanism-design, information-aggregation, market-game, efficiency.
JEL classification numbers: D82, D43

1 Introduction

There is a large literature–some of it theoretical (see below) and some of it experimental (see Axelrod *et al.* [1])—that deals with settings in which there is dispersed and private information that is valuable in the sense that better outcomes could be achieved if the private information is revealed. In such settings, the problem is to induce people to reveal what they know. This paper deals with a pure-exchange setting in which the challenge is to achieve ex post efficient allocations using a mechanism that is simple in two senses. The first is detail-freeness: the mechanism does not rely on specific information about the economy such as the functional form of agents' utilities or the way that private signals relate to the unobservable state (see Hurwicz *et al.* [7]). The second is that the participants' actions are low-dimensional and that there is a simple algorithm that

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computes the outcome for each profile of participants' actions. The mechanism we devise and analyze is a three-stage market-game (trading-post) mechanism. It is simple in those two senses and, as we show, achieves almost ex post efficiency when the finite number of agents is sufficiently large.

Some of the literature has focused solely on efficiency. Gul and Postlewaite [6] and McLean and Postlewaite [8] construct direct mechanisms that achieve almost ex post efficiency in environments that include ours as a special case. However, as is widely recognized, their direct mechanisms are not detail-free. Presumably, that is why McLean and Postlewaite [8] (pages 2,439 and 2,441) do not regard their mechanism as suitable for actual use. Some of the literature has focused on both efficiency and simplicity under particular mechanisms. Reny and Perry [13] and Vives [17] study double-auction mechanisms and show that those mechanisms achieve ex post efficiency in special settings: Reny and Perry [13] assumes unit demands, while Vives [17] assumes quadratic utilities and normally distributed signals. However, except in such special cases, double auctions are not simple in terms of actions or the way the mechanism uses those actions. In order to achieve ex post efficiency in our setting, a general two-divisible-good setting, actions in a double auction would have to be general demand functions, which are not simple objects. Moreover, there is no simple algorithm that computes a market-clearing price in the double auction from arbitrary demand functions. Our mechanism, in contrast, has agents choosing one-dimensional actions and has outcomes produced using simple arithmetical operations (see, also, the remarks in Dubey et. al. [5], page 108).[1]

The environment we study is a (static) two-divisible-good, finite number-of-agents, pure-exchange setting in which there is a role for information aggregation. The information-preference structure is borrowed from Gul and Postlewaite [6] and is closely to related to those in Reny and Perry [13] and Vives [17]. There is an unobserved state-of-the-world, which in our case is drawn from a finite set, and there are private signals, also drawn from a finite set, that are informative about the state. The realized utility of an agent depends both on the state and on the private signal received. The state can be interpreted as a common taste (or quality) shock and the signal as an idiosyncratic taste (or quality) shock. In the language of *auction* theory, the model is a mixed *common-private value* setting. From a more general point of view, the presence of private signals makes it an *adverse-selection* model in the sense that after receiving a private signal, each agent knows something that others do not know. Finally, the setting has agents with endowments that are not under the direct control of the mechanism.

At stage-1, before learning their types, agents choose whether or not to participate when faced with a suitably small entry fee. Then the participating agents learn their types and enter stage-2. At stage-2, each agent names an offer as in most market-game mechanisms, but faces an exogenous price chosen by the mechanism. Then, in a random fashion, the mechanism divides the agents into two groups: a small *inactive* group and a large *active* group. Those in the inactive group participate no further; their stage-2

[1] Peck [12] also obtains information aggregation and efficiency in a two-stage market game, but one with a finite set of large agents and a continuum of small agents. Ritzberger [14] studies a market game with limit orders and obtains competitive outcomes in a setting with "private values," but shows that his mechanism fails in settings with general aggregate uncertainty.

offers are executed at the exogenous price. Next, the histogram of the stage-2 offers of the active agents is announced. Finally, at stage-3, the active group participate in a market game in which the price is determined by their stage-3 offers as in the usual two-sided Cournot or market-game model.[2]

Participating agents face a tradeoff at stage-2. Contingent on becoming inactive, an agent's stage-2 offer determines his final payoff so that it is in the agent's interest to reveal his private information. Contingent on becoming active, the stage-2 offer affects his final payoff only by way of its influence on the actions of other active agents at stage-3. And that influence depends on how beliefs are formulated, both on and off the equilibrium path.

We have results about existence of a Bayesian Nash equilibrium, about *almost* ex post efficiency of the equilibrium, and about uniqueness of the equilibrium. All are related to the ex post competitive equilibria (CE) of a corresponding limit (continuum-of-agents) economy—an economy in which the state is known and in which the fraction of agents with each realization of the private signal is the known probability of receiving that signal conditional on the state.[3] If some mild genericity conditions hold and if the finite number of agents is sufficiently large, then there exists a *fully-revealing* equilibrium, one in which everyone participates and in which stage-2 offers reveal the active agents' types. Moreover, for any CE of the ex post limit economy, there is a fully-revealing equilibrium in which the stage-3 behavior converges almost surely to that CE. A crucial aspect of the argument is the formulation of beliefs so that the stage-2 tradeoff is resolved entirely in favor of the payoff contingent on being inactive.

Our formulation of beliefs is simple and plausible: it associates any stage-2 offer, whether in equilibrium or not, with an agent type and then employs Bayes rule to derive the updated beliefs over the state-of-the-world. In particular, given our formulation of beliefs, in order to misrepresent one's type, the stage-2 offer has to be sufficiently far away from the offer that would be best contingent on being inactive. Then, because the stage-3 effect of the misrepresentation vanishes as the number of agents gets large, the stage-2 tradeoff is resolved in favor of the payoff contingent on being inactive. The assumption that there is a finite number of states and types is crucial for our construction.

Our notion of almost ex post efficiency is the same as that in Gul and Postlewaite [6]. The qualification *almost* arises from two sources: efficiency gains and the randomness of realizations of the profile of types. Because our mechanism, although feasible, is not budget-balanced, we need a notion of efficiency that allows for an arbitrarily small Pareto

[2]As shown in previous work, *one-stage* market games do not, in general, aggregate information in a way that leads to efficiency because agents commit to quantities before the relevant information is revealed. Palfrey [11] uses a Cournot mechanism and obtains ex post optimality, but, as Vives [16] points out, only because marginal cost is common and constant so that it does not matter how production is allocated among the firms in the model. Dubey *et al.* [5] study a dynamic market game with trades in multiple periods, and show that information may be aggregated, but only after trades and consumption at the first-period are observed. As they emphasize, this precludes ex post efficiency.

[3]From the ex ante perspective, it is standard to label any such CE a rational-expectations CE (see, for example, Reny and Perry [13]). From that perspective, our results establish a strategic foundation for rational-expectations CE in a new setting—one with divisible goods, general preferences, a general information structure, and a finite number of agents.

improvement. And, we only get a characterization of equilibrium with high probability because the type-profile is random. We show that the outcome of our fully-revealing equilibrium achieves the same efficiency result as the direct mechanisms constructed in Gul and Postlewaite [6] and McLean and Postlewaite [8].

We also have a uniqueness result. For this result, we assume that the ex post limit economy has monotone competitive demand for each state-of-the-world—which, of course, implies a unique CE for each state—and we make some mild additional assumptions, including a restriction on off-equilibrium beliefs that is similar to the "no-signaling-what-you-don't-know" restriction in Fudenberg and Tirole [4]. Given these assumptions, we show that any equilibrium in our mechanism is fully revealing.

Several difficulties have to be overcome in order to produce the above results. First, because types are distributed *i.i.d.* conditional on the state, we do not have deterministic replication of a given profile of types. Therefore, in order to show almost-sure convergence of stage-3 behavior to any given CE of the limit economy, we use a lower hemi-continuity argument that treats the type-configuration as one of the parameters. Second, at both stage-2 and stage-3, our mechanism uses resources. At stage-2, the trades of the (small) inactive group do not necessarily clear at the exogenous price. At stage-3, in order to avoid the no-trade equilibrium of the market game and for other reasons, the mechanism makes small exogenous trades using a formulation borrowed from Dubey and Shubik [2]. To finance those two uses of resources, we impose an entry fee at stage-1, one that implies feasibility of the mechanism for any actions and one consistent with existence of an equilibrium in which everyone participates.

Finally, a few remarks are in order about seeming limitations of our environment and mechanism. As regards the environment, there are two critical assumptions. One is finite supports for the state and for types and the other is the two-good assumption. While a finite number of states may seem more restrictive than the continuum specifications in Reny and Perry [13] and Vives [17], they make strong distributional assumptions that we do not need. Also, our two-good model may seem limited relative to some multiple-good trading-post models. In particular, Forges and Minelli [3] have a general finite number of goods and use a two-stage mechanism that is somewhat similar to ours: agents send messages about their private information to the mechanism at their first stage, and, after seeing others' messages, all agents engage in a market game at their second stage. However, in order to avoid the well-known difficulty that the proceeds from sales at some trading posts cannot be used to make purchases on other posts in a multi-good static trading-post model, they assume a monetary structure of trading posts and need strong assumptions about the endowments of money and the preferences for it (see Forges and Minelli [3], Assumption 2 on page 397 and footnote 11 on the same page). Such assumptions are not so distant from assuming quasi-linear preferences (transferable utility) and under quasi-linear preferences, our model and results can easily be adapted to a general finite number of goods. Moreover, Forges and Minelli [3] use a continuum-of-agents setting which eliminates all the strategic considerations that are central to what we do.

Our three-stage mechanism may not be the only simple mechanism that achieves ex post efficiency. One alternative mechanism, which more closely resembles pari-mutuel betting, works as follows. The stage-2 offers of the inactive agents are part of the offers that determine the "price" at stage-3, and their payoffs are determined in the same way as those for active agents. However, in such a version, even if agents make stage-2 offers based on the presumption that they will be inactive, they would want to predict the stage-3 price which, itself, is affected by their offers—both directly and by the information revealed by stage-2 offers. Thus, to get an equilibrium for such a mechanism, a mapping that takes both stages into account would have to be studied. Our approach decouples stage-2 payoffs from what happens at stage-3 and, therefore, is simpler. Given that it has good welfare properties, its simplicity is a virtue—both for us in analyzing the properties of the mechanism and for those who play the game.

2 The environment

Our economy is a two-good endowment economy with N ex ante identical agents. Each agent is endowed with the per capita endowment of each good, denoted $(\bar{q}, \bar{r}) \in \mathbb{R}^2_{++}$. First, nature draws a state-of-the-world $z \in Z$ with probability $\tau_0(z)$, a state which no one observes. Then each agent gets a type realization, $x \in X$, with probability $\mu_z(x)$, where x is private to the agent and is *i.i.d.* across people conditional on z. An agent of type $x \in X$ maximizes expected utility with ex post utility function, $u(q, r; x, z)$, where $(q, r) \in \mathbb{R}^2_+$ is the vector of quantities of the two goods consumed. The function $u(\cdot, \cdot; x, z)$ is strictly increasing, strictly concave, continuously twice differentiable, and satisfies Inada conditions.

Both Z and X are finite sets. We assume that $\tau_0(z) > 0$ for each $z \in Z$ and that $\mu_z(x) > 0$ for each $x \in X$ and $z \in Z$.[4] We also assume that x is informative in the sense that $z \neq z'$ implies $\mu_z(x) \neq \mu_{z'}(x)$ for some $x \in X$. (This informativeness assumption is without loss of generality: If $\mu_z(x) = \mu_{z'}(x)$ for all $x \in X$, then we treat z and z' as a single state z'' with utility $u(q, r; x, z'') = \tau_0(z)u(q, r; x, z) + \tau_0(z')u(q, r; x, z')$.) As noted above, our interpretation is that x is an idiosyncratic taste shock and z is a common taste shock. The realized type, x, plays two roles: it serves as private information about z and it is private information about preferences.

We make two additional assumptions about preferences and the underlying uncertainty. These assumptions ensure that there are ex post gains-from-trade among the agents. Although these are numbered so that we can refer to their roles in the discussion, they are maintained throughout. The first is about the way the unique outcome of a static price-taking choice at a particular price depends on types.

A1. For $\overline{p} = \bar{r}/\bar{q}$, there exists (small) $\overline{\kappa} = (\overline{\kappa}_q, \overline{\kappa}_r) > 0$ such that for all $\kappa \leq \overline{\kappa}$, if

$$(\widehat{q}(x, \kappa), \widehat{r}(x, \kappa)) = \arg\max \sum_{z \in Z} \tau_x(z) u(q, r; x, z) \quad (1)$$

[4] This assumption appears in most other related papers. Without this full support assumption, a realization x could perfectly reveal the state even without any information aggregation. If it did, then the information structure would violate the "informational smallness" assumption in McLean and Postlewaite [8].

subject to $\overline{p}q + r \leq \overline{p}(\bar{q} - 2\kappa_q) + (\bar{r} - 2\kappa_r)$, then $x \neq x'$ implies $(\widehat{q}(x,\kappa), \widehat{r}(x,\kappa)) \neq (\widehat{q}(x',\kappa), \widehat{r}(x',\kappa))$. Here, $\tau_x(z)$ denotes the probability that the state is z conditional on x.

Since u is strictly concave, by the Theorem of Maximum, *A1* is satisfied if $x \neq x'$ implies $(\widehat{q}(x,0), \widehat{r}(x,0)) \neq (\widehat{q}(x',0), \widehat{r}(x',0))$. Thus, *A1* is violated only for knife-edge cases for two distinct aspects of the environment: the probabilities, $\tau_x(\cdot)$ and $\tau_y(\cdot)$, and the utilities, $u(\widehat{q}(x,0), \widehat{r}(x,0); x, \cdot)$ and $u(\widehat{q}(x',0), \widehat{r}(x',0); x', \cdot)$. Moreover, under *A1*, $\sum_{z\in Z} \tau_x(z) u(\widehat{q}(x,0), \widehat{r}(x,0); x, z) > \sum_{z\in Z} \tau_x(z) u(\overline{q}, \overline{r}; x, z)$ for some x, and hence, by continuity, $\sum_{z\in Z} \tau_x(z) u(\widehat{q}(x,\kappa), \widehat{r}(x,\kappa); x, z) > \sum_{z\in Z} \tau_x(z) u(\overline{q}, \overline{r}; x, z)$ for small κ.

The second assumption is about an ex post limiting version in which there is a continuum of agents and no uncertainty.

A2. For each $z \in Z$, let $\mathcal{L}^z$ be the corresponding continuum-of-agents economy with known aggregate state z and with fraction of type-x agents equal to $\mu_z(x)$. For each $z \in Z$, $\mathcal{L}^z$ has a finite number of *regular* competitive equilibria (CE) (see Mas-Colell *et al.* [10], Definition 17.D.1) in which every type trades.

As explained further below, *A2*, which is generic, is also used to get differentiability of best responses in our stage-3 market game.

This is a convenient time to introduce a perturbation of the $\mathcal{L}^z$ economy. Let $\mathcal{L}^z(\kappa) = \mathcal{L}^z(\kappa_q, \kappa_r)$ be a continuum-of-agents economy defined by three properties: (i) it has a known aggregate state z and has fraction of type-x agents equal to $\mu_z(x)$; (ii) the endowment for each agent is equal to $(\bar{q} - 2\kappa_q, \bar{r} - 2\kappa_r)$; (iii) there are (small) exogenous per capita *supplies* $\kappa = (\kappa_q, \kappa_r)$. The following lemma is a well known consequence of *regularity* of CE in $\mathcal{L}^z$.

Lemma 0. Fix a regular CE allocation for $\mathcal{L}^z$, denoted $(q^z_x, r^z_x)_{x\in X}$. There exists $\bar{\kappa} > 0$ such that for any $\kappa \in [0, \bar{\kappa}]$, $\mathcal{L}^z(\kappa)$ has a regular CE allocation, denoted $(q^{z,\kappa}, r^{z,\kappa}) = (q^{z,\kappa}_x, r^{z,\kappa}_x)_{x\in X}$, that is continuous in κ.

By *A2,* if κ is small, $(q^{z,\kappa}, r^{z,\kappa})$ is such that every type trades, and, by continuity, satisfies $u(q^{z,\kappa}_x, r^{z,\kappa}_x; x, z) > u(\overline{q}, \overline{r}; x, z)$ for all x.

3 The mechanism

Before realizations of z and x occur, the N agents face a participation fee that consists of small amounts of both goods, $2\kappa = (2\kappa_q, 2\kappa_r) \in (0, \bar{q}) \times (0, \bar{r})$. They simultaneously decide whether or not to participate, which we label stage-1. Those who decide to participate pay the entry fee and participate in the next stage, labelled stage-2. Those who decide not to participate consume their endowments. We assume that the profile of participation decisions is observed by the mechanism, but is otherwise private. The set of participants is denoted $\mathcal{N}'$ and $N' = |\mathcal{N}'|$.

After the participation decisions and after the state and types are realized, each agent

$n \in \mathcal{N}'$ chooses an offer $a_n = (a_{n,q}, a_{n,r}) \in \mathcal{O}$, where

$$\mathcal{O} = \{(o_q, o_r) \in [0, \bar{q} - 2\kappa_q] \times [0, \bar{r} - 2\kappa_r] : o_q o_r = 0\}. \tag{2}$$

This is stage-2. Then the participants are randomly divided into two groups in the following way. Let $\eta \in (0,1)$ and let $\lceil (1-\eta)N' \rceil = M$ denote the smallest integer that is no less than $(1-\eta)N'$. An assignment, which assigns a number n' to each agent n in a one-to-one fashion, is drawn from the uniform distribution over the set of all such assignments, and agent n is called *active* if $n' \leq M$ and is called *inactive* if $n' > M$. (Notice that the identities of the inactive/active are random, but that M is a deterministic function of N'.) The payoff for an inactive agent n who made offer $a_n = (a_{n,q}, a_{n,r})$ is given by trade at the fixed price, $\overline{p} = \bar{r}/\bar{q}$, and his consumption bundle is given by

$$(\bar{q} - 2\kappa_q - a_{n,q} + a_{n,r}/\overline{p}, \bar{r} - 2\kappa_r - a_{n,r} + \overline{p} a_{n,q}) \text{ for } n \in \mathcal{N}' \setminus \mathcal{M}, \tag{3}$$

where $\mathcal{M}$ is the set of active agents. Notice that if a type-x agent makes a stage-2 offer to maximize expected utility contingent on being inactive and if the agent is selected to be inactive, then that offer gives rise to the payoff $(\widehat{q}(x,\kappa), \widehat{r}(x,\kappa))$ in (1).

Next, the mechanism announces the histogram of the stage-2 offers of the active agents, denoted $\nu : \mathcal{O} \to \{0, 1, 2, ..., M\}$.[5] For each $a \in \mathcal{O}$, $\nu(a)$ is the number of active agents whose stage-2 offers are a. Then, given that information, each active agent participates in a market game, which is stage-3.

In stage-3, each active agent n makes an offer $b_n = (b_{n,q}, b_{n,r}) \in \mathcal{O}$ and gets the consumption bundle

$$(\bar{q} - 2\kappa_q - b_{n,q} + b_{n,r}/(R/Q), \bar{r} - 2\kappa_r - b_{n,r} + (R/Q) b_{n,q}) \text{ for } n \in \mathcal{M}, \tag{4}$$

where

$$(Q, R) = \sum_{n' \in \mathcal{M}} b_{n'} + M\kappa. \tag{5}$$

Here, $\kappa = (\kappa_q, \kappa_r)$ are exogenous (small) quantities or offers, a formulation borrowed from Dubey and Shubik [2]. The exogenous supplies in the stage-3 trade play three roles for us: there is no need to define payoffs when there are zero-offers on one side of the market; no-trade is eliminated as an equilibrium; and, most important, there are implied bounds on the "price," R/Q, that help us prove existence of the implied stage-2 game.[6] Notice

[5] We could let the mechanism announce two histograms, one for active agents and one for inactive agents. However, that would complicate the notation and would not change the results.

In a comment on an earlier draft, George Mailath pointed out that if only the histogram for inactive agents were announced, then the ex ante tradeoff facing agents would not be present. However, as explained further below, that would make stage-3 a game of incomplete information and would prevent information aggregation from being achieved. In our specification, it is a game with complete information on the equilibrium path. Our specification also has the virtue that the designer is not hiding information that agents would like to have—namely, the types of the other active agents.

[6] Our market game is a version of what is known as the "buy-sell" game, which respects individual rationality, as opposed to the "sell-all" version (see Shapley-Shubik [15]), which does not. Forges and Minelli [3] use the sell-all version of the market game, which eliminates the first two roles of κ.

that this "price" depends both on the offers of other active agents and on agent n's offer. Moreover, because we require offers to be in the set $\mathcal{O}$, this "price" is not differentiable in agent n's offer at the offer $(0,0)$. That is one reason for assuming that every type trades in at least one regular CE for the ex post limit economy, $\mathcal{L}^z(0)$.

Notice that the stage-3 exogenous offers are exactly half of the entry fees in per capita terms. That allows us to prove the following.[7]

Lemma 1. If $(2\kappa_q, 2\kappa_r) \in (0,\bar{q}) \times (0,\bar{r})$ and if $\eta \leq \min\{\kappa_q/2\bar{q}, \kappa_r/2\bar{r}\}$, then entry fees of the participants cover the stage-2 and stage-3 costs of operating the mechanism.

As this indicates, the fraction of inactive agents, η, must be small relative to the entry fees because the market does not clear for the inactive agents and additional resources may be necessary to cover trades there.

The restriction in $\mathcal{O}$ that agents can only make offers on one side of the market plays a significant role in our analysis. It is used to obtain uniqueness of best responses. The following lemma shows that the restriction is not binding on the agent when there is no private information, which is the case for the stage-3 game in the candidate equilibrium we construct.

Lemma 2. Fix stage-3 offers of all other agents. Given those offers, for any offer $b' \in [0, \bar{q} - 2\kappa_q] \times [0, \bar{r} - 2\kappa_r]$, there exists $b'' \in \mathcal{O}$ that has the same payoff as b'.

Obviously, the restriction is also not binding in the same sense on payoffs for inactive agents.

In what follows, we assume that the mechanism has selected a pair $(\kappa, \eta) > 0$ such that κ is small in the sense that it is less than the thresholds given by A1 and Lemma 0, and such that $\eta \leq \min\{\kappa_q/2\bar{q}, \kappa_r/2\bar{r}\}$ so that Lemma 1 holds. We also let $(\bar{q}', \bar{r}') = (\bar{q} - 2\kappa_q, \bar{r} - 2\kappa_r)$.

4 Existence of fully-revealing equilibrium

We begin with definitions of strategy and equilibrium. We use Perfect Bayesian Equilibrium (PBE) as our solution concept and limit consideration to symmetric equilibria in pure strategies, where symmetry holds for both strategies and beliefs.[8] A *symmetric strategy profile* is a triple $(s_1, s_2, s_3) = s$ defined as follows: (i) $s_1 \in \{yes, no\}$ is the participation decision, where *yes* denotes willingness to participate and pay the entry fee, while *no* denotes unwillingness; (ii) $s_2(x) \in \mathcal{O}$ is the stage-2 strategy (if the agent has decided to participate); (iii) $s_3(x, a, \nu^{-a}) \in \mathcal{O}$ is the stage-3 strategy, where (x, a) denotes the agent's type and stage-2 action and ν^{-a} denotes the announced histogram of offers of active agents *net of the agent's own action.* (For any $a' \in \mathcal{O}$, $\nu^{-a}(a') = \nu(a')$ if $a \neq a'$ and $\nu^{-a}(a) = \nu(a) - 1$.)

Definition 1. A symmetric strategy profile s is a *PBE* if there exists a symmetric belief

[7]All proofs appear in the Appendix.

[8]Of course, this is without loss of generality regarding existence, but our uniqueness result is only relative to this class.

profile φ such that (a) (s_1, s_2) is optimal given s, (b) s_3 is optimal against s and φ, and (c) φ is consistent with s in the sense that it satisfies Bayes rule whenever possible. Here, $\varphi(x, a, \nu^{-a}) \in \Delta(Z \times \Theta)$, where Θ is the set of all configurations θ of type/stage-2-action of the other active agents; i.e., $\theta : X \times \mathcal{O} \to \{0, 1, 2, ..., M-1\}$ with M being the number of active agents.

In our mechanism, agents only partially observe others' participation decisions through the number of active agents from the public announcement ν, which may range from 1 to $\lceil (1-\eta)N \rceil$. The stage-3 strategy, s_3, has to specify an offer for any such possible announcement. Agents also have to form beliefs about others at that stage. Because we restrict our attention to symmetric equilibrium, an agent's expected payoff at stage-3 depends only on his private history (x, a) and the configuration of other active agents' private histories θ. This allows us to formulate the belief as a distribution over $Z \times \Theta$. Moreover, even though all agents use the symmetric function φ to form their beliefs, different agents can have different beliefs after seeing the same announcement ν because they have different private histories and different interpretations of ν^{-a}.

We are interested in PBE that have full participation and full information-aggregation. Such an equilibrium is formally defined as follows.

Definition 2. A *fully-revealing* equilibrium is a PBE in which $s_1 = yes$ and $x \neq x'$ implies $s_2(x) \neq s_2(x')$.

The definition of a fully-revealing equilibrium restricts equilibrium behavior at stages 1 and 2, but not at stage-3, where most agents' payoffs are determined. Proposition 1 establishes existence of fully revealing equilibria whose stage-3 behavior is arbitrarily close to a given profile of ex post CE's. To set the stage for that proposition, we first define the sense in which consumption is random in our model.

Definition 3. An allocation is a mapping $c : X^N \times \Omega^N \times Z \to (\mathbb{R}^2_+)^N$ that maps the profile of agents' types, denoted $\zeta^N = (\zeta_1, ..., \zeta_n, ...\zeta_N) \in X^N$, the profile of agents' activeness statuses, denoted $\omega^N = (\omega_1, ..., \omega_N) \in \Omega^N$, where $\Omega^N = \{\omega^N \in \{0,1\}^N : \sum_{n=1}^N \omega_n = \lceil (1-\eta)N \rceil\}$, and $\omega_n = 0$ means inactive and $\omega_n = 1$ means active, and the state of the world, $z \in Z$, to a profile of consumption-bundles for each agent.

For any symmetric strategy profile, s, with full participation—i.e., with $s_1 = yes$—there is a corresponding allocation, denoted c^s. While there are three sources of such uncertainty in our model, whose realizations are ζ^N, ω^N, and z, c^s does not depend on z directly because no agent ever observes z. However, the distribution of c^s depends on z because the distribution of ζ^N depends on z. Also, because the strategy profile s is symmetric, for any given ζ^N all active agents of the same type have the same stage-3 consumption. Because of this symmetry, we also use $c^s_{1,x}$ to denote the consumption bundle of any active agent of type x.

Proposition 1. Fix an ex post regular CE profile $\{(q^{z,\kappa}, r^{z,\kappa})\}_{z \in Z}$ in which every type trades. Then there exists a number $\bar{N}$ and a sequence of fully-revealing equilibria $\{s^N = (s_1^N, s_2^N, s_3^N)\}_{N=\bar{N}}^{\infty}$ such that

$$\lim_{N \to \infty} c^{s^N}_{1,x} = (q^{z,\kappa}_x, r^{z,\kappa}_x) \text{ for all } x \in X \tag{6}$$

almost surely conditional on z for each $z \in Z$.

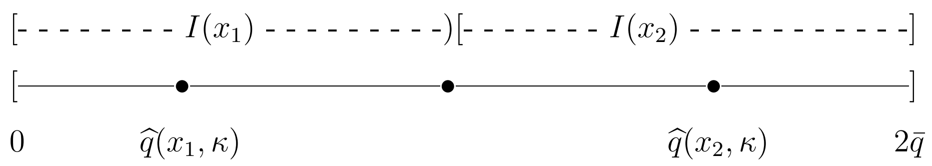
We prove Proposition 1 by first specifying a candidate equilibrium strategy profile, denoted $s^* = (s_1^*, s_2^*, s_3^*)$, and a corresponding belief profile, φ^*, where all but s_3^* are given explicitly. The specification of s_3^* has to be defined for all possible announcements ν; has to satisfy the limiting property (6); and has to be optimal with respect to a belief. We meet these conditions by a belief formulation that reduces optimality to finding an equilibrium in a suitably defined complete-information static game. We give a general existence result for that game that allows us to specify s_3^* for an arbitrary ν and we provide a lower hemi-continuity result for that equilibrium correspondence that allows us to establish the limiting properties required by (6). After specifying s_3^*, we prove the optimality of the following s_1^* and s_2^*: $s_1^* = yes$ (full participation) and $s_2^*(x)$ is the stage-2 offer that is optimal for a type-x agent contingent on being inactive so that the implied payoff if inactive is $(\widehat{q}(x,\kappa), \widehat{r}(x,\kappa))$ as given in (1). It follows from *A1* that $x \neq x'$ implies $s_2^*(x) \neq s_2^*(x')$. Therefore, s^* is fully revealing.

When the observed announcement after stage-2 is consistent with (s_1^*, s_2^*), Bayes rule requires the agent to have a degenerate belief that concentrates on the configuration implied by the announcement. For other announcements, our candidate belief also has a degenerate belief formulated as follows.

First, we order the elements of X so that $\widehat{q}(x_i,\kappa) < \widehat{q}(x_{i+1},\kappa)$ for $i \in \{1, 2, ..., |X|-1\}$, where $|X|$ denotes the cardinality of X and where $\widehat{q}(x,\kappa)$ is given in (1). Next, we partition the interval $[0, 2\bar{q}]$ into $|X|$ subintervals indexed by that ordering as follows:

$$I(x_i) = \begin{cases} \left[0, \frac{\widehat{q}(x_2,\kappa)+\widehat{q}(x_1,\kappa)}{2}\right) \text{ for } i = 1 \\ \left[\frac{\widehat{q}(x_i,\kappa)+\widehat{q}(x_{i-1},\kappa)}{2}, \frac{\widehat{q}(x_{i+1},\kappa)+\widehat{q}(x_i,\kappa)}{2}\right) \text{ for } i = 2, 3, ..., |X|-1 \\ \left[\frac{\widehat{q}(x_i,\kappa)+\widehat{q}(x_{i-1},\kappa)}{2}, 2\bar{q}\right] \text{ for } i = |X| \end{cases}. \tag{7}$$

For $|X| = 2$, $I(x_1)$ and $I(x_2)$ are depicted as follows:



Our candidate for equilibrium beliefs is that each agent forms a degenerate distribution over the type/stage-2-action configuration of the other active agents by treating an observed stage-2 action in $I(x_i)$ as coming from an agent of type x_i. Formally, for any $a \in \mathcal{O}$, let

$$q_1(a) = \bar{q}' - a_q + \frac{a_r}{\overline{p}} \in [0, 2\bar{q}]. \tag{8}$$

If M is the number of active agents in the observed histogram ν, the belief of an agent with private history (x, a), denoted $\varphi^*(x, a, \nu^{-a})$, puts probability 1 on the configuration

$\theta_{\nu^{-a}}$ defined by

$$\theta_{\nu^{-a}}(y,a') = \begin{cases} \nu^{-a}(a') \text{ if } q_1(a') \in I(y) \\ 0 \text{ otherwise} \end{cases} . \quad (9)$$

An associated marginal distribution over Z is given by the posterior derived from Bayes rule using the type-configuration of all active agents $\sigma : X \to \{0, 1, ..., M\}$ defined by

$$\sigma(y) = \begin{cases} \sum_{a' \in \mathcal{O}} \theta_{\nu^{-a}}(y,a') \text{ if } y \neq x \\ \sum_{a' \in \mathcal{O}} \theta_{\nu^{-a}}(x,a') + 1 \text{ if } y = x \end{cases} . \quad (10)$$

Notice that the belief φ^* is consistent with (s_1^*, s_2^*).

We call a type-x agent a *nondefector* if the agent's stage-2 action is in $I(x)$; otherwise, the agent is called a *defector*. (A defector uses the first line of (10), while a nondefector uses the second line.) That is, under our candidate belief, each agent believes that all others are nondefectors, and he believes that all others believe that *all* are nondefectors, even though he may be a defector himself. When all agents are nondefectors, the type-configuration σ given by (10) coincides with the true configuration and is commonly known among active agents. Moreover, there is also a common belief about z derived from σ via Bayes rule. Therefore, the stage-3 continuation game can be regarded as a complete-information game in which the common posterior distribution over z appears as a preference parameter. Once we prove a general existence result for that game under arbitrary σ, we can use it to define s_3^* for nondefectors under any arbitrary ν. We can also easily describe the best response of any defector because the defector believes that all other agents are nondefectors who believe that they are in a game with nondefectors only.

However, in order to establish the limiting behavior of stage-3 strategies as given in (6), we need to show existence of equilibrium behavior that converges to the given CE allocations. To do that, it is convenient to treat the above stage-3 continuation game as a game that depends on three parameters $(1/M, \mu, \phi) \in [0, 1] \times \Delta(X) \times \Delta(Z)$, all determined by σ, a game denoted $\mathcal{E}(1/M, \mu, \phi)$. In this game, there are M players; the action set for each player is $\mathcal{O}$; the known number of players of type-x as a fraction of M is $\mu(x)$; each agent has endowment $(\bar{q}', \bar{r}')$; there are per capita exogenous offers $\kappa = (\kappa_q, \kappa_r)$; and the payoff for type-x players is the expected value of u *w.r.t.* the common $\phi \in \Delta(Z)$ evaluated at the consumption implied by the stage-3 market game (see (4)).

We can now translate limiting property (6) into a characterization of the equilibrium correspondence of $\mathcal{E}(1/M, \mu, \phi)$ for limiting parameters. The parameter values for the limit economy $\mathcal{L}^z(\kappa)$ is given by $(1/M, \mu, \phi) = (0, \mu_z, \delta_z)$ (and, hence, $M = \infty$), where δ_z is the dirac measure concentrated at z, and the competitive allocation of interest in $\mathcal{L}^z(\kappa)$ is $(q^{z,\kappa}, r^{z,\kappa})$ given by Lemma 0. In terms of offers in $\mathcal{O}$, that CE allocation is $\beta^{z,\kappa} = \langle \beta_q^{z,\kappa}(x), \beta_r^{z,\kappa}(x) \rangle_{x \in X}$, where

$$\beta_q^{z,\kappa}(x) = \max\{\bar{q}' - q_x^{z,\kappa}, 0\} \text{ and } \beta_r^{z,\kappa}(x) = \max\{\bar{r}' - r_x^{z,\kappa}, 0\}. \quad (11)$$

Our next lemma gives both the general existence result and the characterization result for parameters close to $(0, \mu_z, \delta_z)$.

Lemma 3. (i) The set of symmetric equilibria for the game $\mathcal{E}(1/M, \sigma/M, \phi)$ is not empty. (ii) Fix $z \in Z$ and a regular CE allocation $(q^{z,\kappa}, r^{z,\kappa})$ for $\mathcal{L}^z(\kappa)$ in which every type trades. Let $\Lambda = [0,1] \times \Delta(X) \times \Delta(Z)$ and let $\lambda_0^z = (0, \mu_z, \delta_z)$. There exists an open neighborhood of λ_0^z, denoted Λ_z, with $\Lambda_z \subset \Lambda$, and a continuous function $f_z : \Lambda_z \to \mathcal{O}^X$ such that $f_z(\lambda_0^z) = \beta^{z,\kappa}$ (see (11)) and such that $(1/M, \sigma/M, \phi^\sigma) \in \Lambda_z$ implies that $f_z(1/M, \sigma/M, \phi^\sigma)$ is a symmetric equilibrium in $\mathcal{E}(M, \sigma/M, \phi^\sigma)$.

The proof of (i) is a routine application of Brouwer's fixed point theorem.[9] The proof of (ii) is an application of the implicit function theorem.[10] This is applicable because positive trade implies differentiability of best responses in the stage-3 game.[11]

Now, let

$$\beta^\sigma = \begin{cases} f_z(1/M, \sigma/M, \phi^\sigma) \text{ if } (1/M, \sigma/M, \phi^\sigma) \in \Lambda_z \text{ for some } z \in Z \\ \text{any Lemma 3 (i) equilibrium otherwise} \end{cases}. \tag{12}$$

Notice that this mapping is well-defined *iff* $\Lambda_z \cap \Lambda_{z'} = \emptyset$ for all $z \neq z'$. Because $\lambda_0^z \neq \lambda_0^{z'}$ for all $z \neq z'$ and Z is finite, we can choose the Λ_z's to be disjoint. We describe the strategies s_3^* in terms of the β^σ mapping.

For a type-x nondefector—i.e., a type-x agent with $a \in I(x)$—his stage-3 action, $s_3^*(x, a, \nu^{-a})$, is β^σ, where σ is derived from ν according to (10). For a defector, the strategy is a bit more complicated. A defector is an agent with private history (x, a, ν^{-a}) and $q_1(a) \in I(x')$ for $x \neq x'$. Let σ^* be this agent's belief about the type-configuration of all active agents under φ^* and let σ' be the type-configuration that he believes other agents believe (see (10)). (That is, $\sigma'(x) = \sigma^*(x) - 1$, $\sigma'(x') = \sigma^*(x') + 1$, and $\sigma'(y) = \sigma^*(y)$ for all $y \notin \{x, x'\}$.) Then, $s_3^*(x, a, \nu^{-a})$, our candidate stage-2 strategy, is

$$s_3^*(x, a, \nu^{-a}) = \arg\max_{b \in \mathcal{O}} \sum_{z \in Z} \phi^{\sigma^*}(z) u(\bar{q}' + \frac{b_r Q_- - b_q R_-}{R_- + b_r}, \bar{r}' + \frac{b_q R_- - b_r Q_-}{Q_- + b_q}; x, z), \tag{13}$$

where $\phi^{\sigma^*}(z)$ is derived from σ^* using Bayes rule and where

$$(Q_-, R_-) = M\kappa + \sum_{y \neq x} \sigma^*(y) \beta^{\sigma'}(y) + (\sigma^*(x) - 1)\beta^{\sigma'}(x).$$

(The candidate $s_3^*(x, a, \nu^{-a})$ for a nondefector is a special case of (13) with $\sigma' = \sigma^*$.) This construction implies that s_3^* is optimal against s_3^* and φ^*. Moreover, (6) follows directly from part (ii) of Lemma 3 and (12).

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[9] It does, however, depend on the constraint $b_q b_r = 0$. With it, the best response, which is the mapping studied in order to get a fixed point, is a function; without that constraint, the mapping is not necessarily a convex correspondence. Dubey-Shubik [2] obtain a similar result.

[10] If we drop the every-type-trades assumption, then this result may go through if we allow agents to use any offer $(o_q, o_r) \in [0, \bar{q}] \times [0, \bar{r}]$. While this would avoid issues with differentiability, it would require us to use a version of the implicit function theorem that applies to correspondences.

[11] The only other lower hemicontinuity result for market games seems to be that in Mas-Colell [9], but he uses a very different model. As he says, his version is not a game because each agent's offers are constrained by ex post budget balance, a constraint which depends on the actions of others.

What remains is to show that both s_1^* and s_2^* are optimal given that other agents follow the candidate equilibrium. First we consider the optimality of s_2^*, assuming that all agents participate. An agent at the stage-2 faces a tradeoff. Conditional on being inactive, playing s_2^* is optimal for any N. Conditional on being active, a type-x agent could gain by playing something not in $I(x)$. By doing that, the agent influences the beliefs and, thereby, the stage-3 actions of other active agents. To demonstrate that the trade-off is resolved in favor of the inactive case, we first show that an agent's expected stage-3 payoff conditional on each state z converges to the payoff according to the CE allocation, $(q^{z,\kappa}, r^{z,\kappa})$ independent of his offer made at stage-2. In this argument only those type configurations, σ's, that are close to the limit configurations are relevant as they occur with probability close to one as the population gets large. Hence, the arbitrariness of the selection in the second line of (12) does not matter. As a result, the gain from manipulating stage-3 beliefs is smaller than the loss implied by playing something that is not in $I(x)$—a play which, by construction, is bounded away from $s_2^*(x)$.

Finally, we need to show that $s_1^* = yes$ is optimal. In order to show this, we use both *A1* and *A2*. Conditional on being inactive, any agent's stage-2 payoff is no less than that implied by no-trade, and, by *A1*, is strictly better for some types than consuming the initial endowment for small κ. Conditional on being active, the agent's stage-3 payoff is close to the CE allocation in $\mathcal{L}^z(\kappa)$ conditional on any z. Because every type trades in that CE, that payoff is also strictly better than no-trade, and, by Lemma 0, strictly better than consuming the initial endowment for small κ. As a result, participation is strictly better than no-trade in both contingencies, and the result follows.

5 Almost ex post efficiency

According to Proposition 1, associated with any regular CE allocation (in which every type trades) in $\mathcal{L}^z(\kappa)$ is a sequence of fully-revealing equilibria whose stage-3 behavior converges almost surely to that CE allocation. This suggests that such equilibria are in some approximate sense ex post efficient. However, because the convergence is probabilistic, the standard definition of Pareto efficiency does not apply. Therefore, we adapt the definition of almost ex post efficiency in McLean and Postlewaite [8] and Gul and Postlewaite [6] who have settings like ours. Using that definition, we show that our mechanism achieves the same kind of efficiency as the direct mechanisms constructed in those papers.

We define almost ex post efficiency in terms of a notion of ex post pareto-superiority.

Definition 4. Let $c : X^N \times \Omega^N \times Z \to (\mathbb{R}^2_+)^N$ be an allocation (see Definition 3).

(i) We say that c is *feasible* if

$$\sum_n c_n(\zeta^N, \omega^N, z) \leq N(\bar{q}, \bar{r}) \tag{14}$$

for all $(\zeta^N, \omega^N, z) \in X^N \times \Omega^N \times Z$.

(ii) Let $\{Y(z, \omega^N)\}$ be a collection of subsets of X^N, one subset for each $(z, \omega^N) \in Z \times \Omega^N$.

We say that c' is *ex post ε-pareto-superior to c w.r.t.* the collection $\{Y(z,\omega^N)\}$ if

$$u[c'_n(\zeta^N,\omega^N,z);\zeta_n,z] > u[c_n(\zeta^N,\omega^N,z);\zeta_n,z] + \varepsilon \text{ for each } n \tag{15}$$

for some (z,ω^N) and some $\zeta^N \in Y(z,\omega^N)$.

(iii) We say that a feasible c is *ex post ε-efficient* if there exists a collection $\{Y(z,\omega^N)\}$ with $\mathbb{P}[Y(z,\omega^N)|z,\omega^N] \geq 1-\varepsilon$ for each (z,ω^N) and such that no other feasible allocation c' is ex post ε-pareto-superior to c *w.r.t.* $\{Y(z,\omega^N)\}$.

When $\varepsilon = 0$, the above definition coincides with the usual definition of ex post efficiency.[12] And, except for the presence of ω^N, this definition coincides with the definitions in McLean and Postlewaite [8] and in Gul and Postlewaite [6].

Here, then, is our efficiency result.

Proposition 2. Let $\varepsilon > 0$ be given. There exists $\bar{\kappa}$ and $N(\kappa,\eta)$ such that if (κ,η) satisfies

$$\bar{\kappa} > \kappa \geq 4\eta(\bar{q},\bar{r}) > 0 \tag{16}$$

and $N > N(\kappa,\eta)$, then there exists a fully-revealing equilibrium whose outcome is ex post ε-efficient.

The proof shows that the fully-revealing equilibrium constructed in Proposition 1 is ex post ε-efficient, provided that the parameters of the mechanism satisfy (16). We begin with a regular CE allocation in $\mathcal{L}^z(0)$ in which every type trades, denoted $(q^{z,0},r^{z,0})$. Proposition 1 shows that there is a sequence of fully-revealing equilibria whose outcome conditional on state z approaches the CE allocation $(q^{z,\kappa},r^{z,\kappa})$, which, by Lemma 0, is close to $(q^{z,0},r^{z,0})$ for small κ. The main challenge is to construct the events $\{Y(z,\omega^N)\}$. The construction uses the type configurations that are close to the limit distribution conditional on z and for which the equilibrium stage-3 offers are close to $(q^{z,\kappa},r^{z,\kappa})$. By Proposition 1, that event has arbitrarily high probability. After providing that construction, we complete the argument by using the first fundamental welfare theorem in the following way.

Consider a type-configuration realization, σ^N, belonging to the event $Y(z,\omega^N)$, and consider the allocation c^{s^N} derived from the equilibrium outcome of the constructed fully-revealing equilibrium $s^N = (s_1^N, s_2^N, s_3^N)$. Given the realization (σ^N,ω^N,z), c^{s^N} gives rise to a deterministic allocation. (Notice that because c^{s^N} is symmetric, σ^N is sufficient to determine the consumption bundle for active agents of each type.) For that same type-configuration and for the state z, consider an alternative N-agent economy in which each agent has the utility function $u(q,r,;x,z)$, but the endowment for each active agent is $(\bar{q}+\kappa_q, \bar{r}+\kappa_r)$, while the endowment for each inactive agent is $(0,0)$. Let c'' be the competitive allocation for this alternative economy that is close to $(q^{z,\kappa},r^{z,\kappa})$ for κ small. (By (16), η is small and existence is guaranteed by regularity.) By contradiction, let c' be feasible and be an allocation that is ex post ε-pareto superior to c^{s^N} under the realization

[12]Notice that when $\varepsilon = 0$, (15) requires the allocation $c'_n(\zeta^N,\omega^N,z)$ to be strictly better than $c_n(\zeta^N,\omega^N,z)$ for all n. This is without loss of generality: any allocation that is weakly better for all n and strictly better for some n can be modified to be strictly better off for all n because goods are divisible and utilities are continuous.

(σ^N, ω^N, z). Also, let $c'' = (c''_0, c''_1)$ and $c^{s^N} = (c_0^{s^N}, c_1^{s^N})$, where in each case the subscript 0 is the part of the allocation that pertains to inactive agents and the subscript 1 is the part that pertains to active agents. By definition, $c''_0 \equiv (0,0)$ because inactive agents have zero endowments in the alternative economy.

By our construction of $Y(z, \omega^N)$, both c''_1 and $c_1^{s^N}$ (under realizations (σ^N, ω^N, z) with $\sigma^N \in Y(z, \omega^N)$) approach the same limit, (q^z, r^z), as $(\kappa, N) \to (0, \infty)$. Therefore, by (15) and the fact that c' is ex post ε-pareto-superior to c^{s^N}, for N sufficiently large and κ sufficiently small, c' is pareto superior to c''. (This uses $c''_0 \equiv (0,0)$.) However, for small enough η, c'' uses more resources than c'. This contradicts the first welfare theorem; an economy with fewer resources and the same strictly increasing utility functions cannot have a feasible allocation that is pareto superior to c'', which is the competitive allocation in the alternative economy.

6 Uniqueness of equilibrium

If there are multiple regular competitive equilibria in the ex post limit economy, $\mathcal{L}^z$, then there are also multiple fully-revealing equilibria in our mechanism. Here we show that if the competitive demand for good-q is monotone in $\mathcal{L}^z$ for each $z \in Z$, and if some mild additional conditions hold, then any equilibrium is fully revealing and can be characterized asymptotically by the unique CE.[13] We split the argument into two parts. We first give sufficient conditions for uniqueness within the class of full-participation equilibria. Then, we give additional conditions under which full participation also obtains.

6.1 Uniqueness of full-participation equilibria

There are three such conditions. The first is a stronger assumption about the informativeness of the type realizations.

U1. Let $\mathcal{Y} = \{Y_1, Y_2\}$ be any bipartition of X and let $\mu_z(Y_i) \equiv \sum_{y \in Y_i} \mu_z(y)$. For any $z \neq z'$, $\mu_z(Y_1) \neq \mu_{z'}(Y_1)$.

This implies that for any partition $\mathcal{Y} = \{Y_1, ..., Y_K\}$ of X with $K \geq 2$ and for any $z \neq z'$, there exists some k such that $\mu_z(Y_k) \neq \mu_{z'}(Y_k)$. Although this assumption is stronger than our original informativeness assumption, parameters for which it does not hold are nongeneric.

The second is a modification of the mechanism.

U2. If all agents make the same stage-2 offer, then the game ends at stage-2.

This says that the market shuts down after stage-2 if all agents announce the same offer at stage-2. Under *A1*, this modification rules out any full-participation equilibrium in which

[13] This monotonicity assumption helps simplify our notation and the statement of results, but is not essential for our uniqueness result. As noted below, uniqueness of CE in $\mathcal{L}^z$ would suffice.

all agents make the same stage-2 offer, but does not change any other symmetric equilibrium with full participation. In particular, this modification does not affect the existence of a fully-revealing equilibrium. Finally, because all agents have the same endowments, in the rare event that every agent receives the same type realization in a fully-revealing equilibrium, such shutting down is costless in terms of realized welfare because in that rare event there is no role for trade.

The third is a restriction on off-equilibrium beliefs.

U3. If a single defecting offer is observed at stage-2, then it is believed to come from some set of types $A \subset X$. Moreover, that belief and the equilibrium play of other agents are used via Bayes rule to form a belief over Z and the type configuration of other active agents.

Along the equilibrium path of a symmetric equilibrium in pure strategies, the equilibrium belief associates each equilibrium stage-2 offer a with a set of types and then applies Bayes rule to derive a belief about the type configuration and the state. *U3* requires off-equilibrium beliefs to be derived using the same procedure, but allows there to be an arbitrary set of types, A, to be associated with an arbitrary deviating offer. The assumption that A is common to all nondefectors is convenient, but not crucial. The crucial part of *U3* is that a set of types is assumed for the defector and that Bayes rule is used based on that set. As a result, *U3* excludes off-equilibrium beliefs that allow the defector to signal something about other agents' types or about the state in a way that is not warranted by the defector's private information. This requirement is essentially the requirement for "reasonable" belief systems in Fudenberg and Tirole [4]. Their requirement says that inferences drawn from a defecting action should be limited to the defector's type (that is, no signaling about what you don't know). We need to augment their requirement with the use of Bayes rule because of the presence in our model of a payoff-relevant state-of-the-world. Doing so is reasonable because an agent is trying to update his belief about the type profile and the state-of-the-world, which are exogenous.[14]

Proposition 3. Assume *U1-U3.* Suppose that the competitive demand for good-q is monotone in $\mathcal{L}^z$ for each $z \in Z$. Then, there exists $\bar{N}$ such that for all $N > \bar{N}$, any symmetric PBE in pure strategies $s^N = (s_1^N, s_2^N, s_3^N)$ with $s_1^N = yes$ (full participation) is fully revealing.

We show by contradiction that any equilibrium with full participation is fully revealing for sufficiently large N. First, we use *U2* to eliminate a complete pooling equilibrium—one in which $s_2(x) = s_2(y)$ for all $x, y \in X$. Next, we consider a semi-pooling equilibrium—one in which there is a partition $\mathcal{Y} = \{Y_1, ..., Y_K\}$ of X with $|X| > K \geq 2$ such that $s_2(y) = s_2(y')$ if $y, y' \in Y_k$ and $s_2(y) \neq s_2(y')$ if $y \in Y_k$ and $y' \in Y_{k'}$ with $k \neq k'$. The main body of the proof shows that, even under a semi-pooling equilibrium, the stage-3 outcome converges to the unique competitive allocation. Of course, this convergence will

[14]A less restrictive extension would allow the off-equilibrium belief to associate a deviating offer with a *distribution* of types and would then employ Bayes rule to pin down the belief about the state and the type configuration of other agents. However, it is rather complicated to formulate the use of Bayes rule under this assumption and doing so does not seem to affect our main results.

fail without uniqueness of the CE. We also make use of *U1* and *U3*: *U1* is used to deal with the asymmetric information that exists in a semi-pooling equilibrium, while *U3* is used to restrict off-equilibrium beliefs. Then we eliminate any semi-pooling equilibrium by an argument that resembles the main idea in the proof of Proposition 1: because a defection by one agent has a vanishing effect on the beliefs of other agents, an agent is induced to defect from a semi-pooling equilibrium and play the stage-2 strategy that is best contingent on becoming inactive.

6.2 Uniqueness of full participation

We show that if we amend the participation stage and if we impose a further refinement, then under U1-U3 all equilibria have full participation. Here is the modification of the participation stage.

U4. After the mechanism receives all the participation decisions, it shuts down and refunds all the entry fees if some agents refuse to participate; otherwise; it proceeds to the next stage.

As with modification U2, U4 does not affect any fully-revealing equilibrium or any of our earlier results.

Now we turn to the refinement. In the modified mechanism under U4, a symmetric pure strategy may still be rewritten as $s = (s_1, s_2, s_3)$, but now (s_2, s_3) only specifies what happens in the two-stage market-game mechanism when all agents participate (and when full participation becomes common knowledge among them). Thus, for any candidate equilibrium (s_1, s_2, s_3), we may regard stage-1 as an N-agent simultaneous game where the action set for each agent n is $\{yes, no\}$ and the payoff under a profile in which some agent says *no* is $(\overline{q}, \overline{r})$ for everyone and can be computed from (s_2, s_3) otherwise.[15] We call this game the stage-1 game induced by s. Our refinement is the following.

U5. The equilibrium s has the property that s_1 does not use a weakly dominated action in the stage-1 game induced by s.

Proposition 4. Assume *U1-U5* and suppose that the competitive demand for good-q is monotone in $\mathcal{L}^z$ for each $z \in Z$. Then, there exists $\bar{N}$ such that for all $N > \bar{N}$, any symmetric PBE in pure strategies is fully revealing.

The logic of the proof is simple. An agent at stage-1 thinks about two alternatives. Either everyone else participates or not. If not, then the agent is indifferent between $\{yes, no\}$ because both leave the agent with the endowment $(\overline{q}, \overline{r})$. If everyone else participates, then, according to the refinement, the agent computes an expected payoff from participating from the unique continuation equilibrium of the previous subsection, a payoff that exceeds that of the endowment, the payoff from playing *no*. Therefore, playing *no* is a weakly dominated action in the sense of U5.

[15] Here we assume that after observing that the mechanism continues after stage-1 (which implies that all agents participate), agents do not update their beliefs about the fundamentals (the state and other agents' types). Because agents observe nothing when making their participation decisions, this assumption is consistent with U3.

Finally, under U1-U5, we can also show that all equilibria are ex post ε-efficient for large N. This result does not directly follow from the statement of Proposition 2 because it requires that the allocations for all equilibria converge to the corresponding limiting CE uniformly. However, the proof of Proposition 2 does establish such uniform convergence. Therefore, the result follows.

7 Concluding remarks

Given the finite population, the threat of being inactive plays a crucial role. Without it, there would be no penalty attached to stage-2 actions that are devoted entirely to manipulating the beliefs of others and such manipulation could be desirable for any finite N. Therefore, we strongly suspect that a fully revealing equilibrium does not exist if $\eta = 0$. In this respect, there is a significant distinction between the model with a finite number of agents and the same model with a continuum of agents. In the continuum version as usually formulated, one agent cannot manipulate the beliefs of others and a fully revealing equilibrium exists even if $\eta = 0$.

We have assumed that agents make their participation decisions before their types are realized. Alternatively, we could assume that agents make that decision after they know their types. All our results would go through without any significant change. In particular, *A2* implies that, for small κ's, all types have positive gains from trade in the CE allocation in $\mathcal{L}^z(\kappa)$ for all z and the expected payoffs from stage-3 trade converges to that allocation conditional on z. Thus, the arguments for willingness to participate go through without any essential changes.

One special assumption that we have not commented on is the assumption that all agents have the same endowment. If not, then we would need a way of describing endowment patterns as we vary the number of agents. One way would be to have a finite number of possible endowment types and to have deterministic replication over that finite-type profile with preference types drawn randomly as we do. Provided that endowments are positive, we could preserve the Lemma-1 feasibility result. We could also handle known preference heterogeneity in the same way. The one place where known heterogeneity would play a role is in our uniqueness argument. If agents' endowments are heterogeneous (or there is another source of known heterogeneity), then assumption U2 would need to be modified. We would require that stage-3 be shut down whenever all agents with the same endowment make the same offer at stage-2. That would weaken our detail-freeness claim.

Finally, two special cases of the information structure deserve comment. All our results hold for the pure-private value case in which the state z does not affect preferences. In that case, the state remains a source of aggregate risk because it determines the proportions of agents' types. Therefore, stage-2 remains useful because information aggregation is important for ex post efficiency. In contrast, the pure common-value case in which types are purely signals and do not affect preferences makes the model irrelevant for two reasons related to the *no-trade theorem*. First, our existence result requires that every type trades at the limit economy with $\kappa = 0$, an assumption that depends on the appearance of types in preferences. Second, if types do not appear in preferences, then trade disappears at stage-3 as $\kappa \to 0$ and agents will not want to enter in the presence of an entry fee.

8 Appendix: Proofs

The proofs are set out by section of the paper.

8.1 The mechanism

Lemma 1. If $(2\kappa_q, 2\kappa_r) \in (0, \bar{q}) \times (0, \bar{r})$ and if $\eta \leq \min\{\kappa_q/2\bar{q}, \kappa_r/2\bar{r}\}$, then entry fees of the participants cover the stage-1 and stage-2 costs of operating the mechanism.

Proof. The mechanism has to supply at most $N'(\kappa_q, \kappa_r)$ units of the two goods for the stage-2 trades. For stage-2 trades, there are two extreme cases: first, all agents offer all of their endowed r good at the first stage, in which case the mechanism has to supply at most $\eta N'\overline{q}$ units of the q good; second, all agents offer all of their endowed q goods in the first stage, in which case the mechanism has to supply at most $\eta N\overline{r}$ units of the r good. Thus, the total supply from the mechanism for the q good is less than (remember that we have $\eta \leq \kappa_q/2\overline{q}$)

$$N'\kappa_q + \eta N'\overline{q} \leq 1.5N'\kappa_q < 2N'\kappa_q.$$

A symmetric argument shows that the the total supply from the mechanism for the r good is less than $2N'\kappa_r$. This guarantees feasibility.■

Lemma 2. Fix stage-3 offers of all other agents. Given those offers, for any offer $b' \in [0, \bar{q} - 2\kappa_q] \times [0, \bar{r} - 2\kappa_r]$, there exists $b'' \in \mathcal{O}$ that has the same payoff as b'.

Proof. Let $(Q_-, R_-) \in \mathbb{R}^2_{++}$ be total offers of other agents (including the exogenous offers). For any $b \in [0, \bar{q} - 2\kappa_q] \times [0, \bar{r} - 2\kappa_r]$, (4) implies that the corresponding payoffs are

$$q(b_q, b_r) = \bar{q} - 2\kappa_q + \frac{b_r Q_- - b_q R_-}{R_- + b_r} \text{ and } r(b_q, b_r) = \bar{r} - 2\kappa_r + \frac{b_q R_- - b_r Q_-}{Q_- + b_q}. \tag{17}$$

Case (i): $b'_r Q_- - b'_q R_- > 0$. In this case, let $b''_q = 0$ and let b''_r be the unique solution to

$$\frac{b''_r Q_-}{R_- + b''_r} = \frac{b'_r Q_- - b'_q R_-}{R_- + b'_r} \equiv \gamma. \tag{18}$$

It follows that $\gamma \in (0, Q_-)$ and that the solution is $b''_r = R_-\gamma/(Q_- - \gamma)$. Therefore, by (18), $q(b''_q, b''_r) = q(b'_q, b'_r)$. Also,

$$r(b''_q, b''_r) - \bar{r} = b''_r = R_-\gamma/(Q_- - \gamma) = r(b'_q, b'_r) - \bar{r},$$

where the last equality follows from the definition of γ.

Case (ii): $b'_r Q_- - b'_q R_- < 0$. This is completely analogous, but with $b''_r = 0$.

Case (iii): $b'_r Q_- - b'_q R_- = 0$. Here, of course, we let $b''_q = b''_r = 0$.■

8.2 Existence of fully-revealing equilibrium

Lemma 3. (i) The set of symmetric equilibria for the game $\mathcal{E}(1/M, \sigma/M, \phi)$ is not empty. (ii) Fix $z \in Z$ and a regular CE allocation $(q^{z,\kappa}, r^{z,\kappa})$ for $\mathcal{L}^z(\kappa)$ in which every type trades. Let $\Lambda = [0,1] \times \Delta(X) \times \Delta(Z)$ and let $\lambda_0^z = (0, \mu_z, \delta_z)$. There exists an open neighborhood of λ_0^z, denoted Λ_z, with $\Lambda_z \subset \Lambda$, and a continuous function $f_z : \Lambda_z \to \mathcal{O}^X$ such that $f_z(\lambda_0^z) = \beta^{z,\kappa}$ (see (11)) and such that $(1/M, \sigma/M, \phi^\sigma) \in \Lambda_z$ implies that $f_z(1/M, \sigma/M, \phi^\sigma)$ is a symmetric equilibrium in $\mathcal{E}(M, \sigma/M, \phi^\sigma)$.

Proof. (i) We apply Brouwer's fixed point theorem. Let $\bar{q}' = \bar{q} - 2\kappa_q$ and let $\bar{r}' = \bar{r} - 2\kappa_r$, and let $S = \{[0, \bar{q}'] \times [0, \bar{r}']\}^X$, which is compact and convex. We let $s = \{s^y\}_{y \in X}$ with $s^y = (s_q^y, s_r^y)$ denote a generic element of S. For $s \in S$ and $x \in X$, let $F : S \to S$ be given by

$$F_x(s) = \arg\max_{b \in \mathcal{O}} H_x(b; Q_-, R_-), \tag{19}$$

where

$$H_x(b; Q_-, R_-) = \sum_{z \in Z} \phi(z) u(\bar{q}' + \frac{b_r Q_- - b_q R_-}{R_- + b_r}, \bar{r}' + \frac{b_q R_- - b_r Q_-}{Q_- + b_q}; x, z) \tag{20}$$

and

$$(Q_-, R_-) = M\kappa + \sum_{y \neq x} \sigma(y) s^y + [\sigma(x) - 1] s^x.$$

Here ϕ is the common posterior on z. By the definition of the game $\mathcal{E}(M, \sigma/M, \phi, \kappa)$, a fixed point of F is an equilibrium. (Although the domain of the mapping, S, does not satisfy $b_q b_r = 0$, the range does. Therefore, any fixed point satisfies $b_q b_r = 0$.)

It remains to show that $F_x(s)$ is unique and is continuous in s. We start with uniqueness. Notice that $(Q_-, R_-) \in \mathbb{R}^2_{++}$ for any $s \in S$.

Because of the $b_q b_r = 0$ constraint in (19), it is helpful to consider $H_x(b_q, 0; Q_-, R_-)$ and $H_x(0, b_r; Q_-, R_-)$ separately, where

$$H_x(b_q, 0; Q_-, R_-) = \sum_{z \in Z} \phi(z) u(\bar{q}' - b_q, \bar{r}' + \frac{b_q R_-}{Q_- + b_q}; x, z) \equiv g(b_q),$$

and

$$H_x(0, b_r; Q_-, R_-) = \sum_{z \in Z} \phi(z) u(\bar{q}' + \frac{b_r Q_-}{R_- + b_r}, \bar{r}' - b_r; x, z) \equiv h(b_r).$$

For any $(Q_-, R_-) \in \mathbb{R}^2_{++}$, the functions $f_q(b_q) = \bar{r}' + \frac{b_q R_-}{Q_- + b_q}$ and $f_r(b_r) = \bar{q}' + \frac{b_r Q_-}{R_- + b_r}$ are strictly concave. Then, because u is strictly concave and because a strictly increasing concave function of a concave function is strictly concave, both g and h are strictly concave. It follows that g has a unique maximum and that h has a unique maximum, denoted $\hat{b}_q$ and $\hat{b}_r$, respectively. Moreover, by the Inada conditions on u, these maxima are characterized by

$$\hat{b}_q = \begin{cases} 0 \text{ if } g'(0) \leq 0 \\ \text{satisfies } g'(\hat{b}_q) = 0 \text{ if } g'(0) > 0 \end{cases}, \tag{21}$$

and

$$\hat{b}_r = \begin{cases} 0 \text{ if } h'(0) \leq 0 \\ \text{satisfies } h'(\hat{b}_r) = 0 \text{ if } h'(0) > 0 \end{cases}. \quad (22)$$

Therefore, a sufficient condition for uniqueness is $\min\{g'(0), h'(0)\} \leq 0$, where

$$g'(0) = \sum_{z \in Z} \phi(z) \left[-u_q\left(\bar{q}'; x, z\right) + u_r\left(\bar{r}'; x, z\right) \frac{R_-}{Q_-} \right],$$

and

$$h'(0) = \sum_{z \in Z} \phi(z) \left[u_q\left(\bar{q}'; x, z\right) \frac{Q_-}{R_-} - u_r\left(\bar{r}'; x, z\right) \right].$$

Note that

$$sign[h'(0)] = sign[\frac{R_-}{Q_-} h'(0)] = sign[-g'(0)] = -sign[g'(0)], \quad (23)$$

which implies $\min\{g'(0), h'(0)\} \leq 0$.

Now we turn to continuity in s, which follows if $(\hat{b}_q, \hat{b}_r)$ is continuous in (Q_-, R_-). By (23), $g'(0) = 0$ *iff* $h'(0) = 0$. That and (21) and (22) imply that $\max\{\hat{b}_q, \hat{b}_r\}$ satisfies a first-order condition with equality. Then, the implicit-function theorem applied to that first-order condition gives the required continuity.

(ii) First, we give a claim that is used to evaluate a determinant that appears when we verify a full-rank condition.

Claim 1. Let $\mathbf{a}_n = (a_1, ..., a_n) \in \mathbb{R}^n$, $\mathbf{k}_n = (k_1, ..., k_n) \in \mathbb{R}^n$, and $\mathbf{P}_n = \mathbf{a}_n' \mathbf{k}_n - \mathbf{I}_n$ (where $\mathbf{I}_n$ is the $n \times n$ identity matrix). Then $|\mathbf{P}_n| = (-1)^{n+1}(\sum_{i=1}^n a_i k_i - 1)$.

Proof. The proof is by induction on n. The claim holds for $n = 1$. Now, suppose it holds for n. By definition, $\mathbf{P}_n = [k_1 \mathbf{a}_n', k_2 \mathbf{a}_n', ..., k_n \mathbf{a}_n'] - \mathbf{I}_n$. If $k_{n+1} = 0$, then $|\mathbf{P}_{n+1}| = -1 \, |\mathbf{P}_n|$. Thus, we can assume that $\Pi_{i=1}^{n+1} k_i \neq 0$. Then,

$$|\mathbf{P}_{n+1}| = \left| [k_1 \mathbf{a}_{n+1}', k_2 \mathbf{a}_{n+1}', ..., k_n \mathbf{a}_{n+1}', k_{n+1} \mathbf{a}_{n+1}'] - \mathbf{I}_n \right| = (\Pi_{i=1}^{n+1} k_i) \left| \begin{pmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{c} & -\frac{1}{k_{n+1}} \end{pmatrix} \right|,$$

where

$$\mathbf{A} = [\mathbf{a}_n', ..., \mathbf{a}_n'] - \text{diag}(\frac{1}{k_1}, ..., \frac{1}{k_n}), \mathbf{b}' = (1/k_1, 0, ..., 0), \text{ and } \mathbf{c} = (a_{n+1}, ..., a_{n+1}).$$

Then,

$$|\mathbf{P}_{n+1}| = (\Pi_{i=1}^{n+1} k_i)[-(1/k_{n+1})|\mathbf{A}| + (-1)^n (1/k_1)|\mathbf{B}|],$$

where

$$\mathbf{B} = \begin{pmatrix} a_2 & a_2 - 1/k_2 & \ldots & a_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_n & \ldots & a_n - 1/k_n \\ a_{n+1} & a_{n+1} & \ldots & a_{n+1} \end{pmatrix}$$

and

$$|\mathbf{B}| = \left| \begin{pmatrix} a_2 & -1/k_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_n & 0 & \dots & -1/k_n \\ a_{n+1} & 0 & \dots & 0 \end{pmatrix} \right| = (-1)^{n-1} a_{n+1} \left(\Pi_{i=2}^{n+1} (-1/k_i) \right) = a_{n+1} \frac{1}{\Pi_{i=2}^{n+1} k_i}.$$

By the induction hypothesis,

$$|\mathbf{A}| = (-1)^{n+1} \frac{1}{\Pi_{i=1}^{n} k_i} (\sum_{i=1}^{n} k_i a_i - 1).$$

Therefore,

$$|\mathbf{P}_{n+1}| = (-1)^{n+2} (\sum_{i=1}^{n} k_i a_i - 1) + (-1)^n a_{n+1} k_{n+1} = (-1)^{n+2} (\sum_{i=1}^{n+1} k_i a_i - 1),$$

as required.□

Now we prove (ii). Let $(q^{z,\kappa}, r^{z,\kappa})$ be the given regular CE where every type trades in $\mathcal{L}^z(\kappa)$ and let $p^{z,\kappa}$ be the corresponding price, and let $\beta^{z,\kappa}$ be the corresponding offers given by (11). Hence, $\beta^{z,\kappa}(x) \neq (0,0)$ for all $x \in X$. Let $X_1 = \{x \in X : \beta_q^{z,\kappa}(x) > 0\}$ and let $X_2 = \{x \in X : \beta_r^{z,\kappa}(x) > 0\}$. Then, $X_1 \cap X_2 = \emptyset$ and $X = X_1 \cup X_2$.

The first step is to define the function to which we apply the Implicit Function Theorem. Let $\bar{q}' = \bar{q} - 2\kappa_q$ and let $\bar{r}' = \bar{r} - 2\kappa_r$. We use β to denote $\langle (\beta_q(x))_{x \in X_1}, (\beta_r(x))_{x \in X_2} \rangle$, the offers, and let $\lambda = (\epsilon, \mu, \phi) \in D \times \Delta(Z)$ be the parameter vector, where

$$D = \{(\epsilon, \mu) \in [-1, 1] \times \Delta(X) : \mu(x) > \epsilon^2 \text{ for all } x \in X\}.$$

(In what follows, $M^{-1/2}$ is one possible magnitude of ϵ.) Let

$$H_x(b_q, b_r; \beta, a) = \sum_{z \in Z} \phi(z) u\left(q, r; x, z\right), \tag{24}$$

where

$$q = \bar{q}' + \frac{b_r q_- - b_q r_-}{r_- + \epsilon^2 b_r} \text{ and } r = \bar{r}' + \frac{b_q r_- - b_r q_-}{q_- + \epsilon^2 b_q},$$

and

$$(q_-, r_-) = \begin{cases} (\sum_{x' \in X_1} \mu(x') \beta_q(x') + \kappa_q - \epsilon^2 \beta_q(x), \sum_{x' \in X_2} \mu(x') \beta_r(x') + \kappa_r) \text{ if } x \in X_1 \\ \\ (\sum_{x' \in X_1} \mu(x') \beta_q(x') + \kappa_q, \sum_{x' \in X_2} \mu(x') \beta_r(x') + \kappa_r - \epsilon^2 \beta_r(x)) \text{ if } x \in X_2 \end{cases}.$$

When $\epsilon^2 = M^{-1}$, H_x is the stage-2 objective function of an agent expressed in terms of average offers of others, (q_-, r_-). Also, because $(\epsilon, \mu) \in D$, $(q_-, r_-) > 0$.

Now, let

$$G_x(\beta, a) = \begin{cases} \arg\max_{b_q \geq 0} H_x(b_q, 0; \beta, a) \text{ if } x \in X_1 \\ \\ \arg\max_{b_r \geq 0} H_x(0, b_r; \beta, a) \text{ if } x \in X_2 \end{cases}.$$

Because each branch of H_x is strictly concave and, hence, has a unique maximum, G_x is a well-defined function. Moreover, by the Theorem of the Maximum, G_x is continuous in all its arguments. Letting $G = (G_x)_{x\in X}$, the function to which we apply the implicit function theorem is $G(\beta, a) - \beta$, whose range is in $\mathbb{R}^X$.

Before doing that, there are several facts about G that we will use. First, let $\lambda_0 = (0, \mu_z, \delta_z)$ and let

$$p(\beta) = \frac{\sum_{y\in X_2} \mu_z(y)\beta_r(y) + \kappa_r}{\sum_{x\in X_1} \mu_z(x)\beta_q(x) + \kappa_q}. \tag{25}$$

Then, for all $x \in X_1$,

$$G_x(\beta, \lambda_0) = \max\{\bar{q}' - q_x^z(p(\beta)), 0\}, \tag{26}$$

and for all $y \in X_2$,

$$G_y(\beta, \lambda_0) = \max\{p(\beta)[q_y^z(p(\beta)) - \bar{q}'], 0\}, \tag{27}$$

where $q_x^z(p)$ is the demand function of good q for type-x under known state z. Because $\beta^{z,\kappa}$ is a CE in $\mathcal{L}^z(\kappa)$, it follows that $G(\beta^{z,\kappa}, \lambda_0) = \beta^{z,\kappa}$. Notice that the first two components of λ_0 lie in the interior of D because $\mu_z(x) > 0$ for all $x \in X$ and for all $z \in Z$. Second, there is an open neighborhood of $(\beta^{z,\kappa}, \lambda_0)$ such that G is continuously differentiable in that neighborhood and has a positive offer of q for all $x \in X_1$ and has a positive offer of r for $x \in X_2$. The proof of this claim follows from the fact that each branch of H_x is strictly concave and continuously differentiable near $(\beta^{z,\kappa}, \lambda_0)$, so that the Implicit Function Theorem can be applied to the first-order conditions that characterize G in that neighborhood. This is where the assumption that all types trade in $\mathcal{L}^z$ is used. Third, if $\sigma : X \to \{0, ..., M\}$ is a type-configuration of M agents and if $\beta \in \mathcal{O}^X$ satisfies $G(\beta, 1/\sqrt{M}, \sigma/M, \phi) = \beta$ with $\beta_q(x) > 0$ for all $x \in X_1$ and $\beta_r(x) > 0$ for all $x \in X_2$, then β is a symmetric equilibrium in $\mathcal{E}(M, \sigma/M, \phi^\sigma, \kappa)$. This follows because H_x defined by (24) is the same as that defined by (20) in the proof of Lemma 2 if $\epsilon^2 = 1/M$ and $\mu = \sigma/M$ and because the strict concavity of each branch of H_x implies that the sign restriction in each branch in (24) is not binding if the maximum is attained at positive offers.

The last preliminary step is to set out the partial derivatives of G *w.r.t.* to β evaluated at $(\beta, \lambda) = (\beta^{z,\kappa}, \lambda_0)$. Notice that, by (11), $p(\beta^{z,\kappa}) = p^{z,\kappa}$. Using equations (25)-(26), we compute the derivatives according to the chain rule and obtain

$$\frac{\partial G_{x'}(\beta^{z,\kappa}, \lambda_0)}{\partial \beta_q(x)} = \begin{cases} -\frac{\partial}{\partial \beta_q(x)} p(\beta^{z,\kappa}) \frac{d}{dp} q_{x'}^z(p^{z,\kappa}) \text{ for } (x, x') \in X_1 \times X_1 \\ \frac{\partial}{\partial \beta_q(x)} p(\beta^{z,\kappa}) \left[p^{z,\kappa} \frac{d}{dp} q_{x'}^z(p^{z,\kappa}) + (q_{x'}^z(p^{z,\kappa}) - \bar{q}') \right] \text{ for } (x, x') \in X_1 \times X_2 \end{cases}$$

and

$$\frac{\partial G_{x'}(\beta^*, \lambda_0)}{\partial \beta_r(x)} = \begin{cases} \frac{\partial}{\partial \beta_r(x)} p(\beta^{z,\kappa}) \left[p^{z,\kappa} \frac{d}{dp} q_{x'}^z(p^{z,\kappa}) + (q_{x'}^z(p^*) - \bar{q}') \right] \text{ for } (x, x') \in X_2 \times X_2 \\ -\frac{\partial}{\partial \beta_q(x)} p(\beta^{z,\kappa}) \frac{d}{dp} q_{x'}^z(p^{z,\kappa}) \text{ for } (x, x') \in X_2 \times X_1 \end{cases}.$$

Now, let $|X| = L$. Then $\left[\frac{\partial G(\beta^{z,\kappa},\lambda_0)}{\partial \beta(x)}\right]_{x\in X} = \mathbf{a}'_L \mathbf{k}_L$ with

$$\mathbf{a}_L = [(\frac{d}{dp} q^z_x(p^{z,\kappa}))_{x\in X_1}, (\frac{d}{dp} q^z_{x'}(p^{z,\kappa}) + (q^z_{x'}(p^{z,\kappa}) - \bar{q}'))_{x'\in X_2}]$$

and

$$\mathbf{k}_L = [(-\frac{\partial}{\partial \beta_q(x)} p(\beta^{z,\kappa}))_{x\in X_1}, (\frac{\partial}{\partial \beta_r(x')} p(\beta^{z,\kappa}))_{x'\in X_2}].$$

Thus, by Claim 1, $\left|\left[\frac{\partial G(\beta^{z,\kappa},\lambda_0)}{\partial \beta(x)}\right]_{x\in X} - \mathbf{I}_L\right| = (-1)^{L+1} C$, where

$$\begin{aligned}
C &= \sum_{x\in X_1} (-\frac{\partial p(\beta^{z,\kappa})}{\partial \beta_q(x)}) \frac{dq^z_x(p^{z,\kappa})}{dp} + \sum_{y\in X_2} \frac{\partial p(\beta^{z,\kappa})}{\partial \beta_r(y)} \left[p^{z,\kappa} \frac{d}{dp} q^z_x(p^{z,\kappa}) + (q^z_y(p^{z,\kappa}) - \bar{q}')\right] - 1 \\
&= \frac{p^{z,\kappa}}{\sum_{x\in X_1} \mu_z(x)\beta^{z,\kappa}_q(x) + \kappa_q} \sum_{x\in X_1} \mu_z(x) \frac{dq^z_x(p^{z,\kappa})}{dp} + \frac{(p^{z,\kappa})^2}{\sum_{y\in X_2} \mu_z(y)\beta^{z,\kappa}_r(y) + \kappa_r} \sum_{y\in X_2} \mu_z(y) \frac{dq^z_y(p^{z,\kappa})}{dp} \\
&+ \frac{p^{z,\kappa}}{\sum_{y\in X_2} \mu_z(y)\beta_r(y) + \kappa_r} \sum_{y\in X_2} \mu_z(y)[q^z_y(p^{z,\kappa}) - \bar{q}'] - 1 \\
&= \left[\frac{(p^{z,\kappa})^2}{\sum_{y\in X_2} \mu_z(y)\beta^{z,\kappa}_r(y) + \kappa_r} \sum_{x\in X} \mu_z(x) \frac{dq^z_x(p^{z,\kappa})}{dp}\right] + \left[\frac{\sum_{y\in X_2} \mu_z(y)\beta_r(y)}{\sum_{y\in X_2} \mu_z(y)\beta_r(y) + \kappa_r} - 1\right] \\
&= \frac{(p^{z,\kappa})^2}{\sum_{y\in X_2} \mu_z(y)\beta^{z\kappa}_r(y) + \kappa_r} \left[\sum_{x\in X} \mu_z(x) \frac{dq^z_x(p^{z,\kappa})}{dp} + \frac{-\kappa_r}{(p^{z,\kappa})^2}\right].
\end{aligned}$$

The last expression differs from zero because the CE corresponding to $\beta^{z,\kappa}$ is regular.

Therefore, by the implicit function theorem, there is a neighborhood $\Psi_z \subset D \times \Delta(Z)$ around λ_0 and a continuously differentiable function $g_z : \Psi_z \to ([0,\bar{q}']^{X_1} \times [0,\bar{r}']^{X_2})$ such that $g_z(\lambda_0) = \beta^{z,\kappa}$, and $G(g_z(\lambda),\lambda) = g_z(\lambda)$ for all $\lambda \in \Psi_z$. Because $\beta^{z,\kappa}_q(x) > 0$ for all $x \in X_1$ and $\beta^{z,\kappa}_r(y) > 0$ for all $y \in X_2$ and because g_z is continuous, there exists an open neighborhood $\Lambda_z \subset \Psi_z$ containing λ_0 such that for all $\lambda \in \Lambda_z$, $g_z(\lambda)$ is strictly positive in all its coordinates. Thus, if $(1/\sqrt{M}, \sigma/M, \phi^\sigma) \in \Lambda_z$ for a type-configuration $\sigma : \{0,...,M\} \to \mathcal{O}$, then $g_z(1/\sqrt{M}, \sigma/M, \phi^\sigma)$ is a symmetric equilibrium for $\mathcal{E}(M, \sigma/M, \phi^\sigma)$. Finally, let $f_z(\epsilon,\mu,\phi) = g_z(\sqrt{\epsilon},\mu,\phi)$.■

Proposition 1. Fix an ex post regular CE profile $\{(q^{z,\kappa}, r^{z,\kappa})\}_{z\in Z}$ in which every type trades. Then there exists a number $\bar{N}$ and a sequence of fully-revealing equilibria $\{s^N = (s^N_1, s^N_2, s^N_3)\}^\infty_{N=\bar{N}}$ such that

$$\lim_{N\to\infty} c^{s^N}_{1,x} = (q^{z,\kappa}_x, r^{z,\kappa}_x) \text{ for all } x \in X \tag{28}$$

almost surely conditional on z for each $z \in Z$.

Proof. We show that for large N's, $((s^*_0, s^*_1, s^*_2), \varphi^*)$ is a PBE, where s^*_2 is given by (12)-(13), and φ^* is given by (9)-(10). Notice that both s^*_2 and φ^* depend on N but not on s^*_1. By construction and Lemma 3, s^*_2 is a best response against s^*_2 w.r.t. φ^* and φ^*

is consistent with Bayes' rule. It remains to show that s_1^* and s_2^* is a best response to (s_1^*, s_2^*, s_3^*) for sufficiently large N.

We begin with optimality of s_2^*. Let $\bar{q}' = \bar{q} - 2\kappa_q$ and let $\bar{r}' = \bar{r} - 2\kappa_r$. Given s_1^*, $M^N = \lceil (1-\eta)N \rceil$ is the number of active agents and consider an agent of type x. Because the assignment into active/inactive categories is drawn independently from the types, conditional on being active, the agent's belief about other agents' types is such that those types are *i.i.d.* with marginal probabilities $(\mu_z(x))_{x\in X}$ conditional on each state z. Let γ_z^N be the *i.i.d.* distribution over X^{M^N-1} generated by $(\mu_z(x))_{x\in X}$. Given s_3^*, the stage-2 problem for the agent of type x is $\max_{a\in\mathcal{O}} G_x^N(a)$, where

$$G_x^N(a) = \eta G_x(a) + (1-\eta) F_x^N(a). \tag{29}$$

Here, $F_x^N(a)$ is the expected payoff contingent on being active and playing offer a at stage-2, and $G_x(a)$ is the stage-2 payoff of playing a contingent on being inactive; namely,

$$G_x(a) = \sum_{z\in Z} \tau_x(z) u(\bar{q}' - a_q + a_r/\overline{p}, \bar{r}' - a_r + \overline{p} a_q; x, z), \tag{30}$$

where $\overline{p} = \overline{r}/\overline{q}$.

Claim 1. Let $q_1(a;x)$ be the consumption of q of a type-x agent who plays a and becomes inactive. There exists $\epsilon > 0$ such that if $q_1(a;x) \notin I(x)$, then $G_x(a) < G_x(s_1^*(x)) - \epsilon$.

Proof. It is easy to verify that $\max_{a\in\mathcal{O}} G_x(a)$ is equivalent to $\max_{q\in[0,2\bar{q}]} L_x(q)$, where

$$L_x(q) = \sum_{z\in Z} \tau_x(z) u(q, \overline{p}\bar{q}' + \bar{r}' - \overline{p} q; x, z).$$

Let $2\delta_x = \min_{y\in X,\ y\neq x} |q(x,\kappa) - q(y,\kappa)|$. Then, $q \notin I(x)$ implies $|q - q(x,\kappa)| \geq \delta_x$. Because $L_x(q)$ is strictly concave in q and has a maximum at $q(x,\kappa)$, it follows that $A_x = \min\{-L_x'(q(x,\kappa) + \frac{\delta_x}{2}), L_x'(q(x,\kappa) - \frac{\delta_x}{2})\} > 0$. Then, for any q such that $|q - q(x,\kappa)| \geq \frac{\delta_x}{2}$, $L_x(q) \leq L_x(q(x,\kappa)) - \frac{\delta_x}{2} A_x$. Take $\epsilon_x = (\delta_x/4)A_x$. Then, $q_1(a;x) \notin I(x)$ implies $G_x(a) = L_x(q_1(a;x)) \leq L_x(q(x,\kappa)) - 2\epsilon_x < G_x(s_1^*(x)) - \epsilon_x$. Finally, let $\epsilon = \min\{\epsilon_x\}_{x\in X}$.□

Claim 2. For all $a \in \mathcal{O}$,

$$\lim_{N\to\infty} F_x^N(a) = \sum_{z\in Z} \tau_x(z) u(q_x^{z,\kappa}, r_x^{z,\kappa}; x, z), \tag{31}$$

uniformly over $\mathcal{O}$. (Recall that $(q_x^{z,\kappa}, r_x^{z,\kappa})_{x\in X}$ is the regular CE allocation of $\mathcal{L}^z(\kappa)$ that was used to construct the strategy profiles β^σ in (12)).

Proof. First we give an explicit expression for $F_x^N(a)$. For each $a \in I(\bar{x})$,

$$F_x^N(a) = \sum_{z\in Z} \tau_x(z) \left[\sum_{\xi\in X^{M^N-1}} \gamma_z^N(\xi) \left[u(q^N(a;z,\xi), r^N(a;z,\xi); x, z) \right] \right], \tag{32}$$

where $\xi = (\xi_1, ..., \xi_{M^N-1}) \in X^{M^N-1}$ is the vector of types of the other active agents. Here,

$$q^N(a;z,\xi) = \bar{q}' + \frac{s_{3,r}^*(x,a,\nu^{\xi,-a}) Q_-^N - s_{3,q}^*(x,a,\nu^{\xi,-a}) R_-^N}{s_{3,r}^*(x,a,\nu^{\xi,-a}) + R_-^N},$$

and

$$r^N(a;z,\xi) = \bar{r}' + \frac{s^*_{3,q}(x,a,\nu^{\xi,-a})R^N_- - s^*_{3,r}(x,a,\nu^{\xi,-a})Q^N_-}{s^*_{3,q}(x,a,\nu^{\xi,-a}) + Q^N_-},$$

where $\nu^{\xi,-a}$ is the announced histogram given that other active agents' types are ξ and that other agents follow s^*_2. Also, Q^N_- and R^N_- are the implied stage-3 offers of other active agents according to the candidate equilibrium. That is, letting $\mathbf{1}_y(y) = 1$ and $\mathbf{1}_y(x) = 0$ if $x \neq y$,

$$\nu^{\xi,-a}(s^*_2(y)) = \sum_{i=1}^{M^N-1} \mathbf{1}_y(\xi_i) \text{ for each } y \in X \text{ and } \nu^{\xi,-a}(a') = 0 \text{ otherwise,} \tag{33}$$

and

$$(Q^N_-, R^N_-) = \sum_{y\in X} \sigma^\xi(y)(\beta^{\sigma^\xi}(y) + \kappa) - \beta^{\sigma^\xi}(\bar{x}), \tag{34}$$

where σ^ξ is the type-configuration believed by other active agents. Recall that for $a \in I(\bar{x}))$,

$$\sigma^\xi(y) = \sum_{i=1}^{M^N-1} \mathbf{1}_y(\xi_i) \text{ for each } y \neq \bar{x} \text{ and } \sigma^\xi(\bar{x}) = \sum_{i=1}^{M^N-1} \mathbf{1}_{\bar{x}}(\xi_i) + 1. \tag{35}$$

We prove the claim by showing that, for any infinite sequence of X-valued random variables that is *i.i.d. w.r.t.* the marginal distribution $(\mu_z(x))_{x\in X}$, $\xi = (\xi_1, ..., \xi_n, ...)$, in which $\xi^{M^N-1} = (\xi_1, ..., \xi_{M^N-1})$ describes the types of the other active agents when there are M^N of them, we have

$$\lim_{N\to\infty} q^N(a;z,\xi^{M^N-1}) = q^{z,\kappa}_x, \quad \lim_{N\to\infty} r^N(a;z,\xi^{M^N-1}) = r^{z,\kappa}_x, \tag{36}$$

almost surely conditional on the state z. Because u is continuous, the claim follows immediately from (32) and (36).

By our construction of off-equilibrium beliefs, (9) and (10), $F^N_x(a)$ depends only on the interval $I(\overline{x})$ for which $a \in I(\overline{x})$. Because the number of intervals is finite, uniformity follows from convergence; namely, (31).

By definition, $\xi^{M^N-1} = (\xi_1, ..., \xi_{M^N-1})$ is distributed according to γ^N_z. For each N, let $\sigma^N = \sigma^{\xi^{M^N-1}}$ as defined in (35) (recall that $a \in I(\overline{x})$) and let $\nu^N = \nu^{\xi^{M^N-1},-a}$, as defined in (33). That is, σ^N is the type-configuration believed by all other agents. Then, the sequence $\{\sigma^N\}$ is such that $\sum_{y\in X} \sigma^N(y) = M^N$ and for each $y \in X$, $\lim_{N\to\infty}(\sigma^N(y)/M^N) = \mu_z(y)$ almost surely. Consider a realization of ξ for which $\lim_{N\to\infty}(\sigma^N(y)/M^N) = \mu_z(y)$. Then, for N sufficiently large, $(1/M^N, \sigma^N/M^N, \phi^{\sigma^N}) \in A_z$ and hence, for such N's, $\beta^{\sigma^N} = f_z(1/M^N, \sigma^N/M^N, \phi^{\sigma^N})$. Notice that $\lim_{N\to\infty} \phi^{\sigma^N} = \delta_z$. Thus, by Lemma 3 (ii), we have $\lim_{N\to\infty} \beta^{\sigma^N} = \beta^{z,\kappa}$ (the offers corresponding to the CE $\langle p^{z,\kappa}, (q^{z,\kappa}_x, r^{z,\kappa}_x)_{x\in X}\rangle$). This implies that

$$\lim_{N\to\infty} (\frac{Q^N_-}{M^N}, \frac{R^N_-}{M^N}) = \sum_{y\in X} \mu_z(y)(\beta^z(y) + \kappa) \text{ and } \lim_{N\to\infty} \frac{R^N_-}{Q^N_-} = p^{z,\kappa}, \tag{37}$$

where Q_-^N and R_-^N are defined in (34) with $\xi = \xi^{M^N-1}$.

Finally, we show that $\lim_{N\to\infty} s_3^*(x, a, \nu^N) = \beta^{z,\kappa}(x)$, where s_3^* is defined in (13). Letting $\phi^N = \text{marg}_Z \varphi^*(x, a, \nu^N)$, where φ^* is defined in (9) and (10), we have

$$\lim_{N\to\infty} \phi^N[z] = 1.$$

Notice that ϕ^N is derived from the type-configuration believed by the agent, which is different from σ^N if $x \neq \bar{x}$. For each N, $s_3^*(x, a, \nu^N)$ solves

$$\max_{b\in\mathcal{O}} H_x^N(b) = \max_{b\in\mathcal{O}} \sum_{z'\in Z} \phi^N[z'] u\left(\bar{q}' + \frac{b_r Q_-^N - b_q R_-^N}{R_-^N + b_r}, \bar{r}' + \frac{b_q R_-^N - b_r Q_-^N}{Q_-^N + b_q}; x, z'\right). \quad (38)$$

Now, let

$$J_x(b; p, c_1, c_2, \phi) = \sum_{z'\in Z} \phi[z'] u\left(q, r; x, z'\right)$$

with $q = \bar{q}' + \frac{b_r}{p(1+c_2 b_r)} - \frac{b_q}{1+c_2 b_r}$ and $r = \bar{r}' - \frac{b_r}{1+c_1 b_q} + \frac{p b_q}{1+c_1 b_q}$, and where the domain for J_x is $\mathcal{O} \times \left[\frac{\kappa_r}{\bar{q}'+\kappa_q}, \frac{\bar{r}'+\kappa_r}{\kappa_q}\right] \times \left[0, \frac{1}{\kappa_r}\right] \times \left[0, \frac{1}{\kappa_q}\right] \times \Delta(Z)$. It follows that $J_x(b; \frac{R_-^N}{Q_-^N}, 1/Q_-^N, 1/R_-^N, \phi^N) = H_x^N(b)$. Therefore, by the argument used in the proof of Lemma 3 (i), $J_x(\cdot; p, c_1, c_2, \phi)$ has a unique maximum, $j_x(p, c_1, c_2, \phi)$. And because J_x is continuous on its domain, the Maximum Theorem implies that $j_x(p, c_1, c_2, \phi)$ is continuous.

Now, for each N, $s_3^*(x, a, \nu^N) = j_x(\frac{R_-^N}{Q_-^N}, 1/Q_-^N, 1/R_-^N, \phi^N)$. By (37) and the continuity of j_x, it follows that

$$b^* = \lim_{N\to\infty} s_3^*(x, a, \nu^N) = \lim_{N\to\infty} j_x\left(\frac{R_-^N}{Q_-^N}, 1/Q_-^N, 1/R_-^N, \phi^N\right) = j_x(p^{z,\kappa}, 0, 0, \delta_z).$$

By the definition of J_x, it follows that b^* maximizes $u\left(\bar{q}' - b_q + \frac{b_r}{p^z}, \bar{r}' - b_r + p^{z,\kappa} b_q; x, z\right)$. Therefore, b^* is the offer for type-x agents corresponding to the CE, $\langle p^{z,\kappa}, (q_x^{z,\kappa}, r_x^{z,\kappa})_{x\in X}\rangle$; that is, $b^* = \beta^{z,\kappa}(x)$. This shows that

$$\lim_{N\to\infty} q^N(a; z, \xi^{M^N-1}) = \bar{q}' + \frac{\beta_r^{z,\kappa}(x)}{p^{z,\kappa}} - \beta_q^{z,\kappa}(x), \quad \lim_{N\to\infty} r^N(a; z, \xi^{M^N-1}) = \bar{r}' + \beta_q^{z,\kappa}(x) p^{z,\kappa} - \beta_r^{z,\kappa}(x),$$

almost surely for all $a \in \mathcal{O}$. This proves (36). □

In order to have any effect on $F_x^N(a)$, the agent must choose an offer sufficiently far from s_2^*, the offer that maximizes $G_x(a)$. Claim 1 shows that the implied loss in terms of $G_x(a)$ is bounded away from zero (and does not depend on N). By Claim 2, any effect on $F_x^N(a)$ goes to zero as $N \to \infty$. Together, they imply that s_2^* is a best response to (s_1^*, s_2^*, s_3^*) for sufficiently large N.

Finally, we need to show that s_1^* is a best response to (s_1^*, s_2^*, s_3^*); that is, that each agent is willing to participate given that all agents participate and their behavior follows

(s_2^*, s_3^*) in stages two and three. By Lemma 0 (and small κ), the CE $(q^{z,\kappa}, r^{z,\kappa})$ for each $z \in Z$ is such that for all $x \in X$ and $z \in Z$, $u(q_x^{z,\kappa}, r_x^{z,\kappa}; x, z) > u(\overline{q}, \overline{r}; x, z)$. Therefore,

$$\Delta_0 = \sum_{z\in Z, x\in X} \tau_0(z)\mu_z(x)[u(q_x^{z,\kappa}, r_x^{z,\kappa}, x, z) - u(\overline{q}, \overline{r}, x, z)] > 0.$$

For each z, let $\mathbb{E}_z[u(q_x^N, r_x^N, x, z)]$ be the expected payoff for an agent of type x, conditional on state z and being active, given that all agents participate and their behavior follows (s_2^*, s_3^*) in stages two and three. Now, by (36), there exists $\overline{N}$ such that if $N \geq \overline{N}$, then

$$\left| \sum_{z\in Z, x\in X} \tau_0(z)\mu_z(x)\{\mathbb{E}_z[u(q_x^N, r_x^N; x, z)] - u(q_x^{z,\kappa}, r_x^{z,,\kappa}; x, z)\} \right| < \Delta_0/2. \tag{39}$$

By following the equilibrium strategy, an agent's ex ante expected payoff (before type is realized) is given by

$$\sum_{z\in Z, x\in X} \tau_0(z)\mu_z(x)\{\eta u(\widehat{q}(x,\kappa), \widehat{r}(x,\kappa); x, z) + (1-\eta)\mathbb{E}_z[u(q_x^N, r_x^N; x, z)]\},$$

and his payoff by not participating is given by $\sum_{z\in Z, x\in X} \tau_0(z)\mu_z(x)u(\overline{q}, \overline{r}; x, z) \equiv A$. Thus, the agent is willing to participate if

$$\sum_{z\in Z, x\in X} \tau_0(z)\mu_z(x)u(\widehat{q}(x,\kappa), \widehat{r}(x,\kappa); x, z) > A \tag{40}$$

and

$$\sum_{z\in Z, x\in X} \tau_0(z)\mu_z(x)\mathbb{E}_z[u(q_x^N, r_x^N; x, z)] > A. \tag{41}$$

Now, by *A1* (and small κ), $\sum_{z\in Z} \tau_x(z)u(\widehat{q}(x,\kappa), \widehat{r}(x,\kappa); x, z) > \sum_{z\in Z} \tau_x(z)u(\overline{q}, \overline{r}; x, z)$ for some x. Hence,

$$\begin{aligned} &\sum_{x\in X}\left[\sum_{z'\in Z} \tau_0(z')\mu_{z'}(x)\right] \sum_{z\in Z} \tau_x(z)u(q(x,\kappa); r(x,\kappa), x, z) \\ > &\sum_{x\in X}\left[\sum_{z'\in Z} \tau_0(z')\mu_{z'}(x)\right] \sum_{z\in Z} \tau_x(z)u(\overline{q}, \overline{r}; x, z). \end{aligned}$$

Because $\tau_x(z) = \tau_0(z)\mu_z(x)/\left[\sum_{z'\in Z} \tau_0(z')\mu_{z'}(x)\right]$, this implies (40).

Moreover, $N \geq \overline{N}$ implies

$$\begin{aligned} &\sum_{z\in Z, x\in X} \tau_0(z)\mu_z(x)\mathbb{E}_z[u(q_x^N, r_x^N; x, z)] - \sum_{z\in Z, x\in X} \tau_0(z)\mu_z(x)u(\overline{q}, \overline{r}; x, z) \\ = &\sum_{z\in Z, x\in X} \tau_0(z)\mu_z(x)\{\mathbb{E}_z[u(q_x^N, r_x^N; x, z)] - u(q_x^{z,\kappa}, r_x^{z,\kappa}; x, z)\} \\ &+ \sum_{z\in Z, x\in X} \tau_0(z)\mu_z(x)[u(q_x^{z,\kappa}, r_x^{z,\kappa}; x, z) - u(\overline{q}, \overline{r}; x, z)] \\ > &\Delta_0 - \Delta_0/2 > 0, \end{aligned}$$

where the inequality, which implies (41), follows from (39).■

8.3 Almost ex post efficiency

Proposition 2. Let $\varepsilon > 0$ be given. There exists $\bar{\kappa}$ and $N(\kappa, \eta)$ such that if (κ, η) satisfies

$$\bar{\kappa} > \kappa \geq 4\eta(\bar{q}, \bar{r}) > 0 \tag{42}$$

and $N > N(\kappa, \eta)$, then there exists a fully revealing equilibrium whose outcome is ex post ε-efficient.

Proof. Let $(q^{z,\kappa}, r^{z,\kappa})$ be a competitive allocation in the economy $\mathcal{L}^z(\kappa)$ which is regular and in which every type trades. Consider another economy $\mathcal{J}^z(\rho, \kappa)$, where $\rho \in \Delta(X)$ is the proportion of agents according to type, and each agent has endowment $(\bar{q}+\kappa_q, \bar{r}+\kappa_r)$. Let $(q^{z,\rho,\kappa}, r^{z,\rho,\kappa})$ denote the competitive allocation for $\mathcal{J}^z(\rho, \kappa)$ under known state-of-the-world z. We omit the proof of the following claim, which only asserts continuity of competitive allocations *w.r.t.* endowment parameters and the proportion of different types.

Claim 1. Let (q^z, r^z) be a regular competitive allocation in $\mathcal{L}^z$, and, let $(q^{z,\kappa}, r^{z,\kappa})$ be the competitive allocation in $\mathcal{L}^z(\kappa)$ that is close to (q^z, r^z). Then, for any $\epsilon > 0$, there exists $\delta^1(\epsilon) > 0$ such that if $|\rho(x) - \mu_z(x)| < \delta^1(\epsilon)$ for each $x \in X$ and if $\max\{\kappa_q, \kappa_r\} < \delta^1(\epsilon)$, then there is a competitive allocation $(q^{z,\rho,\kappa}, r^{z,\rho,\kappa})$ in $\mathcal{J}^z(\rho; \kappa)$ for which

$$|u(q^{z,\kappa}_x, r^{z,\kappa}_x; x, z) - u(q^{z,\rho,\kappa}_x, r^{z,\rho,\kappa}_x; x, z)| < \epsilon.$$

for all $x \in X$.

The next claim constructs the high probability event $E_{z,\omega^N}(\epsilon)$ that we need to establish ε ex post optimality of a Proposition-1 equilibrium. The event E_{z,ω^N} will be the intersection of two events, E^1_{z,ω^N} and E^2_{z,ω^N}, where the first involves only exogenous random variables and the second depends on a selected equilibrium.

Fix some $(\kappa, \eta) > 0$ (recall that η is the probability of being inactive) such that $\mathcal{L}^z(\kappa)$, for each $z \in Z$, has a regular competitive equilibrium allocation, denoted $(q^{z,\kappa}, r^{z,\kappa})$, in which every type trades that is close to the regular competitive allocation in $\mathcal{L}^z$, (q^z, r^z). For any realization ζ^N and ω^N, there is a unique corresponding type-configuration for active agents, denoted $\sigma(\zeta^N, \omega^N) = (\sigma(\zeta^N, \omega^N)(x) : x \in X)$. For each $(z, \omega^N) \in Z \times \nleq^N$ and for any $\epsilon > 0$, define the event $E^1_{z,\omega^N}(\epsilon)$ as

$$E^1_{z,\omega^N}(\epsilon) = \left\{\zeta^N : (\forall x) \left|\sigma(\zeta^N, \omega^N)(x)/M^N - \mu_z(x)\right| < \delta^1(\epsilon)\right\}, \tag{43}$$

where $\delta^1(\epsilon)$ is defined in Claim 1 above (uniformly across all z's). By Proposition 1, for κ small and for η satisfies (42), there exists $\bar{N}(\kappa, \eta)$ such that if $N > \bar{N}(\kappa, \eta)$, then there exists a fully revealing equilibrium (s^N_1, s^N_2, s^N_3) (corresponding to the competitive allocations $\{(q^{z,\kappa}, r^{z,\kappa}) : z \in Z\}$). As above, we use β^σ to denote the stage-3 offers along the corresponding equilibrium path for a realization of type-configuration σ for active agents; we also use $(q^\sigma(x), r^\sigma(x))$ to denote the corresponding payoffs as determined in (4) from offers β^σ. Then, let

$$E^2_{z,\omega^N}(\epsilon) = \left\{\zeta^N : (\forall x) \left|u(q^{\sigma(\zeta^N,\omega^N)}(x), r^{\sigma(\zeta^N,\omega^N)}(x); x, z) - u(q^{z,\kappa}_x, r^{z,\kappa}_x; x, z)\right| < \epsilon\right\}. \tag{44}$$

Claim 2. Let $E_{z,\omega^N}(\epsilon) = E^1_{z,\omega^N}(\epsilon) \cap E^2_{z,\omega^N}(\epsilon)$. There exists $N^2(\kappa, \eta, \epsilon)$ such that if $N > N^2(\kappa, \eta, \epsilon)$, then for any $(z, \omega^N) \in Z \times \Omega^N$, $\mathbb{P}\left[E_{z,\omega^N}(\epsilon) \mid \omega^N, z\right] > 1 - \epsilon$.

Proof. Let ξ be an infinite sequence of *i.i.d.* X-valued random variables with marginal distribution given by μ_z. Because ω^N is independent of the realization of types and the state-of-the-world, for any N the sequence $\{\zeta_n : 1 \leq n \leq N,\ \omega_n = 1\}$ and the sequence $\{\xi_m : m = 1, ..., M^N\}$ have the same distribution conditional on z and ω^N. For each N and $\xi^{M^N} = (\xi_1, ..., \xi_{M^N})$, let β^{σ^N} describe the equilibrium stage-3 offers under (s_1^N, s_2^N, s_3^N) along the equilibrium path with $\sigma^N(x) = \#\{1 \leq m \leq M^N : \xi_m = x\}$ and let $(q^{\sigma^N}, r^{\sigma^N})$ describe the corresponding equilibrium payoffs for active agents. By the law of large numbers, for any z and $x \in X$,

$$\lim_{N\to\infty} \sigma^N(x)/M^N = \mu_z(x)$$

almost surely conditional on z. Thus, by Proposition 1, it follows that $\lim_{N\to\infty} \beta^{\sigma^N} = \beta^z$ almost surely conditional on z. Hence, by continuity of u, for any z,

$$\lim_{N\to\infty} u\left(q^{\sigma^N}(x), r^{\sigma^N}(x); x, z\right) = u(q_x^{z,\kappa}, r_x^{z,\kappa}; x, z) \text{ for all } x,$$

almost surely conditional on z.□

Now we can complete the proof. Given ε, let $\bar{\kappa} = \delta^1(\frac{\varepsilon}{3})$ and consider our mechanism with $\max\{\kappa_q, \kappa_r\} < \bar{\kappa}$ and with $\eta \leq \min\{\frac{\kappa_q}{4\bar{q}}, \frac{\kappa_r}{4\bar{r}}\}$. Let $(\bar{q}', \bar{r}') = (\bar{q} - 2\kappa_q, \bar{r} - 2\kappa_r)$.

Given the fully revealing equilibrium (s_1^N, s_2^N, s_3^N) constructed above for $N > \bar{N}(\kappa, \eta)$ agents, and given a type realization, $\zeta^N = (\zeta_1, ..., \zeta_N) \in X^N$, and an activeness-status realization, $\omega^N = (\omega_1, ..., \omega_N)$, the corresponding allocation is as follows: if $\omega_n = 0$, then

$$c_n^{s^N}(\zeta^N, \omega^N, z) = (\widehat{q}(x, \kappa), \widehat{r}(x, \kappa));$$

if $\omega_n = 1$, then

$$c_n^{s^N}(\zeta^N, \omega^N, z) = (q^{\sigma(\zeta^N,\omega^N)}(\zeta_n, \kappa), r^{\sigma(\zeta^N,\omega^N)}(\zeta_n, \kappa)).$$

Let $N(\kappa, \eta) = N^2(\kappa, \eta, \frac{\varepsilon}{3})$. Now we show that if $\max\{\kappa_q, \kappa_r\} < \bar{\kappa}$ and $\eta \leq \min\{\frac{\kappa_q}{4\bar{q}}, \frac{\kappa_r}{4\bar{r}}\}$, and if $N > N(\kappa, \eta)$, then the allocation $\{c_n^{s^N} : n \in \mathcal{N}\}$ is ex post ε-efficient. For any realization $\zeta^N = (\zeta_1, ..., \zeta_N) \in X^N$ and $\omega^N = (\omega_1, ..., \omega_N)$, let $\mathcal{M}(\zeta^N, \omega^N) = \{m \in \mathcal{N} : \omega_m = 1\}$. By Lemma 1 and $\eta \leq \min\{\frac{\kappa_q}{4\bar{q}}, \frac{\kappa_r}{4\bar{r}}\}$, we know that the allocation c^{s^N} is feasible.

For each z and for each ω^N, by Claim 2, $N > N^2(\kappa, \eta, \frac{\varepsilon}{3})$ implies $\mathbb{P}[E_{z,\omega^N}(\varepsilon/3)|z, \omega^N] > 1 - \varepsilon$. Thus, to show that c^{s^N} is ex post ε-efficient, suppose, by way of contradiction, that c' is feasible and (15) holds for some z, ω^N and for some $\zeta^N \in E_{z,\omega^N}(\varepsilon/3)$. Fix such a (z, ω^N, ζ^N) and let $c'_n = (q'_n, r'_n)$ for each $n \in \mathcal{N}$.

Let $\rho(x) = \sigma(\zeta^N, \omega^N)(x)/M^N$. Then, let $(q^{z,\rho,\kappa}, r^{z,\rho,\kappa})$ be the competitive allocation for a finite economy which has $\sigma(\zeta^N, \omega^N)(x)$ agents of type-x for each x and in which each agent has endowment $(\bar{q} + \kappa_q, \bar{r} + \kappa_r)$. Now, consider the following allocation $\{(q''_n, r''_n) :$

$n \in \mathcal{N}\}$, using $(q^{z,\rho,\kappa}, r^{z,\rho,\kappa})$ as constructed in Claim 1, defined as $(q''_n, r''_n) = (q^{z,\rho,\kappa}_x, r^{z,\rho,\kappa}_x)$ if $\omega_n = 1$ and $\zeta_n = x$; and $(q''_n, r''_n) = (0,0)$ if $\omega_n = 0$. Because $\zeta^N \in E^1_{z,\omega^N}(\varepsilon/3)$ and $\max\{\kappa_q, \kappa_r\} < \delta^1(\varepsilon/3)$, it follows from Claim 1 that

$$|u(q^{z,\kappa}_x, r^{z,\kappa}_x; x, z) - u(q^{z,\rho,\kappa}_x, r^{z,\rho,\kappa}_x; x, z)| < \varepsilon/3.$$

Moreover, because $\zeta^N \in E^2_{z,\omega^N}(\varepsilon/3)$, we have

$$\left| u(q^{\sigma(\zeta^N,\omega^N)}(x), r^{\sigma(\zeta^N,\omega^N)}(x); x, z) - u(q^{z,\kappa}_x, r^{z,\kappa}_x; x, z) \right| < \varepsilon/3.$$

Thus,

$$u(q^{z,\rho,\kappa}_x, r^{z,\rho,\kappa}_x; x, z) < u(q^{\sigma(\zeta^N,\omega^N)}(x), r^{\sigma(\zeta^N,\omega^N)}(x); x, z) + 2\varepsilon/3.$$

Now, for each n such that $\omega_n = 1$ and $\zeta_n = x$,

$$u(q''_n, r''_n; x, z) < u(c^{s^N}_n(\zeta^N, \omega^N, z); x, z) + 2\varepsilon/3 < u(q'_n, r'_n; x, z) - \varepsilon/3; \qquad (45)$$

while for each n such that $\omega_n = 0$ and $\zeta_n = x$,

$$u(q''_n, r''_n; x, z) = u(0, 0; x, z) \leq u(c^{s^N}_n(\zeta^N, \omega^N, z); x, z) < u(q'_n, r'_n; x, z) - \varepsilon, \qquad (46)$$

where the second inequality in each of (45) and (46) follows from (15), the contradicting assumption. Therefore, $\{(q'_n, r'_n) : n \in \mathcal{N}\}$ Pareto dominates $\{(q''_n, r''_n) : n \in \mathcal{N}\}$.

Now, notice that

$$\sum_{n \in \mathcal{N}} (q''_n, r''_n) \geq (1 - \eta) N(\bar{q} + \kappa_q, r + \kappa_r) \geq N(\bar{q}, \bar{r})$$

since $\eta \leq \min\{\frac{\kappa_q}{4\bar{q}}, \frac{\kappa_r}{4\bar{r}}\}$. Thus, $\{(q''_n, r''_n) : n \in \mathcal{N}\}$ is a competitive allocation (with inactive agents having zero endowments) for an economy with total resources no less than that for the allocation $\{(q'_n, r'_n) : n \in \mathcal{N}\}$. By the first fundamental theorem of welfare economics, it follows that $\{(q''_n, r''_n) : n \in \mathcal{N}\}$ cannot be Pareto dominated by $\{(q'_n, r'_n) : n \in \mathcal{N}\}$. ■

8.4 Uniqueness of equilibrium

Proposition 3. Assume *U1-U3*. Suppose that the competitive demand for good-q is monotone in $\mathcal{L}^z$ for each $z \in Z$. Then for some $\bar{\kappa} > 0$, if $\kappa \in (0, \bar{\kappa}]$ and $\eta \leq \min\{\kappa_q/2\bar{q}, \kappa_r/2\bar{r}\}$, then there exists $\bar{N}$ such that for all $N > \bar{N}$, any symmetric PBE in pure strategies $s^N = (s^N_1, s^N_2, s^N_3)$ with $s^N_1 = yes$ is fully revealing.

Proof. First, by *A2* and Lemma 0, we have uniqueness of CE in $\mathcal{L}^z(\kappa)$. Let $\bar{q}' = \bar{q} - 2\kappa_q$ and let $\bar{r}' = \bar{r} - 2\kappa_r$. (Excess demand for good-q in $\mathcal{L}^z(\kappa)$ is $f(p;\kappa) = \sum_{x \in X} \mu_z(x) q_x(p) + \kappa_r/p - \bar{q}' - \kappa_q$. Therefore, $\partial f(p;\kappa)/\partial p = \sum_{x \in X} \mu_z(x) \left[\partial q_x(p)/\partial p\right] - \kappa_r/p^2$. Monotonicity of demand functions implies that the first term is negative. Hence, $\partial f(p;\kappa)/\partial p < 0$, which rules out multiple CE's.) Now, fix some $\kappa > 0$. For each $z \in Z$, let $\beta^{z,\kappa}$ be the offer corresponding to the unique CE in $\mathcal{L}^z(\kappa)$.

First we exclude complete pooling; i.e., an equilibrium s with full participation such that for some $\bar{a} \in \mathcal{O}$, $s_1(x) = \bar{a}$ for all $x \in X$. We use $\alpha(x, \kappa)$ to denote the stage-1 offer that corresponds to stage-1 consumption $\widehat{q}(x, \kappa)$ for all x.

Claim 0. For any equilibrium s with $s_1 = yes$, there exist $x \neq y \in X$ such that $s_2(x) \neq s_2(y)$.

Proof. By way of contradiction, suppose that s is an equilibrium with $s_1 = yes$ and with $s_2(x) = \bar{a}$ for all $x \in X$. By *U2*, this implies that the realized payoff of all active agents is $(\bar{q}', \bar{r}')$, no-trade. However, by *A1*, there exists some x such that $G_x(\bar{a}) < G_x(\alpha(x, \kappa))$. (See (30) for the definition of $G_x(a)$.) Therefore, this agent has a profitable deviation to $\alpha(x, \kappa)$ because no-trade is feasible at stage-3 contingent on being active. $\square$

Claim 0 implies that any candidate equilibrium with full participation that is not fully revealing is associated with a partition $\mathcal{Y} = (Y_1, ..., Y_K)$ of X with $1 < K < |X|$. We denote such a candidate equilibrium for N agents by s^N. We prove, by way of contradiction, that s^N cannot be an equilibrium for sufficiently large N. The contradiction is that one agent, called the *potential defector*, has a profitable deviation (to the stage-2 action described by α).

For a potential defector of type-x, the stage-2 objective function is

$$G_x^N(a) = \eta G_x(a) + (1 - \eta) F_x^N(a),$$

where $F_x^N(a)$ is the expected payoff implied by offer a at stage-2 conditional on being active. We show that for any $\epsilon > 0$, $F_x^N(a)$ does not vary with a by more than ϵ for sufficiently large N. By assumption *U3*, $F_x^N(a)$ depends on the associated set of types, $A \subseteq X$, that is believed by other active agents when a is offered at stage-2. Indeed, this holds for equilibrium beliefs as well: if a is an equilibrium offer with $a = s_2^N(y)$, then a is associated with $A = Y_k$ with $y \in Y_k$. Therefore, we may rewrite $F_x^N(a)$ as $F_x^N(A)$ with $A \subseteq X$ being the set of types associated with a. To calculate $F_x^N(A)$, we need to characterize the stage-3 offers following a public announcement ν_N such that $\nu_N(\tilde{a}) = 1$ for some $\tilde{a}$ that is associated with A and $\nu_N(a) = 0$ if $a \notin \{s_2^N(Y_k) : k = 1, ..., K\} \cup \{\tilde{a}\}$.

We divide the rest of the proof into four claims and a final argument. Each of the first three claims has two similar parts—one part for the potential defector and the other for agents following equilibrium behavior. Claim 1 is concerned with beliefs along the equilibrium path and has nothing to do with behavior. Claim 2 provides bounds on offers that assure that consumption of each good is bounded away from zero—bounds that hold in any equilibrium. Those bounds imply bounds on the derivatives that appear in the first-order conditions that hold at all best responses. Claim 3 establishes uniform convergence of equilibrium offers, β^N, to "price-taking" offers with a known state-of-the-world and with a price that is given by the equilibrium offers of others, where the uniformity is over all possible sequences of equilibria. Claim 4 is closely related to Claim 2 in the proof of Theorem 1 because it says that $F_x^N(a)$ does not vary much with a for sufficiently large N. The final argument follows the logic of the proof of Proposition 1. In what follows, let $M^N = \lceil (1 - \eta) N \rceil$ be the number of active agents.

Claim 1. Let ν_N be the public announcement which includes the offer $\tilde{a}^N$ (made by the potential defector), and let λ^N be the corresponding signal configuration for agents other than the potential defector.

(a) Denote the stage-3 belief of the potential defector of type x, following λ^N, by

$$\varphi^N(x, \tilde{a}, \nu_N^{-\tilde{a}})[z, \xi^1, ..., \xi^K] = \tilde{\phi}_x^N[z]\tilde{\gamma}_z^N[\xi^1, \xi^2, ..., \xi^K],$$

where $\tilde{\phi}_x^N[z]$ is the posterior over states, $\tilde{\gamma}_z^N$ is that over the types of other active agents conditional on z, and $\xi^k = (\xi_1^k, ..., \xi_{\lambda^N(Y_k)}^k) \in Y_k^{\lambda^N(Y_k)}$ describes the types for those agents who made the offer associated with Y_k. Then, for any $\epsilon > 0$, there exist $N_a^1(\epsilon)$ and $\delta_a^1(\epsilon) \leq \epsilon$ such that if $N > N_a^1(\epsilon)$ and if $\left|\frac{\lambda^N(Y_k)}{M^N} - \mu_{z^*}(Y_k)\right| < \delta_a^1(\epsilon)$ for all k, then for each $y \in X$,

$$\tilde{\phi}_x^N[z^*] > 1 - \epsilon \text{ and } \tilde{\gamma}_{z^*}^N\left[\left|\tilde{\rho}^N(y) - \frac{\mu_{z^*}(y)}{\mu_{z^*}(Y_k)}\right| < \epsilon\right] > 1 - \epsilon,$$

where

$$\tilde{\rho}^N(y) = \#\{\xi_i^k : \xi_i^k = y, i = 1, ..., \lambda^N(Y_k)\}/\lambda^N(Y_k),$$

the fraction of agents other than the potential defector with types in Y_k who are of type-y.

(b) Consider an agent of type x other than the potential defector. For such an agent the relevant signal configuration is the offer $\tilde{a}^N$ (made by the potential defector) and λ_-^N defined as follows: If $x \in Y_{\bar{k}}$, then $\lambda_-^N(Y_k) = \lambda^N(Y_k)$ for each $k \neq \bar{k}$ and $\lambda_-^N(Y_{\bar{k}}) = \lambda^N(Y_{\bar{k}}) - 1$. We denote such an agent's stage-3 belief, after observing the signal configuration λ_-^N and $\tilde{a}^N$, by

$$\varphi^N(y, s_1(y), \nu_N^{-s_1(y)})[z, \xi^1, ..., \xi^K, \tilde{\xi}] = \hat{\phi}_y^N[z]\hat{\gamma}_z^N[\xi^1, ..., \xi^K, \tilde{\xi}],$$

where $\hat{\phi}_x^N$ is the posterior distribution over states, $\hat{\gamma}_z^N$ is that over types of the other active agents conditional on z. Then, for any $\epsilon > 0$, there exist $N_b^1(\epsilon)$ and $\delta_b^1(\epsilon) \leq \epsilon$ such that if $N > N_b^1(\epsilon)$ and if $\left|\frac{\lambda^N(Y_k)}{M^N} - \mu_{z^*}(Y_k)\right| < \delta_b^1(\epsilon)$ for all k, then for each $y \in X$,

$$\hat{\phi}_x^N[z^*] > 1 - \epsilon \text{ and } \hat{\gamma}_{z^*}^N\left[\left|\hat{\rho}^N(y) - \frac{\mu_{z^*}(y)}{\mu_{z^*}(Y_k)}\right| < \epsilon\right] > 1 - \epsilon,$$

where

$$\hat{\rho}^N(y) = \#\{\xi_i^k : \xi_i^k = y, i = 1, ..., \lambda_-^N(Y_k)\}/\lambda_-^N(Y_k),$$

the fraction of *other* active (non-defecting) agents with types in Y_k who are of type-y.

Proof. By Bayes' rule, we can derive $\hat{\phi}_x^N$, $\hat{\gamma}_z^N$, $\tilde{\phi}_x^N$, and $\tilde{\gamma}_z^N$ as follows:

$$\hat{\phi}_x^N[\bar{z}] = \frac{\pi(\bar{z})\mu_{\bar{z}}(x)\mu_{\bar{z}}(A)\prod_{k=1}^K[\mu_{\bar{z}}(Y_k)]^{\lambda_-^N(Y_k)}}{\sum_{z\in Z}\pi(z)\mu_z(x)\mu_z(A)\prod_{k=1}^K[\mu_z(Y_k)]^{\lambda_-^N(Y_k)}},$$

(recall that A is associated with the offer $\tilde{a}^N$) and

$$\hat{\gamma}_z^N[\xi^1, ..., \xi^K, \tilde{\xi}] = \frac{\mu_z(\tilde{\xi})}{\mu_z(A)}\prod_{k=1}^K\left[\prod_{i=1}^{\lambda_-^N(Y_k)}\left(\frac{\mu_z(\xi_i^k)}{\mu_z(Y_k)}\right)\right],$$

where $\xi^k = (\xi_1^k, ..., \xi_{\lambda^N(Y_k)}^k) \in Y_k^{\lambda^N(Y_k)}$ describes the types for those who make the offer associated with Y_k and $\tilde{\xi}$ describes the type of the agent who offered $\tilde{a}^N$.

Similarly,

$$\tilde{\phi}_x^N[\bar{z}] = \frac{\pi(\bar{z})\mu_{\bar{z}}(x)\prod_{k=1}^K[\mu_{\bar{z}}(Y_k)]^{\lambda^N(Y_k)}}{\sum_{z\in Z}\pi(z)\mu_z(x)\prod_{k=1}^K[\mu_z(Y_k)]^{\lambda^N(Y_k)}},\ \tilde{\gamma}_z^N[\xi^1, ..., \xi^K] = \prod_{k=1}^K \prod_{i=1}^{\lambda^N(Y_k)} \frac{\mu_z(\xi_i^k)}{\mu_z(Y_k)},$$

where $\xi^k = (\xi_1^k, ..., \xi_{\lambda^N(Y_k)}^k) \in Y_k^{\lambda^N(Y_k)}$ describes the types for those agents other than the potential defector who make the offer associated with Y_k ($\lambda^N(Y_k)$ is the number of such agents).

The convergence result for $\hat{\phi}_x^N$ and $\tilde{\phi}_x^N$ follows immediately from the above expressions. We prove only Claim 1(b) for $\hat{\rho}^N$. (The proof for 1(a) is similar.) For each $z \in Z$, consider K infinite sequences of random variables $(\zeta^1, \zeta^2, ..., \zeta^K)$ such that ζ_i^k is Y_k-valued for all $i \in \mathbb{N}$, the K sequences are independent of each other, and ζ^k is an *i.i.d.* sequence with marginal distribution $(\frac{\mu_z(y)}{\mu_z(Y_k)})_{y\in Y_k}$. Let γ_z denote the joint distribution of $(\zeta^1, \zeta^2, ..., \zeta^K)$. Then, given a sequence of signal-configurations λ^N and a realization $(\zeta^1, \zeta^2, ..., \zeta^K)$, for each $k = 1, ..., K$, each $y \in Y_k$, and each N, define

$$\rho^N(y) = \#\{\zeta_i^k : \zeta_i^k = y, i = 1, ..., \lambda_-^N(Y_k)\}/\lambda_-^N(Y_k).$$

Notice that for each $y \in X$, $\rho^N(y)$ and $\hat{\rho}^N(y)$ have the same distribution. By the *law of large numbers*, for each $y \in Y_k$, $\rho^N(y)$ converges to $\mu_z(y)/\mu_z(Y_k)$ in probability under γ_z for any k as $\lambda_-^N(Y_k)$ converges to infinity. This implies the result.□

Now we turn to equilibrium behavior, where, again, we distinguish between the potential defector and the other agents. We use $\tilde{\beta}^{N,A}(x)$ to denote the equilibrium offer from the type-x potential defector and use $\hat{\beta}^{N,A}(x)$ to denote the equilibrium offer from any other type-x agent. Again, for notational ease, let $\bar{q}' = \bar{q} - 2\kappa_q$ and let $\bar{r}' = \bar{r} - 2\kappa_r$.

Claim 2. There exist $\bar{b} = (\bar{b}_q, \bar{b}_r) < (\bar{q}', \bar{r}')$ such that $\tilde{\beta}^{N,A}(x) \leq \bar{b}$ and $\hat{\beta}^{N,A}(x) \leq \bar{b}$.

Proof. As might be expected, this follows from the Inada conditions and the bounds on prices implied by $\kappa > 0$. First notice that $\tilde{\beta}^{N,A}(x)$ and $\hat{\beta}^{N,A}(x)$ solve the following problems.

Following a signal configuration λ^N for agents other than the potential defector, the type-x potential defector has the stage-3 objective function,

$$\tilde{H}_x^{\lambda^N}(b_q, b_r) = \sum_{z\in Z}\tilde{\phi}_x^N[z]\mathbb{E}_{\tilde{\gamma}_z^N}\left[u\left(\bar{q}' + \frac{b_r\tilde{Q}_-^N - b_q\tilde{R}_-^N}{\tilde{R}_-^N + b_r}, \bar{r}' + \frac{b_q\tilde{R}_-^N - b_r\tilde{Q}_-^N}{\tilde{Q}_-^N + b_q}; x, z\right)\right], \quad (47)$$

where

$$(\tilde{Q}_-^N, \tilde{R}_-^N) = \sum_{k=1,..,K}\sum_{y\in Y_k}\lambda^N(Y_k)\tilde{\rho}^N(y)\hat{\beta}^{N,A}(y) + M^N\kappa.$$

Following a signal configuration λ_-^N of other (non-defecting) agents, a type-x nondefector has the stage-3 objective function,

$$\hat{H}_x^{\lambda^N}(b_q, b_r) = \sum_{z \in Z} \hat{\phi}_x^N[z] \mathbb{E}_{\hat{\gamma}_z^N} \left[u \left(\bar{q}' + \frac{b_r \hat{Q}_-^N - b_q \hat{R}_-^N}{\hat{R}_-^N + b_r}, \bar{r}' + \frac{b_q \hat{R}_-^N - b_r \hat{Q}_-^N}{\hat{Q}_-^N + b_q}; x, z \right) \right], \quad (48)$$

where

$$(\hat{Q}_-^N, \hat{R}_-^N) = \sum_{k=1,..,K} \sum_{y \in Y_k} \lambda_-^N(Y_k) \hat{\rho}^N(y) \hat{\beta}^{N,A}(y) + \tilde{\beta}^{N,A}(\tilde{\xi}) + M^N \kappa,$$

and where $\tilde{\xi}$ denotes the potential defector's type.

We prove the claim for $\tilde{\beta}^{N,A}$; the other case is exactly the same. Obviously, we are only concerned with positive offers. We spell out the details for $\tilde{\beta}_q^{N,A}(x) > 0$. To abbreviate notation, denote $\tilde{\beta}_q^{N,A}(x)$ by b_q^*.

Being positive, b_q^* satisfies the first-order condition (u_q and u_r denote the first-order derivatives of u),

$$\sum_{z \in Z} \tilde{\phi}_x^N[z] \left\{ \mathbb{E}_{\tilde{\gamma}_z^N} \left[-u_q(q(b_q^*), r(b_q^*); x, z) + u_r(q(b_q^*), r(b_q^*); x, z) \frac{\tilde{Q}_-^N \tilde{R}_-^N}{(\tilde{Q}_-^N + b_q)^2} \right] \right\} = 0 \quad (49)$$

where

$$(q(b_q), r(b_q)) = \left(\bar{q}' - b_q, \bar{r}' + \frac{b_q \tilde{R}_-^N}{\tilde{Q}_-^N + b_q} \right). \quad (50)$$

For any $b_q \in [0, \bar{q}']$, $\tilde{Q}_-^N \in [M^N \kappa_q, (M^N - 1)\bar{q}' + M^N \kappa_q)]$, and $\tilde{R}_-^N \in [M^N \kappa_r, (M^N - 1)\bar{r}' + M^N \kappa_r)]$,

$$u_r(q(b_q), r(b_q); x, z) \frac{\tilde{Q}_-^N \tilde{R}_-^N}{(\tilde{Q}_-^N + b_q)^2} \le \max_{q \in [0, \bar{q}], z \in Z} u_r(q, \bar{r}; x, z) \frac{(\bar{q}' + \kappa_q)(\bar{r}' + \kappa_r)}{\kappa_q^2} \equiv A.$$

For each $b_q \in [0, \bar{q}')$, let

$$J(b_q) = \min_{r \in [\bar{r}, \bar{r} + \frac{\bar{q}(\bar{r} + \kappa_r)}{\kappa_q}], z \in Z} u_q(\bar{q}' - b_q, r; x, z).$$

Because $[\bar{r}', \bar{r}' + \frac{\bar{q}'(\bar{r}' + \kappa_r)}{\kappa_q}]$ is compact and X and Z are both finite, J is well-defined, positive, strictly increasing, $\lim_{b_q \to \bar{q}'} J(b_q) = \infty$, and, of course, $J(b_q) \le u_q(q(b_q), r(b_q); x, z)$.

Let $\gamma > 1$ be such that there is a solution for b_q to $J(b_q) = \gamma A$. Denote the solution, which is unique, $\tilde{b}_q(x)$. We next show that $\tilde{\beta}_q^{N,A}(x) = b_q^* \le \tilde{b}_q(x) < \bar{q}'$. The second inequality follows from $\gamma A < \infty$. Suppose the first inequality does not hold. Then, by (49), for some $(z, \tilde{Q}_-^N, \tilde{R}_-^N)$, we must have

$$u_q(q(b_q^*), r(b_q^*); x, z) - u_r(q(b_q^*), r(b_q^*); x, z) \frac{\tilde{Q}_-^N \tilde{R}_-^N}{(\tilde{Q}_-^N + b_q)^2} \le 0$$

with $(q(b_q), r(b_q))$ as in (50). Because $b_q^* > \tilde{b}_q(x)$,

$$u_q(q(b_q^*)), r(b_q^*); x, z) > \gamma A \text{ and } u_r(q(b_q^*), r(b_q^*); x, z)\frac{\tilde{Q}_-^N \tilde{R}_-^N}{(\tilde{Q}_-^N + b_q)^2} \le \gamma A,$$

a contradiction. The argument for $\tilde{\beta}_r^{N,A}(x) > 0$ is exactly analogous.

Finally, take $\bar{b}_q = \max\{\tilde{b}_q(x), \hat{b}_q(x) : x \in X\}$ and $\bar{b}_r = \max\{\tilde{b}_r(x), \hat{b}_r(x) : x \in X\}$, where $\hat{b}_q(x)$ and $\hat{b}_r(x)$ are the analogous bounds for $\hat{\beta}^{N,A}(x)$. □

Claim 3. Recall that $\tilde{\beta}^{N,A}$ denotes the equilibrium offer from the type-x potential defector and $\hat{\beta}^{N,A}$ denotes the equilibrium offer from other type-x agents. Fix a state $z \in Z$. For any $p > 0$ let $\chi(x;p) = (\chi_q(x;p), \chi_r(x;p))$ be the unique solution to

$$\max_{b \in \mathcal{O}} H_x(b;p) = \max_{b \in \mathcal{O}} u(\bar{q}' - b_q + \frac{b_r}{p}, \bar{r}' - b_r + b_q p; x, z).$$

For any $\epsilon > 0$, there exists $N^3(\epsilon)$ and $\delta^3(\epsilon) \le \epsilon$ such that if $N > N^3(\epsilon)$ and if $\left|\frac{\lambda^N(Y_k)}{M^N} - \mu_z(Y_k)\right| < \delta^3(\epsilon)$ for all k, then for each $x \in X$,

$$|\tilde{\beta}_q^{N,A}(x) - \chi_q(x;p^N)| + |\tilde{\beta}_r^{N,A}(x) - \chi_r(x;p^N)| < \epsilon$$

and

$$|\hat{\beta}_q^{N,A}(x) - \chi_q(x;p^N)| + |\hat{\beta}_r^{N,A}(x) - \chi_r(x;p^N)| < \epsilon,$$

where

$$p^N = \frac{\sum_{y \in X} \mu_z(y)\hat{\beta}_r^{N,A}(y) + \kappa_r}{\sum_{y \in X} \mu_z(y)\hat{\beta}_q^{N,A}(y) + \kappa_q}.$$

Proof. We first prove the claim for $\tilde{\beta}^{N,A}$. The objective $\tilde{H}_x^{\lambda^N}(b)$, defined in (47), for which $\tilde{\beta}^{N,A}(x)$ is a best response, differs from $H_x(b;p^N)$, for which $\chi(x;p^N)$ is a best response, in two respects. In $H_x(b;p^N)$, offers of others are weighted by limit weights, while in $\tilde{H}_x^{\lambda^N}(b)$ they are weighted by the agent's posterior over the types of others. And, in $H_x(b;p^N)$, the price is unaffected by the agent's own offer, while in $\tilde{H}_x^{\lambda^N}(b)$ it responds to the agent's offer as in the market game. The proof of the claim shows that both differences disappear for sufficiently large N.

Let $d = \frac{1}{2}\min\{\bar{q}' - \bar{b}_q, \bar{r}' - \bar{b}_r\}$, where $(\bar{b}_q, \bar{b}_r)$ is given by Claim 2. First we show that, for any $\epsilon > 0$, there exist $N^2(\epsilon)$ and $\delta^2(\epsilon)$ such that if $N > N^2(\epsilon)$ and if $\left|\frac{\lambda^N(Y_k)}{M^N} - \mu_z(Y_k)\right| < \delta^2(\epsilon)$ for all k, then for all $x \in X$ and for all $b_q \in [0, \bar{q}' - d]$ and all $b_r \in [0, \bar{r}' - d]$,

$$\left|\frac{\partial}{\partial b_q} H_x(b_q, 0; p^N) - \frac{\partial}{\partial b_q}\tilde{H}_x^{\lambda^N}(b_q, 0)\right| < \epsilon \text{ and } \left|\frac{\partial}{\partial b_r} H_x(0, b_r; p^N) - \frac{\partial}{\partial b_r}\tilde{H}_x^{\lambda^N}(0, b_r)\right| < \epsilon. \tag{51}$$

Because the arguments are essentially the same, we only prove the first of these.

Fix some $x \in X$ and let

$$L_z(b_q, p_1, p_2) = u_q(\bar{q}' - b_q, \bar{r}' + b_q p_1; x, z) - u_r(\bar{q}' - b_q, \bar{r}' + b_q p_1; x, z)p_2.$$

$L_z(b_q, \overline{p}, p_2)$ is continuous over $[0, \overline{q}' - d] \times \left[\frac{\kappa_r}{\overline{q}'+\kappa_q}, \frac{\kappa_r+\overline{r}'}{\kappa_q}\right]^2$ and, hence, is uniformly continuous. Therefore, for any $\epsilon > 0$, there exists some $\hat{\delta}(\epsilon) \leq \epsilon$ such that

$$|p_1 - p_1'| < \hat{\delta}(\epsilon) \text{ and } |p_2 - p_2'| < \hat{\delta}(\epsilon) \Rightarrow |L_z(b_q, p_1, p_2) - L_z(b_q, p_1', p_2')| < \epsilon. \tag{52}$$

Notice that $\frac{\partial}{\partial b_q} H_x(b_q, 0; p^N) = L_z(b_q, p^N, p^N)$ and that

$$\frac{\partial}{\partial b_q} \tilde{H}_x^{\lambda^N}(b_q, 0) = \sum_{z' \in Z} \tilde{\phi}_x^N[z'] \mathbb{E}_{\tilde{\gamma}_{z'}^N}\left[L_{z'}\left(b_q, \frac{\tilde{R}_-^N}{\tilde{Q}_-^N + b_q}, \frac{\tilde{Q}_-^N \tilde{R}_-^N}{(\tilde{Q}_-^N + b_q)^2}\right)\right].$$

Hence, it is sufficient to show that p^N is close to both $\frac{\tilde{R}_-^N}{\tilde{Q}_-^N+b_q}$ and $\frac{\tilde{Q}_-^N \tilde{R}_-^N}{(\tilde{Q}_-^N+b_q)^2}$ as N becomes large and as λ^N/M^N converges uniformly.

Because

$$p^N - \frac{\tilde{R}_-^N}{\tilde{Q}_-^N} = \frac{\sum_{y \in X} \mu_z(y) \hat{\beta}_r^{N,A}(y) + \kappa_r}{\sum_{y \in X} \mu_z(y) \hat{\beta}_q^{N,A}(y) + \kappa_q} - \frac{\sum_{k=1,..,K} \sum_{y \in Y_k} \frac{\lambda_-^N(Y_k)}{M^N} \tilde{\rho}^N(y) \hat{\beta}_r^{N,A}(y) + \kappa_r}{\sum_{k=1,..,K} \sum_{y \in Y_k} \frac{\lambda_-^N(Y_k)}{M^N} \tilde{\rho}^N(y) \hat{\beta}_q^{N,A}(y) + \kappa_q},$$

we have

$$\left|p^N - \frac{\tilde{R}_-^N}{\tilde{Q}_-^N}\right| \leq \frac{2(\overline{q}' + \kappa_q)(\overline{r}' + \kappa_r)}{\kappa_q^2} \sum_{k=1}^{K} \left|\mu_{z^*}(Y_k) - \frac{\lambda_-^N(Y_k)}{M^N} \tilde{\rho}^N(y)\right|.$$

Moreover,

$$\left|\frac{\tilde{Q}_-^N \tilde{R}_-^N}{(\tilde{Q}_-^N + b_q)^2} - \frac{\tilde{R}_-^N}{\tilde{Q}_-^N}\right| \leq \frac{\overline{r}' + \kappa_r}{\kappa_q} \frac{\overline{q}'(2\kappa_q + \frac{\overline{q}}{M^N})}{\kappa_q^2 M^N} \text{ and } \left|\frac{\tilde{R}_-^N}{\tilde{Q}_-^N + b_q} - \frac{\tilde{R}_-^N}{\tilde{Q}_-^N}\right| \leq \frac{\overline{r}' + \kappa_r}{\kappa_q} \frac{\overline{q}'}{\kappa_q M^N}.$$

Hence, for any $\epsilon > 0$ there exist $\tilde{N}(\epsilon)$ and $\tilde{\delta}(\epsilon) \leq \epsilon$ such that if $N > \tilde{N}(\epsilon)$ and if $\left|\tilde{\rho}^N(y) - \frac{\mu_z(y)}{\mu_z(Y_k)}\right| < \tilde{\delta}(\epsilon)$ and $\left|\frac{\lambda^N(Y_k)}{M^N} - \mu_z(Y_k)\right| < \tilde{\delta}(\epsilon)$ for all k, then

$$\left|\frac{\tilde{Q}_-^N \tilde{R}_-^N}{(\tilde{Q}_-^N + b_q)^2} - p^N\right| < \epsilon \text{ and } \left|\frac{\tilde{R}_-^N}{\tilde{Q}_-^N + b_q} - p^N\right| < \epsilon \text{ for all } b_q. \tag{53}$$

Let $B = 2\max\{1, u_q(d, r; x, z'), u_r(q, d; x, z') : r \in \left[0, \overline{r}' + 2\overline{q}' \frac{\overline{r}'+\kappa_r}{\kappa_q}\right], q \in [0, \overline{q}'], z' \in Z\}$. Then, for all $z' \in Z$, $|L_{z'}(b_q, \overline{p}, p_2)| \leq \frac{1}{2}B$ for all $(b_q, \overline{p}, p_2) \in [0, \overline{q}' - d] \times [\frac{\kappa_r}{\overline{q}'+\kappa_q}, \frac{\kappa_r+\overline{r}'}{\kappa_q}]^2$. Let $\delta' = \hat{\delta}(\frac{\epsilon}{10B})$ (see (52)). Let

$$\delta^2(\epsilon) = \min\{\delta^1(\frac{\epsilon}{10B}), \delta^1(\tilde{\delta}(\delta'))\} \text{ and } N^2(\epsilon) = \max\{N^1(\frac{\epsilon}{10B}), \tilde{N}(\delta^2(\epsilon))\},$$

where $\tilde{N}$ and $\tilde{\delta}$ are given in (53) and $\delta^1(\epsilon) = \min\{\delta_a^1(\epsilon), \delta_b^1(\epsilon)\}$ and $N^1(\epsilon) = \max\{N_a^1(\epsilon), N_b^1(\epsilon)\}$ with $\delta_a^1(\epsilon), \delta_b^1(\epsilon), N_a^1(\epsilon), N_b^1(\epsilon)$ given in Claim 1.

Suppose that $N > N^2(\epsilon)$ and that $\left|\frac{\lambda^N(Y_k)}{M^N} - \mu_{z^*}(Y_k)\right| < \delta^2(\epsilon)$ for all k. By Claim 1a, because $N > N^2(\epsilon) \geq N^1(\epsilon/10B)$ and for all k, $\left|\frac{\lambda^N(Y_k)}{M^N} - \mu_z(Y_k)\right| < \delta^2(\epsilon) \leq \delta^1(\epsilon/10B)$, we have $\tilde{\phi}_x^N[z] > 1 - \epsilon/10B$.

Moreover, because $N > N^1(\tilde{\delta}(\delta'))$ and because $\left|\frac{\lambda^N(Y_k)}{M^N} - \mu_z(Y_k)\right| < \delta^1(\tilde{\delta}(\delta'))$ for all k, it follows from Claim 1a that

$$\tilde{\gamma}_z^N\left[\left|\tilde{\rho}^N(y) - \frac{\mu_z(y)}{\mu_z(Y_k)}\right| < \tilde{\delta}(\delta')\right] > 1 - \tilde{\delta}(\delta') \geq 1 - \frac{\epsilon}{10B}.$$

Now, by (53), it follows that if $\left|\tilde{\rho}^N(y) - \frac{\mu_z(y)}{\mu_z(Y_k)}\right| < \tilde{\delta}(\delta')$, if $\left|\frac{\lambda^N(Y_k)}{M^N} - \mu_z(Y_k)\right| < \tilde{\delta}(\delta')$, and if $N > \tilde{N}(\delta')$, then

$$\max\left\{\left|p^N - \frac{\tilde{R}_-^N}{\tilde{Q}_-^N + b_q}\right|, \left|p^N - \frac{\tilde{Q}_-^N\tilde{R}_-^N}{(\tilde{Q}_-^N + b_q)^2}\right|\right\} < \delta' = \hat{\delta}(\frac{\epsilon}{10B}).$$

This and (52) imply

$$\left|L_z\left(b_q, \frac{\tilde{R}_-^N}{\tilde{Q}_-^N + b_q}, \frac{\tilde{Q}_-^N\tilde{R}_-^N}{(\tilde{Q}_-^N + b_q)^2}\right) - L_z(b_q, p^N, p^N)\right| < \frac{\epsilon}{10B}.$$

Therefore, $\left|\frac{\lambda^N(Y_k)}{M^N} - \mu_z(Y_k)\right| < \tilde{\delta}(\delta')$ and $N > \tilde{N}(\delta')$ imply that

$$\tilde{\gamma}_z^N\left[\left|L_z\left(b_q, \frac{\tilde{R}_-^N}{\tilde{Q}_-^N + b_q}, \frac{\tilde{Q}_-^N\tilde{R}_-^N}{(\tilde{Q}_-^N + b_q)^2}\right) - L_z(b_q, p^N, p^N)\right| < \frac{\epsilon}{10B}\right] > 1 - \frac{\epsilon}{10B}.$$

Combining these results we have

$$\begin{aligned}
&\left|\frac{\partial}{\partial b_q}H_x(b_q, 0; p^N) - \frac{\partial}{\partial b_q}\tilde{H}_x^{\lambda^N}(b_q, 0)\right| \\
\leq\ & \mathbb{E}_{\tilde{\gamma}_z^N}\left[L_z\left(b_q, \frac{\tilde{R}_-^N}{\tilde{Q}_-^N + b_q}, \frac{\tilde{Q}_-^N\tilde{R}_-^N}{(\tilde{Q}_-^N + b_q)^2}\right) - L_z(b_q, p^N, p^N)\right] + (1 - \tilde{\phi}_x^N[z])B \\
<\ & [\frac{\epsilon}{10} + \frac{\epsilon}{10B}(2B)] + [\epsilon/10B]B < \epsilon.
\end{aligned}$$

This establishes (51).

Now we complete the proof of Claim 3 for $\tilde{\beta}^{N,A}$. Let $Q(b_q, b_r; p) = \bar{q}' - b_q + \frac{b_r}{p}$. It is straightforward to check that there exists a $D_1 > 0$ such that

$$|b_q - b_q'| + |b_r - b_r'| < D_1|Q(b;p) - Q(b';p)| \text{ for all } (b, b', p) \in \mathcal{O}^2 \times \left[\frac{\kappa_r}{\bar{q}' + \kappa_q}, \frac{\kappa_r + \bar{r}'}{\kappa_q}\right].$$

Also, letting

$$M(q;p) = u_q(q, p\bar{q}' + \bar{r}' - pq; x, z) - u_r(q, p\bar{q}' + \bar{r}' - pq; x, z)p,$$

we have

$$\frac{\partial}{\partial b_q} H_x(b_q, 0; p^N) = -M(Q(b_q, 0; p^N); p^N) \text{ and } \frac{\partial}{\partial b_r} H_x(0, b_r; p^N) = M(Q(0, b_r; p^N); p^N)/p^N.$$

Now, let D_2 satisfies

$$\begin{aligned} 1/D_2 &= -\max\{u_{qq}(q,r;x,z) - 2pu_{qr}(q,r;x,z) + p^2 u_{rr}(q,r;x,z) : \\ &\quad (q,r,p) \in \left[d, \bar{q}' + \frac{\bar{r}'(\bar{q}' + \kappa_q)}{\kappa_r}\right] \times \left[d, \bar{r}' + \frac{\bar{q}'(\bar{r}' + \kappa_r)}{\kappa_q}\right] \times \left[\frac{\kappa_r}{\bar{q}' + \kappa_q}, \frac{\kappa_r + \bar{r}'}{\kappa_q}\right]\}, \end{aligned}$$

where u_{qq}, u_{qr}, and u_{rr} denote second-order derivatives of u. Because u is strictly concave and continuously twice differentiable, D_2 is well-defined and $D_2 > 0$. Moreover,

$$M'(q; p^N) = u_{qq}(q,r;x,z) - 2p^N u_{qr}(q,r;x,z) + (p^N)^2 u_{rr}(q,r;x,z)$$

with $r = p^N \bar{q}' + \bar{r}' - p^N q$. Hence, $M'(q;p^N) < -1/D_2$ for all $q = Q(b;p^N)$ with $(b, p^N) \in \mathcal{O}^2 \times \left[\frac{\kappa_r}{\bar{q}'+\kappa_q}, \frac{\kappa_r+\bar{r}'}{\kappa_q}\right]$.

Because the offer $\chi(x;p^N)$ is a "price-taking" offer, it satisfies the first-order conditions at equality, i.e., $M(Q(\chi(x;p^N);p^N);p^N) = 0$. Therefore, by the Mean Value Theorem, for any $\epsilon > 0$, if $|M(\bar{q} - b_q + \frac{b_r}{p^N}; p^N)| < \epsilon/D_1 D_2$ with $b_q b_r = 0$, then

$$|b_q - \chi_q(x;p^N)| + |b_r - \chi_r(x;p^N)| < \epsilon. \tag{54}$$

Let $D = 2D_1 D_2 \frac{\bar{r}'+\kappa_r}{\kappa_q}$. Then, for any $p^N \in \left[\frac{\kappa_r}{\bar{q}'+\kappa_q}, \frac{\kappa_r+\bar{r}'}{\kappa_q}\right]$, $p^N D_1 D_2 < D$.

Now, let $N^3(\epsilon) = N^2(\epsilon/D)$ and $\delta^3(\epsilon) = \delta^2(\epsilon/D)$, where N^2 and δ^2 are given by (51). Suppose that $N > N^2(\epsilon)$ and that $\left|\frac{\lambda^N(Y_k)}{M^N} - \mu_z(Y_k)\right| < \delta^2(\epsilon)$ for all k. We consider three cases.

(a) $\tilde{\beta}_q^{N,A}(x) > 0$.

Then, $\frac{\partial}{\partial b_q}\tilde{H}_x^{\lambda^N}(\tilde{\beta}_q^{N,A}(x), 0) = 0$. By (51), we have $\left|\frac{\partial}{\partial b_q} H_x(\tilde{\beta}_q^{N,A}(x), 0; p^N)\right| < \epsilon/D$ and hence $|M(Q(\tilde{\beta}_q^{N,A}(x), 0; p^N); p^N)| < \epsilon/D$. This, by (54), implies that

$$|\tilde{\beta}_q^{N,A}(x) - \chi_q(x;p^N)| + |\chi_r(x;p^N)| < \epsilon.$$

(b) $\tilde{\beta}_r^{N,A}(x) > 0$.

Then, $\frac{\partial}{\partial b_r}\tilde{H}_x^{\lambda^N}(0, \tilde{\beta}_r^{N,A}(x)) = 0$. By (51), we have $\left|\frac{\partial}{\partial b_r} H_x(0, \tilde{\beta}_r^{N,A}(x); p^N)\right| < \epsilon/D$ and hence $|M(Q(\tilde{\beta}_q^{N,A}(x), 0; p^N))/p^N| < \epsilon/D$. This, (54), and $D/p^N > D_1 D_2$ for all $p^N \in \left[\frac{\kappa_r}{\bar{q}+\kappa_q}, \frac{\kappa_r+\bar{r}}{\kappa_q}\right]$ imply

$$|\chi_q(x;p^N)| + |\tilde{\beta}_r^{N,A}(x) - \chi_r(x;p^N)| < \epsilon.$$

(c) $\tilde{\beta}_q^{N,A}(x) = 0 = \tilde{\beta}_r^{N,A}(x)$.

Then, $\frac{\partial}{\partial b_q}\tilde{H}_x^{\lambda^N}(0,0) \leq 0$ and $\frac{\partial}{\partial b_r}\tilde{H}_x^{\lambda^N}(0,0) \leq 0$. By (51), we have

$$-M(Q(0,0;p^N);p^N) = \frac{\partial}{\partial b_q}H_x(0,0;p^N) < \frac{\partial}{\partial b_q}H_x^{\lambda^N}(0,0) + \epsilon/D \leq \epsilon/D$$

and

$$M(Q(0,0;p^N);p^N)/p^N = \frac{\partial}{\partial b_r}H_x(0,0;p^N) < \frac{\partial}{\partial b_r}H_x^{\lambda^N}(0,0) + \epsilon/D \leq \epsilon/D.$$

It then follows that $|M(Q(0,0;p^N);p^N)| < \epsilon/D_1D_2$ and hence

$$|\chi_q(x;p^N)| + |\chi_r(x;p^N)| < \epsilon.$$

This concludes the proof of Claim 3 for $\tilde{\beta}^{N,A}$.

The argument is identical for $\hat{\beta}^{N,A}$, except that we need an additional argument to show that for any $\epsilon > 0$, there exists $N^2(\epsilon)$ and $\delta^2(\epsilon)$ such that if $N > N^2(\epsilon)$ and if $\left|\frac{\lambda^N(Y_k)}{M^N} - \mu_z(Y_k)\right| < \delta^2(\epsilon)$ for all k, then for all $b_q \in [0, \bar{q}' - d]$ and all $b_r \in [0, \bar{r}' - d]$,

$$\left|\frac{\partial}{\partial b_q}H_x(b_q,0;p^N) - \frac{\partial}{\partial b_q}\hat{H}_x^{\lambda^N}(b_q,0)\right| < \epsilon \text{ and } \left|\frac{\partial}{\partial b_r}H_x(0,b_r;p^N) - \frac{\partial}{\partial b_r}\hat{H}_x^{\lambda^N}(0,b_r)\right| < \epsilon. \tag{55}$$

Although (55) is completely analogous to (51), an additional argument is required because $\tilde{\beta}^{N,A}$ appears in $(\hat{Q}_-^N, \hat{R}_-^N)$, while p^N only involves $\hat{\beta}^{N,A}$.

Notice that

$$p^N - \frac{\hat{R}_-^N}{\hat{Q}_-^N} = \frac{\sum_{y\in X}\mu_z(y)\hat{\beta}_r^{N,A}(y) + \kappa_r}{\sum_{y\in X}\mu_z(y)\hat{\beta}_q^{N,A}(y) + \kappa_q} - \frac{\sum_{k=1,..,K}\sum_{y\in Y_k}\frac{\lambda_-^N(Y_k)}{M^N}\hat{\rho}^N(y)\hat{\beta}_r^{N,A}(y) + \frac{1}{M^N}\tilde{\beta}_r^{N,A}(\tilde{\xi}) + \kappa_r}{\sum_{k=1,..,K}\sum_{y\in Y_k}\frac{\lambda_-^N(Y_k)}{M^N}\hat{\rho}^N(y)\hat{\beta}_q^{N,A}(y) + \frac{1}{M^N}\tilde{\beta}_q^{N,A}(\tilde{\xi}) + \kappa_q}.$$

Therefore,

$$\left|p^N - \frac{\hat{R}_-^N}{\hat{Q}_-^N}\right| \leq \frac{2(\bar{q}' + \kappa_q)(\bar{r}' + \kappa_r)}{\kappa_q^2}\left[\sum_{k=1}^{K}\left|\mu_z(Y_k) - \frac{\lambda_-^N(Y_k)}{M^N}\hat{\rho}^N(y)\right| + \left|\frac{\bar{q}}{M^N} + \frac{\bar{r}}{M^N}\right|\right].$$

Also,

$$\left|\frac{\hat{Q}_-^N\hat{R}_-^N}{(\hat{Q}_-^N + b_q)^2} - \frac{\hat{R}_-^N}{\hat{Q}_-^N}\right| \leq \frac{\bar{r}' + \kappa_r}{\kappa_q}\frac{\bar{q}'(2\kappa_q + \frac{\bar{q}'}{M^N})}{\kappa_q^2 M^N} \text{ and } \left|\frac{\hat{R}_-^N}{\hat{Q}_-^N + b_q} - \frac{\hat{R}_-^N}{\hat{Q}_-^N}\right| \leq \frac{\bar{r}' + \kappa_r}{\kappa_q}\frac{\bar{q}'}{\kappa_q M^N}.$$

Thus, for any $\epsilon > 0$ there exist $\tilde{N}(\epsilon)$ and $\tilde{\delta}(\epsilon) \leq \epsilon$ such that if $N > \tilde{N}(\epsilon)$ and if $\left|\hat{\rho}^N(y) - \frac{\mu_z(y)}{\mu_z(Y_k)}\right| < \tilde{\delta}(\epsilon)$ and $\left|\frac{\lambda^N(Y_k)}{M^N} - \mu_z(Y_k)\right| < \tilde{\delta}(\epsilon)$, then

$$\left|p^N - \frac{\hat{R}_-^N}{\hat{Q}_-^N}\right| < \epsilon, \; \left|\frac{\hat{Q}_-^N\hat{R}_-^N}{(\hat{Q}_-^N + b_q)^2} - p^N\right| < \epsilon, \text{ and } \left|\frac{\hat{R}_-^N}{\hat{Q}_-^N + b_q} - p^N\right| < \epsilon$$

for all b_q. Given these results, the rest of the argument is exactly the same as for $\tilde{\beta}^{N,A}$.□

Claim 4. For any $\epsilon > 0$, there exists $N^4(\epsilon)$ such that if $N > N^4(\epsilon)$, then for all $x \in X$,

$$|F_x^N(A) - F_x^N(s_2^N(x))| < \epsilon \text{ for any } A \subseteq X, \tag{56}$$

where, recall, $F_x^N(A)$ is the expected payoff from offer a that is associated with the set A at stage-2 conditional on being active.

Proof. Consider a state z. First we show that for any $\epsilon > 0$, there exist $N^5(\epsilon)$ and $\delta^5(\epsilon)$ such that if $N \geq N^5(\epsilon)$ and if $\left|\lambda^N(Y_k)/M^N - \mu_z(Y_k)\right| < \delta^5(\epsilon)$, then for each $x \in X$,

$$\| \hat{\beta}^{N,A}(x) - \beta^{z,\kappa}(x) \| < \epsilon \text{ and } \| \tilde{\beta}^{N,A}(x) - \beta^{z,\kappa}(x) \| < \epsilon, \tag{57}$$

where $\| b - b' \| = |b_q - b'_q| + |b_r - b'_r|$ for all $b, b' \in \mathcal{O}$, and recall that $\beta^{z,\kappa}$ is the offer corresponding to the CE in $\mathcal{L}^z(\kappa)$. We establish (57) for $\hat{\beta}^{N,A}$ and a fixed state z^*. The other case is exactly the same.

If we set

$$\tilde{\delta} = (\tilde{\delta}_q, \tilde{\delta}_r) = \sum_{x\in X} \mu_{z^*}(x)[\beta^N(x) - \chi(x; p^N)],$$

then, by construction, $(\chi(x; p^N))_{x\in X}$ satisfies

$$p^N = \frac{\sum_{x\in X} \mu_{z^*}(x)\chi_r(x; p^N) + \kappa_r + \tilde{\delta}_r}{\sum_{x\in X} \mu_{z^*}(x)\chi_q(x; p^N) + \kappa_q + \tilde{\delta}_q}.$$

That is, $(\chi(x; p^N))_{x\in X}$ is an offer corresponding to the unique CE in the limit economy with $\mu_{z^*}(x)$ proportion of type-x agents, each with endowment $(\overline{q}', \overline{r}')$, and with exogenous supplies $(\kappa_q + \tilde{\delta}_q, \kappa_r + \tilde{\delta}_r)$. Moreover, the CE is continuous in $\tilde{\delta}$. Thus, there exists a $\delta^P(\epsilon) \leq \epsilon$ such that if $\max\{|\tilde{\delta}_q|, |\tilde{\delta}_r|\} \leq \delta^P(\epsilon)$, then

$$\|\beta^{z^*,\kappa}(x) - \chi(x; p^N)\| < \epsilon \text{ for all } x \in X. \tag{58}$$

Now for any $\epsilon > 0$, let $\delta' = \delta^P(\epsilon/2)$ and let $\delta^5(\epsilon) = \delta^3(\delta')$. Let $N^5(\epsilon) = N^3(\delta')$. By Claim 3, if $\left|\frac{\lambda^N(Y_k)}{M^N} - \mu_z(Y_k)\right| < \delta^3(\delta')$ for all k and if $N > N^3(\delta')$, then

$$\|\tilde{\beta}^{N,A}(x) - \chi(x; p^N)\| < \delta' \leq \frac{\epsilon}{2} \text{ for all } x \in X.$$

This then implies that

$$|\tilde{\delta}_q| + |\tilde{\delta}_r| \leq \sum_{x\in X} \mu_{z^*}(x)|\tilde{\beta}_q^{N,A}(x) - \chi_q(x; p^N)| + \sum_{x\in X} \mu_{z^*}(x)|\tilde{\beta}_r^{N,A}(x) - \chi_r(x; p^N)| < \delta' = \delta^P(\epsilon/2).$$

By (58), this implies that $\|\beta^{z,\kappa}(x) - \chi(x; p)\| < \epsilon/2$. Thus, for each x,

$$\|\tilde{\beta}^{N,A}(x) - \beta^{z^*,\kappa}(x)\| \leq \|\tilde{\beta}^{N,A}(x) - \chi(x; p^N)\| + \|\beta^{z^*,\kappa}(x) - \chi(x; p^N)\| < \epsilon,$$

which is (57).

Let $\bar{\gamma}_z^N$ be the probability distribution over other active agents' types conditional on state z; that is, $\bar{\gamma}_z^N[\xi_1, ..., \xi_{M^N-1}] = \prod_{t=1}^{M^N} \mu_z(\xi_t)$. For any nonempty $A \subseteq X$,

$$F_x^N(A) = \sum_{z \in Z} \tau_x[z] \mathbb{E}_{\bar{\gamma}_z^N} \left[u \left(\bar{q}' + \frac{\tilde{\beta}_r^{N,A}(x) Q_-^N - \tilde{\beta}_q^{N,A}(x) R_-^N}{\tilde{\beta}_r^{N,A}(x) + R_-^N}, \bar{r}' + \frac{\tilde{\beta}_q^{N,A}(x) R_-^N - \tilde{\beta}_r^{N,A}(x) Q_-^N}{\tilde{\beta}_q^{N,A}(x) + Q_-^N}; x, z \right) \right],$$

where

$$(Q_-^N, R_-^N) = \sum_{y \in X} \sigma(y) \hat{\beta}^{N,A}(y) + M^N \kappa \text{ and } \sigma(y) = \#\{\xi_t : t = 1, ..., M^N - 1, \xi_t = y\}.$$

By the *law of large numbers*, $\sigma(y)/M^N$ converges to $\mu_z(y)$ in probability under $\bar{\gamma}_z^N$. As a result, $\lambda^N(Y_k)/M^N$ converges to $\mu_z(Y_k)$ in probability under $\bar{\gamma}_z^N$. Therefore, by (57), for any $\epsilon' > 0$ there exists $N^z(\epsilon')$ such that if $N > N^z(\epsilon')$, then

$$\bar{\gamma}_z^N \left[\left| \left(\frac{\tilde{\beta}_r^{N,A}(x) Q_-^N - \tilde{\beta}_q^{N,A}(x) R_-^N}{\tilde{\beta}_r^{N,A}(x) + R_-^N} \right) - \left(\frac{\beta_r^{z,\kappa}(x)}{p^{z,\kappa}} - \beta_q^{z,\kappa}(x) \right) \right| < \epsilon' \right] > 1 - \epsilon'$$

and

$$\bar{\gamma}_z^N \left[\left| \left(\frac{\tilde{\beta}_q^{N,A}(x) R_-^N - \tilde{\beta}_r^{N,A}(x) Q_-^N}{\tilde{\beta}_q^{N,A}(x) + Q_-^N} \right) - \left(\beta_q^{z,\kappa}(x) p^{z,\kappa} - \beta_r^{z,\kappa}(x) \right) \right| < \epsilon' \right] > 1 - \epsilon'.$$

With appropriate ϵ''s, this implies that

$$\left| F_x^N(A) - \sum_{z \in Z} \tau_x[z] u \left(\bar{q}' + \frac{\beta_r^{z,\kappa}(x)}{p^{z,\kappa}} - \beta_q^{z,\kappa}(x), \bar{r}' + \beta_q^{z,\kappa}(x) p^{z,\kappa} - \beta_r^{z,\kappa}(x); x, z \right) \right| < \epsilon.$$

Claim 4 follows from the fact that there are only finitely many nonempty $A \subseteq X$. □

Now we complete the proof. Recall that we begin with a candidate semi-pooling equilibrium associated with the partition $\mathcal{Y} = \{Y_1, ..., Y_K\}$ with $1 < K < |X|$. Because such an equilibrium does not exist when $|X| = 2$, we may assume that for some $y^1, y^2 \in X$, $y^1 \neq y^2 \in Y_1$. We use $\alpha(x, \kappa)$ to denote the stage-2 offer that corresponds to stage-2 consumption $\widehat{q}(x, \kappa)$ for all x. Recall that $G_x(a)$ is the objective function for a type-x agent at stage-2 conditional on being inactive (see (30)). Because $y^1 \neq y^2$, there exists $C > 0$ such that for any $a \in \mathcal{O}$, either $G_{y^1}(a) < G_{y^1}(\alpha(y^1, \kappa)) - C$ or $G_{y^2}(a) < G_{y^2}(\alpha(y^1, \kappa)) - C$. Assume without loss of generality that $G_{y^1}(s_1^N(Y_1)) < G_{y^1}(\alpha(y^1, \kappa)) - C$ so that $s_2^N(y^1) \neq \alpha(y^1, \kappa)$. Consider then a potential defector of type y_1.

Let $\bar{N} = N^4(\frac{\eta C}{2(1-\eta)})$. Then, by Claim 4, if $N \geq \bar{N}$,

$$|F_{y^1}^N(s_2^N(Y_1)) - F_{y^1}^N(\alpha(y^1, \kappa))| < \frac{\eta C}{2(1-\eta)} \text{ and } |F_{y^2}^N(s_2^N(Y_1)) - F_{y^2}^N(\alpha(y^2, \kappa))| < \frac{\eta C}{2(1-\eta)}.$$

Then, for $N \geq \bar{N}$,

$$\begin{aligned}
& \eta G_{y^1}(s_1^N(Y_1)) + (1-\eta) F_{y^1}^N(s_2^N(Y_1)) \\
< \; & \eta (G_{y^1}(\alpha(y^1, \kappa)) - C) + (1-\eta)(F_{y^1}^N(\alpha(y^1, \kappa)) + \frac{\eta C}{2(1-\eta)}) \\
= \; & \eta G_{y^1}(\alpha(y^1, \kappa)) + (1-\eta) F_{y^1}^N(\alpha(y^1, \kappa)) - \frac{\eta C}{2}.
\end{aligned}$$

Hence, deviating from $s_2^N(y^1)$ to $\alpha(y^1, \kappa)$ is profitable, a contradiction. This shows that s^N is a fully-revealing equilibrium.■

Proposition 4. Assume *U1-U5* and suppose that the competitive demand for good-q is monotone in $\mathcal{L}^z$ for each $z \in Z$. Then, there exists $\bar{N}$ such that for all $N > \bar{N}$, any symmetric PBE in pure strategies $s^N = (s_1^N, s_2^N, s_3^N)$ is fully revealing.

Proof. Under U4, (s_2^N, s_3^N) describes equilibrium behavior conditional on full participation, and, by Proposition 3, we have the required uniqueness conditional on full-participation for N large. It remains to establish that full-participation is the unique stage-1 equilibrium.

An agent at stage-1 considers two alternatives. Either everyone else participates or not. If not, then the agent is indifferent between $\{yes, no\}$ because both leave the agent with the endowment $(\overline{q}, \overline{r})$. If everyone else participates, then, according to U5, the agent computes an expected payoff from participating from the unique continuation equilibrium of Proposition 3. As shown in the final step of the proof of Proposition 1, for N large that payoff exceeds the utility of the endowment $(\overline{q}, \overline{r})$, the payoff from playing *no*. Therefore, playing *no* is a weakly dominated action in the sense of U5.■

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