A Quantum–Classical Bracket from p-Mechanics

VLADIMIR V. KISIL
School of Mathematics, University of Leeds, Leeds LS2 9JT, UK
On leave from Odessa University

PACS. 03.65.Sq – Semiclassical theories and applications.
PACS. 03.65.Db – Functional analytical methods.
PACS. 03.65.Fd – Algebraic methods.

Abstract. – We provide an answer to the long standing problem of mixing quantum and classical dynamics within a single formalism. The construction is based on p-mechanical derivation of quantum and classical dynamics from the representation theory of the Heisenberg group. To achieve a quantum-classical mixing we take the product of two copies of the Heisenberg group which represent two different Planck's constants. In comparison with earlier guesses our answer contains an extra term of analytical nature, which was not obtained before in purely algebraic setup.

В одну телегу впрячь неможно коня и трепетную лань. *А.С. Пушкин*

Introduction. – There is a strong and persistent interest over decades in *a self-consistent model for an aggregate system, which combines components with both quantum and classical behaviour* (see [1, 2, 4, 7, 8, 17, 18] and references therein). There are various reasons for such an interest. Firstly, there are many questions where considerations of quantum-classical aggregates are unavoidable, e.g. measurement of a quantum system by a classical apparatus, a quantum particle in the classical gravity field, etc. Secondly, even for purely quantum conglomerates we expect that a quantum-classical approximation may be easier for investigation than the purely quantum picture. Thus it is natural that models of quantum-classical interaction became of separate theoretical interest.

The discussion is typically linked to a search of *quantum-classical bracket* which should combine properties of the quantum commutator $[\cdot,\cdot]$ and Poisson's bracket $\{\cdot,\cdot\}$ in the corresponding sectors. Some simple algebraic combinations like

$$\frac{1}{\mathrm{i}\hbar}[A,B]+\frac{1}{2}(\{A,B\}-\{B,A\}) \tag{1}$$

were guessed during the last twenty five years [1, 2, 4] but neither of them turned to be completely satisfactory. Moreover several "no-go" theorems in that direction were proved over the last ten years [7, 18, 19]. Thus the prevailing opinion now is that no consistent quantum-classical bracket is possible. However the explicit similarity between the Hamiltonian descriptions of quantum and classical dynamics repeatedly undermine such a believe.

This paper builds a consistent quantum-classical bracket within the framework of p-mechanics [12–15]. This approach is based on the representation theory of nilpotent Lie groups (the Heisenberg group $\mathbb{H}^n$ in the first instance) and naturally embeds both quantum and classical descriptions. p-Mechanical observables are convolutions on a nilpotent group G and contain both classical and quantum pictures for all values of Planck's constants at the same time. These pictures can be separated by a restriction of p-observables to irreducible representations of G, e.g. by considering their actions on p-mechanical states [5, 6].

The important step [13, 14] is the definition of the universal bracket between convolutions on the Heisenberg group, which are transformed by the above mentioned representations into the quantum commutator (Moyal bracket [21]) and the Poisson bracket correspondingly. Consequently it is sufficient to solve the dynamic equation in p-mechanics in order to obtain both quantum and classical dynamics. Since the universal bracket is based on the usual commutator of convolutions (i.e. inner derivations of the convolution algebra) they satisfy all important requirements, i.e. linearity, antisymmetry, Leibniz and Jacoby identities [7]. Moreover due to presence of antiderivative operator (12) the universal bracket with a Hamiltonian has the dimensionality of time derivative [14]. This approach was extended to quantum field theory in [15]. A brief account of p-mechanics is provided in the first part of this paper.

To construct quantum-classical bracket we develop p-mechanics on the group $\mathbb{D}^n$, which is the product of two copies of the Heisenberg group $\mathbb{H}^n$. The group $\mathbb{D}^n$ was already used to this end in our earlier paper [17] but the right bracket was not derived there due to several reasons: the derivation followed the notorious semiclassical limit procedure; the universal bracket [13, 14] was not known at that time. A correct derivation of quantum-classical bracket in the consistent p-mechanical framework is given in the second part of the present note. This bracket (26) includes as a part the Aleksandrov's bracket (1) together with an extra term of analytical nature, which involves derivative with respect to the second Planck's constant, see (26). This analytic term escapes all previous purely algebraic considerations and "no-go" theorems [7, 18, 19] for the obvious reasons.

Future investigations of these new quantum-classical bracket will be given elsewhere.

The Heisenberg group and p-mechanical bracket. –

The Heisenberg group and its representations. Let (s, x, y), where $x, y \in \mathbb{R}^n$ and $s \in \mathbb{R}$, be an element of the Heisenberg group $\mathbb{H}^n$ [9, 10]. The group law on $\mathbb{H}^n$ is given as follows:

$$(s, x, y) * (s', x', y') = (s + s' + \tfrac{1}{2}\omega(x, y; x', y'), x + x', y + y'), \tag{2}$$

where the non-commutativity is due to ω—the *symplectic form* $\omega(x, y; x', y') = xy' - x'y$ on $\mathbb{R}^{2n}$ [3, § 37]. The Lie algebra $\mathfrak{h}^n$ of $\mathbb{H}^n$ is spanned by left-invariant vector fields

$$S = \partial_s, \quad X_j = \partial_{x_j} - \tfrac{1}{2}y_j\partial_s, \quad Y_j = \partial_{y_j} + \tfrac{1}{2}x_j\partial_s \tag{3}$$

on $\mathbb{H}^n$ with the Heisenberg *commutator relations* $[X_i, Y_j] = \delta_{i,j}S$ and all other commutators vanishing. There is the *co-adjoint representation* [11, § 15.1] $\mathrm{Ad}^*: \mathfrak{h}_n^* \to \mathfrak{h}_n^*$ of $\mathbb{H}^n$:

$$\mathrm{ad}^*(s, x, y) : (h, q, p) \mapsto (h, q + hy, p - hx), \tag{4}$$

where $(h, q, p) \in \mathfrak{h}_n^*$ in bi-orthonormal coordinates to the exponential ones on $\mathfrak{h}^n$. There are two types of orbits in (4) for Ad^*, i.e. Euclidean spaces $\mathbb{R}^{2n}$ and single points:

$$\mathcal{O}_h = \{(h, q, p) : \text{ for } h \neq 0, (q, p) \in \mathbb{R}^{2n}\}, \qquad \mathcal{O}_{(q,p)} = \{(0, q, p) : \text{ for } (q, p) \in \mathbb{R}^{2n}\}. \tag{5}$$

All representations are *induced* [11, § 13] by a character $\chi_h(s, 0, 0) = e^{2\pi \mathrm{i} hs}$ of the centre of $\mathbb{H}^n$ generated by $(h, 0, 0) \in \mathfrak{h}_n^*$ and shifts (4) from the *left* on orbits (5). The explicit formula respecting *physical units* [14] is:

$$\rho_h(s, x, y) : f_h(q, p) \mapsto e^{-2\pi \mathrm{i}(hs + qx + py)} f_h(q - \tfrac{h}{2}y, p + \tfrac{h}{2}x). \tag{6}$$

The Stone–von Neumann theorem [11, § 18.4], [9, Chap. 1, § 5] describes all unitary irreducible representations of $\mathbb{H}^n$ parametrised up to equivalence by two classes of orbits (5):

- The infinite dimensional representations by transformation ρ_h (6) for $h \neq 0$ in Fock [9,10] space $F_2(\mathcal{O}_h) \subset L_2(\mathcal{O}_h)$ of null solutions of Cauchy–Riemann type operators [14].
- The one-dimensional representations which drops out from (6) for $h = 0$:

$$\rho_{(q,p)}(s,x,y) : c \mapsto e^{-2\pi \mathrm{i}(qx+py)}c. \tag{7}$$

Commutative representations (7) are oftenly forgotten, however their union naturally (see the appearance of Poisson bracket in (15)) act as the classic *phase space*: $\mathcal{O}_0 = \bigcup_{(q,p)\in\mathbb{R}^{2n}} \mathcal{O}_{(q,p)}$.

Convolutions (observables) on $\mathbb{H}^n$ and commutator. Using a left invariant measure dg on $\mathbb{H}^n$ the linear space $L_1(\mathbb{H}^n, dg)$ can be upgraded to an algebra with the convolution:

$$(k_1 * k_2)(g) \;=\; \int_{\mathbb{H}^n} k_1(g_1)\, k_2(g_1^{-1}g)\, dg_1. \tag{8}$$

Convolutions on $\mathbb{H}^n$ are *observables* in p-mechanic [12,14]. Inner *derivations* D_k of the convolution algebra $L_1(\mathbb{H}^n)$ are given by the *commutator*:

$$D_k : f \mapsto [k,f] \;=\; k*f - f*k \int_{\mathbb{H}^n} k(g_1)\left(f(g_1^{-1}g) - f(gg_1^{-1})\right)\, dg_1. \tag{9}$$

A unitary representation ρ_h of $\mathbb{H}^n$ extends to $L_1(\mathbb{H}^n, dg)$:

$$\rho_h(k) = \int_{\mathbb{H}^n} k(g)\rho_h(g)\, dg. \tag{10}$$

Thus $\rho_h(k)$ for a fixed $h \neq 0$ depends only on $\hat{k}_s(h,x,y) = \int k(s,x,y)\, e^{-2\pi \mathrm{i} hs}\, ds$—the partial Fourier transform $s \mapsto h$ of $k(s,x,y)$. Consequently the representation of commutator (9) depends only on:

$$[k',k]\hat{_s} \;=\; 2\mathrm{i}\int_{\mathbb{R}^{2n}} \sin(\pi h(xy'-yx'))\, \hat{k}'_s(h,x',y')\, \hat{k}_s(h,x-x',y-y')\, dx'dy', \tag{11}$$

which is exactly the Moyal bracket [21] for the full Fourier transforms of k' and k. Also it vanishes for $h = 0$ as can be expected from the commutativity of representations (7).

p-Mechanical bracket on $\mathbb{H}^n$. An antiderivative $\mathcal{A}$ is a scalar multiple of a right inverse operator to the vector field $S \in \mathfrak{h}^n$ (3):

$$S\mathcal{A} = 4\pi^2 I, \text{ or } \mathcal{A}e^{2\pi \mathrm{i} hs} = \begin{cases} \frac{2\pi}{\mathrm{i}h} e^{2\pi \mathrm{i} hs}, & \text{if } h \neq 0, \\ 4\pi^2 s, & \text{if } h = 0. \end{cases} \tag{12}$$

It can be extended by the linearity to $L_1(\mathbb{H}^n)$. We introduce p-mechanical bracket [13,14] on $L_1(\mathbb{H}^n)$ as a modified commutator of observables:

$$\{[k_1,k_2]\} = (k_1 * k_2 - k_2 * k_1)\mathcal{A}. \tag{13}$$

Then from (10) one gets $\rho_h(\mathcal{A}k) = (ih)^{-1}\rho_h(k)$ for $h \neq 0$. Consequently the modification of (11) for $h \neq 0$ is only slightly different from the original one:

$$\{[k',k]\}\hat{_s} \;=\; \int_{\mathbb{R}^{2n}} \frac{2\pi}{h} \sin(\pi h(xy'-yx'))\, \hat{k}'_s(h,x',y')\, \hat{k}_s(h,x-x',y-y')\, dx'dy'. \tag{14}$$

However the last expression for $h = 0$ is significantly distinct from (11), which vanishes as noted above. From the natural assignment $\frac{4\pi}{h}\sin(\pi h(xy' - yx')) = 4\pi^2(xy' - yx')$ for $h = 0$ we get the Poisson bracket for the Fourier transforms of k' and k defined on $\mathcal{O}_0$:

$$\rho_{(q,p)}\,\{[k',k]\} = \frac{\partial \hat{k}'}{\partial q}\frac{\partial \hat{k}}{\partial p} - \frac{\partial \hat{k}'}{\partial p}\frac{\partial \hat{k}}{\partial q}. \tag{15}$$

Furthermore the dynamical equation [13, 14]

$$\dot{f} = \{[H, f]\} \tag{16}$$

based on the bracket (13) with a Hamiltonian $H(g)$ for an observable $f(g)$ is reduced [13, 14] to Moyal's and Poisson's equations by ρ_h with $h \neq 0$ and $h = 0$ correspondingly. The same connections are true for the solutions of these three equations, see [6, 13, 14].

Mixed Quantum-Classical Bracket. –

A nilpotent group with two dimensional centre. To derive quantum-classical bracket we again use the "quantum-classical" group $\mathbb{D}^n = \mathbb{H}^n \oplus \mathbb{H}^n$ [17]. This is a step 2 nilpotent Lie group of the (real) dimension $4n + 2$. The group law is given by the formula:

$$(g_1; g_2) * (g_1'; g_2') = (g_1 * g_1'; g_2 * g_2'), \tag{17}$$

where $g_i^{(\prime)} = (s_i^{(\prime)}, x_i^{(\prime)}, y_i^{(\prime)}) \in \mathbb{H}^n$, $i = 1, 2$ and products $g_i * g_i'$ are the same as in (2).

The group $\mathbb{D}^n$ has a two-dimensional centre $\mathbb{Z} = \{(s_1, 0, 0; s_2, 0, 0) \mid s_1, s_2 \in \mathbb{R}\}$. The irreducible representations of a nilpotent group $\mathbb{D}^n$ are induced [11, § 13.4], [20, § 6.2] by the characters of the centre: $\mu : (s_1, s_2) \mapsto \exp(-2\pi\mathrm{i}(h_1 s_1 + h_2 s_2))$. For $h_1 h_2 \neq 0$ the induced representation coincides with the irreducible representation of $\mathbb{H}^{n+n}$:

$$\rho_{(h_1;h_2)}(g_1; g_2) = \rho_{h_1}(g_1)\,\rho_{h_2}(g_2) \tag{18}$$

This corresponds to *purely quantum* behavior of both sets of variables (x_1, y_1) and (x_2, y_2). The trivial character $h_1 = h_2 = 0$ gives the family of one-dimensional (*purely classical*) representations parametrised by points of $\mathbb{R}^{4n}$:

$$\rho_{(q_1,p_1;q_2,p_2)}(s_1, x_1, y_1; s_2, x_2, y_2) = e^{-2\pi\mathrm{i}(x_1 p_1 + y_1 q_1 + x_2 p_2 + y_2 q_2)} \tag{19}$$

These cases for $\mathbb{H}^n$ were described above and studied in details in [6,13,14]. A new situation appears when $h_1 \neq 0$ and $h_2 = 0$ corresponding to quantum behavior for (x_1, y_1) and classical behavior for (x_2, y_2). The choice $h_1 = 0$, $h_2 \neq 0$ swaps the quantum and classical parts. The quantum-classical representation is given by

$$\rho_{(h;q,p)}(g_1; g_2) \;=\; \rho_h(g_1)\rho_{(q,p)}(g_2) = \rho_h(s_1, x_1, y_1)e^{-2\pi\mathrm{i}(qx_2 + py_2)}, \tag{20}$$

where $q, p \in \mathbb{R}^n$ and $h \in \mathbb{R} \setminus \{0\}$. In this representation a convolution (observable) on $\mathbb{D}^n$ generates a function on the classic phase space $\mathbb{R}^{2n}$ with values in space of quantum operators acting of $L_2(\mathbb{R}^n)$, cf. [1], or explicitly:

$$\begin{aligned} \rho_{(h;q,p)}k &= \int_{\mathbb{D}^n} k(g_1; g_2)\,\rho_h(g_1)e^{-2\pi\mathrm{i}(qx_2 + py_2)}\,dg_1\,dg_2 \\ &= \int_{\mathbb{H}^n} \hat{k}_2(s_1, x_1, y_1; 0, q, p)\,\rho_h(s_1, x_1, y_1)\,dg_1 \end{aligned} \tag{21}$$

where $\hat{k}_2$ is the partial Fourier transform of k with respect to variables $(s_2, x_2, y_2) \mapsto (h, q, p)$.

The Mixed Bracket. We define p-bracket in the case of $\mathbb{D}^n$ similarly to (13). Although this is not a unique option, some other similar definitions may be of interest as well.

Definition 1. The *p-mechanical bracket* of two convolutions (observables) $k_1(g_1; g_2)$ and $k_2(g_1; g_2)$ on the group $\mathbb{D}^n$ is defined as follows:

$$\{[k_1, k_2]\} = (k_1 * k_2 - k_2 * k_1)(\mathcal{A}_1 + \mathcal{A}_2), \tag{22}$$

where $*$ denotes the group convolution on $\mathbb{D}^n$. $\mathcal{A}_1$ and $\mathcal{A}_2$ are antiderivatives with respect to the variable s_1 and s_2 correspondingly, cf. (12).

Consistence of this definition, cf. [7], is given by:

Lemma 2. *The p-mechanical bracket* (22) *is linear, antisymmetric, satisfy Leibniz and Jacoby identities. Moreover p-mechanical bracket with a Hamiltonian has the dimensionality of time derivative.*

We define *p-mechanisation* [14] of a classical observable $f(q,p)$ is given by the Weyl (symmetrized) calculus [9] defined on the generators as follows:

$$q_j \mapsto Q_j = \delta'_{x_j}(g_1; g_2), \qquad p_1 \mapsto P_j = \chi'_{s_k}(s_1+s_2) * \delta'_{y_j}(g_1; g_2), \quad j = 1,2 \text{ and } k = 3-j, \tag{23}$$

where δ'_z is the derivative of the Dirac delta function with respect to the variable z and χ'_{s_k} is the derivative of the Heaviside step function such that $\chi'_z(z) = \delta(z)$. Using the identity

$$\int_{\mathbb{R}} \chi'_{s_k}(s_1 + s_2)\, e^{-2\pi \mathrm{i}(h_1 s_1 + h_2 s_2)}\, dz = \frac{h_k}{h_1 + h_2} \qquad \text{we get that} \qquad \{[Q_i, P_j]\} = \delta_{ij} I, \tag{24}$$

and all other brackets vanish. Representations of distributions (23) and the bracket (22) are:

	$\rho_{(h_1;h_2)}$	$\rho_{(h;q,p)}$	$\partial_{h_2}\rho_{(h;q,p)}\big\vert_{h_2=0}$	$\rho_{(q_1,p_1;q_2,p_2)}$
Q_j	$\partial_{x_j} - \frac{\mathrm{i}h_j}{2} y_j$	$\partial_{x_1} - \mathrm{i}hy_1/2$ $\mathrm{i}q$	$0, \quad \text{if } j=1$ $\partial_p/2, \quad \text{if } j=2$	q_j
P_j	$\frac{h_k}{h_1+h_2}\left(\partial_{y_j} + \frac{\mathrm{i}h_j}{2} x_j\right)$	0 $\mathrm{i}p$	$\partial_{y_1}/h + \mathrm{i}x_1/2, \quad \text{if } j=1$ $-\mathrm{i}p/h - \partial_q/2, \quad \text{if } j=2$	p_j
$\{[K_1, K_2]\}$	$\left(\frac{1}{\mathrm{i}h_1} + \frac{1}{\mathrm{i}h_2}\right)[K_1, K_2]$	$[K_1, K_2]_{qc}$		$\{\hat{k}_1, \hat{k}_2\}$

(25)

where the bracket $[\cdot,\cdot]_{qc}$ in the quantum-classical case is defined by the expression:

$$[K_1, K_2]_{qc} \;=\; \frac{1}{\mathrm{i}h}[K_1, K_2] + \frac{1}{2}\left(\{K_1, K_2\} - \{K_2, K_1\}\right) - \mathrm{i}\partial_{h_2}[K_1, K_2]\big\vert_{h_2=0}. \tag{26}$$

Calculations of the two first terms in (26) is similar to p-bracket [13, 14], the third term is:

$$\int_{\mathbb{D}^n}\int_{\mathbb{D}^n} (k_1(g'_1; g'_2)\, k_2(g'^{-1}_1 g_1; g''_2) - k_2(g'_1; g'_2)\, k_1(g'^{-1}_1 g_1; g''_2))$$
$$\times (s''_2 + s'_2)\, e^{-2\pi \mathrm{i}(qx'_2 + py'_2 + qx''_2 + py''_2)}\, dg'_2\, dg''_2 dg'_1\, \rho_h(g_1)\, dg_1.$$

The complete derivation will be given elsewhere. The derivative ∂_{h_2} in (26) highlights the important difference between Aleksandrov's [1] and our approach: *quantum-classical observables are not operator valued functions on the classical phase space but rather first jets* [16] *of such functions.* This explain the appearance of the fourth column in (25).

By algebraic inheritance [13] the quantum-classic bracket (26) enjoys all the properties from Lem. 2. Moreover quantum-classical bracket coincides with the Moyal bracket for purely quantum observables and the Poisson bracket for purely classical ones. Let a p-mechanical observable $f(t; g_1; g_2)$, which is a function on $\mathbb{R} \times \mathbb{D}^n$, be a solution of the equation:

$$\frac{d}{dt} f(t; g_1; g_2) = \{[f, H]\} \tag{27}$$

with a Hamiltonian $H(g_1, g_2)$ on $\mathbb{D}^n$. Then $f(t; g_1; g_2)$ provide consistent dynamics (in the sense of [7]) under either representation (18)–(20).

Example 3 *(Dynamics with two different Planck's constants, cf.* [18]*).* Let p-mechanical Hamiltonian is defined by such a distribution on $\mathbb{D}^n$ (see definitions (23)):

$$H \;=\; Q_1P_2 - Q_2P_1 = \chi'_{s_1}(s_1+s_2) * \delta^{(2)}_{x_1,y_2}(g_1; g_2) - \chi'_{s_2}(s_1+s_2) * \delta^{(2)}_{x_2,y_1}(g_1; g_2). \tag{28}$$

In the classic-classical representation (19) it produces (see the last column of (25)) the quadratic Hamiltonian $H_{cc} = q_1p_2 - q_2p_1$, which generates a simple rotational dynamic:

$$q_1(t) = \cos t\, q_1(0) + \sin t\, q_2(0), \quad q_2(t) = -\sin t\, q_1(0) + \cos t\, q_2(0), \tag{29}$$

$$p_1(t) = \cos t\, p_1(0) + \sin t\, p_2(0), \quad p_2(t) = -\sin t\, p_1(0) + \cos t\, p_2(0). \tag{30}$$

In the quantum-quantum representation (18) defined by two Planck's constants h_1 and h_2 ($h_1h_2 \neq 0$) the Hamiltonian becomes (see the first column of (25)):

$$H_{qq} = \tfrac{h_1}{h_1+h_2}\left(\partial_{x_1} - \tfrac{\mathrm{i}h_1}{2}y_1\right)\left(\partial_{y_2} + \tfrac{\mathrm{i}h_2}{2}x_2\right) - \tfrac{h_2}{h_1+h_2}\left(\partial_{x_2} - \tfrac{\mathrm{i}h_2}{2}y_2\right)\left(\partial_{y_1} + \tfrac{\mathrm{i}h_1}{2}x_1\right).$$

The dynamic from the bracket $\left(\frac{1}{\mathrm{i}h_1} + \frac{1}{\mathrm{i}h_2}\right)[H_{qq}, f]$ in (25) is the coordinate map on $\mathbb{D}^n$:

$$x_1(t) = \cos t\, x_1(0) + \sin t\, x_2(0), \qquad x_2(t) = -\sin t\, x_1(0) + \cos t\, x_2(0), \tag{31}$$

$$h_2y_1(t) = h_2\cos t\, y_1(0) + h_1 \sin t\, y_2(0), \quad h_1y_2(t) = -h_2\sin t\, y_1(0) + h_1\cos t\, y_2(0). \tag{32}$$

For $h_1 = h_2$ this coincides with the standard quantisation of the classical dynamics (29)–(30).

The quantum-classic Hamiltonian is the 1-jet (see the middle column of (25)):

$$H_{qc} = \left(\mathrm{i}(\partial_{x_1} - \tfrac{ih}{2}y_1)p, \quad -\tfrac{\mathrm{i}}{h}\left(\partial_{x_1} - \tfrac{ih}{2}y_1\right)\left(p - \tfrac{ih}{2}\partial_q\right) - \tfrac{\mathrm{i}}{h}q\left(\partial_{y_1} + \tfrac{ih}{2}x_1\right)\right).$$

Note, that Aleksandrov's bracket (1) of H_{qc} with $\rho_{(h;q,p)}(Q_1)$ vanish and thus do not generate any dynamics for this observable. However $[H_{qq}, \rho_{(h_1;h_2)}(Q_1)] = \frac{\mathrm{i}h_1h_2}{h_1+h_2}\rho_{(h_1;h_2)}(Q_2)$ and thus the third term in the bracket (26) of H_{qc} and $\rho_{(h;q,p)}(Q_1)$ is equal to $\rho_{(h;q,p)}(Q_2) = \mathrm{i}q$ (this is also the value of the entire bracket (26)). Together with the value of $[H_{qc}, \rho_{(h;q,p)}(Q_2)]_{qc} = -\rho_{(h;q,p)}(Q_1)$ this defines quantum-classic dynamics of coordinates as the partial Fourier transform $x_2 \mapsto q$ of the quantum-quantum coordinate map (31).

Similarly we calculate that $[H_{qc}, \rho_{(h;q,p)}(P_1)]_{qc} = \rho_{(h;q,p)}(P_2) = \mathrm{i}p$ and $[H_{qc}, \rho_{(h;q,p)}(P_2)]_{qc} = -\rho_{(h;q,p)}(P_1)$. Note that $\rho_{(h;q,p)}(P_1)$ is the 1-jet with the value $(0, \frac{1}{h}\partial_{y_1} + \frac{\mathrm{i}}{2}x_1)$ according to the (25) and the quantum-classic bracket depends from its both components. The quantum-classic dynamic of momenta is obtained from (32) by prolongation [16] into the 1-jet space with respect to the variable h_2 at point $h_2 = 0$.

Conclusion. – The sum (1) of first two terms in (26) was proposed [1, 4] as a version of quantum-classical bracket. It was also obtained by approximation arguments within p-mechanical approach in [17] as a part of the true bracket unknown at that time. However the expression (1) violates the Jacobi identity and Leibniz rule (i.e. is not a derivative), as a consequence it could not be used for a consistent dynamic equation [7]. Our new bracket (26) has one extra term which makes it satisfactory to this end. This term is of an analytical nature (i.e. involves a derivative in Planck's constant) and is hard to guess from algebraic manipulations with the quantum commutator and Poisson's bracket. For the same reasons our bracket (26) are immunised against the "no-go" theorem of the type proved in [7, 18, 19]. We present an example of a dynamics (31)–(32), which mixes two quantum sectors with different Planck's constants, and demonstrate the quantum-classic dynamics in the 1-jet space.

I am grateful to Prof. G. Ingold and anonymous referees who helped to improve this letter.

References

[1] I.V. Aleksandrov, Z. Naturforsch. A **36** (1981), no. 8, 902–908. **MR** # 82k:82034
[2] Arlen Anderson, *Coupling "classical" and quantum variables*, Quantum classical correspondence (Philadelphia, PA, 1994), Internat. Press, Cambridge, MA, 1997, `quant-ph/9511014`, pp. 213–227. **MR** # MR1610315
[3] V. I. Arnol'd, *Mathematical methods of classical mechanics*, Graduate Texts in Mathematics, vol. 60, Springer-Verlag, New York, 1991. **MR** # 96c:70001
[4] Wayne Boucher and Jennie Traschen, Phys. Rev. **D37** (1988), 3522.
[5] Alastair Brodlie, Internat. J. Theoret. Phys. **42** (2003), no. 8, 1707–1731, `quant-ph/0303142`.
[6] Alastair Brodlie and Vladimir V. Kisil, *Observables and states in p-mechanics*, Advances in mathematics research. Vol. 5, Nova Sci. Publ., Hauppauge, NY, 2003, `quant-ph/0304023`, pp. 101–136. **MR** # MR2117375
[7] J. Caro and L. L. Salcedo, Phys. Rev. **A60** (1999), 842–852, `quant-ph/9812046`.
[8] L. Diosi, N. Gisin, and W.T. Strunz, Phys. Rev. A, At. Mol. Opt. Phys. **61** (2000), no. 2, 022108, `quant-ph/9902069`.
[9] Gerald B. Folland, *Harmonic analysis in phase space*, Annals of Mathematics Studies, vol. 122, Princeton University Press, Princeton, NJ, 1989. **MR** # 92k:22017
[10] Roger Howe, J. Funct. Anal. **38** (1980), no. 2, 188–254. **MR** # 83b:35166
[11] A. A. Kirillov, *Elements of the theory of representations*, Springer-Verlag, Berlin, 1976, Grundlehren der Mathematischen Wissenschaften, Band 220. **MR** # 54 #447
[12] Vladimir V. Kisil, J. Natur. Geom. **9** (1996), no. 1, 1–14, **MR** # 96m:81112. `funct-an/9405002`.
[13] Vladimir V. Kisil, Internat. J. Theoret. Phys. **41** (2002), no. 1, 63–77, `math-ph/0007030`. **MR** # 2003b:81105
[14] Vladimir V. Kisil, J. Phys. A **37** (2004), no. 1, 183–204, `quant-ph/0212101`. **MR** # 2005c:81078. **Zbl** # 1045.81032.
[15] Vladimir V. Kisil, Rep. Math. Phys. **56** (2005), no. 2, 161–174, `quant-ph/0402035`.
[16] Peter J. Olver, *Applications of Lie groups to differential equations*, second ed., Springer-Verlag, New York, 1993. **MR** # 94g:58260
[17] Oleg V. Prezhdo and Vladimir V. Kisil, Phys. Rev. A (3) **56** (1997), no. 1, 162–175, **MR** # 99j:81010. `quant-ph/9610016`.
[18] Debendranath Sahoo, J. Phys. A: Math. Gen. **37** (2004), 997–1010, `quant-ph/0301044`.
[19] L.L. Salcedo, Phys. Rev. **A54** (1996), 3657–3660, `hep-th/9509089`.
[20] Michael E. Taylor, *Noncommutative harmonic analysis*, Mathematical Surveys and Monographs, vol. 22, American Mathematical Society, Providence, RI, 1986. **MR** # 88a:22021
[21] Cosmas Zachos, Internat. J. Modern Phys. A **17** (2002), no. 3, 297–316, `hep-th/0110114`. **MR** # 1 888 937