Booker, A. R. (2018). Finite connected components of the aliquot graph. *Mathematics of Computation*, *87*(314), 2891-2902. https://doi.org/10.1090/mcom/3299

FINITE CONNECTED COMPONENTS OF THE ALIQUOT GRAPH

ANDREW R. BOOKER

Abstract. Conditional on a strong form of the Goldbach conjecture, we determine all finite connected components of the aliquot graph containing a number less than 10^9, as well as those containing an amicable pair below 10^{14} or one of the known perfect or sociable cycles below 10^{17}. Along the way we develop a fast algorithm for computing the inverse image of an even number under the sum-of-proper-divisors function.

1. Introduction

For $n \in \mathbb{N}$, let $s(n) = \sum_{\substack{d|n \\ d \neq n}} d$ denote the sum of the proper divisors of n. Ancient Greek mathematicians studied the forward orbits n, $s(n)$, $s(s(n))$, $\ldots$, now called *aliquot sequences*, and noted that they sometimes enter cycles, such as 6, 6, $\ldots$ and 220, 284, 220, $\ldots$. In the modern computer era, more than a billion examples of such *aliquot cycles* have been found [3, 9]; most of these, like $\{220, 284\}$, have order 2, and are termed *amicable pairs*. A long-standing conjecture posits that there are infinitely many aliquot cycles.

One can also ask about the inverse orbits $\{n\} \cup s^{-1}(\{n\}) \cup s^{-1}(s^{-1}(\{n\})) \cup \cdots$. Although questions concerning the inverse image $s^{-1}(\{n\})$ of a given n go back at least 1000 years [13], the idea of iterating the inverse map appears to have been considered only recently (see [5, Theorem 5.3] and [4], for instance). In relation to this, Garambois [7] has conducted many numerical studies, focusing in particular on *isolated cycles*, i.e. cycles that are their own inverse orbits. For instance, $s^{-1}(\{28\}) = \{28\}$, so $\{28\}$ is an isolated cycle of order 1.

In this paper, we seek to generalize this concept. To do so, following Delahaye [4], we introduce the *aliquot graph*, which packages all of the aliquot sequences together into a single directed graph. Precisely, every natural number is a node of the graph, and for any $m, n \in \mathbb{N}$, there is a directed edge from m to n if and only if $n = s(m)$. As the examples noted above demonstrate, the aliquot graph is not connected; in fact any two distinct aliquot cycles lie in distinct connected components, so presumably the graph has infinitely many components. In these terms, we see that Garambois' isolated cycles are examples of finite connected components.

Our objectives are (1) to find examples of finite connected components beyond simple cycles, and (2) to determine a comprehensive list of all finite connected components with at least one small node. Toward the first objective, in Section 2 we present an algorithm for computing the inverse image $s^{-1}(\{n\})$ of a given even number n in time $O(n^{1/2+\varepsilon})$; as a corollary, we obtain the estimate $\#s^{-1}(\{n\}) \ll n^{1/2+\varepsilon}$, which improves on a recent result of Pomerance [12, Corollary 3.6]. In Section 3.1 we apply the algorithm to all even amicable pairs with smaller element below 10^{14}, and to all known[1] aliquot cycles of order $\neq 2$ with smallest element below 10^{17}. In this way we identify many interesting examples of finite connected components.

Concerning the second objective, note first that if p and q are distinct primes then $s(pq) = p + q + 1$. As a slight strengthening of the Goldbach conjecture, we have the following:

Hypothesis G. *Every even number at least* 8 *is the sum of two distinct primes.*

The author was partially supported by EPSRC Grant EP/K034383/1.

[1]The list of known cycles is likely complete up to at least 10^{14}. However, there are many open-ended aliquot sequences beginning with a number below that bound, so it is impossible to say for sure that the list is complete without imposing an upper bound on the cycle length. It might even be the case that the completeness of the list is undecidable and cannot be certified with a finite computation!

There is ample evidence in favor of Hypothesis G: Lu [8] showed that it holds for all but at most $O(x^{0.879})$ even numbers $\le x$, and Oliveira e Silva et al. [10] ran a large distributed computation to verify it for all even $n \in [8, 4 \times 10^{18}]$.[2]

Assuming Hypothesis G, for any odd number $n \ge 9$, we have $n = p + q + 1 = s(pq)$ for distinct odd primes p, q. Since $pq > n$ and is again odd, we can repeat this construction to see that the inverse orbit $\{n\} \cup s^{-1}(\{n\}) \cup s^{-1}(s^{-1}(\{n\})) \cup \cdots$ is infinite; in particular, n has infinite connected component. Note also that $1 = s(11)$, $3 = s(s(9))$ and $7 = s(s(49))$. Thus, under Hypothesis G, every odd number except 5 has infinite inverse orbit and, since $s(5) = 1$, every odd number has infinite connected component.[3] Unconditionally, Erdős et al. [5, Theorem 5.3] showed that infinite inverse orbits exist; in fact all but a density zero subset of the odd numbers have infinite inverse orbit, although the method of proof does not enable one to exhibit a specific such number.

We say that a connected component of the aliquot graph is *potentially infinite* if it contains an odd number. Absent a proof of Hypothesis G (including the Goldbach conjecture), we cannot prove that a given potentially infinite connected component is actually infinite, unless it is shown to contain 1. However, we will take Hypothesis G for granted in what follows, so our numerical results will be conditional upon it. With this caveat, in Section 3.2 we describe a computation determining the complete list of finite connected components of the aliquot graph that contain a number below 10^9.

Finally, in Section 4 we conclude with some related questions and speculations suggested by the numerics.

Notation. We shall make frequent use of the following symbols for arithmetic functions:

$$\omega(n) = \sum_{\substack{p \mid n \\ p \text{ prime}}} 1 \quad \text{is the number of distinct prime factors of } n,$$

$$\Omega(n) = \sum_{p^k \| n} k \quad \text{is the number of prime factors of } n\text{, counted with multiplicity,}$$

$$\sigma_k(n) = \sum_{d \mid n} d^k \quad \text{is the sum of } k\text{th powers of the divisors of } n,$$

$$\sigma(n) = \sigma_1(n) = s(n) + n.$$

Acknowledgements. This paper was inspired by the work of Jean-Luc Garambois [7], as well as posts by David Stevens and another user, who wishes to remain anonymous, on `mersenneforum.org`. I thank them for raising interesting questions. I also thank Carl Pomerance for helpful comments and for pointing out the related results in [5] and [12].

2. An algorithm for s^{-1}

Suppose that $n \in \mathbb{N}$ is given, and we wish to find $m \in \mathbb{N}$ satisfying $s(m) = n$. If $n \ge 9$ is odd, then searching through small primes p, we expect to find one quickly (polynomial time in $\log n$) such that $q = n - 1 - p$ is prime, so that $n = s(pq)$; although a proof of this seems far off, that does not prevent it from working well in practice to find an element of $s^{-1}(\{n\})$, even for very large odd n. On the other hand, it is conjectured that all large odd n have $\gg n/\log^2 n$ representations as $p + q + 1$ (and this certainly holds for at least some arbitrarily large n, by the prime number theorem and pigeonhole principle), and it follows that no algorithm can compute all of $s^{-1}(\{n\})$ in fewer than $O(n/\log n)$ bit operations. In light of this, and since our application requires only even values, we assume henceforth that n is even.

Let us first consider the possibility of odd m. If $n \in 2\mathbb{N}$ and $m \in 1 + 2\mathbb{N}$, then it is easy to see that m must be a square. Let p be the largest prime factor of m, and write $m = a^2 p^{2k}$, with $p \nmid a$. Then we have

$$n = s(m) = s(a^2)p^{2k} + \sigma(a^2)(1 + p + \ldots + p^{2k-1}), \tag{2.1}$$

[2] Strictly speaking, they only verified the Goldbach conjecture, which is weaker than Hypothesis G for numbers of the form $2p$ for p prime. However, for every even $n \in [6, 4 \times 10^{18}]$, they found a Goldbach partition $n = p + q$ with $p \le 9781$. Hence, it suffices to verify Hypothesis G for $n = 2p$ for all primes $p \in [5, 9781]$.

[3] Note that $1 = s(p)$ for every prime p, so its connected component is trivially infinite under our definition. Some authors prefer to exclude 1 from the aliquot graph to avoid this triviality. Fortunately, under Hypothesis G, the only difference that this makes to our question of finite connected components is that 2 and 5 become singleton components.

so that $a^2 \le n/(1+\ldots+p^{2k-1})$ and $k \le \frac{1}{2}[1+\log_p(n/\sigma(a^2))]$. For $a=1$ and each odd $a \in [3,\sqrt{n/6}]$, we run through all $k \le \frac{1}{2}[1+\log_q(n/\sigma(a^2))]$, where q is the smallest odd number ≥ 3 exceeding every prime factor of a, perform a binary search for integral $p \ge q$ satisfying (2.1), and apply a primality test. (For our implementation, which was limited to $n < 2^{64}$, we used a strong Fermat test to base 2, together with the classification [6] of small strong pseudoprimes.)

Next we consider $m \in 2\mathbb{N}$. In this case, since $m/2$ is a proper divisor of m, we have $s(m) \ge m/2$, whence $m \le 2n$. We write m in the form ab, where we think of $a \in 2\mathbb{N}$ as the "smooth" part of m, with only small prime factors, and b as the rest. Then we have

$$n = s(m) = \sigma(a)s(b) + s(a)b. \tag{2.2}$$

For a fixed choice of a, we view (2.2) as a linear equation constraining the pair $(s(b), b)$. First note that if $g = \gcd(\sigma(a), s(a)) = \gcd(a, s(a))$, then (2.2) has no solutions unless $g \mid n$. When $g \mid n$, we put $u = \sigma(a)/g$ and $v = s(a)/g$, so that $(x, y) = (s(b), b)$ is a solution to $ux + vy = n/g$. Using the Euclidean algorithm, we can determine a particular solution $(x_0, y_0) \in \mathbb{Z}^2$, and the general solution in positive integers is then $(x, y) = (x_0 + rv, y_0 - ru)$ for $r \in \mathbb{Z} \cap (-\frac{x_0}{v}, \frac{y_0}{u})$. Our algorithm proceeds by working recursively through all possible prime factorizations of a. For the base case of the recursion, once the number of possibilities for b is small enough, we test all of them to see if the equality $n = s(ab)$ is satisfied.

As described, this method is only a little more efficient than directly considering every possible even $m \le 2n$, but fortunately there are a few ways in which we can reduce the search space. First, we can detect the cases $b = p$ or $b = p^2$ for a prime p by solving (2.2), which gives a linear or quadratic equation for p, and applying a primality test. Second, in the typical case when b has no small prime factors, we can narrow the range for $s(b)$ using the following estimate:

Lemma 2.1. *Let $b > 1$ be an integer with smallest prime factor p. Then $s(b) \in [b/p, b\Omega(b)/p]$.*

Proof. Since b/p is a proper divisor of b, we have $s(b) \ge b/p$, directly from the definition. For the upper bound, let $\prod_{i=1}^{\omega(b)} p_i^{e_i}$ be the prime factorization of b, consider the set

$$S = \bigcup_{i=1}^{\omega(b)} \{p_i, p_i^2, \ldots, p_i^{e_i}\},$$

and write $S = \{q_1, \ldots, q_{\Omega(b)}\}$, with $q_1 < \cdots < q_{\Omega(b)}$ in increasing order. Next set $b_0 = 1$ and $b_j = \operatorname{lcm}(q_1, \ldots, q_j)$ for $j = 1, \ldots, \Omega(b)$. Then

$$\sigma_{-1}(b) = \prod_{j=1}^{\Omega(b)} \frac{\sigma_{-1}(b_j)}{\sigma_{-1}(b_{j-1})}.$$

Consider $j \in \{1, \ldots, \Omega(b)\}$, and suppose that $q_j = p_i^k$. Then $b_j = p_i b_{j-1}$ and

$$\frac{\sigma_{-1}(b_j)}{\sigma_{-1}(b_{j-1})} = \frac{\sigma_{-1}(p_i^k)}{\sigma_{-1}(p_i^{k-1})} = 1 + \frac{1}{p_i + \ldots + p_i^k} \le 1 + \frac{1}{q_j}.$$

Since $q_1 = p$ and the q_j are strictly increasing, we thus have

$$\sigma_{-1}(b) \le \prod_{j=1}^{\Omega(b)} \left(1 + \frac{1}{q_j}\right) \le \prod_{j=1}^{\Omega(b)} \left(1 + \frac{1}{p+j-1}\right) = 1 + \frac{\Omega(b)}{p}.$$

Hence

$$\frac{s(b)}{b} = \sigma_{-1}(b) - 1 \le \frac{\Omega(b)}{p}.$$

□

Although we do not know p in advance, we will know a lower bound for it in the course of the recursion. Supposing that $p \ge p_1$ and that we have already checked the cases $b = 1$, $b = p$ and $b = p^2$, we have

$$b \ge b_1 := p_1 p_1' \quad \text{and} \quad s(b) \ge s_1 := 1 + p_1 + p_1', \tag{2.3}$$

where p_1' denotes the smallest prime exceeding p_1. Thus, defining

$$b_2 = \frac{n - \sigma(a)s_1}{s(a)}, \quad k = \left\lfloor \frac{\log b_2}{\log p_1} \right\rfloor \quad \text{and} \quad s_2 = \frac{kn}{k\sigma(a) + p_1 s(a)}, \tag{2.4}$$

we have

$$s(b)(\sigma(a) + s(a)p_1/k) \le \sigma(a)s(b) + s(a)b = n,$$

so that $b \in [b_1, b_2]$ and $s(b) \in [s_1, s_2]$. We stop the recursion and test every value of b once the number of $(x, y) \in [s_1, s_2] \times [b_1, b_2]$ satisfying $ux + vy = n/g$ falls below p_1.

Third, the solutions with $b = pq$ for distinct primes p and q can also be determined without searching, since in this case we have

$$(vp + u)(vq + u) = v^2pq + uv(p + q) + u^2 = v(n/g - u) + u^2 = (au + nv)/g.$$

Thus, factoring $(au+nv)/g$ and testing all of its divisors $\equiv u \pmod v$ will reveal p and q. Since $(au+nv)/g$ is potentially quite large, this test is more expensive than that for $b = p$ or p^2, so we use it only when p_1 is large enough to guarantee that b is a product of two primes.

Our procedure is described in more detailed pseudocode in Algorithm 2.1. We turn now to the running time analysis. First, by either using a sieve to amortize the factorization of a or working recursively through the possible factorizations, we see that it takes at most $O_\varepsilon(n^{1/2+\varepsilon})$ bit operations to find all odd m with $s(m) = n$. For even m, note that each prime p_1 considered before the recursion is stopped satisfies

$$p_1 \le \#\{r \in \mathbb{Z} : x_0 + rv \in [s_1, s_2] \text{ and } y_0 - ru \in [b_1, b_2]\} \le \frac{s_2}{v} + 1,$$

and together with (2.4) this implies the bound $p_1 \le \frac{\sqrt{gn \log_3 n}}{s(a)}$. To simplify the analysis, we consider a modified version of the algorithm in which we omit the checks for $b = p$, $b = p^2$ and $b = pq$, and stop the recursion once $p_1 > \sqrt{n}/a$. (These simplifications make the algorithm slightly less efficient, but one can see that they increase the running time by a factor of $O_\varepsilon(n^\varepsilon)$ at most.)

Suppose that the recursive procedure is called with input a, and let p denote the largest prime factor of a, with $p^k \parallel a$. Then either $p = 2$ or the criterion for stopping the recursion was not satisfied when considering a/p^k, so that $p \le \sqrt{n}/(a/p^k)$. We may assume that $n \ge 4$, so in either case, writing $f(a) = a/p^{k-1}$, we have $f(a) \le \sqrt{n}$. Note that $f(a)$ is again an even integer with largest prime factor p. Thus,

$$\begin{aligned} \#\{a \in 2\mathbb{N} : a \le 2n,\ f(a) \le \sqrt{n}\} &= \sum_{\substack{t \in 2\mathbb{N} \\ t \le \sqrt{n}}} \#\{a \in 2\mathbb{N} : a \le 2n,\ f(a) = t\} \\ &\le \sum_{\substack{t \in 2\mathbb{N} \\ t \le \sqrt{n}}} \left(1 + \log_2\left(\tfrac{2n}{t}\right)\right) \le \tfrac{1}{2}\sqrt{n}\log_2(2n), \end{aligned}$$

and this gives an upper bound for the number of times that the recursive procedure is called.

Next, let p_1 denote the smallest prime exceeding both $\sqrt{n}/a$ and every prime factor of a. Then by (2.4), the values of b that we consider in the base case of the recursion for a satisfy

$$s(b) < \frac{kn}{p_1 s(a)} \le \frac{n \log_{p_1} n}{p_1 s(a)} \le \frac{n \log_3 n}{(\sqrt{n}/a)(a/2)} = 2\sqrt{n}\log_3 n.$$

Moreover, $s(b)$ is determined modulo $v = s(a)/g$, so the number of possibilities to consider is at most

$$1 + \frac{2\sqrt{n}\log_3 n}{s(a)/g} \le 1 + \frac{4g\sqrt{n}\log_3 n}{a}.$$

Summing over all $g \mid n$ and a satisfying $\gcd(s(a), \sigma(a)) = g$, we see that the total number of candidate values for b is bounded by

$$\begin{aligned} \sum_{g \mid n} \sum_{\substack{a \in 2\mathbb{N} \cap [2,2n] \\ f(a) \le \sqrt{n} \\ \gcd(a, \sigma(a)) = g}} \left(1 + \frac{4g\sqrt{n}\log_3 n}{a}\right) &\le \sum_{\substack{a \in 2\mathbb{N} \cap [2,2n] \\ f(a) \le \sqrt{n}}} 1 + \sum_{g \mid n} \sum_{\substack{a \le 2n \\ g \mid a}} \frac{4\sqrt{n}\log_3 n}{a/g} \\ &\ll \sigma_0(n)\sqrt{n}\log^2 n \ll_\varepsilon n^{1/2+\varepsilon}. \end{aligned}$$

Algorithm 2.1 Procedure to compute $s^{-1}(\{n\})$ for $n \in 2\mathbb{N}$

```
  function S_INVERSE(n)
Input: n ∈ 2ℕ
Output: list of m ∈ ℕ such that s(m) = n
    initialize the output list
    for each a ∈ {1} ∪ [3, √(n/6)] ∩ (1 + 2ℕ) do
        compute s(a^2) and the smallest odd number q ≥ 3 exceeding every prime factor of a
        for each k ∈ ℕ such that q^(2k−1) ≤ n/σ(a^2) do
            solve (2.1) for p
            if p is a prime ≥ q then append a^2 p^(2k) to the output list end if
        end for
    end for
    for each k ∈ ℕ such that 2^k < n do
        call S_INVERSE_EVEN_RECURSION(2^k)
    end for
    return the output list
  end function

  procedure S_INVERSE_EVEN_RECURSION(a)
Input: a ∈ 2ℕ
Ensure: appends to the output list all m = ab such that s(m) = n, b > 1 and the smallest prime factor of
  b exceeds the largest prime factor of a
    compute g = gcd(s(a), σ(a)), and return if g ∤ n
    check for solutions to (2.2) with b = p and b = p^2, and append them to the output list
    compute u = σ(a)/g, v = s(a)/g, and (x_0, y_0) such that u x_0 + v y_0 = n/g
    for primes p_1 exceeding the largest prime factor of a, in increasing order, do
        compute the intervals [s_1, s_2] and [b_1, b_2] defined in (2.3)–(2.4)
        if #{r ∈ ℤ : x_0 + rv ∈ [s_1, s_2] and y_0 − ru ∈ [b_1, b_2]} < p_1 then
            for each such r do
                compute b = y_0 − ru and s(b)
                if every prime factor of b is at least p_1 and s(b) = x_0 + rv then
                    append ab to the output list
                end if
            end for
            return
        end if
        if s(a p_1^3) > n then
            factor N = (au + nv)/g and find all of its divisors d < √N satisfying d ≡ u (mod v)
            for each such d do
                compute p = (d − u)/v and q = (N/d − u)/v
                if p and q are primes ≥ p_1 then append apq to the output list end if
            end for
            return
        end if
        for each k ∈ ℕ such that s(a p_1^k) ≤ n do
            if s(a p_1^k) < n then
                call S_INVERSE_EVEN_RECURSION(a p_1^k)
            else if k ≥ 3 then
                append a p_1^k to the output list
            end if
        end for
    end for
  end procedure
```

The largest prime factor of a and the value of $s(a)$ can be carried along as extra state information during the recursion, so no work is required to factor a. On the other hand, we can expect the b values that arise to occur sparsely throughout $(0, n)$, and we need to factor them in order to compute $s(b)$. In practice, one can use a generic factoring algorithm with good average-case performance. To get a provable estimate for the running time, it suffices to record all of the candidate pairs (a, b) in a list and apply Bernstein's batch factorization algorithm [1] to the b values. Since there are $O_\varepsilon(n^{1/2+\varepsilon})$ pairs and each b is bounded by n, the total time to factor all of them is still $O_\varepsilon(n^{1/2+\varepsilon})$.

Thus, we have shown the following.

Theorem 2.2. *The algorithm described in this section computes $s^{-1}(\{n\})$ for a given $n \in 2\mathbb{N}$ in time at most $O_\varepsilon(n^{1/2+\varepsilon})$.*

Corollary 2.3. *For $n \in 2\mathbb{N}$, $\#s^{-1}(\{n\}) \ll_\varepsilon n^{1/2+\varepsilon}$.*

3. Numerical results

3.1. **Examples of finite connected components.** For any given $n \in \mathbb{N}$, the forward orbit of n under s either terminates with 1, grows without bound, or enters a cycle. In the first two cases, n must have infinite connected component. Hence, to find finite connected components, it suffices to consider only those n contained in a cycle, and compute their inverse orbits. For each even amicable pair with smaller element below 10^{14}, as well as the known perfect or sociable cycles with smallest element below 10^{17}, we started with the smallest n in the cycle and iteratively computed $s^{-1}(\{n\})$, $s^{-1}(s^{-1}(\{n\}))$, $\ldots$ until reaching either the empty set or a set containing an odd number. In the former case, n has finite connected component, and our computation determines it entirely; in the latter case, assuming Hypothesis G, the connected component is infinite.

It is also conceivable that there are n for which neither case occurs, and the procedure does not terminate. However, for any n, the elements of $s^{-1}(\{n\}) \cap 2\mathbb{N}$ are bounded by $2n$, so chains of even numbers in the inverse orbit of n grow at most exponentially in the iteration count. Moreover, for any m of the form $p + 1$ for prime p, we have $m = s(p^2)$. We see no reason why numbers of this form should not occur among the elements of the inverse orbit of n with the same frequency as for random numbers of the same size. Thus, provided that the kth iterate of s^{-1} applied to $\{n\}$ is non-empty, we expect it to contain an odd number with probability $\gg 1/k$. Since the harmonic series diverges, we therefore expect to reach an odd number eventually, as long as the inverse orbit is infinite. This was borne out by our numerics, as every connected component that we considered was found to be either finite or potentially infinite.[4]

Of the 24003 even amicable pairs that we considered, 7438 pairs were found to belong to a finite connected component, and of those, 2394 were isolated cycles. The average size of the components was $37968/7438 \approx 5.1$, and the largest was of size 58, corresponding to the amicable pair $\{29215166389256, 31021462090744\}$; it is shown in Figure 3.1.

For even aliquot cycles of size other than 2, only 75 are known with smallest element below 10^{17}. We found 12 belonging to a finite connected component, of which three are isolated cycles (including the perfect numbers 28 and 137438691328); they are shown in Figure 3.2.

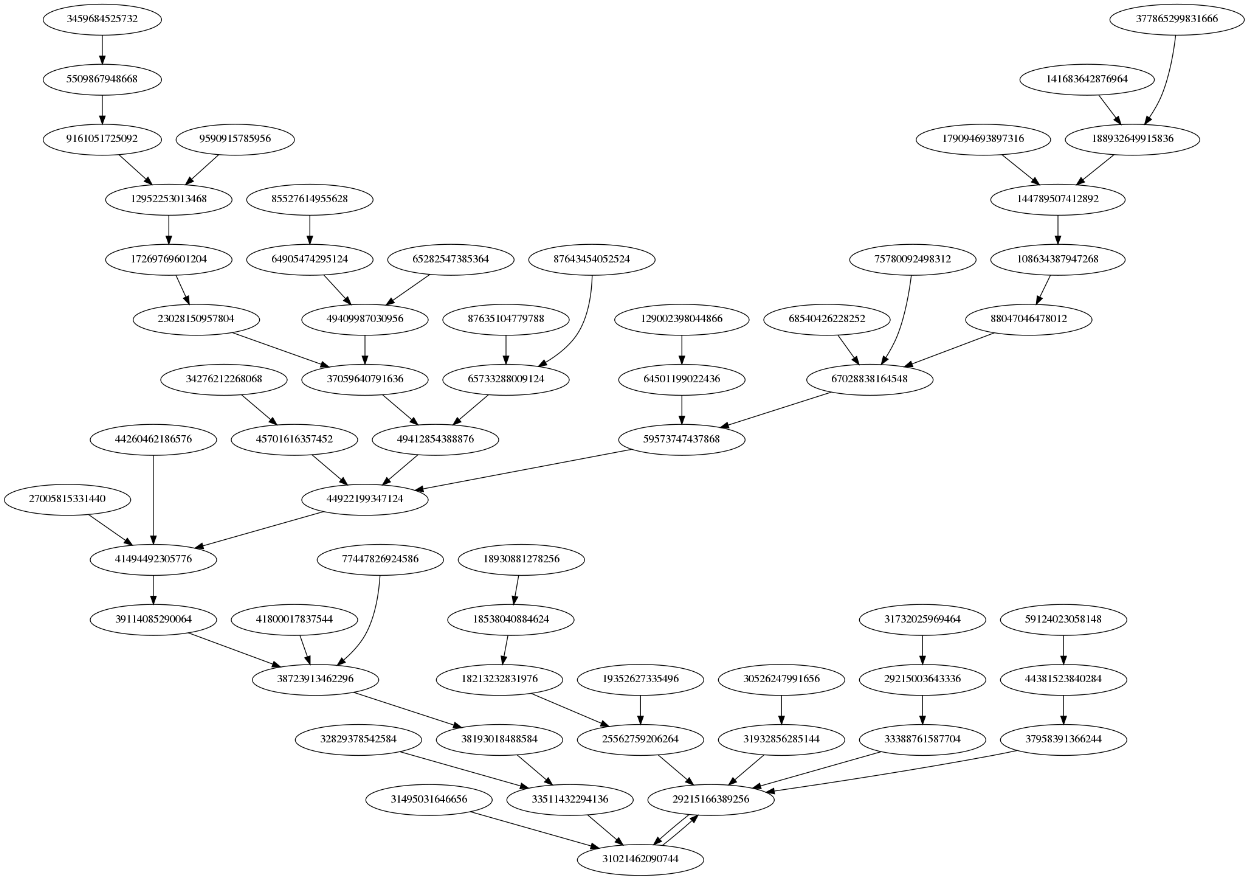
3.2. **Finite connected components containing a small node.** Towards our second objective, we found, conditional on Hypothesis G, the complete list of finite connected components of the aliquot graph containing a number $\leq 10^9$. As it turns out, there are 101 such components, compared to 453 known cycles of even numbers in that range. They are comprised of 462 nodes, 88 of which exceed 10^9. The 14 examples containing a number below 10^7 are shown in Figure 3.3.

Our computation proceeded as follows. First, beginning with each even number $n \leq 10^9$, we used PARI/GP [14] to compute the forward orbit n, $s(n)$, $\ldots$, until arriving at a number $m = s^k(n)$ satisfying one of the following conditions:

(1) m is odd;

[4] However, in the case of the amicable pair $\{48569114359984, 49074636040016\}$, the numbers exceeded the 64-bit limit of our implementation without reaching an odd number. We wrote a special-purpose routine to continue the search in this case, looking for m of the form ap with $a < 2 \times 10^{10}$ and p prime, and fortunately that sufficed to prove that $s^{25}(18471983707171354573^2) = 49074636040016$.

Figure 3.1. The largest finite connected component containing an amicable pair with smaller element below 10^{14}

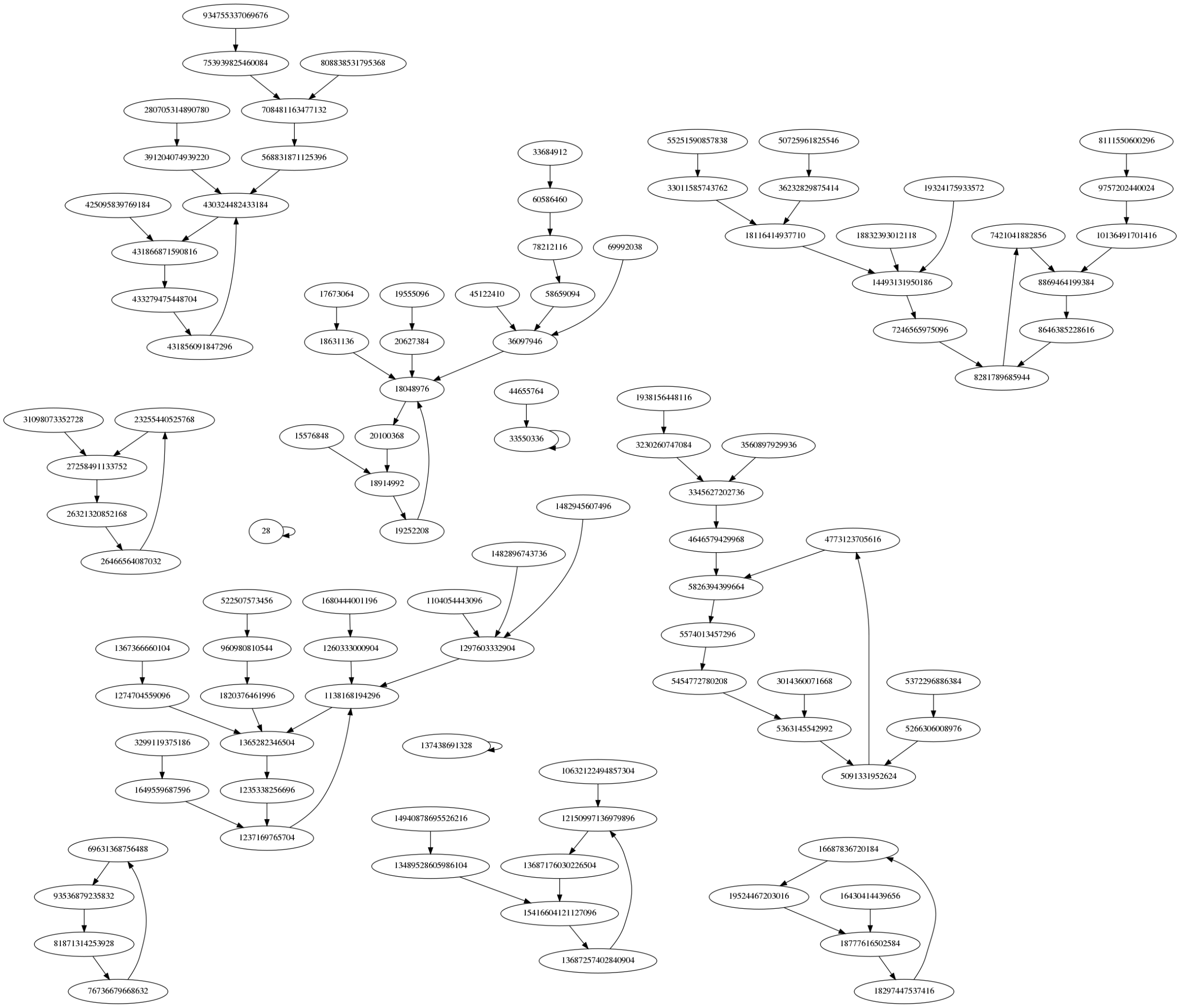


(2) $m - 1$ is prime;
(3) $m = s^j(n)$ for some $j < k$;
(4) $m \geq 10^{50}$.

In the first two cases, n is connected to an odd number (m in case (1), $(m-1)^2$ in case (2)), so its connected component is potentially infinite. In the third case, the forward orbit enters a cycle. We determined the minimum number in each cycle and collated the cycles discovered for all $n \leq 10^9$. It turned out that they were all among the cycles considered in Section 3.1, so we could readily classify each connected component as either finite or potentially infinite. That left 1053 numbers in the indeterminate case (4), to which we applied the algorithm from Section 2 to search for odd numbers in the inverse orbit of n, then of $s(n)$, $s(s(n))$, $\ldots$, until reaching a value of $s^k(n)$ in excess of 2^{48}. With this method we succeeded in finding an odd number for all but nine values of n, whose forward orbits merged into just four distinct aliquot sequences. Finally, we resolved these by continuing the forward orbits with a larger cutoff of 10^{70}.

For each n with potentially infinite connected component, we recorded, as a certificate, an odd number $m \in \mathbb{N}$ and indices $j, k \geq 0$ such that $s^k(n) = s^j(m)$. The interested reader may find these at [2], along with the data pertaining to the finite connected components.

FIGURE 3.2. The finite connected components containing a known cycle of order $\neq 2$ with a node $\leq 10^{17}$



4. Related questions

Recall that a number $n \in \mathbb{N}$ is called *non-aliquot* (or *untouchable*) if $s^{-1}(\{n\}) = \emptyset$. Pollack and Pomerance [11] have conjectured that the non-aliquot numbers have asymptotic density

$$\lim_{y \to \infty} \frac{\sum_{\substack{a \in 2\mathbb{N} \\ a \leq y}} a^{-1} e^{-a/s(a)}}{\sum_{\substack{a \in \mathbb{N} \\ a \leq y}} a^{-1}} \approx 17\%$$

in the natural numbers, and this is supported by the available numerical evidence. Their analysis relies heavily on some heuristics for the typical behavior of s over the natural numbers. The heuristics do not apply to amicable numbers, which are atypical in this respect (e.g., for any amicable number a, the sequence a, $s(a)$, $s(s(a))$ is not monotonic, which is a rare event among all natural numbers). Nevertheless, one can ask whether the amicable pairs that form isolated cycles have a density within the set of all amicable pairs (ordered by smaller element, say). Empirically almost all aliquot cycles have order 2, so this density, if it exists, should agree with that of the isolated cycles among all cycles. Table 4.1 shows the frequency of

Figure 3.3. All finite connected components containing a node $\leq 10^7$

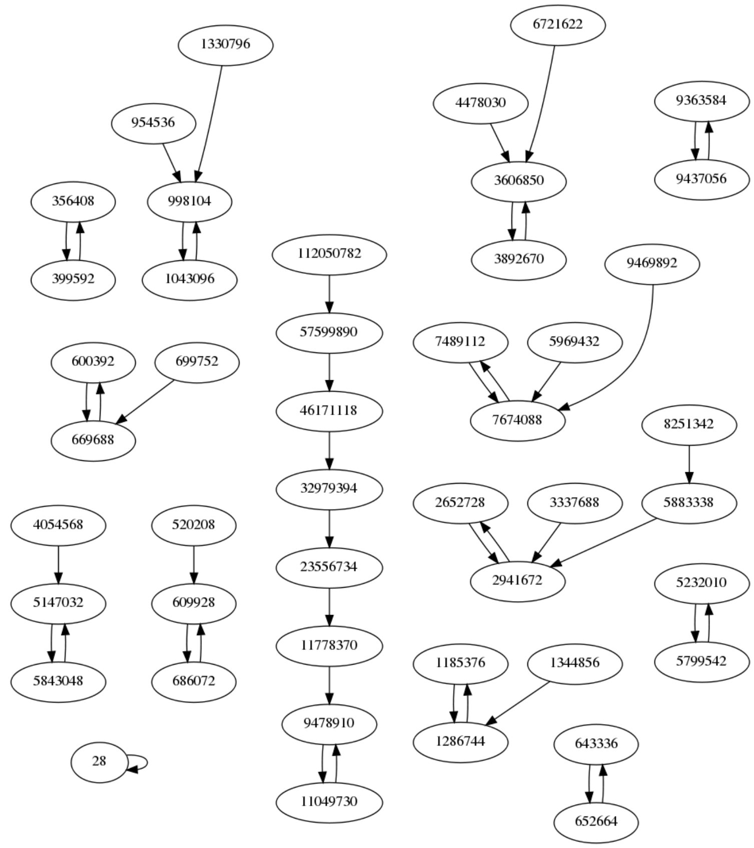


Table 4.1. Frequency of isolated cycles and cycles with finite connected component

x	number of cycles with smallest element $\leq x$	number that are isolated	number with finite connected component
10^{10}	1462	98 (6.70%)	249 (17.03%)
10^{11}	3385	214 (6.32%)	613 (18.11%)
10^{12}	7692	471 (6.12%)	1445 (18.79%)
10^{13}	17583	1052 (5.98%)	3309 (18.82%)
10^{14}	39457	2397 (6.07%)	7448 (18.88%)

isolated cycles among all known cycles in various ranges up to 10^{14}. Based on this limited evidence, we speculate that the limiting density does exist and is approximately 6%.

Similarly, one might ask whether there are infinitely many finite connected components, and whether the cycles with finite connected component have a density among all cycles. Table 4.1 also shows data relevant to these questions. Again we speculate that the answer to both is yes, with the limiting density approximately 19%.

Finally, we found finite connected components of every size ≤ 41. Table 4.2 shows the ones of record size when ordered by smallest element. It seems plausible that every positive integer is the cardinality of a finite connected component; in particular, we conjecture that there are arbitrarily large finite components.

References

1. Daniel J. Bernstein, *How to find small factors of integers*, 2002, `http://cr.yp.to/papers.html#sf`.
2. Andrew R. Booker, *Web page with extended data*, `http://people.maths.bris.ac.uk/~maarb/aliquot/`.

Table 4.2. Numbers with finite connected component of record size

n	size	n	size
28	1	7651954416	24
356408	2	10238969536	35
520208	3	97624271600	36
954536	4	757688279778	37
2652728	5	944013126176	38
9478910	8	1164087362100	41
15576848	16	1336635061736	52
932913124	21	3459684525732	58

3. Sergei Chernykh, *Amicable pairs list*, `http://sech.me/ap/`.
4. Jean-Paul Delahaye, *Nombres amiables et suites aliquotes*, Pour la Science (2002), no. 292, 98–103.
5. P. Erdős, A. Granville, C. Pomerance, and C. Spiro, *On the normal behavior of the iterates of some arithmetic functions*, Analytic number theory (Allerton Park, IL, 1989), Progr. Math., vol. 85, Birkhäuser Boston, Boston, MA, 1990, pp. 165–204. MR 1084181
6. Jan Feitsma, *Pseudoprimes*, `http://www.janfeitsma.nl/math/psp2/index`.
7. Jean-Luc Garambois, *Les suites aliquotes*, `http://www.aliquotes.com/`.
8. Wen Chao Lu, *Exceptional set of Goldbach number*, J. Number Theory **130** (2010), no. 10, 2359–2392. MR 2660899
9. David Moews, *A list of aliquot cycles of length greater than* 2, `http://djm.cc/sociable.txt`.
10. Tomás Oliveira e Silva, Siegfried Herzog, and Silvio Pardi, *Empirical verification of the even Goldbach conjecture and computation of prime gaps up to* $4 \cdot 10^{18}$, Math. Comp. **83** (2014), no. 288, 2033–2060. MR 3194140
11. Paul Pollack and Carl Pomerance, *Some problems of Erdős on the sum-of-divisors function*, Trans. Amer. Math. Soc. Ser. B **3** (2016), 1–26. MR 3481968
12. Carl Pomerance, *The first function and its iterates*, Connections in Discrete Mathematics: A Celebration of the Work of Ron Graham (S. Butler, J. Cooper, and G. Hurlbert, eds.), Cambridge University Press, 2017, `https://math.dartmouth.edu/~carlp/aliquot8.pdf`.
13. J. Sesiano, *Two problems of number theory in Islamic times*, Arch. Hist. Exact Sci. **41** (1991), no. 3, 235–238. MR 1107382
14. The PARI Group, Bordeaux, *PARI/GP version* `2.8.0`, 2016, available from `http://pari.math.u-bordeaux.fr/`.

School of Mathematics, University of Bristol, University Walk, Bristol, BS8 1TW, United Kingdom
E-mail address: `andrew.booker@bristol.ac.uk`