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TRANSSERIES AS GERMS OF SURREAL FUNCTIONS

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Abstract. We show that Écalle's transseries and their variants (LE and EL-series) can be interpreted as functions from positive infinite surreal numbers to surreal numbers. The same holds for a much larger class of formal series, here called omega-series. Omega-series are the smallest subfield of the surreal numbers containing the reals, the ordinal omega, and closed under the exp and log functions and all possible infinite sums. They form a proper class, can be composed and differentiated, and are surreal analytic. The surreal numbers themselves can be interpreted as a large field of transseries containing the omega-series, but, unlike omega-series, they lack a composition operator compatible with the derivation introduced by the authors in an earlier paper.

Contents

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1. Introduction

Fields of transseries are an important tool in asymptotic analysis and played a crucial role in Écalle's approach to the problem of Dulac [Dul23, É92]. They appear in various versions, see for instance [DG87, DMM97, Hoe97, Kuh00, DMM01, Sch01, KS05, Hoe06, Hoe09] and the bibliography therein. In [BM] we proved that Conway's field **No** of surreal numbers [Con76] admits the structure of a field of transseries (in the sense of [Sch01]) and a compatible derivation (in fact more than one). We also proved the existence of "integrals", in the sense of anti-derivatives, for the "simplest" surreal derivation on **No**. This makes **No** into a Liouville closed H-field in the sense of [AD02]. We recall that an H-field is an ordered differential field with some compatibility properties between the derivation ∂ and the order; in particular if f is greater than any constant, then $\partial f > 0$. A basic example is the field of rational functions $\mathbb{R}(x)$, ordered by $x > \mathbb{R}$, with constant field $\mathbb{R} = \ker \partial$ and $\partial x = 1$. The notion of H-field arises as an attempt to axiomatize some of the properties of Hardy fields, where a Hardy field is a field of germs at $+\infty$ of eventually C^1-functions $f : \mathbb{R} \to \mathbb{R}$ closed under derivation. Such fields have been studied since the 70's, see for instance [Bou76, Ros83b, Ros83a, Ros87]. Any o-minimal structure on the reals gives rise to an H-field, namely the field of germs at $+\infty$ of its definable unary functions. In [ADH] van den Dries, Aschenbrenner and van der Hoeven proved that, with the "simplest" derivation ∂ introduced in [BM], the surreals are a universal H-field; more precisely, every H-field with "small derivations" and constant field $\mathbb{R}$ embeds in **No** as a differential field. Moreover, they proved that $(\mathbf{No}, \partial)$ satisfies the complete first order theory of the logarithmic-exponential series of [DMM97, DMM01] and therefore, by the model completeness of the theory [ADH], it admits solutions to all the differential equations that can be solved in a bigger model.

Another approach to derivation and integration on the surreal numbers was taken by Costin, Ehrlich and Friedman [CEF15] in a more analytic vein, possibly suitable

for asymptotic analysis, namely they consider derivatives and definite integrals of functions, rather than derivatives of "numbers" (elements of **No**).

This paper is a first attempt to reconcile the algebraic and the analytic approach to surreal derivation and integration through a notion of composition. The special session on surreal numbers at the joint AMS-MAA meeting in Seattle (6-9 Jan. 2016) was a timely occasion to discuss these developments and some of the results of this paper were presented during that meeting.

To discuss our contribution in more detail, we need some definitions. We recall that in **No**, as in any Hahn field, there is a formal notion of summability, and one can associate to each summable family $(x_i)_{i\in I}$ its "sum" $\sum_{i\in I} x_i \in \mathbf{No}$. We can thus define the field of **omega-series** $\mathbb{R}\langle\!\langle\omega\rangle\!\rangle$ as the smallest subfield of **No** containing $\mathbb{R}(\omega)$ and closed under exp, log and sums of summable families. Here ω is the first infinite ordinal and plays the role of a formal variable with derivative 1. It turns out that $\mathbb{R}\langle\!\langle\omega\rangle\!\rangle$ is a very big exponential field (in fact a proper class) properly containing an isomorphic copy of the logarithmic-exponential series of [DMM97, DMM01] (LE-series) and their variants, such as the exponential-logarithmic series of [Kuh00, KT12] (EL-series). More precisely, we can isolate two subfields $\mathbb{R}((\omega))^{LE} \subset \mathbb{R}((\omega))^{EL}$ of $\mathbb{R}\langle\!\langle\omega\rangle\!\rangle$ which are isomorphic to the LE and EL-series respectively. The field $\mathbb{R}((\omega))^{LE}$ is a countable union $\bigcup_{n\in\mathbb{N}} X_n \subseteq \mathbf{No}$, where $X_0 := \mathbb{R}(\omega)$ and X_{n+1} is the set of all sums of summable sequences of elements in $X_n \cup \exp(X_n) \cup \log(X_n)$. In other words, a surreal number is a LE-series if it can be obtained from $\mathbb{R}(\omega)$ by finitely many applications of $\sum, \exp, \log$ (Theorem 4.11). This remarkably simple characterization of the LE-series, which should be compared with the original definition, is made possible by working inside the surreals, with its notion of summability and exponential structure. The EL-series admit a similar characterization (Proposition 4.12).

We show that each omega-series $f \in \mathbb{R}\langle\!\langle\omega\rangle\!\rangle$, hence in particular each LE or EL-series, can be interpreted as a function from positive infinite surreal numbers to surreal numbers (Corollary 5.23). The idea is simply to substitute ω with a positive infinite surreal and evaluate the resulting expression, but the proof of summability (Lemma 5.21) is rather long and technical and it is carried out in Section 9. Similar problems were tackled in [Sch01] and in some of the cited works by van der Hoeven, although not in the context of surreal numbers. We shall borrow from those papers the idea of isolating the contributions coming from different "trees", but with enough differences to warrant an independent treatment. This will give rise to a natural composition operator $\circ : \mathbb{R}\langle\!\langle\omega\rangle\!\rangle \times \mathbf{No}^{>\mathbb{R}} \to \mathbf{No}$ (Theorem 6.3) which restricts to a composition $\circ : \mathbb{R}\langle\!\langle\omega\rangle\!\rangle \times \mathbb{R}\langle\!\langle\omega\rangle\!\rangle^{>\mathbb{R}} \to \mathbb{R}\langle\!\langle\omega\rangle\!\rangle$ extending the usual composition of ordinary power series. Formally, we define a **composition** on $\mathbb{R}\langle\!\langle\omega\rangle\!\rangle$ to be a function $\circ : \mathbb{R}\langle\!\langle\omega\rangle\!\rangle \times \mathbf{No}^{>\mathbb{R}} \to \mathbf{No}$ satisfying the following conditions for all $f, g \in \mathbb{R}\langle\!\langle\omega\rangle\!\rangle$ and $x \in \mathbf{No}^{>\mathbb{R}}$:

(1) if $f = \sum_{i<\alpha} r_i e^{\gamma_i}$, then $f \circ x = \left(\sum_{i<\alpha} r_i e^{\gamma_i}\right) \circ x = \sum_{i<\alpha} r_i e^{\gamma_i \circ x}$;
(2) $f \circ g \in \mathbb{R}\langle\!\langle\omega\rangle\!\rangle$ and $(f \circ g) \circ x = f \circ (g \circ x)$;
(3) $f \circ \omega = f$, $\omega \circ x = x$.

We then prove the following.

Theorem 6.3. *There is a (unique) composition* $\circ : \mathbb{R}\langle\!\langle\omega\rangle\!\rangle \times \mathbf{No}^{>\mathbb{R}} \to \mathbf{No}$.

In the last part of the paper we study the interaction between the derivation $\partial : \mathbf{No} \to \mathbf{No}$ introduced in [BM] and the composition on $\mathbb{R}\langle\!\langle\omega\rangle\!\rangle$. Let us recall that

in [BM] we proved the existence of several "surreal derivations" $\partial : \mathbf{No} \to \mathbf{No}$ and we studied in detail the "simplest" such derivation [BM, Def. 6.21]. It is easy to see that all surreal derivations coincide on the subfield $\mathbb{R}((\omega))$, so the latter admits a unique surreal derivation $\partial : \mathbb{R}((\omega)) \to \mathbb{R}((\omega))$. The derivation ∂ on $\mathbb{R}((\omega))$ makes it into a H-field, although not a Liouville closed one because $\partial : \mathbb{R}((\omega)) \to \mathbb{R}((\omega))$ is not surjective. There are, however, many subfields of $\mathbb{R}((\omega))$ which are Liouville closed, among which $\mathbb{R}((\omega))^{LE}$.

We will show that the formal derivative ∂f of an omega-series $f \in \mathbb{R}((\omega))$ can be interpreted as the derivative of the function $\hat{f} : \mathbf{No}^{>\mathbb{R}} \to \mathbf{No}$ defined by $\hat{f}(x) = f \circ x$, namely we have

$$\partial f \circ x = \lim_{\varepsilon \to 0} \frac{f \circ (x + \varepsilon) - f \circ x}{\varepsilon},$$

where x and ε range in $\mathbf{No}$ (Corollary 7.6). Since $\partial f \circ \omega = \partial f$, this shows in particular that the derivative can be defined in terms of the composition: $\partial f = \lim_{\varepsilon \to 0} \frac{f \circ (\omega + \varepsilon) - f \circ \omega}{\varepsilon}$. Other compatibility conditions then follow, such as the chain rule $\partial(f \circ g) = (\partial f \circ g) \cdot \partial g$ (Corollary 7.7).

These results tells us that any omega-series $f \in \mathbb{R}((\omega))$, hence in particular every logarithmic-exponential series, can be interpreted as a differentiable function $\hat{f} : \mathbf{No}^{>\mathbb{R}} \to \mathbf{No}$ from positive infinite surreal numbers to surreal numbers. We shall prove that all such functions are surreal analytic in the following sense.

Theorem 7.14. *Every $f \in \mathbb{R}((\omega))$ is surreal analytic, namely for every $x \in \mathbf{No}^{>\mathbb{R}}$ and every sufficiently small $\varepsilon \in \mathbf{No}$ we have*

$$f \circ (x + \varepsilon) = \sum_{n \in \mathbb{N}} \frac{1}{n!} (\partial^n f \circ x) \cdot \varepsilon^n.$$

It is tempting to raise the conjecture that the exponential field $\mathbf{No}$, enriched with all the functions $\hat{f} : \mathbf{No}^{>\mathbb{R}} \to \mathbf{No}$ for $f \in \mathbb{R}((\omega))$ (possibly restricted to some interval $(a, +\infty)$) has a good model theory. For instance, the restricted version could yield an o-minimal structure on $\mathbf{No}$. Indeed, note that the family of all functions $\hat{f} : \mathbf{No}^{>\mathbb{R}} \to \mathbf{No}$ (for $f \in \mathbb{R}((\omega))$) yields a sort of non-standard Hardy field on $\mathbf{No}$, namely a field of functions closed under differentiation (it is also closed under exp, log and composition).

We do not know up to what extent the above results can be extended beyond $\mathbb{R}((\omega))$, namely whether we can introduce a composition operator on the whole of $\mathbf{No}$, thus giving a functional interpretation to all surreal numbers. Concerning this problem, we have a negative result. Say that a derivation ∂ and a composition $\circ$ are **compatible** if the function $x \mapsto f \circ x$ is constant when $\partial f = 0$ and strictly increasing when $\partial f > 0$, and if the chain rule $\partial(f \circ g) = (\partial f \circ g) \cdot \partial g$ holds for all $f, g \in \mathbf{No}$ (see Definition 8.1).

Theorem 8.4. *The simplest derivation $\partial : \mathbf{No} \to \mathbf{No}$ of* [BM] *cannot be compatible with a composition on* $\mathbf{No}$.

We conclude with some questions. The first is to study possible notions of compositions and compatible derivations on the whole of $\mathbf{No}$ (see Question 8.3). This is also connected with the long-standing question of the existence of trans-exponential o-minimal structures; a good composition on $\mathbf{No}$ may provide a non-archimedean example. Another related question is to understand whether $\mathbf{No}$ has non-trivial field automorphisms preserving infinite sums and the function exp.

2. Preliminaries

In this section, we recall a few well known constructions and facts regarding ordered fields and surreal numbers, and above all, we shall establish some of the notations that will be used throughout the rest of the paper. Since surreal numbers form a proper class, we implicitly work in a set theoretic framework which allows to talk about classes as first class objects, such as NBG. Therefore, in the following definitions all objects are allowed to be proper classes, unless specified otherwise. Given a class C, we shall say that C is **small** if it is a set and not a proper class.

2.1. Hahn fields.

Definition 2.1. Let K be an ordered field, $R \subseteq K$ a subfield, and $f, g \in K$. We let:

(1) $f \preceq_R g$, or $f \in \mathcal{O}_R(g)$, if there is $c \in R$ such that $|f| \leq c|g|$, and we say that f is R-**dominated** by g;
(2) $f \prec_R g$, or $f \in o_R(g)$, if $c|f| < |g|$ for every $c \in R$, and we say that f is R-**strictly dominated** by g;
(3) f is R-**finite** (or R-**bounded**) if $f \preceq_R 1$;
(4) f is R-**infinitesimal** if $f \prec_R 1$;
(5) $f \asymp_R g$ if $f \preceq_R g$ and $g \preceq_R f$, namely f/g is R-finite and not R-infinitesimal, and we say that f is R-**comparable** to g;
(6) $f \sim_R g$ if $f - g \prec_R g$, and we say that f is R-**asymptotic** to g.

When $R \subseteq \mathbb{R}$ we suppress the "R". For instance we write $f \preceq g$ if there is $c \in \mathbb{Q}$ such $|f| \leq c|g|$ and we say that f is **dominated** by g, or we write $f \in \mathcal{O}(1)$ if f is **finite**, namely f is dominated by 1. We say that f and g are in the **same Archimedean class** if $f \asymp g$, namely $f \preceq g$ and $g \preceq f$.

Finally, we say that $\Gamma \subseteq K^{>0}$ is a **group of monomials for** K if it is a multiplicative subgroup and for every $x \in K$ there is a unique $\mathfrak{m} \in \Gamma$ such that $x \asymp \mathfrak{m}$. It can be proved that any real closed field admits a group of monomials.

Example 2.2. The field of Laurent series $\mathbb{R}((x^{\mathbb{Z}}))$ consists of all formal series of the form $\sum_{n \geq n_0} a_n x^n$, where $a_n \in \mathbb{R}$ and $n_0 \in \mathbb{Z}$, ordered according to the sign of the leading coefficient a_{n_0}. The multiplicative subgroup $x^{\mathbb{Z}} := \{x^n : n \in \mathbb{Z}\}$ is a group of monomials for $\mathbb{R}((x^{\mathbb{Z}}))$.

Remark 2.3. Given two monomials $\mathfrak{m}, \mathfrak{n}$, we have $\mathfrak{m} < \mathfrak{n}$ if and only if $\mathfrak{m} \prec \mathfrak{n}$.

Definition 2.4. Let $(\Gamma, \cdot, <)$ be an ordered abelian group written in multiplicative notation. Let R be an ordered field. The **Hahn field** $R((\Gamma))$ consists of all formal sums $x = \sum_{\mathfrak{m} \in \Gamma} x_{\mathfrak{m}} \mathfrak{m}$ with **coefficients** $x_{\mathfrak{m}} \in R$, whose **support** $\operatorname{Supp}(x) := \{\mathfrak{m} \in \Gamma : x_{\mathfrak{m}} \neq 0\}$ is **reverse well-ordered**, namely every non-empty subset of the support has a maximal element. If $x_{\mathfrak{m}} \neq 0$ we say that $x_{\mathfrak{m}}\mathfrak{m}$ is a **term** of x. We denote by $R((\Gamma))_{\text{small}} \subseteq R((\Gamma))$ the subclass of all formal sums $x = \sum_{\mathfrak{m} \in \Gamma} x_{\mathfrak{m}} \mathfrak{m}$ whose support is small (it coincides with $R((\Gamma))$ when Γ is small).

The addition in $R((\Gamma))$ is defined component-wise and the multiplication is given by the usual convolution formula: $\left(\sum_{\mathfrak{m}} x_{\mathfrak{m}} \mathfrak{m}\right)\left(\sum_{\mathfrak{n}} y_{\mathfrak{n}} \mathfrak{n}\right) = \left(\sum_{\mathfrak{o}} z_{\mathfrak{o}} \mathfrak{o}\right)$ where $z_{\mathfrak{o}} = \sum_{\mathfrak{m}\mathfrak{n} = \mathfrak{o}} x_{\mathfrak{m}} y_{\mathfrak{n}} \in R$. The fact that the supports are reverse well-ordered ensures that the latter sum is finite.

The **leading monomial** $\operatorname{LM}(x)$ of x is the maximal monomial in $\operatorname{Supp}(x)$. The **leading term** $\operatorname{LT}(x)$ is the leading monomial multiplied by its coefficient, and the

leading coefficient is the coefficient of the leading monomial. $R((\Gamma))$ is ordered as follows: x is positive if and only if its leading coefficient is positive. We denote by $\mathrm{Term}(x) := \{x_{\mathfrak{m}}\mathfrak{m} : \mathfrak{m} \in \mathrm{Supp}(x)\}$ the class of the terms of x.

Fact 2.5. Both $R((\Gamma))$ and $R((\Gamma))_{\mathrm{small}}$ are ordered fields.

Remark 2.6. Note that Γ is a multiplicative subgroup of $R((\Gamma))$, where we identify $\mathfrak{m} \in \Gamma$ with $1\mathfrak{m} \in R((\Gamma))$. It follows from the definitions that $\Gamma \subseteq R((\Gamma))$ contains one and only one representative for each equivalence class modulo $\asymp_R$. In particular, taking $R = \mathbb{R}$, we have that Γ is a group of monomials for $\mathbb{R}((\Gamma))$. The same is true for $\mathbb{R}((\Gamma))_{\mathrm{small}}$.

2.2. **Surreal numbers.** We denote by **No** the ordered field of surreal numbers [Con76, Gon86]. A minimal introduction to **No**, containing all the prerequisites for this paper, is contained in [BM]. However, there is no need to assume a prior knowledge of the surreal numbers (the definition itself will not be needed), if one is willing to take for granted the following fact.

Fact 2.7. We have:

(1) **No** is an ordered real closed field equipped with an exponential function $\exp : \mathbf{No} \to \mathbf{No}$, $x \mapsto e^x := \exp(x)$, making it into an elementary extension of $(\mathbb{R}, <, +, \cdot, \exp)$ [DE01a]; in particular, $\exp : \mathbf{No} \to \mathbf{No}$ is an increasing isomorphism from the additive to the positive multiplicative group.
(2) **No** contains an isomorphic copy of the ordered class **On** of all ordinal numbers (hence **No** is a proper class). The addition and multiplication restricted to **On** coincide with the Hessenberg sum and product.
(3) There is a representation of surreal numbers as binary sequences of any ordinal length. The relation of being an initial segment, called **simplicity**, is well founded and makes **No** into a binary tree. This gives us a canonical choice for a group $\mathfrak{M} \subseteq \mathbf{No}^{>0}$ of monomials: the **monomials** are the simplest positive representatives of the Archimedean classes (they form a proper class).
(4) The ordinal ω belongs to $\mathfrak{M}$ (it will later play the role of a formal variable with derivative 1). If $1 \prec \mathfrak{m} \in \mathfrak{M}$, then $e^{\mathfrak{m}} \in \mathfrak{M}$. In particular e^{ω} and $e^{-\omega}$ are monomials, but $e^{1/\omega}$ is not.
(5) There is a canonical isomorphism (written as an identification)
$$\mathbf{No} = \mathbb{R}((\mathfrak{M}))_{\mathrm{small}} \subset \mathbb{R}((\mathfrak{M})).$$
(6) A surreal number $\sum_{m \in \mathfrak{M}} x_{\mathfrak{m}}\mathfrak{m}$ is **purely infinite** if all the monomials $\mathfrak{m}$ in its support are infinite, namely $\mathfrak{m} \succ 1$. Letting $\mathbb{J} \subseteq \mathbf{No}$ be the class of all purely infinite surreal numbers, there is a direct sum decomposition of $\mathbb{R}$-vector spaces
$$\mathbf{No} = \mathbb{J} \oplus \mathbb{R} \oplus o(1).$$
(7) We have $\mathfrak{M} = \exp(\mathbb{J}) = \{e^{\gamma} : \gamma \in \mathbb{J}\}$, so we can write
$$\mathbf{No} = \mathbb{R}((e^{\mathbb{J}}))_{\mathrm{small}}.$$
In other words, every surreal number $x \in \mathbf{No}$ can be uniquely written in the form
$$x = \sum_{i<\alpha} r_i e^{\gamma_i}$$

where $\alpha \in \mathbf{On}$, $r_i \in \mathbb{R}^*$, and $(\gamma_i)_{i<\alpha}$ is a decreasing sequence in $\mathbb{J}$ indexed by an ordinal $\alpha \in \mathbf{On}$. We call this the **Ressayre normal form** of x.

(8) The exponential function on $o(1)$ can be calculated using the Taylor series of exp, namely

$$\exp(\varepsilon) = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!}$$

for all $\varepsilon \in o(1)$ (see Subsection 2.3 for the meaning of the above infinite sum). Likewise, the inverse log satisfies

$$\log(1+\varepsilon) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\varepsilon^n}{n}.$$

Remark 2.8. For infinite x, the equality $\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ does *not* hold. In fact, the right-hand side does not even represent a surreal number (see Subsection 2.3). Likewise for $\log(1+x)$.

Definition 2.9. By the decomposition $\mathbf{No} = \mathbb{J} \oplus \mathbb{R} \oplus o(1)$, for every surreal number $x \in \mathbf{No}$ we can write uniquely

$$x = x^{\uparrow} + x^{=} + x^{\downarrow}$$

where $x^{\uparrow} \in \mathbb{J}$, $x^{=} \in \mathbb{R}$ and $x^{\downarrow} \prec 1$. We also write $x^{\uparrow =}$ for $x^{\uparrow} + x^{=}$.

Definition 2.10. Thanks to Fact 2.7(5) we can apply to **No** the definitions already introduced for Hahn fields (support, leading term, etc.). In particular, if $x = \sum_{i<\alpha} r_i e^{\gamma_i}$ is in normal form, its leading monomial is e^{γ_0} and its leading term is $r_0 e^{\gamma_0}$; in this case we define

$$\log^{\uparrow}(x) := \gamma_0.$$

Note that $\log^{\uparrow}(x) = \log(x)^{\uparrow}$, as in fact $\log(x) = \log(r_0 e^{\gamma_0}(1+\varepsilon)) = \gamma_0 + \log(r_0) + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\varepsilon^n}{n}$ where $\varepsilon \prec 1$. Moreover, $x \prec y$ if and only if $\log^{\uparrow}(x) < \log^{\uparrow}(y)$ (so $-\log^{\uparrow}$ is a Krull valuation).

Definition 2.11. If $x = \sum_{i<\alpha} r_i e^{\gamma_i}$ and $\beta \leq \alpha$, the number $\sum_{i<\beta} r_i e^{\gamma_i}$ is called a **truncation** of x. A subclass $A \subseteq \mathbf{No}$ is **truncation closed** if for every x in A, all truncations of x are also in A.

Note that $x^{\uparrow}$ is a truncation of x and it coincides with the sum of all the terms $r_i e^{\gamma_i}$ of x with $\gamma_i > 0$ (if there are no such terms, then $x^{\uparrow} = 0$).

Notation 2.12. Given $A, B \subseteq \mathbf{No}$ we shall use some self-explanatory notations like the following:

- $A^{>0}$ is the set of positive elements of A;
- $A^{\succ 1}$ is the set of elements $a \in A$ satisfying $a \succ 1$;
- $A < B$ means $a < b$ for all $a \in A$ and $b \in B$;
- $\exp(A) := \{\exp(x) : x \in A\}$ and $\log(A) := \{\log(x) : x \in A^{>0}\}$, where $\log : \mathbf{No}^{>0} \to \mathbf{No}$ is the inverse of exp.

Example 2.13. Since $\mathfrak{M} = \exp(\mathbb{J})$, we have $\mathfrak{M}^{\succ 1} = \exp(\mathbb{J}^{>0})$ and $\mathfrak{M}^{\prec 1} = \exp(\mathbb{J}^{<0})$.

2.3. **Summability.** Any Hahn field, and in particular **No** by Fact 2.7(5), admits a natural notion of infinite sum, as follows.

Definition 2.14. Let I be a set (not a proper class) and $(x_i : i \in I)$ be an indexed family of elements of **No**.

We say that $(x_i : i \in I)$ is **summable** if $\bigcup_{i\in I} \mathrm{Supp}(x_i)$ is reverse well-ordered and for each $\mathfrak{m} \in \bigcup_{i\in I} \mathrm{Supp}(x_i)$, there are only finitely many $i \in I$ such that $\mathfrak{m} \in \mathrm{Supp}(x_i)$. In this case, the **sum** $\sum_{i\in I} x_i$ is the unique surreal number $y = \sum_{\mathfrak{m}} y_{\mathfrak{m}}\mathfrak{m}$ such that $\mathrm{Supp}(y) \subseteq \bigcup_{i\in I} \mathrm{Supp}(x_i)$ and, for every $\mathfrak{m} \in \mathfrak{M}$, $y_{\mathfrak{m}} = \sum_{i\in I}(x_i)_{\mathfrak{m}}$ (note that there are finitely many $i \in I$ with $x_i \neq 0$ by the hypothesis of summability). Similar definitions apply replacing **No** with any field of the form $R((\Gamma))_{\mathrm{small}}$.

We shall also say that $\sum_{i\in I} x_i$ **exists** to mean that $(x_i)_{i\in I}$ is summable.

Remark 2.15. A family $(x_i : i \in I)$ is summable if and only if there are no injective sequences $(i_n)_{n\in\mathbb{N}}$ in I and monomials $\mathfrak{m}_n \in \mathrm{Supp}(x_{i_n})$ (not necessarily distinct) such that $\mathfrak{m}_n \preceq \mathfrak{m}_{n+1}$ for each $n \in \mathbb{N}$ (where $\mathbb{N}$ is the set of non-negative integers). Equivalently, for every injective sequence $(i_n)_{n\in\mathbb{N}}$ in I and for any choice of monomials $\mathfrak{m}_n \in \mathrm{Supp}(x_{i_n})$, there is a subsequence $(i_{f(n)})_{n\in\mathbb{N}}$ such that $\mathfrak{m}_{i_{f(n)}} \succ \mathfrak{m}_{i_{f(n+1)}}$ for every $n \in \mathbb{N}$.

2.4. **Hahn fields embedded in No.** Given a subfield R of **No** and a multiplicative subgroup Γ of the monomials $\mathfrak{M} = e^{\mathbb{J}}$, we will sometimes be interested in the class of all surreal numbers that can be written as a sum $\sum r_{\mathfrak{m}}\mathfrak{m}$ for $r_{\mathfrak{m}} \in R$ and $\mathfrak{m} \in \Gamma$. Under suitable assumptions on R and Γ, this subclass of **No** can be identified with the Hahn field $R((\Gamma))$.

Proposition 2.16. *Let Γ be a small multiplicative subgroup of $\mathfrak{M} = e^{\mathbb{J}}$ and R be a truncation closed subfield of* **No**. *If $R < \Gamma^{>1}$, there is a unique field embedding $R((\Gamma)) \to$ **No** sending $r\mathfrak{m}$ (as an element of $R((\Gamma))$) to $r\mathfrak{m}$ (as an element of* **No***) and preserving infinite sums.*

Proof. Suppose that $R < \Gamma^{>1}$. It suffices to check that the embedding exists. Without loss of generality, we may assume that $\mathbb{R} \subseteq R$, as the compositum $\mathbb{R} \cdot R$ is clearly truncation closed and it also satisfies $\mathbb{R} \cdot R < \Gamma^{>1}$.

Let $\sum r_{\mathfrak{m}}\mathfrak{m}$ be an element of $R((\Gamma))$. We wish to prove that $(r_{\mathfrak{m}}\mathfrak{m} \in \mathbf{No} : r_{\mathfrak{m}} \neq 0)$ is summable. Take an injective sequence $(r_{\mathfrak{m}_n}\mathfrak{m}_n)_{n\in\mathbb{N}}$ and a choice of $\mathfrak{n}_n \in \mathrm{Supp}(r_{\mathfrak{m}_n}\mathfrak{m}_n)$. We can write $\mathfrak{n}_n = \mathfrak{m}_n\mathfrak{o}_n$, where $\mathfrak{o}_n \in \mathrm{Supp}(r_{\mathfrak{m}_n})$. Note that $\mathfrak{o}_n \in R$, since R contains $\mathbb{R}$ and is closed under truncation.

After extracting a subsequence, we may assume that $(\mathfrak{m}_n)_{n\in\mathbb{N}}$ is strictly decreasing. We can now easily check that $(\mathfrak{n}_n)_{n\in\mathbb{N}}$ is also strictly decreasing: indeed,

$$\frac{\mathfrak{n}_n}{\mathfrak{n}_{n+1}} = \frac{\mathfrak{m}_n}{\mathfrak{m}_{n+1}} \cdot \frac{\mathfrak{o}_n}{\mathfrak{o}_{n+1}} > 1,$$

as $\frac{\mathfrak{o}_{n+1}}{\mathfrak{o}_n} \in R < \Gamma^{>1}$. □

Notation 2.17. By Proposition 2.16, given a small multiplicative group Γ of $\mathfrak{M} = e^{\mathbb{J}}$ (the class of monomials of **No**) and a truncation closed subfield $R \subseteq$ **No** such that $R < \Gamma^{>1}$, we can identify the field $R((\Gamma))$ with the class of surreal numbers that are of the form $\sum r_{\mathfrak{m}}\mathfrak{m}$ with $r_{\mathfrak{m}} \in R$ and $\mathfrak{m} \in \Gamma$.

Lemma 2.18. *Let Γ_1 and Γ_2 be subgroups of a given ordered abelian multiplicative group. Suppose $\Gamma_1 < \Gamma_2^{>1}$. Then $\Gamma_1\Gamma_2$ is naturally isomorphic, as an ordered group, to the direct product $\Gamma_1 \times \Gamma_2$ with the reverse lexicographic order.*

Proof. Clearly, $\Gamma_1 \cap \Gamma_2 = \{1\}$, so the map sending $ab \in \Gamma_1\Gamma_2$ to $(a, b) \in \Gamma_1 \times \Gamma_2$ is a well-defined isomorphism of abelian groups. We can easily verify that it preserves the ordering. Indeed, let $a, a' \in \Gamma_1$ and $b, b' \in \Gamma_2$ be such that $b < b'$. It suffices to show that $ab < a'b'$. This can be rewritten as $a/a' < b'/b$. Since $b'/b > 1$, the desired result follows by the hypothesis $\Gamma_1 < \Gamma_2^{>1}$. □

Using the above notation, Proposition 2.16, and Lemma 2.18, we can then deduce the following well-known result (see for instance [DMM01, 1.4]). However, note that the result contains an equality rather than just an isomorphism, thanks to the identifications of Notation 2.17.

Corollary 2.19. *Let Γ_1, Γ_2 be small subgroups of $\mathfrak{M}$. If $\Gamma_1 < \Gamma_2^{>1}$, then we have $\mathbb{R}((\Gamma_1))((\Gamma_2)) = \mathbb{R}((\Gamma_1\Gamma_2)) \cong \mathbb{R}((\Gamma_1 \times \Gamma_2))$.*

Proof. We first note that $\mathbb{R}((\Gamma_1)) < \Gamma_2^{>1}$, from which it follows at once that $\mathbb{R}((\Gamma_1))((\Gamma_2)) \subseteq \mathbb{R}((\Gamma_1\Gamma_2))$ by Proposition 2.16. On the other hand, let $x = \sum_{\mathfrak{m} \in \Gamma_1\Gamma_2} r_{\mathfrak{m}}\mathfrak{m}$ be an element of $\mathbb{R}((\Gamma_1\Gamma_2))$. Since $\Gamma_1\Gamma_2 \cong \Gamma_1 \times \Gamma_2$, each $\mathfrak{m} \in \Gamma_1\Gamma_2$ decomposes uniquely as a product $\mathfrak{m} = \mathfrak{n}\mathfrak{o}$ with $\mathfrak{n} \in \Gamma_1$ and $\mathfrak{o} \in \Gamma_2$. But then it is easy to verify that

$$x = \sum_{\mathfrak{m}} r_{\mathfrak{m}}\mathfrak{m} = \sum_{\mathfrak{o} \in \Gamma_2} \left(\sum_{\mathfrak{n} \in \Gamma_1} r_{\mathfrak{n}\mathfrak{o}}\mathfrak{n} \right) \mathfrak{o} \in \mathbb{R}((\Gamma_1))((\Gamma_2)).$$

□

Remark 2.20. If one drops the assumption that Γ is small, then the conclusion of 2.16 holds with $R((\Gamma))_{\text{small}}$ in place of $R((\Gamma))$. In particular, we may canonically identify $R((\Gamma))_{\text{small}}$ with a subfield of **No**, as in Notation 2.17. As a special case, one recovers the already mentioned identification $\mathbf{No} = \mathbb{R}((\mathfrak{M}))_{\text{small}}$ of Fact 2.7(5). The conclusion of Corollary 2.19 also holds, provided one uses $R((\Gamma_i))_{\text{small}}$ instead of $R((\Gamma_i))$ for $i = 1, 2$.

3. Surreal analytic functions

A real function is analytic at a point in its domain if there is a neighborhood of the point in which it coincides with the limit of a power series. Such notion does not generalize directly to surreal numbers, as **No** does not have a good notion of limit for series. However, we can replace the limit with the natural notion of infinite sum from Definition 2.14. This leads to a theory of "surreal analytic function" developed in [All87]. In this section we isolate and extend some of those results in a form suitable for our goals.

Infinite sum bears some resemblance with the usual notion of absolute convergence. On the one hand, like absolute convergence, it enjoys some good algebraic properties, such as being independent on the "order" in which we sum the elements of the family. On the other hand, it is not related to the order topology; for instance, even if a family $(x_i)_{i \in I}$ is summable, and $(y_i)_{i \in I}$ is such that $|y_i| \leq |x_i|$, it does not necessarily follow that $(y_i)_{i \in I}$ is summable.

Lemma 3.1. *Let $(a_i : i \in I)$ be a summable family of surreal numbers. Then for any partition $I = \bigsqcup_{j \in J} I_j$ of the set I, each sum $\sum_{i \in I_j} a_i$ exists, the family*

$(\sum_{i\in I_j} a_i : j \in J)$ *is summable, and*

$$\sum_{j\in J}\sum_{i\in I_j} a_i = \sum_{i\in I} a_i.$$

Proof. Clearly, since $(a_i : i \in I)$ is summable, so is each $(a_i : i \in I_j)$ for $j \in J$. Moreover, it also follows easily that $(\sum_{i\in I_j} a_i : j \in J)$ is summable, as each monomial $\mathfrak{m}$ in $\mathrm{Supp}(\sum_{i\in I_j} a_i)$ must appear in $\mathrm{Supp}(a_i)$ for some $i \in I_j$. To check that its sum is indeed equal to $\sum_{i\in I} a_i$, for a given monomial $\mathfrak{m}$, let $a_{i,\mathfrak{m}}$be the coefficient of $\mathfrak{m}$ in a_i. Then the coefficient of $\mathfrak{m}$ in $\sum_{j\in J}\sum_{i\in I_j} a_i$ is $\sum_{j\in J}\sum_{i\in I_j} a_{i,\mathfrak{m}} = \sum_{i\in I} a_{i,\mathfrak{m}}$, which in turn is the coefficient of $\mathfrak{m}$ in $\sum_{i\in I} a_i$, proving the conclusion. □

Corollary 3.2. *Let $(a_{i,j} : (i,j) \in I \times J)$ be a summable family of surreal numbers. Then both $\sum_{i\in I}\sum_{j\in J} a_{i,j}$ and $\sum_{j\in J}\sum_{i\in I} a_{i,j}$ exist and*

$$\sum_{i\in I}\sum_{j\in J} a_{i,j} = \sum_{j\in J}\sum_{i\in I} a_{i,j} = \sum_{(i,j)\in I\times J} a_{i,j}.$$

Remark 3.3. The assumption of summability of $(a_{i,j} : (i,j) \in I \times J)$ is necessary, or the equality may not hold. For instance, take $a_{i,i} = \omega$, $a_{i,i+1} = -\omega$, and $a_{i,j} = 0$ otherwise for $i, j \in \mathbb{N}$, which is clearly not summable. Then $\sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} a_{i,j} = 0$ while $\sum_{j\in\mathbb{N}}\sum_{i\in\mathbb{N}} a_{i,j} = \omega$. Moreover, one of the two sums may not even exists; for instance, $\sum_{i\in\mathbb{N}}\sum_{j=0}^{1}(-1)^j\omega$ clearly exists and is equal to 0, while $\sum_{j=0}^{1}\sum_{i\in\mathbb{N}}(-1)^j\omega$ does not exist. It can also happen that the two sums $\sum_i\sum_j$ and $\sum_j\sum_i$ exists and are equal, but the sum $\sum_{(i,j)\in I\times J}$ does not exists: take $a_{i,i} = 2\omega$ and $a_{i+1,i} = a_{i,i+1} = -\omega$, with all other terms $a_{i,j}$ being zero.

3.1. **Products and powers of summable families.** The following is well known.

Remark 3.4. If $(x_i)_{i\in I}$ and $(y_j)_{i\in J}$ are summable, then so is $(x_i y_j : (i,j) \in I \times J)$. Its sum $\sum_{(i,j)\in I\times J} x_i y_j$ coincides with the product $(\sum_{i\in I} x_i)(\sum_{j\in J} y_j)$.

Using Remark 3.4, one can easily express the n-th power of a sum as follows.

Proposition 3.5. *Let $(x_i)_{i\in I}$ be a summable family of surreal numbers and let $n \in \mathbb{N}$. Then the family $\left(\prod_{m<n} x_{\tau(m)} : \tau : n \to I\right)$ is summable and*

$$\left(\sum_{i\in I} x_i\right)^n = \sum_{\tau:n\to I}\prod_{m<n} x_{\tau(m)}.$$

Proof. By induction on $n \in \mathbb{N}$ based on Remark 3.4. □

Corollary 3.6. *If $(a_i\varepsilon^i)_{i\in\mathbb{N}}$ is summable, then for every $n \in \mathbb{N}$,*

$$\left(\sum_{i\in\mathbb{N}} a_i\varepsilon^i\right)^n = \sum_{k\in\mathbb{N}}\left(\sum_{i_1+\ldots+i_n=k} a_{i_1}a_{i_2}\ldots a_{i_n}\right)\varepsilon^k.$$

Proof. By Proposition 3.5, $\left(\sum_{i\in\mathbb{N}} a_i\varepsilon^i\right)^n = \sum_{\tau:n\to\mathbb{N}}\prod_{m<n} a_{\tau(m)}\varepsilon^{\tau(m)}$, and the result follows by setting $\tau(m) = i_m$ and isolating the coefficient of ε^k in the second member. □

3.2. **Sums of power series.** We shall now define how to evaluate a surreal power series on a surreal number, and the corresponding notion of surreal analytic function. This is similar to how real analytic functions are extended to **No**, with the difference that we now allow power series to have surreal coefficients.

Definition 3.7. Given a surreal power series $P(X) = \sum_{i=0}^{\infty} a_i X^i \in \mathbf{No}[[X]]$, we define

$$P(\varepsilon) := \sum_{i \in \mathbb{N}} a_i \varepsilon^i$$

for any $\varepsilon \in \mathbf{No}$ such that the sum on the right hand side exists.

Given a function $f : U \to \mathbf{No}$ from an open subset U of **No**, we say that f is **surreal analytic at** x if there are a neighborhood $V \subseteq U$ of x and a power series $P(X) \in \mathbf{No}[[X]]$ such that $f(y) = P(y - x)$ for all $y \in V$.

Unlike the case of real analytic functions, in which some power series are not convergent and thus do not yield analytic functions, we shall now verify that *every* power series with surreal coefficients induces a surreal analytic function.

By Neumann's lemma [Neu49], if $(a_i)_{i\in\mathbb{N}}$ is a sequence of real coefficients and $\varepsilon \prec 1$, then $(a_i\varepsilon^i)_{i\in\mathbb{N}}$ is summable. Therefore, for every power series $P(X) \in \mathbb{R}[[X]]$, $P(\varepsilon)$ is well defined for any $\varepsilon \prec 1$. We can easily extend this result to series with surreal coefficients. We start with the following variant of Neumann's lemma. Its proof is an adaptation of a similar argument in [Gon86, p. 52].

Lemma 3.8. *Let R be a subfield of* **No** *and $\varepsilon \prec_R 1$. Let $(\mathfrak{n}_i)_{i\in\mathbb{N}}$, $(\mathfrak{m}_{i,j})_{i\in\mathbb{N},j\leq k_i}$ be sequences of monomials in respectively R and* $\mathrm{Supp}(\varepsilon)$*, where $(k_i)_{i\in\mathbb{N}}$ is a sequence of natural numbers with $\lim_{i\to\infty} k_i = \infty$. Then the sum $\sum_{i\in\mathbb{N}} \mathfrak{n}_i\mathfrak{m}_{i,0}\dots\mathfrak{m}_{i,k_i}$ exists.*

Proof. Suppose by contradiction that there are two family as in the hypothesis such that $\sum_{i\in\mathbb{N}} \mathfrak{n}_i\mathfrak{m}_{i,0}\dots\mathfrak{m}_{i,k_i}$ does not exist. By taking a subsequence, we may assume that $(\mathfrak{n}_i\mathfrak{m}_{i,0}\dots\mathfrak{m}_{i,k_i})_{i\in\mathbb{N}}$ is weakly increasing. We may picture $\mathfrak{m}_{i,j}$ as the (i,j)-entry of an infinite table, where i is the row index and j is the column index. Rearranging the terms, we can assume that each row is weakly increasing, namely $\mathfrak{m}_{i,0} \leq \mathfrak{m}_{i,1} \leq \ldots \leq \mathfrak{m}_{i,k_i}$ for all $i \in \mathbb{N}$.

Taking a subsequence we may further assume that $(k_i)_{i\in\mathbb{N}}$ is strictly increasing, so in particular $k_i \geq i$. Choosing a further subsequence we can assume that the first column $(\mathfrak{m}_{i,0})_{i\geq 0}$ is weakly decreasing, since all these monomials are in the support of ε. Similarly we can assume that $(\mathfrak{m}_{i,1})_{i\geq 1}$ is weakly decreasing. Continuing in this fashion, by a diagonalization argument we can assume that, for any fixed k, the k-th column $(\mathfrak{m}_{i,k})_{i\in\mathbb{N}}$ becomes weakly decreasing after its k-th entry, namely $\mathfrak{m}_{k,k} \geq \mathfrak{m}_{k+1,k} \geq \mathfrak{m}_{k+2,k} \geq \ldots$. Note that these terms exist since $k_i \geq k$ for all $i \geq k$.

Now fix $i \in \mathbb{N}$ and let $j > i$ (so $k_j > k_i$). By construction, $\mathfrak{n}_i\mathfrak{m}_{i,0}\dots\mathfrak{m}_{i,k_i} \leq \mathfrak{n}_j\mathfrak{m}_{j,0}\dots\mathfrak{m}_{j,k_i}\mathfrak{m}_{j,k_i+1}\dots\mathfrak{m}_{j,k_j}$. Since $\mathfrak{m}_{j,k_i+1}\dots\mathfrak{m}_{j,k_j} \prec_R 1$, we must have $\mathfrak{n}_i > \mathfrak{n}_j\mathfrak{m}_{j,k_i+1}\dots\mathfrak{m}_{j,k_j}$. It follows that $\mathfrak{m}_{i,0}\dots\mathfrak{m}_{i,k_i} < \mathfrak{m}_{j,0}\dots\mathfrak{m}_{j,k_i}$, so in particular there is some $k \leq k_i$ with $\mathfrak{m}_{i,k} < \mathfrak{m}_{j,k}$. Now recall that the k-th column is weakly decreasing after its k-th entry, hence necessarily $i < k$. We have thus proved that for each $i \in \mathbb{N}$ and $j > i$ there is some k with $i < k \leq k_i$ such that $\mathfrak{m}_{i,k} < \mathfrak{m}_{j,k}$.

Taking $j = k_i$, and recalling that all the rows are weakly increasing, we obtain $\mathfrak{m}_{i,i} \leq \mathfrak{m}_{i,k} < \mathfrak{m}_{k_i,k} \leq \mathfrak{m}_{k_i,k_i}$ for all $i \in \mathbb{N}$. Iterating we obtain an infinite increasing chain of elements of the form $\mathfrak{m}_{l,l}$, contradicting the fact that $\{\mathfrak{m}_{i,j} : i \in \mathbb{N}, j \leq k_i\}$ is in $\mathrm{Supp}(\varepsilon)$. □

Corollary 3.9. *Let R be a truncation closed subfield of* **No** *and $\varepsilon \prec_R 1$. Let $(a_i)_{i\in\mathbb{N}}$ be a sequence of coefficients in R. Then $(a_i\varepsilon^i)_{i\in\mathbb{N}}$ is summable.*

Proof. Without loss of generality, we may assume that $\mathbb{R} \subseteq R$. Indeed, we may replace R with the compositum $\mathbb{R} \cdot R$, which is also closed under truncation, as $\varepsilon \prec_R 1$ trivially implies $\varepsilon \prec_{\mathbb{R}\cdot R} 1$. In particular, we may assume that $\mathrm{Supp}(a_i) \subseteq R$ for all $a_i \in R$. Note that for all $i \in \mathbb{N}$, any monomial in the support of $a_i\varepsilon^i$ has the form $\mathfrak{n}_i\mathfrak{m}_{i,0}\dots\mathfrak{m}_{i,i-1}$ where $\mathfrak{n}_i \in \mathrm{Supp}(a_i) \subseteq R$ and $\mathfrak{m}_{i,j} \in \mathrm{Supp}(\varepsilon)$ for $j \le i-1$. The conclusion then follows easily from Lemma 3.8. □

Corollary 3.10. *For every power series $P(X) \in \mathbf{No}[[X]]$, the partial function $\varepsilon \mapsto P(\varepsilon)$ is surreal analytic at 0.*

Proof. Given a power series $P(X) = \sum_{i=0}^{\infty} a_i X^i$, it suffices to apply Corollary 3.9 with the ring R generated by the monomials in the supports $\mathrm{Supp}(a_i)$. The function $\varepsilon \mapsto P(\varepsilon)$ is then defined at least on $o_R(1)$, which is a nonempty convex subclass containing 0 as R is necessarily small. □

Proposition 3.11. *Suppose that f is a surreal analytic function at some $x \in \mathbf{No}$. Then f is infinitely differentiable at x and*

$$f(x+\varepsilon) = \sum_{i=0}^{\infty} \frac{f^{(i)}(x)}{i!}\varepsilon^i.$$

Proof. Let f be surreal analytic at x, with power series $P(X) = \sum_{i=0}^{\infty} a_i X^i$. Then for every sufficiently small δ we have

$$\begin{aligned} f'(x+\delta) &= \lim_{\varepsilon\to 0} \frac{f(x+\delta+\varepsilon) - f(x+\delta)}{\varepsilon} = \lim_{\varepsilon\to 0} \sum_{i=0}^{\infty} a_i \frac{(\delta+\varepsilon)^i - \delta^i}{\varepsilon} \\ &= \lim_{\varepsilon\to 0} \sum_{i=0}^{\infty} a_i \cdot \frac{\delta^i + i\delta^{i-1}\varepsilon + \binom{i}{2}\delta^{i-2}\varepsilon^2 + \dots + i\delta\varepsilon^{i-1} + \varepsilon^i - \delta^i}{\varepsilon} \\ &= \sum_{i=1}^{\infty} i a_i \delta^{i-1} + \lim_{\varepsilon\to 0} \varepsilon \cdot \sum_{i=2}^{\infty} \left(\binom{i}{2}\delta^{i-2} + \dots + \varepsilon^i \right) = \sum_{i=1}^{\infty} i a_i \delta^{i-1}. \end{aligned}$$

Therefore, f is differentiable at x and its derivative f' is surreal analytic at x. Moreover, the above equation also shows that $f'(x) = a_1$. By induction, it follows that f is infinitely differentiable, and that $a_i = \frac{f^{(i)}(x)}{i!}$, as desired. □

Moreover, we also observe that Neumann's lemma, already in its original formulation, implies the following statement for power series with real coefficients, which will prove useful later on.

Corollary 3.12. *Let $(\varepsilon_i)_{i\in I}$ be a summable family such that $\varepsilon_i \prec 1$ for all $i \in \mathbb{N}$. Let $P_i(X) = \sum_{n=1}^{\infty} a_{i,n} X^n \in \mathbb{R}[[X]]$ be real power series for $i \in I$. Then the family $(P_i(\varepsilon_i) : i \in I)$ is summable.*

Proof. Suppose by contradiction that there is a weakly increasing sequence of monomials $(\mathfrak{m}_n)_{n\in\mathbb{N}}$ such that $\mathfrak{m}_n \in \mathrm{Supp}(P_{i_n}(\varepsilon_{i_n}))$. Then for all $n \in \mathbb{N}$ there is a positive integer k_n such that $\mathfrak{m}_n \in \mathrm{Supp}(a_{i_n,k_n}\varepsilon_{i_n}^{k_n})$. After extracting a subsequence, we may either assume that $\lim_{n\to\infty} k_n = \infty$, and we reach a contradiction by Lemma 3.8, or we may assume that the sequence $(k_n)_{n\in\mathbb{N}}$ is constant, so that $\mathfrak{m}_n \in \mathrm{Supp}(\varepsilon_{i_n}^k)$ for some fixed $k \in \mathbb{N}$ and all $n \in \mathbb{N}$.

In the latter case, write $\mathfrak{m}_n = \mathfrak{n}_{n,1} \cdot \cdots \cdot \mathfrak{n}_{n,k}$ with $\mathfrak{n}_{n,j} \in \mathrm{Supp}(\varepsilon_{i_n})$. Since $(\varepsilon_i)_{i\in I}$ is summable, we may extract a subsequence and assume that $(\mathfrak{n}_{n,j})_{n\in\mathbb{N}}$ is strictly decreasing for each $j = 1, \dots, k$. But then $(\mathfrak{m}_n)_{n\in\mathbb{N}}$ is strictly decreasing, a contradiction. □

Remark 3.13. Since **No** is totally disconnected, the present notion of surreal analyticity does not have a good theory of analytic continuation. For instance, one can define a surreal analytic function on all finite numbers by choosing a power series $P_r(X) \in \mathbb{R}[[X]]$ for each $r \in \mathbb{R}$ and defining $f(r + \varepsilon) = P_r(\varepsilon)$ for each $r \in \mathbb{R}$ and $\varepsilon \prec 1$. Moreover, one can choose the series P_r such that the restriction of f to $\mathbb{R}$ is itself a real analytic function, but with yet other Taylor expansions. It would be interesting to develop an analogous of rigid analytic geometry for surreal numbers that prevents such pathological behavior.

3.3. **Composition of power series.** By Corollary 3.10, there is a morphism from $\mathbf{No}[[X]]$ to germs at zero of surreal functions defined by evaluating a formal power series $P(x) = \sum_{i\in\mathbb{N}} a_i X^i \in \mathbf{No}[[X]]$ at $X = \varepsilon$ for any sufficiently small $\varepsilon \in \mathbf{No}$. As for traditional power series, we can show that this morphism behaves well with respect to composition of power series.

Definition 3.14. Let R be a subfield of **No**. Given two formal power series $P(X) := \sum_{n=0}^{\infty} a_n X^n$ and $Q(X) := \sum_{m=1}^{\infty} b_m X^m$ in $R[[X]]$, where $Q(X)$ has no constant term, their composition $(P \circ Q)(X)$ is defined as the power series $\sum_{k\in\mathbb{N}} c_k X^k \in R[[X]]$ where $c_0 = a_0$ and, for $k > 0$,

$$c_k = \sum_{n=1}^{k} a_n \sum_{m_1+\ldots+m_n=k} b_{m_1} \cdots b_{m_n}.$$

Lemma 3.15. *Let R be a truncation closed subfield of* **No** *and $\varepsilon \prec_R 1$. Let $(a_{i,j} : (i,j) \in I \times J)$ be a family of surreal numbers in R such that, for any fixed $j \in J$, $\sum_{i\in I} a_{i,j}$ exists. Then $\sum_{(i,j)\in I\times J} a_{i,j}\varepsilon^j$ exists.*

Proof. As in the proof of Corollary 3.9, we may assume that $\mathrm{Supp}(a_{i,j}) \subseteq R$ for all $(i,j) \in I\times J$. For a contradiction, suppose that there is an injective sequence of pairs $(i_n, j_n)_{n\in\mathbb{N}}$ and a weakly increasing sequence of monomials $\mathfrak{m}_n \in \mathrm{Supp}(a_{i_n,j_n}\varepsilon^{j_n})$. After extracting a subsequence, we may assume that either $\lim_{n\in\mathbb{N}} j_n = +\infty$, in which case we reach a contradiction by Corollary 3.9, or the sequence $(j_n)_{n\in\mathbb{N}}$ is constant, so that there is some $j \in J$ such that $\mathfrak{m}_n \in a_{i_n,j}\varepsilon^j$ for every $n \in \mathbb{N}$. In this case, it follows that $(a_{i,j}\varepsilon^j)_{i\in I}$ is not summable, which is absurd since $\sum_{i\in I} a_{i,j}$ exists, hence so does $\varepsilon^j(\sum_{i\in I} a_{i,j}) = \sum_{i\in I} a_{i,j}\varepsilon^j$. □

Proposition 3.16. *Let R be a truncation closed subfield of* **No** *and $\varepsilon \prec_R 1$. Let $P(X) := \sum_{n=0}^{\infty} a_n X^n$ and $Q(X) := \sum_{m=1}^{\infty} b_m X^m$ be two power series in $R[[X]]$ (where $Q(X)$ has no constant term). Then $(P \circ Q)(\varepsilon) = P(Q(\varepsilon))$.*

Proof. The three sums $P(\varepsilon)$, $Q(\varepsilon)$ and $(P \circ Q)(\varepsilon)$ exist by Corollary 3.9. Since $Q(\varepsilon) \prec_R 1$, $P(Q(\varepsilon))$ exists as well. Let $d_{n,k} = \sum_{m_1+\ldots+m_n=k} b_{m_1} \cdots b_{m_n}$ for $k \in \mathbb{N}^*$. By Corollary 3.6,

$$P(Q(\varepsilon)) = \sum_{n=0}^{\infty} a_n \left(\sum_{m=1}^{\infty} b_m \varepsilon^m\right)^n = a_0 + \sum_{n=1}^{\infty} a_n \sum_{k=1}^{\infty} d_{n,k}\varepsilon^k.$$

Note that $d_{n,k} = 0$ for $k < n$, so the family $(a_n d_{n,k} : n \in \mathbb{N})$ is summable for any $k \in \mathbb{N}^*$. By Lemma 3.15, the family $(a_n d_{n,k} \varepsilon^k : (n,k) \in \mathbb{N} \times \mathbb{N}^*)$ is summable. Therefore, by Corollary 3.2 we have

$$a_0 + \sum_{n=1}^{\infty} a_n \sum_{k=1}^{\infty} d_{n,k} \varepsilon^k = a_0 + \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} a_n d_{n,k} \varepsilon^k = (P \circ Q)(\varepsilon).$$

□

4. Transseries

With the help of the surreal numbers we shall attempt a general definition of "field of transseries".

Definition 4.1. We say that T is a **transserial subfield** of **No** if T is a truncation closed subfield of **No** (Definition 2.11) containing $\mathbb{R}$ and such that $\log(T^{>0}) \subseteq T$.

More generally, let F be an ordered logarithmic field (not necessarily included in **No**) containing $\mathbb{R}$ and endowed with a partial operator $\sum$ from small indexed families of elements of F to F. We say that F is a **field of transseries** if it is isomorphic to a transserial subfield T of **No** through a field isomorphism $f : F \to T$ preserving $\mathbb{R}$, log and $\sum$ (the latter condition means that $(x_i : i \in I)$ is the domain of $\sum$ if and only if $(f(x_i))_{i \in I}$ is summable in **No** and $\sum_{i \in I} f(x_i) = f(\sum_{i \in I} x_i)$). We shall call f an **isomorphism of transseries**.

In [Sch01] an axiomatic definition of transseries field is given. The critical axiom, there called "T4", is rather technical. One of the main results in [BM] is that **No** satisfies T4, hence it is a field of transseries in the sense of [Sch01]. More generally, since T4 is inherited by taking subfields, it follows that a field of transseries in the sense of Definition 4.1 is also a field of transseries in the sense of [Sch01] (we also expect the converse to be true, but it is beyond the scope of this paper).

4.1. **Log-atomic numbers.** We write $\log_n(x)$ for the n-fold iterate of $\log(x)$, namely $\log_0(x) = x$, $\log_{n+1}(x) = \log(\log_n(x))$. Likewise, we write $\exp_0(x) = x$, $\exp_{n+1}(x) = \exp(\exp_n(x))$.

Definition 4.2. A positive infinite surreal number $x \in$ **No** is **log-atomic** if for every $n \in \mathbb{N}$, $\log_n(x)$ is an infinite monomial. We call $\mathbb{L}$ the class of all log-atomic numbers. Note that $\log(\mathbb{L}) = \exp(\mathbb{L}) = \mathbb{L}$.

A subclass of the log-atomic numbers, the so called κ-numbers, was isolated by [KM15]. The ordinal ω is a κ-number, hence in particular it is log-atomic. In [BM] we gave a parametrization $\{\lambda_x : x \in \mathbf{No}\}$ of $\mathbb{L}$ and we proved that there is exactly one log-atomic numbers in each "level" of **No**.

Definition 4.3. Given $x, y > \mathbb{R}$ we write $x \asymp^L y$, and we say that x, y are in the same **level** if for some $n \in \mathbb{N}$ we have $\log_n(x) \asymp \log_n(y)$.

Remark 4.4. For all $x, y > \mathbb{R}$, $x \asymp y$ implies $\log(x) \sim \log(y)$, so in the above definition we can equivalently require $\log_n(x) \sim \log_n(y)$.

Fact 4.5 ([BM])**.** We have:

(1) for each $x \in$ **No** with $x > \mathbb{R}$, there are $n \in \mathbb{N}$ and $\lambda \in \mathbb{L}$ such that $\log_n(x) \asymp \lambda$ [BM, Prop. 5.8]; in particular, every level contains a log-atomic number;

(2) for each $\lambda, \mu \in \mathbb{L}$, if $\lambda \asymp^L \mu$, then $\lambda = \mu$; in particular, every level contains a *unique* log-atomic number;
(3) for every $x > \mathbb{R}$ and every positive $n \in \mathbb{N}$, we have $x \asymp^L x^n$, but $x \not\asymp^L e^x$;
(4) in particular, for $\lambda, \mu \in \mathbb{L}$, if $\lambda < \mu$, then $\lambda^n < \mu$ for every $n \in \mathbb{N}$;
(5) there are log-atomic numbers strictly between ω and e^ω; there are also log-atomic numbers smaller than $\log_n(\omega)$ for every $n \in \mathbb{N}$ or bigger than $\exp_n(\omega)$ for every $n \in \mathbb{N}$, such as the ordinal ε_0.

4.2. **Omega-series, LE-series, EL-series .** In this section we shall introduce three subfields $\mathbb{R}((\omega))^{LE} \subset \mathbb{R}((\omega))^{EL} \subset \mathbb{R}\langle\!\langle\omega\rangle\!\rangle$ of **No**. We shall see that first two are naturally isomorphic to the exponential fields of respectively the LE-series of [DMM97, DMM01] and the EL-series generated by logarithmic words of [Kuh00, KT12], while the third one is a very big field properly containing both (the ordinal ω plays the role of a formal variable $> \mathbb{R}$).

Definition 4.6. Given a subclass X of **No**, we write $\sum X$ for the family of all surreal numbers $x \in$ **No** which can be written in the form $x = \sum_{i\in I} y_i$ for some summable family $(y_i)_{i\in I}$ of elements of X indexed by a set I. Note that $\sum$ is a closure operator, as $X \subseteq \sum X = \sum\sum X$.

Definition 4.7. We define $\mathbb{R}\langle\!\langle\omega\rangle\!\rangle$, the field of **omega-series**, as the smallest subfield of **No** containing $\mathbb{R} \cup \{\omega\}$ and closed under $\sum$, exp and log.

We shall prove later that $\mathbb{R}\langle\!\langle\omega\rangle\!\rangle$ is a proper class.

Definition 4.8. Let $\mathbb{R}((\omega))^{LE} \subset \mathbb{R}\langle\!\langle\omega\rangle\!\rangle$ be the union $\bigcup_{n\in\mathbb{N}} X_n$, where $X_0 = \mathbb{R}\cup\{\omega\}$ and $X_{n+1} = \sum(X_n \cup \exp(X_n) \cup \log(X_n))$. In other words, a surreal number x belongs to $\mathbb{R}((\omega))^{LE}$ if and only if x can be obtained in finitely many steps starting from $\mathbb{R} \cup \{\omega\}$ and using the set-operations $\sum, \exp, \log$.

Definition 4.9. Let $\mathbb{R}((\omega))^{EL}$ be defined as $\mathbb{R}((\omega))^{LE}$ but starting with $X_0' = \mathbb{R} \cup \{\omega, \log(\omega), \log_2(\omega), \ldots\}$ instead of $X_0 = \mathbb{R} \cup \{\omega\}$. In other words, a surreal number belongs to $\mathbb{R}((\omega))^{EL}$ if and only if it can be obtained in finitely many steps from X_0' using $\sum, \exp, \log$ (in this case it turns out that log is not actually necessary).

Remark 4.10. Unlike $\mathbb{R}\langle\!\langle\omega\rangle\!\rangle$, the subfields $\mathbb{R}((\omega))^{LE}$ and $\mathbb{R}((\omega))^{EL}$ are not closed under $\sum$; for instance $\sum_{n\in\mathbb{N}} \log_n(\omega)$ belongs to $\mathbb{R}\langle\!\langle\omega\rangle\!\rangle$ but not to $\mathbb{R}((\omega))^{LE}$. Indeed, one needs k steps to generate $\log_k(\omega)$ starting from $\mathbb{R} \cup \{\omega\}$, so the whole sum $\sum_{n\in\mathbb{N}} \log_n(\omega)$ cannot be generated in finitely many steps. The same example witnesses that the inclusion $\mathbb{R}((\omega))^{LE} \subset \mathbb{R}((\omega))^{EL}$ is proper, as the latter field does contain $\sum_{n\in\mathbb{N}} \log_n(\omega)$. Finally note that $\sum_{n\in\mathbb{N}} 1/\exp_n(\omega)$ belongs to $\mathbb{R}\langle\!\langle\omega\rangle\!\rangle$ but not to $\mathbb{R}((\omega))^{EL}$.

Both $\mathbb{R}((\omega))^{LE}$ and $\mathbb{R}((\omega))^{EL}$ are elementary extensions of the real exponential field $(\mathbb{R}, +, \cdot, \exp)$, but they are no longer elementary equivalent if we add the differential operator ∂ of [BM] to the language (see Subsection 7.1): indeed in $\mathbb{R}((\omega))^{LE}$ (and in **No** itself) the derivation ∂ is surjective, while in $\mathbb{R}((\omega))^{EL}$ it is not. For instance one can show that $\exp(-\sum_{n\in\mathbb{N}} \log_n(\omega))$ is an element of $\mathbb{R}((\omega))^{EL}$ without anti-derivative in $\mathbb{R}((\omega))^{EL}$, and in fact not even in $\mathbb{R}\langle\!\langle\omega\rangle\!\rangle$. Indeed, for the simplest surreal derivation ∂ ([BM, Def. 6.7]), which has anti-derivatives, we have $\partial\kappa_{-1} = \exp(-\sum_{n\in\mathbb{N}} \log_n(\omega))$, where $\kappa_{-1} \in$ **No** is the simplest log-atomic number smaller than $\log_n(\omega)$ for each $n \in \mathbb{N}$. Such a number cannot belong to $\mathbb{R}\langle\!\langle\omega\rangle\!\rangle$, and since $\ker\partial = \mathbb{R}$, there cannot be any $x \in \mathbb{R}\langle\!\langle\omega\rangle\!\rangle$ with $\partial x = \exp(-\sum_{n\in\mathbb{N}} \log_n(\omega))$.

There are many interesting subfields between $\mathbb{R}((\omega))^{LE}$ and $\mathbb{R}\langle\!\langle\omega\rangle\!\rangle$ whose domain is a set, for instance the series in $\mathbb{R}\langle\!\langle\omega\rangle\!\rangle$ with hereditarily countable support.

The definition of $\mathbb{R}((\omega))^{LE}$ as a union $\bigcup_{n\in\mathbb{N}} X_n$ suggests the possibility of prolonging the sequence X_n along the transfinite ordinals, setting $X_0 = \mathbb{R}\cup\{\omega\}$, $X_{\alpha+1} = \sum(X_\alpha \cup \exp(X_\alpha)\cup\log(X_\alpha))$ and $X_\lambda = \bigcup_{i<\lambda} X_i$ for each limit ordinal α. One can verify that the union $\bigcup_{\alpha\in\mathbf{On}} X_\alpha$ along all the ordinals would then coincide with $\mathbb{R}\langle\!\langle\omega\rangle\!\rangle$.

4.3. **Isomorphism with classical LE-series.** It is well known that there is a unique embedding of the field of LE-series into **No** sending x to ω [1], $\mathbb{R}$ to $\mathbb{R}$, and preserving exp and infinite sums (see [ADH]). This subsection will be devoted to the long, but straightforward proof that $\mathbb{R}((\omega))^{LE}$ is naturally isomorphic to the field of LE-series, so in particular it is the image of such embedding. This provides a simple characterization of the LE-series, which should be compared with the original definition.

Theorem 4.11. *$\mathbb{R}((\omega))^{LE}$ is a field of transseries and it is isomorphic to the field of logarithmic-exponential series $\mathbb{R}((x))^{LE}$ of* [DMM97, DMM01]*; the isomorphism sending ω to x is unique.*

Similarly we have:

Proposition 4.12. *The field $\mathbb{R}((\omega))^{EL}$ is naturally isomorphic to the field of EL-series generated by logarithmic words* [KT12, Def. 6.2, Example 4.6] *(see also Remark 4.33).*

We leave the verification of Proposition 4.12 to the reader, but we shall give a detailed proof of Theorem 4.11. To this aim we shall first give an equivalent description of $\mathbb{R}((\omega))^{LE}$ (recall from Notation 2.17 that we are identifying Hahn fields $R((\Gamma))$ with subfields of **No**).

Definition 4.13. Let $\lambda\in\mathbb{L}$ (a log-atomic number). We define:

(1) $\mathfrak{M}_{0,\lambda} := \lambda^{\mathbb{R}}$, $\quad K_{0,\lambda} := \mathbb{R}((\mathfrak{M}_{0,\lambda}))$, $\quad \mathbb{J}_{0,\lambda} := \mathbb{R}((\mathfrak{M}_{0,\lambda}^{\succ 1}))$;
(2) $\mathfrak{M}_{n+1,\lambda} := e^{\mathbb{J}_{n,\lambda}}$, $\quad K_{n+1,\lambda} = K_{n,\lambda}((\mathfrak{M}_{n+1,\lambda}))$, $\quad \mathbb{J}_{n+1,\lambda} := K_{n,\lambda}((\mathfrak{M}_{n+1,\lambda}^{\succ 1}))$;
(3) $\mathbb{R}((\lambda))^E := \bigcup_{n\in\mathbb{N}} K_{n,\lambda}$.

The next Lemma shows that the above definition is well posed, namely at each step $\mathfrak{M}_{n+1,\lambda}$ is a subgroup of $\mathfrak{M}$ and $K_{n,\lambda} < \mathfrak{M}_{n+1,\lambda}^{>1}$, so that under the conventions of Notation 2.17, each $K_{n+1,\lambda}$ is again in **No**; in particular, in clause (2) we are allowed to use the exponential function of **No** to define $e^{\mathbb{J}_{n+1,\lambda}}$. Note moreover that $\mathbb{J}_{n,\lambda}\subseteq\mathbb{J}$, as we shall verify in a moment.

Lemma 4.14. *For each $n\in\mathbb{N}$, $\mathfrak{M}_{n,\lambda}$ is a well defined divisible subgroup of $\mathfrak{M}$ and moreover $\mathfrak{M}_{n+1,\lambda}^{\succ 1} > K_{n,\lambda}$.*

Proof. We proceed by induction on n. Trivially, $\mathfrak{M}_{0,\lambda}$ is a well defined divisible subgroup of $\mathfrak{M}$. Now fix n and assume that $\mathfrak{M}_{n,\lambda}$ is well defined and that $\mathfrak{M}_{n,\lambda}^{\succ 1} > K_{n-1,\lambda}$ (an empty condition if $n=0$). Then $\mathbb{J}_{n,\lambda}$ is a well defined subset of **No** by Proposition 2.16, and in particular it is a divisible additive subgroup of $K_{n,\lambda}$.

[1] In [DMM01], the field of logarithmic-exponential series is denoted either by $\mathbb{R}((x^{-1}))^{LE}$ or by $\mathbb{R}((t))^{LE}$, where $x>\mathbb{R}$ and $t=x^{-1}$ is infinitesimal. We prefer here to use the notation $\mathbb{R}((x))^{LE}$ for the LE-series, with $x>\mathbb{R}$, as in [ADH17], to better match the notation $\mathbb{R}((\omega))^{LE}$.

We claim that $\mathbb{J}_{n,\lambda}$ is consists only of purely infinite numbers. Indeed, let $\mathfrak{m}$ be a monomial in the support of $\mathbb{J}_{n,\lambda}$. Then $\mathfrak{m} = \mathfrak{n}\mathfrak{o}$ for some $\mathfrak{n} \in \mathfrak{M}_{n,\lambda}^{\succ 1}$ and $\mathfrak{o} \in K_{n-1,\lambda}$ (with $\mathfrak{o} = 1$ if $n = 0$). By inductive hypothesis, $\mathfrak{o}^{-1} \in K_{n-1,\lambda} < \mathfrak{n}$, so $\mathfrak{m} > 1$, proving the claim. It follows that $\mathfrak{M}_{n+1,\lambda}$ is a divisible multiplicative subgroup of $\mathfrak{M}$.

Finally, let $e^\gamma \in \mathfrak{M}_{n+1,\lambda}^{\succ 1}$. We wish to prove that $e^\gamma > \mathfrak{M}_{n,\lambda}$. Let $\mathfrak{m}$ be the leading monomial of γ. As before, we can write $\mathfrak{m} = \mathfrak{n}\mathfrak{o}$ for some $\mathfrak{n} \in \mathfrak{M}_{n,\lambda}^{\succ 1}$ and $\mathfrak{o} \in K_{n-1,\lambda}$ (with $\mathfrak{o} = 1$ if $n = 0$). By inductive hypothesis, we also know that $\mathfrak{n}^{\frac{1}{2}} > K_{n-1,\lambda}$. Since $\gamma > \mathfrak{n}^{\frac{1}{2}}$, it follows that $\mathfrak{m} = e^\gamma > e^{K_{n-1,\lambda}}$, so in particular, $\mathfrak{m} > e^{\mathbb{J}_{n-1,\lambda}} = \mathfrak{M}_{n,\lambda}$, as desired. □

Remark 4.15. By Corollary 2.19 we have $K_{n,\lambda} = \mathbb{R}((\mathfrak{M}_{0,\lambda}\mathfrak{M}_{1,\lambda}\dots\mathfrak{M}_{n,\lambda}))$.

Lemma 4.16. *For all $n \in \mathbb{N}$ we have:*

(1) $\exp(K_{n,\lambda}) \subseteq K_{n+1,\lambda}$*;*
(2) $K_{n,\lambda} \subseteq K_{n+1,\log(\lambda)}$*;*
(3) $\log(K_{n,\lambda}^{>0}) \subseteq K_{n+1,\log(\lambda)}$*.*

In particular, $\mathbb{R}((\lambda))^E$ *is closed under* exp *and* $\log(\mathbb{R}((\lambda))^E) \subseteq \mathbb{R}((\log(\lambda)))^E$*.*

Proof. We work by induction on n.

For (1), let $x \in K_{n,\lambda}$. We can write uniquely $x = \gamma + r + \varepsilon$ where $\gamma \in \mathbb{J}_{n,\lambda}$, and if $n > 0$, $r \in K_{n-1,\lambda}$ and $\varepsilon \prec_{K_{n-1,\lambda}} 1$, otherwise simply $r \in \mathbb{R}$ and $\varepsilon \prec 1$. In any case, $e^x = e^\gamma \cdot e^r \cdot \sum_{i=0}^\infty \frac{\varepsilon^i}{i!}$. But then it suffices to note that $e^\gamma \in \mathfrak{M}_{n+1,\lambda} \subseteq K_{n+1,\lambda}$ by definition, while e^r is either already in $\mathbb{R}$ or in $K_{n,\lambda}$ by inductive hypothesis, and the remaining sum is in $K_{n,\lambda}$ because $K_{n,\lambda}$ is a Hahn field. Therefore, $e^x \in K_{n+1,\lambda}$, as desired.

Concerning (2), note that $\mathfrak{M}_{0,\lambda} = \lambda^{\mathbb{R}} = e^{\mathbb{R}\log(\lambda)} \subseteq e^{\mathbb{J}_{0,\log(\lambda)}} = \mathfrak{M}_{1,\log(\lambda)}$. It follows that $\mathbb{J}_{0,\lambda} \subseteq \mathbb{J}_{1,\log(\lambda)}$ and $K_{0,\lambda} \subseteq K_{1,\log(\lambda)}$. By a straightforward induction, it follows that $\mathfrak{M}_{n,\lambda} \subseteq \mathfrak{M}_{n+1,\log(\lambda)}$, $\mathbb{J}_{n,\lambda} \subseteq \mathbb{J}_{n+1,\log(\lambda)}$ and $K_{n,\lambda} \subseteq K_{n+1,\log(\lambda)}$, proving the desired conclusion.

Finally, for (3), let $x \in K_{n,\lambda}^{>0}$. We can write uniquely $x = \mathfrak{m} \cdot r \cdot (1 + \varepsilon)$ where $\mathfrak{m} \in \mathfrak{M}_{n,\lambda}$, and if $n > 0$, $r \in K_{n-1,\lambda}^{>0}$ and $\varepsilon \prec_{K_{n-1,\lambda}} 1$, otherwise simply $r \in \mathbb{R}$ and $\varepsilon \prec 1$. We have $\log(x) = \log(\mathfrak{m}) + \log(r) + \sum_{i=1}^\infty (-1)^{i+1}\frac{\varepsilon^i}{i}$. Since $K_{n,\lambda}$ is a Hahn field, the rightmost sum is in $K_{n,\lambda}$, which is contained in $K_{n+1,\log(\lambda)}$ by (2), while $\log(r)$ is either already in $\mathbb{R}$ or in $K_{n,\log(\lambda)}$ by inductive hypothesis. For $\log(\mathfrak{m})$, we simply note that if $n = 0$, then $\log(\mathfrak{m}) = s \cdot \log(\lambda) \in K_{0,\log(\lambda)}$ for some $s \in \mathbb{R}$, otherwise $\log(\mathfrak{m}) \in K_{n-1,\lambda}$, which is contained in $K_{n,\log(\lambda)}$ by (2). Therefore, $\log(x) \in K_{n+1,\log(\lambda)}$, as desired. □

Proposition 4.17. *For each $\lambda \in \mathbb{L}$, $\mathbb{R}((\lambda))^E$ is (uniquely) isomorphic to the exponential field $\mathbb{R}((x))^E$ defined in* [DMM97, DMM01] *through an isomorphism sending λ to x and preserving* exp, $\sum$ *and* $\mathbb{R}$*.*

Proof. It suffices to note that Definition 4.13 is formally identical to the definition of $\mathbb{R}((x))^E$, except that in our case the various Hahn fields are identified with subfields of **No** (Notation 2.17) and the role of the formal variable is taken by λ. The uniqueness follows trivially. □

Proposition 4.18. *For each $\lambda \in \mathbb{L}$, $\bigcup_{k\in\mathbb{N}} \mathbb{R}((\log_k(\lambda)))^E$ is (uniquely) isomorphic to the exponential field $\mathbb{R}((x))^{LE}$ defined in* [DMM97, DMM01] *through an isomorphism sending λ to x and preserving* exp, $\sum$ *and* $\mathbb{R}$.

Proof. In [DMM01], $\mathbb{R}((x))^{LE}$ is defined as a direct limit of a suitable system of self-embeddings $\Phi_k : \mathbb{R}((x))^E \to \mathbb{R}((x))^E$. The embedding Φ_k sends x to $\exp_k(x)$. In turn, when composed with the isomorphism $\mathbb{R}((x))^E \cong \mathbb{R}((\log_k(\lambda)))^E$ of Proposition 4.17, it gives the embedding of $\mathbb{R}((x))^E$ into $\mathbb{R}((\log_k(\lambda)))^E$ sending x to λ. Therefore, the image of such direct limit is the directed union $\bigcup_{k\in\mathbb{N}} \mathbb{R}((\log_k(\lambda)))^E$, as desired. The uniqueness follows trivially. □

Proposition 4.19. *$\bigcup_{k\in\mathbb{N}} \mathbb{R}((\log_k(\omega)))^E$ is equal to $\mathbb{R}((\omega))^{LE}$. In particular, there is a unique isomorphism of transseries from $\bigcup_{k\in\mathbb{N}} \mathbb{R}((\log_k(\omega)))^E$ to $\mathbb{R}((x))^{LE}$ sending ω to x.*

Proof. Note that each $K_{n,\lambda}$ is closed under infinite sums, while by Lemma 4.16, $\exp(K_{n,\lambda}) \subseteq K_{n+1,\lambda}$ and $\log(K^{>0}_{n,\lambda}) \subseteq K_{n+1,\log(\lambda)}$. Since $X_0 \subseteq K_{0,\omega}$, it follows at once that $X_n \subseteq K_{n,\log_n(\omega)}$ for all $n \in \mathbb{N}$, so in particular $\mathbb{R}((\omega))^{LE} \subseteq \bigcup_{k\in\mathbb{N}} \mathbb{R}((\log_k(\omega)))^E$.

Conversely, it is clear that each element of $K_{n,\log_n(\omega)}$ can be obtained from $X_0 = \mathbb{R} \cup \{\omega\}$ by finitely many applications of exp, log and infinite sums. It follows at once that $\bigcup_{k\in\mathbb{N}} \mathbb{R}((\log_k(\omega)))^E \subseteq \mathbb{R}((\omega))^{LE}$. □

Theorem 4.11 then follows at once by Propositions 4.19 and 4.18.

Remark 4.20. If we modify Definition 4.13 putting $\mathfrak{M}_{0,\lambda} := \lambda^{\mathbb{Z}}$ instead of $\lambda^{\mathbb{R}}$, the union $\bigcup_{k\in\mathbb{N}} \mathbb{R}((\log_k(\omega)))^E$ will be the same, since $\lambda^{\mathbb{R}} = \exp(\mathbb{R}\log\lambda)$. So in the definition of the LE-series in [DMM01] one may start with $x^{\mathbb{Z}}$ instead of $x^{\mathbb{R}}$.

4.4. **Adding more log-atomic numbers.**

Definition 4.21. Consider the class $\mathbb{L} \subseteq \mathbf{No}$ of log-atomic numbers and let $\mathbb{R}\langle\!\langle \mathbb{L} \rangle\!\rangle$ be the smallest subfield of **No** containing $\mathbb{R} \cup \mathbb{L}$ and closed under exp, log and $\sum$ (in the sense of Definition 4.6).

In [BM, Thm. 8.6] we showed that $\mathbb{R}\langle\!\langle \mathbb{L} \rangle\!\rangle$ is the largest subfield of transseries satisfying axiom ELT4 of [KM15, Def. 5.1]. We also showed that **No** itself does not satisfy ELT4, hence $\mathbb{R}\langle\!\langle \mathbb{L} \rangle\!\rangle \neq \mathbf{No}$ [BM, Thm. 8.7]. The derivative $\partial : \mathbf{No} \to \mathbf{No}$ introduced in [BM, Def. 6.21] can be restricted to $\mathbb{R}\langle\!\langle \mathbb{L} \rangle\!\rangle$ and remains surjective on this subfield. We thus have the inclusions

$$\mathbb{R}((\omega))^{LE} \subset \mathbb{R}((\omega))^{EL} \subset \mathbb{R}\langle\!\langle \omega \rangle\!\rangle \subset \mathbb{R}\langle\!\langle \mathbb{L} \rangle\!\rangle \subset \mathbf{No}$$

with $\mathbb{R}((\omega))^{LE}$, $\mathbb{R}\langle\!\langle \mathbb{L} \rangle\!\rangle$ and **No** having a surjective derivation, while the derivation on $\mathbb{R}((\omega))^{EL}$ and $\mathbb{R}\langle\!\langle \omega \rangle\!\rangle$ is not surjective. It would be interesting to study the complete first order theories of these structures, both as differential fields, and as differential fields with an exponentiation. The only known result so far is that **No** and $\mathbb{R}((\omega))^{LE}$ are elementary equivalent as differential fields [ADH], and probably the same proof can be used to deduce that $\mathbb{R}\langle\!\langle \mathbb{L} \rangle\!\rangle$ has the same first order theory as well.

4.5. **Inductive generation of transseries fields and associated ranks.** For the purposes of Section 5, it is useful to inductively construct $\mathbb{R}\langle\langle\omega\rangle\rangle$ and other subfields of $\mathbb{R}\langle\langle\mathbb{L}\rangle\rangle$ with a limited use of the log function, and to introduce a rank function reflecting the stages of the inductive construction. We need the following definition.

Definition 4.22. Let $\Delta \subseteq \mathbb{L}$ be a subclass with $\log(\Delta) \subseteq \Delta$ and let $\mathbb{R}\langle\langle\Delta\rangle\rangle$ be the smallest subfield of **No** containing $\mathbb{R} \cup \Delta$ and closed under $\sum$, exp and log.

As we shall see Corollary 4.30, $\mathbb{R}\langle\langle\Delta\rangle\rangle$ coincides with the smallest subclass of **No** containing $\mathbb{R} \cup \Delta$ and closed under $\sum$ and exp (or even just $\exp_{\restriction\mathbb{J}}$); the closure under log can be automatically deduced. Taking $\Delta = \mathbb{L}$, we obtain the field $\mathbb{R}\langle\langle\mathbb{L}\rangle\rangle$ seen in Subsection 4.4. On the other hand, when $\Delta = \{\log_n(\omega) : n \in \mathbb{N}\}$, we obtain $\mathbb{R}\langle\langle\omega\rangle\rangle$ (Definition 4.7).

Notation 4.23. Given a subclass $A \subseteq \mathfrak{M}$, we denote by $\mathbb{R}((A))_{\text{small}}$ (or just $\mathbb{R}((A))$ if A is a set) the class of all surreal numbers with support contained in A. Notice that if A is a group, $\mathbb{R}((A))_{\text{small}}$ is a field, but we occasionally use the notation without assuming that A is a group.

Definition 4.24. Let $\log(\Delta) \subseteq \Delta \subseteq \mathbb{L}$. We define by induction on the ordinal $\alpha \in \mathbf{On}$ a subclass $\Delta_\alpha \subseteq \mathbf{No}$ as follows: $\Delta_0 = \emptyset$, $\Delta_1 = \Delta \cup \{0\}$; $\Delta_{\alpha+1} = \mathbb{R}((e^{\Delta_\alpha \cap \mathbb{J}}))_{\text{small}}$ for $\alpha \geq 1$; $\Delta_\lambda = \bigcup_{\alpha<\lambda} \Delta_\alpha$ for λ a limit ordinal. Given $x \in \bigcup_{\alpha\in\mathbf{On}} \Delta_\alpha$, we define the **(exponential) rank** $\mathrm{ER}_\Delta(x)$ as the least ordinal β such that $x \in \Delta_{\beta+1}$.

Remark 4.25. Note that Δ_1 is not an additive group. For $\alpha \geq 2$, Δ_α is an $\mathbb{R}$-linear subspace of **No** (and it is closed under $\sum$); for $\alpha \geq 3$, Δ_α is a field, and a Hahn field when α is a successor ordinal. Moreover, all the classes Δ_α are truncation closed.

Proposition 4.26. *For all $\alpha < \beta$ we have $\Delta_\alpha \subseteq \Delta_\beta$.*

Proof. It suffices to prove that $\Delta_\alpha \subseteq \Delta_{\alpha+1}$ for all $\alpha \in \mathbf{On}$. This is clear for $\alpha = 0$. Since $\log(\Delta) \subseteq \Delta$, we have $\Delta \subseteq e^\Delta \subseteq \mathbb{R}((e^\Delta))_{\text{small}}$, thus $\Delta_1 \subseteq \Delta_2$, proving the case $\alpha = 1$. We then proceed by induction. If $\alpha = \beta + 1$, then $\Delta_\beta \subseteq \Delta_{\beta+1}$ holds by inductive hypothesis, so $\Delta_\alpha = \mathbb{R}((e^{\Delta_\beta \cap \mathbb{J}}))_{\text{small}} \subseteq \mathbb{R}((e^{\Delta_{\beta+1} \cap \mathbb{J}}))_{\text{small}} = \Delta_{\alpha+1}$. If α is a limit ordinal, take some $x \in \Delta_\alpha$. By definition of Δ_α, there is some $\beta < \alpha$ such that $x \in \Delta_\beta$, so by inductive hypothesis, $x \in \Delta_{\beta+1} = \mathbb{R}((e^{\Delta_\beta \cap \mathbb{J}}))_{\text{small}} \subseteq \mathbb{R}((e^{\Delta_\alpha \cap \mathbb{J}}))_{\text{small}} = \Delta_{\alpha+1}$. Since x is arbitrary, we obtain $\Delta_\alpha \subseteq \Delta_{\alpha+1}$, as desired. □

The following corollary provides an equivalent definition of the rank. Its proof is easy and left to the reader.

Corollary 4.27. *For $x \in \bigcup_{\alpha\in\mathbf{On}} \Delta_\alpha$ we have*

(1) if $x \in \Delta \cup \{0\}$, then $\mathrm{ER}_\Delta(x) = 0$;
(2) otherwise, $\mathrm{ER}_\Delta(x) = \sup\{\mathrm{ER}_\Delta(\gamma) + 1 : e^\gamma \in \mathrm{Supp}(x)\}$.

Moreover, $x \in \Delta_\beta$ if and only if $\mathrm{ER}_\Delta(x) < \beta$.

Proposition 4.28. *We have:*

(1) for all $\alpha \geq 1$, $\sum \Delta_{\alpha+1} \subseteq \Delta_{\alpha+1}$ (in particular, $\sum \Delta_\alpha \subseteq \Delta_{\alpha+2}$ for all α);
(2) for all $\alpha \geq 3$, $\log(\Delta_\alpha^{>0}) \subseteq \Delta_\alpha$;
(3) for all $\alpha \in \mathbf{On}$, $e^{\Delta_\alpha} \subseteq \Delta_{\alpha+1}$ (in particular, $e^{\Delta_\alpha} \subseteq \Delta_\alpha$ for all limit α).

In particular, Δ_α is a transserial subfield of **No** *for all $\alpha \geq 3$, and $\bigcup_{\alpha\in\mathbf{On}} \Delta_\alpha$ is closed under* exp*,* log *and infinite sums.*

Proof. (1) Trivial, since by definition $\Delta_{\alpha+1} = \mathbb{R}((e^{\Delta_\alpha \cap \mathbb{J}}))_{\mathrm{small}}$ for $\alpha \geq 1$.

(2) Without loss of generality, we may assume that α is of the form $\beta + 1$ with $\beta \geq 2$, so that Δ_α is a Hahn field (see Remark 4.25). Take any $x \in \Delta_\alpha^{>0}$. We can write uniquely $x = re^\gamma(1+\varepsilon)$, where $r \in \mathbb{R}^{>0}$, $\gamma \in \Delta_\beta \cap \mathbb{J}$ and $\varepsilon \in \Delta_\alpha \cap o(1)$. Then $\log(x) = \gamma + \log(r) + \sum_{n=1}^\infty (-1)^n \frac{\varepsilon^n}{n} \in \Delta_\beta + \Delta_\alpha = \Delta_\alpha$ by Proposition 4.26. It follows that $\log(\Delta_\alpha^{>0}) \subseteq \Delta_\alpha$, as desired.

(3) Note that the conclusion is trivially true for $\alpha = 0, 1$, so we may assume that $\alpha \geq 2$. Take any $x \in \Delta_\alpha$. Since Δ_α is closed under truncation (see again Remark 4.25), we can write uniquely $x = \gamma + r + \varepsilon$, with $\gamma \in \Delta_\alpha \cap \mathbb{J}$, $r \in \mathbb{R}$ and $\varepsilon \in \Delta_\alpha \cap o(1)$. Since $\Delta_{\alpha+1}$ is a Hahn field, we have $e^x = e^\gamma \cdot e^r \cdot \sum_{n=0}^\infty \frac{\varepsilon^n}{n!} \in \mathbb{R}((e^{\Delta_\alpha \cap \mathbb{J}}))_{\mathrm{small}} = \Delta_{\alpha+1}$, as desired. □

Corollary 4.29. $\mathbb{R}\langle\!\langle \Delta \rangle\!\rangle = \bigcup_{\alpha\in\mathbf{On}} \Delta_\alpha$.

Proof. By Proposition 4.28, $\bigcup_{\alpha\in\mathbf{On}} \Delta_\alpha$ contains $\mathbb{R}$ and Δ (as both are contained in Δ_2) and it is closed under exp, log and infinite sums. It follows that $\mathbb{R}\langle\!\langle \Delta \rangle\!\rangle \subseteq \bigcup_{\alpha\in\mathbf{On}} \Delta_\alpha$. On the other hand, one can easily verify by induction that $\Delta_\alpha \subseteq \mathbb{R}\langle\!\langle \Delta \rangle\!\rangle$ for all $\alpha \in \mathbf{On}$, and the conclusion follows. □

Corollary 4.30. $\bigcup_{\alpha\in\mathbf{On}} = \mathbb{R}\langle\!\langle \Delta \rangle\!\rangle$ *is the smallest class containing $\Delta \cup \{0\}$ and such that whenever the exponents $\gamma_i \in \mathbb{J}$ of $x = \sum_{i<\alpha} r_i e^{\gamma_i}$ are in the class, then also x is in the class. The ordinal $\mathrm{ER}_\Delta(x)$ measures the number of steps needed to obtain x with this inductive construction.*

Corollary 4.31. $\mathbb{R}\langle\!\langle \Delta \rangle\!\rangle$ *is truncation closed, so it is a field of transseries in the sense of Definition 4.1.*

Proof. Immediate from the equality $\mathbb{R}\langle\!\langle \Delta \rangle\!\rangle = \bigcup_{\alpha\in\mathbf{On}} \Delta_\alpha$. □

Corollary 4.32. $\mathbb{R}\langle\!\langle \Delta \rangle\!\rangle$ *is a proper class. In particular, $\mathbb{R}\langle\!\langle \omega \rangle\!\rangle$ is a proper class.*

Proof. Let $\Gamma = \mathfrak{M} \cap \mathbb{R}\langle\!\langle \Delta \rangle\!\rangle$ be the class of monomials of $\mathbb{R}\langle\!\langle \Delta \rangle\!\rangle$. Since $\mathbb{R}\langle\!\langle \Delta \rangle\!\rangle$ is closed under $\sum$ and truncations, we have $\mathbb{R}\langle\!\langle \Delta \rangle\!\rangle = \mathbb{R}((\Gamma))_{\mathrm{small}}$. If for a contradiction $\mathbb{R}\langle\!\langle \Delta \rangle\!\rangle$ were a set, then $\mathbb{R}\langle\!\langle \Delta \rangle\!\rangle = \mathbb{R}((\Gamma))$. Since on the other hand $\mathbb{R}\langle\!\langle \Delta \rangle\!\rangle$ is an exponential subfield of **No**, $\mathbb{R}((\Gamma))$ would then carry a compatible exponential function, contradicting [KKS97]. □

The following remark is implicit in our previous observations, but it is worth to record it:

Remark 4.33. Let $\Delta = \{\log_n(\omega) : n \in \mathbb{N}\}$. Then $\mathbb{R}((\omega))^{EL} = \bigcup_{n\in\mathbb{N}} \Delta_n = \Delta_\omega$.

5. Substitutions

Before defining the full notion of composition, we first define *substitutions* (also called right-compositions in [Sch01]).

Definition 5.1. Let T be a field of transseries. We say that $f : T \to \mathbf{No}$ is **strongly additive** if for every summable sequence $(x_i : i \in I)$ in T, the sequence $(f(x_i) : i \in I)$ in **No** is summable and $f(\sum_{i\in I} x_i) = \sum_{i\in I} f(x_i)$.

Definition 5.2. Let T a field of transseries. A **substitution** $c : T \to \mathbf{No}$ is a strongly additive map which is the identity on $\mathbb{R}$ and preserves log, namely $c(\log(x)) = \log(c(x))$ for all $x \in T$.

It is fairly easy to check that the substitutions are well behaved functions.

Proposition 5.3. *Let $c : T \to \mathbf{No}$ be a substitution. Then c is an ordered field isomorphism fixing $\mathbb{R}$. In particular, for all $x, y \in T$ we have $x < y \to c(x) < c(y)$ and therefore $x \prec y \to c(x) \prec c(y)$.*

Proof. Fix some $x, y \in T$. Clearly, c is additive. Moreover, if $x > 0$, then $\log(x) \in T$, so $c(\log(x)) = \log(c(x))$, so $c(x) > 0$, and in particular, c preserves the ordering. If $x, y > 0$, then $c(xy) = c(e^{\log(xy)}) = e^{c(\log(x)+\log(y))} = e^{c(\log(x))} \cdot e^{c(\log(y))} = c(x)c(y)$, and it follows easily that c is multiplicative. Therefore, c is an ordered field isomorphism which by definition fixes $\mathbb{R}$. In particular, if $x < y$, then $c(x) < c(y)$. Moreover, if $x \prec y$, then $r|x| < |y|$ for all $r \in \mathbb{R}$, so $r|c(x)| < |c(y)|$ for all $r \in \mathbb{R}$, so $c(x) \prec c(y)$. □

In this section, we show how to construct inductively substitutions on fields of the form $\mathbb{R}\langle\!\langle\Delta\rangle\!\rangle$ starting from their values on some subclass $\Delta \subseteq \mathbb{L}$. The proof that the construction is well defined is fairly complicated and technical; for the sake of readability, the proof of one of the intermediate statement, the "summability lemma" 5.21, will be postponed to Section 9.

5.1. **Pre-substitutions.** To build a substitution on $\mathbb{R}\langle\!\langle\Delta\rangle\!\rangle$, we start with a certain assignment of values to each element of Δ satisfying some suitable compatibility conditions. We call such assignment a pre-substitution.

Definition 5.4. A map $c_0 : \Delta \to \mathbf{No}$ is a **pre-substitution** if

(1) the domain Δ is a subclass of $\mathbb{L}$ closed under log;
(2) $c_0(\lambda) > 0$ and $c_0(\log(\lambda)) = \log(c_0(\lambda))$ for all $\lambda \in \Delta$;
(3) for any decreasing sequence $(\lambda_i \in \Delta)_{i\in\mathbb{N}}$, the family $(c_0(\lambda_i))_{i\in\mathbb{N}}$ is summable;
(4) for any increasing sequence $(\lambda_i \in \Delta)_{i\in\mathbb{N}}$, the family $\left(c_0(\lambda_i)^{-1}\right)_{i\in\mathbb{N}}$ is summable;
(5) for all $\lambda, \mu \in \Delta$, if $\lambda < \mu$, then $c_0(\lambda) \prec c_0(\mu)$.

Remark 5.5. By (1) and (2) it follows by induction on $n \in \mathbb{N}$ that $c_0(\lambda) > \exp_n(0)$ for every $\lambda \in \Delta$, and therefore for all $\lambda \in \Delta$ we have $1 \prec c_0(\lambda)$. Moreover, if $\lambda < \mu$, then $c_0(\lambda) < c_0(\mu)$ and $c_0(\lambda)^n \prec c_0(\mu)$ for all $n \in \mathbb{N}$ (since $\log(c_0(\lambda)) = c_0(\log(\lambda)) \prec c_0(\log(\mu)) = \log(c_0(\mu))$).

Clearly, if $\Delta \subseteq \mathbb{L}$ is a class closed under log and $c : \mathbb{R}\langle\!\langle\Delta\rangle\!\rangle \to \mathbf{No}$ is a substitution, then $c_{\restriction\Delta}$ is a pre-substitution. We shall prove that the converse holds, namely that every pre-substitution with domain Δ extends to a (unique) substitution with domain $\mathbb{R}\langle\!\langle\Delta\rangle\!\rangle$ (Theorem 5.22), and as a corollary we shall deduce the existence of substitutions on $\mathbb{R}\langle\!\langle\omega\rangle\!\rangle$ (Corollary 5.23). We first give an explicit example of pre-substitution on $\Delta = \{\log_i(\omega) : i \in \mathbb{N}\}$.

Proposition 5.6. *Let $x \in \mathbf{No}^{>\mathbb{N}}$. Then the sequence $(\log_i(x))_{i\in\mathbb{N}}$ is summable.*

Proof. By [BM, Prop. 5.8], there is an integer $k \in \mathbb{N}$ and some log-atomic number $\mu \in \mathbb{L}$ such that $\log_k(x) = \mu + \varepsilon$ for some $\varepsilon \prec 1$. Thus, it suffices to show that the

sequence $(\log_i(\mu+\varepsilon))_{i\in\mathbb{N}}$ is summable. Let $P(y)$ be the Taylor series of $\log(1+y)$, namely $P(y) := \sum_{n=1}^{\infty} \frac{(-1)^n}{n} y^n$. Then

$$\begin{aligned}\log(\mu+\varepsilon) &= \log(\mu) + \log\left(1+\frac{\varepsilon}{\mu}\right) = \log(\mu) + P\left(\frac{\varepsilon}{\mu}\right)\\ &= \mu_1 + \varepsilon_1\end{aligned}$$

where $\mu_1 := \log(\mu) \in \mathbb{L}$ and $\varepsilon_1 = P\left(\frac{\varepsilon}{\mu}\right) \prec 1$. We define inductively $\mu_0 := \mu$, $\mu_{i+1} := \log(\mu_i)$, $\varepsilon_0 := \varepsilon$ and $\varepsilon_{i+1} := P\left(\frac{\varepsilon_i}{\mu_i}\right)$. By construction, $\varepsilon_i \prec 1$ and $\log_i(\mu+\varepsilon) = \mu_i + \varepsilon_i$ for all $i \in \mathbb{N}$. Since $(\mu_i)_{i\in\mathbb{N}}$ is a decreasing sequence of monomials, $\sum_i \mu_i$ exists. To finish the proof it suffices to show that $\sum_i \varepsilon_i$ exists. Let $\mathfrak{m}$ be a monomial in the support of $\varepsilon_{i+1} = P\left(\frac{\varepsilon_i}{\mu_i}\right)$. Then there is an integer $m \geq 1$ such that $\mathfrak{m} \in \operatorname{Supp}\left(\left(\frac{\varepsilon_i}{\mu_i}\right)^m\right) \subseteq \frac{1}{\mu_i^m}\operatorname{Supp}(\varepsilon_i)^m$. By an easy induction it follows that

$$\mathfrak{m} = \frac{1}{\mu_0^{n_0}\cdot\dots\cdot\mu_i^{n_i}}\cdot\mathfrak{o}$$

where $n_0 \geq \ldots \geq n_i \geq 1$ and $\mathfrak{o}$ is a product of finitely many elements of $\operatorname{Supp}(\varepsilon)$. Note that $\mathfrak{o}$ varies in the set $\bigcup_{m=1}^{\infty}\operatorname{Supp}(\varepsilon)^m$, which is reverse well-ordered by Lemma 3.8. Therefore, it suffices to prove that the family $\left(\frac{1}{\mu_0^{n_0}\cdot\dots\cdot\mu_i^{n_i}} : i\in\mathbb{N}\right)$ is summable.

Letting $\delta = \sum_{i\in\mathbb{N}} \frac{1}{\mu_0\mu_1\cdots\mu_i}$, we have that $\frac{1}{\mu_0^{n_0}\cdot\dots\cdot\mu_i^{n_i}}$ is in the support of δ^{n_0}. Since $\delta \prec 1$, by Corollary 3.9, $(\delta^n : n\in\mathbb{N})$ is summable, so $\left(\frac{1}{\mu_0^{n_0}\cdot\dots\cdot\mu_i^{n_i}} : i\in\mathbb{N}\right)$ is summable, hence $(\varepsilon_i)_{i\in\mathbb{N}}$ is summable, as desired. □

Corollary 5.7. *Let $x \in \mathbf{No}^{>\mathbb{R}}$, let $\Delta = \{\log_i(\omega) : i\in\mathbb{N}\}$ and let $c_0^x : \Delta \to \mathbf{No}$ be the map that sends $\log_i(\omega)$ to $\log_i(x)$. Then c_0^x is a pre-substitution.*

5.2. **Trees.** We now aim at extending each pre-substitution $c_0 : \Delta \to \mathbf{No}$ to a substitution $c : \mathbb{R}\langle\!\langle\Delta\rangle\!\rangle \to \mathbf{No}$. For this, we introduce the notion of *tree*, whose aim is to keep track of the monomials that may appear in the support of $c(x)$ by expressing $c(x)$ in terms of the values of c_0. To justify the definition of tree, consider the following heuristic argument.

Suppose we wish to calculate $c(x)$ for some $x \in \mathbb{R}\langle\!\langle\Delta\rangle\!\rangle$. If $\mathrm{ER}_\Delta(x) = 0$, then we simply use the equations $c(\lambda) = c_0(\lambda) = \sum_{t\in\mathrm{Term}(c_0(\lambda))} t$ and $c(0) = 0$. Now assume $\mathrm{ER}_\Delta(x) > 0$ and write $x = \sum_{i<\alpha} r_i e^{\gamma_i}$ in normal form. First, we observe that we must have $c(x) = \sum_{i<\alpha} c(r_i e^{\gamma_i})$, so our problem reduces to calculating $c(r_i e^{\gamma_i})$ for each $i < \alpha$. Fix one $\gamma = \gamma_i$ and consider the following equation:

$$c(re^\gamma) = re^{c(\gamma)} = re^{c(\gamma)^{\uparrow=}}\cdot\exp(c(\gamma)^\downarrow) = re^{c(\gamma)^{\uparrow=}}\cdot\sum_{n=0}^{\infty}\frac{\left(c(\gamma)^\downarrow\right)^n}{n!}.$$

Note that $\mathrm{ER}_\Delta(\gamma) < \mathrm{ER}_\Delta(x)$, so we may assume to already have obtained $c(\gamma)$, and that $c(\gamma)^\downarrow$ is presented as a sum $c(\gamma)^\downarrow = \sum_{j\in J} t_j$ for some family $(t_j)_{j\in J}$ of terms (i.e. elements of $\mathbb{R}^*\mathfrak{M}$), where $J = J_i$ is some index set. Using Proposition 3.5, we get

$$c(re^\gamma) = re^{c(\gamma)^{\uparrow=}}\cdot\sum_{n=0}^{\infty}\sum_{\tau:n\to J}\prod_{m<n} t_{\tau(m)} = \sum_{n=0}^{\infty}\sum_{\tau:n\to J} re^{c(\gamma)^{\uparrow=}}\cdot\prod_{m<n} t_{\tau(m)}.$$

Note that the right-hand side can be seen as a sum of *terms*. We use the above equation to present $c(re^\gamma)$ as a sum of terms indexed by the set $\{(n,\tau) : n \in \mathbb{N}, \tau : n \to J\}$. By taking the sum over all terms $r_i e^{\gamma_i}$, we obtain a presentation of $c(x)$ as a sum of terms indexed by the set $\{(r_i e^{\gamma_i}, n, \tau) : i < \alpha, n \in \mathbb{N}, \tau : n \to J_i\}$.

We then proceed inductively and assume that the index sets J_i are themselves constructed in the same way (unless $\mathrm{ER}_\Delta(\gamma_i) = 0$, in which case we use the equations $c(\lambda) = c_0(\lambda) = \sum_{t \in \mathrm{Term}(c_0(\lambda))} t$ and $c(0) = 0$). One can then picture the index $(r_i e^{\gamma_i}, n, \tau)$ as a tree with root $r_i e^{\gamma_i}$ and children $\tau(0), \dots, \tau(n-1)$, as in the following definition.

Definition 5.8. Fix a pre-substitution $c_0 : \Delta \to \mathbf{No}$. We define inductively the class of trees as follows. A **tree** is an ordered triple $T = \langle \mathrm{R}(T), n, \tau\rangle$ where $\mathrm{R}(T) \in \mathbb{R}\langle\!\langle \Delta \rangle\!\rangle \cap \mathbb{R}^*\mathfrak{M}$ is a term, called the **root** of T, and n, τ are defined as follows:

(1) if $\mathrm{R}(T) = \lambda \in \Delta$, then $n = 0$ and τ is a term of $c_0(\lambda)$, so in this case $T = \langle \lambda, 0, t\rangle$ with $t \in \mathrm{Term}(c_0(\lambda))$;
(2) if $\mathrm{R}(T) = re^\gamma \notin \Delta$, then $n \in \mathbb{N}$ and τ is a function with domain $n = \{0, 1, \dots, n-1\}$ such that $\tau(0), \dots, \tau(n-1)$ are trees, called the **children** of T (n can be zero, in which case T has no children); we also require that, for each $i < n$, the root $\mathrm{R}(\tau(i))$ of $\tau(i)$ is a term of $\gamma = \log^\uparrow(\mathrm{R}(T))$ (where $\log^\uparrow$ is as in Definition 2.10).

The **descendants** of T are T itself, its children, and the descendants of its children. The **proper descendants** are the descendants different from T itself. The **leaves** of T are the descendants U of T without children (for instance the descendants with root in Δ).

Note that by induction on ER_Δ, the above definition of tree is well founded.

Definition 5.9. Let $T = \langle \mathrm{R}(T), n, \tau\rangle$ be a tree. We define $\mathrm{size}(T) \in \mathbb{N}$ as the number of descendants of T, namely:

(1) $\mathrm{size}(T) := 1$ if T has no children, namely $n = 0$;
(2) $\mathrm{size}(T) := 1 + \sum_{i<n} \mathrm{size}(\tau(i))$ otherwise.

5.3. **Extending pre-substitutions to substitutions.** Fix a pre-substitution $c_0 : \Delta \to \mathbf{No}$. We shall now define a substitution $c : \mathbb{R}\langle\!\langle \Delta \rangle\!\rangle \to \mathbf{No}$ extending the given

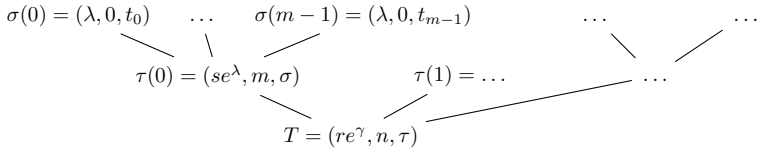

FIGURE 5.1. An example of tree with root $R(T) = re^\gamma$, where se^λ is a term of γ, $\lambda \in \Delta$, $t_0, \dots, t_{m-1}$ are terms of $c_0(\lambda)$, and the contribution $\overline{c}(T)$ of T is

$$\begin{aligned}\overline{c}(T) &= re^{c(\gamma)^{\uparrow=}} \tfrac{1}{n!} \overline{c}(\tau(0))\overline{c}(\tau(1)) \dots \\ &= re^{c(\gamma)^{\uparrow=}} \tfrac{1}{n!} se^{c_0(\lambda)^{\uparrow=}} \tfrac{1}{m!} \overline{c}(\sigma(0)) \dots \overline{c}(\sigma(m-1))\overline{c}(\tau(1)) \dots \\ &= re^{c(\gamma)^{\uparrow=}} \tfrac{1}{n!} se^{c_0(\lambda)^{\uparrow=}} \tfrac{1}{m!} t_0 \dots t_{m-1} \overline{c}(\tau(1)) \dots\end{aligned}$$

pre-substitution c_0. To this aim, we shall define simultaneously by induction on $\alpha \in \mathbf{On}$ the following objects:

- the set of **admissible trees** $\mathrm{A}(x)$ of each $x \in \Delta_\alpha$ (which are trees in the sense of Definition 5.8 with root $\mathrm{R}(T) \in \mathrm{Term}(x)$ and some further requirements);
- the **contribution** $\overline{c}(T) \in \mathbb{R}^*\mathfrak{M}$ of each $T \in \mathrm{A}(x)$;
- the extension $c : \Delta_\alpha \to \mathbf{No}$ (which is obtained by summing the contributions of the admissible trees in $\mathrm{A}(x)$, that is $c(x) = \sum_{T\in \mathrm{A}(x)} \overline{c}(T)$).

The main difficulty will be in proving that each family $(\overline{c}(T) : T \in \mathrm{A}(x))$ is summable, which is needed to show that $c(x) = \sum_{T\in \mathrm{A}(x)} \overline{c}(T)$ is well defined (Lemma 5.21).

Definition 5.10. Let $\alpha \in \mathbf{On}$ be given. Let $I(\alpha)$ be the hypothesis

For all $x \in \Delta_\alpha$, $(\overline{c}(T) : T \in \mathrm{A}(x))$ *is summable*

where $\mathrm{A}(x)$ and $\overline{c}(T)$ for $T \in \mathrm{A}(x)$ are inductively defined as in Definition 5.11 (assuming $I(\beta)$ for $\beta < \alpha$).

Definition 5.11. First, we let $\mathrm{A}(0) := \emptyset$, and for $\lambda \in \Delta$, we define:

(1) $\mathrm{A}(\lambda) := \{\langle \lambda, 0, t\rangle : t \in \mathrm{Term}(c_0(\lambda))\}$ (namely every tree with root in Δ is admissible);
(2) $\overline{c}(\langle \lambda, 0, t\rangle) := t$ (the value of a tree with root in Δ is its third component);
(3) $c(\lambda) := \sum_{T\in \mathrm{A}(\lambda)} \overline{c}(T)$.

This defines $\mathrm{A}(x)$, $\overline{c}(T)$ and $c(x)$ for all $x \in \Delta_1$ and $T \in \mathrm{A}(x)$.

Now let $\alpha > 1$ and assume $I(\beta)$ for all $\beta < \alpha$. For general $x \in \Delta_\alpha$ we define:

(4) $\mathrm{A}(x) := \bigcup_{t\in \mathrm{Term}(x)} \mathrm{A}(t)$;
(5) $\mathrm{A}^\circ(x) := \{T \in \mathrm{A}(x) : \overline{c}(T) \prec 1\}$.

When $x = t = re^\gamma$ is a term in $\Delta_\alpha \setminus \Delta_1$, let $\beta < \alpha$ be such that $re^\gamma \in \Delta_{\beta+1} \setminus \Delta_\beta$. We observe that $\gamma \in \Delta_\beta$, and we define:

(6) $\mathrm{A}(re^\gamma) := \{\langle re^\gamma, n, \tau\rangle : n \in \mathbb{N}, \tau : n \to \mathrm{A}^\circ(\gamma)\}$;
(7) for $T = \langle re^\gamma, n, \tau\rangle \in \mathrm{A}(re^\gamma)$,

$$\overline{c}(T) := re^{c(\gamma)^{\uparrow=}} \cdot \frac{1}{n!} \prod_{i<n} \overline{c}(\tau(i)).$$

Finally, for any $x \in \Delta_\alpha$, if $I(\alpha)$ holds, we define:

(8) $c(x) := \sum_{T\in \mathrm{A}(x)} \overline{c}(T)$.

Remark 5.12. It is important to note that points (1)-(7) only require $I(\beta)$ for $\beta < \alpha$, while (8) does require $I(\alpha)$. The inductive hypothesis $I(\alpha)$ itself is defined by induction on α! We also remark that $I(0), I(1)$ are trivially true.

Definition 5.13. Assuming that $I(\alpha)$ holds for every $\alpha \in \mathbf{On}$, we define $c : \mathbb{R}\langle\!\langle\Delta\rangle\!\rangle \to \mathbf{No}$ as the union of the functions $c : \Delta_\alpha \to \mathbf{No}$ for $\alpha \in \mathbf{On}$.

Remark 5.14. The present notion of tree should be compared with the similar notion of labeled trees in [Sch01]. In this comparison, the admissible trees play the same role as the well-labeled trees.

We shall now prove that $c : \mathbb{R}\langle\!\langle\Delta\rangle\!\rangle \to \mathbf{No}$ is well defined and that it is the unique substitution on $\mathbb{R}\langle\!\langle\Delta\rangle\!\rangle$ extending c_0. The most technical and difficult part will be proving that if $I(\alpha)$ holds, then $(\overline{c}(T) : T \in \mathrm{A}(x))$ is summable for all

$x \in \Delta_{\alpha+1} \setminus \Delta_\alpha$ (Lemma 5.21). As anticipated, the proof of this fact is postponed to Section 9.

First, we check that c is indeed an extension of c_0, and that it fixes $\mathbb{R}$.

Proposition 5.15. *For all $\lambda \in \Delta$, $(\overline{c}(T) : T \in \mathrm{A}(\lambda))$ is summable and*

$$c(\lambda) = \sum_{T \in \mathrm{A}(\lambda)} \overline{c}(T) = c_0(\lambda).$$

In particular, $I(0)$ and $I(1)$ hold, and c extends c_0.

Proof. For any $\lambda \in \Delta$ and $T \in \mathrm{A}(\lambda)$, we have $T = \langle \lambda, 0, t\rangle$ for some $t \in \mathrm{Term}(\lambda)$ and $\overline{c}(T) = t$. Moreover, $(\overline{c}(T) : T \in \mathrm{A}(\lambda))$ coincides with $(t : t \in \mathrm{Term}(c_0(\lambda)))$, hence it is summable and by definition

$$c(\lambda) = \sum_{T \in \mathrm{A}(\lambda)} \bar{c}(T) = \sum_{t \in \mathrm{Term}(c_0(\lambda))} t = c_0(\lambda).$$

□

Proposition 5.16. *If $r \in \mathbb{R}$, then $(\overline{c}(T) : T \in \mathrm{A}(r))$ is summable and $c(r) = r$.*

Proof. Note first that $I(1)$ holds by 5.15, and that $\mathbb{R} \subseteq \Delta_2$, so $\mathrm{A}(r)$ is well defined for each $r \in \mathbb{R}$. Now observe that $\mathrm{A}(0) = \emptyset$, so $c(0) = \sum_{T \in \mathrm{A}(0)} \overline{c}(T)$ is an empty sum (equal to zero) and we get $c(0) = 0$. For $r \neq 0$ the only admissible tree $T \in \mathrm{A}(r)$ is given by $T = \langle re^0, 0, \emptyset\rangle$. By definition, $\overline{c}(T) = re^{c(0)} = re^0 = r$, hence $c(r) = r$. □

We now prove that assuming $I(\alpha)$, the extension $c : \Delta_\alpha \to \mathbf{No}$ preserves log and infinite sums. For $\alpha \geq 3$, since Δ_α is a field of transseries, this says that $c : \Delta_\alpha \to \mathbf{No}$ is a substitution. Note that in the following statement the hypothesis is $I(\alpha)$, but the conclusion is about terms in $\Delta_{\alpha+1}$.

Proposition 5.17. *Assume $I(\alpha)$. Let $re^\gamma \in \Delta_{\alpha+1}$ be a term. Then $(\overline{c}(T) : T \in \mathrm{A}(re^\gamma))$ is summable, so $c(re^\gamma) = \sum_{T \in \mathrm{A}(re^\gamma)} \overline{c}(T)$ is well defined and*

$$c(re^\gamma) = re^{c(\gamma)}.$$

Proof. The result is clear if $re^\gamma \in \Delta_1 = \Delta \cup \{0\}$, for in that case c coincides with c_0 by Proposition 5.15, so we can assume $re^\gamma \notin \Delta_1$. Then $\mathrm{ER}_\Delta(\gamma) < \mathrm{ER}_\Delta(re^\gamma)$, and by the inductive hypothesis, $c(\gamma) = \sum_{T' \in \mathrm{A}(\gamma)} \overline{c}(T')$. By definition of $\mathrm{A}^\circ(\gamma)$ we have

$$c(\gamma)^\downarrow = \sum_{T' \in \mathrm{A}^\circ(\gamma)} \overline{c}(T').$$

Unraveling the definitions we have:

$$\begin{aligned} re^{c(\gamma)} &= re^{c(\gamma)^{\uparrow=}}e^{c(\gamma)^{\downarrow}} \\ &= \sum_{n\in\mathbb{N}} re^{c(\gamma)^{\uparrow=}}\frac{1}{n!}\left(c(\gamma)^{\downarrow}\right)^n \\ &= \sum_{n\in\mathbb{N}} re^{c(\gamma)^{\uparrow=}}\frac{1}{n!}\left(\sum_{T'\in \mathrm{A}^\circ(\gamma)}\bar{c}(T')\right)^n \\ &= \sum_{n\in\mathbb{N}} re^{c(\gamma)^{\uparrow=}}\frac{1}{n!}\sum_{\tau:n\to \mathrm{A}^\circ(\gamma)}\prod_{i<n}\overline{c}(\tau(i)) \\ &= \sum_{T\in\mathrm{A}(re^\gamma)}\bar{c}(T) = c(re^\gamma), \end{aligned}$$

where in the fourth line we used Proposition 3.5 (which also shows the summability of the relevant sequences) and in the fifth line we used the definition of $\overline{c}(T)$ for $T \in \mathrm{A}(re^\gamma)$. □

Proposition 5.18. *Assume $I(\alpha)$. Let $\sum_{i<\beta} r_i e^{\gamma_i} \in \Delta_\alpha$. Then*

$$c\left(\sum_{i<\beta} r_i e^{\gamma_i}\right) = \sum_{i<\beta} r_i e^{c(\gamma_i)}.$$

Proof. It follows at once from Proposition 5.17 and the equality $\mathrm{A}(\sum_{i<\beta} r_i e^{\gamma_i}) = \bigcup_{i<\beta}\mathrm{A}(r_i e^{\gamma_i})$. □

Corollary 5.19. *Assume $I(\alpha)$, with $\alpha \geq 3$. Then $c : \Delta_\alpha \to \mathbf{No}$ is a substitution.*

Proof. Since $\alpha \geq 3$, Δ_α is a transserial subfield of **No** by Proposition 4.28. By Proposition 5.16, c fixes $\mathbb{R}$, and by Proposition 5.18, it is strongly additive. Moreover, c preserves log. Indeed, let $x = re^\gamma(1+\varepsilon) \in \Delta_\alpha$, where $r \in \mathbb{R}$, $\gamma \in \mathbb{J}$ and $\varepsilon \prec 1$. We have

$$c(\log(x)) = c\left(\gamma + \log(r) + \sum_{i=1}^{\infty}(-1)^n\frac{\varepsilon^n}{n}\right) = c(\gamma) + \log(r) + \sum_{i=1}^{\infty}(-1)^n\frac{c(\varepsilon)^n}{n}.$$

By Proposition 5.17, $c(\gamma) = \log(c(e^\gamma))$, so the right hand side is $\log(c(x))$, as desired. □

Corollary 5.20. *Assume $I(\alpha)$ for all $\alpha \in \mathbf{On}$. Then $c : \mathbb{R}\langle\!\langle\Delta\rangle\!\rangle \to \mathbf{No}$ is a substitution extending c_0.*

Finally, we need to prove inductively that $I(\alpha)$ holds for all $\alpha \in \mathbf{On}$. The main difficulty is in proving the successor stage, namely that $I(\alpha)$ implies $I(\alpha+1)$. This is contained in the following lemma, the proof of which is postponed to Section 9.

Lemma 5.21 (Summability)**.** *Assume $I(\alpha)$. Then $(\overline{c}(T) : T \in \mathrm{A}(x))$ is summable for all $x \in \Delta_{\alpha+1} \setminus \Delta_\alpha$. In particular, $I(\alpha)$ implies $I(\alpha+1)$.*

Proof. Postponed to Section 9. □

Theorem 5.22. *Any pre-substitution $c_0 : \Delta \to \mathbf{No}$ extends uniquely to a substitution $c : \mathbb{R}\langle\!\langle\Delta\rangle\!\rangle \to \mathbf{No}$.*

Proof. Fix a pre-substitution $c_0 : \Delta \to \mathbf{No}$. By Proposition 5.15, $I(0)$ and $I(1)$ hold. It is also clear by the definition of $I(\alpha)$ that whenever α is a limit ordinal, $I(\alpha)$ is implied by, and in fact equivalent to, $\bigwedge_{\beta<\alpha} I(\beta)$. Moreover, by Lemma 5.21, $I(\alpha)$ implies $I(\alpha+1)$. Therefore, $I(\alpha)$ holds for all $\alpha \in \mathbf{On}$. By Corollary 5.20, c is a substitution extending c_0. The uniqueness follows by an easy induction on ER_Δ. □

Corollary 5.23. *Given $x \in \mathbf{No}^{>0}$, there is a unique substitution $c^x : \mathbb{R}\langle\!\langle\omega\rangle\!\rangle \to \mathbf{No}$ sending ω to x.*

Proof. Let $\Delta = \{\log_i(\omega) : i \in \mathbb{N}\}$ and let $c_0^x : \Delta \to \mathbf{No}$ be the map that sends $\log_i(\omega)$ to $\log_i(x)$. Then c_0^x is a pre-substitution by Corollary 5.7. By Theorem 5.22, there is a unique substitution $c^x : \mathbb{R}\langle\!\langle\Delta\rangle\!\rangle = \mathbb{R}\langle\!\langle\omega\rangle\!\rangle \to \mathbf{No}$ extending c_0^x. □

6. Composition

We prove that omega-series can be composed in a meaningful way. Intuitively, for $f, g \in \mathbb{R}\langle\!\langle\omega\rangle\!\rangle$, with $g > \mathbb{R}$, $f \circ g$ is the result of substituting g for ω in f. For instance, we will have

$$\left(\sum_{i\in\mathbb{N}} \log_i(\omega)\right) \circ \left(\sum_{i\in\mathbb{N}} \log_i(\omega)\right) = \sum_{i\in\mathbb{N}} \log_i \left(\sum_{j\in\mathbb{N}} \log_j(\omega)\right).$$

Note that the right-hand side exists in **No** by the results in Section 5 and it is in fact an element of $\mathbb{R}\langle\!\langle\omega\rangle\!\rangle$.

Definition 6.1. Let $T \subseteq \mathbf{No}$ be a transseries subfield containing ω. A **composition** on T is a function $\circ : T \times \mathbf{No}^{>\mathbb{R}} \to \mathbf{No}$ which satisfies the following axioms:

(1) for all $x \in \mathbf{No}^{>\mathbb{R}}$, the map $f \mapsto f \circ x$ is a substitution, namely:
   (a) for any summable $(f_i)_{i\in I}$ in T, the family $(f_i \circ x)_{i\in I}$ is summable and
   $$\left(\sum_{i\in I} f_i\right) \circ x = \sum_{i\in I} (f_i \circ x);$$
   (b) $r \circ x = r$ for all $r \in \mathbb{R}$;
   (c) $\log(f) \circ x = \log(f \circ x)$ for all $f \in T$;
(2) T is closed under composition: for all $f \in T$, $g \in T^{>\mathbb{R}}$ we have $f \circ g \in T$;
(3) associativity: $(f \circ g) \circ x = f \circ (g \circ x)$ for all $f \in T$, $g \in T^{>\mathbb{R}}$, $x \in \mathbf{No}^{>\mathbb{R}}$;
(4) ω is the identity: for all $x \in \mathbf{No}^{>\mathbb{R}}$ and $f \in T$ we have $\omega \circ x = x$, $f \circ \omega = f$.

The axioms are modeled on the usual composition of real valued functions, where we interpret ω as the identity function. The restriction on the second argument to be positive infinite is necessary for a composition to exist; for instance we cannot hope to define $\sum_{n\in\mathbb{N}} \omega^{-n} \circ (1/2)$ in any reasonable way, as the axioms imply that the result should be $\sum_{n\in\mathbb{N}} 2^n$. Recall that by Proposition 5.3, for all $x \in \mathbf{No}^{>\mathbb{N}}$, the map $f \circ x$ is increasing and it preserves the dominance relation $\preceq$.

When $T \subseteq \mathbb{R}\langle\!\langle\omega\rangle\!\rangle$, the list of axioms can be shortened. More precisely, we have:

Proposition 6.2. *If T is a transserial field included in $\mathbb{R}\langle\!\langle\omega\rangle\!\rangle$, there is at most one function $\circ : T \times \mathbf{No}^{>\mathbb{R}} \to \mathbf{No}$ satisfying the following conditions:*

(1) for all $x \in \mathbf{No}^{>\mathbb{R}}$, the map $f \mapsto f \circ x$ is a substitution;
(2) for all $x \in \mathbf{No}^{>\mathbb{R}}$, $\omega \circ x = x$.

If any such function $\circ$ exists, it satisfies $f \circ \omega = f$ for any $f \in T$. If moreover T is closed under $\circ$, then $\circ$ is associative, so it is a composition.

Proof. Suppose that $\circ$ is a function satisfying the above properties. Let $\Delta = \{\log_i(\omega) : i \in \mathbb{N}\} \subseteq T$, and fix some $x \in \mathbf{No}^{>\mathbb{R}}$. We claim that the values of the substitution $f \mapsto f \circ x$ for $f \in \Delta$ are uniquely determined by the requirement $\omega \circ x = x$. We shall prove this by induction on $\mathrm{ER}_\Delta(f)$; at the same time, we will also verify associativity when T is closed under $\circ$.

Note first that $\log_i(\omega) \circ x = \log_i(x)$ by definition of substitution. Moreover,

$$\log_i(\omega) \circ (g \circ x) = \log_i(g \circ x) = \log_i(g) \circ x = (\log_i(\omega) \circ g) \circ x$$

for any $g \in T^{>\mathbb{N}}$, and also $\log_i(\omega) \circ \omega = \log_i(\omega)$. It now follows by induction on $\mathrm{ER}_\Delta(f)$ that the value of $f \circ x$ is also uniquely determined, $f \circ \omega = f$, and if T is closed under $\circ$, then $f \circ (g \circ x) = (f \circ g) \circ x$ for any $g \in T^{>\mathbb{R}}$. Indeed, if $f = \sum_{i<\alpha} r_i e^{\gamma_i}$, where $\mathrm{ER}_\Delta(f) > 0$, then we must have

$$f \circ x = \sum_{i<\alpha} r_i e^{\gamma_i \circ x}$$

where $\mathrm{ER}_\Delta(\gamma_i) < \mathrm{ER}_\Delta(f)$. The value of $f \circ x$ is then uniquely determined by the values $\gamma_i \circ x$, which are themselves uniquely determined by inductive hypothesis, and clearly $f \circ \omega = f$ as again by induction $\gamma_i \circ \omega = \gamma_i$. Moreover, if T is closed under $\circ$, then

$$f \circ (g \circ x) = \sum_{i<\alpha} r_i e^{\gamma_i \circ (g \circ x)} = \sum_{i<\alpha} r_i e^{(\gamma_i \circ g) \circ x} = \left(\sum_{i<\alpha} r_i e^{\gamma_i \circ g} \right) \circ x = (f \circ g) \circ x.$$

Therefore, $\circ$ is unique, $f \circ \omega = f$ for any $f \in T$, and if T is closed under $\circ$, then it is associative, so it is a composition. □

Theorem 6.3. *There is a unique composition $\circ : \mathbb{R}\langle\langle\omega\rangle\rangle \times \mathbf{No}^{>\mathbb{R}} \to \mathbf{No}$.*

Proof. Let $\Delta = \{\log_i(\omega) : i \in \mathbb{N}\}$. Fix $x \in \mathbf{No}^{>\mathbb{R}}$ and $f \in \mathbb{R}\langle\langle\omega\rangle\rangle$. By Corollary 5.23, there exists a unique substitution c^x on $\mathbb{R}\langle\langle\Delta\rangle\rangle = \mathbb{R}\langle\langle\omega\rangle\rangle$ such that $c^x(\log_i(\omega)) = \log_i(x)$ for all $i \in \mathbb{N}$. We then define $f \circ x := c^x(f)$. Clearly, this function is the unique one satisfying the hypothesis of Proposition 6.2. One can easily verify by induction on ER_Δ that $\mathbb{R}\langle\langle\omega\rangle\rangle$ is closed under $\circ$, so it is a composition. □

7. Taylor expansions

In this section, let $\circ$ be the unique composition on $\mathbb{R}\langle\langle\omega\rangle\rangle$. We shall now prove that for every $f \in \mathbb{R}\langle\langle\omega\rangle\rangle$, the function $x \mapsto f \circ x$ is surreal analytic in the sense of Definition 3.7. Moreover, the coefficients will coincide with the iterated derivatives of f divided by $n!$, when using the unique surreal derivation on $\mathbb{R}\langle\langle\omega\rangle\rangle$.

7.1. Transserial derivations.

Recall the notion of derivation from [Sch01, BM].

Definition 7.1. Given a field T, we recall that a map $\partial : T \to T$ is a **derivation** if it is additive ($\partial(x+y) = \partial x + \partial y$) and satisfies the Leibniz rule ($\partial(xy) = x \cdot \partial y + \partial x \cdot y$). If T is a field of transseries we say that $\partial : T \to T$ is a **transserial derivation** if it is a derivation satisfying the following additional properties:

(1) ∂ is strongly additive;
(2) $\partial e^x = e^x \cdot \partial x$;
(3) $\partial \omega = 1$;

(4) $\partial r = 0$ if $r \in \mathbb{R}$.

As in [BM], we call **surreal derivation** a transserial derivation with $\ker \partial = \mathbb{R}$.

In [BM], the authors proved that there exist surreal derivations on **No**, and in fact several of them. However, just like we proved that there is a unique composition on $\mathbb{R}\langle\!\langle\omega\rangle\!\rangle$, we can easily verify that there exists a unique transserial derivation on $\mathbb{R}\langle\!\langle\omega\rangle\!\rangle$.

Proposition 7.2. *The field of omega-series admits a unique transserial derivation $\partial : \mathbb{R}\langle\!\langle\omega\rangle\!\rangle \to \mathbb{R}\langle\!\langle\omega\rangle\!\rangle$, which is in fact a surreal derivation.*

Proof. Suppose first that there exists a transserial derivation $\partial : \mathbb{R}\langle\!\langle\omega\rangle\!\rangle \to \mathbb{R}\langle\!\langle\omega\rangle\!\rangle$. Since $\partial\omega = 1$, an easy induction on ER_Δ shows that in fact the values of ∂ are uniquely determined, and that $\ker(\partial) = \mathbb{R}$. Therefore, if there is one such derivation, it is unique, and it is a surreal derivation.

For the existence, let ∂ be any surreal derivation, which exists by the results of [BM]. By the same argument as above, since $\partial\omega = 1 \in \mathbb{R}\langle\!\langle\omega\rangle\!\rangle$, an easy induction on ER_Δ shows that $\partial(\mathbb{R}\langle\!\langle\omega\rangle\!\rangle) \subseteq \mathbb{R}\langle\!\langle\omega\rangle\!\rangle$. Therefore, the restriction of ∂ to $\mathbb{R}\langle\!\langle\omega\rangle\!\rangle$ is the unique transserial derivation on $\mathbb{R}\langle\!\langle\omega\rangle\!\rangle$. □

Remark 7.3. Unlike the subfield $\mathbb{R}((\omega))^{LE}$, but like $\mathbb{R}((\omega))^{EL}$, the field of omega-series $\mathbb{R}\langle\!\langle\omega\rangle\!\rangle$ is not closed under anti-derivatives. For instance, it contains no integral for the monomial $\exp(-\sum_{n\in\mathbb{N}} \log_n(\omega))$.

7.2. **A Taylor theorem.** From now on, let $\partial : \mathbb{R}\langle\!\langle\omega\rangle\!\rangle \to \mathbb{R}\langle\!\langle\omega\rangle\!\rangle$ be the unique transserial derivation on $\mathbb{R}\langle\!\langle\omega\rangle\!\rangle$. Recall that for any $x \prec 1$ we have $\exp(x) = \sum_{n\in\mathbb{N}} \frac{x^n}{n!}$. When $x \succ 1$, the equality does not hold, as the right hand side clearly does not exist. However, we can still approximate $\exp(x)$ with Taylor polynomials. In particular we have the following:

Proposition 7.4. *Given $x \in \mathbf{No}$, there are $A \in \mathbf{No}$ and $\varepsilon_0 \in \mathbf{No}^{>0}$ (depending on x) such that, for every $\varepsilon \in \mathbf{No}$ smaller in modulus than ε_0, we have*

$$\exp(x+\varepsilon) = \exp(x) + \exp'(x)\varepsilon + \mathcal{O}(A\varepsilon^2)$$

where $\exp'(x) := \exp(x)$ and $\mathcal{O}(A\varepsilon^2)$ is a surreal number $\preceq A\varepsilon^2$. Similarly, we can write

$$\log(x+\varepsilon) = \log(x) + \log'(x)\varepsilon + \mathcal{O}(A\varepsilon^2)$$

where $\log'(x) := \frac{1}{x}$.

Proof. Immediate from the fact that **No** is an elementary extension of $\mathbb{R}_{\exp}$. □

The next theorem extends the above remark to a much larger class of functions.

Theorem 7.5. *Given $f \in \mathbb{R}\langle\!\langle\omega\rangle\!\rangle$ and $x \in \mathbf{No}^{>\mathbb{R}}$, there are $A \in \mathbf{No}$ and $\varepsilon_0 \in \mathbf{No}^{>0}$ (both depending on f and x) such that, for every $\varepsilon \in \mathbf{No}$ smaller in modulus than ε_0, we have*

$$f \circ (x+\varepsilon) = f \circ x + (\partial f \circ x)\cdot\varepsilon + \mathcal{O}(A\varepsilon^2),$$

where $\mathcal{O}(A\varepsilon^2)$ is a surreal number $\preceq A\varepsilon^2$.

Proof. We reason by induction on the ordinal $\mathrm{ER}_\Delta(f)$, where $\Delta = \{\log_i(\omega) : i \in \mathbb{N}\}$.

Case 1. The theorem is clear if $f \in \mathbb{R}$ or $f = \omega$, as in this case $f \circ (x+\varepsilon) = f \circ x + (\partial(f) \circ x)\varepsilon$ for every ε and we can take $A = 0$.

Case 2. Now consider the case when $f = \log(g)$ where $g > 0$, and assume that conclusion holds for g. Then there are $B \in \mathbf{No}$ and $\varepsilon_1 \in \mathbf{No}^{>0}$ (depending on g, x) such that

$$g \circ (x + \varepsilon) = g \circ x + (\partial(g) \circ x)\varepsilon + \mathcal{O}(B\varepsilon^2)$$

whenever $|\varepsilon| \leq |\varepsilon_1|$. Taking the log of both sides, and recalling that $\log(g \circ (x+\varepsilon)) = \log(g) \circ (x + \varepsilon) = f \circ (x + \varepsilon)$, we obtain

$$f \circ (x + \varepsilon) = \log(g \circ x + (\partial(g) \circ x)\varepsilon + \mathcal{O}(B\varepsilon^2)).$$

Using the second order Taylor expansion of log at $g \circ x$, we can find $A \in \mathbf{No}$, depending on g and x, such that, for all sufficiently small ε,

$$\begin{aligned}\log(g \circ x + (\partial(g) \circ x)\varepsilon + \mathcal{O}(B\varepsilon^2)) &= \log(g \circ x) + \frac{1}{g \circ x}(\partial(g) \circ x)\varepsilon + \mathcal{O}(A\varepsilon^2) \\ &= \log(g) \circ x + \left(\frac{\partial(g)}{g} \circ x\right)\varepsilon + \mathcal{O}(A\varepsilon^2) \\ &= f \circ x + (\partial(f) \circ x)\,\varepsilon + \mathcal{O}(A\varepsilon^2).\end{aligned}$$

Combining the equations we obtain $f \circ (x + \varepsilon) = f \circ x + (\partial(f) \circ x)\,\varepsilon + \mathcal{O}(A\varepsilon^2)$, as desired.

Case 3. When $f = \log_n(\omega)$ for some $n \in \mathbb{N}$, the desired result follows from the previous cases by induction on n. We have thus established the conclusion when $\mathrm{ER}_\Delta(f) = 0$, namely $f \in \Delta_1 = \Delta \cup \{0\}$.

Case 4. Consider now the case when $f = \exp(g)$ and assume that the conclusion holds for g. We can then proceed as in case 2 using the second order Taylor expansion of exp at $g \circ x$.

Case 5. Consider the case when $f = \sum_{i \in I} f_i$ and assume by induction that the result holds for each f_i. By definition $f \circ (x+\varepsilon) = \sum_{i \in I}(f_i \circ (x+\varepsilon))$. By induction there are $\varepsilon_{i,x} \in \mathbf{No}^{>0}$ and $A_{i,x} \in \mathbf{No}$ such that

$$f_i \circ (x + \varepsilon) = f_i \circ x + (\partial(f_i) \circ x)\varepsilon + \mathcal{O}(A_{i,x}\varepsilon^2)$$

for all $\varepsilon < \varepsilon_{i,x}$. Now let $\varepsilon_0 \in \mathbf{No}^{>0}$ be smaller than $\varepsilon_{i,x}$ for every $i \in I$ and let $A \succeq A_{i,x}$ for every $i \in I$. Then for every ε smaller in modulus than ε_0 we have $f \circ (x + \varepsilon) = f \circ x + (\partial(f) \circ x) \cdot \varepsilon + \mathcal{O}(A\varepsilon^2)$, as desired.

Finally, observe that the above cases suffices to establish inductively the theorem for every $f \in \mathbb{R}\langle\!\langle\omega\rangle\!\rangle$. □

Corollary 7.6. *For every $f \in \mathbb{R}\langle\!\langle\omega\rangle\!\rangle$ and every $x \in \mathbf{No}^{>\mathbb{R}}$ we have*

$$\partial f \circ x = \lim_{\varepsilon \to 0} \frac{f \circ (x + \varepsilon) - f \circ x}{\varepsilon}$$

In particular, taking $x = \omega$, we obtain $\partial f = \lim_{\varepsilon \to 0} \frac{f \circ (\omega + \varepsilon) - f \circ \omega}{\varepsilon}$, so the derivative is definable in terms of the composition.

Corollary 7.7. *The unique composition on $\mathbb{R}\langle\!\langle\omega\rangle\!\rangle$ satisfies $\partial(f \circ g) = (\partial f \circ g) \cdot \partial g$.*

Proof. Thanks to Corollary 7.6, it suffices to show that that for all sufficiently small ε we have

$$(f \circ g) \circ (x + \varepsilon) = (f \circ g) \circ x + ((\partial f \circ g) \cdot \partial g)\,\varepsilon + \mathcal{O}(A\varepsilon^2)$$

where $A \in \mathbf{No}$ depends on f,g, x but not on ε. Applying Theorem 7.5 first to g and then to f, there are $C, D \in \mathbf{No}$, not depending on ε, such that

$$\begin{aligned}(f \circ g) \circ (x+\varepsilon) &= f \circ (g \circ (x+\varepsilon)) \\ &= f \circ (g \circ x + (\partial g \circ x)\varepsilon + \mathcal{O}(C \cdot \varepsilon^2)) \\ &= f \circ (g \circ x) + (\partial f \circ (g \circ x)) \cdot (\partial g \circ x)\varepsilon + \mathcal{O}(D \cdot \varepsilon^2),\end{aligned}$$

and we conclude by noting that $(\partial f \circ (g \circ x)) \cdot (\partial g \circ x) = ((\partial f \circ g) \cdot \partial g) \circ x$. □

7.3. **Surreal analyticity.** We now extend in the obvious way the notion of surreal analyticity of Definition 3.7 to the numbers in $\mathbb{R}\langle\!\langle \omega \rangle\!\rangle$.

Definition 7.8. Let $f \in \mathbb{R}\langle\!\langle \omega \rangle\!\rangle$. We say that f is **surreal analytic at** $x \in \mathbf{No}^{>\mathbb{R}}$ if the function $y \mapsto f \circ y$ is surreal analytic in a neighborhood of x is the sense of Definition 3.7. We say that f is **surreal analytic** if $y \mapsto f \circ y$ is surreal analytic at every $x \in \mathbf{No}^{>\mathbb{R}}$.

For instance, $\exp(\omega)$ and $\log(\omega)$ are surreal analytic.

Proposition 7.9. *Let $x \in \mathbf{No}^{>\mathbb{R}}$. Then for every $\varepsilon \prec 1$ we have* $\exp(x+\varepsilon) = \sum_{i=0}^{\infty} \frac{e^x}{i!}\varepsilon^i$. *In particular,* $\exp(\omega)$ *is surreal analytic.*

Proof. Indeed, $\exp(x+\varepsilon) = \exp(x) \cdot \exp(\varepsilon) = \exp(x) \cdot \sum_{i=0}^{\infty} \frac{\varepsilon^i}{i!}$. □

Proposition 7.10. *Let $x \in \mathbf{No}^{>\mathbb{R}}$. Then for every $\varepsilon \prec x$ we have* $\log(x+\varepsilon) = \log(x) + \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{ix^i}\varepsilon^i$. *In particular,* $\log(\omega)$ *is surreal analytic.*

Proof. It suffices to write $x + \varepsilon = x\left(1 + \frac{\varepsilon}{x}\right)$, so that $\delta := \frac{\varepsilon}{x} \prec 1$, and recall that

$$\log(x+\varepsilon) = \log\left(x\left(1+\frac{\varepsilon}{x}\right)\right) = \log(x) + \log(1+\delta) = \log(x) + \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i}\delta^i.$$

□

Moreover, surreal analyticity is preserved under compositions.

Lemma 7.11. *If $g \in \mathbb{R}\langle\!\langle \omega \rangle\!\rangle$ is surreal analytic at $x \in \mathbf{No}^{>\mathbb{R}}$ and $f \in \mathbb{R}\langle\!\langle \omega \rangle\!\rangle$ is surreal analytic at $y := g \circ x$, then $f \circ g$ is surreal analytic at x.*

Proof. Fix f, g, x, y as in the hypothesis. By assumption there are two sequences $(a_i)_{i\in\mathbb{N}}$ and $(b_j)_{j\in\mathbb{N}}$ in $\mathbf{No}$ such that, for every sufficiently small ε, δ we have

$$g \circ (x+\varepsilon) = g \circ x + \sum_{j=1}^{\infty} b_j \varepsilon^j$$

and

$$f \circ (y+\delta) = \sum_{i\in\mathbb{N}} a_i \delta^i.$$

Note that $(f \circ g) \circ (x+\varepsilon) = f \circ (y + \sum_{j=1}^{\infty} b_j\varepsilon^j) = \sum_{i\in\mathbb{N}} a_i (\sum_{j=1}^{\infty} b_j\varepsilon^j)^i$ for every sufficiently small ε. To finish the proof it suffices to observe that, by Proposition 3.16, there is a sequence $(c_m)_{m\in\mathbb{N}}$ in $\mathbf{No}$ such that, for every sufficiently small ε, we have

$$\sum_{k\in\mathbb{N}} a_k (\sum_{n=1}^{\infty} b_n \varepsilon^n)^k = \sum_{m\in\mathbb{N}} c_m \varepsilon^m.$$

□

Corollary 7.12. *For all $i \in \mathbb{N}$, $\log_i(\omega)$ is surreal analytic.*

We can also verify that if $f \in \mathbb{R}\langle\!\langle\omega\rangle\!\rangle$ is surreal analytic, the coefficients of its Taylor expansions can be calculated using the derivation ∂ just like with classical analytic functions.

Proposition 7.13. *If $f \in \mathbb{R}\langle\!\langle\omega\rangle\!\rangle$ is surreal analytic at $x \in \mathbf{No}^{>\mathbb{R}}$, then for every sufficiently small $\varepsilon \in \mathbf{No}$ we have*

$$f \circ (x + \varepsilon) = \sum_{n\in\mathbb{N}} \frac{1}{n!}(\partial^n f \circ x) \cdot \varepsilon^n$$

where $\partial^0 f = f$ and $\partial^{n+1} f = \partial(\partial^n f)$.

Proof. Let $f \in \mathbb{R}\langle\!\langle\omega\rangle\!\rangle$ be analytic at $x \in \mathbf{No}^{>\mathbb{R}}$. Let $\hat{f}$ be associated function $x + \varepsilon \mapsto f \circ (x + \varepsilon)$, which by assumption is also surreal analytic (in the sense of Definition 3.7). By Proposition 3.11, we know that

$$f \circ (x + \varepsilon) = \hat{f}(x + \varepsilon) = \sum_{i=0}^{\infty} \frac{\hat{f}^{(i)}(x)}{i!}\varepsilon^i.$$

By Corollary 7.6, it follows by induction on i that in fact $\hat{f}^{(i)}(x) = \partial^i f \circ x$, proving the desired conclusion. □

We can then conclude that *every* omega-series is surreal analytic.

Theorem 7.14. *Every $f \in \mathbb{R}\langle\!\langle\omega\rangle\!\rangle$ is surreal analytic, and for every $x \in \mathbf{No}^{>\mathbb{R}}$ and every sufficiently small $\varepsilon \in \mathbf{No}$ we have*

$$f \circ (x + \varepsilon) = \sum_{i\in\mathbb{N}} \frac{1}{i!}(\partial^i f \circ x) \cdot \varepsilon^i.$$

Proof. Let $f \in \mathbb{R}\langle\!\langle\omega\rangle\!\rangle$. We reason by induction on $\mathrm{ER}_\Delta(f)$, where $\Delta = \{\log_i(\omega) : i \in \mathbb{N}\}$.

The case $f = 0$ is trivial, while the case $f = \log_n(\omega)$ follows from Corollary 7.12 and Proposition 7.13. This shows the conclusion for $\mathrm{ER}_\Delta(f) = 0$, namely for $f \in \Delta_1 = \Delta \cup \{0\}$.

Now suppose $\mathrm{ER}_\Delta(f) > 0$. Write $f = \sum_{j<\alpha} r_j e^{\gamma_j}$, and recall that by definition $\mathrm{ER}_\Delta(\gamma_j) < \mathrm{ER}_\Delta(f)$ for all $j < \alpha$. Therefore, by inductive hypothesis, we can assume that γ_j is surreal analytic for every $j < \alpha$. Since $\exp(\omega)$ is surreal analytic by Proposition 7.9, it follows that $\exp(\omega) \circ \gamma_j = \exp(\gamma_j)$ is surreal analytic by Lemma 7.11, hence so is $f_j := r_j e^{\gamma_j}$. This means that for each x, there is some $\varepsilon_j > 0$ such that for all ε smaller than ε_j in absolute value, we have $f_j \circ (x + \varepsilon) = \sum_{i\in\mathbb{N}} \frac{1}{i!}(\partial^i f_j \circ x) \cdot \varepsilon^i$.

Since ∂ is strongly additive, and $(f_j : j < \alpha)$ is summable, the family $(\partial f_j : j < \alpha)$ is also summable and $\sum_j \partial f_j = \partial\left(\sum_j f_j\right)$. In turn, $(\partial f_j \circ x : j < \alpha)$ must be summable, and $\sum_j (\partial f_j \circ x) = (\sum_j \partial f_j) \circ x = \partial\left(\sum_j f_j\right) \circ x = \partial f \circ x$. Similarly, by induction on $i \in \mathbb{N}$, $(\partial^i f_j \circ x : j < \alpha)$ is summable and $\sum_j (\partial^i f_j \circ x) = \partial^i f \circ x$. By Lemma 3.15, for every sufficiently small ε, $\big((\partial^i f_j \circ x) \cdot \varepsilon^i : (i, j) \in \mathbb{N} \times \alpha\big)$ is summable and therefore, by Corollary 3.2, we have

$$\sum_j \sum_i \frac{1}{i!}(\partial^i f_j \circ x) \cdot \varepsilon^i = \sum_i \sum_j \frac{1}{i!}(\partial^i f_j \circ x) \cdot \varepsilon^i = \sum_i \frac{1}{i!}\left(\partial^i f \circ x\right) \cdot \varepsilon^i.$$

Recalling that $f_j \circ (x+\varepsilon) = \sum_{i\in\mathbb{N}} \frac{1}{i!}(\partial^i f_j \circ x)\cdot \varepsilon^i$, it follows that

$$f\circ(x+\varepsilon) = \sum_j (f_j\circ(x+\varepsilon)) = \sum_j \sum_i \frac{1}{i!}(\partial^i f_j\circ x)\cdot\varepsilon^i = \sum_n \frac{1}{i!}(\partial^i f\circ x)\cdot\varepsilon^n$$

thus proving that f is surreal analytic. □

Remark 7.15. When $f \in \mathbb{R}((\omega))^{LE}$ and $x \in \mathbb{R}((\omega))^{LE}$, one can verify that there exists an $n\in\mathbb{N}$ such that the equation of Theorem 7.14 holds for any $\varepsilon \preceq e^{-\exp_n(\omega)}$. Indeed, note that the subfields $K_{m,\log_i(\omega)}$ (see Definition 4.13) are closed under the derivation ∂, and that there is some $k\in\mathbb{N}$ such that $g\circ x \in K_{m+k,\log_{i+k}(\omega)}$ for any $g\in K_{m,\log_i(\omega)}$. Then all the coefficients $\partial^i f\circ x/i!$ live in some fixed $K_{n,\log_n(\omega)}$, and it suffices to apply Corollary 3.9 to get the desired conclusion. In particular, one can give a meaningful definition of analyticity for LE-series by staying inside the field of LE-series, without resorting to **No**.

In full generality, Corollary 3.9 guarantees that the equation of Theorem 7.14 holds for any ε that is infinitesimal with respect to any non-zero omega-series $g\in\mathbb{R}\langle\!\langle\omega\rangle\!\rangle$. In some cases, this is the best we can do. Take for instance $f = \sum_{n=0}^{\infty} e^{-\exp_n(\omega)}$. Then one can easily verify that $(\partial^i f\circ\omega)_{i\in\mathbb{N}} = (\partial^i f)_{i\in\mathbb{N}}$ is not summable, and in fact that $(\partial^i f\cdot\varepsilon)_{i\in\mathbb{N}}$ is not summable for any ε such that $\varepsilon \succeq e^{-\exp_n(\omega)}$ for some $n\in\mathbb{N}$, and in particular for any $\varepsilon\in\mathbb{R}\langle\!\langle\omega\rangle\!\rangle^*$. Therefore, the expansion of $f\circ(\omega+\varepsilon)$ given by Theorem 7.14 only exists for the numbers ε with absolute value smaller than any omega-series.

Corollary 7.16. *Given $f\in\mathbb{R}\langle\!\langle\omega\rangle\!\rangle$ and $x\in\mathbf{No}^{>\mathbb{R}}$, we have*

$$f\circ(x+\varepsilon) = f\circ x + (\partial f\circ x)\cdot\varepsilon + \mathcal{O}((\partial^2 f\circ x)\cdot\varepsilon^2)$$

whenever $\varepsilon\in\mathbf{No}$ satisfies $(\partial^{i+2}f\circ x)\cdot\varepsilon^i \preceq \partial^2 f\circ x$ for all $i\in\mathbb{N}$.

8. A negative result

The interaction between the unique composition $\circ$ on $\mathbb{R}\langle\!\langle\omega\rangle\!\rangle$ and the unique transserial derivation on $\mathbb{R}\langle\!\langle\omega\rangle\!\rangle$ suggests looking for compositions that are compatible with a transserial derivation.

Definition 8.1. Given a transserial subfield $T\subseteq\mathbf{No}$, a transserial derivation $\partial: T\to T$, and a composition $\circ: T\times\mathbf{No}^{>\mathbb{R}}\to\mathbf{No}$, we say that ∂ and $\circ$ are **compatible** if the following holds:

(1) if $\partial f = 0$, then $f\circ x = f$ for every x;
(2) $\partial f > 0 \implies f\circ x < f\circ y$ whenever $x<y$;
(3) $\partial(f\circ g) = (\partial f\circ g)\cdot\partial g$.

Theorem 8.2. *The unique surreal derivation ∂ on $\mathbb{R}\langle\!\langle\omega\rangle\!\rangle$ is compatible with the unique composition on $\mathbb{R}\langle\!\langle\omega\rangle\!\rangle$.*

Proof. Condition (1) follow at once from $\ker(\partial)=\mathbb{R}$.

For condition (2), let $f\in\mathbb{R}\langle\!\langle\omega\rangle\!\rangle$. We reason by induction on $\mathrm{ER}_\Delta(f)$, where $\Delta = \{\log_i(\omega) : i\in\mathbb{N}\}$. If $\mathrm{ER}_\Delta(f)=0$, then the conclusion is easy: for instance if $f=\log_i(\omega)$, then $f\circ g = \log_i(g)$ and the chain rule in (3) can be verified as in the classical case, recalling also Corollary 7.6. Now suppose that $\mathrm{ER}_\Delta(f)>0$. Write $f=\sum_{i<\alpha} r_i e^{\gamma_i}$, where $\mathrm{ER}_\Delta(\gamma_i)<\mathrm{ER}_\Delta(f)$ for all $i<\alpha$. Suppose that $f\circ x\geq f\circ y$ for some $x<y$. Since the maps $g\mapsto g\circ x$, $g\mapsto g\circ y$ are substitutions, they

preserve the relation $\preceq$ (Proposition 5.3), so we must have $(r_0 e^{\gamma_0}) \circ x \geq (r_0 e^{\gamma_0}) \circ y$, so $r_0 e^{\gamma_0 \circ x} \geq r_0 e^{\gamma_0 \circ y}$.

Without loss of generality, we may assume that $\gamma_0 \neq 0$ (by replacing f with $f - r_0$) and that $r_0 > 0$ (by replacing f with $-f$). Under these assumptions, we must have $\gamma_0 \circ x \geq \gamma_0 \circ y$, so by inductive hypothesis $\partial\gamma_0 \leq 0$. Note moreover that since $\gamma_0 \in \mathbb{J}^{\neq 0}$, we must have $\partial\gamma_0 \neq 0$. In turn, since $\partial f \sim r_0 e^{\gamma_0} \partial\gamma_0$, it follows that $\partial f \geq 0$, as desired.

Point (3) is Corollary 7.7. □

Question 8.3. We do not know whether there is a composition and a compatible transserial derivation (possibly with $\ker(\partial)$ bigger than $\mathbb{R}$) on the whole of **No**.

Note that the present notion of compatibility is rather weak, and for instance it does not require the conclusion of Theorem 7.14 to hold, or even just Theorem 7.5. However, even such a weak notion does not allow the "simplest" derivation $\partial : \mathbf{No} \to \mathbf{No}$ of [BM] to be compatible with a composition.

Theorem 8.4. *The "simplest" surreal derivation $\partial : \mathbf{No} \to \mathbf{No}$ in* [BM] *cannot be compatible with a composition $\circ : \mathbf{No} \times \mathbf{No}^{>\mathbb{R}} \to \mathbf{No}$.*

Proof. Let $y \in \mathbf{No}$, and observe that the rules of transserial derivations yield $\partial(\log_n(y)) = \frac{1}{\prod_{i<n} \log_i(y)}$. Taking $y = \omega$ we obtain $\partial(\lambda_{-n}) = \frac{1}{\prod_{i<n} \lambda_{-i}}$, where $\lambda_{-n} = \log_n(\omega)$. Now let $\partial : \mathbf{No} \to \mathbf{No}$ be the "simplest derivation" in [BM]. In that paper we showed that ∂ is surjective, so in particular there is an anti-derivative of $\frac{1}{\prod_{n\in\mathbb{N}} \lambda_{-n}}$. In fact we proved that there is a log-atomic number $\lambda_{-\omega} \in \mathbb{L}$ such that $\partial(\lambda_{-\omega}) = \frac{1}{\prod_{n\in\mathbb{N}} \lambda_{-n}}$. With a suggestive notation $\lambda_{-\omega}$ is denoted $\log_\omega(\omega)$ in [ADH], suggesting that it should be considered as an infinite compositional iterate of $\log(\omega)$. In [BM] we showed that, if λ is a log-atomic number bigger than $\exp_n(\omega)$ for every $n \in \mathbb{N}$, then

$$\partial(\lambda) = \prod_{n\in\mathbb{N}} \log_n(\lambda).$$

Note that there is a proper class of log-atomic numbers λ satisfying $\lambda > \exp_n(\omega)$ for all $n \in \mathbb{N}$, so the above differential equation has a proper class of solutions. Now fix such a solution λ and suppose for a contradiction that ∂ is compatible with a composition on the whole of **No**. By the rules for ∂ and $\circ$ we obtain

$$\begin{aligned}\partial(\lambda_{-\omega} \circ \lambda) &= (\partial(\lambda_{-\omega}) \circ \lambda) \cdot \partial(\lambda) \\ &= \left(\frac{1}{\prod_n \log_n(\omega)} \circ \lambda\right) \cdot \partial(\lambda) \\ &= \left(\frac{1}{\prod_n \log_n(\lambda)}\right) \cdot \partial(\lambda) = 1.\end{aligned}$$

Since $\partial(\lambda_{-\omega}) > 0$, by the compatibility conditions the function $x \mapsto \lambda_{-\omega} \circ x$ is strictly increasing, so there is a proper class of elements of the form $\lambda_{-\omega} \circ x$ with derivative 1. This however contradicts the fact that $\ker(\partial) = \mathbb{R}$ is a set. □

Remark 8.5. The above result can be interpreted in different ways. The first is that there could be no reasonable composition on the whole of **No**. The second is that, despite the positive results in [ADH, BM], the simplest derivation in [BM] may have some shortcomings. It is conceivable that, in order to be able to give positive solution to Question 8.3, we should allow a proper class as the kernel of ∂.

9. Proof of the summability lemma (Lemma 5.21)

We will now give a proof of Lemma 5.21. We work under the notations of Section 5. Suppose that $c_0 : \Delta \to \mathbf{No}$ is a given pre-substitution. Then we wish to prove the following:

Lemma 5.21. *Assume $I(\alpha)$. Then $(\overline{c}(T) : T \in \mathrm{A}(x))$ is summable for all $x \in \Delta_{\alpha+1} \setminus \Delta_\alpha$. In particular, $I(\alpha)$ implies $I(\alpha+1)$.*

For the rest of this section, *let $c_0 : \Delta \to \mathbf{No}$ be a pre-substitution, and assume that the inductive hypothesis $I(\alpha)$ holds.* Then $c : \Delta_\alpha \to \mathbf{No}$ is well defined, and the objects $\mathrm{A}(x)$, $\overline{c}(T)$ and $\mathrm{A}^\circ(x)$ are clearly well defined for all $x \in \Delta_{\alpha+1}$ and all $T \in \mathrm{A}(x)$. Moreover, recall that by Proposition 5.17, $c(t)$ is also well defined for all terms $t \in \Delta_{\alpha+1} \cap \mathbb{R}^*\mathfrak{M}$.

9.1. **A property of pre-substitutions.** We start by observing a rather technical, but crucial fact on pre-substitutions.

Lemma 9.1. *Let $x \in \mathbf{No}$ and $\mathfrak{m}$ be the leading monomial of x. Then*

$$\mathrm{Supp}(x) \subseteq \bigcup_{n=0}^{\infty} \mathfrak{m}^{n+1} \cdot \mathrm{Supp}(x^{-1})^n.$$

Proof. Let $t = r\mathfrak{m}$ be the leading term of x. Write $x^{-1} = t^{-1}(1+\varepsilon)$, where $\varepsilon \prec 1$. Then

$$x = \frac{t}{(1+\varepsilon)} = t \cdot \sum_{n=0}^{\infty} (-1)^n \varepsilon^n,$$

hence every element in the support of x has the form $\mathfrak{m} \cdot \mathfrak{n}_1 \cdot \ldots \cdot \mathfrak{n}_n$ with $n \geq 0$ and $\mathfrak{n}_i \in \mathrm{Supp}(\varepsilon)$. On the other hand, since $\varepsilon = tx^{-1} - 1 = r\mathfrak{m}x^{-1} - 1$ and $\varepsilon \prec 1$, we have $\mathrm{Supp}(\varepsilon) \subseteq \mathfrak{m} \cdot \mathrm{Supp}(x^{-1})$, and the conclusion follows. □

Lemma 9.2. *Let $c_0 : \Delta \to \mathbf{No}$ be a pre-substitution. Let $(\lambda_i)_{i\in\mathbb{N}}$, $(\mathfrak{m}_i)_{i\in\mathbb{N}}$ be two sequences such that $\lambda_i \in \Delta$ and $\mathfrak{m}_i \in \mathrm{Supp}(c_0(\lambda_i))$ for all $i \in \mathbb{N}$. Then there is an increasing sequence of indexes $(i_j)_{j\in\mathbb{N}}$ such that one of the following holds:*

(1) the subsequence $(\lambda_{i_j})_{j\in\mathbb{N}}$ is decreasing and for all $j \in \mathbb{N}$

$$\frac{\mathfrak{m}_{i_{j+1}}}{\mathfrak{m}_{i_j}} \prec \frac{c_0(\lambda_{i_{j+1}})}{c_0(\lambda_{i_j})} \prec 1;$$

(2) the subsequence $(\lambda_{i_j})_{j\in\mathbb{N}}$ is increasing and for all $j \in \mathbb{N}$

$$\frac{\mathfrak{m}_{i_{j+1}}}{\mathfrak{m}_{i_j}} \prec c_0(\lambda_{i_{j+1}})^2;$$

(3) the subsequence $(\lambda_{i_j})_{j\in\mathbb{N}}$ is constant and for all $j \in \mathbb{N}$

$$\frac{\mathfrak{m}_{i_{j+1}}}{\mathfrak{m}_{i_j}} \preceq 1.$$

Note that in all three cases we have $\frac{\mathfrak{m}_{i_{j+1}}}{\mathfrak{m}_{i_j}} \prec c_0(\lambda_{i_{j+1}})^2$.

Proof. Let $\lambda_i =: e^{\mu_i}$. Note that $\mu_i \in \Delta$. We have

$$c_0(\lambda_i) = e^{c_0(\mu_i)} = e^{c_0(\mu_i)^{\uparrow=}} e^{c_0(\mu_i)^{\downarrow}}.$$

Thus $\mathfrak{n}_i := \mathfrak{m}_i e^{-c_0(\mu_i)^\uparrow} \in \mathrm{Supp}\left(\exp(c_0(\mu_i)^{\downarrow=})\right) = \mathrm{Supp}(\exp(c_0(\mu_i)^\downarrow)$, and therefore there is some $n_i \in \mathbb{N}$ such that $\mathfrak{n}_i \in \mathrm{Supp}((c_0(\mu_i)^\downarrow)^{n_i})$. After extracting a subsequence we may assume that $(\lambda_i)_{i\in\mathbb{N}}$ is monotone, so either increasing, decreasing, or constant.

(1) Suppose that $(\lambda_i)_{i\in\mathbb{N}}$ is decreasing. Then $(\mu_i)_{i\in\mathbb{N}}$ is also decreasing, hence the family $(c_0(\mu_i) : i \in \mathbb{N})$ is summable. In particular, $(c_0(\mu_i)^\downarrow : i \in \mathbb{N})$ is summable, and by Corollary 3.12, $(\exp(c_0(\mu_i)^\downarrow) : i \in \mathbb{N})$ is summable. We may therefore extract a subsequence and assume that $(\mathfrak{n}_i)_{i\in\mathbb{N}}$ is decreasing, so that

$$\mathfrak{m}_{i+1} e^{-c_0(\mu_{i+1})^\uparrow} \prec \mathfrak{m}_i e^{-c_0(\mu_i)^\uparrow}.$$

Since $c_0(\lambda_i) = e^{c_0(\mu_i)}$, it follows that

$$\frac{\mathfrak{m}_{i+1}}{\mathfrak{m}_i} \prec \frac{e^{c_0(\mu_{i+1})^\uparrow}}{e^{c_0(\mu_i)^\uparrow}} \asymp \frac{c_0(\lambda_{i+1})}{c_0(\lambda_i)} \prec 1.$$

(2) Consider now the case when $(\lambda_i)_{i\in\mathbb{N}}$ is increasing. Let $\mathfrak{o}_i := \mathrm{LM}(c_0(\mu_i))$. By Lemma 9.1, applied with $x = c_0(\mu_i)$, we deduce that

$$\mathrm{Supp}(c_0(\mu_i)) \subseteq \bigcup_{m=0}^{\infty} \mathfrak{o}_i^{m+1} \cdot \mathrm{Supp}(c_0(\mu_i)^{-1})^m.$$

Since $\mathfrak{n}_i = \frac{\mathfrak{m}_i}{e^{c_0(\mu_i)^\uparrow}} \in \mathrm{Supp}((c_0(\mu_i)^\downarrow)^{n_i})$, it follows that there is an $m_i \in \mathbb{N}$ such that

$$\frac{\mathfrak{m}_i}{e^{c_0(\mu_i)^\uparrow}} \in \mathfrak{o}_i^{n_i(m_i+1)} \cdot \mathrm{Supp}(c_0(\mu_i)^{-1})^{n_i m_i}$$

and therefore $\mathfrak{m}_i \cdot e^{-c_0(\mu_i)^\uparrow} \cdot \mathfrak{o}_i^{-n_i(m_i+1)} \in \mathrm{Supp}(c_0(\mu_i)^{-1})^{n_i m_i}$.

Now observe that $c_0(\mu_i)^{-1} \prec 1$ and that the family $(c_0(\mu_i)^{-1} : i \in \mathbb{N})$ is summable because $(\mu_i^{-1})_{i\in\mathbb{N}}$ is decreasing. By Corollary 3.12, applied with $\varepsilon_i = c_0(\mu_i)^{-1}$, the family $\left(\mathfrak{m}_i \cdot e^{-c_0(\mu_i)^\uparrow} \cdot \mathfrak{o}_i^{-n_i(m_i+1)} : i \in \mathbb{N}\right)$ is summable. We may therefore extract a subsequence and assume that

$$\frac{\mathfrak{m}_i}{e^{c_0(\mu_i)^\uparrow} \cdot \mathfrak{o}_i^{n_i(m_i+1)}} \succ \frac{\mathfrak{m}_{i+1}}{e^{c_0(\mu_{i+1})^\uparrow} \cdot \mathfrak{o}_{i+1}^{n_{i+1}(m_{i+1}+1)}}.$$

Since $c_0(\mu_i)$ is positive infinite, $e^{c_0(\mu_i)^\uparrow} \succ c_0(\mu_i)^n \asymp \mathfrak{o}_i^n$ for any $n \in \mathbb{N}$, so

$$\frac{\mathfrak{m}_{i+1}}{e^{2c_0(\mu_{i+1})^\uparrow}} \prec \frac{\mathfrak{m}_{i+1}}{e^{c_0(\mu_{i+1})^\uparrow} \cdot \mathfrak{o}_{i+1}^{n_{i+1}(m_{i+1}+1)}} \prec \frac{\mathfrak{m}_i}{e^{c_0(\mu_i)^\uparrow}}.$$

Therefore,

$$\frac{\mathfrak{m}_{i+1}}{\mathfrak{m}_i} \prec \frac{e^{2c_0(\mu_{i+1})^\uparrow}}{e^{c_0(\mu_i)^\uparrow}} \asymp \frac{c_0(\lambda_{i+1})^2}{c_0(\lambda_i)} \preceq c_0(\lambda_{i+1})^2.$$

(3) Finally, suppose that there is a $\lambda \in \Delta$ such that $\lambda_i = \lambda$ for all $i \in \mathbb{N}$. In this case all the monomials $\mathfrak{m}_i$ are in the support of $c_0(\lambda) \in \mathbf{No}$, hence obviously we may extract a subsequence and assume that $\mathfrak{m}_{i+1} \preceq \mathfrak{m}_i$ for all $i \in \mathbb{N}$. □

9.2. **Further properties of the extensions.** Recall that $I(\alpha)$ implies that $c : \Delta_\alpha \to \mathbf{No}$ is a substitution when $\alpha \geq 3$ (Corollary 5.20). In particular, c preserves the ordering and the dominance relation $\prec$ by Proposition 5.3. We observe that $I(\alpha)$ implies similar monotonicity properties for $\alpha < 3$, and also for terms in $\Delta_{\alpha+1}$.

Proposition 9.3. *For all $x, y \in \Delta_\alpha$, and for all $x, y \in \Delta_{\alpha+1} \cap \mathbb{R}^*\mathfrak{M}$, we have $x < y \to c(x) < c(y)$ and $x \prec y \to c(x) \prec c(y)$.*

Proof. If α is 0 or 1, then for all $x, y \in \Delta_\alpha$ we have $x < y \to c(x) < c(y)$ and $x \prec y \to c(x) \prec c(y)$ by definition of pre-substitution. The same conclusion holds for $\alpha \geq 3$ by Corollary 5.19 and Proposition 5.3. For $\alpha = 2$, note that by Proposition 5.18, if we expand some $x \in \Delta_2 \setminus \mathbb{R}$ as $x = r_0 e^{\lambda_0} + \sum_{1 \leq i < \beta} r_i e^{\lambda_i} + s$ (where $r_i, s \in \mathbb{R}$, $\lambda_i \in \Delta$, and $\lambda_i > \lambda_j$ for all $i \leq j < \beta$), we have

$$c(x) = r_0 e^{c_0(\lambda_0)} + \sum_{1 \leq i < \beta} r_i e^{c_0(\lambda_i)} + s,$$

while $c(r) = r$ for all $r \in \mathbb{R}$ by Proposition 5.16. By definition of pre-substitution, it follows at once that $c(x) \sim r_0 e^{c_0(\lambda_0)}$, and in turn, that $c(x) > 0$ if and only if $x > 0$ (and obviously $c(r) > 0$ if and only if $r > 0$). Since Δ_2 is an additive group, we have $x < y \to c(x) < c(y)$ for all $x, y \in \Delta_2$. By the same argument, it also follows that $x \prec y \to c(x) \prec c(y)$ for all $x, y \in \Delta_2$.

Now take some $x, y \in \Delta_{\alpha+1} \cap \mathbb{R}^* \mathfrak{M}$. Write $x = re^\gamma, y = se^\delta$, with $r, s \in \mathbb{R}^*$ and $\gamma, \delta \in \mathbb{J}$. By Proposition 5.17, $c(re^\gamma)$ and $c(se^\delta)$ are well defined and equal to respectively $re^{c(\gamma)}$, $se^{c(\delta)}$. We observe that if $\gamma < \delta$, then $c(\gamma) < c(\delta)$, and if moreover $0 < \gamma$, then $0 < c(\gamma)$ and $\gamma \prec \delta$, so $c(\gamma) \prec c(\delta)$. This easily implies that $x < y \to c(x) < c(y)$ and $x \prec y \to c(x) \prec c(y)$. □

We also need the following properties of admissible trees.

Lemma 9.4. *Let $x \in \Delta_{\alpha+1}$ and $T = \langle re^\gamma, n, \tau \rangle \in \mathrm{A}(x)$. We have:*

(1) $re^{c(\gamma)^{\uparrow=}} \asymp re^{c(\gamma)} = c(re^\gamma) = c(\mathrm{R}(T))$;
(2) if $re^\gamma = \mathrm{R}(T) \notin \Delta$, then $\overline{c}(T) \asymp c(\mathrm{R}(T)) \cdot \prod_{i<n} \overline{c}(\tau(i))$;
(3) if $U = \tau(i)$ is a child of T, then $\overline{c}(U)$ is infinitesimal;
(4) if U is a proper descendant of T, then $\overline{c}(U)$ is infinitesimal;
(5) $\overline{c}(T) \preceq c(\mathrm{R}(T))$;
(6) if $\mathrm{size}(T) > 1$, then all the leaves of T have root in Δ.

Proof. (1) follows from Proposition 5.17.

(2), (3), (4) follow at once from the definitions and (1).

(5) If $\lambda = \mathrm{R}(T) \in \Delta$, then $\overline{c}(T) \in \mathrm{Term}(c_0(\lambda))$, so $\overline{c}(T) \preceq c_0(\lambda) = c(\mathrm{R}(T))$ as desired. If $\mathrm{R}(T) \notin \Delta$, then $\overline{c}(T) \asymp c(\mathrm{R}(T)) \cdot \prod_{i<n} \overline{c}(\tau(i))$ by (2), and since $\overline{c}(\tau(i)) \prec 1$ for each $i < n$ by (3), we reach the same conclusion.

(6) Assume $\mathrm{size}(T) > 1$, and let L be a leaf of T. Then L is a leaf of some child of T. Reasoning by induction, we may directly assume, without loss of generality, that L is a child of T. Write $L = \langle se^\delta, 0, \sigma \rangle$. Note that se^δ is a term of $\gamma = \log^\uparrow(\mathrm{R}(T)) \in \mathbb{J}$, so $\mathrm{R}(L) = se^\delta \succ 1$. By Proposition 9.3, it follows that $c(\mathrm{R}(L)) \succ 1$. Now suppose by contradiction that $se^\delta \notin \Delta$. Then (2) implies that $\overline{c}(L) \asymp c(\mathrm{R}(T)) \succ 1$, but by (4) we must have $\overline{c}(L) \prec 1$. Therefore, $se^\delta \in \Delta$, as desired. □

9.3. **Bad sequences.** In order to prove that the family $(\overline{c}(T) : T \in \mathrm{A}(x))$ is summable for any $x \in \Delta_{\alpha+1}$, by Remark 2.15, one could try to verify that there is no injective sequence $(T_i)_{i \in \mathbb{N}}$ of trees in $\mathrm{A}(x)$ such that $\overline{c}(T_i) \preceq \overline{c}(T_{i+1})$ for all $i \in \mathbb{N}$. However, we will actually prove the stronger statement that there are *no bad sequences*, which are defined as follows:

Definition 9.5. Let $x \in \Delta_{\alpha+1}$ and let $(T_i)_{i \in \mathbb{N}}$ be a sequence of trees in $\mathrm{A}(x)$. We say that the sequence is **bad** if it is injective, $\mathrm{R}(T_i) \succeq \mathrm{R}(T_{i+1})$ for each $i \in \mathbb{N}$, and

$$\left(\frac{\overline{c}(T_i)}{\overline{c}(T_{i+1})} \right)^n \preceq \frac{c(\mathrm{R}(T_i))}{c(\mathrm{R}(T_{i+1}))}$$

for all $i, n \in \mathbb{N}$.

For instance, Lemma 9.2(1) and (3) immediately imply that there are no bad sequences in $\mathrm{A}(x)$ for any $x \in \Delta_2$. The non-existence of bad sequences in a given $\mathrm{A}(x)$ quickly implies the desired summability.

Proposition 9.6. *Let $x \in \Delta_{\alpha+1}$. If there are no bad sequences in $\mathrm{A}(x)$, then $(\overline{c}(T) : T \in \mathrm{A}(x))$ is summable.*

Proof. Suppose that $(\overline{c}(T) : T \in \mathrm{A}(x))$ is not summable. Then there is an injective sequence of trees $(T_i)_{i\in\mathbb{N}}$ in $\mathrm{A}(x)$ such that

$$\overline{c}(T_i) \preceq \overline{c}(T_{i+1})$$

for all $i \in \mathbb{N}$. After extracting a subsequence, we may assume that $\mathrm{R}(T_i) \succeq \mathrm{R}(T_{i+1})$ for every $i \in \mathbb{N}$, as all these roots are terms of x. Therefore, $c(\mathrm{R}(T_i)) \succeq c(\mathrm{R}(T_{i+1}))$ for all $i \in \mathbb{N}$ by Proposition 9.3. It follows that for all $i, n \in \mathbb{N}$ we have

$$\left(\frac{\overline{c}(T_i)}{\overline{c}(T_{i+1})}\right)^n \preceq 1 \preceq \frac{c(\mathrm{R}(T_i))}{c(\mathrm{R}(T_{i+1}))},$$

so the sequence $(T_i)_{i\in\mathbb{N}}$ is bad. □

Remark 9.7. If $(T_i)_{i\in\mathbb{N}}$ is a bad sequence, then all its subsequences are bad. This follows from the fact that for all $i, k, n \in \mathbb{N}$ we have

$$\left(\frac{\overline{c}(T_i)}{\overline{c}(T_{i+k+1})}\right)^n = \left(\prod_{j=0}^{k} \frac{\overline{c}(T_{i+j})}{\overline{c}(T_{i+j+1})}\right)^n \preceq \prod_{j=0}^{k} \frac{c(\mathrm{R}(T_{i+j}))}{c(\mathrm{R}(T_{i+j+1}))} = \frac{c(\mathrm{R}(T_i))}{c(\mathrm{R}(T_{i+k+1}))}.$$

We start with a few special cases in which it is easy to prove that sequences of trees are not bad.

Proposition 9.8. *Let $x \in \Delta_{\alpha+1}$. Let $(T_i)_{i\in\mathbb{N}}$ be a sequence of distinct trees in $\mathrm{A}(x)$. If $\mathrm{R}(T_i) \in \Delta$ for all $i \in \mathbb{N}$, then $(T_i)_{i\in\mathbb{N}}$ is not bad.*

Proof. Write $T_i = \langle \lambda_i, 0, t_i \rangle$, where $t_i = \overline{c}(T_i)$ is a term of $c_0(\lambda_i)$. Since $\lambda_i \in \mathrm{Term}(x)$ for each $i \in \mathbb{N}$, after extracting a subsequence, we may assume that $(\lambda_i : i \in \mathbb{N})$ is either constant or decreasing. In the former case, all the contributions $\overline{c}(T_i)$ are distinct elements of $\mathrm{Term}(c_0(\lambda))$ for some fixed $\lambda \in \Delta$, so after extracting a subsequence we may assume $\overline{c}(T_i) \succ \overline{c}(T_{i+1})$ for all $i \in \mathbb{N}$, so the sequence is not bad. In the latter case, by Lemma 9.2, we may extract a further subsequence and assume that

$$\frac{\overline{c}(T_i)}{\overline{c}(T_{i+1})} = \frac{t_i}{t_{i+1}} \succ \frac{c_0(\lambda_i)}{c_0(\lambda_{i+1})} = \frac{c(\mathrm{R}(T_i))}{c(\mathrm{R}(T_{i+1}))}.$$

Therefore, $(T_i)_{i\in\mathbb{N}}$ is not bad. □

Proposition 9.9. *Let t be a term in $\Delta_{\alpha+1}$. Then there are no bad sequences in $\mathrm{A}(t)$.*

Proof. Let $(T_i)_{i\in\mathbb{N}}$ be a sequence of distinct trees in $\mathrm{A}(t)$. We want to prove that $(T_i)_{i\in\mathbb{N}}$ is not bad. Since t is a term, by Proposition 5.17 $(\overline{c}(T) : T \in \mathrm{A}(t))$ is summable. Thus, extracting a subsequence, we can assume that $\overline{c}(T_i) \succ \overline{c}(T_{i+1})$ for every $i \in \mathbb{N}$. Observing that $\mathrm{R}(T_i) = t$ for every $i \in \mathbb{N}$, it follows that $\frac{\overline{c}(T_i)}{\overline{c}(T_{i+1})} \succ 1 = \frac{c(\mathrm{R}(T_i))}{c(\mathrm{R}(T_{i+1}))}$, and therefore $(T_i)_{i\in\mathbb{N}}$ is not bad. □

9.4. **Two types of sequences of trees.** We now distinguish two special types of sequences of trees, and verify that every injective sequences of trees in some given $\mathrm{A}(x)$ has at least one subsequence of one of the two types.

Definition 9.10. Let $x \in \Delta_{\alpha+1}$ and let $T_i = \langle r_i e^{\gamma_i}, n_i, \tau_i \rangle \in \mathrm{A}(x)$ be distinct trees for $i \in \mathbb{N}$ such that $(\gamma_i)_{i\in\mathbb{N}}$ is weakly decreasing.

We say that the sequence $(T_i)_{i\in\mathbb{N}}$ has type:

(A) if $\mathrm{R}(\tau_i(j)) \succ \gamma_0 - \gamma_i$ for all $i \in \mathbb{N}$, $j < n_i$;

(B) if $n_0 \geq 1$ and for all $i \in \mathbb{N}^{>0}$ there is $k < n_i$ such that $\mathrm{R}(\tau_i(k)) \preceq \gamma_{i-1} - \gamma_i$.

Note that a sequence $(T_i)_{i\in\mathbb{N}}$ may be of neither type. A sequence with $n_i = 0$ for all $i \in \mathbb{N}$, or with $(\gamma_i)_{i\in\mathbb{N}}$ constant, is vacuously of type (A). Moreover, for a sequence of type (B), $(\gamma_i)_{i\in\mathbb{N}}$ is necessarily strictly decreasing and $n_i \geq 1$ for all $i \in \mathbb{N}$.

Lemma 9.11. *If $(T_i)_{i\in\mathbb{N}}$ is a sequence of type (A) or (B), then all its subsequences have type (A) or (B) respectively.*

Proof. Suppose $(T_i)_{i\in\mathbb{N}}$ is of type (A) and let $(T_{i_j})_{j\in\mathbb{N}}$ be a subsequence. Since $(\gamma_i)_{i\in\mathbb{N}}$ is weakly decreasing, for all $k < n_{i_j}$ we have

$$\mathrm{R}(\tau_{i_j}(k)) \succ \gamma_0 - \gamma_{i_j} \succeq \gamma_{i_0} - \gamma_{i_j},$$

so the subsequence is of type (A).

Now let $(T_i)_{i\in\mathbb{N}}$ be a sequence of type (B). Write $T_i = \langle r_i e^{\gamma_i}, n_i, \tau_i \rangle$. Using again the fact that $(\gamma_i)_{i\in\mathbb{N}}$ is weakly decreasing, if k is such that $\mathrm{R}(\tau_i(k)) \preceq \gamma_{i-1} - \gamma_i$, then $\mathrm{R}(\tau_i(k)) \preceq \gamma_j - \gamma_i$ for all $j < i$, so any any subsequence of $(T_i)_{i\in\mathbb{N}}$ is of type (B). □

Proposition 9.12. *Let $x \in \Delta_{\alpha+1}$ and let $T_i = \langle r_i e^{\gamma_i}, n_i, \tau_i \rangle \in \mathrm{A}(x)$ be distinct trees for $i \in \mathbb{N}$. Then $(T_i)_{i\in\mathbb{N}}$ has a subsequence of type (A) or (B).*

Proof. After extracting a subsequence, we may assume that $(\gamma_i)_{i\in\mathbb{N}}$ is weakly decreasing. If $n_i = 0$ for every $i \in \mathbb{N}$, then $(T_i)_{i\in\mathbb{N}}$ is of type (A) and we are done. We can therefore suppose without loss of generality that $n_0 \geq 1$.

We proceed by trying to construct a subsequence $(T_{i_j})_{j\in\mathbb{N}}$ of type (B), and check that when the construction fail we find a subsequence of type (A). We define T_{i_j} by induction on $j \in \mathbb{N}$. For $j = 0$, we let $T_{i_j} = T_{i_0} := T_0$.

Assuming that T_{i_j} has been defined, we have two cases. If $\mathrm{R}(\tau_i(k)) \succ \gamma_{i_j} - \gamma_i$ for all $i > i_j$, $k < n_i$, then the sequence $T_{i_j}, T_{i_j+1}, T_{i_j+2}, \ldots$, has type (A), and we are done. Otherwise, we let i_{j+1} be the minimum i for which there exists k such that $\mathrm{R}(\tau_i(k)) \preceq \gamma_{i_j} - \gamma_i$.

Clearly, either the procedure fails after a finite number of steps, and we find a subsequence of type (A), or it defines a subsequence $(T_{i_j})_{j\in\mathbb{N}}$ of type (B), as desired. □

9.5. **No bad sequences of type (A).** As a start, it is fairly easy to see that bad sequences of type (A) do not exist.

Proposition 9.13. *Let $x \in \Delta_{\alpha+1}$. Then $\mathrm{A}(x)$ contains no bad sequences of type (A).*

Proof. For a contradiction let $(T_i)_{i\in\mathbb{N}}$ be a bad sequence in $\mathrm{A}(x)$ of type (A). By Proposition 9.9 the sequence of terms $(\mathrm{R}(T_i) : i \in \mathbb{N})$ cannot be constant, so by

taking a subsequence we can assume that the terms $\mathrm{R}(T_i)$ are distinct, and since they are all terms of x, we may also assume (taking another subsequence) that $\mathrm{R}(T_0) \succ \mathrm{R}(T_1) \succ \mathrm{R}(T_2) \succ \dots$. By Proposition 9.3 it then follows that $c(\mathrm{R}(T_n)) \succ c(\mathrm{R}(T_{n+1}))$ for every $n \in \mathbb{N}$.

Let $i \in \mathbb{N}$ and write $T_i = \langle r_i e^{\gamma_i}, n_i, \tau_i \rangle$. By assumption, for any child $U = \tau_i(j)$ of T_i we have $\mathrm{R}(U) \succ \gamma_0 - \gamma_i$ (this holds vacuously if T_i has no children). We claim that for any such U we must have $\mathrm{R}(U) \in \mathrm{Term}(\gamma_0)$. Indeed by construction $\mathrm{R}(U) \in \mathrm{Term}(\gamma_i)$; therefore, if $\mathrm{R}(U) \notin \mathrm{Term}(\gamma_0)$, then $\mathrm{R}(U)$ would be a term of the difference $\gamma_0 - \gamma_i$, contradicting the assumption $\mathrm{R}(U) \succ \gamma_0 - \gamma_i$.

We have thus proved that all the roots of the children of the trees T_i are terms of $\gamma_0 = \log^\uparrow(\mathrm{R}(T_0))$; hence, we can replace the root of each T_i with e^{γ_0} obtaining a new sequence $T_i' := \langle e^{\gamma_0}, n_i, \tau_i \rangle$ in $\mathrm{A}(e^{\gamma_0})$. Since T_i and T_i' have the same children, by Lemma 9.4(2) we have:

$$\frac{\overline{c}(T_i)}{\overline{c}(T_i')} \asymp \frac{c(\mathrm{R}(T_i))}{c(\mathrm{R}(T_i'))} \asymp \frac{c(e^{\gamma_i})}{c(e^{\gamma_0})}.$$

By Proposition 5.17, the family $(\overline{c}(T') : T' \in \mathrm{A}(e^{\gamma_0}))$ is summable. Therefore, after extracting a subsequence we may assume that $\frac{\overline{c}(T_i')}{\overline{c}(T_{i+1}')} \succeq 1$ (note that the inequality is not necessarily strict, because the trees T_i' might not be distinct). It follows that

$$\frac{\overline{c}(T_i)}{\overline{c}(T_{i+1})} \asymp \frac{c(\mathrm{R}(T_i))}{c(\mathrm{R}(T_{i+1}))} \cdot \frac{\overline{c}(T_i')}{\overline{c}(T_{i+1}')} \succeq \frac{c(\mathrm{R}(T_i))}{c(\mathrm{R}(T_{i+1}))} \succ 1.$$

Therefore, $(T_i)_{i \in \mathbb{N}}$ is not bad. □

9.6. **Pruning trees.** In the sequel we consider trees in $\mathrm{A}(x)$ for some $x \in \Delta_{\alpha+1}$. We establish a procedure to "prune" a tree T, that is, to remove some descendants, in such a way that its contribution $\overline{c}(T)$ changes only by a small amount.

Definition 9.14. Let $T = \langle re^\gamma, n, \tau \rangle$ be an admissible tree (i.e. $T \in \mathrm{A}(re^\gamma)$), U be a child of T (necessarily admissible), and U' be an admissible tree with the same root as U. Let j be the *minimum* integer such that $\tau(j) = U$.

(1) We define $T[U'/U]$ as T with U replaced by U'. More precisely,

$$T[U'/U] := \langle re^\gamma, n, \tau^* \rangle$$

where $\tau^*(i) := \tau(i)$ for $i \neq j$ and $\tau^*(j) := U'$. Note that if $\overline{c}(U') \prec 1$, then $T[U'/U]$ is again an admissible tree.

(2) We define $T \setminus U$ as the admissible tree obtained from T by removing the child U. More precisely,

$$T \setminus U := \langle re^\gamma, n-1, \tau^* \rangle$$

where $\tau^*(i) := \tau(i)$ for $i < j$ and $\tau^*(i) := \tau(i+1)$ for $i \geq j$.

Definition 9.15. Let $T = \langle re^\gamma, n, \tau \rangle \in \mathrm{A}(x)$ with $\mathrm{size}(T) > 1$. If L is a leaf of T, we define the **minimal child of T with leaf L** to be the child $U = \tau(j)$ of T such that:

(1) L is a leaf of U (possibly $L = U$);
(2) among such children, $\mathrm{R}(U)$ is minimal with respect to $\preceq$;
(3) among such children, j is minimal.

Definition 9.16. Let $T = \langle re^\gamma, n, \tau\rangle \in \mathrm{A}(x)$ with $\mathrm{size}(T) > 1$ and let L be a leaf of T. We define T^L by induction on $\mathrm{size}(T)$ as follows. Let U be the minimal child of T with leaf L. We define:

(1) if $\mathrm{size}(U) = 1$ (namely $L = U$), let $T^L := T \setminus L$;
(2) if $\mathrm{size}(U) > 1$ and $\overline{c}(U^L) \prec 1$, let $T^L := T[U^L/U]$;
(3) if $\mathrm{size}(U) > 1$ and $\overline{c}(U^L) \succeq 1$, let $T^L := T \setminus U$.

Remark 9.17. Note that in all three cases, T^L is still an admissible tree; in particular, in (2) this is guaranteed by the condition $\overline{c}(U^L) \prec 1$, as for all children S of an admissible tree the contribution $\overline{c}(S)$ must be infinitesimal.

Lemma 9.18. *Let L be a leaf in $T \in \mathrm{A}(x)$, with $\mathrm{size}(T) > 1$, and let U be the minimal child of T with leaf L. We have:*

(1) $\mathrm{size}(T^L) < \mathrm{size}(T)$ and $\mathrm{R}(T^L) = \mathrm{R}(T)$;
(2) $T^L \in \mathrm{A}(x)$;
(3) if $T^L = T \setminus U$, then $\overline{c}(T) \asymp \overline{c}(T^L) \cdot \overline{c}(U)$;
(4) if $T^L := T[U^L/U]$, then $\overline{c}(T) = \overline{c}(T^L) \cdot \frac{\overline{c}(U)}{\overline{c}(U^L)}$;
(5) $\overline{c}(T^L) \succ \overline{c}(T)$;

Proof. We work by induction on $\mathrm{size}(T)$. Point (1) is straightforward and point (2) is Remark 9.17.

For (3), let $T =: \langle re^\gamma, n, \tau\rangle$ and let $j < n$ be minimal such that $U = \tau(j)$. By definition we have

$$\overline{c}(T) = re^{c(\gamma)^{\uparrow=}} \cdot \overline{c}(U) \cdot \frac{1}{n!} \prod_{\substack{i<n \\ i\neq j}} \overline{c}(\tau(i))$$

while

$$\overline{c}(T \setminus U) = re^{c(\gamma)^{\uparrow=}} \cdot \frac{1}{(n-1)!} \cdot \prod_{\substack{i<n \\ i\neq j}} \overline{c}(\tau(i))$$

Thus clearly $\overline{c}(T \setminus U) \asymp \frac{\overline{c}(T)}{\overline{c}(U)}$ and (3) follows.

A similar argument shows that if $T^L = T[U^L/U]$, then $\overline{c}(T^L) = \overline{c}(T) \cdot \frac{\overline{c}(U^L)}{\overline{c}(U)}$ and we obtain (4).

For (5), just note that if $T^L = T \setminus U$, then $\overline{c}(T) \asymp \overline{c}(T^L)\overline{c}(U)$, and since $\overline{c}(U) \prec 1$ we obtain $\overline{c}(T) \prec \overline{c}(T^L)$; if instead $T^L = T[U^L/U]$, by induction we have $\overline{c}(U) \prec \overline{c}(U^L)$ and we reach the same conclusion using (4). □

Lemma 9.19. *Let T be an admissible tree and U be a proper descendant of T. Then $\mathrm{R}(U) \succ 1$, and if U' is a proper descendant of U we have $1 \prec \mathrm{R}(U')^n \prec \mathrm{R}(U)$ for every $n \in \mathbb{N}$.*

Proof. Suppose first that U is a child of T. Write $\mathrm{R}(T) = re^\gamma$, so that $\mathrm{R}(U)$ is a term of $\gamma = \log^\uparrow(\mathrm{R}(T))$. Since $\gamma \in \mathbb{J}$, $\mathrm{R}(U)$ is of the form se^δ with $0 < \delta \in \mathbb{J}$, so $\mathrm{R}(U) \succ 1$, proving the first conclusion. Moreover, it follows that $\delta^n \prec e^\delta \asymp \mathrm{R}(U)$ for all $n \in \mathbb{N}$. If now U' is a child of U, then $\mathrm{R}(U')$ is a term of δ, so $\mathrm{R}(U')^n \preceq \delta^n \prec \mathrm{R}(U)$, while by the previous argument $\mathrm{R}(U') \succ 1$. The general conclusion with U a descendant of T and U' a descendant of U now follows by transitivity of $\preceq$. □

Proposition 9.20. *Let L be a leaf in a tree T of size > 1 and let U be the minimal child of T with leaf L. Then*

$$\overline{c}(T) \asymp \overline{c}(T^L) \cdot \overline{c}(L) \cdot t \quad \textit{where} \quad 1 \preceq t \preceq c(\mathrm{R}(U))^2.$$

Proof. We work by induction on $\mathrm{size}(T)$.

Case 1. If $\mathrm{size}(U) = 1$ (namely $U = L$), then $T^L = T \setminus L$ and $\overline{c}(T) \asymp \overline{c}(T^L) \cdot \overline{c}(L)$, so it suffices to take $t = 1$.

Case 2. Assume $\mathrm{size}(U) > 1$ and $\overline{c}(U^L) \succeq 1$. Then $T^L = T \setminus U$, and therefore $\overline{c}(T) \asymp \overline{c}(T^L) \cdot \overline{c}(U)$. We may assume by induction that $\overline{c}(U) \asymp \overline{c}(U^L) \cdot \overline{c}(L) \cdot u$, where $1 \preceq u \preceq c(\mathrm{R}(U'))^2$ and U' is the minimal child of U with leaf L. Substituting we obtain

$$\overline{c}(T) \asymp \overline{c}(T^L) \cdot \overline{c}(L) \cdot \overline{c}(U^L) \cdot u.$$

By Lemma 9.4 we have $\overline{c}(U^L) \preceq c(\mathrm{R}(U^L)) = c(\mathrm{R}(U))$, and by Lemma 9.19 $u \preceq c(\mathrm{R}(U'))^2 \prec c(\mathrm{R}(U))$, hence we can take $t := \overline{c}(U^L) \cdot u$.

Case 3. Finally, assume $\mathrm{size}(U) > 1$ and $\overline{c}(U^L) \prec 1$. Then $T^L = T[U^L/U]$, and by Lemma 9.18 we have $\overline{c}(T) = \overline{c}(T^L) \cdot \frac{\overline{c}(U)}{\overline{c}(U^L)}$. By inductive hypothesis we have $\overline{c}(U) \asymp \overline{c}(U^L) \cdot \overline{c}(L) \cdot u$, where reasoning as above we have $1 \preceq u \prec c(\mathrm{R}(U))$. Substituting we get

$$\overline{c}(T) \asymp \overline{c}(T^L) \cdot \overline{c}(L) \cdot u,$$

hence we can take $t = u$. □

9.7. **No bad sequences.** We can finally prove that there are no bad sequences at all in any $\mathrm{A}(x)$.

Proposition 9.21. *Let $x \in \Delta_{\alpha+1}$. If $(T_i)_{i\in\mathbb{N}}$ is a bad sequence in $\mathrm{A}(x)$, then there are a bad sequence $(S_j)_{j\in\mathbb{N}}$ in $\mathrm{A}(x)$ and some $k \in \mathbb{N}$ such that $\mathrm{size}(S_0) < \mathrm{size}(T_k)$, $\mathrm{R}(S_0) = \mathrm{R}(T_k)$ and $\overline{c}(S_0) \succ \overline{c}(T_k)$.*

Proof. By Proposition 9.12 and Proposition 9.13, there is a subsequence $(P_j)_{j\in\mathbb{N}}$ of $(T_i)_{i\in\mathbb{N}}$ of type (B). Recall that by definition of type (B), $\mathrm{size}(P_j) > 0$ for all $j \in \mathbb{N}$.

Write $P_j = \langle r_j e^{\gamma_j}, n_j, \tau_j \rangle$. Let L_0 be a leaf of P_0. For $j \geq 1$, let U_j be a child of P_j with $\mathrm{R}(U_j) \preceq \gamma_{j-1} - \gamma_j$, which exists by definition of type (B), and let L_j be a leaf of U_j. We may then assume that U_j is the *minimal* child with leaf L_j (if not, just replace U_j with the minimal child U with leaf L_j, and observe that the condition $\mathrm{R}(U) \preceq \gamma_{j-1} - \gamma_j$ is still satisfied because $\mathrm{R}(U) \preceq \mathrm{R}(U_j)$).

We can write $L_j = \langle \lambda_j, 0, s_j \rangle$, where $\lambda_j \in \Delta$ and $s_j = \overline{c}(L_j) \in \mathrm{Term}(c_0(\lambda_j))$. By Lemma 9.19 we have $\lambda_j \preceq \mathrm{R}(U_j)$; therefore, since c preserves $\preceq$ by Proposition 9.3,

$$c(\lambda_j) \preceq c(\mathrm{R}(U_j)) \preceq c(\gamma_{j-1} - \gamma_j)$$

for all $j \geq 1$.

By Lemma 9.2, we may extract a further subsequence of $(P_j)_{j\in\mathbb{N}}$ and assume that for all $j \in \mathbb{N}$ we have $\left(\frac{s_{j+1}}{s_j}\right) \prec c(\lambda_{j+1})^2$, so

$$\left(\frac{s_{j+1}}{s_j}\right) \preceq c(\gamma_j - \gamma_{j+1})^2.$$

Now let $S_j := P_j^{L_j}$, which is well defined since $\mathrm{size}(P_j) > 0$ for all $j \in \mathbb{N}$. We shall prove that $(S_j)_{j\in\mathbb{N}}$ has the desired properties.

By Proposition 9.20, for all $j \in \mathbb{N}$ we have

$$\overline{c}(P_j) = \overline{c}(P_j^{Lj}) \cdot \overline{c}(L_j) \cdot t_j = \overline{c}(P_j^{L_j}) \cdot s_j \cdot t_j$$

where $1 \preceq t_j \preceq c(\mathrm{R}(U_j))^2$ for all $j \in \mathbb{N}$. In particular, $\frac{t_{j+1}}{t_j} \preceq t_{j+1} \preceq c(\mathrm{R}(U_{j+1}))^2$, so

$$\frac{t_{j+1}}{t_j} \preceq c(\gamma_j - \gamma_{j+1})^2.$$

It follows that

$$\frac{\overline{c}(P_j^{L_j})}{\overline{c}(P_{j+1}^{L_{j+1}})} = \frac{\overline{c}(P_j)}{\overline{c}(P_{j+1})} \cdot \frac{s_{j+1}}{s_j} \cdot \frac{t_{j+1}}{t_j} \preceq c(\gamma_j - \gamma_{j+1})^4 \cdot \frac{\overline{c}(P_j)}{\overline{c}(P_{j+1})}.$$

Since $(P_j)_{j\in\mathbb{N}}$ is bad, for all $j, n \in \mathbb{N}$ we have

$$\left(\frac{\overline{c}(P_j)}{\overline{c}(P_{j+1})}\right)^n \preceq \frac{c(\mathrm{R}(P_j))}{c(\mathrm{R}(P_{j+1}))}.$$

Likewise, for all $j, n \in \mathbb{N}$ we also have

$$(c(\gamma_j - \gamma_{j+1}))^n \preceq e^{c(\gamma_j - \gamma_{j+1})} \asymp \frac{c(\mathrm{R}(P_j))}{c(R(P_{j+1}))}$$

using Lemma 9.4, Proposition 9.3 and the fact that $\gamma_j - \gamma_{j+1} \succ 1$. It follows that for all $j, n \in \mathbb{N}$ we have

$$\left(\frac{\overline{c}(P_j^{L_j})}{\overline{c}(P_{j+1}^{L_{j+1}})}\right)^n \preceq \frac{c(\mathrm{R}(P_j))}{c(\mathrm{R}(P_{j+1}))}.$$

Recalling that $\mathrm{R}(P_j^{L_j}) = \mathrm{R}(P_j)$ for all $j \in \mathbb{N}$, it follows that $(S_j)_{j\in\mathbb{N}} = (P_j^{L_j})_{j\in\mathbb{N}}$ is another bad sequence in $\mathrm{A}(x)$.

To conclude, let $k \in \mathbb{N}$ be such that $T_k = P_0$. By construction, $\mathrm{size}(S_0) = \mathrm{size}(P_0^{L_0}) < \mathrm{size}(P_0) = \mathrm{size}(T_k)$, and by Lemma 9.18, $\overline{c}(S_0) = \overline{c}(P_0^{L_0}) \succ \overline{c}(P_0) = \overline{c}(T_k)$, as desired. □

Proposition 9.22. *Let $x \in \Delta_{\alpha+1}$. Then $\mathrm{A}(x)$ contains no bad sequences.*

Proof. Suppose by contradiction that there is a bad sequence of trees in $\mathrm{A}(x)$. Among all such bad sequences, let $(T_i)_{i\in\mathbb{N}}$ be the one such that $\mathrm{size}(T_0)$ is minimal, and fixed T_0, $\mathrm{size}(T_1)$ is minimal, and so on. By Proposition 9.21, there is another bad sequence $(S_j)_{j\in\mathbb{N}}$ in $\mathrm{A}(x)$ and some $k \in \mathbb{N}$ such that $\mathrm{size}(S_0) < \mathrm{size}(T_k)$, $\mathrm{R}(S_0) = \mathrm{R}(T_k)$ and $\overline{c}(S_0) \succ \overline{c}(T_k)$.

We observe that

$$T_0, T_1, \ldots, T_{k-1}, S_0, S_1, \ldots$$

is again a bad sequence in $\mathrm{A}(x)$. Indeed, it suffices to note that for all $n \in \mathbb{N}$ we have

$$\left(\frac{\overline{c}(T_{k-1})}{\overline{c}(S_0)}\right)^n \prec \left(\frac{\overline{c}(T_{k-1})}{\overline{c}(T_k)}\right)^n \preceq \frac{c(\mathrm{R}(T_{k-1}))}{c(\mathrm{R}(T_k))} = \frac{c(\mathrm{R}(T_{k-1}))}{c(\mathrm{R}(S_0))}.$$

However, since $\mathrm{size}(S_0) < \mathrm{size}(T_k)$, this contradicts our minimality assumption. Therefore, there are no bad sequences in $\mathrm{A}(x)$, as desired. □

By Proposition 9.6, this completes the proof of Lemma 5.21, as desired.

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