This is a repository copy of *Cherednik algebras and Zhelobenko operators*.

Article:

CHEREDNIK ALGEBRAS AND
ZHELOBENKO OPERATORS

SERGEY KHOROSHKIN

MAXIM NAZAROV

Institute for Theoretical and
Experimental Physics
Moscow 117259, Russia

Department of Mathematics
University of York
York YO10 5DD, England

maxim.nazarov@york.ac.uk

National Research University
Higher School of Economics
Moscow 101000, Russia

khor@itep.ru

Abstract. We study canonical intertwining operators between induced modules of the trigonometric Cherednik algebra. We demonstrate that these operators correspond to the Zhelobenko operators for the affine Lie algebra $\widehat{\mathfrak{sl}}_m$. To establish the correspondence, we use the functor of Arakawa, Suzuki and Tsuchiya which maps certain $\widehat{\mathfrak{sl}}_m$-modules to modules of the Cherednik algebra.

Introduction

0.1. In the present article we study the trigonometric Cherednik algebra $\mathfrak{C}_N$ corresponding to the general linear Lie algebra $\mathfrak{gl}_N$. The complex associative algebra $\mathfrak{C}_N$ is generated by the symmetric group ring $\mathbb{C}\mathfrak{S}_N$, by the ring $\mathfrak{P}_N$ of Laurent polynomials in N variables $x_1, \ldots, x_N$ and by another family of pairwise commuting elements denoted by $u_1, \ldots, u_N$. The subalgebra of $\mathfrak{C}_N$ generated by the first two rings is the crossed product $\mathfrak{S}_N \ltimes \mathfrak{P}_N$ where the symmetric group $\mathfrak{S}_N$ permutes the variables $x_1, \ldots, x_N$. The subalgebra generated by $\mathfrak{S}_N$ and $u_1, \ldots, u_N$ is the degenerate affine Hecke algebra $\mathfrak{H}_N$ introduced by Drinfeld [5] and Lusztig [11]. The other defining relations in $\mathfrak{C}_N$ are given in Subsection 2.1 of our article. In particular, the algebra $\mathfrak{C}_N$ depends on a parameter $\kappa \in \mathbb{C}$.

The degenerate affine Hecke algebra $\mathfrak{H}_N$ has a distinguished family of modules which are called *standard*. These modules are determined by pairs of sequences $\lambda = (\lambda_1, \ldots, \lambda_m)$ and $\mu = (\mu_1, \ldots, \mu_m)$ of length m of complex numbers such that for every $a = 1, \ldots, m$ the difference $\lambda_a - \mu_a$ is a positive integer, while $\lambda_1 - \mu_1 + \cdots + \lambda_m - \mu_m = N$. We denote the corresponding standard module of $\mathfrak{H}_N$ by S_μ^λ. It is induced from a one-dimensional module of the subalgebra of $\mathfrak{H}_N$ generated by $u_1, \ldots, u_N$ and by the subgroup of $\mathfrak{S}_N$ preserving the partition of

the sequence $1, \ldots, N$ to segments of lengths $\lambda_1 - \mu_1, \ldots, \lambda_m - \mu_m$. This subgroup of $\mathfrak{S}_N$ acts on the one-dimensional module trivially, while u_p acts as $\mu_a - a + h$ where a is the number of the segment of the sequence $1, \ldots, N$ which the index p belongs to, and h is the number of the place of the index p within that segment.

Now consider the symmetric group $\mathfrak{S}_m$ which acts on sequences of length m of complex numbers by permutations. We denote by the symbol $\circ$ the corresponding *shifted* action of $\mathfrak{S}_m$. To define the latter action, one takes a sequence of length m, subtracts the sequence $(1, \ldots, m)$ from it, permutes the resulting sequence and adds the sequence $(1, \ldots, m)$ back. If $\lambda_a - \lambda_b \notin \mathbb{Z}$ or equivalently $\mu_a - \mu_b \notin \mathbb{Z}$ for all $a \neq b$, then the standard $\mathfrak{H}_N$-module S_μ^λ is irreducible. Moreover, then for every permutation $\sigma \in \mathfrak{S}_m$ the standard module $S_{\sigma \circ \mu}^{\sigma \circ \lambda}$ is isomorphic to S_μ^λ. Hence there exists an intertwining mapping $S_\mu^\lambda \to S_{\sigma \circ \mu}^{\sigma \circ \lambda}$ of $\mathfrak{H}_N$-modules, unique up to scalar multiplier. These mappings were already used by Rogawski [12].

The standard $\mathfrak{H}_N$-module S_μ^λ has another realization due to Arakawa, Suzuki and Tsuchiya [1]. Take the complex Lie algebra $\mathfrak{gl}_m$. The sequences of length m of complex numbers can be regarded as weights of $\mathfrak{gl}_m$. There we employ the Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{gl}_m$ described in Subsection 1.3 of our article. Then the above defined action $\circ$ becomes the shifted action of the Weyl group $\mathfrak{S}_m$ of $\mathfrak{gl}_m$ on weights. Take the Verma module M_μ of $\mathfrak{gl}_m$ corresponding to the weight μ.

The tensor product $(\mathbb{C}^m)^{\otimes N} \otimes M_\mu$ of $\mathfrak{gl}_m$-modules can be also equipped with an action of the algebra $\mathfrak{H}_N$, which commutes with the action of $\mathfrak{gl}_m$. Here the symmetric group $\mathfrak{S}_N \subset \mathfrak{H}_N$ acts on $(\mathbb{C}^m)^{\otimes N} \otimes M_\mu$ by permutations of the N tensor factors $\mathbb{C}^m$, while the elements (1.2) of $\mathfrak{H}_N$ act as the operators (1.7) respectively. Let $\mathfrak{n}$ be the nilpotent subalgebra of $\mathfrak{gl}_m$ defined in Subsection 1.3. The space $((\mathbb{C}^m)^{\otimes N} \otimes M_\mu)_\mathfrak{n}^\lambda$ of $\mathfrak{n}$-coinvariants of weight λ inherits an action of the algebra $\mathfrak{H}_N$. As a $\mathfrak{H}_N$-module it is isomorphic to S_μ^λ; see our Proposition 1.3.

Following Zhelobenko [16], for any λ and μ obeying the above non-integrality conditions, and for any permutation $\sigma \in \mathfrak{S}_m$, one can define a canonical linear map

$$((\mathbb{C}^m)^{\otimes N} \otimes M_\mu))_\mathfrak{n}^\lambda \to ((\mathbb{C}^m)^{\otimes N} \otimes M_{\sigma \circ \mu}))_\mathfrak{n}^{\sigma \circ \lambda}. \qquad (0.1)$$

See also the work of Khoroshkin and Ogievetsky [10]. In particular, the linear map (0.1) is $\mathfrak{H}_N$-intertwining. Using Proposition 1.3, the map (0.1) determines an $\mathfrak{H}_N$-intertwining map $S_\mu^\lambda \to S_{\sigma \circ \mu}^{\sigma \circ \lambda}$. By the irreducibility of the source and of target standard $\mathfrak{H}_N$-modules here, the latter map coincides with the intertwining map from [12] up to a scalar multiplier.

We will work with the special linear Lie algebra $\mathfrak{sl}_m$ alongside of $\mathfrak{gl}_m$. Our $\mathfrak{n}$ is a subalgebra of $\mathfrak{sl}_m$. Further, the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{sl}_m$ described in Subsection 1.3 is contained in $\mathfrak{t} \subset \mathfrak{gl}_m$. Let us denote by α and β the weights of $\mathfrak{sl}_m$ corresponding to the weights λ and μ of $\mathfrak{gl}_m$ by restriction. Hence α and β are elements of the space dual to $\mathfrak{h}$.

The Verma module M_β is isomorphic to the restriction of M_μ to the subalgebra $\mathfrak{sl}_m \subset \mathfrak{gl}_m$. However, another action of $\mathfrak{H}_N$ on $(\mathbb{C}^m)^{\otimes N} \otimes M_\beta$ can be defined by using only the structure of M_β as a module of $\mathfrak{sl}_m$. Namely, he symmetric group $\mathfrak{S}_N \subset \mathfrak{H}_N$ acts on $(\mathbb{C}^m)^{\otimes N} \otimes M_\beta$ again by permutations of the N tensor factors $\mathbb{C}^m$, but the elements (1.2) of $\mathfrak{H}_N$ act as the operators (1.9) respectively. The space

$((\mathbb{C}^m)^{\otimes N} \otimes M_\beta))_\mathfrak{n}^\alpha$ of $\mathfrak{n}$-coinvariants of weight α inherits an action of $\mathfrak{H}_N$. As a $\mathfrak{H}_N$-module it is isomorphic to the pullback of S_μ^λ relative to the automorphism (1.5) of $\mathfrak{H}_N$ where $f = -(\mu_1 + \cdots + \mu_m)/m$; see Corollary 1.4. This automorphism acts trivially on the elements of the subalgebra $\mathfrak{S}_m \subset \mathfrak{H}_N$. Note that the latter space of $\mathfrak{n}$-coinvariants can be naturally identified with the space $((\mathbb{C}^m)^{\otimes N} \otimes M_\mu))_\mathfrak{n}^\lambda$.

The shifted action of $\mathfrak{S}_m$ on the weights of $\mathfrak{gl}_m$ factors to an action on the weights of $\mathfrak{sl}_m$. By again following [10] and [16], one defines a canonical linear map

$$((\mathbb{C}^m)^{\otimes N} \otimes M_\beta))_\mathfrak{n}^\alpha \rightarrow ((\mathbb{C}^m)^{\otimes N} \otimes M_{\sigma\circ\beta}))_\mathfrak{n}^{\sigma\circ\alpha}. \tag{0.2}$$

If we identify the source vector spaces of the maps (0.1) and (0.2) as above, and also identify the target vector spaces, then the two maps become the same. Note that the shifted action $\circ$ of $\mathfrak{S}_m$ on μ preserves the sum $\mu_1 + \cdots + \mu_m$. Hence the map (0.2) is also $\mathfrak{H}_N$-intertwining.

Via the Drinfeld duality between $\mathfrak{H}_N$-modules and modules of the Yangians of general linear Lie algebras [5] this interpretation of intertwining maps for the standard modules of $\mathfrak{H}_N$ goes back to the work of Tarasov and Varchenko [15]; see also our work [7]. In the present work we extend this interpretation to intertwining maps for certain $\mathfrak{C}_N$-modules. Instead of $\mathfrak{gl}_m$ and $\mathfrak{sl}_m$ above, we will use the corresponding affine Lie algebras $\widehat{\mathfrak{gl}}_m$ and $\widehat{\mathfrak{sl}}_m$.

0.2. We will regard $\widehat{\mathfrak{sl}}_m$ as a one-dimensional central extension of the current Lie algebra $\mathfrak{sl}_m[t, t^{-1}]$. We choose a basis element C in the extending one-dimensional vector space. For any $\ell \in \mathbb{C}$, a module of $\widehat{\mathfrak{sl}}_m$ is said to be of level ℓ if C acts as the scalar ℓ on that module. Let us extend the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{sl}_m$ by the one-dimensional space spanned by C, and denote by $\widehat{\mathfrak{h}}$ the Abelian subalgebra of $\widehat{\mathfrak{sl}}_m$ so obtained. For $\ell = \kappa - m$ we will denote by $\widehat{\alpha}$ and $\widehat{\beta}$ the extensions of the weights α and β from $\mathfrak{h}$ to $\widehat{\mathfrak{h}}$, determined by setting $\widehat{\alpha}(C) = \widehat{\beta}(C) = \ell$. We will use the Verma module $M_{\widehat{\beta}}$ of $\widehat{\mathfrak{sl}}_m$ as defined in Subsection 2.3.

Let us now regard $\mathfrak{H}_N$ as a subalgebra of $\mathfrak{C}_N$. Denote by $\widehat{S}_\mu^\lambda$ the module of $\mathfrak{C}_N$ induced from the standard module S_μ^λ of $\mathfrak{H}_N$. The induced module also has another realization [1]. Take the vector space $\mathfrak{P}_N \otimes (\mathbb{C}^m)^{\otimes N}$. It can be naturally identified with the tensor product of N copies of the vector space $\mathbb{C}^m[t, t^{-1}]$. By regarding the latter space as a module of $\widehat{\mathfrak{sl}}_m$ of level zero, $\mathfrak{P}_N \otimes (\mathbb{C}^m)^{\otimes N}$ becomes a zero level module of $\widehat{\mathfrak{sl}}_m$. Further, the vector space

$$\mathfrak{P}_N \otimes (\mathbb{C}^m)^{\otimes N} \otimes M_{\widehat{\beta}} \tag{0.3}$$

can be equipped with an action of the algebra $\mathfrak{C}_N$. The symmetric group $\mathfrak{S}_N \subset \mathfrak{C}_N$ acts on (0.3) by simultaneous permutations of the variables $x_1, \ldots, x_N$ and of the N tensor factors $\mathbb{C}^m$ while the subalgebra $\mathfrak{P}_N \subset \mathfrak{C}_N$ acts on (0.3) via multiplication in the first tensor factor. The elements (1.2) of $\mathfrak{H}_N \subset \mathfrak{C}_N$ act on (0.3) as the operators (2.7) respectively.

In general, the action of $\mathfrak{C}_N$ on the vector space (0.3) does *not* commute with the action of $\widehat{\mathfrak{sl}}_m$. However, let $\widehat{\mathfrak{n}}$ be the nilpotent subalgebra of $\widehat{\mathfrak{sl}}_m$ defined in

Subsection 2.1. For $\ell = \kappa - m$ the action of the algebra $\mathfrak{C}_N$ on (0.3) preserves the image of the action of $\widehat{\mathfrak{n}}$; see Corollary 2.2. Therefore the space

$$(\mathfrak{P}_N \otimes (\mathbb{C}^m)^{\otimes N} \otimes M_{\widehat{\beta}})_{\widehat{\mathfrak{n}}}^{\widehat{\alpha}} \tag{0.4}$$

of $\widehat{\mathfrak{n}}$-coinvariants of (0.3) of weight $\widehat{\alpha}$ inherits an action of the algebra $\mathfrak{C}_N$. The automorphism (1.5) of $\mathfrak{H}_N$ extends to $\mathfrak{C}_N$ so that it acts on the elements of the subalgebra $\mathfrak{P}_N \subset \mathfrak{C}_N$ trivially. As a $\mathfrak{C}_N$-module, (0.4) is isomorphic to the pullback of $\widehat{S}_\mu^\lambda$ relative to the extended automorphism (1.5) where $f = -(\mu_1 + \cdots + \mu_m)/m$; see our Corollary 2.2 and Proposition 2.3.

Now consider the semidirect group product $\mathfrak{S}_m \ltimes \mathbb{Z}^m$. Extend the permutation action of the group $\mathfrak{S}_m$ on sequences of length m of complex numbers to an action of $\mathfrak{S}_m \ltimes \mathbb{Z}^m$ so that the elements of $\mathbb{Z}^m$ act by addition of the respective elements of $\ell \mathbb{Z}^m \subset \mathbb{C}^m$. Here we set $\ell = \kappa - m$ as above. The action $\circ$ of $\mathfrak{S}_m$ on the sequences also extends to an action of the group $\mathfrak{S}_m \ltimes \mathbb{Z}^m$, where the elements of $\mathbb{Z}^m$ however act by addition of the respective elements of $\kappa \mathbb{Z}^m \subset \mathbb{C}^m$. Let us denote by the symbol $\circ$ the latter action of $\mathfrak{S}_m \ltimes \mathbb{Z}^m$ on the sequences. If $\lambda_a - \lambda_b \notin \mathbb{Z} + \kappa \mathbb{Z}$ or equivalently $\mu_a - \mu_b \notin \mathbb{Z} + \kappa \mathbb{Z}$ for $a \neq b$, then the induced $\mathfrak{C}_N$-module $\widehat{S}_\mu^\lambda$ is irreducible. Moreover, then for every element $\omega \in \mathfrak{S}_m \ltimes \mathbb{Z}^m$ the $\mathfrak{C}_N$-module $\widehat{S}_{\omega \circ \mu}^{\omega \circ \lambda}$ is isomorphic to $\widehat{S}_\mu^\lambda$. Hence there is an intertwining mapping $\widehat{S}_\mu^\lambda \to \widehat{S}_{\omega \circ \mu}^{\omega \circ \lambda}$ of $\mathfrak{C}_N$-modules, unique up to a scalar multiplier. These mappings were used by Suzuki [13]. Recently they were further used by Balagović [2].

The group $\mathfrak{S}_m \ltimes \mathbb{Z}^m$ is isomorphic to the extended affine Weyl group of $\mathfrak{gl}_m$, for details see Subsection 2.4. This group is $\mathbb{Z}$-graded so that the degree of any element of $\mathfrak{S}_m$ is zero, while the degree of any element of $\mathbb{Z}^m$ is the sum of its m components. All the elements of degree zero make a subgroup of $\mathfrak{S}_m \ltimes \mathbb{Z}^m$ isomorphic to the proper affine Weyl group of $\mathfrak{gl}_m$. Note that this subgroup is also isomorphic to the Weyl group of the affine Lie algebra $\widehat{\mathfrak{sl}}_m$.

Again regard λ and μ as weights of $\mathfrak{gl}_m$. Restrict them to the weights α and β of $\mathfrak{sl}_m$. Extend the latter two to the weights $\widehat{\alpha}$ and $\widehat{\beta}$ of $\widehat{\mathfrak{sl}}_m$ as above. The action $\circ$ of $\mathfrak{S}_m \ltimes \mathbb{Z}^m$ on λ and μ determines its action on $\widehat{\alpha}$ and $\widehat{\beta}$. We will still denote by $\circ$ the action of $\mathfrak{S}_m \ltimes \mathbb{Z}^m$ so determined. It can also be described as a shifted action of the group $\mathfrak{S}_m \ltimes \mathbb{Z}^m$ on those weights of $\widehat{\mathfrak{sl}}_m$ which take the value $\ell = \kappa - m$ at $C \in \widehat{\mathfrak{h}}$; see Subsection 3.1 for details.

By following [10] and [16], for every element $\omega \in \mathfrak{S}_m \ltimes \mathbb{Z}^m$ we can define a canonical linear map from the vector space (0.4) to the vector space

$$(\mathfrak{P}_N \otimes (\mathbb{C}^m)^{\otimes N} \otimes M_{\omega \circ \widehat{\beta}})_{\widehat{\mathfrak{n}}}^{\omega \circ \widehat{\alpha}}. \tag{0.5}$$

Details of this definition are given in Subsection 4.1. Denote by g the $\mathbb{Z}$-degree of ω. Our linear map is $\mathfrak{C}_N$-intertwining only if $\kappa = 0$ or $g = 0$. In general, it becomes $\mathfrak{C}_N$-intertwining if we pull the action of $\mathfrak{C}_N$ on the target space (0.5) back through the automorphism (1.5) where $f = \kappa g/m$. Here we use Proposition 2.4 and its Corollary 2.5 which seem to be new.

By using Proposition 2.3 we can replace the source and the target modules of this $\mathfrak{C}_N$-intertwining linear map by their isomorphic modules. The source module

can be replaced by the pullback of $\widehat{S}_\mu^\lambda$ relative to the automorphism (1.5) where $f = -(\mu_1 + \cdots + \mu_m)/m$. Note that the sum of the terms of the sequence $\omega \circ \mu$ is equal to $\mu_1 + \cdots + \mu_m + \kappa\, g$ by the definition of the action $\circ$ of $\mathfrak{S}_m \ltimes \mathbb{Z}^m$ on the sequences. Therefore the target module here can be replaced by the pullback of $\widehat{S}_{\omega \circ \mu}^{\omega \circ \lambda}$ relative to the automorphism (1.5) where

$$f = -(\mu_1 + \cdots + \mu_m + \kappa\, g)/m + \kappa\, g/m = -(\mu_1 + \cdots + \mu_m)/m.$$

Since the values of f for the source and the target modules are the same, our canonical linear map from (0.4) to (0.5) also determines a $\mathfrak{C}_N$-intertwining linear map $\widehat{S}_\mu^\lambda \to \widehat{S}_{\omega \circ \mu}^{\omega \circ \lambda}$. By the irreducibility of the source and target modules here, the latter map coincides with the intertwining map from [13] up to a scalar multiplier.

0.3. Let us now briefly survey our article. In Section 1 we collect basic facts about the degenerate affine Hecke algebra $\mathfrak{H}_N$, including the realisation of its standard modules [1]. In Section 2 we recall the definition of the trigonometric Cherednik algebra $\mathfrak{C}_N$, and describe the action of $\mathfrak{C}_N$ on the spaces of $\widehat{\mathfrak{n}}$-coinvariants. By using this action, we give the realisation of induced modules of $\mathfrak{C}_N$ mentioned above. Towards the end of Section 2 we introduce the extended affine Weyl group of $\mathfrak{gl}_m$, and describe its action on the spaces of $\widehat{\mathfrak{n}}$-coinvariants. Our Proposition 2.4 relates this action to the action of $\mathfrak{C}_N$ on the same spaces. This relation is the key technical result of our article. In Section 3 we define the Zhelobenko operators for the affine Lie algebra $\widehat{\mathfrak{sl}}_m$. Theorem 3.6 relates these operators to the algebra $\mathfrak{C}_N$. Further details of this relation are worked out in Section 4.

1. Hecke algebras

1.1. We begin with the definition of the *degenerate affine Hecke algebra* $\mathfrak{H}_N$ corresponding to the general linear group GL_N over a local non-Archimedean field. This algebra has been introduced by Drinfeld [D2]; see also the work of Lusztig [L]. The complex associative algebra $\mathfrak{H}_N$ is generated by the symmetric group algebra $\mathbb{C}\mathfrak{S}_N$ and by pairwise commuting elements $u_1, \ldots, u_N$ with the cross relations for $p = 1, \ldots, N-1$ and $q = 1, \ldots, N$

$$\sigma_p\, u_q = u_q\, \sigma_p \quad \text{for} \quad q \neq p, p+1;$$
$$\sigma_p\, u_p = u_{p+1}\, \sigma_p - 1.$$

Here and in what follows $\sigma_p \in \mathfrak{S}_N$ denotes the transposition of numbers p and $p+1$. More generally, $\sigma_{pq} \in \mathfrak{S}_N$ will denote the transposition of the numbers p and q. The group algebra $\mathbb{C}\mathfrak{S}_N$ can be then regarded as a subalgebra in $\mathfrak{H}_N$. Furthermore, it follows from the defining relations of $\mathfrak{H}_N$ that a homomorphism $\mathfrak{H}_N \to \mathbb{C}\mathfrak{S}_N$, identical on the subalgebra $\mathbb{C}\mathfrak{S}_N \subset \mathfrak{H}_N$, can be defined by mapping

$$u_p \mapsto \sigma_{1p} + \cdots + \sigma_{p-1,p} \quad \text{for} \quad p = 1, \ldots, N. \tag{1.1}$$

We will also use the elements of the algebra $\mathfrak{H}_N$

$$z_p = u_p - \sigma_{1p} - \cdots - \sigma_{p-1,p} \quad \text{where} \quad p = 1, \ldots, N. \tag{1.2}$$

Notice that $z_p \mapsto 0$ under the homomorphism $\mathfrak{H}_N \to \mathbb{C}\mathfrak{S}_N$ defined by (1.1). For every $\sigma \in \mathfrak{S}_N$ we have

$$\sigma \, z_p \, \sigma^{-1} = z_{\sigma(p)}. \tag{1.3}$$

The elements $z_1, \ldots, z_N$ do not commute, but satisfy the commutation relations

$$[\, z_p, z_q \,] = \sigma_{pq} \, (z_p - z_q). \tag{1.4}$$

The relations (1.3) and (1.4) easily follow from the above definition of the algebra $\mathfrak{H}_N$; see, for instance, [7, Sect. 1]. Obviously, the algebra $\mathfrak{H}_N$ is generated by $\mathbb{C}\mathfrak{S}_N$ and the elements $z_1, \ldots, z_N$. Together with relations in $\mathbb{C}\mathfrak{S}_N$, (1.3) and (1.4) are defining relations of $\mathfrak{H}_N$ too.

It follows from the definition of $\mathfrak{H}_N$ that for any $f \in \mathbb{C}$ an automorphism of this algebra, identical on the subalgebra $\mathbb{C}\mathfrak{S}_N \subset \mathfrak{H}_N$, can be defined by mapping

$$u_p \mapsto u_p + f \quad \text{for} \quad p = 1, \ldots, N. \tag{1.5}$$

Then by (1.2)

$$z_p \mapsto z_p + f \quad \text{for} \quad p = 1, \ldots, N.$$

By pulling the trivial one-dimensional module of the algebra $\mathbb{C}\mathfrak{S}_N$ back through the homomorphism (1.1), and further back through the automorphism (1.5), we get a one-dimensional module of $\mathfrak{H}_N$. On the latter module each of the elements $z_1, \ldots, z_N \in \mathfrak{H}_N$ acts as multiplication by f. Let us denote this module by S_f^{f+N}; this peculiar choice of notation will be justified next.

Now fix a postive integer m. Take any two sequences $\lambda = (\lambda_1, \ldots, \lambda_m)$ and $\mu = (\mu_1, \ldots, \mu_m)$ of length m of complex numbers. For each $a = 1, \ldots, m$ denote $\nu_a = \lambda_a - \mu_a$ and suppose that ν_a is a non-negative integer. Note that unlike in the Introduction, here we allow the equality $\nu_a = 0$. We still suppose that $\nu_1 + \cdots + \nu_m = N$. Denote $\nu = (\nu_1, \ldots, \nu_m)$. Let $\mathfrak{S}_\nu$ be the corresponding subgroup of the symmetric group $\mathfrak{S}_N$. This subgroup is naturally isomorphic to the direct product $\mathfrak{S}_{\nu_1} \times \cdots \times \mathfrak{S}_{\nu_m}$. The tensor product $\mathfrak{H}_{\nu_1} \otimes \cdots \otimes \mathfrak{H}_{\nu_m}$ can be naturally identified with the subalgebra of $\mathfrak{H}_N$ generated by the subgroup $\mathfrak{S}_\nu \subset \mathfrak{S}_N$ and by all the pairwise commuting elements $u_1, \ldots, u_N$. Denote by $\mathfrak{H}_\nu$ this subalgebra. The induced module of $\mathfrak{H}_N$

$$\mathrm{Ind}_{\mathfrak{H}_\nu}^{\mathfrak{H}_N} \, S_{\mu_1}^{\lambda_1} \otimes S_{\mu_2 - 1}^{\lambda_2 - 1} \otimes \cdots \otimes S_{\mu_m - m + 1}^{\lambda_m - m + 1}$$

is called *standard* and denoted by S_μ^λ. If $m = 1$ and $\mu_1 = f$ then $\lambda_1 = f + N$ and $S_\mu^\lambda = S_f^{f+N}$. The reason to use, in the definition of S_μ^λ, the numbers $\lambda_a - a + 1$ and $\mu_a - a + 1$ rather than λ_a and μ_a will become clear in Subsection 1.3.

1.2. Let us now recall a construction due to Cherednik [4, Example 2.1]. It has been further developed by Arakawa, Suzuki and Tsuchiya [1, Subsect. 5.3]. Let U be any module over the complex general linear Lie algebra $\mathfrak{gl}_m$. Let $E_{ab} \in \mathfrak{gl}_m$ with $a, b = 1, \ldots, m$ be the standard matrix units. We will also regard the matrix units E_{ab} as elements of the algebra $\mathrm{End}(\mathbb{C}^m)$; this should not cause any confusion. Let us consider the tensor product $(\mathbb{C}^m)^{\otimes N} \otimes U$ of $\mathfrak{gl}_m$-modules. Here each of the N tensor factors $\mathbb{C}^m$ is a copy of the natural $\mathfrak{gl}_m$-module. We shall use the indices $1, \ldots, N$ to label these N tensor factors. For any index $p = 1, \ldots, N$ we will denote by $E_{ab}^{(p)}$ the operator on the vector space $(\mathbb{C}^m)^{\otimes N}$ acting as

$$\mathrm{id}^{\otimes(p-1)} \otimes E_{ab} \otimes \mathrm{id}^{\otimes(N-p)}. \tag{1.6}$$

Proposition 1.1. (i) *By using the $\mathfrak{gl}_m$-module structure of U, an action of the algebra $\mathfrak{H}_N$ on the vector space $(\mathbb{C}^m)^{\otimes N} \otimes U$ is defined as follows: the symmetric group $\mathfrak{S}_N \subset \mathfrak{H}_N$ acts by permutations of the N tensor factors $\mathbb{C}^m$, and the element $z_p \in \mathfrak{H}_N$ with $p = 1, \ldots, N$ acts as*

$$\sum_{a,b=1}^{m} E_{ab}^{(p)} \otimes E_{ba}. \tag{1.7}$$

(ii) *This action of $\mathfrak{H}_N$ commutes with the diagonal action of $\mathfrak{gl}_m$ on $(\mathbb{C}^m)^{\otimes N} \otimes U$.*

For a proof of this proposition see [7, Sect. 1]. By using Proposition 1.1 we obtain a functor $\mathcal{E}_N : U \mapsto (\mathbb{C}^m)^{\otimes N} \otimes U$ from the category of all $\mathfrak{gl}_m$-modules to the category of bimodules over $\mathfrak{gl}_m$ and $\mathfrak{H}_N$. We will also use a version of this proposition for the special linear Lie algebra $\mathfrak{sl}_m$ instead of $\mathfrak{gl}_m$. Let us denote $I = E_{11} + \cdots + E_{mm}$ so that $\mathfrak{gl}_m = \mathfrak{sl}_m \oplus \mathbb{C} I$. Moreover, we have

$$\sum_{a,b=1}^{m} E_{ab} \otimes E_{ba} \in \frac{1}{m} I \otimes I + \mathfrak{sl}_m \otimes \mathfrak{sl}_m. \tag{1.8}$$

Hence an action of

$$\sum_{a,b=1}^{m} E_{ab}^{(p)} \otimes E_{ba} - \frac{1}{m} \mathrm{id}^{\otimes N} \otimes I \tag{1.9}$$

can be defined on the vector space $(\mathbb{C}^m)^{\otimes N} \otimes U$ by using only the $\mathfrak{sl}_m$-module structure of U. Because the element $I \in \mathfrak{gl}_m$ is central, the operators (1.9) with $p = 1, \ldots, N$ satisfy the same commutation relations (1.4) as the operators (1.7) respectively instead of $z_1, \ldots, z_N$.

Corollary 1.2. (i) *By using the $\mathfrak{sl}_m$-module structure of U, an action of the algebra $\mathfrak{H}_N$ on the vector space $(\mathbb{C}^m)^{\otimes N} \otimes U$ is defined as follows: the group $\mathfrak{S}_N \subset \mathfrak{H}_N$ acts by permutations of the N tensor factors $\mathbb{C}^m$, and the element $z_p \in \mathfrak{H}_N$ with $p = 1, \ldots, N$ acts as (1.9).*
(ii) *This action of $\mathfrak{H}_N$ commutes with the diagonal action of $\mathfrak{sl}_m$ on $(\mathbb{C}^m)^{\otimes N} \otimes U$.*

Using Corollary 1.2 we get a functor $\mathcal{F}_N : U \mapsto (\mathbb{C}^m)^{\otimes N} \otimes U$ from the category of all $\mathfrak{sl}_m$-modules to the category of bimodules over $\mathfrak{sl}_m$ and $\mathfrak{H}_N$. Our principal

tool will be an analogue of this functor for the affine Lie algebra $\widehat{\mathfrak{sl}}_m$ instead of $\mathfrak{sl}_m$. The role of the degenerate affine algebra $\mathfrak{H}_N$ will be then played by the trigonometric Cherednik algebra $\mathfrak{C}_N$.

1.3. Consider the *triangular decomposition* of the Lie algebra $\mathfrak{gl}_m$,

$$\mathfrak{gl}_m = \mathfrak{n} \oplus \mathfrak{t} \oplus \mathfrak{n}'.$$

Here $\mathfrak{t}$ is the Cartan subalgebra of $\mathfrak{gl}_m$ with the basis vectors $E_{11}, \ldots, E_{mm}$. Every element of the vector space $\mathfrak{t}^*$ dual to $\mathfrak{t}$ is called a *weight* of $\mathfrak{gl}_m$. We will regard any sequence $\mu = (\mu_1, \ldots, \mu_m)$ of length m of complex numbers as such a weight, by setting $\mu(E_{aa}) = \mu_a$ for $a = 1, \ldots, m$. For any $\mathfrak{gl}_m$-module U, its subspace consisting of all vectors of weight μ is denoted by U^μ. In the above display $\mathfrak{n}$ is the nilpotent subalgebra of $\mathfrak{gl}_m$ spanned by all the elements E_{ab} with $a > b$, while $\mathfrak{n}'$ is spanned by all E_{ab} with $a < b$. We will denote by $U_\mathfrak{n}$ the vector space $U/\mathfrak{n}\,U$ of the *coinvariants* of the action of the subalgebra $\mathfrak{n}$ on U. Note that the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{gl}_m$ acts on the vector space $U_\mathfrak{n}$.

Consider the Verma module M_μ of the Lie algebra $\mathfrak{gl}_m$. It can be described as the quotient of the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_m)$ by the left ideal generated by all the elements E_{ab} with $a < b$ and by the elements $E_{aa} - \mu_a$. The elements of the Lie algebra $\mathfrak{gl}_m$ act on this quotient via left multiplication. Let us apply the functor $\mathcal{E}_N$ to the $\mathfrak{gl}_m$-module $U = M_\mu$. By using Proposition 1.2 we obtain a bimodule $\mathcal{E}_N(M_\mu)$ of $\mathfrak{gl}_m$ and $\mathfrak{H}_N$. For any $\lambda = (\lambda_1, \ldots, \lambda_m)$ consider the space $\mathcal{E}_N(M_\mu)_\mathfrak{n}^\lambda$ of those coinvariants of this bimodule relative to $\mathfrak{n}$ which are of the weight λ. This space comes with an action of the algebra $\mathfrak{H}_N$.

Proposition 1.3. *The module $\mathcal{E}_N(M_\mu)_\mathfrak{n}^\lambda$ of the algebra $\mathfrak{H}_N$ is isomorphic to the standard module S_μ^λ.*

Proof. By repeatedly using [8, Thm. 1.3] the proof reduces to its particular case when $m = 1$. In the latter case the proposition is immediate. $\square$

Let us now give a counterpart [1, Prop. 5.3.1] of Proposition 1.3 for the Lie algebra $\mathfrak{sl}_m$ instead of $\mathfrak{gl}_m$. Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{sl}_m$ with the basis vectors $E_{11} - E_{22}, \ldots, E_{m-1,m-1} - E_{mm}$. Note that $\mathfrak{h} \subset \mathfrak{t}$. We have the triangular decomposition

$$\mathfrak{sl}_m = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}'$$

where $\mathfrak{n}$ and $\mathfrak{n}'$ are the same as above. We will denote respectively by α and β the restrictions of the weights λ and μ of $\mathfrak{gl}_m$ to the subspace $\mathfrak{h} \subset \mathfrak{t}$. Thus α and β will be weights of $\mathfrak{sl}_m$. Note that the restriction of the $\mathfrak{gl}_m$-module M_μ to the subalgebra $\mathfrak{sl}_m \subset \mathfrak{gl}_m$ is isomorphic to the Verma module M_β, while the central element $I \in \mathfrak{gl}_m$ acts on M_μ as multiplication by $\mu_1 + \cdots + \mu_m$. Therefore by using the definition of $\mathcal{F}_N$ we get a corollary to Proposition 1.3.

Corollary 1.4. *The $\mathfrak{H}_N$-module $\mathcal{F}_N(M_\beta)_\mathfrak{n}^\alpha$ is isomorphic to the pullback of the standard module S_μ^λ relative to the automorphism (1.5) of $\mathfrak{H}_N$ where*

$$f = -(\mu_1 + \cdots + \mu_m)/m.$$

Note that by pulling the standard module S_μ^λ back through the automorphism (1.5) with any f we get another standard module, corresponding to the sequences $(\lambda_1 + f, \ldots, \lambda_m + f)$ and $(\mu_1 + f, \ldots, \mu_m + f)$ instead of λ and μ. However, we will use Corollary 1.4 as stated.

2. Cherednik algebras

2.1. Let $\mathfrak{P}_N = \mathbb{C}[x_1, x_1^{-1}, \ldots, x_N, x_N^{-1}]$ be the ring of of Laurent polynomials in N variables $x_1, \ldots, x_N$ with complex coefficients. We will denote by $\partial_1, \ldots, \partial_N$ the derivation operators in $\mathfrak{P}_N$ relative to these variables. The *trigonometric Cherednik algebra* $\mathfrak{C}_N$ depending on a parameter $\kappa \in \mathbb{C}$ is the complex associative algebra generated by $\mathfrak{H}_N$ and $\mathfrak{P}_N$, subject to the relations $\sigma\, x_p\, \sigma^{-1} = x_{\sigma(p)}$ for all $\sigma \in \mathfrak{S}_N$ and to the commutation relations

$$[\,z_p, x_q\,] = -\, x_p\, \sigma_{pq} \quad \text{for} \quad q \neq p;$$
$$[\,z_p, x_p\,] = \kappa\, x_p + \sum_{r \neq p} x_p\, \sigma_{pr}.$$

We can also employ the pairwise commuting generators $u_1, \ldots, u_N \in \mathfrak{H}_N$ instead of $z_1, \ldots, z_N$; see (1.2). Then instead of the above displayed relations in $\mathfrak{C}_N$ we get

$$[\,u_p, x_q\,] = -\, x_q\, \sigma_{pq} \quad \text{for} \quad q < p;$$
$$[\,u_p, x_q\,] = -\, x_p\, \sigma_{pq} \quad \text{for} \quad q > p;$$
$$[\,u_p, x_p\,] = \kappa\, x_p + \sum_{r < p} x_r\, \sigma_{pr} + \sum_{r > p} x_p\, \sigma_{pr}.$$

The latter set of defining relations shows that (1.5) extends to an automorphism of the algebra $\mathfrak{C}_N$ identical in the subalgebras $\mathbb{C}\mathfrak{S}_N$ and $\mathfrak{P}_N$. By [6, Thm. 1.3] multiplication in the algebra $\mathfrak{C}_N$ yields a bijective linear map

$$\mathfrak{P}_N \otimes \mathbb{C}\mathfrak{S}_N \otimes \mathbb{C}[u_1, \ldots, u_N] \to \mathfrak{C}_N.$$

Next we will state the generalizations of Proposition 1.1 and Corollary 1.2 to $\mathfrak{C}_N$. They go back to the work of Cherednik [3].

2.2. First consider the affine Lie algebra $\widehat{\mathfrak{gl}}_m$ over the field $\mathbb{C}$. We will define it as a central extension of the *current Lie algebra* $\mathfrak{gl}_m[t, t^{-1}]$ by a one-dimensional complex vector space with a fixed basis element which will be denoted by C. Here t is a formal variable. Choose the basis of $\mathfrak{gl}_m[t, t^{-1}]$ consisting of the elements $E_{cd}\, t^j$ where $c, d = 1, \ldots, m$ whereas j ranges over $\mathbb{Z}$. The commutators in the Lie algebra $\mathfrak{gl}_m[t, t^{-1}]$ are taken pointwise so that

$$[\,E_{ab}\, t^i, E_{cd}\, t^j\,] = (\delta_{bc} E_{ad} - \delta_{da}\, E_{cb})\, t^{i+j}$$

for the basis elements. In the extended Lie algebra $\widehat{\mathfrak{gl}}_m$ we have the relations

$$[\,E_{ab}\, t^i, E_{cd}\, t^j\,] = (\delta_{bc}\, E_{ad} - \delta_{da}\, E_{cb})\, t^{i+j} + i\, \delta_{i,-j}\, \delta_{bc}\, \delta_{da}\, C. \tag{2.1}$$

We will also work with the affine Lie algebra $\widehat{\mathfrak{sl}}_m$. This is a subalgebra of $\widehat{\mathfrak{gl}}_m$ spanned by the subspace $\mathfrak{sl}_m[t, t^{-1}] \subset \mathfrak{gl}_m[t, t^{-1}]$ and by the central element C. Let $\widehat{\mathfrak{h}}$ be the Abelian subalgebra of $\widehat{\mathfrak{sl}}_m$ spanned by C and by the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{sl}_m$. The vector spaces

$$\widehat{\mathfrak{n}} = \mathfrak{n} \oplus t^{-1}\,\mathfrak{sl}_m[t^{-1}] \quad \text{and} \quad \widehat{\mathfrak{n}}' = \mathfrak{n}' \oplus t\,\mathfrak{sl}_m[t]$$

are also Lie subalgebras of $\widehat{\mathfrak{sl}}_m$ by the relations (2.1). As a vector space,

$$\widehat{\mathfrak{sl}}_m = \widehat{\mathfrak{n}} \oplus \widehat{\mathfrak{h}} \oplus \widehat{\mathfrak{n}}'.$$

Let V be any module of $\widehat{\mathfrak{gl}}_m$ such that for any given vector in V, there exists a degree i such that the subspace $t^i\,\mathfrak{gl}_m[t] \subset \widehat{\mathfrak{gl}}_m$ annihilates the vector. Consider the vector space

$$W = \mathfrak{P}_N \otimes (\mathbb{C}^m)^{\otimes N} \otimes V. \tag{2.2}$$

Due to our condition on V for any $p = 1, \ldots, N$ there is a well-defined linear operator on W

$$\sum_{i=0}^{\infty} \sum_{a,b=1}^{m} x_p^{-i} \otimes E_{ab}^{(p)} \otimes E_{ba}\, t^i. \tag{2.3}$$

Here $E_{ab}^{(p)}$ is the operator (1.6) acting on $(\mathbb{C}^m)^{\otimes N}$. Further, the symmetric group $\mathfrak{S}_N$ acts on the tensor factor $\mathfrak{P}_N$ of W by permutations of the variables $x_1, \ldots, x_N$. There is another copy of the group $\mathfrak{S}_N$ acting on the N tensor factors $\mathbb{C}^m$ of W by permutation. Using these two actions of $\mathfrak{S}_N$ for any $p = 1, \ldots, N$ introduce the *Cherednik operator* on W

$$\kappa\, x_p\, \partial_p \otimes \mathrm{id}^{\otimes N} \otimes \mathrm{id} + \sum_{r \neq p} \frac{x_p}{x_p - x_r} (1 - \sigma_{pr}) \otimes \sigma_{pr} \otimes \mathrm{id}$$
$$+ \sum_{i=0}^{\infty} \sum_{a,b=1}^{m} x_p^{-i} \otimes E_{ab}^{(p)} \otimes E_{ba}\, t^i. \tag{2.4}$$

The vector space $\mathfrak{P}_N \otimes (\mathbb{C}^m)^{\otimes N}$ can be naturally identified with the tensor product of N copies of the space $\mathbb{C}^m[t, t^{-1}]$. The latter space can be regarded as a $\widehat{\mathfrak{gl}}_m$-module where the central element C acts as zero. By taking the tensor product of N copies of this module with V we turn the vector space W to a $\widehat{\mathfrak{gl}}_m$-module. The element $E_{cd}\, t^j \in \widehat{\mathfrak{gl}}_m$ acts on W as

$$\mathrm{id} \otimes \mathrm{id}^{\otimes N} \otimes E_{cd}\, t^j + \sum_{q=1}^{N} x_q^j \otimes E_{cd}^{(q)} \otimes \mathrm{id}. \tag{2.5}$$

For any complex number ℓ, a module of the Lie algebra $\widehat{\mathfrak{gl}}_m$ or $\widehat{\mathfrak{sl}}_m$ is said to be of *level* ℓ if the element C acts on this module as that complex number. In particular, the $\widehat{\mathfrak{gl}}_m$-module $\mathbb{C}^m[t, t^{-1}]$ used above is of level zero. We can now state the main properties of Cherednik operators on W from [1], [14]. These properties immediately follow from [8, Prop. 2.3].

Proposition 2.1. (i) *By using the $\widehat{\mathfrak{gl}}_m$-module structure on V, an action of the algebra $\mathfrak{C}_N$ on the vector space W is defined as follows: the elements $x_p, x_p^{-1} \in \mathfrak{C}_N$ act via mutiplication in $\mathbb{C}[x_1, x_1^{-1}, \ldots, x_N, x_N^{-1}]$, the group $\mathfrak{S}_N \subset \mathfrak{H}_N$ acts by simultaneous permutations of the variables $x_1, \ldots, x_N$ and of the N tensor factors $\mathbb{C}^m$, and the element $z_p \in \mathfrak{H}_N$ acts as (2.4).*
(ii) *This action of $\mathfrak{C}_N$ on W commutes with that of the Lie subalgebra $\mathfrak{gl}_m \subset \widehat{\mathfrak{gl}}_m$.*
(iii) *If V has level $\kappa - m$ then the action of $\mathfrak{C}_N$ preserves the subspace $\widehat{\mathfrak{n}} \, W \subset W$.*

Below is a version of this proposition in the case when V is a module not of $\widehat{\mathfrak{gl}}_m$ but only of $\widehat{\mathfrak{sl}}_m$, also due to [1]. There for any vector in V we assume the existence of i such that the subspace $t^i \, \mathfrak{sl}_m[t] \subset \widehat{\mathfrak{sl}}_m$ annihilates the vector. Let $I = E_{11} + \cdots + E_{mm}$ as before. By (1.8) an action of

$$\sum_{i=0}^{\infty} x_p^{-i} \otimes \left(\sum_{a,b=1}^{m} E_{ab}^{(p)} \otimes E_{ba} \, t^i - \frac{1}{m} \, \mathrm{id}^{\otimes N} \otimes I \, t^i \right) \tag{2.6}$$

can be defined on (2.2) by using only the $\widehat{\mathfrak{sl}}_m$-module structure of V. Then for every $p = 1, \ldots, N$ we have a modification of the Cherednik operator (2.4) on W,

$$\kappa \, x_p \, \partial_p \otimes \mathrm{id}^{\otimes N} \otimes \mathrm{id} + \sum_{r \neq p} \frac{x_p}{x_p - x_r} \, (1 - \sigma_{pr}) \otimes \sigma_{pr} \otimes \mathrm{id}$$
$$+ \sum_{i=0}^{\infty} x_p^{-i} \otimes \left(\sum_{a,b=1}^{m} E_{ab}^{(p)} \otimes E_{ba} \, t^i - \frac{1}{m} \, \mathrm{id}^{\otimes N} \otimes I \, t^i \right). \tag{2.7}$$

Here we use the sum (2.6) instead of (2.3) used in (2.4). Further, we can turn the vector space (2.2) into another $\widehat{\mathfrak{sl}}_m$-module by regarding $\mathfrak{P}_N \otimes (\mathbb{C}^m)^{\otimes N}$ as $\widehat{\mathfrak{sl}}_m$-module of level zero.

Corollary 2.2. (i) *Using the $\widehat{\mathfrak{sl}}_m$-module structure on V, an action of the algebra $\mathfrak{C}_N$ on the vector space W can be defined as follows: the elements $x_p, x_p^{-1} \in \mathfrak{C}_N$ act via mutiplication in $\mathbb{C}[x_1, x_1^{-1}, \ldots, x_N, x_N^{-1}]$, the group $\mathfrak{S}_N \subset \mathfrak{H}_N$ acts by simultaneous permutations of the variables $x_1, \ldots, x_N$ and of the N tensor factors $\mathbb{C}^m$, and the element $z_p \in \mathfrak{H}_N$ acts as (2.7).*
(ii) *This action of $\mathfrak{C}_N$ on W commutes with that of the Lie subalgebra $\mathfrak{sl}_m \subset \widehat{\mathfrak{sl}}_m$.*
(iii) *If V has level $\kappa - m$ then the action of $\mathfrak{C}_N$ preserves the subspace $\widehat{\mathfrak{n}} \, W \subset W$.*

Using Corollary 2.2(i) and the definition (2.2), we get a functor $\mathcal{A}_N : V \mapsto W$ from the category of all $\widehat{\mathfrak{sl}}_m$-modules satisfying the annihilation condition stated just before (2.6). Note that the resulting actions of $\widehat{\mathfrak{sl}}_m$ and $\mathfrak{C}_N$ on W do not commute in general. However, this will be our analogue for $\widehat{\mathfrak{sl}}_m$ of the functor $\mathcal{F}_N$ introduced in the end of Subsection 1.2.

2.3. Let λ and μ be same sequences of length m of complex numbers as in Subsection 1.1. Take the standard module S_μ^λ over the algebra $\mathfrak{H}_N$. By regarding $\mathfrak{H}_N$ as a subalgebra of $\mathfrak{C}_N$, consider the induced module

$$\mathrm{Ind}_{\mathfrak{H}_N}^{\mathfrak{C}_N} \, S_\mu^\lambda.$$

Denote the latter module by $\widehat{S}_\mu^\lambda$. Its underlying vector space can be identified with that of $\mathfrak{P}_N \otimes S_\mu^\lambda$ whereon the subalgebra $\mathfrak{P}_N \subset \mathfrak{C}_N$ acts via multiplication in the first tensor factor. Notice that by transitivity of induction and by the definition of S_μ^λ, the $\mathfrak{C}_N$-module $\widehat{S}_\mu^\lambda$ is isomorphic to

$$\operatorname{Ind}_{\mathfrak{H}_\nu}^{\mathfrak{C}_N} S_{\mu_1}^{\lambda_1} \otimes S_{\mu_2-1}^{\lambda_2-1} \otimes \cdots \otimes S_{\mu_m-m+1}^{\lambda_m-m+1}.$$

Now suppose that $\ell = \kappa - m$, so that our Corollary 2.2(iii) applies. Regard λ and μ as weights of $\mathfrak{gl}_m$. Their restrictions to the subspace $\mathfrak{h} \subset \mathfrak{t}$ are denoted by α and β respectively. Define the weight $\widehat{\beta}$ of $\widehat{\mathfrak{sl}}_m$ as the element of the space dual to $\widehat{\mathfrak{h}}$ such that

$$\widehat{\beta}(C) = \ell \quad \text{and} \quad \widehat{\beta}(X) = \beta(X) \quad \text{for all} \quad X \in \widehat{\mathfrak{h}}. \tag{2.8}$$

Consider the Verma module $M_{\widehat{\beta}}$ of $\widehat{\mathfrak{sl}}_m$. By definition, this is the quotient of the universal enveloping algebra $\operatorname{U}(\widehat{\mathfrak{sl}}_m)$ by the left ideal generated by $\widehat{\mathfrak{n}}'$ and by all the elements $X - \widehat{\beta}(X)$ where X ranges over $\widehat{\mathfrak{h}}$. Since the element $C \in \widehat{\mathfrak{sl}}_m$ is central, the first equality in (2.8) implies that the $\widehat{\mathfrak{sl}}_m$-module $M_{\widehat{\beta}}$ is of level ℓ. Moreover, $V = M_{\widehat{\beta}}$ satisfies the annihilation condition stated just before (2.6). Therefore we can apply the functor $\mathcal{A}_N$ to this V.

Further, let us define the weight $\widehat{\alpha}$ of $\widehat{\mathfrak{sl}}_m$ similarly to $\widehat{\beta}$ and consider the space $\mathcal{A}_N(M_{\widehat{\beta}})_{\widehat{\mathfrak{n}}}^{\widehat{\alpha}}$ of those coinvariants of $\mathcal{A}_N(M_{\widehat{\beta}})$ relative to $\widehat{\mathfrak{n}}$ which have the weight $\widehat{\alpha}$. This space comes with an action of the algebra $\mathfrak{C}_N$ due to Corollary 2.2(iii). The latter action is described by the next proposition [1, Prop. 5.2.3]. The proof given in [1] was different, however.

Proposition 2.3. *The $\mathfrak{H}_N$-module $\mathcal{A}_N(M_{\widehat{\beta}})_{\widehat{\mathfrak{n}}}^{\widehat{\alpha}}$ is isomorphic to the pullback of $\widehat{S}_\mu^\lambda$ relative to the automorphism (1.5) of the algebra $\mathfrak{C}_N$ where*

$$f = -(\mu_1 + \cdots + \mu_m)/m.$$

Proof. By the transitivity of induction, the module $M_{\widehat{\beta}}$ of the Lie algebra $\widehat{\mathfrak{sl}}_m$ is isomorphic to the module of level ℓ parabolically induced from the Verma module M_β of $\mathfrak{sl}_m$. To define the parabolically induced module, we first extend the action of $\mathfrak{sl}_m$ on M_β to the subalgebra $\mathfrak{p}$ of $\widehat{\mathfrak{sl}}_m$ spanned by $\mathfrak{sl}_m[t]$ and C. Namely, we let the elements of $t\,\mathfrak{sl}_m[t]$ act on M_β as zero, while C acts as multiplication by ℓ. Then we induce the resulting action from $\mathfrak{p}$ to $\widehat{\mathfrak{sl}}_m$.

Denote by $\mathfrak{q}$ the subspace $t^{-1}\,\mathfrak{sl}_m[t^{-1}] \subset \mathfrak{sl}_m[t,t^{-1}]$. This is a Lie subalgebra of $\widehat{\mathfrak{n}}$, and moreover $\widehat{\mathfrak{n}} = \mathfrak{n} \oplus \mathfrak{q}$ as a vector space. Consider the space of coinvariants of $\mathcal{A}_N(M_{\widehat{\beta}})$ relative to $\mathfrak{q}$. This space comes with mutually commuting actions of $\mathfrak{C}_N$ and $\mathfrak{sl}_m$; see again Corollary 2.2. By [8, Thm. 2.5] the so obtained bimodule of $\mathfrak{C}_N$ and $\mathfrak{sl}_m$ is isomorphic to

$$\operatorname{Ind}_{\mathfrak{H}_N}^{\mathfrak{C}_N} \mathcal{F}_N(M_\beta).$$

Proposition 2.3 now follows from Corollary 1.4 and from the definition of $\widehat{S}_\mu^\lambda$. $\square$

2.4. Denote by $\mathfrak{T}_m$ the *affine Weyl group* of the Lie algebra $\mathfrak{gl}_m$. This group is generated by the elements τ_c where $c = 0, 1, \ldots, m - 1$. However, we will let the indices of the generators τ_c run through $\mathbb{Z}$, assuming that $\tau_{c+m} = \tau_c$ for $c \in \mathbb{Z}$. Then the defining relations of $\mathfrak{T}_m$ are

$$\tau_c^2 = 1; \quad \tau_c\,\tau_{c+1}\,\tau_c = \tau_{c+1}\,\tau_c\,\tau_{c+1}; \quad \tau_c\,\tau_d = \tau_d\,\tau_c \quad \text{for} \quad c - d \neq \pm 1 \bmod m.$$

The corresponding *extended affine Weyl group* is generated by $\mathfrak{T}_m$ and an element π such that

$$\pi\,\tau_c = \tau_{c+1}\,\pi.$$

Let us denote the extended group by $\mathfrak{R}_m$. The group $\mathfrak{R}_m$ acts on the set $\mathbb{Z}$ by permutations of period m. Namely, each generator τ_c of $\mathfrak{T}_m$ exchanges $c + j\,m$ with $c + 1 + j\,m$ for each $j \in \mathbb{Z}$, leaving all other integers fixed. The extra generator π maps any integer d to $d + 1$.

The group $\mathfrak{R}_m$ is $\mathbb{Z}$-graded so that the element π has degree one, while all elements of $\mathfrak{T}_m$ have degree zero. Further, the group $\mathfrak{R}_m$ is isomorphic to the semidirect product $\mathfrak{S}_m \ltimes \mathbb{Z}^m$. We will use the isomorphism $\mathfrak{R}_m \to \mathfrak{S}_m \ltimes \mathbb{Z}^m$ defined by mapping

$$\pi \mapsto (1, 0, \ldots, 0)\,\sigma_1 \ldots \sigma_{m-1} \quad \text{and} \quad \tau_0 \mapsto (1, 0, \ldots, 0, -1)\,\sigma_{1m}$$

while $\tau_c \mapsto \sigma_c$ for $c = 1, \ldots, m - 1$. Here $\mathfrak{S}_m$ and $\mathbb{Z}^m$ are regarded as subgroups of $\mathfrak{S}_m \ltimes \mathbb{Z}^m$. In particular, here $(1, 0, \ldots, 0)$ and $(1, 0, \ldots, 0, -1)$ are elements of $\mathbb{Z}^m \subset \mathfrak{S}_m \ltimes \mathbb{Z}^m$. Via this isomorphism, the $\mathbb{Z}$-grading on $\mathfrak{R}_m$ defined here corresponds to that on $\mathfrak{S}_m \ltimes \mathbb{Z}^m$ as defined in the Introduction. Relative to the latter $\mathbb{Z}$-grading, the degree of any element of $\mathfrak{S}_m$ is zero, while the degree of any element of $\mathbb{Z}^m$ is the sum of its m components.

The group $\mathfrak{R}_m$ acts by automorphisms of the Lie algebra $\widehat{\mathfrak{gl}}_m$ so that the central element C is invariant,

$$\tau_c : E_{ab}\,t^i \mapsto E_{\tau_c(a), \tau_c(b)}\,t^i \quad \text{for} \quad c = 1, \ldots, m - 1,$$

while

$$\tau_0 : E_{ab}\,t^i \mapsto E_{\tau_0(a), \tau_0(b)}\,t^{i + \delta_{a1} - \delta_{am} - \delta_{b1} + \delta_{bm}} + \delta_{i0}\,\delta_{ab}\,(\delta_{a1} - \delta_{am})\,C$$

and

$$\pi : E_{ab}\,t^i \mapsto E_{a+1, b+1}\,t^{i - \delta_{am} + \delta_{bm}} - \delta_{i0}\,\delta_{ab}\,\delta_{am}\,C.$$

Then

$$\pi^{-1} : E_{ab}\,t^i \mapsto E_{a-1, b-1}\,t^{i + \delta_{a1} - \delta_{b1}} + \delta_{i0}\,\delta_{ab}\,\delta_{a1}\,C.$$

Here we let $a, b = 1, \ldots, m$. If any of the indices of the matrix units appearing in the last three displayed formulas is 0 or $m + 1$, it should be then replaced respectively by m or 1.

Take the level zero module $\mathbb{C}^m[t, t^{-1}]$ of $\widehat{\mathfrak{gl}}_m$. Let $e_1, \ldots, e_m$ be the standard basis vectors of $\mathbb{C}^m$. The group $\mathfrak{R}_m$ acts on the vector space $\mathbb{C}^m[t, t^{-1}]$ so that

$$\tau_c : e_a \, t^{\,i} \mapsto e_{\tau_c(a)} \, t^{\,i} \quad \text{for} \quad c = 1, \ldots, m-1$$

while

$$\tau_0 : e_a \, t^{\,i} \mapsto e_{\tau_0(a)} \, t^{\,i+\delta_{a1}-\delta_{am}} \quad \text{and} \quad \pi : e_a \, t^{\,i} \mapsto e_{a+1} \, t^{\,i-\delta_{am}}.$$

Then

$$\pi^{-1} : e_a \, t^{\,i} \mapsto e_{a-1} \, t^{\,i+\delta_{a1}}.$$

Here we use the same interpretation of the indices of the standard basis vectors of $\mathbb{C}^m$ as of the indices of the matrix units above. One can easily verify that the actions of $\widehat{\mathfrak{gl}}_m$ and of $\mathfrak{R}_m$ on $\mathbb{C}^m[t, t^{-1}]$ extend to an action of the crossed product algebra $\mathfrak{R}_m \ltimes \mathrm{U}(\widehat{\mathfrak{gl}}_m)$. This algebra is defined by the above described action of the group $\mathfrak{R}_m$ on $\widehat{\mathfrak{gl}}_m$. In the crossed product algebra,

$$\pi \, X \pi^{-1} = \pi(X) \quad \text{for} \quad X \in \widehat{\mathfrak{gl}}_m.$$

2.5. Suppose that the $\widehat{\mathfrak{gl}}_m$-module V is also equipped with an action of the extended affine Weyl group $\mathfrak{R}_m$. Moreover, suppose that the actions of both $\widehat{\mathfrak{gl}}_m$ and $\mathfrak{R}_m$ on V extend to an action of the crossed product algebra $\mathfrak{R}_m \ltimes \mathrm{U}(\widehat{\mathfrak{gl}}_m)$. By identifying the tensor product of N copies of $\mathbb{C}^m[t, t^{-1}]$ with $\mathfrak{P}_N \otimes (\mathbb{C}^m)^{\otimes N}$ we define an action of the group $\mathfrak{R}_m$ on the latter vector space, and hence on its tensor product (2.2) with V.

By the definition given in Subsection 2.4, the action of the element $\pi \in \mathfrak{R}_m$ on the Lie algebra $\widehat{\mathfrak{gl}}_m$ preserves the subalgebra $\widehat{\mathfrak{n}}$. Therefore the element π acts on the space $W_{\widehat{\mathfrak{n}}}$ of $\widehat{\mathfrak{n}}$-coinvariants of the $\widehat{\mathfrak{gl}}_m$-module W. On the other hand, under the assumption $\ell = \kappa - m$, the Cherednik operator (2.4) also acts on $W_{\widehat{\mathfrak{n}}}$ due to Proposition 2.1. Let us denote by ζ_p the operator on $W_{\widehat{\mathfrak{n}}}$ corresponding to (2.4). The next property of ζ_p will be crucial for us.

Proposition 2.4. *If the $\widehat{\mathfrak{gl}}_m$-module V has level $\kappa - m$ then for $p = 1, \ldots, N$ we have an equality of operators on $W_{\widehat{\mathfrak{n}}}$*

$$\pi \, \zeta_p \, \pi^{-1} = \zeta_p + \mathrm{id}.$$

Proof. Extend the vector space W in (2.2) by replacing its first tensor factor $\mathfrak{P}_N$ by the space of all complex valued rational functions in the variables $x_1, \ldots, x_N$ with the permutation action of the symmetric group $\mathfrak{S}_N$. Extend the action of the element π on W accordingly. To this end, identify the tensor product $\mathfrak{P}_N \otimes (\mathbb{C}^m)^{\otimes N}$ in (2.2) with the tensor product of N copies of $\mathbb{C}^m[t, t^{-1}]$ as above. Then restate the definition of the action of π on the vector space $\mathbb{C}^m[t, t^{-1}]$ by regarding the latter as the tensor product $\mathbb{C}^m \otimes \mathbb{C}[t, t^{-1}]$. Here we also use the given action of the group element $\pi \in \mathfrak{R}_m$ on the tensor factor V of (2.2).

For $p = 1, \ldots, N$ consider the following operators on the extended vector space,

$$D_p = \kappa \, x_p \, \partial_p \otimes \mathrm{id}^{\otimes N} \otimes \mathrm{id},$$

$$R_p = \sum_{r \neq p} \frac{x_p}{x_p - x_r} \, \sigma_{pr} \otimes \sigma_{pr} \otimes \mathrm{id},$$

$$T_p = \sum_{r \neq p} \frac{x_p}{x_p - x_r} \otimes \sigma_{pr} \otimes \mathrm{id} \; + \sum_{i=0}^{\infty} \sum_{a,b=1}^{m} x_p^{-i} \otimes E_{ab}^{(p)} \otimes E_{ba} \, t^i.$$

Then (2.4) is the restriction of the operator $D_p - R_p + T_p$ to the space (2.2).

By identifying $\mathfrak{P}_N \otimes (\mathbb{C}^m)^{\otimes N}$ with the tensor product of N copies of $\mathbb{C}^m[t, t^{-1}]$ and using the action of the element π on the pth of these N copies as defined in Subsection 2.4,

$$\pi \, D_p \, \pi^{-1} = D_p + \mathrm{id} \otimes \kappa \, E_{11}^{(p)} \otimes \mathrm{id}. \tag{2.9}$$

The action of π on the tensor product $\mathfrak{P}_N \otimes (\mathbb{C}^m)^{\otimes N}$ commutes with the multiplication by any element of $\mathfrak{P}_N$ in the first tensor factor. It also commutes with the operator $\sigma_{pr} \otimes \sigma_{pr}$ for any $r \neq p$. Therefore

$$\pi \, R_p \, \pi^{-1} = R_p.$$

Consider the operator T_p. In its definition, the summand corresponding to any $r \neq p$ can be rewritten as

$$\sum_{i=0}^{\infty} \sum_{a,b=1}^{m} x_p^{-i} x_r^{i} \otimes E_{ab}^{(p)} E_{ba}^{(r)} \otimes \mathrm{id}.$$

Hence $\pi \, T_p \, \pi^{-1}$ equals the sum over the indices $i = 0, 1, \ldots$ and $a, b = 1, \ldots, m$ of

$$\sum_{r \neq p} x_p^{-i - \delta_{am} + \delta_{bm}} x_r^{i + \delta_{am} - \delta_{bm}} \otimes E_{a+1,b+1}^{(p)} E_{b+1,a+1}^{(r)} \otimes \mathrm{id}$$

$$+ \, x_p^{-i - \delta_{am} + \delta_{bm}} \otimes E_{a+1,b+1}^{(p)} \otimes (E_{b+1,a+1} \, t^{i + \delta_{am} - \delta_{bm}} - \delta_{i0} \, \delta_{ab} \, \delta_{bm} \, \ell). \tag{2.10}$$

Here we use the action of $\pi \in \mathfrak{R}_m$ on $\mathrm{U}(\widehat{\mathfrak{gl}}_m)$ as defined in Subsection 2.4. By the definition of T_p, the sum over the indices i and a, b of the expressions displayed in the two lines (2.10) equals

$$T_p + \sum_{r \neq p} \sum_{a \neq m} x_p \, x_r^{-1} \otimes E_{a+1,1}^{(p)} E_{1,a+1}^{(r)} \otimes \mathrm{id} - \sum_{r \neq p} \sum_{b \neq m} \mathrm{id} \otimes E_{1,b+1}^{(p)} E_{b+1,1}^{(r)} \otimes \mathrm{id}$$

$$+ \sum_{a \neq m} x_p \otimes E_{a+1,1}^{(p)} \otimes E_{1,a+1} \, t^{-1} - \sum_{b \neq m} \mathrm{id} \otimes E_{1,b+1}^{(p)} \otimes E_{b+1,1} - \mathrm{id} \otimes \ell \, E_{11}^{(p)} \otimes \mathrm{id}.$$

By adding to this result the right-hand side of (2.9) and by subtracting R_p, we get back the Cherednik operator (2.4) plus the sum

$$\sum_{r \neq p} \sum_{a \neq 1} x_p \, x_r^{-1} \otimes E_{a1}^{(p)} E_{1a}^{(r)} \otimes \mathrm{id} - \sum_{r \neq p} \sum_{b \neq 1} \mathrm{id} \otimes E_{1b}^{(p)} E_{b1}^{(r)} \otimes \mathrm{id}$$

$$+ \sum_{a \neq 1} x_p \otimes E_{a1}^{(p)} \otimes E_{1a} \, t^{-1} - \sum_{b \neq 1} \mathrm{id} \otimes E_{1b}^{(p)} \otimes E_{b1} + \mathrm{id} \otimes m \, E_{11}^{(p)} \otimes \mathrm{id}. \tag{2.11}$$

Here we used the equality $\kappa - \ell = m$ and replaced the indices $a+1, b+1$ by a, b. The sum displayed in the two lines (2.11) can be rewritten as

$$\sum_{a \neq 1} \Big(\sum_{r=1}^{N} x_r^{-1} \otimes E_{1a}^{(r)} \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{id}^{\otimes N} \otimes E_{1a} t^{-1} \Big) \cdot x_p \otimes E_{a1}^{(p)} \otimes \mathrm{id}$$

$$+ \sum_{b=1}^{m} \mathrm{id} \otimes E_{bb}^{(p)} \otimes \mathrm{id} - \sum_{b \neq 1} \Big(\sum_{r=1}^{N} \mathrm{id} \otimes E_{b1}^{(r)} \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{id}^{\otimes N} \otimes E_{b1} \Big) \cdot \mathrm{id} \otimes E_{1b}^{(p)} \otimes \mathrm{id}.$$

Here the sum over $b = 1, \ldots, m$ is the identity operator on W. For any $a \neq 1$ the element $E_{1a} \, t^{-1} \in \widehat{\mathfrak{n}}$ acts on W as the sum in the brackets in the first of the last two displayed lines; see (2.5). Hence the whole expression displayed in the first line vanishes on the quotient $W_{\widehat{\mathfrak{n}}}$. Further, for any $b \neq 1$ the element $E_{b1} \in \widehat{\mathfrak{n}}$ acts on W as the sum in the brackets in the second of the last two displayed lines. Hence the whole expression displayed in the second line acts on $W_{\widehat{\mathfrak{n}}}$ as the identity operator. $\quad \square$

Below is a version of Proposition 2.4 in the case when V is a module not of $\widehat{\mathfrak{gl}}_m$ but only of $\widehat{\mathfrak{sl}}_m$. Here we regard W as $\widehat{\mathfrak{sl}}_m$-module, and use the action of the element $\pi \in \mathfrak{R}_m$ on the corresponding space $W_{\widehat{\mathfrak{n}}}$ of $\widehat{\mathfrak{n}}$-coinvariants. Under the assumption $\ell = \kappa - m$, the modified Cherednik operator (2.7) acts on $W_{\widehat{\mathfrak{n}}}$ due to Corollary 2.2. Let us denote by θ_p the operator on $W_{\widehat{\mathfrak{n}}}$ corresponding to (2.7).

Corollary 2.5. *If the $\widehat{\mathfrak{sl}}_m$-module V has level $\kappa - m$ then for $p = 1, \ldots, N$ we have an equality of operators on $W_{\widehat{\mathfrak{n}}}$*

$$\pi \, \theta_p \, \pi^{-1} = \theta_p + \frac{\kappa}{m} \, \mathrm{id}.$$

Proof. The modified Cherednik operator (2.7) is obtained by subtracting from (2.4) the sum

$$\frac{1}{m} \sum_{i=0}^{\infty} x_p^{-i} \otimes \mathrm{id}^{\otimes N} \otimes I \, t^i.$$

But the action of $\pi \in \mathfrak{R}_m$ on the latter sum amounts to subtracting from it the operator

$$\frac{1}{m} \, \mathrm{id} \otimes \mathrm{id}^{\otimes N} \otimes C;$$

see Subsection 2.4. Since the module V has level $\kappa - m$, Proposition 2.4 implies that

$$\pi \, \theta_p \, \pi^{-1} = \theta_p + \frac{\kappa - m}{m} \, \mathrm{id} + \mathrm{id} = \theta_p + \frac{\kappa}{m} \, \mathrm{id}. \quad \square$$

3. Zhelobenko operators

3.1. Let $\widehat{\mathfrak{t}}$ be the subalgebra of $\widehat{\mathfrak{gl}}_m$ with the basis vectors C and $E_{11}, \ldots, E_{mm}$. Note that $\widehat{\mathfrak{t}}$ contains the subalgebra $\mathfrak{h}$ of $\widehat{\mathfrak{sl}}_m$. Consider the action of the extended affine Weyl group $\mathfrak{R}_m$ on $\widehat{\mathfrak{gl}}_m$ defined in Subsection 2.4. This action preserves

the subalgebra $\hat{\mathfrak{t}} \subset \widehat{\mathfrak{gl}}_m$. By definition, we have $\pi(C) = C$ and $\tau(C) = C$ for all $\tau \in \mathfrak{T}_m$. Further, we have

$$\pi(E_{dd}) = E_{d+1,d+1} \quad \text{for} \quad 1 \leqslant d < m, \quad \pi(E_{mm}) = E_{11} - C,$$
$$\tau_0(E_{11}) = E_{mm} + C, \quad \tau_0(E_{dd}) = E_{dd} \quad \text{for} \quad 1 < d < m, \quad \tau_0(E_{mm}) = E_{11} - C,$$

while the generators $\tau_1, \ldots, \tau_{m-1}$ act on the basis vectors $E_{11}, \ldots, E_{mm}$ naturally, that is by transpositions of the indices $1, \ldots, m$. Note that here we also have

$$\pi^{-1}(E_{dd}) = E_{d-1,d-1} \quad \text{for} \quad 1 < d \leqslant m, \quad \pi^{-1}(E_{11}) = E_{mm} + C.$$

We will also use the action of the group $\mathfrak{R}_m$ on the vector space $\hat{\mathfrak{t}}^*$, dual to the above action on $\hat{\mathfrak{t}}$. To describe the dual action explicitly, let C^* and $E_{11}^*, \ldots, E_{mm}^*$ be the basis vectors of $\hat{\mathfrak{t}}^*$ dual to our chosen basis vectors of $\hat{\mathfrak{t}}$. Then

$$\pi(C^*) = C^* + E_{11}^* \quad \text{and} \quad \tau_0(C^*) = C^* + E_{11}^* - E_{mm}^*,$$
$$\pi(E_{dd}^*) = E_{d+1,d+1}^* \quad \text{for} \quad 1 \leqslant d < m, \quad \pi(E_{mm}^*) = E_{11}^*,$$
$$\tau_0(E_{11}^*) = E_{mm}^*, \quad \tau_0(E_{dd}^*) = E_{dd}^* \quad \text{for} \quad 1 < d < m, \quad \tau_0(E_{mm}^*) = E_{11}^*$$

while the generators $\tau_1, \ldots, \tau_{m-1}$ leave C^* invariant and act on the vectors $E_{11}^*, \ldots$ $\ldots, E_{mm}^*$ by transpositions of the indices $1, \ldots, m$.

Now for any given $\ell \in \mathbb{C}$ and for any weight $\mu \in \mathfrak{t}^*$ define an element $\widehat{\mu} \in \hat{\mathfrak{t}}^*$ by setting

$$\widehat{\mu}(C) = \ell \quad \text{and} \quad \widehat{\mu}(X) = \mu(X) \quad \text{for all} \quad X \in \mathfrak{t}.$$

Equivalently,

$$\widehat{\mu} = \ell\, C^* + \mu_1 E_{11}^* + \cdots + \mu_m E_{mm}^*.$$

Then

$$\pi(\widehat{\mu}) = \ell\, C^* + (\mu_m + \ell)E_{11}^* + \mu_1 E_{22}^* + \cdots + \mu_{m-1} E_{mm}^*,$$
$$\tau_0(\widehat{\mu}) = \ell\, C^* + (\mu_m + \ell)E_{11}^* + \mu_2 E_{22}^* + \cdots + \mu_{m-1} E_{m-1,m-1}^* + (\mu_1 - \ell)E_{mm}^*.$$

In particular, for any given $\ell \in \mathbb{C}$ the action of the group $\mathfrak{R}_m$ on $\hat{\mathfrak{t}}^*$ preserves the set of weights of the form $\widehat{\mu}$. So we get an action of $\mathfrak{R}_m$ on the set of sequences of length m of complex numbers,

$$\pi : (\mu_1, \ldots, \mu_m) \mapsto (\mu_m + \ell, \mu_1, \ldots, \mu_{m-1}),$$
$$\tau_0 : (\mu_1, \ldots, \mu_m) \mapsto (\mu_m + \ell, \mu_2, \ldots, \mu_{m-1}, \mu_1 - \ell),$$

while $\tau_1, \ldots, \tau_{m-1}$ naturally act on these sequences by transpositions of the indices $1, \ldots, m$. Note that via the isomorphism $\mathfrak{R}_m \to \mathfrak{S}_m \ltimes \mathbb{Z}^m$ chosen in Subsection 2.4, the same action of $\mathfrak{R}_m$ on the sequences can be obtained by letting the elements of $\mathfrak{S}_m$ act by permutations, while the elements of $\mathbb{Z}^m$ act by addition of the respective elements of $\ell\,\mathbb{Z}^m \subset \mathbb{C}^m$.

We will also use the *shifted action* of $\mathfrak{R}_m$ on $\hat{\mathfrak{t}}^*$. It is defined by adding

$$m\, C^* - E_{11}^* - 2E_{22}^* - \cdots - m\, E_{mm}^* \tag{3.1}$$

to the elements of $\hat{\mathfrak{t}}^*$, then applying the above described action of $\mathfrak{R}_m$, and then subtracting (3.1). We will employ the symbol $\circ$ to denote the shifted action. Put $\varepsilon_0 = E_{mm}^* - E_{11}^*$ while $\varepsilon_c = E_{cc}^* - E_{c+1,c+1}^*$ for $c = 1, \ldots, m-1$. Note that then

$$\tau_c \circ \widehat{\mu} = \tau_c(\widehat{\mu} + \varepsilon_c) \quad \text{for} \quad c = 0, 1, \ldots, m-1. \tag{3.2}$$

For any given ℓ the shifted action of $\mathfrak{R}_m$ on $\hat{\mathfrak{t}}^*$ preserves the set of weights of the form $\widehat{\mu}$. Hence we get a shifted action of $\mathfrak{R}_m$ on the set of sequences of length m of complex numbers. We will use the symbol $\circ$ to denote it as well. Then for any sequence $\mu = (\mu_1, \ldots, \mu_m)$

$$\pi \circ \mu = (\mu_m + \ell + 1, \mu_1 + 1, \ldots, \mu_{m-1} + 1),$$
$$\tau_0 \circ \mu = (\mu_m + \ell + 1, \mu_2, \ldots, \mu_{m-1}, \mu_1 - \ell - 1),$$
$$\tau_c \circ \mu = (\mu_1, \ldots, \mu_{c-1}, \mu_{c+1} - 1, \mu_c + 1, \mu_{c+2}, \ldots, \mu_m) \quad \text{for} \quad c = 1, \ldots, m-1.$$

Note that via our isomorphism $\mathfrak{R}_m \to \mathfrak{S}_m \ltimes \mathbb{Z}^m$, the same shifted action of the group $\mathfrak{R}_m$ on the sequences can be obtained by using the last displayed formula and by letting the elements of the subgroup $\mathbb{Z}^m \subset \mathfrak{S}_m \ltimes \mathbb{Z}^m$ act by addition of the respective elements of $\kappa \mathbb{Z}^m$ where $\kappa = \ell + m$. Indeed, because the group $\mathfrak{R}_m$ is generated by $\tau_1, \ldots, \tau_{m-1}$ and π, it suffices to check the coincidence of two actions of the element π only. Its image under the isomorphism $\mathfrak{R}_m \to \mathfrak{S}_m \ltimes \mathbb{Z}^m$ is the product $(1, 0, \ldots, 0)\, \sigma_1 \ldots \sigma_{m-1}$; see Subsection 2.4. But by our definition of the shifted action of the group $\mathfrak{S}_m \ltimes \mathbb{Z}^m$ on the sequences we have

$$\sigma_1 \cdots \sigma_{m-1} \circ \mu = (\mu_m - m + 1, \mu_1 + 1, \ldots, \mu_{m-1} + 1),$$
$$(1, 0, \ldots, 0)\, \sigma_1 \cdots \sigma_{m-1} \circ \mu = (\mu_m - m + \kappa + 1, \mu_1 + 1, \ldots, \mu_{m-1} + 1)$$
$$= (\mu_m + \ell + 1, \mu_1 + 1, \ldots, \mu_{m-1} + 1).$$

3.2. Consider the tensor product of N copies of the $\widehat{\mathfrak{sl}}_m$-module $\mathbb{C}^m[t, t^{-1}]$. In Subsection 2.2 we identified the vector space of this tensor product with $\mathfrak{P}_N \otimes (\mathbb{C}^m)^{\otimes N}$. Put

$$B = \mathfrak{P}_N \otimes (\mathbb{C}^m)^{\otimes N} \otimes U(\widehat{\mathfrak{sl}}_m).$$

Following [9] regard B as bimodule over the associative algebra $U(\widehat{\mathfrak{sl}}_m)$ by setting

$$X\,(P \otimes A) = XP \otimes A + P \otimes XA \quad \text{and} \quad (P \otimes A)\,X = P \otimes AX$$

for $X \in \widehat{\mathfrak{sl}}_m$ while $P \in \mathfrak{P}_N \otimes (\mathbb{C}^m)^{\otimes N}$ and $A \in U(\widehat{\mathfrak{sl}}_m)$. So the left module structure on B is defined by regarding $U(\widehat{\mathfrak{sl}}_m)$ as a module over itself via left multiplication, and then taking its tensor product with the module $\mathfrak{P}_N \otimes (\mathbb{C}^m)^{\otimes N}$ by using the standard comultiplication on $U(\widehat{\mathfrak{sl}}_m)$. The right module structure on B is defined by using only the right multiplication in the tensor factor $U(\widehat{\mathfrak{sl}}_m)$ of B. We will also use the *adjoint action* of $U(\widehat{\mathfrak{sl}}_m)$ on B. Here

$$\mathrm{ad}_X(P \otimes A) = X\,(P \otimes A) - (P \otimes A)\,X = XP \otimes A + P \otimes [X, A].$$

The action of the group $\mathfrak{R}_m$ on the Lie algebra $\widehat{\mathfrak{gl}}_m$ preserves the subalgebra $\widehat{\mathfrak{sl}}_m$. By again identifying the tensor product of N copies of $\mathbb{C}^m[t, t^{-1}]$ with $\mathfrak{P}_N \otimes (\mathbb{C}^m)^{\otimes N}$, we get an action of the group $\mathfrak{R}_m$ on the former vector space, and hence on the vector space of B.

Take the universal enveloping algebra $\mathrm{U}(\widehat{\mathfrak{h}})$ of the Abelian Lie algebra $\widehat{\mathfrak{h}} \subset \widehat{\mathfrak{sl}}_m$. Let $\overline{\mathrm{U}}(\widehat{\mathfrak{h}})$ be the ring of fractions of the commutative algebra $\mathrm{U}(\widehat{\mathfrak{h}})$ with the set of denominators generated by

$$\{\, E_{aa} - E_{bb} + i\,C + j \mid 1 \leqslant a < b \leqslant m \ \text{and} \ i,j \in \mathbb{Z}\,\}. \tag{3.3}$$

The elements of this ring can also be regarded as rational functions on the vector space $\widehat{\mathfrak{h}}^*$. The elements of $\mathrm{U}(\widehat{\mathfrak{h}}) \subset \overline{\mathrm{U}}(\widehat{\mathfrak{h}})$ are then regarded as polynomial functions on $\widehat{\mathfrak{h}}^*$. Further, let $\overline{\mathrm{U}}(\widehat{\mathfrak{sl}}_m)$ be the ring of fractions of the algebra $\mathrm{U}(\widehat{\mathfrak{sl}}_m)$ with the same set of denominators.

Let us denote

$$\overline{\mathrm{B}} = \mathfrak{P}_N \otimes (\mathbb{C}^m)^{\otimes N} \otimes \overline{\mathrm{U}}(\widehat{\mathfrak{sl}}_m).$$

Using the right multiplication in $\overline{\mathrm{U}}(\widehat{\mathfrak{sl}}_m)$, the right action of $\mathrm{U}(\widehat{\mathfrak{sl}}_m)$ on B extends to a right action of $\overline{\mathrm{U}}(\widehat{\mathfrak{sl}}_m)$ on $\overline{\mathrm{B}}$. To extend the left action of $\mathrm{U}(\widehat{\mathfrak{sl}}_m)$ on B to a right action of $\overline{\mathrm{U}}(\widehat{\mathfrak{sl}}_m)$ on $\overline{\mathrm{B}}$, note that the vector space of B has a basis of elements Y such that for any a, b and i as in (3.3) there exists $k \in \mathbb{Z}$ also depending on Y, such that

$$\mathrm{ad}_{E_{aa} - E_{bb} + i\,C}\,(Y) = k\,Y.$$

Set

$$(E_{aa} - E_{bb} + i\,C + j)^{-1}\,Y = Y(E_{aa} - E_{bb} + i\,C + j + k)^{-1}.$$

Hence the vector space $\overline{\mathrm{B}}$ becomes a bimodule over the algebra $\overline{\mathrm{U}}(\widehat{\mathfrak{sl}}_m)$.

The action of the group $\mathfrak{R}_m$ on the Lie algebra $\widehat{\mathfrak{gl}}_m$ preserves the subalgebra $\widehat{\mathfrak{h}} \subset \widehat{\mathfrak{sl}}_m$. Moreover, the resulting action of $\mathfrak{R}_m$ on $\mathrm{U}(\widehat{\mathfrak{h}})$ preserves the set of denominators generated by (3.3). So the action of $\mathfrak{R}_m$ extends from B to $\overline{\mathrm{B}}$. We will use the extended action later.

3.3. The Lie algebra $\widehat{\mathfrak{sl}}_m$ is generated by the elements

$$E_0 = E_{m1}\,t, \quad F_0 = E_{1m}\,t^{-1}, \quad H_0 = C - E_{11} + E_{mm};$$
$$E_c = E_{c,c+1}, \quad F_c = E_{c+1,c}, \quad H_c = E_{cc} - E_{c+1,c+1} \quad \text{where} \quad c = 1, \ldots, m-1.$$

For each $c = 0, 1, \ldots, m-1$ the elements E_c, F_c, H_c span a subalgebra of $\widehat{\mathfrak{sl}}_m$ isomorphic to $\mathfrak{sl}_2$. We will also use the element $\varepsilon_c \in \widehat{\mathfrak{t}}^*$ defined in Subsection 3.1.

Consider the vector spaces B and $\overline{\mathrm{B}}$ introduced in Subsection 3.2. For every $c = 0, 1, \ldots, m-1$ define a linear map $\xi_c : \mathrm{B} \to \overline{\mathrm{B}}$ by setting for any $Y \in \mathrm{B}$

$$\xi_c\,(Y) = Y + \sum_{n=1}^{\infty} (n!\,H_c^{(n)})^{-1} E_c^{\,n}\,\mathrm{ad}_{F_c}^{\,n}\,(Y) \tag{3.4}$$

where

$$H_c^{(n)} = H_c(H_c - 1) \cdots (H_c - n + 1)$$

and we take the nth power of the adjoint operator corresponding to the element $F_c \in \widehat{\mathfrak{sl}}_m$. For any given $Y \in \mathrm{B}$ only finitely many terms of the sum (3.4) differ from zero, so the map ξ_c is well defined. The definition (3.4) and the next proposition go back to [16, Sect. 2]. By using the left action of the Lie subalgebra $\widehat{\mathfrak{n}} \subset \widehat{\mathfrak{sl}}_m$, introduce the vector subspaces

$$\mathrm{J} = \widehat{\mathfrak{n}}\,\mathrm{B} \subset \mathrm{B} \quad \text{and} \quad \bar{\mathrm{J}} = \widehat{\mathfrak{n}}\,\bar{\mathrm{B}} \subset \bar{\mathrm{B}}.$$

Proposition 3.1. *For any* $X \in \widehat{\mathfrak{h}}$ *and* $Y \in \mathrm{B}$ *we have*

$$\xi_c(XY) \in (X + \varepsilon_c(X))\,\xi_c(Y) + \bar{\mathrm{J}}, \tag{3.5}$$
$$\xi_c(YX) \in \xi_c(Y)(X + \varepsilon_c(X)) + \bar{\mathrm{J}}. \tag{3.6}$$

Proof. It suffices to verify the properties (3.5) and (3.6) only for $X = H_c$ and for all $X \in \widehat{\mathfrak{h}}$ such that $\varepsilon_c(X) = 0$. In the latter case we have the relations $[E_c, X] = [F_c, X] = 0$ in $\widehat{\mathfrak{sl}}_m$. Then $\xi_c(XY) = X\,\xi_c(Y)$ and $\xi_c(YX) = \xi_c(Y)\,X$ by (3.4). Hence we get (3.5) and (3.6).

For $X = H_c$ the proof of (3.5) is based on the following commutation relations in the subalgebra of $\mathrm{U}(\widehat{\mathfrak{sl}}_m)$ generated by the three elements E_c, F_c, H_c:

$$[\,E_c^n, H_c\,] = -\,2\,n\,E_c^n \quad \text{and} \quad [\,E_c^n, F_c\,] = n\,(H_c - n + 1)\,E_c^{n-1}. \tag{3.7}$$

Let us use the symbol $\equiv$ to indicate equalities in $\bar{\mathrm{B}}$ modulo the subspace $\bar{\mathrm{J}}$. By (3.7), for any element $Y \in \mathrm{B}$ we get

$$\xi_c\,(H_c\,Y) \equiv (H_c + 2)\,\xi_c\,(Y) = (H_c + \varepsilon_c(H_c))\,\xi_c\,(Y).$$

Here the relation $\equiv$ is obtained as in the proof of [7, Proposition 3.1]. By following another calculation, as given in the end of the proof of [7, Proposition 3.1],

$$\xi_c\,(Y\,H_c) \equiv \xi_c\,(Y)(H_c + 2) = \xi_c\,(Y)(H_c + \varepsilon_c(H_c)). \qquad \square$$

3.4. The property (3.5) allows us to define a linear map $\bar{\xi}_c : \bar{\mathrm{B}} \to \bar{\mathrm{B}}/\bar{\mathrm{J}}$ by setting

$$\bar{\xi}_c(Y A) = \xi_c(Y)\,Z + \bar{\mathrm{J}} \quad \text{for} \quad A \in \bar{\mathrm{U}}(\widehat{\mathfrak{h}}) \quad \text{and} \quad Y \in \mathrm{B}$$

where the element $Z \in \bar{\mathrm{U}}(\widehat{\mathfrak{h}})$ is obtained from A by regarding it as a rational function on the dual vector space $\widehat{\mathfrak{h}}^*$, and then adding ε_c to the argument of that rational function. Recall that in the end of Subsection 3.2 we defined an action of the extended affine Weyl group $\mathfrak{R}_m$ on the vector space $\bar{\mathrm{B}}$. For any index $c = 0, 1, \ldots, m - 1$ consider the image $\tau_c(\bar{\mathrm{J}}) \subset \bar{\mathrm{B}}$.

Proposition 3.2. *We have* $\tau_c(\bar{\mathrm{J}}) \subset \ker \bar{\xi}_c$.

Proof. Note that $\tau_c(F_c) = E_c$. If $c > 0$ then let $\widehat{\mathfrak{n}}_c$ be the subspace of $\widehat{\mathfrak{sl}}_m$ spanned by all the elements $E_{ab}\, t^i$ where $i < 0$, and by those elements E_{ab} where $a > b$ but $(a, b) \neq (c+1, c)$. Further, let $\widehat{\mathfrak{n}}_0$ be the subspace of $\widehat{\mathfrak{sl}}_m$ spanned by the elements E_{ab} where $a > b$, and by those elements $E_{ab}\, t^i$ where $i < 0$ but $(a, b, i) \neq (1, m, -1)$. Then for any $c = 0, 1, \ldots, m-1$ the image $\tau_c(\bar{\mathrm{J}}) \subset \bar{\mathrm{B}}$ is spanned by the subspaces $\widehat{\mathfrak{n}}_c \bar{\mathrm{B}}$ and $E_c \bar{\mathrm{B}}$.

By using the relations (2.1) one can check that the subspace $\widehat{\mathfrak{n}}_c \subset \widehat{\mathfrak{sl}}_m$ is preserved by the adjoint action of the elements E_c, F_c, H_c. So we have $\xi_c(XY) \in \bar{\mathrm{J}}$ for any $X \in \widehat{\mathfrak{n}}_c$ and any $Y \in \mathrm{B}$; see (3.4). To prove Proposition 3.2 it remains to show that $\xi_c(E_c\, Y) \in \bar{\mathrm{J}}$ for any $Y \in \mathrm{B}$. By using the relations (3.7), this can be shown by the same calculation as in the proof of [7, Prop. 3.2]. $\quad\square$

Proposition 3.2 allows us to define for $c = 0, 1, \ldots, m-1$ a linear map

$$\eta_c : \bar{\mathrm{B}}/\bar{\mathrm{J}} \to \bar{\mathrm{B}}/\bar{\mathrm{J}}$$

as the composition $\bar{\xi}_c\, \tau_c$ applied to the elements of $\bar{\mathrm{B}}$ which are taken modulo the subspace $\bar{\mathrm{J}}$. This definition also goes back to [16], and we will call $\eta_0, \eta_1, \ldots, \eta_{m-1}$ the *Zhelobenko operators* on $\bar{\mathrm{B}}/\bar{\mathrm{J}}$. The next proposition states their key property; for its proof see [10, Sect. 6]. As in the beginning of Subsection 2.4, here we will let the indices c of the operators η_c run through $\mathbb{Z}$, assuming that $\eta_{c+m} = \eta_c$.

Proposition 3.3. *The operators $\eta_0, \eta_1, \ldots, \eta_{m-1}$ on $\bar{\mathrm{B}}/\bar{\mathrm{J}}$ satisfy the relations*

$$\eta_c\, \eta_{c+1}\, \eta_c = \eta_{c+1}\, \eta_c\, \eta_{c+1}; \quad \eta_c\, \eta_d = \eta_d\, \eta_c \quad for \quad c - d \neq \pm 1 \bmod m.$$

Corollary 3.4. *For any reduced decomposition $\tau = \tau_c \ldots \tau_d$ in the group $\mathfrak{T}_m$ the composition $\eta_c \ldots \eta_d$ of operators on $\bar{\mathrm{B}}/\bar{\mathrm{J}}$ does not depend on the choice of the decomposition of τ.*

By the definition given in Subsection 2.4, the action of the element $\pi \in \mathfrak{R}_m$ on the Lie algebra $\widehat{\mathfrak{sl}}_m$ maps E_c, F_c, H_c respectively to $E_{c+1}, F_{c+1}, H_{c+1}$. If the index $c + 1$ here is m, it should be then replaced by 0. Furthermore, the action of the element π on $\widehat{\mathfrak{sl}}_m$ preserves the subalgebra $\widehat{\mathfrak{n}}$. Hence the action of π on $\bar{\mathrm{B}}$ determines its action on the quotient $\bar{\mathrm{B}}/\bar{\mathrm{J}}$. It now follows from the definition (3.4) that on $\bar{\mathrm{B}}/\bar{\mathrm{J}}$ we have

$$\pi\, \eta_c = \eta_{c+1}\, \pi. \tag{3.8}$$

3.5. Using the right action of the Lie subalgebra $\widehat{\mathfrak{n}}' \subset \widehat{\mathfrak{sl}}_m$, introduce the vector subspaces

$$\mathrm{J}' = \mathrm{B}\, \widehat{\mathfrak{n}}' \subset \mathrm{B} \quad \text{and} \quad \bar{\mathrm{J}}' = \bar{\mathrm{B}}\, \widehat{\mathfrak{n}}' \subset \bar{\mathrm{B}}.$$

For any $c = 0, 1, \ldots, m-1$ consider the image $\tau_c(\bar{\mathrm{J}}') \subset \bar{\mathrm{B}}$.

Proposition 3.5. *We have $\bar{\xi}_c(\tau_c(\bar{\mathrm{J}}')) \subset \bar{\mathrm{J}} + \bar{\mathrm{J}}'$.*

Proof. Note that $\tau_c(E_c) = F_c$. If $c > 0$ then let $\widehat{\mathfrak{n}}'_c$ be the subspace of $\widehat{\mathfrak{sl}}_m$ spanned by all the elements $E_{ab}\, t^i$ where $i > 0$, and by those elements E_{ab} where $a < b$ but $(a, b) \neq (c, c+1)$. Further, let $\widehat{\mathfrak{n}}'_0$ be the subspace of $\widehat{\mathfrak{sl}}_m$ spanned by all the elements E_{ab} where $a < b$, and by those elements $E_{ab}\, t^i$ where $i > 0$ but

$(a, b, i) \neq (m, 1, 1)$. Then for any $c = 0, 1, \ldots, m - 1$ the image $\tau_c(\bar{J}') \subset \bar{B}$ is spanned by the subspaces

$$\bar{B}\,\widehat{\mathfrak{n}}'_c \quad \text{and} \quad \bar{B}F_c.$$

By using the relations (2.1) one can check that the subspace $\widehat{\mathfrak{n}}'_c \subset \widehat{\mathfrak{sl}}_m$ is preserved by the adjoint action of the element F_c. Hence we have $\xi_c(XY) \in \bar{J}'$ for any $X \in \widehat{\mathfrak{n}}'_c$ and any $Y \in B$; see the definition (3.4). Further, note that $\xi_c(YF_c) = \xi_c(Y)\,F_c$ for any $Y \in B$, because $\mathrm{ad}_{F_c}(YF_c) = \mathrm{ad}_{F_c}(Y)\,F_c$. The proof of Proposition 3.5 can be now completed by showing that here $\xi_c(Y)F_c \in \bar{J}$. By using (3.7), the latter inclusion is obtained by the same calculation as in the proof of [7, Prop. 3.5]. $\quad\square$

By Proposition 3.5 for $c = 0, 1, \ldots, m-1$ the Zhelobenko operator η_c determines a linear map

$$\bar{B}/(\bar{J} + \bar{J}') \to \bar{B}/(\bar{J} + \bar{J}').$$

Recall that $\ell = \kappa - m$ by an assumption made in Subsection 2.3. Denote by I the subspace $B\,(C - \ell) \subset B$. Similarly, denote by $\bar{I}$ the subspace $\bar{B}\,(C - \ell) \subset \bar{B}$. Since $C \in \widehat{\mathfrak{sl}}_m$ is central, the Zhelobenko operator η_c also determines a linear map

$$\bar{B}/(\bar{J} + \bar{J}' + \bar{I}) \to \bar{B}/(\bar{J} + \bar{J}' + \bar{I}). \tag{3.9}$$

Observe that the vector space $\bar{B}/(\bar{J} + \bar{J}' + \bar{I})$ coincides with the space of $\widehat{\mathfrak{n}}$-coinvariants of the $\widehat{\mathfrak{sl}}_m$-module (2.2) where the tensor factor V is the *universal* Verma module of level ℓ. Namely, here V is the quotient of the universal enveloping algebra $\mathrm{U}(\widehat{\mathfrak{sl}}_m)$ by the left ideal generated by $\widehat{\mathfrak{n}}'$ and by the element $C - \ell$. This V satisfies the annihilation condition stated before (2.6). By applying Corollary 2.2 to this $\widehat{\mathfrak{sl}}_m$-module V, we define an action of the Cherednik algebra $\mathfrak{C}_N$ on the quotient vector space $\bar{B}/(\bar{J} + \bar{J}' + \bar{I})$.

Note that the action of the element π on $\bar{B}$ also determines a linear map (3.9). For $\kappa \neq 0$ this map does not commute with the action of $\mathfrak{C}_N$; see Corollary 2.5. However, we still have the following theorem.

Theorem 3.6. *For $c = 0, 1, \ldots, m - 1$ and $\ell = \kappa - m$ the linear map (3.9) determined by the Zhelobenko operator η_c commutes with the action of $\mathfrak{C}_N$.*

Proof. First consider the linear map (3.9) determined by the Zhelobenko operator η_c for any $c > 0$. This map commutes with the action of $\mathfrak{C}_N$ by the definition (3.4) of corresponding operator $\xi_c : B \to \bar{B}$; see Corollary 2.2(ii). Here we also use the observation that for any $c > 0$ the action of τ_c on B commutes with multiplications by the variables $x_1, \ldots, x_N$ and with permutations of these variables in the tensor factor $\mathfrak{P}_N$ of B, commutes with permutations of the N tensor factors $\mathbb{C}^m$ of B, and commutes with (2.7) if (2.7) is regarded as an operator on B using the left multiplication by elements of $\widehat{\mathfrak{sl}}_m$ in the tensor factor $\mathrm{U}(\widehat{\mathfrak{sl}}_m)$ of B.

Consider the linear map (3.9) determined by the Zhelobenko operator η_0. We can write $\eta_0 = \pi^{-1}\eta_1\pi$; see the end of Subsection 3.4. The action of the element $\pi \in \mathfrak{R}_m$ on B commutes with multiplication by the variables $x_1, \ldots, x_N$ in the tensor factor $\mathfrak{P}_N$ of B, and also commutes with simultaneous permutations of these variables and of the corresponding N tensor factors $\mathbb{C}^m$ of B. By using the

argument from the previous paragraph when $c = 1$, and by applying Corollary 2.5 when V is the universal Verma module of level ℓ, we can now complete the proof of our theorem. If we denote simply by υ the linear map (3.9) determined by the Zhelobenko operator η_1, then for any $p = 1, \ldots, N$ we have

$$\pi^{-1} \upsilon \pi \theta_p = \pi^{-1} \upsilon \left(\theta_p + \frac{\kappa}{m} \operatorname{id} \right) \pi = \pi^{-1} \left(\theta_p + \frac{\kappa}{m} \operatorname{id} \right) \upsilon \pi = \theta_p \pi^{-1} \upsilon \pi. \quad \square$$

The group $\mathfrak{T}_m$ generated by $\tau_0, \tau_1, \ldots, \tau_{m-1}$ can be regarded as the Weyl group of the affine Lie algebra $\widehat{\mathfrak{sl}}_m$. The quotient of $\mathfrak{R}_m$ by the relation $\pi^m = 1$ can be then regarded as the extended Weyl group of $\widehat{\mathfrak{sl}}_m$. These two facts underline our definition of the operators $\eta_0, \eta_1, \ldots, \eta_{m-1}$. In the next section we will apply Theorem 3.6 when the universal Verma module V of level ℓ appearing above is replaced by the usual Verma module $M_{\widehat{\beta}}$ of $\widehat{\mathfrak{sl}}_m$.

4. Intertwining operators

4.1. Using the right action of the Lie subalgebra $\widehat{\mathfrak{h}} \subset \widehat{\mathfrak{sl}}_m$, take the vector subspace

$$\mathrm{B} \left(E_{aa} - E_{bb} - \mu_a + \mu_b \right) \subset \mathrm{B} \quad \text{where} \quad 1 \leqslant a < b \leqslant m.$$

This subspace depends on the weight $\mu \in \mathfrak{t}^*$ via its restriction β to $\mathfrak{h} \subset \mathfrak{t}$. Let $\mathrm{I}_{\widehat{\beta}}$ be the sum of all these subspaces and of the subspace $\mathrm{I} \subset \mathrm{B}$ introduced just before stating Theorem 3.6. As an $\widehat{\mathfrak{sl}}_m$-module, the quotient $\mathrm{B}/(\mathrm{J}' + \mathrm{I}_{\widehat{\beta}})$ can be identified with

$$\mathfrak{P}_N \otimes (\mathbb{C}^m)^{\otimes N} \otimes M_{\widehat{\beta}} = \mathcal{A}_N(M_{\widehat{\beta}}). \tag{4.1}$$

Here $\widehat{\mathfrak{sl}}_m$ acts on the quotient via the left action of the algebra $\mathrm{U}(\widehat{\mathfrak{sl}}_m)$ on its bimodule B.

Now suppose that the sequence $(\mu_1, \ldots, \mu_m)$ of complex numbers obeys the conditions

$$\mu_a - \mu_b \notin \mathbb{Z} + \ell \mathbb{Z} \quad \text{for} \quad 1 \leqslant a < b \leqslant m. \tag{4.2}$$

Note that then the sequence $\lambda = (\lambda_1, \ldots, \lambda_m)$ obeys the same conditions, since $\lambda_a - \mu_a \in \mathbb{Z}$ for $a = 1, \ldots, m$ by our assumption. Similarly to $\mathrm{I}_{\widehat{\beta}}$, denote by $\bar{\mathrm{I}}_{\widehat{\beta}}$ the sum of all subspaces

$$\bar{\mathrm{B}} \left(E_{aa} - E_{bb} - \mu_a + \mu_b \right) \subset \bar{\mathrm{B}} \quad \text{where} \quad 1 \leqslant a < b \leqslant m,$$

and of the subspace $\bar{\mathrm{I}}$. As a module of $\widehat{\mathfrak{sl}}_m$, the quotient $\bar{\mathrm{B}}/(\bar{\mathrm{J}}' + \bar{\mathrm{I}}_{\widehat{\beta}})$ can be also identified with (4.1). The quotient $\bar{\mathrm{B}}/(\bar{\mathrm{J}} + \bar{\mathrm{J}}' + \bar{\mathrm{I}}_{\widehat{\beta}})$ is then identified with the space $\mathcal{A}_N(M_{\widehat{\beta}})_{\widehat{\mathfrak{n}}}$ of $\widehat{\mathfrak{n}}$-coinvariants of (4.1).

The shifted action of the affine Weyl group $\mathfrak{R}_m$ on $\widehat{\mathfrak{t}}^*$ determines an action of $\mathfrak{R}_m$ on $\widehat{\mathfrak{h}}^*$. We use the same symbol $\circ$ to denote the latter action. Then by (3.2)

$$\tau_c \circ \widehat{\beta} = \tau_c(\widehat{\beta} + \varepsilon_c) \quad \text{for} \quad c = 0, 1, \ldots, m - 1.$$

Here the summand ε_c defined in Subsection 3.1 is regarded as a linear function on the vector space $\widehat{\mathfrak{h}}$ by restriction from $\widehat{\mathfrak{t}}$. Due to (3.6) and to the last displayed equality, we have

$$\bar{\xi}_c(\,\tau_c(\bar{I}_{\widehat{\beta}}))\subset \bar{J}+\bar{I}_{\tau_c\circ\widehat{\beta}}.$$

So the Zhelobenko operator η_c defined in Subsection 3.4 determines a linear map

$$\bar{B}/(\,\bar{J}+\bar{J}'+\bar{I}_{\widehat{\beta}}\,)\ \to\ \bar{B}/(\,\bar{J}+\bar{J}'+\bar{I}_{\tau_c\circ\widehat{\beta}});\tag{4.3}$$

see also Subsection 3.5. Via the identifications described above, (4.3) becomes a linear map

$$\mathcal{A}_N(M_{\widehat{\beta}})_{\widehat{\mathfrak{n}}}\ \to\ \mathcal{A}_N(M_{\tau_c\circ\widehat{\beta}})_{\widehat{\mathfrak{n}}}.$$

Then by (3.5) the restriction of (4.3) to the subspace of vectors of weight $\widehat{\alpha}$ becomes a linear map

$$\mathcal{A}_N(M_{\widehat{\beta}})_{\widehat{\mathfrak{n}}}^{\widehat{\alpha}}\ \to\ \mathcal{A}_N(M_{\tau_c\circ\widehat{\beta}})_{\widehat{\mathfrak{n}}}^{\tau_c\circ\widehat{\alpha}}.\tag{4.4}$$

Now consider the action of the element $\pi\in\mathfrak{R}_m$ on $\bar{B}$. This action preserves the subspaces $\bar{J}$ and $\bar{J}'$ of $\bar{B}$. Further, by the definition of the subspace $\bar{I}_{\widehat{\beta}}$ of $\bar{B}$ we have

$$\pi(\bar{I}_{\widehat{\beta}})=\bar{I}_{\pi(\widehat{\beta})}.$$

Here we employ the usual, not shifted action of $\mathfrak{R}_m$ on $\widehat{\mathfrak{t}}^*$. But we also have $\pi(\widehat{\beta})=\pi\circ\widehat{\beta}$; see again Subsection 3.1. Hence the action of π on $\bar{B}$ determines a linear map

$$\bar{B}/(\,\bar{J}+\bar{J}'+\bar{I}_{\widehat{\beta}}\,)\ \to\ \bar{B}/(\,\bar{J}+\bar{J}'+\bar{I}_{\pi\circ\widehat{\beta}}).\tag{4.5}$$

Via the identifications described above, (4.5) becomes a linear map

$$\mathcal{A}_N(M_{\widehat{\beta}})_{\widehat{\mathfrak{n}}}\ \to\ \mathcal{A}_N(M_{\pi\circ\widehat{\beta}})_{\widehat{\mathfrak{n}}}$$

Then the restriction of (4.5) to the subspace of vectors of weight $\widehat{\alpha}$ becomes a linear map

$$\mathcal{A}_N(M_{\widehat{\beta}})_{\widehat{\mathfrak{n}}}^{\widehat{\alpha}}\ \to\ \mathcal{A}_N(M_{\pi\circ\widehat{\beta}})_{\widehat{\mathfrak{n}}}^{\pi\circ\widehat{\alpha}}.\tag{4.6}$$

4.2. In Subsection 2.3 we did already assume that $\ell=\kappa-m$. Under this assumption, by Theorem 3.6 the action of the algebra $\mathfrak{C}_N$ on the vector space $\bar{B}/(\bar{J}+\bar{J}'+\bar{I})$ commutes with the linear map (3.9) determined by the Zhelobenko operator η_c for $c=0,1,\dots,m-1$. Further, since the action of $\mathfrak{C}_N$ on $\bar{B}$ commutes with the right action of $\bar{U}(\widehat{\mathfrak{sl}}_m)$, the algebra $\mathfrak{C}_N$ acts on the source and target vector spaces of the linear map (4.3). Moreover, the map (4.3) intertwines these two actions.

Therefore the linear map (4.4) corresponding to (4.3) is also $\mathfrak{C}_N$-intertwining. Indeed, the action of $\mathfrak{C}_N$ on (4.1) defined in Subsection 2.2 corresponds to the action of $\mathfrak{C}_N$ on $B/(J'+I_{\widehat{\beta}})$. Similar correspondence holds for $\tau_c\circ\widehat{\beta}$ instead of $\widehat{\beta}$.

We can replace the source and target $\mathfrak{C}_N$-modules in (4.4) by their isomorphic modules, using Proposition 2.3. The value of f appearing in that proposition is

the same for the sequence μ and for the sequence $\tau_c \circ \mu$ instead of μ; see the end of Subsection 3.2. Hence our replacement modules in (4.4) will be pullbacks of respectively $\widehat{S}_\mu^\lambda$ and $\widehat{S}_{\tau_c \circ \mu}^{\tau_c \circ \lambda}$ relative to the same automorphism (1.5). By applying the inverse of this automorphism, the Zhelobenko operator η_c now determines an $\mathfrak{C}_N$-intertwining linear map

$$\widehat{S}_\mu^\lambda \to \widehat{S}_{\tau_c \circ \mu}^{\tau_c \circ \lambda}. \tag{4.7}$$

Note that because $\ell = \kappa - m$, the conditions (4.2) on the sequence μ can be restated as

$$\mu_a - \mu_b \notin \mathbb{Z} + \kappa \mathbb{Z} \quad \text{for} \quad 1 \leqslant a < b \leqslant m.$$

Under the latter conditions both the source and target $\mathfrak{C}_N$-modules in (4.7) are irreducible by [1, Prop. 2.4.3]. Hence any intertwining linear map between them is unique up to a factor from $\mathbb{C}$. In the next subsection we will determine this scalar factor for the intertwining map determined by the Zhelobenko operator η_c.

Now consider the map (4.5) and the corresponding map (4.6), which are determined by the action of the element $\pi \in \mathfrak{R}_m$ on $\overline{B}$. The map (4.6) is not $\mathfrak{C}_N$-intertwining unless $\kappa = 0$; see Corollary 2.5. However, it will become intertwining if we replace the target $\mathfrak{C}_N$-module in (4.6) by its pullback via the automorphism (1.5) of $\mathfrak{C}_N$ where $f = \kappa/m$.

We can now replace the source and target $\mathfrak{C}_N$-modules of the latter intertwining operator by their isomorphic modules, again using Proposition 2.3. The source module can be replaced by the pullback of $\widehat{S}_\mu^\lambda$ relative to the automorphism (1.5) where $f = -(\mu_1 + \cdots + \mu_m)/m$. The target module here can be replaced by the pullback of $\widehat{S}_{\pi \circ \mu}^{\pi \circ \lambda}$ relative to (1.5) where

$$f = -(\mu_1 + \cdots + \mu_m + \ell + m)/m + \kappa/m = -(\mu_1 + \cdots + \mu_m)/m.$$

Here we used the formula for $\pi \circ \mu$ given in Subsection 3.1. Since the values of f for the source and the target replacement modules are the same, the action of π on $\overline{B}$ now determines a $\mathfrak{C}_N$-intertwining operator

$$\widehat{S}_\mu^\lambda \to \widehat{S}_{\pi \circ \mu}^{\pi \circ \lambda}. \tag{4.8}$$

By the irreducibility of the source and of target induced $\mathfrak{C}_N$-modules here, the latter operator must coincide with the intertwining operator from [13] up to a scalar multiplier.

Any element of $\mathfrak{R}_m$ has a reduced decomposition of the form $\pi^g \tau_c \cdots \tau_d$ where g is the degree of this element relative to the $\mathbb{Z}$-grading defined in Subsection 2.4. By using Corollary 2.4 and the relation (3.8), the composition of linear maps

$$\pi^g \eta_c \cdots \eta_d \colon \overline{B}/\overline{J} \to \overline{B}/\overline{J}$$

now determines a $\mathfrak{C}_N$-intertwining operator $\widehat{S}_\mu^\lambda \to \widehat{S}_{\pi^g \tau_c \cdots \tau_d \circ \mu}^{\pi^g \tau_c \cdots \tau_d \circ \lambda}$. It is defined as a composition of intertwining operators of the form (4.7),(4.8) corresponding to generators of $\mathfrak{R}_m$. This is the intertwining operator mentioned in the Introduction, where ω is now the image of the element $\pi^g \tau_c \cdots \tau_d$ under the isomorphism of groups $\mathfrak{R}_m \to \mathfrak{S}_m \ltimes \mathbb{Z}^m$. Indeed, under this isomorphism the shifted actions of the two groups on λ and μ correspond to each other.

4.3. Here we will provide explicit formulas for the maps (4.3) and (4.5) determined by the Zhelobenko operator η_c with $c = 0, 1, \ldots, m - 1$ and by the action of the element $\pi \in \mathfrak{R}_m$. For $1 \leqslant a_1, \ldots, a_N \leqslant m$ and $i_1, \ldots, i_N \in \mathbb{Z}$ consider the element

$$Y^{\,i_1 \ldots i_N}_{a_1 \ldots a_N} = x_1^{i_1} \cdots x_N^{i_N} \otimes e_{a_1} \otimes \cdots \otimes e_{a_N} \otimes 1 \in \mathrm{B}.$$

Due to the Poincaré-Birkhoff-Witt theorem for the universal enveloping algebra $\mathrm{U}(\widehat{\mathfrak{sl}}_m)$, the images of all these elements in the source quotient vector space of the maps (4.3) and (4.5) make a basis in this quotient. Now suppose that the numbers $1, \ldots, m$ occur respectively $\nu_1, \ldots, \nu_m$ times in the sequence $a_1, \ldots, a_N$. Then the image of the element $Y^{\,i_1 \ldots i_N}_{a_1 \ldots a_N}$ in the source quotient has weight $\widehat{\alpha}$ relative to the left $\widehat{\mathfrak{h}}$-module structure on the quotient. This is because the image of the element $1 \in \mathrm{U}(\widehat{\mathfrak{sl}}_m)$ in the Verma module $M_{\widehat{\beta}}$ has weight $\widehat{\beta}$.

Proposition 4.1. *For $1 \leqslant c < m$, the Zhelobenko operator η_c maps the image of $Y^{\,i_1 \ldots i_N}_{a_1 \ldots a_N}$ in the source quotient in (4.3) to the image of the next sum of elements of* B *in the target quotient:*

$$\sum_{h=0}^{\min(\nu_c, \nu_{c+1})} \ \sum_{b_1, \ldots, b_N} h! \, (\lambda_{c+1} - \lambda_c - 1) \, Y^{\,i_1 \ldots i_N}_{b_1 \ldots b_N} \prod_{s=0}^{h} \frac{1}{\mu_{c+1} - \lambda_c + s - 1}$$

where $b_1, \ldots, b_N$ is a sequence obtained from $a_1, \ldots, a_N$ by changing $\nu_c - h$ terms c to $c + 1$, and also changing $\nu_{c+1} - h$ terms $c + 1$ to c.

Proof. Applying the action of τ_c on B to the element $Y^{\,i_1 \ldots i_N}_{a_1 \ldots a_N}$ amounts to replacing every c in the sequence $a_1, \ldots, a_N$ by $c+1$, and the other way round. Let $d_1, \ldots, d_N$ be the sequence so obtained. Apply the operator ξ_c to the resulting element of B by using (3.4). Note that for any $n = 0, 1, 2, \ldots$ we have an equality in B

$$E_c^{\,n} \operatorname{ad}_{F_c}^n \left(Y^{\,i_1 \ldots i_N}_{d_1 \ldots d_N} \right) = x_1^{i_1} \cdots x_N^{i_N} \otimes E_c^{\,n} \left(F_c^n (e_{d_1} \otimes \cdots \otimes e_{d_N}) \otimes 1 \right).$$

Modulo the subspace $\mathrm{J}' \subset \mathrm{B}$, the element of B displayed here at the right-hand side equals

$$x_1^{i_1} \cdots x_N^{i_N} \otimes E_c^{\,n} F_c^n (e_{d_1} \otimes \cdots \otimes e_{d_N}) \otimes 1.$$

In its turn, the last displayed element equals the sum of elements of the form $Y^{\,i_1 \ldots i_N}_{b_1 \ldots b_N}$ taken with certain multiplicities. Here the multiplicity is the number of ways the sequence $b_1, \ldots, b_N$ can be obtained from $d_1, \ldots, d_N$ by consecutively replacing any n occurences of c by $c + 1$, and then consecutively replacing any n occurences of $c + 1$ by c in the result.

Denote by h the number of those terms of the sequence $d_1, \ldots, d_N$ which are equal to c, but change to $c + 1$ in the sequence $b_1, \ldots, b_N$. Obviously $h \leqslant \nu_{c+1}$. When passing from $d_1, \ldots, d_N$ to $b_1, \ldots, b_N$ as above, the numbers of occurences of $1, \ldots, m$ remain the same. Therefore h is also the number of those terms of the sequence $d_1, \ldots, d_N$ which are equal to $c + 1$, but change to c in $b_1, \ldots, b_N$. Hence

$h \leqslant \nu_c$. The above stated multiplicity is not zero only if $h \leqslant n \leqslant \nu_{c+1}$. In this case it is equal to

$$\frac{n!\, n!\, (\nu_{c+1} - h)!}{(n-h)!\, (\nu_{c+1} - n)!}.$$

It follows that modulo the subspace $\bar{J}' \subset \bar{B}$, the element $\xi_c(Y_{d_1 \ldots d_N}^{i_1 \ldots i_N})$ of $\bar{B}$ equals

$$\sum_{h=0}^{\min(\nu_c,\nu_{c+1})} \sum_{b_1,\ldots,b_N} \sum_{n=h}^{\nu_{c+1}} \frac{n!\, (\nu_{c+1} - h)!}{(n-h)!\, (\nu_{c+1} - n)!\, H_c^{(n)}}\, (Y_{b_1 \ldots b_N}^{i_1 \ldots i_N}) \tag{4.9}$$

where $b_1, \ldots, b_N$ range as in Proposition 4.1. Here the sum over $n = h, \ldots, \nu_{c+1}$ can be computed by the Gauss formula for the hypergeometric function $F(u, v, w; z)$ at $z = 1$,

$$F(u, v, w; 1) = \frac{\Gamma(w)\, \Gamma(w - u - v)}{\Gamma(w - u)\, \Gamma(w - v)}$$

which is valid for any $u, v, w \in \mathbb{C}$ where $w \neq 0, -1, \ldots$ and $\mathrm{Re}(w - u - v) > 0$. By setting $u = h - \nu_{c+1}$ and $v = h + 1$ in that formula, we obtain an equality of rational functions in w

$$\sum_{n=h}^{\nu_{c+1}} \frac{(-1)^{n-h}\, n!\, (\nu_{c+1} - h)!}{h!\, (n-h)!\, (\nu_{c+1} - n)!} \prod_{s=0}^{n-h-1} \frac{1}{w + s} = \prod_{s=0}^{\nu_{c+1}-h-1} \frac{w - h + s - 1}{w + s}.$$

Replacing the complex variable w by the element $h - H_c \in U(\mathfrak{h})$ in this equality, the sum of the fractions in (4.9) taken over the indices $n = h, \ldots, \nu_{c+1}$ equals

$$h! \prod_{s=0}^{h-1} \frac{1}{H_c - s} \cdot \prod_{s=0}^{\nu_{c+1}-h-1} \frac{H_c - s + 1}{H_c - s - h} = h!\, (H_c + 1) \prod_{s=0}^{h} \frac{1}{H_c - \nu_{c+1} + s + 1}.$$

On the other hand, we also know that the image of the element (4.9) of $\bar{B}$ in the target quotient in (4.3) has weight $\tau_c \circ \widehat{\alpha}$ relative to the left $\widehat{\mathfrak{h}}$-module structure on the quotient. So when computing the image, we can replace the element $H_c \in \mathfrak{h}$ in (4.9) by the weight value

$$(\tau_c \circ \widehat{\alpha})(H_c) = \lambda_{c+1} - \lambda_c - 2.$$

Using the relation $\lambda_{c+1} - \nu_{c+1} = \mu_{c+1}$, we now complete the proof. □

The action of π on B maps the element $Y_{a_1 \ldots a_N}^{i_1 \ldots i_N}$ to $Y_{b_1 \ldots b_N}^{j_1 \ldots j_N}$ where $b_p = a_p + 1$ and $j_p = i_p - \delta_{a_p m}$ for $p = 1, \ldots, N$. Using this observation and (3.8) for $c = m$, the next result can be derived from Proposition 4.1. It can also be obtained by directly following the arguments employed in the proof of that proposition.

Proposition 4.2. *For $c = 0$, the Zhelobenko operator η_0 maps the image of $Y_{a_1 \ldots a_N}^{i_1 \ldots i_N}$ in the source quotient in (4.3) to the image of the next sum of elements of* B *in the target quotient:*

$$\sum_{h=0}^{\min(\nu_1,\nu_m)} \sum_{b_1,\ldots,b_N} h!\, (\lambda_1 - \lambda_m - \ell - 1)\, Y_{b_1 \ldots b_N}^{j_1 \ldots j_N} \prod_{s=0}^{h} \frac{1}{\mu_1 - \lambda_m - \ell + s - 1}$$

where

$$j_p = i_p + \delta_{a_p 1} - \delta_{b_p 1} = i_p - \delta_{a_p m} + \delta_{b_p m} \quad for \quad p = 1, \ldots, N$$

whereas $b_1, \ldots, b_N$ is a sequence obtained from $a_1, \ldots, a_N$ by changing $\nu_1 - h$ terms 1 to m, and also changing $\nu_m - h$ terms m to 1.

References

[1] T. Arakawa, T. Suzuki, A. Tsuchiya, *Degenerate double affine Hecke algebras and conformal field theory*, in: *Topological Field Theory, Primitive Forms and Related Topics* (Kyoto, 1996), Progress in Mathematics, Vol. 160, Birkhäuser Boston, Boston, MA, 1998, pp. 1–34.

[2] M. Balagović, *Irreducible modules for the degenerate double affine Hecke algebra of type A as submodules of Verma modules*, J. Comb. Theory **A 133** (2015), 97–138.

[3] I. Cherednik, *A unification of Knizhnik–Zamolodchikov and Dunkl operators via affine Hecke algebras*, Invent. Math. **106** (1991), 411–431.

[4] I. Cherednik, *Lectures on Knizhnik–Zamolodchikov equations and Hecke algebras*, in: *Quantum Many-Body Problems and Representation Theory*, MSJ Mem., Vol. 1, Math. Soc. Japan, Tokyo, 1998, pp. 1–96.

[5] В. Г. Дринфельд, *Вырожденные аффинные алгебры Гекке и янгианы*, Функц. анализ и его прил. **20** (1986), вып. 1, 69–70. Engl. transl.: V. Drinfeld, *Degenerate affine Hecke algebras and Yangians*, Funct. Anal. Appl. **20** (1986), no. 1, 58–60.

[6] P. Etingof, V. Ginzburg, *Symplectic reflection algebras, Calogero–Moser space, and deformed Harish-Chandra homomorphism*, Invent. Math. **147** (2002), 243–348.

[7] S. Khoroshkin, M. Nazarov, *Yangians and Mickelsson algebras* I, Transformation Groups **11** (2006), no. 4, 625–658.

[8] S. Khoroshkin, M. Nazarov, *On the functor of Arakawa, Suzuki and Tsuchiya*, in: *Representation Theory, Special Functions and Painlevé Equations* (Kyoto, 2015), Advanced Studies in Pure Mathematics, Vol. 76, Math. Soc. Japan, Tokyo (to appear).

[9] S. Khoroshkin, M. Nazarov, E. Vinberg, *A generalized Harish-Chandra isomorphism*, Adv. Math. **226** (2011), 1168–1180.

[10] S. Khoroshkin, O. Ogievetsky, *Mickelsson algebras and Zhelobenko operators*, J. Algebra **319** (2008), 2113–2165.

[11] G. Lusztig, *Affine Hecke algebras and their graded version*, J. Amer. Math. Soc. **2** (1989), 599–635.

[12] J. Rogawski, *On modules over the Hecke algebras of a p-adic group*, Invent. Math. **79** (1985), 443–465.

[13] T. Suzuki, *Classification of simple modules over degenerate double affine Hecke algebras of type* A, Int. Math. Res. Notices (2003), 2313–2339.

[14] T. Suzuki, *Double affine Hecke algebras, conformal coinvariants and Kostka polynomials*, C. R. Acad. Sci. Paris **I 343** (2006), 383–386.

[15] V. Tarasov, A. Varchenko, *Duality for Knizhnik–Zamolodchikov and dynamical equations*, Acta Appl. Math. **73** (2002), 141–154.

[16] Д. П. Желобенко, *Экстремальные коциклы на группах Вейля*, Функц. анализ и его прил. **21** (1987), вып. 3, 11–21. Engl. transl.: D. Zhelobenko, *Extremal cocycles of Weyl groups*, Funct. Anal. Appl. **21** (1987), no. 3, 183–192.

CHEREDNIK ALGEBRAS AND ZHELOBENKO OPERATORS