DETAILED ERROR ANALYSIS FOR A FRACTIONAL ADAMS METHOD WITH GRADED MESHES

YANZHI LIU, JASON ROBERTS, AND YUBIN YAN *

Abstract. We consider a fractional Adams method for solving the nonlinear fractional differential equation ${}_0^C D_t^\alpha y(t) = f(t, y(t))$, $\alpha > 0$, equipped with the initial conditions $y^{(k)}(0) = y_0^{(k)}, k = 0, 1, \dots, \lceil \alpha \rceil - 1$. Here α may be an arbitrary positive number and $\lceil \alpha \rceil$ denotes the smallest integer no less than α and the differential operator is the Caputo derivative. Under the assumption ${}_0^C D_t^\alpha y \in C^2[0, T]$, Diethelm et al. [8, Theorem 3.2] introduced a fractional Adams method with the uniform meshes $t_n = T(n/N), n = 0, 1, 2, \dots, N$ and proved that this method has the optimal convergence order uniformly in t_n, that is $O(N^{-2})$ if $\alpha > 1$ and $O(N^{-1-\alpha})$ if $\alpha \le 1$. They also showed that if ${}_0^C D_t^\alpha y(t) \notin C^2[0, T]$, the optimal convergence order of this method cannot be obtained with the uniform meshes. However, it is well known that for $y \in C^m[0, T]$ for some $m \in \mathbb{N}$ and $0 < \alpha < m$, the Caputo fractional derivative ${}_0^C D_t^\alpha y(t)$ takes the form "${}_0^C D_t^\alpha y(t) = ct^{\lceil \alpha \rceil - \alpha} + \textit{smoother terms}$" [8, Theorem 2.2], which implies that ${}_0^C D_t^\alpha y$ behaves as $t^{\lceil \alpha \rceil - \alpha}$ which is not in $C^2[0, T]$. By using the graded meshes $t_n = T(n/N)^r, n = 0, 1, 2, \dots, N$ with some suitable $r > 1$, we show that the optimal convergence order of this method can be recovered uniformly in t_n even if ${}_0^C D_t^\alpha y$ behaves as $t^\sigma, 0 < \sigma < 1$. Numerical examples are given to show that the numerical results are consistent with the theoretical results.

Key words. Fractional differential equations, Caputo derivative, Adams method

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1. Introduction. In this paper, we will consider a numerical method for solving the following fractional nonlinear differential equation, with $\alpha > 0$,

$$ {}_0^C D_t^\alpha y(t) = f(t, y(t)),\ t > 0,\ y^{(k)}(0) = y_0^{(k)},\ k = 0, 1, \dots, \lceil \alpha \rceil - 1, \tag{1.1} $$

where the $y_0^{(k)}$ may be arbitrary real numbers and ${}_0^C D_t^\alpha y(t)$ denotes the Caputo fractional derivative defined by

$$ {}_0^C D_t^\alpha y(t) = \frac{1}{\Gamma(\lceil \alpha \rceil - \alpha)} \int_0^t (t - s)^{\lceil \alpha \rceil - \alpha - 1} y^{\lceil \alpha \rceil}(s)\, ds, \tag{1.2} $$

where $\lceil \alpha \rceil$ is the smallest integer $\ge \alpha$. As usual we demand that the function f is continuous and fulfills a Lipschitz condition with respect to its second argument with Lipschitz constant L on a suitable set G. Under these assumptions, Diethelm et al. [7, Theorems 2.1, 2.2] showed that (1.1) has a unique solution y on some interval $[0, T]$.

It is well-known that (1.1) is equivalent to [7, Lemma 2.3]

$$ y(t) = \sum_{\nu=0}^{\lceil \alpha \rceil - 1} y_0^{(\nu)} \frac{t^\nu}{\nu!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} f(s, y(s))\, ds. \tag{1.3} $$

Equations of this type arise in a number of applications where models based on fractional calculus are used, such as viscoelastic materials, anomalous diffusion, signal

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processing and control theory, etc., see Oldham and Spanier [16], Kilbas et al. [10], Podlubny [20].

The analytic solution of (1.1) for the general function f is not known. Therefore we have to apply some numerical methods for solving (1.1). Stability and convergence of such numerical methods are analyzed under certain smoothness assumptions for the solutions of (1.1), see, for example, [7], [2], [1], [14], [23], [26], [17], [18], [11].

Most analysis of the numerical methods for solving (1.1) is deduced under the assumptions that the meshes are uniform, see, for example, [7], [8], [9], [13], [14], [26]. To obtain a higher order numerical method with uniform meshes, the solutions or data of (1.1) are required to be sufficiently smooth, for example, ${}_0^C D_t^\alpha y \in C^m[0,T], m \geq 2$ in [8, Theorem 3.2]. However, as we will see below in Theorem 1.2, although $y \in C^m[0,T]$ for some $m \in \mathbb{N}, 0 < \alpha < m$, the Caputo fractional derivative ${}_0^C D_t^\alpha y$ behaves as $t^{\lceil \alpha \rceil - \alpha}$ when $y^{\lceil \alpha \rceil}(0) \neq 0, \alpha > 0$. Therefore it is interesting to design some numerical methods which have the optimal convergence orders when ${}_0^C D_t^\alpha y$ behaves as $t^{\lceil \alpha \rceil - \alpha}, \alpha > 0$. Diethelm [4, Theorem 3.1] used the graded meshes to recover the optimal convergence order for the approximation of the Hadamard finite-part integral. Recently Stynes et al. [22], [21] applied the graded meshes to recover the convergence order of the finite difference method for solving a time-fractional diffusion equation when the solution is not sufficiently smooth. This excellent approach in [22], [21] allows to obtain a (relatively) high convergence order without the otherwise required very unnatural smoothness assumptions on the given solution. Other works for solving fractional differential equations with non-uniform meshes may be found in, for example, [12], [19], [24], [25].

Motivated by the ideas in Diethelm [4] and Stynes et al. [22] we will introduce a numerical method for solving (1.1) with the graded meshes and we prove that the optimal convergence order uniformly in t_n for the proposed numerical method can be recovered when ${}_0^C D_t^\alpha y(t), \alpha > 0$ behaves as $t^\sigma, 0 < \sigma < 1$.

Before we introduce our numerical method, we recall some well-known smoothness properties of the solution y of (1.1) under some assumptions of f.

THEOREM 1.1. *[15, Lubich, 1983, Theorem 2.1]*

1. *Let $\alpha > 0$. Assume that $f \in C^2(G)$. Define $\hat{v} := \lceil \frac{1}{\alpha} \rceil - 1$. Then there exist a function $\psi \in C^1[0,T]$ and some $c_1, c_2, \dots, c_{\hat{\nu}} \in \mathbb{R}$ such that the solution y of* (1.1) *can be expressed in the form*

$$y(t) = \psi(t) + c_1 t^\alpha + c_2 t^{2\alpha} + c_3 t^{3\alpha} + \cdots + c_{\hat{\nu}} t^{\hat{\nu}\alpha}.$$

2. *Let $\alpha > 0$. Assume that $f \in C^3(G)$. Define $\hat{v} := \lceil \frac{2}{\alpha} \rceil - 1$ and $\tilde{v} := \lceil \frac{1}{\alpha} \rceil - 1$. Then there exist a function $\psi \in C^2[0,T]$ and some $c_1, c_2, \dots, c_{\hat{\nu}} \in \mathbb{R}$ and $d_1, d_2, \dots, d_{\tilde{\nu}} \in \mathbb{R}$ such that the solution y of* (1.1) *can be expressed in the form*

$$y(t) = \psi(t) + \sum_{\nu=1}^{\hat{\nu}} c_\nu t^{\nu\alpha} + \sum_{\nu=1}^{\tilde{\nu}} d_\nu t^{1+\nu\alpha}.$$

For example, when $0 < \alpha < 1, f \in C^2(G)$, we have $\hat{v} = \lceil \frac{1}{\alpha} \rceil - 1 \geq 1$ and

$$y = ct^\alpha + \text{smoother terms},$$

which implies that the solution y of (1.1) behaves as $t^\alpha, 0 < \alpha < 1$ when $f \in C^2(G)$.

THEOREM 1.2. *[8, Theorem 2.2] If $y \in C^m[0,T]$ for some $m \in \mathbb{N}$ and $0 < \alpha < m$,* *then*

$$ {}_0^C D_t^\alpha y(t) = \varphi(t) + \sum_{l=0}^{m-\lceil\alpha\rceil-1} \frac{y^{(l+\lceil\alpha\rceil)}(0)}{\Gamma(\lceil\alpha\rceil-\alpha+l+1)} t^{\lceil\alpha\rceil-\alpha+l}, $$

with some function $\varphi \in C^{m-\lceil\alpha\rceil}[0,T]$. *Moreover, the* $(m-\lceil\alpha\rceil)$*th derivative of* φ *satisfies a Lipschitz condition of order* $\lceil\alpha\rceil-\alpha$.

For example, when $0<\alpha<1$, $y \in C^m[0,T], m \geq 2,$, we have

$$ {}_0^C D_t^\alpha y(t) = \varphi(t) + \frac{y'(0)}{\Gamma(2-\alpha)} t^{1-\alpha} + \text{smoother terms}, $$

where $\varphi \in C^{m-1}[0,T]$ which implies that the Caputo fractional derivative ${}_0^C D_t^\alpha y(t), 0 < \alpha < 1$ behaves as $t^{1-\alpha}$ when $y'(0) \neq 0$. Similarly when $1<\alpha<2$, $y \in C^m[0,T], m \geq 3$, we have

$$ {}_0^C D_t^\alpha y(t) = \varphi(t) + \frac{y''(0)}{\Gamma(3-\alpha)} t^{2-\alpha} + \text{smoother terms}, $$

where $\varphi \in C^{m-2}[0,T]$.

In view of Theorems 1.1 and 1.2, we see that smoothness of one of the functions y and ${}_0^C D_t^\alpha y$ will imply nonsmoothness of the other unless some special conditions are fulfilled. Based on Theorems 1.1 and 1.2, we introduce the following assumption. The similar assumption for the smoothness of the solution u of the time-fractional diffusion equation are introduced in Stynes et al. [22, Theorem 2.1].

Assumption 1. *Let* $0<\sigma<1$ *and let* $g := {}_0^C D_t^\alpha y$ *with* $\alpha > 0$. *There exists a constant* $c>0$ *such that*

$$ |g'(t)| \leq c t^{\sigma-1}, \quad |g''(t)| \leq c t^{\sigma-2}. \tag{1.4} $$

Remark 1.3. *It is easy to see that* (1.4) *does not imply* $g \in C^2[0,T]$, *but* $g \in C^2(0,T]$.

Let N be a positive integer and let $0 = t_0 < t_1 < \cdots < t_N = T$ be the graded meshes on $[0,T]$ defined by

$$ t_j = T(j/N)^r, j = 0,1,2,\dots,N, \quad \text{with } r \geq 1. \tag{1.5} $$

For simplicity, we assume that $T=1$ in this paper.

Let us now introduce the fractional Adams method with the graded meshes (1.5). This method has been introduced and analyzed in Diethelm [5, Appendix C] and Diethelm et al. [8] for the uniform meshes.

Denote $y_j \approx y(t_j), j = 0,1,2,\dots,n+1$ with $n = 0,1,2,\dots,N-1$, the approximation of $y(t_j)$, we define the following predictor-corrector Adams method for solving (1.3), with $\alpha > 0$:

$$ \begin{aligned} y_{n+1}^P &= \sum_{\nu=0}^{\lceil\alpha\rceil-1} y_0^{(\nu)} \frac{t_{n+1}}{\nu!} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{j,n+1} f(t_j, y_j), \\ y_{n+1} &= \sum_{\nu=0}^{\lceil\alpha\rceil-1} y_0^{(\nu)} \frac{t_{n+1}}{\nu!} + \frac{1}{\Gamma(\alpha)} \Big(\sum_{j=0}^{n} a_{j,n+1} f(t_j, y_j) + a_{n+1,n+1} f(t_{n+1}, y_{n+1}^P) \Big), \\ y_0^{(\nu)} &\ \text{is given}, \end{aligned} \tag{1.6} $$

where the weights $b_{j,n+1}, j = 0, 1, 2, \dots, n$ satisfy

$$b_{j,n+1} = \frac{N^{-r\alpha}}{\alpha}\big(((n+1)^r - j^r)^\alpha - ((n+1)^r - (j+1)^r)^\alpha\big), \tag{1.7}$$

and the weights $a_{j,n+1}, j = 0, 1, 2, \dots, n+1$ satisfy

$$\begin{aligned}
a_{0,n+1} &= \frac{N^{-r\alpha}}{\alpha(1+\alpha)}\big((n+1)^{r\alpha}(\alpha+1) + ((n+1)^r - 1)^{\alpha+1} - (n+1)^{r(\alpha+1)}\big),\\
a_{j,n+1} &= \frac{N^{-r\alpha}}{\alpha(1+\alpha)}\Big(\frac{[(n+1)^r - (j-1)^r]^{\alpha+1} - [(n+1)^r - j^r]^{\alpha+1}}{j^r - (j-1)^r}\\
&\quad + \frac{[(n+1)^r - (j+1)^r]^{\alpha+1} - [(n+1)^r - j^r]^{\alpha+1}}{(j+1)^r - j^r}\Big), \; j = 1, 2, \dots, n,\\
a_{n+1,n+1} &= \frac{N^{-r\alpha}}{\alpha(1+\alpha)}\big((n+1)^r - n^r\big)^\alpha.
\end{aligned} \tag{1.8}$$

The predictor term y_{n+1}^P in (1.6) is obtained by approximating the integral $\int_0^{t_{n+1}}(t_{n+1}-s)^{\alpha-1}f(s,y(s))\,ds$ in (1.3) with $\int_0^{t_{n+1}}(t_{n+1}-s)^{\alpha-1}P_0(s)\,ds$, where $P_0(s)$ is the piecewise constant function defined on $[0, t_{n+1}]$, i.e.,

$$P_0(s) = f(t_j, y(t_j)), \; s \in [t_j, t_{j+1}], \; j = 0, 1, 2, \dots, n.$$

Similarly, the corrector term y_{n+1} in (1.6) is obtained by approximating the integral $\int_0^{t_{n+1}}(t_{n+1}-s)^{\alpha-1}f(s,y(s))\,ds$ in (1.3) with $\int_0^{t_{n+1}}(t_{n+1}-s)^{\alpha-1}P_1(s)\,ds$, where $P_1(s)$ is the piecewise linear function defined on $[0, t_{n+1}]$, i.e.,

$$P_1(s) = \frac{s - t_{j+1}}{t_j - t_{j+1}}f(t_j, y(t_j)) + \frac{s - t_j}{t_{j+1} - t_j}f(t_{j+1}, y(t_{j+1})), \; s \in [t_j, t_{j+1}], \; j = 0, 1, 2, \dots, n.$$

We remark that when $r = 1$, the weights in (1.8) reduce to the weights in Diethelm et al. [8, (1.14)] with the uniform meshes.

Under the assumption that $g(t) := {}_0^C D_t^\alpha y(t) \in C^2[0, T]$ and $r = 1$ (i.e., uniform meshes), Diethelm et al. [8] proved the following error estimates, i.e. [8, Theorem 3.2]:

THEOREM 1.4. *Let $\alpha > 0$ and assume that $g := {}_0^C D_t^\alpha y \in C^2[0, T]$ for some suitable T. Assume that $y(t_j)$ and y_j are the solutions of (1.3) and (1.6), respectively. Let $r = 1$ (uniform meshes). Then*

$$\max_{0\le j\le N} |y(t_j) - y_j| \le \begin{cases} CN^{-(1+\alpha)}, & \text{if } 0 < \alpha \le 1,\\ CN^{-2}, & \text{if } \alpha > 1.\end{cases}$$

In this work, under the Assumption 1, and $r > 1$, we shall prove the following error estimates:

THEOREM 1.5. *Let $\alpha > 0$ and assume that $g := {}_0^C D_t^\alpha y$ satisfies Assumption 1.*

1. *If $0 < \alpha \le 1$, assume that $y(t_j)$ and y_j are the solutions of (1.3) and (1.6), respectively, then we have*

$$\max_{0\le j\le N} |y(t_j) - y_j| \le \begin{cases} CN^{-r(\sigma+\alpha)}, & \text{if } r(\alpha+\sigma) < 1+\alpha,\\ CN^{-r(\sigma+\alpha)}\ln(N), & \text{if } r(\alpha+\sigma) = 1+\alpha,\\ CN^{-(1+\alpha)}, & \text{if } r(\alpha+\sigma) > 1+\alpha.\end{cases}$$

2. If $\alpha > 1$, then we have

$$\max_{0\le j\le N} |y(t_j) - y_j| \le \begin{cases} CN^{-r(1+\sigma)}, & \text{if } r(1+\sigma) < 2, \\ CN^{-2}\ln(N), & \text{if } r(1+\sigma) = 2, \\ CN^{-2}, & \text{if } r(1+\sigma) > 2. \end{cases}$$

Remark 1.6. *By Theorem 1.1, assume that $f \in C^m(G), m \ge 2$ and $\alpha \in (0,1)$, then, with some constants $c_1, c_2, \dots, c_{\hat{\nu}} \in \mathbb{R}$,*

$$y = c_1 t^{\alpha} + c_2 t^{2\alpha} + \cdots + c_{\hat{\nu}} t^{\hat{\nu}\alpha} + \text{smoother terms},$$

which implies that, with some constants $d_1, d_2, \dots, d_{\hat{\nu}} \in \mathbb{R}$,

$$\begin{aligned} g := {}_0^C D_t^{\alpha} y &= d_1 t^{\alpha-\alpha} + d_2 t^{2\alpha-\alpha} + \cdots + d_{\hat{\nu}} t^{\hat{\nu}\alpha-\alpha} + \text{smoother terms} \\ &= d_1 + d_2 t^{\alpha} + \cdots + d_{\hat{\nu}} t^{(\hat{\nu}-1)\alpha} + \text{smoother terms}. \end{aligned}$$

We see $g := {}_0^C D_t^{\alpha} y$ behaves as $c + ct^{\alpha}$, therefore we may apply Theorem 1.5 with $\sigma = \alpha$ in this case.

Remark 1.7. *If one uses M corrector iterations instead of just one, the order in Theorem 1.5 can be improved to $O(N^{-\min\{2,1+M\alpha\}})$, see Diethelm [6].*

Remark 1.8. *The modification of the basic Adams-Bashforth-Moulton method suggested by Deng [3] for the case of a uniform grid can be applied for the graded mesh used in this paper as well. This should lead to a reduction of the computational cost without an increased error.*

We remark that the optimal convergence order $O(N^{-\min(1+\alpha,2)})$, $\alpha > 0$ obtained in Theorem 1.4 for the numerical method (1.6) for the smooth g with the uniform meshes with $r = 1$ can be recovered in Theorem 1.5 for the nonsmooth g with the graded meshes (1.5) with $r > 1$.

The paper is organized as follows. In Section 1 we introduce the predictor-corrector method for solving (1.1) with the graded meshes. In Section 2, we prove our main result Theorem 1.5. Finally in Section 3, we give some numerical examples which show that the numerical results are consistent with the theoretical results.

Throughout, the notations C and c, with or without a subscript, denote generic constants, which may differ at different occurrences, but are always independent of the mesh size.

2. Proof of Theorem 1.5. In this section, we will give the proof of Theorem 1.5. To do this, we need some preliminary lemmas.

Lemma 2.1. *Let $\alpha > 0$. Assume that g satisfies Assumption 1.*

1. If $0 < \alpha \le 1$, then

$$\Big| \int_0^{t_{n+1}} (t_{n+1} - s)^{\alpha-1} \big(g(s) - P_1(s)\big)\, ds \Big| \le \begin{cases} CN^{-r(\alpha+\sigma)}, & \text{if } r(\alpha+\sigma) < 2, \\ CN^{-2}\ln(N), & \text{if } r(\alpha+\sigma) = 2, \\ CN^{-2}, & \text{if } r(\alpha+\sigma) > 2. \end{cases}$$

2. If $\alpha > 1$, then

$$\Big| \int_0^{t_{n+1}} (t_{n+1} - s)^{\alpha-1} \big(g(s) - P_1(s)\big)\, ds \Big| \le \begin{cases} CN^{-r(1+\sigma)}, & \text{if } r(1+\sigma) < 2, \\ CN^{-2}\ln(N), & \text{if } r(1+\sigma) = 2, \\ CN^{-2}, & \text{if } r(1+\sigma) > 2, \end{cases}$$

where $P_1(s)$ *is the piecewise linear function defined by, with* $j = 0, 1, 2, \dots, n$,

$$P_1(s) = \frac{s - t_{j+1}}{t_j - t_{j+1}} g(t_j) + \frac{s - t_j}{t_{j+1} - t_j} g(t_{j+1}), \quad s \in [t_j, t_{j+1}].$$

Proof. Note that, with $n = 0, 1, 2, \dots, N-1$,

$$\begin{aligned}
&\int_0^{t_{n+1}} (t_{n+1} - s)^{\alpha-1} \big(g(s) - P_1(s)\big)\, ds \\
&= \Big(\int_0^{t_1} + \sum_{j=1}^{n-1} \int_{t_j}^{t_{j+1}} + \int_{t_n}^{t_{n+1}} \Big) (t_{n+1} - s)^{\alpha-1} \big(g(s) - P_1(s)\big)\, ds \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

For I_1, we have, by Assumption 1,

$$\begin{aligned}
I_1 &= \frac{1}{\Gamma(\alpha)} \Big| \int_0^{t_1} (t_{n+1} - s)^{\alpha-1} \Big[\frac{s - t_1}{-t_1} \int_0^s g'(\tau)\, d\tau - \frac{s}{t_1} \int_s^{t_1} g'(\tau)\, d\tau \Big]\, ds \Big| \\
&\leq C \int_0^{t_1} (t_{n+1} - s)^{\alpha-1} s^{\sigma}\, ds + C \int_0^{t_1} (t_{n+1} - s)^{\alpha-1} t_1^{\sigma}\, ds.
\end{aligned}$$

Note that there exists a constant $c > 0$ such that

$$t_{n+1} \geq t_{n+1} - t_1 \geq c t_{n+1}, \; n = 1, 2, \dots, N-1,$$

which follows from

$$1 \leq \frac{t_{n+1}}{t_{n+1} - t_1} = \frac{(\frac{n+1}{N})^r}{(\frac{n+1}{N})^r - (\frac{1}{N})^r} = 1 + \frac{1}{(n+1)^r - 1} \leq 1 + \frac{1}{2^r - 1} \leq C.$$

If $0 < \alpha \leq 1$, then we have

$$\begin{aligned}
|I_1| &\leq C (t_{n+1} - t_1)^{\alpha-1} \int_0^{t_1} s^{\sigma}\, ds + C (t_{n+1} - t_1)^{\alpha-1} (t_1)^{\sigma+1} \\
&\leq C (t_{n+1} - t_1)^{\alpha-1} (t_1)^{\sigma+1} \leq C (t_{n+1})^{\alpha-1} (t_1)^{\sigma+1} \\
&\leq C (t_n)^{\alpha-1} (t_1)^{\sigma+1} = C (n^{r(\alpha-1)} N^{-r(\alpha+\sigma)}) \leq C N^{-r(\alpha+\sigma)}. \qquad (2.1)
\end{aligned}$$

If $\alpha > 1$, then we have

$$\begin{aligned}
|I_1| &\leq C (t_{n+1})^{\alpha-1} \int_0^{t_1} s^{\sigma}\, ds + C (t_{n+1})^{\alpha-1} (t_1)^{\sigma+1} \\
&\leq C (t_{n+1})^{\alpha-1} (t_1)^{\sigma+1} \leq C (t_n)^{\alpha-1} (t_1)^{\sigma+1} \\
&= C (n^{r(\alpha-1)} N^{-r(\alpha+\sigma)}) \leq C N^{-r(1+\sigma)}. \qquad (2.2)
\end{aligned}$$

For I_2, we have, with $\xi_j \in (t_j, t_{j+1})$, $j = 1, 2, \dots, n-1$ and $n = 2, 3, \dots, N-1$,

$$|I_2| = \Big| \sum_{j=1}^{n-1} \int_{t_j}^{t_{j+1}} (t_{n+1} - s)^{\alpha-1} g''(\xi_j)(s - t_j)(s - t_{j+1})\, ds \Big|.$$

By Assumption 1 and utilizing Stynes et al. [22, Section 5.2], with $n \geq 4$, we have

$$\begin{aligned}
|I_2| &\leq C\Big|\sum_{j=1}^{n-1}(t_{j+1}-t_j)^2(t_j)^{\sigma-2}\int_{t_j}^{t_{j+1}}(t_{n+1}-s)^{\alpha-1}\,ds\Big| \\
&\leq C\Big|\sum_{j=1}^{\lceil\frac{n-1}{2}\rceil-1}(t_{j+1}-t_j)^2(t_j)^{\sigma-2}\int_{t_j}^{t_{j+1}}(t_{n+1}-s)^{\alpha-1}\,ds\Big| \\
&\quad + C\Big|\sum_{j=\lceil\frac{n-1}{2}\rceil}^{n-1}(t_{j+1}-t_j)^2(t_j)^{\sigma-2}\int_{t_j}^{t_{j+1}}(t_{n+1}-s)^{\alpha-1}\,ds\Big| \\
&= I_{21}+I_{22},
\end{aligned}$$

where $\lceil\frac{n-1}{2}\rceil$ is the smallest integer $\geq \frac{n-1}{2}$.

For I_{21}, we first consider the case $0 < \alpha \leq 1$, we have, with $n \geq 4$,

$$\begin{aligned}
I_{21} &\leq C\sum_{j=1}^{\lceil\frac{n-1}{2}\rceil-1}(t_{j+1}-t_j)^2(t_j)^{\sigma-2}(t_{n+1}-t_{j+1})^{\alpha-1}(t_{j+1}-t_j) \\
&\leq C\sum_{j=1}^{\lceil\frac{n-1}{2}\rceil-1}(t_{j+1}-t_j)^3(t_j)^{\sigma-2}(t_{n+1}-t_{j+1})^{\alpha-1}.
\end{aligned}$$

Note that, with $\xi_j \in [j, j+1]$, $j = 1, 2, \dots, \lceil\frac{n-1}{2}\rceil - 1$,

$$t_{j+1}-t_j = ((j+1)^r - j^r)N^{-r} = r\xi_j^{r-1}N^{-r} \leq r(j+1)^{r-1}N^{-r} \leq Cj^{r-1}N^{-r}, \tag{2.3}$$

and

$$\begin{aligned}
(t_{n+1}-t_{j+1})^{\alpha-1} &= \Big(\frac{N^r}{(n+1)^r-(j+1)^r}\Big)^{1-\alpha} \leq \Big(\frac{N^r}{(n+1)^r-\lceil\frac{n+1}{2}\rceil^r}\Big)^{1-\alpha} \\
&\leq C\big(N^r(n+1)^{-r}\big)^{1-\alpha} \leq C(N/n)^{r(1-\alpha)}. \qquad (2.4)
\end{aligned}$$

Thus, with $n \geq 4$,

$$\begin{aligned}
I_{21} &\leq C\sum_{j=1}^{\lceil\frac{n-1}{2}\rceil-1}\big(j^{r-1}N^{-r}\big)^3(j/N)^{r(\sigma-2)}(N/n)^{r(1-\alpha)} \\
&= C\sum_{j=1}^{\lceil\frac{n-1}{2}\rceil-1}j^{r(\alpha+\sigma)-3}N^{-r(\sigma+\alpha)}(j/n)^{r(1-\alpha)} = CN^{-r(\sigma+\alpha)}\sum_{j=1}^{\lceil\frac{n-1}{2}\rceil-1}j^{r(\alpha+\sigma)-3}.
\end{aligned}$$

Case 1, if $r(\sigma+\alpha) < 2$, we have

$$I_{21} \leq CN^{-r(\sigma+\alpha)}\sum_{j=1}^{\lceil\frac{n-1}{2}\rceil-1}j^{r(\sigma+\alpha)-3} \leq CN^{-r(\sigma+\alpha)}.$$

Case 2, if $r(\sigma+\alpha) = 2$, we have

$$I_{21} \leq CN^{-2}\sum_{j=1}^{\lceil\frac{n-1}{2}\rceil-1}j^{-1} \leq CN^{-2}\Big(1+\frac{1}{2}+\cdots+\frac{1}{N}\Big) \leq CN^{-2}\ln(N).$$

Case 3, if $r(\sigma+\alpha) > 2$, we have

$$I_{21} \le CN^{-r(\sigma+\alpha)} \sum_{j=1}^{\lceil \frac{n-1}{2} \rceil -1} j^{r(\sigma+\alpha)-3} \le CN^{-r(\sigma+\alpha)} n^{r(\sigma+\alpha)-2} = C(n/N)^{r(\sigma+\alpha)-2} N^{-2} \le CN^{-2}.$$

Thus we have, with $0 < \alpha \le 1$,

$$I_{21} \le \begin{cases} CN^{-r(\sigma+\alpha)}, & \text{if } r(\sigma+\alpha) < 2, \\ CN^{-2}\ln(N), & \text{if } r(\sigma+\alpha) = 2, \\ CN^{-2}, & \text{if } r(\sigma+\alpha) > 2. \end{cases}$$

We next consider the case $\alpha > 1$, we have, with $n \ge 4$,

$$\begin{aligned} I_{21} &\le C \sum_{j=1}^{\lceil \frac{n-1}{2} \rceil -1} (t_{j+1}-t_j)^2 (t_j)^{\sigma-2} (t_{n+1}-t_j)^{\alpha-1} (t_{j+1}-t_j) \\ &\le C \sum_{j=1}^{\lceil \frac{n-1}{2} \rceil -1} (t_{j+1}-t_j)^3 (t_j)^{\sigma-2} (t_{n+1})^{\alpha-1} \\ &\le C \sum_{j=1}^{\lceil \frac{n-1}{2} \rceil -1} \big(j^{r-1}N^{-r}\big)^3 (j/N)^{r(\sigma-2)} (n/N)^{r(\alpha-1)} \\ &\le CN^{-r-r\sigma} \sum_{j=1}^{\lceil \frac{n-1}{2} \rceil -1} j^{r(1+\sigma)-3}. \end{aligned}$$

Thus we have, with $\alpha > 1$,

$$I_{21} \le \begin{cases} CN^{-r(1+\sigma)}, & \text{if } r(1+\sigma) < 2, \\ CN^{-2}\ln(N), & \text{if } r(1+\sigma) = 2, \\ CN^{-2}, & \text{if } r(1+\sigma) > 2. \end{cases}$$

For I_{22}, by (2.3) and noting that, with $\lceil \frac{n-1}{2} \rceil \le j \le n-1$, $n \ge 2$,

$$(t_j)^{\sigma-2} = (j/N)^{r(\sigma-2)} = (N/j)^{r(2-\sigma)} \le C(N/n)^{r(2-\sigma)},$$

we have

$$\begin{aligned} I_{22} &\le C\Big| \sum_{j=\lceil \frac{n-1}{2} \rceil}^{n-1} (n^{r-1}N^{-r})^2 (N/n)^{r(2-\sigma)} \int_{t_j}^{t_{j+1}} (t_{n+1}-s)^{\alpha-1}\, ds \Big| \\ &\le Cn^{r\sigma-2} N^{-r\sigma} \int_{t_{\lceil \frac{n-1}{2} \rceil}}^{t_n} (t_{n+1}-s)^{\alpha-1}\, ds. \end{aligned}$$

Note that

$$\begin{aligned} &\int_{t_{\lceil \frac{n-1}{2} \rceil}}^{t_n} (t_{n+1}-s)^{\alpha-1}\, ds = \frac{1}{\alpha}\Big[(t_{n+1} - t_{\lceil \frac{n-1}{2} \rceil})^\alpha - (t_{n+1}-t_n)^\alpha \Big] \\ &\le \frac{1}{\alpha}(t_{n+1} - t_{\lceil \frac{n-1}{2} \rceil})^\alpha \le \frac{1}{\alpha}(t_{n+1})^\alpha = \frac{1}{\alpha}\big((n+1)/N\big)^{r\alpha} \le C(n/N)^{r\alpha}, \end{aligned} \tag{2.5}$$

we get, with $n \geq 2$ and $\alpha > 0$,

$$I_{22} \leq Cn^{r\sigma-2}N^{-r\sigma}(n/N)^{r\alpha} = CN^{-r(\sigma+\alpha)}n^{r(\sigma+\alpha)-2} \leq \begin{cases} CN^{-r(\sigma+\alpha)}, & \text{if } r(\sigma+\alpha) < 2, \\ CN^{-2}, & \text{if } r(\sigma+\alpha) \geq 2. \end{cases}$$

For I_3, we have, with $\xi_n \in (t_n, t_{n+1})$, $n = 1, 2, \dots, N-1$,

$$\begin{aligned} |I_3| &= \Big| \int_{t_n}^{t_{n+1}} (t_{n+1}-s)^{\alpha-1}\big(g(s) - P_1(s)\big)\, ds \Big| \\ &= \Big| \int_{t_n}^{t_{n+1}} (t_{n+1}-s)^{\alpha-1} g''(\xi_n)(s-t_n)(s-t_{n+1})\, ds \Big|. \end{aligned}$$

By Assumption 1 and (2.3), we have, with $\alpha > 0$,

$$\begin{aligned} |I_3| &\leq C(t_{n+1}-t_n)^2 (t_n)^{\sigma-2} \int_{t_n}^{t_{n+1}} (t_{n+1}-s)^{\alpha-1}\, ds \\ &= C(t_{n+1}-t_n)^2 (t_n)^{\sigma-2} \frac{1}{\alpha}(t_{n+1}-t_n)^{\alpha} = C(t_{n+1}-t_n)^{2+\alpha}(t_n)^{\sigma-2} \\ &\leq C(n^{r-1}N^r)^{2+\alpha}(n/N)^{r(\sigma-2)} = Cn^{r(\alpha+\sigma)-2-\alpha}N^{-r(\alpha+\sigma)} \\ &\leq \begin{cases} CN^{-r(\sigma+\alpha)}, & \text{if } r(\sigma+\alpha) < 2+\alpha, \\ CN^{-(2+\alpha)}, & \text{if } r(\sigma+\alpha) \geq 2+\alpha. \end{cases} \end{aligned}$$

Obviously the bound for I_3 is stronger than the bound for I_{21}.

Together these estimates complete the proof of Lemma 2.1. □

LEMMA 2.2. *Let $\alpha > 0$. We have*

1. $a_{j,n+1} > 0, j = 0, 1, 2, \dots, n+1$ *where $a_{j,n+1}$ are the weights defined in* (1.8).
2. $b_{j,n+1} > 0$, $j = 0, 1, 2, \dots, n$, *where $b_{j,n+1}$ are the weights defined in* (1.7).

Proof. It is obvious that $a_{0,n+1} > 0$, $a_{n+1,n+1} > 0$. For $j = 1, 2, \dots, n$, we have

$$a_{j,n+1} = \int_{t_{j-1}}^{t_j} (t_{n+1}-s)^{\alpha-1} \frac{s-t_{j-1}}{t_j - t_{j-1}} ds + \int_{t_j}^{t_{j+1}} (t_{n+1}-s)^{\alpha-1} \frac{s-t_{j+1}}{t_j - t_{j+1}} ds,$$

which is also positive obviously. Further we have, with $j = 0, 1, 2, \dots, n$,

$$b_{j,n+1} = \frac{N^{-r\alpha}}{\alpha}\big(((n+1)^r - j^r)^{\alpha} - ((n+1)^r - (j+1)^r)^{\alpha}\big) > 0.$$

The proof of Lemma 2.2 is complete. □

LEMMA 2.3. *Let $\alpha > 0$. We have, with $n = 0, 1, 2, \dots, N-1$,*

$$a_{n+1,n+1} \leq CN^{-r\alpha}n^{(r-1)\alpha},$$

where $a_{n+1,n+1}$ is defined in (1.8).

Proof. We have, by (1.8), with $\xi_n \in (n, n+1)$,

$$\begin{aligned} a_{n+1,n+1} &\leq C(t_{n+1}-t_n)^{\alpha} = CN^{-r\alpha}\big((n+1)^r - n^r\big)^{\alpha} = CN^{-r\alpha}(r\xi_n^{r-1})^{\alpha} \\ &\leq CN^{-r\alpha}\big(r(n+1)^{r-1}\big)^{\alpha} \leq CN^{-r\alpha}n^{(r-1)\alpha}. \end{aligned}$$

The proof of Lemma 2.3 is complete.
□

LEMMA 2.4. *Let $\alpha > 0$. Assume that $g(t)$ satisfies Assumption 1.*

1. If $0 < \alpha \le 1$, we have

$$\Big| a_{n+1,n+1} \int_0^{t_{n+1}} (t_{n+1}-s)^{\alpha-1} \big(g(s) - P_0(s) \big) \, ds \Big| \le \begin{cases} CN^{-r(\alpha+\sigma)}, & \text{if } r(\alpha+\sigma) < 1+\alpha, \\ CN^{-r(\alpha+\sigma)} \ln(N), & \text{if } r(\alpha+\sigma) = 1+\alpha, \\ CN^{-1-\alpha}, & \text{if } r(\alpha+\sigma) > 1+\alpha. \end{cases}$$

2. If $\alpha > 1$, we have

$$\Big| a_{n+1,n+1} \int_0^{t_{n+1}} (t_{n+1}-s)^{\alpha-1} \big(g(s) - P_0(s) \big) \, ds \Big| \le \begin{cases} CN^{-r(\alpha+\sigma)}, & \text{if } r(\alpha+\sigma) < 1+\alpha, \\ CN^{-1-\alpha}, & \text{if } r(\alpha+\sigma) \ge 1+\alpha, \end{cases}$$

where $P_0(s)$ is the piecewise constant function defined by, with $j = 0, 1, 2, \dots, n$,

$$P_0(s) = g(t_j), \quad s \in [t_j, t_{j+1}].$$

Proof. The proof is similar to the proof of Lemma 2.1. Note that

$$\begin{aligned} & a_{n+1,n+1} \int_0^{t_{n+1}} (t_{n+1} - s)^{\alpha-1} \big(g(s) - P_0(s) \big) \, ds \\ &= a_{n+1,n+1} \Big(\int_0^{t_1} + \sum_{j=1}^{n-1} \int_{t_j}^{t_{j+1}} + \int_{t_n}^{t_{n+1}} \Big) (t_{n+1} - s)^{\alpha-1} (g(s) - P_0(s)) \, ds \\ &= I_1' + I_2' + I_3'. \end{aligned}$$

For I_1', we have, by Assumption 1 and Lemma 2.3

$$\begin{aligned} |I_1'| &\le a_{n+1,n+1} \Big(\int_0^{t_1} (t_{n+1} - s)^{\alpha-1} |g(s)| \, ds + \int_0^{t_1} (t_{n+1} - s)^{\alpha-1} |P_0(s)| \, ds \Big) \\ &\le \big(CN^{-r\alpha} n^{(r-1)\alpha} \big) \Big(\int_0^{t_1} (t_{n+1} - s)^{\alpha-1} s^\sigma \, ds + \int_0^{t_1} (t_{n+1} - s)^{\alpha-1} 0^\sigma \, ds \Big) \\ &= \big(CN^{-r\alpha} n^{(r-1)\alpha} \big) \int_0^{t_1} (t_{n+1} - s)^{\alpha-1} s^\sigma \, ds. \end{aligned}$$

If $0 < \alpha \le 1$, by (2.1), we have

$$\begin{aligned} |I_1'| &\le \big(CN^{-r\alpha} n^{(r-1)\alpha} \big) (t_{n+1} - t_1)^{\alpha-1} (t_1)^{\sigma+1} \\ &\le \big(CN^{-r\alpha} n^{(r-1)\alpha} \big) (CN^{-r(\alpha+\sigma)}) = C(n/N)^{r\alpha} n^{-\alpha} (CN^{-r(\alpha+\sigma)}) \le CN^{-r(\alpha+\sigma)}. \end{aligned}$$

If $\alpha > 1$, by (2.2), we have

$$\begin{aligned} |I_1'| &\le \big(CN^{-r\alpha} n^{(r-1)\alpha} \big) (t_{n+1})^{\alpha-1} (t_1)^{\sigma+1} \\ &\le \big(CN^{-r\alpha} n^{(r-1)\alpha} \big) (CN^{-r(1+\sigma)}) = C(n/N)^{(r-1)\alpha} N^{-\alpha} N^{-r(1+\sigma)} \\ &\le CN^{-r(1+\sigma)-\alpha} \le CN^{-1-\alpha}. \end{aligned}$$

For I_2', we have, with $\xi_j \in (t_j, t_{j+1})$, $j = 1, 2, \dots, n-1$,

$$|I_2'| \le a_{n+1,n+1} \sum_{j=1}^{n-1} \int_{t_j}^{t_{j+1}} (t_{n+1} - s)^{\alpha-1} |f'(\xi_j)| (s - t_j) \, ds.$$

Hence, by Assumption 1,

$$|I_2'| \le Ca_{n+1,n+1}\Big(\sum_{j=1}^{\lceil\frac{n-1}{2}\rceil-1}+\sum_{j=\lceil\frac{n-1}{2}\rceil}^{n-1}\Big)(t_{j+1}-t_j)(t_j)^{\sigma-1}\int_{t_j}^{t_{j+1}}(t_{n+1}-s)^{\alpha-1}\,ds$$
$$= I_{21}'+I_{22}'.$$

For I_{21}', if $0<\alpha\le 1$, then we have, by Lemma 2.3, (2.3), (2.4), with $n\ge 4$,

$$\begin{aligned}
I_{21}' &\le \big(CN^{-r\alpha}n^{(r-1)\alpha}\big)\sum_{j=1}^{\lceil\frac{n-1}{2}\rceil-1}\big((t_{j+1}-t_j)^2(t_j)^{\sigma-1}(t_{n+1}-t_{j+1})^{\alpha-1}\big)\\
&\le \big(CN^{-r\alpha}n^{(r-1)\alpha}\big)\sum_{j=1}^{\lceil\frac{n-1}{2}\rceil-1}(j^{r-1}N^{-r})^2(j/N)^{r(\sigma-1)}(N/n)^{r(1-\alpha)}\\
&= C(n/N)^{r\alpha}\sum_{j=1}^{\lceil\frac{n-1}{2}\rceil-1}j^{r(\alpha+\sigma)-2-\alpha}(j/n)^{\alpha}(j/n)^{r(1-\alpha)}N^{-r(\alpha+\sigma)}\\
&\le CN^{-r(\alpha+\sigma)}\sum_{j=1}^{\lceil\frac{n-1}{2}\rceil-1}j^{r(\alpha+\sigma)-2-\alpha}\le\begin{cases} CN^{-r(\alpha+\sigma)}, & \text{if } r(\alpha+\sigma)<1+\alpha,\\ CN^{-r(\alpha+\sigma)}\ln(N), & \text{if } r(\alpha+\sigma)=1+\alpha,\\ CN^{-1-\alpha}, & \text{if } r(\alpha+\sigma)>1+\alpha.\end{cases}
\end{aligned}$$

If $\alpha>1$, we have

$$\begin{aligned}
I_{21}' &\le \big(CN^{-r\alpha}n^{(r-1)\alpha}\big)\sum_{j=1}^{\lceil\frac{n-1}{2}\rceil-1}\big((t_{j+1}-t_j)^2(t_j)^{\sigma-1}(t_{n+1})^{\alpha-1}\big)\\
&\le \big(CN^{-r\alpha}n^{(r-1)\alpha}\big)\sum_{j=1}^{\lceil\frac{n-1}{2}\rceil-1}(j^{r-1}N^{-r})^2(j/N)^{r(\sigma-1)}(N/n)^{r(1-\alpha)}\\
&= C(n/N)^{(r-1)\alpha}N^{-\alpha}N^{-r\sigma-r}\sum_{j=1}^{\lceil\frac{n-1}{2}\rceil-1}j^{r+r\sigma-2}\\
&\le CN^{-\alpha-r\sigma-r}\sum_{j=1}^{\lceil\frac{n-1}{2}\rceil-1}j^{r+r\sigma-2}.
\end{aligned}$$

Note that $r+r\sigma-2>-1$ for any $r\ge 1$. Hence we have

$$I_{21}'\le CN^{-\alpha-r\sigma-r}n^{r+r\sigma-1}=C(n/N)^{r+r\sigma-1}N^{-1-\alpha}\le CN^{-1-\alpha}.$$

For I_{22}', we have

$$I_{22}'\le (CN^{-r\alpha}n^{(r-1)\alpha})\sum_{j=\lceil\frac{n-1}{2}\rceil}^{n-1}\Big((t_{j+1}-t_j)(t_j)^{\sigma-1}\int_{t_j}^{t_{j+1}}(t_{n+1}-s)^{\alpha-1}\,ds\Big).$$

By (2.3) and noting that, with $\lceil\frac{n-1}{2}\rceil\le j\le n-1$, $n\ge 2$,

$$(t_j)^{\sigma-1}=(j/N)^{r(\sigma-1)}=(N/j)^{r(1-\sigma)}\le C(N/n)^{r(1-\sigma)},$$

we have, by (2.5), with $\alpha > 0$,

$$
\begin{aligned}
I'_{22} &\le (CN^{-r\alpha}n^{(r-1)\alpha}) \sum_{j=\lceil \frac{n-1}{2} \rceil}^{n-1} \Big((Cn^{r-1}N^{-r})(N/n)^{r(1-\sigma)} \int_{t_j}^{t_{j+1}} (t_{n+1}-s)^{\alpha-1}\, ds \Big) \\
&\le (CN^{-r\alpha}n^{(r-1)\alpha}) n^{r-1-r+\sigma} N^{-r+r-r\sigma} (n/N)^{r\alpha} \le Cn^{r(\sigma+\alpha)-1-\alpha} N^{-r(\sigma+\alpha)} \\
&\le \begin{cases} CN^{-r(\sigma+\alpha)}, & \text{if } r(\sigma+\alpha) < 1+\alpha, \\ CN^{-1-\alpha}, & \text{if } r(\sigma+\alpha) \ge 1+\alpha. \end{cases}
\end{aligned}
$$

For I'_3, we have, with $\alpha > 0$,

$$
\begin{aligned}
|I'_3| &\le (CN^{-r\alpha}n^{(r-1)\alpha})(t_{n+1}-t_n)(t_n)^{\sigma-1}(t_{n+1}-t_n)^{\alpha} \\
&\le (CN^{-r\alpha}n^{(r-1)\alpha})(t_{n+1}-t_n)^{1+\alpha}(t_n)^{\sigma-1}.
\end{aligned}
$$

By (2.3), we have

$$
\begin{aligned}
|I'_3| &\le (CN^{-r\alpha}n^{(r-1)\alpha})(n^{r-1}N^{-r})^{1+\alpha}(n/N)^{r(\sigma-1)} \\
&= C(n/N)^{r\alpha} n^{-\alpha} n^{r(\alpha+\sigma)-\alpha-1} N^{-r(\alpha+\sigma)} \\
&\le Cn^{r(\alpha+\sigma)-\alpha-1} N^{-r(\alpha+\sigma)} \\
&\le \begin{cases} CN^{-r(\alpha+\sigma)}, & \text{if } r(\alpha+\sigma) < 1+\alpha, \\ CN^{-1-\alpha}, & \text{if } r(\alpha+\sigma) \ge 1+\alpha. \end{cases}
\end{aligned}
$$

Together these estimates complete the proof of Lemma 2.4. ∎

LEMMA 2.5. *Let $\alpha > 0$. There exists a positive constant C such that*

$$\sum_{j=0}^{n} a_{j,n+1} \le CT^{\alpha}, \tag{2.6}$$

$$\sum_{j=0}^{n} b_{j,n+1} \le CT^{\alpha}, \tag{2.7}$$

where $\alpha_{j,n+1}$ and $b_{j,n+1}, j = 0, 1, 2, \dots, n$ are defined by (1.8) *and* (1.7)*, respectively.*

Proof. We only prove (2.6). The proof of (2.7) is similar. Note that

$$\int_0^{t_{n+1}} (t_{n+1}-s)^{\alpha-1} g(s)ds = \sum_{j=0}^{n+1} a_{j,n+1} g(t_j) + R_1,$$

where R_1 is the remainder term. Let $g(s) = 1$, we have

$$\sum_{j=0}^{n+1} a_{j,n+1} = \int_0^{t_{n+1}} (t_{n+1}-s)^{\alpha-1} \cdot 1\, ds = \frac{1}{\alpha}(t_{n+1})^{\alpha} \le CT^{\alpha}.$$

Thus (2.6) follows by the fact $a_{n+1,n+1} > 0$ in Lemma 2.2.
∎

Now we turn to the proof of Theorem 1.5.

Proof. [Proof of Theorem 1.5] Subtracting (1.3) from (1.6), we have

$$\begin{aligned}
&y(t_{n+1}) - y_{n+1} \\
&= \frac{1}{\Gamma(\alpha)}\Big\{ \int_0^{t_{n+1}} (t_{n+1}-s)^{\alpha-1}\big(f(s,y(s)) - P_1(s)\big)\, ds \\
&\quad + \sum_{j=0}^{n} a_{j,n+1}\big(f(t_j,y(t_j)) - f(t_j,y_j)\big) + a_{n+1,n+1}\big(f(t_{n+1},y(t_{n+1})) - f(t_{n+1},y_{n+1}^P)\big)\Big\} \\
&= \frac{1}{\Gamma(\alpha)}(I + II + III).
\end{aligned}$$

The term I is estimated by Lemma 2.1. For II, we have, by Lemma 2.2 and the Lipschitz condition of f,

$$\begin{aligned}
|II| &= \Big|\sum_{j=0}^{n} a_{j,n+1}\big(f(t_j,y(t_j)) - f(t_j,y_j)\big)\Big| \le \sum_{j=0}^{n} a_{j,n+1}\big|f(t_j,y(t_j)) - f(t_j,y_j)\big| \\
&\le L\sum_{j=0}^{n} a_{j,n+1}|y(t_j) - y_j|.
\end{aligned}$$

For III, we have, by Lemma 2.2 and the Lipschitz condition for f,

$$|III| = \big|a_{n+1,n+1}\big(f(t_{n+1},y(t_{n+1})) - f(t_{n+1},y_{n+1}^P)\big)\big| \le a_{n+1,n+1}L|y(t_{n+1}) - y_{n+1}^P|.$$

Note that,

$$\begin{aligned}
y(t_{n+1}) - y_{n+1}^P = \frac{1}{\Gamma(\alpha)}\Big\{ &\int_0^{t_{n+1}} (t_{n+1}-s)^{\alpha-1}\big(f(s,y(s)) - P_0(s)\big)\, ds \\
&+ \sum_{j=0}^{n} b_{j,n+1}\big(f(t_j,y(t_j)) - f(t_j,y_j)\big)\Big\}.
\end{aligned}$$

Thus

$$\begin{aligned}
|III| &\le Ca_{n+1,n+1}L\int_0^{t_{n+1}} (t_{n+1}-s)^{\alpha-1}\big|f(s,y(s)) - P_0(s)\big|\, ds \\
&\quad + Ca_{n+1,n+1}L\sum_{j=0}^{n} b_{j,n+1}\big|f(t_j,y(t_j)) - f(t_j,y_j)\big| \\
&= III_1 + III_2.
\end{aligned}$$

The term III_1 is estimated by Lemma 2.4. For III_2, we have, by Lemmas 2.2, 2.3,

$$\begin{aligned}
III_2 &\le Ca_{n+1,n+1}\sum_{j=0}^{n} b_{j,n+1}|y(t_j) - y_j| \le \big(CN^{-r\alpha}n^{(r-1)\alpha}\big)\sum_{j=0}^{n} b_{j,n+1}|y(t_j) - y_j| \\
&\le C(n/N)^{(r-1)\alpha}N^{-\alpha}\sum_{j=0}^{n} b_{j,n+1}|y(t_j) - y_j| \le CN^{-\alpha}\sum_{j=0}^{n} b_{j,n+1}|y(t_j) - y_j|.
\end{aligned}$$

Hence we obtain

$$
\begin{aligned}
|y(t_{n+1}) - y_{n+1}| &\le C|I| + C\sum_{j=0}^{n} a_{j,n+1}|y(t_j) - y_j| \\
&\quad + C|III_1| + CN^{-\alpha}\sum_{j=0}^{n} b_{j,n+1}|y(t_j) - y_j|. \qquad (2.8)
\end{aligned}
$$

To complete the proof of Theorem 1.5, we shall use the mathematical induction.

We first consider the case $0 < \alpha \le 1$. In this case, we discuss the error estimates in the following four cases.

Case 1. Let $r(\alpha+\sigma) > \max\{2, 1+\alpha\} = 2$. Assume that there exists a constant $C_0 > 0$ such that, with $j = 0, 1, 2, \dots, n,\ n = 0, 1, 2, \dots, N-1$,

$$|y(t_j) - y_j| \le C_0 N^{-1-\alpha},$$

we shall show that

$$|y(t_{n+1}) - y_{n+1}| \le C_0 N^{-1-\alpha}.$$

In fact, by Lemmas 2.1 and 2.4, we have

$$
\begin{aligned}
|y(t_{n+1}) - y_{n+1}| &\le CN^{-2} + C\sum_{j=0}^{n} a_{j,n+1}|y(t_j) - y_j| \\
&\quad + CN^{-1-\alpha} + CN^{-\alpha}\sum_{j=0}^{n} b_{j,n+1}|y(t_j) - y_j| \\
&\le CN^{-2} + C_0 C T^{\alpha} N^{-1-\alpha} + CN^{-1-\alpha} + T^{\alpha} C_0 N^{-\alpha} N^{-1-\alpha}. \qquad (2.9)
\end{aligned}
$$

Following the idea of the proof for [8, Lemma 3.1, pp.41], we may first choose T sufficiently small such that the second term of the right hand side of (2.9) is less than $\frac{C_0}{2}N^{-1-\alpha}$, then choose N sufficiently large and C_0 sufficiently large such that the summation of the other terms in the right hand side of (2.9) is also less than $\frac{C_0}{2}N^{-1-\alpha}$. Thus we get

$$|y(t_{n+1}) - y_{n+1}| \le C_0 N^{-1-\alpha}.$$

Case 2. Let $r(\alpha+\sigma) \le \min\{2, 1+\alpha\} = 1+\alpha$. Assume that there exists a constant $C_0 > 0$ such that, with $j = 0, 1, 2, \dots, n,\ n = 0, 1, 2, \dots, N-1$,

$$|y(t_j) - y_j| \le C_0 N^{-r(\alpha+\sigma)}.$$

Following the similar argument as in Case 1, we may show that

$$|y(t_{n+1}) - y_{n+1}| \le C_0 N^{-r(\alpha+\sigma)}.$$

Case 3. Let $1+\alpha < r(\alpha+\sigma) \le 2$. We may show that

$$|y(t_{n+1}) - y_{n+1}| \le C_0 N^{-1-\alpha}.$$

Case 4. Let $r(\alpha+\sigma) = 1+\alpha$. We may show that

$$|y(t_{n+1}) - y_{n+1}| \le C_0 N^{-1-\alpha}\ln(N).$$

We next consider the case $\alpha > 1$. In this case, we also discuss the error estimates in the following four cases.

Case 1. Let $r > \max\{\frac{1+\alpha}{\alpha+\sigma}, \frac{2}{1+\sigma}\} = \frac{2}{1+\sigma}$. Assume that there exists a constant $C_0 > 0$ such that, with $j = 0, 1, 2, \dots, n,\ n = 0, 1, 2, \dots, N-1$,

$$|y(t_j) - y_j| \leq C_0 N^{-2},$$

we shall show that

$$|y(t_{n+1}) - y_{n+1}| \leq C_0 N^{-2}.$$

In fact, we have, using the same argument as in the proof of (2.8),

$$\begin{aligned} |y(t_{n+1}) - y_{n+1}| &\leq CN^{-2} + C \sum_{j=0}^{n} a_{j,n+1} |y(t_j) - y_j| \\ &\quad + CN^{-1-\alpha} + CN^{-\alpha} \sum_{j=0}^{n} b_{j,n+1} |y(t_j) - y_j| \\ &\leq CN^{-2} + C_0 T^{\alpha} N^{-2} + CN^{-1-\alpha} + T^{\alpha} C_0 N^{-\alpha} N^{-2}. \end{aligned} \tag{2.10}$$

Following the idea of the proof for [8, Lemma 3.1, pp.41], we may first choose T sufficiently small such that the second term of the right hand side of (2.10) is less than $\frac{C_0}{2} N^{-2}$, then choose N sufficiently large and C_0 sufficiently large such that the summation of the other terms in the right hand side of (2.10) is also less than $\frac{C_0}{2} N^{-2}$. Thus we get

$$|y(t_{n+1}) - y_{n+1}| \leq C_0 N^{-2}.$$

Case 2. Let $r < \min\{\frac{1+\alpha}{\alpha+\sigma}, \frac{2}{1+\sigma}\} = \frac{1+\alpha}{\alpha+\sigma}$. Assume that there exists a constant $C_0 > 0$ such that, with $j = 0, 1, 2, \dots, n,\ n = 0, 1, 2, \dots, N-1$,

$$|y(t_j) - y_j| \leq C_0 N^{-r(1+\sigma)}.$$

Following the similar argument as in Case 1, we may show that,

$$|y(t_{n+1}) - y_{n+1}| \leq C_0 N^{-r(1+\sigma)}.$$

Case 3. Let $\frac{1+\alpha}{\alpha+\sigma} \leq r < \frac{2}{1+\sigma}$. We may show that

$$|y(t_{n+1}) - y_{n+1}| \leq C_0 N^{-r(1+\sigma)}.$$

Case 4. Let $r = \frac{2}{1+\sigma}$. We may show that

$$|y(t_{n+1}) - y_{n+1}| \leq C_0 N^{-2} \ln(N).$$

Together these estimates complete the proof of Theorem 1.5. □

3. Numerical examples. In this section, we will give some numerical examples to illustrate the convergence orders of the numerical method (1.6) under the different smoothness assumptions of ${}_0^C D_t^{\alpha} y$ in (1.3). For simplicity, we only present the numerical results for the case $\alpha \in (0, 1)$. Similarly we may obtain the numerical results for $\alpha > 1$.

EXAMPLE 3.1. *Consider, with $0 < \alpha < 1$, $0 < \beta < 1$ and $\alpha < \beta$,*

$${}_0^C D_t^{\alpha} y(t) = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} t^{\beta-\alpha} + t^{2\beta} - y^2, \quad t \in (0, T], \tag{3.1}$$

$$y(0) = y_0, \tag{3.2}$$

where $y_0 = 0$*, and the exact solution is* $y(t) = t^{\beta}$*, and* ${}_0^C D_t^{\alpha} y(t) = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} t^{\beta-\alpha}$*, which implies that the regularity of* ${}_0^C D_t^{\alpha} y(t)$ *behaves as* $t^{\beta-\alpha}$*. Thus we see that* ${}_0^C D_t^{\alpha} y(t)$ *satisfies the Assumption 1.*

Let N *be a positive integer. Let* $0 = t_0 < t_1 < \cdots < t_N = T$ *be the graded meshes on* $[0, T]$*, where* $t_j = T(j/N)^r, j = 0, 1, 2, \dots, N$ *with* $r \geq 1$*. For simplicity, we choose* $T = 1$*. Assume that* $y(t_j)$ *and* $y_j, j = 0, 1, 2, \dots N$ *are the solutions of* (1.3) *and* (1.6)*, respectively. We have, by Theorem 1.5 with* $\sigma = \beta - \alpha$*,*

$$\|e_N\| := \max_{0 \leq j \leq N} |y(t_j) - y_j| \leq \begin{cases} CN^{-r\beta}, & \text{if } r < \frac{1+\alpha}{\beta}, \\ CN^{-r\beta} \ln(N), & \text{if } r = \frac{1+\alpha}{\beta}, \\ CN^{-(1+\alpha)}, & \text{if } r > \frac{1+\alpha}{\beta}. \end{cases}$$

For the different $\alpha \in (0, 1)$*, we choose the different* r *and the different* $N = 20 \times 2^l, l = 1, 2, 3, 4, 5$*. We obtain the maximum nodal errors* $\|e_N\|_\infty$ *defined above with respect to the different* N*. We also calculate the experimental order of convergence (EOC) by* $\log 2\left(\frac{\|e_N\|_\infty}{\|e_{2N}\|_\infty}\right)$*.*

In Tables 1-3, we choose $\beta = 0.9$ *and we obtain the experimental orders of convergence (EOC) and the maximum nodal errors with respect to the different* N*. We see that the experimental orders of convergence (EOC) are almost* $O(N^{-r\beta}) = O(N^{-(1+\alpha)}$ *if we choose* $r = \frac{1+\alpha}{\beta}$*.*

	N=40	N=80	N=160	N=320	N=640
$r = 1$	1.43E-2	7.68E-3	4.12E-3	2.21E-3	1.18E-3
	0.897	0.899	0.899	0.899	
$r = \frac{1+\alpha}{\beta}$	5.18E-4	1.49E-4	4.27E-5	1.23E-5	3.51E-6
	1.800	1.800	1.800	1.800	

TABLE 1
Maximum nodal errors and orders of convergence for Example 3.1 with $\alpha = 0.8$ *and* $\beta = 0.9$

	N=40	N=80	N=160	N=320	N=640
$r = 1$	8.95E-3	4.83E-3	2.60E-3	1.39E-3	7.46E-4
	0.889	0.896	0.898	0.899	
$r = \frac{1+\alpha}{\beta}$	1.29E-3	3.88E-4	1.20E-4	3.82E-5	1.23E-5
	1.727	1.687	1.655	1.635	

TABLE 2
Maximum nodal errors and orders of convergence for Example 3.1 with $\alpha = 0.6$ *and* $\beta = 0.9$

	N=40	N=80	N=160	N=320	N=640
$r = 1$	4.33E-3	2.40E-3	1.30E-3	7.01E-4	3.76E-4
	0.853	0.881	0.892	0.897	
$r = \frac{1+\alpha}{\beta}$	4.64E-3	1.46E-3	4.84E-4	1.66E-4	5.87E-5
	1.667	1.595	1.541	1.503	

TABLE 3
Maximum nodal errors and orders of convergence for Example 3.1 with $\alpha = 0.4$ *and* $\beta = 0.9$

EXAMPLE 3.2. *Consider, with* $0 < \alpha < 1$,

$$ {}_0^C D_t^\alpha y(t) + y(t) = 0, \quad t \in (0, T], \tag{3.3} $$

$$ y(0) = y_0, \tag{3.4} $$

where $y_0 = 1$. *The exact solution is* $y(t) = E_{\alpha,1}(-t^\alpha)$, *and* ${}_0^C D_t^\alpha y(t) = -E_{\alpha,1}(-t^\alpha)$, *where* $E_{\alpha,\gamma}(z)$ *is the Mittag-Leffler function defined by*

$$ E_{\alpha,\gamma}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \gamma)}, \quad \alpha, \gamma > 0. $$

Hence we have

$$ {}_0^C D_t^\alpha y(t) = -1 - \frac{(-t^\alpha)}{\Gamma(\alpha+1)} - \frac{(-t^\alpha)^2}{\Gamma(2\alpha+1)} - \dots, $$

which implies that the regularity of ${}_0^C D_t^\alpha y(t)$ *behaves as* $c + ct^\alpha, 0 < \alpha < 1$. *By Theorem 1.5 with* $\sigma = \alpha$, *we have*

$$ \|e_N\|_\infty := \max_{0 \le j \le N} |y(t_j) - y_j| \le \begin{cases} CN^{-r(2\alpha)}, & \text{if } r < \frac{1+\alpha}{2\alpha}, \\ CN^{-r(2\alpha)} \ln(N), & \text{if } r = \frac{1+\alpha}{2\alpha}, \\ CN^{-(1+\alpha)}, & \text{if } r > \frac{1+\alpha}{2\alpha}. \end{cases} $$

In Tables 4-6, we obtain the experimental orders of convergence (EOC) and the maximum nodal errors with respect to the different N. *We see that the experimental orders of convergence (EOC) are almost* $O(N^{-r(2\alpha)}) = O(N^{-(1+\alpha)}$ *if we choose* $r = \frac{1+\alpha}{\alpha}$.

	N=40	N=80	N=160	N=320	N=640
$r=1$	1.14E-4	4.43E-5	1.59E-5	5.47E-6	1.85E-6
	1.370	1.481	1.535	1.564	
$r=\frac{1+\alpha}{2\alpha}$	9.48E-5	2.75E-5	8.05E-6	2.36E-6	6.94E-7
	1.784	1.773	1.768	1.767	

TABLE 4

Maximum nodal errors and orders of convergence for Example 3.2 with $\alpha = 0.8$

	N=40	N=80	N=160	N=320	N=640
$r=1$	7.57E-4	4.34E-4	2.20E-4	1.05E-4	4.83E-5
	0.803	0.981	1.069	1.118	
$r=\frac{1+\alpha}{2\alpha}$	2.58E-4	9.66E-5	3.41E-5	1.17E-5	3.93E-6
	1.418	1.503	1.547	1.570	

TABLE 5

Maximum nodal errors and orders of convergence for Example 3.2 with $\alpha = 0.6$

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	N=40	N=80	N=160	N=320	N=640
$r=1$	5.12E-4	8.33E-4	1.00E-3	8.17E-4	5.78E-4
	-0.702	-0.268	0.296	0.498	
$r=\frac{1+\alpha}{2\alpha}$	2.58E-4	9.66E-5	3.41E-5	1.17E-5	3.93E-6
	1.123	1.233	1.304	1.343	

Table 6

Maximum nodal errors and orders of convergence for Example 3.2 with $\alpha = 0.4$

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