Do, A. L., Boccaletti, S., Epperlein, J., Siegmund, S., & Gross, T. (2016). Topological stability criteria for networking dynamical systems with Hermitian Jacobian. *European Journal of Applied Mathematics*, *27*(6), 888-903. https://doi.org/10.1017/S0956792516000425

Topological stability criteria for networking dynamical systems with Hermitian Jacobian

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The central theme of complex systems research is understanding the emergent macroscopic properties of a system from the interplay of its microscopic constituents. The emergence of macroscopic properties is often intimately related to the structure of the microscopic interactions. Here, we present an analytical approach for deriving necessary conditions an interaction network has to obey in order to support a given type of macroscopic behavior. The approach is based on a graphical notation, which allows rewriting Jacobi's signature criterion in an interpretable form and which can be applied to many systems of symmetrically coupled units. The derived conditions pertain to structures on all scales, ranging from individual nodes to the interaction network as a whole. For the purpose of illustration, we consider the example of synchronization, specifically the (heterogeneous) Kuramoto model and an adaptive variant. The results complete and extend the previous analysis of Do et al. [Phys. Rev. Lett. 108: 194102 (2012)].

Key Words: 34D06, 47B15, 05C83

Characterizing the behavior of complex systems, and discovering the critical boundaries in parameter space at which qualitative changes occur are central interests of statistical physics. Many studies have revealed that, in specific models, this can be achieved by the instruments of nonlinear dynamics [25, 5, 42, 15]. However, applying these tools to large heterogeneous systems, such as complex networks of dynamical units poses significant challenges [6, 32, 33, 14]. In particular, there is no simple way to capture the structural aspects of such systems, which are known to crucially influence the emergent dynamics [8, 17].

The interplay of dynamics and structure in networking systems has extensively been studied using the example of synchronization [37, 8, 4]. The paradigmatic model proposed by Kuramoto [24, 1] revealed that the synchronizability of a network depends on global topological measures such as the clustering coefficient, the diameter, and the degree or weight distribution [30, 43, 36, 21, 10, 34, 29], but also on local and mesoscale details [4, 35, 31]. This indicates that a conceptual understanding of collective phenomena on

dynamical networks can only be achieved if structural properties on all scales are taken into account.

In a recent paper, we proposed a nonlinear dynamics approach, which allows to study the dependence of dynamics on structure in networks of symmetrically coupled units [13]. We showed that Jacobi's signature criterion (JSC) can be used to determine necessary conditions an interaction network has to obey in order to support a given type of macroscopic behavior. These conditions pertain to subgraphs on all scales, from few node subgraphs to the entire network. In this paper, we complete and extend the analysis of [13]. We present the full derivation of the proposed approach and discuss a number of new applications including an adaptive Kuramoto model.

1 Stability in networks of phase-oscillators

The approach outlined below is applicable for all dynamical systems with Hermitian Jacobian matrices, and in particular to all networks of symmetrically-coupled one-dimensional dynamical units. However, for the sake of illustration, we will be considering specifically the example of N coupled phase oscillators

$$\dot{x_i} = \omega_i + \sum_{j \neq i} A_{ij} \sin(x_j - x_i) \ , \quad \forall i \in 1 \dots N \ . \tag{1.1}$$

Here, x_i and ω_i denote the phase and the intrinsic frequency of node i, while $\mathbf{A} \in \mathbb{R}^{N \times N}$ is the coupling matrix of an undirected, weighted interaction network. Two oscillators i, j are thus connected if $A_{ij} = A_{ji} \neq 0$. Equation (1.1) represents the so-called Kuramoto model, that is today considered to be a paradigm for the study of synchronization phenomena in coupled discrete systems [1], and is, therefore, used as the natural benchmark for comparative evaluations of performances of methods and tools. The model can exhibit phase-locked states, which correspond to steady states of the governing system of equations. The local stability of such states is determined by the eigenvalues of the Jacobian matrix $\mathbf{J} \in \mathbb{R}^{N \times N}$ defined by $J_{ik} = \partial \dot{x_i} / \partial x_k$. If all eigenvalues of $\mathbf{J}$ are negative, then the state under consideration is asymptotically stable.

In systems of symmetrically coupled phase oscillators, the Jacobian $\mathbf{J}$ is symmetric and thus admits analysis by Jacobi's signature criterion (JSC). The JSC (also known as Sylvester criterion) states that a hermitian or symmetric matrix $\mathbf{J}$ with rank r has r negative eigenvalues if and only if all principal minors of order $q \leqslant r$ have the sign of $(-1)^q$ [44]. Here, the principal minor of order q is defined as $D_q := \det(J_{ik})$, $i, k = s_1, \dots, s_q$.

Stability analysis by means of JSC is well-known in control theory [28] and has been applied to problems of different fields from fluid- and thermodynamics to offshore engineering [7, 41, 9]. However, the applicability of JSC is presently limited to systems with few degrees of freedom. For system with many degrees of freedom the analytical evaluation of JSC is impeded by the growth of both, a) the number of determinants that have to be checked, and b) the number of terms in each determinant.

Let us first consider difficulty (a) stated above. Applying the sufficient condition is impracticable for most larger systems. Note, however, that demanding $\operatorname{sgn}(D_q) = (-1)^q$ for some q already yields a necessary condition for stability.

The necessary stability condition that is found by considering a principal minor of

given order q depends on the ordering of variables, i.e., the ordering of rows and columns in the Jacobian. By considering different orderings the number of conditions obtained for a given q can therefore be increased [12]. To distinguish minors that are based on different orderings of the variables, we define $S = \{s_1, \ldots, s_q\}$ as a set of $|S| = q$ indices and $D_{|S|,S}$ as the determinant of the submatrix of $\mathbf{J}$, which is spanned by the variables $x_{s_1}, \ldots, x_{s_q}$. Therewith, the conditions for stability read

$$\operatorname{sgn}\left(D_{|S|,S}\right) = (-1)^{|S|}, \quad \forall S,\ |S| = 1, \ldots, r. \tag{1.2}$$

2 Graphical notation

Considering necessary rather than sufficient conditions evades difficulty (b) mentioned above, which leaves us to deal with difficulty (a), i.e., the combinatorial explosion of terms that are needed to write out the conditions for increasing $|S|$. In the common notation, more than 700 terms are necessary for expressing the minors of order 6. For circumventing this problem we employ a graphical notation that captures basic intuition and allows for expressing the minors in a concise way [13].

The graphical notation relies on a topological reading of the minors. We interpret the Jacobian $\mathbf{J}$ as the weight matrix of an undirected, weighted graph $\mathcal{G}$. A Jacobian element J_{ij} then corresponds to the weight of a link connecting nodes i and j. We can now relate products of the Jacobian elements to subgraphs of $\mathcal{G}$ spanned by the respective links. For instance $J_{ij}J_{jk}$ is interpreted as path $i - j - k$, $J_{ij}J_{jk}J_{ki}$ as a closed path from i to j to k and back to i. Thus, the minors of $\mathbf{J}$ can be expressed as sums over subgraphs of $\mathcal{G}$.

In Appendix A, we show that every term occurring in a minor $D_{|S|,S}$ of $\mathbf{J}$ corresponds to a subgraph that can be decomposed in cycles of $\mathcal{G}$. This allows expressing the index structure of every term by a combination of symbols denoting cycles of a given length. The idea is now to supplement the basis of symbols with a summation convention; This convention is designed such that all algebraic terms that are structurally identical and only differ by index permutations can be captured in one symbolic term, which drastically reduces the complexity of the minors.

Below, we use the following definitions: The basis of symbols is given by $\times, |, \triangle, \square, \pentagon, \ldots$ denoting cycles of length $n = 1, 2, 3, 4, 5, \ldots$. The summation convention stipulates that in a minor $D_{|S|,S}$, every product of symbols denotes the sum over all non-equivalent possibilities to build the depicted subgraph with the nodes $\in S$ (cf. Fig. 1). With these conventions the first 4 principal minors can be written as

$$D_{1,S} = \times \tag{2.1 a}$$

$$D_{2,S} = \times \cdot \times - | \tag{2.1 b}$$

$$D_{3,S} = \times \cdot \times \cdot \times - \times \cdot | + 2\triangle \tag{2.1 c}$$

$$D_{4,S} = \times \cdot \times \cdot \times \cdot \times - \times \cdot \times \cdot | + | \cdot | + 2 \times \cdot \triangle - 2\square \tag{2.1 d}$$

More generally

$$D_{|S|,S} = \sum \text{all combinations of symbols with } \sum n = |S|\,, \tag{2.2}$$

where symbols with $n > 2$ appear with a factor of 2 that reflects the two possible orientations in which the corresponding subgraphs can be paced out. Symbols with an

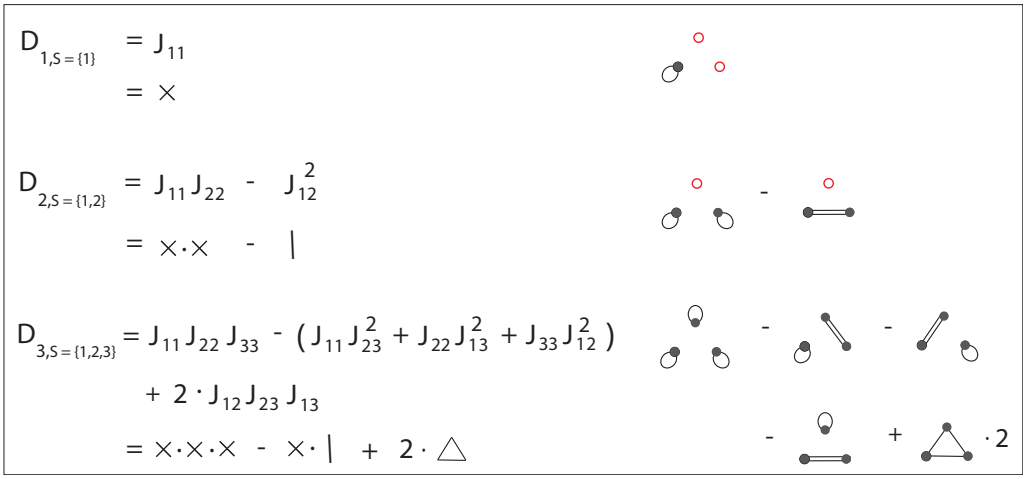

Figure 1. Example for the graphical notation. Shown are the minors of the matrix (2.3) in algebraic, and graphical notation, and, for each term, the corresponding subgraph of a three-node graph $\mathcal{G}$. Here, as well as in the next figures, filled symbols correspond to nodes $\in S$, open symbols to nodes $\notin S$.

even (odd) number of links carry a negative (positive) sign related to the sign of the respective index permutation in the Leibniz formula for determinants [2].

An example for the graphical notation is presented in Fig. 1. The figure displays the three principal minors of the symmetric 3×3 matrix

$$\mathbf{J} = \begin{pmatrix} J_{11} & J_{12} & J_{13} \\ J_{12} & J_{22} & J_{23} \\ J_{13} & J_{23} & J_{33} \end{pmatrix} \tag{2.3}$$

in algebraic, and graphical notation. Moreover, it displays for each term the corresponding subgraph of a three-node graph $\mathcal{G}$.

3 Zero row sum

In many systems, including the standard Kuramoto model, fundamental conservation laws impose a zero-row-sum condition, such that $J_{ii} = -\sum_{j \neq i} J_{ij}$. Using this relation, we can remove all occurrences of elements J_{ii} from the Jacobian and its minors [13]. In the topological reading, this substitution changes the graph $\mathcal{G}$ by replacing a self-loop at a node i by the negative sum over all links that connect to i.

The simplification of the minors due to the zero-row-sum condition can be understood using the example of the Eqs. (2.1). Replacing the self-loops, the first term of every minor $D_{|S|,S}$, $\times^{|S|}$, is $(-1)^{|S|}$ times the sum over all subgraphs meeting the following conditions: (i) the subgraph contains exactly $|S|$ links, and (ii) it can be drawn by starting each (undirected) link at a different node in S. By means of elementary combinatorics it can be verified that all other terms of $D_{|S|,S}$ cancel exactly those subgraphs in $\times^{|S|}$ that

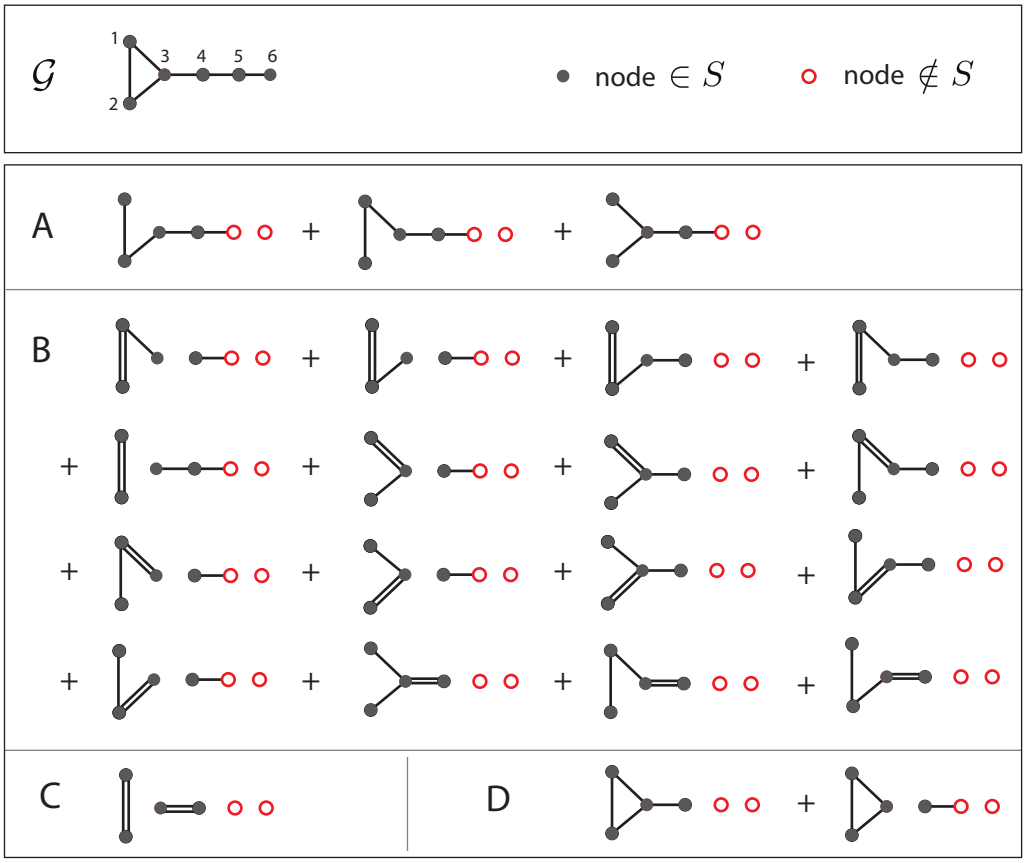

Figure 2. Symbolic calculation of a minor using the zero-row-sum condition. Shown is the graph $\mathcal{G}$, defined by the off-diagonal entries of Eq. (3.3). The terms of the minor $D_{4,S}$ can be written as $\times\cdot\times\cdot\times = A + B + C + 2D$, $-\times\cdot\times\cdot| = -(B + 2C)$, $|\cdot| = C$, $2\times\cdot\triangle = -2D$ and $-2\square = 0$ (cf. Eq. (2.1 *d*)). It thus follows that $D_{4,S} \equiv \Phi_{4,S} = A$ is the sum over all acyclic subgraphs of $\mathcal{G}$ meeting conditions (i)–(iv).

contain cycles. This enables us to express the minors in another way: Defining

$$\Phi_{|S|,S} = \sum \text{all acyclic subgraphs of } \mathcal{G} \text{ meeting (i) and (ii)} \tag{3.1}$$

we can write

$$D_{|S|,S} = (-1)^{|S|}\Phi_{|S|,S} \ . \tag{3.2}$$

We remark that Kirchhoff's Theorem [22], which has previously been used for the analysis of dynamical systems [38, 27], appears as the special case of Eq. (3.2), in which $|S| = N - 1$.

The simplification of the minors due to the zero-row-sum condition as well as the relation between the $D_{|S|,S}$ and their topological equivalents $\Phi_{|S|,S}$ can be illustrated by

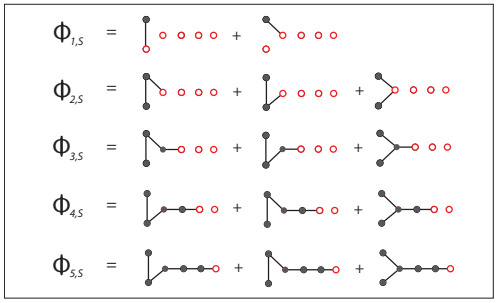

Figure 3. Topological equivalents of minors. Shown is the complete sequence of $\Phi_{|S|,S}$, $|S| = 1, \ldots, 5$ for the graph $\mathcal{G}$ from Fig. 2.

means of a simple example. Consider the symmetric zero-row-sum 6×6 matrix

$$\mathbf{J} = \begin{pmatrix} (J_{11}) & J_{12} & J_{13} & 0 & 0 & 0 \\ J_{12} & (J_{22}) & J_{23} & 0 & 0 & 0 \\ J_{13} & J_{23} & (J_{33}) & J_{34} & 0 & 0 \\ 0 & 0 & J_{34} & (J_{44}) & J_{45} & 0 \\ 0 & 0 & 0 & J_{45} & (J_{55}) & J_{56} \\ 0 & 0 & 0 & 0 & J_{56} & (J_{66}) \end{pmatrix} . \tag{3.3}$$

Here, (J_{ii}) abbreviates $-\sum_{j \neq i} J_{ij}$. In Fig. 2, we calculate the minor $D_{4,S=\{1,\ldots,4\}}$ in terms of the subgraphs of the corresponding graph $\mathcal{G}$. The calculation illustrates the reasoning that lead to the Eqs. (3.1) and (3.2).

The complete sequence of minors $D_{|S|,S}$, $|S| = 1, \ldots, r$ can be calculated as

$$\begin{aligned} D_{1,S} &= (-1)\,(J_{12} + J_{13}) \\ D_{2,S} &= (-1)^2\,(J_{12}J_{13} + J_{12}J_{23} + J_{13}J_{23}) \\ D_{3,S} &= (-1)^3\,(J_{12}J_{13} + J_{12}J_{23} + J_{13}J_{23})\,J_{34} \\ D_{4,S} &= (-1)^4\,(J_{12}J_{13} + J_{12}J_{23} + J_{13}J_{23})\,J_{34}J_{45} \\ D_{5,S} &= (-1)^5\,(J_{12}J_{13} + J_{12}J_{23} + J_{13}J_{23})\,J_{34}J_{45}J_{56} \end{aligned}$$

where $S = \{1, \ldots, |S|\}$, and $r = 5$ due to the zero-row-sum condition.

On the other hand, we can use the definition (3.1) to construct the sequence $\Phi_{|S|,S}$, $|S| = 1, \ldots, r$, directly from the graph $\mathcal{G}$ (cf. Fig. 3). A comparison of both, the algebraic and the topological results, reproduces Eq. (3.2). Note that labeling the nodes in different order would have yielded different algebraic as well as topological expressions.

4 Topological stability conditions

Let us shortly summarize what we obtained so far. The topological reading of determinants maps a Hermitian Jacobian $\mathbf{J}$ with zero row sum onto a graph $\mathcal{G}$, whose weight matrix is given by the off-diagonal part of $\mathbf{J}$. The minors of $\mathbf{J}$ can then be interpreted as sums over values associated with subgraphs of $\mathcal{G}$. Combining the Eqs. (1.2) and (3.2), the algebraic stability constraints on the minors of $\mathbf{J}$ translate into

$$\Phi_{|S|,S} > 0, \quad \forall S,\ |S| = 1, \ldots, r. \tag{4.1}$$

We emphasize that the graph $\mathcal{G}$ is not an abstract construction, but combines information about the physical interaction topology and the dynamical units. For example, if a graph $\mathcal{G}$ has disconnected components, there is a reordering of the variables x_i, such that $\mathbf{J}$ is block diagonal. This implies that the spectra of different topological components of $\mathcal{G}$ decouple and can thus be treated independently.

From Eq. (4.1) we can immediately read off a weak *sufficient* condition for stability: Because $\Phi_{|S|,S}$ is a sum over products of the J_{ij}, a Jacobian with $J_{ij} \geqslant 0\ \forall i,j$ is a solution to Eq. (4.1) irrespective of the specific structure of $\mathcal{G}$ [46]. By contrast, if $J_{ij} < 0$ for some i,j, then the existence of solutions of Eq. (4.1) is dependent on the topology.

The Φ-notation allows to investigate which combinations of negative J_{ij} in a graph $\mathcal{G}$ lead to the violation of at least one of the Eqs. (4.1): The definition of Φ implies that any given $\Phi_{|S|,S}$ can be represented as a function over a set of $\Phi_{|S_i|,S_i}$

$$\Phi_{|S|,S} = f\left(\left\{\Phi_{|S_i|,S_i}\right\}\right), \tag{4.2}$$

where the S_i are subsets of S, and f depends on the choice of the S_i.

Now the key point: For showing that a given topological arrangement of negative J_{ij} is incompatible with the stability conditions Eqs. (4.1), it suffices to show that it exist some $\Phi_{|S|,S}$ and some expansion (4.2) thereof, for which there is no solution with all occurring $\Phi > 0$. Note that this dispenses with the specific calculation of any $\Phi_{|S|,S}$, and hence with addressing the up to $|S|!$ terms they subsume.

To illustrate the application of the topological stability criteria we use this approach to prove that stability requires that every component of $\mathcal{G}$ has a spanning tree, whose links all have positive weights [13]. The proof uses the Φ-notation to show that at least one of the necessary conditions must be violated when no spanning tree of positive connections exists.

We start by assuming that no positive spanning tree exists. In this case it must be possible to partition the nodes into two nonempty sets I_1, I_2 such that

$$J_{ij} \leqslant 0 \quad \forall\ i,j \mid i \in I_1,\ j \in I_2\ . \tag{4.3}$$

The idea is now to evaluate the stability conditions (4.1) for different $S \subseteq I_1$ thereby exploiting that all links leading out of I_1 have negative weights. For this purpose, it is convenient to define E^* as the set of links connecting I_1 and I_2, and $X = \{x_1, \ldots, x_m\}$ as the subset of nodes $\in I_1$ incident to at least one link from E^* ('boundary vertices'). Further, we define σ_i as the sum over all elements of E^* incident to x_i, and, for any subset Y of X, $\sigma_Y := \prod_{m\in Y} \sigma_m$. Finally, for any subset Y of X, we define τ_Y as the sum over all forests of $\mathcal{G}$ that (i) span I_1, and (ii) consists of $|Y|$ trees each of which contains exactly one element from Y.

With the above definitions we can write

$$\Phi_{I_1\setminus C} = \sum_{B\subseteq X\setminus C} \sigma_B \tau_{B\cup C}\ , \tag{4.4}$$

where B and C are disjoint subsets of X and $\Phi_{I_1\setminus C} := \Phi_{q,I_1\setminus C}$. The rather abstract equation is illustrated in Fig. 4 by means of an example.

Equation (4.4) can be read as an expansion of $\Phi_{I_1\setminus C}$ in contributions from E^*. This expansion has the advantage that the sign of one of the two factors in each term is known:

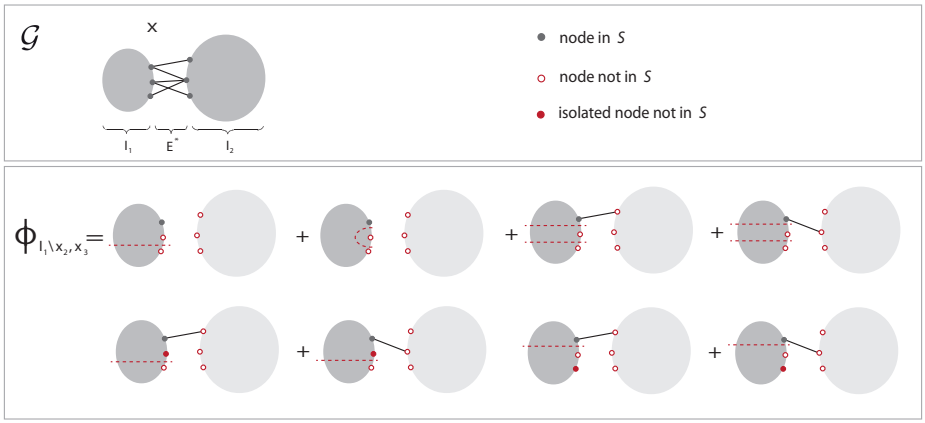

Figure 4. Expanding Φ_S. Upper panel: Consider a graph $\mathcal{G}$ without positive spanning tree. The grey shaded areas represent the subgraphs of $\mathcal{G}$ induced by I_1, I_2 respectively. The nodes in X shall be labeled x_1, x_2, x_3 with x_1 being the upmost and x_3 being the bottommost node. Lower panel: We now want to symbolically construct all terms that contribute to $\Phi_{I_1\setminus\{x_2,x_3\}}$. To this end, we use a definition of Φ_S, that is equivalent to the one given in the main text: Φ_S is the sum over all forests of $\mathcal{G}$, in which each tree contains exactly one vertex $\notin S$. The panel displays the different realizations of such forests with two trees (term 1–2), three trees respectively (term 3–8). Here, a grey shaded area divided by n dashed lines symbolizes all possibilities to span $I_1\setminus\{x_2, x_3\}$ with n trees such that no two nodes in X, which are divided by a dashed line, are part of the same tree. Comparing with the definition of τ_Y and σ_Y reveals that the sum over the first two symbolic terms equals the $B = \emptyset$ term of Eq. (4.4), while the sum over all other symbolic terms equals the $B = \{x_1\}$ term of Eq. (4.4).

$\operatorname{sgn}(\sigma_B) = (-1)^{|B|}$. The signs of the factors $\tau_{B\cup C}$, however, are still indeterminate. Nevertheless, we can show that Eq. (4.4) is incompatible with the stability condition (4.1) by considering a linear combination of $\Phi_{I_1\setminus C}$

$$\begin{aligned}\sum_{C\subseteq X} \underbrace{(-1)^{|C|}\sigma_C}_{\substack{>0 \text{ per}\\ \text{construction}}} \underbrace{\Phi_{I_1\setminus C}}_{>0} &= \sum_{C\subseteq X} (-1)^{|C|}\sigma_C \sum_{B\subseteq X\setminus C} \sigma_B\tau_{B\cup C}\\ &= \sum_{\substack{C\subseteq X\\ B\subseteq X\setminus C}} (-1)^{|C|}\sigma_{B\cup C}\tau_{B\cup C}\\ &= \sum_{\substack{A\subseteq X\\ C\subseteq A}} (-1)^{|C|}\sigma_A\tau_A\\ &= \sum_{A\subseteq X} \sigma_A\tau_A \underbrace{\sum_{C\subseteq A}(-1)^{|C|}}_{=0} = 0\ ,\end{aligned}$$

which is a contradiction. Therefore the existence of a spanning tree of positive elements is a necessary condition for stability, which completes the proof.

In addition to the spanning tree criterion for stability, which pertains to a global

Let C be an unbranched segment of a cycle of $\mathcal{G}$ that consists of d links c_i, $i \in \{1, \ldots, d\}$. Let further w_i denote the weight of c_i, and c_x denote the only link in C with negative weight w_x. Then stability requires that

$$|w_x| < \frac{\prod_{i \in I^*} w_i}{\sum \text{all distinct products of } (d-2) \text{ factors } w_i,\ i \in I^*} \quad (C1) \ ,$$

where $I^* = \{1, \ldots, d\} \setminus x$.

Figure 5. Stability sets an upper bound on the absolute value of a negative link weight. The bounding relation is derived in Appendix B.

property of $\mathcal{G}$, Eq. (4.1) implies further restrictions, which pertain to mesoscale properties of $\mathcal{G}$. Thus, the weights of all links that are not part of any cycle of $\mathcal{G}$ have to be positive. Moreover, at most one of the links that build an unbranched segment of a cycle of $\mathcal{G}$ may have a negative weight. Finally, the weight of a 'negative link' in an unbranched segment of a cycle of $\mathcal{G}$ is bounded below by a value that depends on the weights of the other links in the segment (cf. Fig 5).

Note that while the mesoscale criteria restricting the number and position of negative links can be subsumed under the global spanning tree criterion, the mesoscale criteria restricting the weights of possible negative links are inherently bound to the mesoscale. In particular, the latter can only be derived by considering $\Phi_{|S|,S}$ with small and intermediate $|S|$. This highlights the benefits of the proposed approach: The JSC provide stability criteria on all scales $|S|$, which are made accessible by means of the graphical notation and symbolic calculus.

Applied to the Kuramoto model defined in Eq. (1.1), the spanning tree criterion for stability allows to deduce properties shared by the configuration of all possible phase-locked systems [13]. In a phase-locked state, $J_{ij} = A_{ij} \cos(x_j - x_i)$, where $x_j - x_i$ is the stationary phase difference between oscillator j and i. Given that all link weights $A_{ij} \geqslant 0$, stability of the phase-locked state thus requires that the coupling network has a spanning tree of oscillators obeying $|x_j - x_i| < \pi/2$.

5 Other applications

The proposed approach is applicable to a wide range of questions from different fields. In addition to stability analysis, the graphical notation introduced here allows for instance exploring the isospectrality of Hermitian or symmetric matrices [16, 40], which differ with respect to the signs of some off-diagonal entries. The key idea is that the characteristic polynomial χ of a Hermitian matrix $\mathbf{A} \in \mathbb{C}^{n \times n}$ can be expressed as $\chi(\lambda) = D_n(\mathbf{A} - \lambda \mathbf{I})$. Considering the structure of the graph $\mathcal{G}$ associated to $\mathbf{A} - \lambda \mathbf{I}$ allows to determine which symbols contribute to χ. One can then determine which changes of sign of off-diagonal

entries leave all contributing symbols and thus the characteristic polynomial and the spectrum invariant (see Appendix C).

The derivation of stability criteria by graphical interpretation of the JSC is applicable to all systems with Hermitian Jacobian matrix. Further, the simplification leading up to the spanning tree criterion is possible, whenever the Jacobian has zero row sums. This condition is satisfied for instance by all systems of the form

$$\dot{x_i} = C_i + \sum_{j \neq i} A_{ij} \cdot O_{ij}(x_j - x_i) \ , \quad \forall i \in 1 \dots N \ , \tag{5.1}$$

where the A_{ij} are the weights of a symmetric interaction network and the O_{ij} odd functions. We emphasize that the approach remains applicable in heterogeneous networks containing different link strengths A_{ij}, coupling functions O_{ij} or intrinsic parameters C_i.

The class of systems to which the present results are directly applicable include general networks of phase oscillators as well as other models such as continuous-time variants of the Deffuant model of opinion formation [11] or a class of ecological metapopulation models [26].

Although the specific criteria derived above are contingent on the zero-row sum condition, it can be expected that the general approach proposed here is also applicable to situations where this condition is violated, such as in the model of cooperation among interacting agents studied in [12]. A simple extension of Eq. (5.1) which violates the zero row sum condition is found by replacing C_i by a function of x_i, which allows dynamical retuning of the intrinsic frequency, e.g. for modeling homeostatic feedback in neural networks. For illustrating the application of the proposed method to models in which the zero row sum condition is violated in some rows, we consider another model inspired by neuroscience in the subsequent section.

6 Adaptive Kuramoto model

We now apply the proposed approach to an example of an adaptive network, i.e., a system in which the topology of the network coevolves with the dynamics of oscillators [20, 45, 18]. In the context of the Kuramoto model, adaptive coupling has recently attracted keen interest as it allowed to show that the emergence of synchronous motion can be intimately related to a selection mechanism of specific network topologies [39], and to the identification of complex hierarchical structures in the graph connectivity [45, 3].

We consider a system of N phase oscillators that evolve according to Eq. (1.1), while the coupling strength A_{ij} evolves according to

$$\frac{\mathrm{d}}{\mathrm{dt}} A_{ij} = \cos(x_j - x_i) - b \cdot A_{ij}. \tag{6.1}$$

The first term in Eq. (6.1) states that the more similar the phases of two nodes the stronger reinforced is their connection, the second term guarantees convergence. In a stationary, phase-locked state state $A_{ij} = \cos(x_j - x_i)/b$ and all oscillators oscillate with a common frequency $\Omega = \frac{1}{N} \sum_i \omega_i$. The stability of this state is governed by a symmetric

Jacobian

$$\mathbf{J} = \left(\begin{array}{ccc|ccc} -b & 0 & 0 & s_{21} & s_{12} & 0 \\ 0 & -b & 0 & s_{31} & 0 & s_{13} \\ 0 & 0 & -b & 0 & s_{32} & s_{23} \\ \hline s_{21} & s_{31} & 0 & m_1 & o_{12} & o_{13} \\ s_{12} & 0 & s_{32} & o_{12} & m_2 & o_{23} \\ 0 & s_{13} & s_{23} & o_{13} & o_{23} & m_3 \end{array}\right), \tag{6.2}$$

where $o_{ij} := \frac{1}{b}\cos^2(x_j - x_i)$, $m_i := -\sum_{j\neq i} o_{ij}$, $s_{ji} := \sin(x_j - x_i)$ and we have chosen $N = 3$ for illustration. The marked partitioning separates two blocks on the diagonal. The upper one is a diagonal submatrix of size $L \times L$, $L := N(N-1)/2$, the lower one is a $N \times N$ symmetric submatrix with zero row sum, which we denote as $\mathbf{j}$.

Let us start our analysis by focusing on the upper left block of $\mathbf{J}$. In the chosen ordering of variables, the first L minors $D_{|S|}$ satisfy the stability condition Eq. (1.2) if and only if $b > 0$. Concerning the minors of order $|S| > L$, the following conventions prove advantageous: We define $D_{L+n,S}$ as the determinant of the submatrix of $\mathbf{J}$, which is spanned by all variables A_{ij} and the n variables x_{s_i}. Further, $D_{0+n,S}$ denotes the determinant of the submatrix of $\mathbf{j}$, which is solely spanned by the n variables x_{s_i}.

We find that

$$D_{L+n,S} = (-1)^L b^{L-n} \cdot F(D_{0+n,S}), \tag{6.3}$$

where F is the linear mapping $F : o_{ij} \to \cos(2(x_j - x_i))$. As the submatrix $\mathbf{j}$ is symmetric and has a zero row sum, its minors, $D_{0+n,S}$, can be rewritten using Eq. (3.2)

$$D_{L+n,S} = (-1)^{L+n} b^{L-n} F(\Phi_{n,S}), \tag{6.4}$$

where $\Phi_{n,S}$ refers to subgraphs of the graph $\mathcal{G}$ defined by the off-diagonal entries of $\mathbf{j}$. Stability requires that $\operatorname{sgn}(D_{L+n,S}) = \operatorname{sgn}\left((-1)^{L+n}\right)$. As the necessary stability condition $b > 0$ implies $b^{L-n} > 0$, it follows that in a stable system

$$F(\Phi_{n,S}) > 0, \quad \forall S, n = 1 \ldots r. \tag{6.5}$$

Comparison with Eq. 4.1 reveals that a necessary condition for stability is that the graph $F(\mathcal{G})$ has a positive spanning tree. Revisiting the definitions of the linear map F and the graph $\mathcal{G}$, we find that the weight of a links ij of $F(\mathcal{G})$ is given by $\cos(2(x_j - x_i))$. Hence, $F(\mathcal{G})$ has a positive spanning tree if and only if the adaptive coupling network has a spanning tree of oscillators obeying $|x_j - x_i| < \pi/4$. The restriction on the stationary phase-differences in a stable, phase-locked state are thus more strict in the adaptive than in the non-adaptive case.

7 Conclusions

In the present paper we presented a general approach for the analysis of symmetrically coupled systems and a specific simplifications for systems, in which the Jacobian matrix $\mathbf{J}$ additionally has a zero row sum.

Using a graphical interpretation of Jacobi's signature criterion we showed that for all non-trivial eigenvalues of $\mathbf{J}$ to be negative, the graph $\mathcal{G}$, whose adjacency matrix is given by the off-diagonal part of $\mathbf{J}$, has to obey necessary topological conditions. Thus, $\mathcal{G}$

must have a spanning tree of links with positive weights, which restricts the number and position of potential negative entries in the Jacobian matrix. Moreover, the absolute value of potential negative link weights is bounded above by a topology dependent relation.

We used the approach for deriving necessary conditions for local asymptotic stability of stationary and phase-locked states in networks of phase oscillators. Our results provide an analytical angle that is complementary to statistical analysis of network synchronizability. Where statistical approaches reveal global features impinging on the propensity to synchronize, our approach can pinpoint specific defects precluding synchronization. We note that such defects can occur on all scales, corresponding to the violation of the signature criterion in subgraphs of different size. This highlights synchronization of phase oscillators as a simple but intriguing example in which instabilities can arise from local, global or mesoscale structures. In the future, the approach proposed here may provide a basis for further investigation of these instabilities.

A limitation of the present approach is that the Jacobian matrix of the state under consideration has to be available in some form. However, this limitation is inherent to all forms of linear stability and bifurcation analysis and several strategies have been developed for mitigating it. These include for instance the equation-free approach [23], where the desired information on the Jacobian is generated on-demand from simulations or experiments, and generalized modeling [19], which reveals a parameterized presentation of the Jacobian for very general classes of systems. Even if these specific techniques are not applied, researchers are often aware of the structure of the Jacobian in the system under consideration, which can be sufficient for gaining some fundamental insights by the proposed approach.

It is apparent that applying the stability analysis proposed here may require overcoming specific problems. In particular, the requirements of symmetry and zero-row sum may necessitate extending the basic scheme or focusing on specific cases, as we did in the adaptive network example. However, if these problems can be overcome, the approach can reveal hard analytical conditions that all stable steady states must meet. Further, it can identify topological structures (on all scales) that prevent the existence of such states and pinpoint them in networks. Because this type of information is complementary to insights revealed by common statistical approaches, it is potentially of high value for the analysis of the system.

Appendix A Decomposition of minors in cyclic subgraphs

The Leibniz formula for determinants [2] implies that (i) a minor $D_{q,S}$ is a sum over $q!$ elementary products $J_{i_1 j_1} \cdot \ldots \cdot J_{i_q j_q}$; and that (ii) in each of these products every index $s_i \in S$ occurs exactly twice.

In the topological reading this translates to the following statements: Because of property (i), each term of a minor $D_{q,S}$ corresponds to a subgraph with q links. Because of property (ii), these subgraphs are composed of sets of cycles in $\mathcal{G}$: Every index $s_i \in S$ occurs either with multiplicity two on a diagonal element of $\mathbf{J}$, or, with multiplicity one, on two off-diagonal elements of $\mathbf{J}$. In the former case, the respective factor corresponds to a self-loop of $\mathcal{G}$, i.e., to a cycle of length $n = 1$; in the latter case, there is a set of factors J_{ij} $i \neq j$ corresponding to a closed path of links, i.e., a cycle of length $n > 1$.

Appendix B Lower bound on the negative link weights

Consider a path of $d-1$ degree-two nodes v_i, $i \in \{1,\dots,d-1\}$, that are part of at least one cycle of $\mathcal{G}$. Together with the d egdes c_i that are incident to at least one of the nodes v_i, the path constitutes an unbranched segment C of a cycle of $\mathcal{G}$ (cf. Fig. 3).

According to the spanning tree criterion, C can have at most one link with negative weight. Below, we consider the case where C has exactly one such link, and show that Eq. (4.1) imposes an upper bound on the absolute value of the negative link weight.

We use following conventions: Let the nodes be labeled such that the indices i occur in an increasing order if C is paced out. And let the links be labeled such that v_i is incident to c_i and c_{i+1} (cf. Fig. 2S). Further, let w_i denote the weight of c_i. And lastly, let c_x denote the only link in C with negative weight w_x. Below, show that stability requires that

$$|w_x| < \frac{\prod \text{all } w_i,\ i \in I^*}{\sum \text{all distinct products of } (d-2) \text{ factors } w_i,\ i \in I^*} \ , \tag{B 1}$$

where $I^* = \{1,\dots,d\} \setminus x$.

For deriving Eq. (B 1), we consider a sequence of conditions $\Phi_{S_1} \dots \Phi_{S_{d-1}}$. The sequence is constructed as follows: we choose $S_1 = x$, $S_2 = \{x, x+1\}$, $S_3 = \{x, x+1, x+2\}$, and so forth until $S_{d-x} = \{x, x+1, x+2, \dots, d-1\}$. The remaining elements of the sequence are then constructed as $S_{d-x+1} = S_{d-x} \cup \{x-1\}$, $S_{d-x+2} = S_{d-x} \cup \{x-1, x-2\}$, and so forth until $S_{d-1} = \{v_i\}_{i\in\{1,\dots,d-1\}}$.

The first element of the sequence, $\Phi_{S_1} > 0$, stipulates that $w_x + w_{x+1} > 0$, and thus that $-w_x < w_{x+1}$.

The second element of the sequence $\Phi_{S_2} > 0$ stipulates that $w_x w_{x+1} + w_x w_{x+2} + w_{x+1} w_{x+2} > 0$ and thus that $-w_x < (w_{x+1} w_{x+2})/(w_{x+1} + w_{x+2})$.

More generally, every element Φ_{S_i} of the sequence $\Phi_{S_1}, \dots, \Phi_{S_{d-1}}$ modifies the upper bound on $-w_x$ as per

$$-w_x < \frac{\prod \text{all } w_i,\ i \in S_i^*}{\sum \text{all distinct products of } (|S_i|-1) \text{ factors } w_i,\ i \in S_i^*} \ , \tag{B 2}$$

where $S_i^* = (S_i \setminus x) \cup (\max(S_i) + 1)$.

The right hand side of Eq. (B 2) is monotonously decreasing with increasing $|S_i|$. We can thus conclude that C can have at most one link with negative weight, whose absolute value $|w_x|$ is bounded above by Eq. (B 1).

Appendix C Applicability of the graphical notation to isospectrality problems

The graphical notation provides a straight forward approach to explore the isospectrality of Hermitian or symmetric matrices which differ with respect to the signs of some off-diagonal entries. To see this consider a Hermitian matrix $\mathbf{A} \in \mathbb{C}^{n\times n}$. Its characteristic polynomial χ can be calculated as

$$\chi(\lambda) = D_n(\mathbf{A} - \lambda \mathbf{I})$$

Considering the structure of the graph $\mathcal{G}$ associated to $\mathbf{A} - \lambda\mathbf{I}$ allows to determine which symbols contribute to χ. One can then determine which changes of sign of off-diagonal

entries leave all contributing symbols and thus the characteristic polynomial and the spectrum invariant.

For instance, if the graph $\mathcal{G}$ is a tree, then the only symbols that contribute to the characteristic polynomial χ are the symbols $\times$ and $|$. Thus, χ is a polynomial of factors $(A_{ii} - \lambda)$ (corresponding to symbols $\times$), and factors $A_{ij}A_{ji}$ (corresponding to symbols $|$). Due to the hermiticity of $\mathbf{A}$, $A_{ij}A_{ji} = |A_{ij}|^2$. It follows that χ, and therewith the spectrum, is invariant under any operation that changes the sign of a pair of off-diagonal entries $A_{ij} \to -A_{ij},\ A_{ji} \to -A_{ji}$.

Along the same line, we can infer isospectrality relations for matrices $\mathbf{A}$, whose corresponding graphs $\mathcal{G}$ are composed only of tree-like subgraphs and isolated cycles. For such matrices, the spectrum is invariant under symmetry preserving sign changes of

- an arbitrary number of off-diagonal entries that do not belong to cyclic subgraphs of $\mathcal{G}$.
- an even number of off-diagonal entries that belong to the same cyclic subgraph of $\mathcal{G}$.

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