Immersion and Invariance Adaptive Control for Discrete-Time Systems in Strict Feedback Form with Input Delay and Disturbances

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Abstract: This work presents a new adaptive control algorithm for a class of discrete-time systems in strict feedback form with input delay and disturbances. The Immersion and Invariance formulation is employed to estimate the disturbances and to compensate the effect of the input delay, resulting in a recursive control law. The stability of the closed-loop system is studied employing Lyapunov functions and guidelines for tuning the controller parameters are presented. An explicit expression of the control law in case of multiple simultaneous disturbances is provided for the tracking problem of a pneumatic drive. The effectiveness of the control algorithm is demonstrated with numerical simulations considering disturbances and input delay representative of the application.

Keywords: Nonlinear adaptive control; Input delay; Discrete-time systems

1. Introduction

Systems with input delay and disturbances pose remarkable challenges from a control perspective and have been the object of extensive research [1]–[3]. Traditionally, most research has been focusing on continuous-time linear systems [4]–[6] and on systems with state delays and disturbances [7]–[9]. Notably, an adaptive algorithm for a continuous-time linear system with input delay and sinusoidal disturbance was developed in [1]. The case of linear time invariant systems with unknown disturbance and input delay was studied in [5] and an adaptive controller for time-varying nonlinear systems with input delay and bounded parametric uncertainties was proposed in [3]. In parallel, control methods for discrete-time linear systems with input delay or state delay were also developed: the stabilisation of linear time-varying system with input delay was studied in [10]; a recursive control law for discrete-time systems with both input delay and state delays was presented in [11]; the effect of the input delay on a linear system was treated as an additive disturbance in [12]; in a different stream of research, model-based prediction methods were proposed for uncertain discrete-time linear systems in [13], [14]. However, the case of discrete-time nonlinear systems with input delay and disturbances has remained comparatively unexplored. Notably, a discrete-time predictor combined with Lyapunov redesign was proposed in [2] for discrete-time systems with input delay and bounded uncertainties. Recently, the stabilization of a nonlinear system with input delay, but without disturbances, was studied in [15] employing the Immersion and Invariance (I&I) formulation [16].

The main contribution of this work is a new adaptive control algorithm for nonlinear discrete-time systems in strict feedback form with input delay and disturbances. In particular, the algorithm consists of a recursive control law that includes the adaptive compensation of the disturbances based on the I&I formulation [17], hence extending the scope of [15]. Differently from [2], the algorithm is applicable to multiple disturbances without assumptions on their bounds. After outlining the control design, the stability analysis is conducted employing Lyapunov functions and guidelines for tuning the controller parameters are provided. Subsequently, an explicit expression of the control law for multiple simultaneous disturbances is presented for the tracking problem of a pneumatic drive, which has motivated this study. The effectiveness of the proposed approach is demonstrated with numerical simulations considering disturbances and input delay representative of the chosen application.

The rest of the paper is organized as follows. The problem formulation is detailed in Section 2. The control design is outlined in Section 3 and the stability analysis is presented in Section 4. The control design for a discrete-time model representative of a pneumatic drive is outlined in Section 5. The simulation results are reported in Section 6, while Section 7 contains the concluding remarks.

2. Problem Formulation

In this work the following discrete time nonlinear system in strict feedback form is considered:

$$x(t+1) = f_0(x) + g(x)u(t-N) + f_1(x)\theta \tag{1}$$

with $f_0(x), f_1(x), g(x)$ smooth functions of the state (i.e. infinitely differentiable in x), so that $g(x) \neq 0, \forall x \in \Re^n$, the term $f_1(x)\theta$ represents a disturbance where θ is an unknown constant parameter, and the integer $N > 0$ indicates the input delay. An important property of finite dimensional discrete-time systems in strict feedback form which this work relies upon is that they retain their structure (i.e. finite dimensional strict feedback form) in the presence of input delay [2]. Consequently, we can express (1) in strict feedback form:

$$\begin{cases} x^+ = f_0(x) + g(x)y_1 + f_1(x)\theta \\ y_1^+ = y_2 \\ \quad \cdots \\ y_N^+ = u \end{cases} \tag{2}$$

where the terms $y_1, y_2, .., y_N$ represent the control input at previous sampling intervals so that $y_1 = u(t-N)$. For notational simplicity the time dependency is indicated with a superscript: $x^+ = x(t+1)$; $x^{+N} = x(t+N)$. Notably, (1) is similar to the class of systems considered in [2], but with the addition of the disturbance $f_1(x)\theta$. The aim of this work is finding a control law u that stabilises system (1). To this end the following assumptions are made:

Assumption 1: The sampling interval, here assumed unitary without loss of generality, is constant, while the input delay is a known multiple of the sampling interval (i.e. $N \in \mathbb{N}^+$).

Assumption 2: The disturbance $f_1(x)\theta$ can be variable and nonlinear, while θ is assumed constant and no restrictions are imposed on its bounds. This assumption is additional to those made in [15].

Assumption 3: The un-delayed version of system (1) with known parameter θ^* ($N = 0, \theta = \theta^*$) is stabilizable with an appropriate control law $u = \gamma(x, \theta^*)$ according to a given Lyapunov function $V_1 \geq 0$, and the system state converges to zero. For instance, this assumption is trivially satisfied by $V_1 = kx^2$, where $k \in \Re^+$ is a positive constant, and $u = -f_0(x)/g(x) - f_1(x)\theta^*/g(x)$.

3. Control Design

In this section the control design is outlined for system (2) initially considering the case $N = 1$ and then extending the result to the general case $N > 1$. The parameter θ is estimated adaptively using the I&I method in its discrete-time form [17]. In case $N = 1$ the stabilising control law for the un-delayed system with known parameter θ^* (ref. *Assumption 3*) is indicated in what follows as $\tilde{y}_1 = \gamma(x, \theta^*)$. Two estimation errors z_1, z_2 are defined as:

$$z_1 = \hat{\theta}+\beta_1(x^-, x) - \theta = \hat{\theta} + \beta_1(x^-)x - \theta$$
$$z_2 = \tilde{y}_1 - y_1 \tag{3}$$

The terms $\hat{\theta}, \beta_1(x^-)x$ in z_1 are the state-independent part and the state-dependent part of the disturbance estimate, with β_1 representing the first design parameter of the adaptive algorithm. The term z_2 can be interpreted as a prediction error and represents the discrepancy between the control input of the un-delayed system $\tilde{y}_1$ and the new control input at the previous time step $y_1 = u(t-1)$. Computing (3) at the next time step and substituting x^+ from (2) and θ, y_1 from (3) we obtain:

$$z_1^+ = \hat{\theta}^+ + \beta_1(x)\left(f_0(x) + g(x)(\tilde{y}_1 - z_2) + f_1(x)\big(\hat{\theta} + \beta_1(x^-)x - z_1\big)\right) - \theta$$
$$z_2^+ = \tilde{y}_1^+ - u \tag{4}$$

The update law for $\hat{\theta}$ and the control law u are chosen in order to enforce the convergence of z_1, z_2 to zero. Introducing the second design parameter β_2 we define:

$$\hat{\theta}^+ = \hat{\theta} - \beta_1(x)\left(f_0(x) + g(x)\tilde{y}_1 + f_1(x)\big(\hat{\theta} + \beta_1(x^-)x\big)\right) + \beta_1(x^-)x$$
$$u = \tilde{y}_1^+ - \ \beta_2 z_2 \tag{5}$$

In summary, the control law for system (2) with $N = 1$ is:

$$\begin{gathered}
\tilde{y}_1 = \gamma(x, \hat{\theta} + \beta_1(x^-)x) \\
x^+ = f_0(x) + g(x)\tilde{y}_1 + f_1(x)(\hat{\theta} + \beta_1(x^-)x) \\
\hat{\theta}^+ = \hat{\theta} - \beta_1(x)\left(f_0(x) + g(x)\tilde{y}_1 + f_1(x)\big(\hat{\theta} + \beta_1(x^-)x\big)\right) + \beta_1(x^-)x \\
\tilde{y}_1^+ = \gamma(x^+, \hat{\theta}^+ + \beta_1(x)x^+) \\
u = \tilde{y}_1^+ - \beta_2 z_2
\end{gathered} \tag{6}$$

The first two equations in (6) represent the control input for the un-delayed system and the predicted state at the next time step. The third equation represents the update law for $\hat{\theta}$. The last equation expresses the control input, consisting of the recursive part $\tilde{y}_1^+$ and of a corrective term dependent on the estimation error z_2, as defined in (3).

The extension of (6) to the general case $N > 1$ is immediate and involves defining additional estimation and prediction errors z_1, z_2, z_{N+1} as:

$$\begin{gathered}
z_1 = (\hat{\theta} + \beta_1(x^-)x) - \theta \\
z_2 = \tilde{y}_1 - y_1 \\
\dots \\
z_{N+1} = \tilde{y}_N - y_N
\end{gathered} \tag{7}$$

Computing (7) at the next time step we obtain:

$$\begin{gathered}
z_1^+ = \hat{\theta}^+ + \beta_1(x)\left(f_0(x) + g(x)(\tilde{y}_1 - z_2) + f_1(x)\big(\hat{\theta} + \beta_1(x^-)x - z_1\big)\right) - \theta \\
z_2^+ = \tilde{y}_1^+ - y_2 \\
\dots \\
z_{N+1}^+ = \tilde{y}_N^+ - u
\end{gathered} \tag{8}$$

The update law (5) becomes in this case:

$$\begin{gathered}\hat{\theta}^{+} = \hat{\theta} - \beta_1(x)\left(f_0(x) + g(x)\tilde{y}_1 + f_1(x)\left(\hat{\theta} + \beta_1(x^-)x\right)\right) + \beta_1(x^-)x \\ \tilde{y}_2 = \tilde{y}_1^{+} - \beta_2 z_2 \\ \dots \\ u = \tilde{y}_N^{+} - \beta_2 z_{N+1}\end{gathered} \tag{9}$$

In conclusion, the control law (6) for the general case $N > 1$ is:

$$\begin{gathered}\tilde{y}_1 = \gamma(x, \hat{\theta} + \beta_1(x^-)x) \\ x^{+} = f_0(x) + g(x)\tilde{y}_1 + f_1(x)(\hat{\theta} + \beta_1(x^-)x) \\ \hat{\theta}^{+} = \hat{\theta} - \beta_1(x)\left(f_0(x) + g(x)\tilde{y}_1 + f_1(x)\left(\hat{\theta} + \beta_1(x^-)x\right)\right) + \beta_1(x^-)x \\ \tilde{y}_1^{+} = \gamma\left(x^{+}, \hat{\theta}^{+} + \beta_1(x)x^{+}\right) \\ \tilde{y}_2 = \tilde{y}_1^{+} - \beta_2 z_2 \\ \dots \\ u = \tilde{y}_N^{+} - \beta_2 z_{N+1}\end{gathered} \tag{10}$$

Remark 1: Similarly to the case $N = 1$, the update law (9) only employs two design parameters β_1, β_2. While introducing additional parameters β_i for each z_i with $i > 2$ is possible, in practice the same value (i.e. β_2) can be used for all z_2, z_N considering that the sampling interval is constant (ref. *Assumption 1*). Finally, while the design parameter $\beta_1(x^-)$ can be set constant, in general it is a nonlinear function of the state to be chosen as part of the design. Conversely, the parameter β_2 is typically constant. Guidelines for tuning the parameters $\beta_1(x^-), \beta_2$ are discussed in the next section.

Remark 2: The corrective term $\beta_2 z_{N+1}$ in (10) is computed recursively from (7),(9) and contains the contributions of each time step:

$$\begin{gathered}z_{N+1} = \tilde{y}_N - y_N \\ \tilde{y}_N = \tilde{y}_{N-1}^{+} - \beta_2 z_N \\ \dots \\ \tilde{y}_3 = \tilde{y}_2^{+} - \beta_2 z_3 \\ \tilde{y}_2 = \tilde{y}_1^{+} - \beta_2 z_2\end{gathered} \tag{11}$$

Substituting (9) into (8) the recursive relation becomes apparent:

$$\begin{gathered}\tilde{y}_2 = \tilde{y}_1^{+} - \beta_2 z_2 \\ \tilde{y}_3 = \tilde{y}_2^{+} - \beta_2 z_3 = \tilde{y}_1^{++} - \beta_2(\beta_2 z_2 + 2z_3) \\ \tilde{y}_4 = \tilde{y}_3^{+} - \beta_2 z_4 = \tilde{y}_1^{+++} - \beta_2(\beta_2^2 z_2 + 3\beta_2 z_3 + 3z_4) \\ \tilde{y}_5 = \tilde{y}_4^{+} - \beta_2 z_5 = \tilde{y}_1^{++++} - \beta_2(\beta_2^3 z_2 + 4\beta_2^2 z_3 + 6\beta_2 z_4 + 4z_5) \\ \dots\end{gathered} \tag{12}$$

Notably, the coefficients of z_2, z_N in (12) correspond to the binomial coefficients. Consequently, the control law in (10) can be rewritten in a more compact form as:

$$u = \tilde{y}_1^{+N} - \sum_{i=1}^{N} \beta_2^{N+1-i}\left(\frac{N!}{(i-1)!\,(N-i+1)!}\right) z_{i+1} \tag{13}$$

Remark 3: For comparison purposes, a variation of the above design is defined as follows.

$$\begin{gathered}\hat{\theta}^{+} = \hat{\theta} - \beta_1(x)\left(f_0(x) + g(x)y_1 + f_1(x)\left(\hat{\theta} + \beta_1(x^-)x\right)\right) + \beta_1(x^-)x \\ \tilde{y}_2 = \tilde{y}_1^{+} = \gamma\left(x^{+}, \hat{\theta} + \beta_1(x^-)x\right) \\ \dots \\ u = \tilde{y}_1^{+N} - \beta_2 z_2\end{gathered} \tag{14}$$

Differently from (10) where the adaptation law $\hat{\theta}^+$ is updated N times at each time step based on the predicted state x^{+i}, the update only occurs once in (14) and it is based on the previous value of the control input $y_1 = u^{-N}$. This alternative approach results in decoupling the dynamics of z_1 and z_2, z_{N+1}. Additionally, only introducing the error z_2 once in the last step of the recursive algorithm (14) further simplifies the stability analysis (ref. Section 4). In practice, (10) and (14) coincide for $N = 1$ while their difference becomes more substantial for larger N.

4. Stability Analysis

Since system (1) can be expressed in strict feedback form (2), the stability of the closed-loop system (2)-(6) can be studied in a similar way to [2]. As initial result, the convergence of the errors z_1, z_2 to zero, which represents a necessary condition for the stability of the closed-loop system, is studied for the case $N = 1$.

Lemma 1: Considering the discrete-time system (2) with disturbance $f_1(x)\theta$ and input delay $N = 1$, the update law (5) ensures that z_1, z_2 in (3) are bounded and converge to zero for some $(\beta_1, \beta_2) \in \mathfrak{R}^2$ that satisfy the inequalities $|1 - \beta_1 f_1| < 1,\ \beta_2^2 + (\beta_1 g)^2 < 1$.

Proof: Exploiting the similarity to Proposition 1 in [17], the following Lyapunov function candidate is chosen, where the dependency of f_1, g, β_1 on the state x is omitted for brevity (i.e. β_1 stands for $\beta_1(x)$, while β_1^- stands for $\beta_1(x^-)$):

$$V_2 = z_1^2 + z_2^2 \geq 0 \tag{15}$$

For z_1, z_2 to converge to zero, V_2 in (15) should decrease at each time step. To confirm this, (15) is computed for the next time step substituting (5) and (4). Computing the squares and regrouping the terms we obtain:

$$V_2^+ = z_1^2(1 - \beta_1 f_1)^2 + z_2^2(\beta_1 g)^2 - 2z_1 z_2(1 - \beta_1 f_1)\beta_1 g + z_2^2\beta_2^2 \tag{16}$$

At this point the following inequality, which holds $\forall\varepsilon > 0, \forall(z_1, z_2) \in \mathfrak{R}^2$, is introduced:

$$-2z_1 z_2(1 - \beta_1 f_1)\beta_1 g \leq z_1^2(1 - \beta_1 f_1)^2\varepsilon + z_2^2(\beta_1 g)^2/\varepsilon \tag{17}$$

Substituting (17) back into (16) we can rewrite it as:

$$V_2^+ \leq z_1^2(1 - \beta_1 f_1)^2(1 + \varepsilon) + z_2^2((\beta_1 g)^2(1 + 1/\varepsilon) + \beta_2^2) \tag{18}$$

Comparing the corresponding terms in (18) and (15) and simplifying the common factors results in the following inequalities:

$$\begin{gathered} z_1^2(1 - \beta_1 f_1)^2(1 + \varepsilon) < z_1^2 \\ z_2^2((\beta_1 g)^2(1 + 1/\varepsilon) + \beta_2^2) < z_2^2 \end{gathered} \tag{19}$$

which hold for some $\varepsilon > 0$ if $|1 - \beta_1 f_1| < 1,\ \beta_2^2 + (\beta_1 g)^2 < 1$: for instance, choosing the values $\beta_1 f_1 = 0.1, (\beta_1 g)^2 = 0.1$, verifies (18) for some $\varepsilon > 0$ if simultaneously $\beta_2^2 < 0.9$. Consequently, $V_2 < V_2^+$ which implies that z_1, z_2 are bounded and converge to zero ■

Remark 4: In case $N = 2$ the Lyapunov function candidate (15) becomes:

$$V_2 = z_1^2 + z_2^2 + z_3^2 \tag{20}$$

Computing (20) at the next time step and substituting (9) we obtain:

$$V_2^+ = (z_1(1 - \beta_1 f_1) - z_2\beta_1 g)^2 + (\beta_2 z_2 + z_3)^2 + (\beta_2 z_3)^2 \tag{21}$$

In this case, an obvious choice of the parameters that ensures convergence of the errors to zero is $0 < \beta_1 f_1 < 1, \beta_2 = 0$. It is trivial to show that the same applies to $N > 1$. Notably, a consequence of the update law (9) is that the dynamics of z_1, z_2, z_N are coupled, which introduces cross terms in the Lyapunov function V_2^+. This is due to the simultaneous presence of multiple disturbances affecting the same system state (ref. Section 5). Conversely, this does not occur if the disturbances affect different states, as for the class of systems considered in [17].

Proposition 1: Given the discrete-time system (2) in strict feedback form with disturbance $f_1(x)\theta$ and input delay $N = 1$ under *Assumption 1-3* let us define a stabilising control law $u = \gamma(x, \theta^*)$ and a corresponding Lyapunov function $V_1 = kx^2 \geq 0$ for the un-delayed system with known parameter θ^* $(N = 0, \theta = \theta^*)$ so that $V_1^+ \leq \lambda V_1, 0 < \lambda < 1$.
Then, the closed-loop system (2)-(6) is stable for some $(\beta_1, \beta_2) \in \Re^2$, for which $|1 - \beta_1 f_1| < 1,\ \beta_2^2 + (\beta_1 g)^2 < 1, |1 - \beta_1 f_1||f_1^+| < |f_1|, |g^+\beta_2 - g\beta_1 f_1^+| < |g|$, the system state is bounded and converges to zero.

Proof: We define a new Lyapunov function as:

$$\bar{V}_1 = V_1 + V_1^+ \geq 0 \tag{22}$$

Computing (22) for the next time step we obtain:

$$\bar{V}_1^+ = V_1^+ + V_1^{+2} \tag{23}$$

Substituting (6) into (2) and computing V_1^+, V_1^{+2}gives:

$$\begin{gathered} V_1^+ = k(f_1 z_1 + g z_2)^2 \\ V_1^{+2} = k(f_1^+ z_1(1 - \beta_1 f_1) - \beta_1 f_1^+ g z_2 + g^+ \beta_2 z_2)^2 \end{gathered} \tag{24}$$

According to *Lemma 1* the errors z_1, z_2 converge to zero if $|1 - \beta_1 f_1| < 1,\ \beta_2^2 + (\beta_1 g)^2 < 1$. Consequently, substituting (24) into (23) and considering that $V_1^+ \leq \lambda V_1$by hypothesis we obtain:

$$\begin{gathered} \bar{V}_1 = V_1 + k(f_1 z_1 + g z_2)^2 \\ \bar{V}_1^+ \leq \lambda V_1 + k(f_1^+ z_1(1 - \beta_1 f_1) - \beta_1 f_1^+ g z_2 + g^+ \beta_2 z_2)^2 \end{gathered} \tag{25}$$

Comparing the corresponding terms of $\bar{V}_1, \bar{V}_1^+$ in (25) gives:

$$\begin{gathered} \lambda V_1 < V_1 \\ (f_1^+ z_1(1 - \beta_1 f_1) + z_2(g^+\beta_2 - g\beta_1 f_1^+))^2 < (f_1 z_1 + g z_2)^2 \end{gathered} \tag{26}$$

Computing the squares in the second part of (26), regrouping the terms in a similar way to (17),(19) and simplifying common factors we obtain:

$$\begin{gathered} \lambda V_1 < V_1 \\ z_1^2(1 - \beta_1 f_1)^2 (f_1^+)^2 (1 + \varepsilon) < z_1^2 f_1^2 (1 + \varepsilon) \\ z_2^2 (g^+\beta_2 - g\beta_1 f_1^+)^2 (1 + 1/\varepsilon) < z_2^2 g^2 (1 + 1/\varepsilon) \end{gathered} \tag{27}$$

which holds for some $\varepsilon > 0, \forall (z_1, z_2) \in \Re^2$ with $0 < \lambda < 1,\ |1 - \beta_1 f_1||f_1^+| < |f_1|$, $|g^+\beta_2 - g\beta_1 f_1^+| < |g|$. From (27) we conclude that $\bar{V}_1^+ < \bar{V}_1$, which implies that the closed-loop system (2)-(6) is stable, the system state is bounded and converges to zero for some$(\beta_1, \beta_2) \in \Re^2$, for which $|1 - \beta_1 f_1| < 1,\ \beta_2^2 + (\beta_1 g)^2 < 1, |1 - \beta_1 f_1||f_1^+| < |f_1|, |g^+\beta_2 - g\beta_1 f_1^+| < |g|$■

Remark 5: Due to the disturbance $f_1(x)\theta$ in (2), *Proposition 1* extends the results of Theorem 3.1 in [15] for $N = 1$. In accordance with [2] and by analogy with Theorem 3.2 in [15], the stability of the closed-loop system (2)-(10) for $N > 1$ can be proved by induction based on the case $N = 1$ employing a Lyapunov function candidate of the form $\bar{V}_1 = V_1 + \sum_{i=1}^{N} V_1^{+i}$.

Remark 6: If $g^+ = g, f_1^+ = f_1$, the bounds on β_1, β_2 expressed by *Proposition 1* can be simplified as $|1 - \beta_1 f_1| < 1,\ \beta_2^2 + (\beta_1 g)^2 < 1, |\beta_2 - \beta_1 f_1| < 1$. Consequently, a suitable choice of β_2 for $N = 1$ is $\beta_2 = \beta_1 f_1 < 1$, with $\beta_1^2(f_1^2 + g^2) < 1$, which results in the convergence of z_2 to zero in one step. Considering the case $N = 2$ we assume $g^+ = g, f_1^+ = f_1$ and define $\bar{V}_1$ as:

$$\bar{V}_1 = V_1 + V_1^+ + V_1^{+2} \geq 0 \tag{28}$$

Computing (28) for the next time step under the above assumption we obtain:

$$\bar{V}_1^+ = V_1^+ + V_1^{+2} + V_1^{+3} \geq 0 \tag{29}$$

From *Proposition 1*, we are left with comparing V_1^{+2}, V_1^{+3}which are computed from (2),(10) as:

$$\begin{gathered} V_1^{+2} = (f_1 z_1(1 - \beta_1 f_1) - \beta_1 f_1 g z_2 + g\beta_2 z_2 + g z_3)^2 \\ V_1^{+3} = (f_1 z_1(1 - \beta_1 f_1)^2 - \beta_1 f_1(1 - \beta_1 f_1) g z_2 - \beta_1 f_1 g \beta_2 z_2 - \beta_1 f_1 g z_3 + g\beta_2^2 z_2 + 2g\beta_2 z_3)^2 \end{gathered} \tag{30}$$

Computing the squares in (30), regrouping the terms in a similar way to (27), and simplifying common factors we obtain:

$$\begin{gathered} z_1^2(1 - \beta_1 f_1)^2(1 + \varepsilon_1 + \varepsilon_2) < z_1^2(1 + \varepsilon_1 + \varepsilon_2) \\ z_2^2(\beta_2^2 - \beta_1 f_1 \beta_2 - \beta_1 f_1(1 - \beta_1 f_1))^2(1 + 1/\varepsilon_1 + \varepsilon_3) < z_2^2(\beta_2 - \beta_1 f_1)^2(1 + 1/\varepsilon_1 + \varepsilon_3) \\ z_3^2(2\beta_2 - \beta_1 f_1)(1 + 1/\varepsilon_2 + 1/\varepsilon_3) < z_3^2(1 + 1/\varepsilon_2 + 1/\varepsilon_3) \end{gathered} \tag{31}$$

which hold for some $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0, \forall(z_1, z_2, z_3) \in \Re^3$ if $|1 - \beta_1 f_1| < 1,\ \beta_2^2 + (\beta_1 g)^2 < 1$, $|2\beta_2 - \beta_1 f_1| < 1$. From the last part of (31) choosing $\beta_2 = \beta_1 f_1/2 < 1$ results in z_3 converging to zero in one step if $\beta_1^2(f_1^2/4 + g^2) < 1$. Generalising to the case $N > 1$ we can infer that a suitable choice of the parameter β_2 for the closed-loop system (2)-(10) assuming that $g^+ \cong g, f_1^+ \cong f_1$ is $\beta_2 = \beta_1 f_1/N$. This is in agreement with the fact that the relative weight of the corrective term in (13) increases with the input delay, as pointed out in *Remark 2*. Notably, different values of β_2 from the ones proposed are possible: in particular, setting $\beta_2 = 0$ recovers the standard predictive control [18], in a similar way in which [2] is recovered as a special case of [19]. Furthermore, *Lemma 1*, does not pose constraints on the sign of β_2: for instance, choosing $\beta_2 = -\beta_1 f_1$ for $N = 1$ also satisfies (27) as long as the remaining inequalities in *Proposition 1* are verified.

Remark 7: It follows from (27) and (31) that introducing in the control law the corrective term (13) with $\beta_2 \neq 0$ gives a further degree of freedom in shaping the dynamics of z_2: for instance, choosing $\beta_2 = \beta_1 f_1$ in (27) achieves the convergence of z_2 to zero in one step. Conversely, with $\beta_2 = 0$ this is only possible if $\beta_1 = 0$, which would however prevent the convergence of z_1 to zero. This represents an advantage of the proposed approach in comparison to more traditional predictive control algorithms [18].

For comparison purposes, the counterpart of *Proposition 1* is detailed for the control law (14) considering the case $N = 2$.

Proposition 2: Consider closed-loop system (2)-(14) with input delay $N = 2$ under *Assumption 1-3* and $u = \gamma(x, \theta^*)$ that stabilizes the un-delayed system with known parameter θ^* $(N = 0, \theta = \theta^*)$ according to a Lyapunov function candidate $V_1 = kx^2 \geq 0$ so that $V_1^+ \leq \lambda V_1, 0 < \lambda < 1$.

i. The errors z_1, z_2, z_3 are bounded and converge to zero for some $(\beta_1, \beta_2) \in \Re^2$ that satisfy the inequalities $|1-\beta_1 f_1| < 1,\ |\beta_2| < 1$

ii. If in addition $|f_1^{+2}| < |f_1|, |g^{+2}\beta_2| < |g|$, then the closed-loop system (2)-(14) is stable, the system state is bounded and converges to zero

Proof: To prove the first claim we choose the Lyapunov function candidate (20) and compute the difference over one time step. In particular, substituting (14) into (8) we observe that the dynamics of z_1 is decoupled from z_2, z_3.

$$V_2^+ - V_2 = z_1^2(1-\beta_1 f_1)^2 + z_3^2 + z_2^2\beta_2^2 - (z_1^2 + z_2^2 + z_3^2) \le 0 \tag{32}$$

The above inequality is verified for some $(\beta_1, \beta_2) \in \Re^2, |1-\beta_1 f_1| < 1, |\beta_2| < 1$. Consequently, z_1, z_2, z_3 are bounded and converge to zero.

To prove the second claim we employ the Lyapunov function candidate (28). Substituting (14) into (2) and then into (28) we obtain:

$$\bar{V}_1 = V_1 + V_1^+ + V_1^{+2} = V_1 + (f_1 z_1 + g z_2)^2 + (f_1^+ z_1 + g^+ z_3)^2 \ge 0 \tag{33}$$

Computing (33) at the next time step, gives:

$$\bar{V}_1^+ \le \lambda V_1 + (f_1^+ z_1 + g^+ z_3)^2 + (f_1^{+2} z_1 + g^{+2}\beta_2 z_2)^2 \tag{34}$$

Comparing (33) and (34) we observe that the term V_1^{+2} is common to both, while $\lambda V_1 < V_1$ by hypothesis. Computing the squares in the remaining term and simplifying common factors in a similar way to (27), we obtain:

$$\begin{gathered} z_1^2(f_1^{+2})^2(1+\varepsilon) < z_1^2 f_1^2(1+\varepsilon) \\ z_2^2(g^{+2}\beta_2)^2(1+1/\varepsilon) < z_2^2 g^2(1+1/\varepsilon) \end{gathered} \tag{35}$$

which holds for some $\varepsilon > 0,\ \forall(z_1, z_2) \in \Re^2$ for which $|f_1^{+2}| < |f_1|, |g^{+2}\beta_2| < |g|$. From (35) we conclude that $\bar{V}_1^+ < \bar{V}_1$, which concludes the proof ■

Remark 8: Similarly to *Proposition 1*, the stability of the closed-loop system (2)-(14) for $N > 1$ can be proved by induction based on the case $N = 2$ (ref. *Remark 5*). Notably, if g is constant the condition on β_2 simplify to $|\beta_2| < 1$ regardless of N. Finally, the fact that z_1 is not incremented in (33), (34) is a consequence of $\hat{\theta}^+$ only being updated once at each time step within (14).

5. Pneumatic Drive

While nonlinear systems with input delay and disturbances are ubiquitous, the specific scenario that motivated this study is the control of a pneumatic drive for percutaneous intervention under MRI-guidance [20]. The main challenges associated with this type of devices are the input delay due to the long supply lines, and the high friction forces. A simplified discrete-time model of a pneumatic cylinder based on the one proposed in [18] is:

$$\begin{cases} x_1^+ = x_1 + x_2 T \\ x_2^+ = x_2 + \dfrac{1}{m}(y_1 A - \theta_1 x_2 - \theta_2 \mathrm{sign}(x_2) - \theta_3 \delta(x_2) - p_0 a)T \\ y_1^+ = y_2 \\ \quad \cdots \\ y_N^+ = u \end{cases} \tag{36}$$

The states x_1, x_2 are the position and the velocity of the piston, while $\theta_1, \theta_2, \theta_3$ represent the unknown viscous friction coefficient, the Coulomb friction coefficient, and the stiction coefficient. The functions $\text{sign}(\cdot), \delta(\cdot)$ are defined as:

$$\text{sign}(x_2) = \begin{cases} -1 & x_2 < 0 \\ 0 & x_2 = 0 \\ 1 & x_2 > 0 \end{cases} \quad \delta(x_2) = \begin{cases} 1 & x_2 = 0 \\ 0 & x_2 \neq 0 \end{cases} \tag{37}$$

The delayed control input y_1 corresponds to the pressure relative to atmosphere acting on one side of the piston, while p_0 is the pressure acting on the opposite side. For notational simplicity the areas A, a of the piston and the mass m of piston and payload are assumed unitary and are omitted in what follows. Furthermore, the pressure p_0 is set constant as in [21]. Finally, T is the sampling period (typically $T = 10^{-3}$ seconds, hence $T \ll 1$), which is much smaller compared to the dynamics of the system (typical closed-loop bandwidth < 10 Hz), while the input delay NT is in the range of tens of milliseconds. Comparing (36) with system (2) we have that $g(x) = AT/m$ is constant, and $\theta f_1(x) = (-\theta_1 x_2 - \theta_2 \text{sign}(x_2) - \theta_3 \delta(x_2))T/m$, is the sum of a linear term and two switching terms. The control aim for system (36) is tracking a prescribed trajectory (38) which at any instant is known N time steps in advance:

$$\begin{cases} x_{1D}^+ = x_{1D} + x_{2D}T \\ x_{2D}^+ = x_{2D} + x_{3D}T \end{cases} \tag{38}$$

The terms x_{1D}, x_{2D}, x_{3D} in (38) represent the desired position, velocity and acceleration. It is straightforward to show that *Assumptions 1-3* are satisfied for system (36): the sampling interval is constant for microcontrollers, while the input delay can be calculated knowing the length of the supply pipes; considering the bandwidth of the pneumatic drive (< 10 Hz) and the magnitude of the input delay ($< 30 \times 10^{-3}$ seconds), the disturbances can be considered constant over the input delay; from [18], a suitable control law for the un-delayed version of system (36) with known friction coefficients ($N = 0, \theta_1 = \theta_1^*, \theta_2 = \theta_2^*, \theta_3 = \theta_3^*$) that satisfies *Assumption 3* is:

$$u = \theta_1^* x_2 + \theta_2^* \text{sign}(x_2) + \theta_3^* \delta(x_2) + p_0 + x_{3D} + c_1(x_{2D} - x_2) + c_2 S \tag{39}$$

The variable S represents a combination of position and velocity errors [22] and is defined as:

$$S = c_1(x_{1D} - x_1) + (x_{2D} - x_2) \tag{40}$$

The terms $c_1, c_2 > 0$ in (39),(40) are design parameters defining the responsiveness of the system (i.e. larger values result in a more responsive control action). *Assumption 3* is verified considering the Lyapunov function candidate $V_1 = kS^2 \geq 0$, with $|1 - c_2 T| < 1$:

$$\begin{aligned} V_1^+ = k(c_1(x_{1D} - x_1) + c_1 T(x_{2D} - x_2) + (x_{2D} - x_2) + x_{3D}T \\ - (u - \theta_1^* x_2 - \theta_2^* \text{sign}(x_2) - \theta_3^* \delta(x_2) - p_0)T)^2 = kS^2(1 - c_2 T)^2 < V_1 \end{aligned} \tag{41}$$

Proceeding to the control design for system (36) with $N = 1$, which is characterised by multiple disturbances, the terms z_1, z_2, z_3, z_4 are defined as:

$$\begin{aligned} z_1 &= (\hat{\theta}_1 + \beta_1^- x_2) - \theta_1 \\ z_2 &= (\hat{\theta}_2 + \beta_2^- x_2) - \theta_2 \\ z_3 &= (\hat{\theta}_3 + \beta_3^- x_2) - \theta_3 \\ z_4 &= \tilde{y}_1 - y_1 \end{aligned} \tag{42}$$

The derivation of the adaptation law is reported in the Appendix and results in the following expression:

$$\hat{\theta}_1^+ = \hat{\theta}_1 + c_3 T x_2\big(\tilde{y}_1 - (\hat{\theta}_1 - c_3 x_2^2)x_2 - (\hat{\theta}_2 - c_3|x_2|)\mathrm{sign}(x_2) - p_0\big) + c_3 x_2(x_2 - x_2^-)$$
$$\begin{aligned}\hat{\theta}_2^+ = \hat{\theta}_2 &+ c_3 T\mathrm{sign}(x_2)\big(\tilde{y}_1 - (\hat{\theta}_1 - c_3 x_2^2)x_2 - (\hat{\theta}_2 - c_3|x_2|)\mathrm{sign}(x_2) - p_0\big)\\ &+ c_3 x_2(\mathrm{sign}(x_2) - \mathrm{sign}(x_2^-))\end{aligned} \tag{43}$$
$$\hat{\theta}_3^+ = \hat{\theta}_3 + c_3 T\delta(x_2)\big(\tilde{y}_1 - \hat{\theta}_3\delta(x_2) - p_0\big) - c_3 x_2 \delta(x_2^-)$$
$$u = \tilde{y}_1^+ - \beta_4 z_4$$

By analogy with (10), (13) the control law for system (36) with $N > 1$ and with respect to the tracking problem (38) is:

$$\tilde{y}_1^{+i} = \left(\hat{\theta}_1^{+i} - c_3\left(x_2^{+i}\right)^2\right)x_2^{+i} + \left(\hat{\theta}_2^{+i} - c_3\left|x_2^{+i}\right|\right)\mathrm{sign}(x_2^{+i}) + \hat{\theta}_3^{+i}\delta(x_2^+) + p_0 + x_{3D}^{+i} + c_2 S^{+i} + c_1\left(x_{2D}^{+i} - x_2^{+i}\right)$$

$$\tilde{y}_{i+1}^+ = \tilde{y}_1^{+i} - \sum_{j=1}^{i}\beta_4^{i+1-j}\left(\frac{i!}{(j-1)!\,(i-j+1)!}\right)z_{j+3} \tag{44}$$

$$u = \tilde{y}_1^{+N} - \sum_{i=1}^{N}\beta_4^{N+1-i}\left(\frac{N!}{(i-1)!\,(N-i+1)!}\right)z_{i+3}$$

where $1 \le j \le i \le N$ and the predicted values of the states x_1^{+i}, x_2^{+i} are computed recursively from their current values x_1, x_2.

In order to define tuning guidelines for β_4, the following Lyapunov function candidate is chosen for $N = 1$ in accordance with *Proposition 1*, here adapted for the tracking problem (38):

$$\bar{V}_1 = V_1 + k(S^+ - (1 - c_2 T)S)^2 > 0 \tag{45}$$

Substituting (44) into (45) we obtain:

$$\bar{V}_1 = V_1 + kT^2(-z_1 x_2 - z_2\mathrm{sign}(x_2) - z_3\delta(x_2) + z_4)^2 \tag{46}$$

Evaluating (46) for the next time step, substituting (43), and neglecting the terms in T^2 since $T \ll 1$, we obtain:

$$\begin{aligned}\bar{V}_1^+ = V_1^+ + kT^2\big(&-(z_1(1 - c_3 T x_2^2) - z_2 c_3 T|x_2| + z_4 c_3 T x_2)x_2\\ &- \big(-z_1 c_3 T|x_2| + z_2(1 - c_3 T\mathrm{sign}(x_2)^2) + z_4 c_3 T\mathrm{sign}(x_2)\big)\mathrm{sign}(x_2)\\ &- (z_3(1 - c_3 T\delta(x_2)^2) + z_4 c_3 T\delta(x_2))\delta(x_2) + z_4\beta_4\big)^2\end{aligned} \tag{47}$$

We can split (47) according to (37) and regroup the terms as:

$$\begin{aligned} x_2 = 0 \quad & \bar{V}_1^+ = V_1^+ + kT^2(-z_3(1 - c_3 T) + z_4(-c_3 T + \beta_4))^2\\ x_2 \neq 0 \quad & \bar{V}_1^+ = V_1^+ + kT^2\big(-z_1(1 - c_3 T x_2^2 - c_3 T)x_2 - z_2(c_3 T|x_2|x_2 + (1 - c_3 T)\mathrm{sign}(x_2))\\ &\qquad + z_4(\beta_4 - c_3 T x_2^2 - c_3 T)\big)^2\end{aligned} \tag{48}$$

Observing the coefficients of z_4 suggests $\beta_4 = c_3 T(1 + x_2^2)$ as suitable value for this design parameter. Considering that $|x_2| \ll 1$, the above value can be further simplified as $\beta_4 = c_3 T$ with $c_3 T < 1$. Drawing a parallel to *Remark 6*, we infer $\beta_4 = (c_3 T)/N$ for the general case $N > 1$. It is important to highlight here the effects of the sampling period: a larger T requires smaller values of c_3, β_4 in order to ensure $c_3 T < 1$; finally, larger errors are introduced within the Lyapunov function $\bar{V}_1^+$ (47),(48) where all terms in T^2 and higher order have been neglected.

For comparison purposes, the control law (14) is implemented for system (36) resulting in:

$$\tilde{y}_1^{+i} = \left(\hat{\theta}_1 - c_3 x_2^2\right) x_2^{+i} + \left(\hat{\theta}_2 - c_3 |x_2|\right) \mathrm{sign}\left(x_2^{+i}\right) + \hat{\theta}_3 \delta\left(x_2^{+i}\right) + p_0 + x_{3D}^{+i} + c_2 S^{+i} + c_1\left(x_{2D}^{+i} - x_2^{+i}\right) \tag{49}$$

$$u = \tilde{y}_1^{+N} - \beta_4 z_2$$

Differently from (44), the parameters $\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3$ are only updated once at each time step. Additionally, the parameter $|\beta_4| < 1$ can be chosen independently of N (ref. *Remark 8*). Finally, the difference between (44) and (49) vanishes if $N = 1$.

6. Simulation Results

System (36) was simulated in Matlab® with $T = 0.001$ seconds. The parameters c_1, c_2, c_3 were tuned on the un-delayed system in order to achieve a settling time shorter than 0.2 s with an overshoot smaller than 0.5% (i.e. $c_1 = 25, c_2 = 40, c_3 = 30$). Two versions of the proposed control law were compared considering an input delay representative of the application ($N = 20, N = 25$): control scheme (44) with $\beta_4 = c_3 T/N$; control scheme (49) with $\beta_4 = 0.3$. The disturbances were chosen in accordance with experimental data from needle insertions in silicone phantoms [21] as: $\theta_1 = 2x_2$; $\theta_2 = 0.2\mathrm{sign}(x_2)$; $\theta_3 = 0.5\delta(x_2)$.

Simulations were conducted using a step signal with N =20 (Figure 1) and a ramp trajectory with N =25 (Figure 2). The plots show that omitting the adaptive disturbance compensation (i.e. $c_3 = \beta_4 = 0$), which corresponds to the standard predictive control [2], results in large tracking errors that increase with the input delay. Conversely, the adaptive algorithms (44) and (49) both achieve superior performance. In particular, the algorithm (44) results in higher responsiveness which makes it the most appropriate for tracking tasks. Nevertheless, in all cases the piston

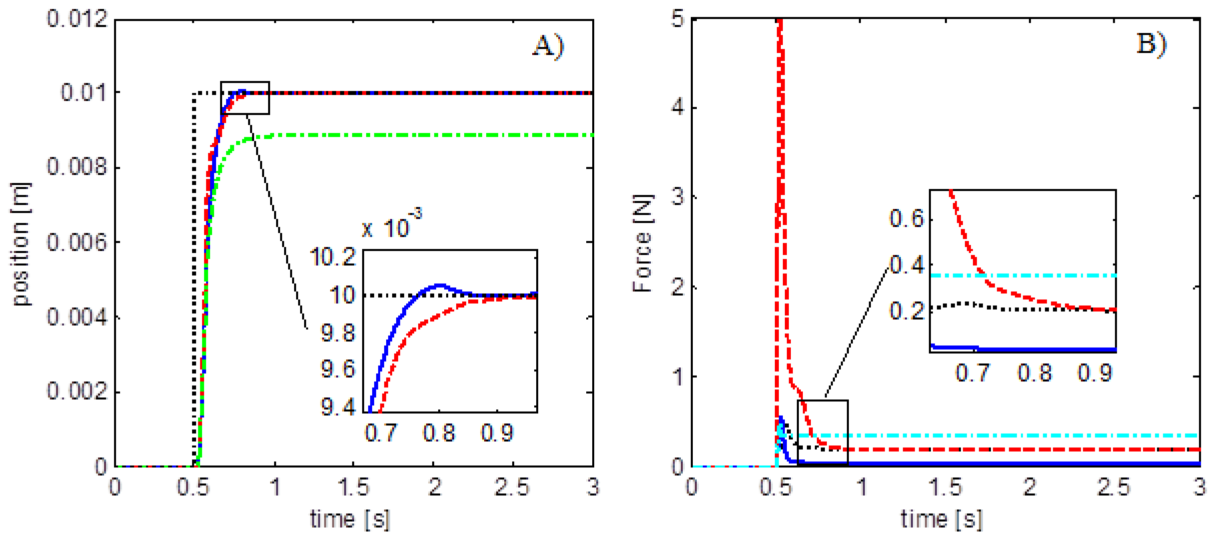

Fig. 1 Step response (A): dashed red line refers to control scheme (44); solid blue line refers to control scheme (49); green centreline refers to standard predictor ($\beta_4 = 0, c_3 = 0$); reference position x_{1D} is in dotted black. Corresponding disturbance estimate (B) for control scheme (44): solid blue line refers to $\hat{\theta}_1$ (viscous friction); red dashed line refers to $\hat{\theta}_2$ (Coulomb friction); cyan centreline refers to $\hat{\theta}_3$ (stiction); the cumulative disturbance is in dotted black.

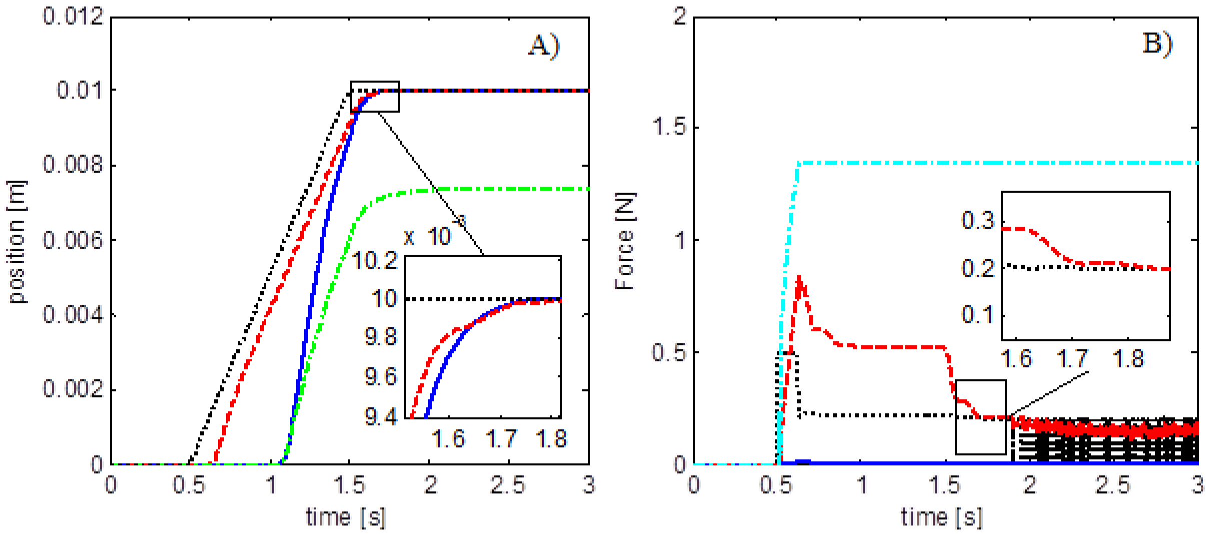

Fig. 2 Response to ramp trajectory (A): dashed red line refers to control scheme (44); solid blue line refers to control scheme (49); green centreline refers to standard predictor ($\beta_4 = 0, c_3 = 0$); reference position x_{1D} is in dotted black. Corresponding disturbance estimate (B) for control scheme (44): solid blue line refers to $\hat{\theta}_1$ (viscous friction); red dashed line refers to $\hat{\theta}_2$ (Coulomb friction); cyan centreline refers to $\hat{\theta}_3$ (stiction); the cumulative disturbance is in dotted black.

trajectory becomes less smooth with increasing input delay and larger values of T suggesting that a less aggressive tuning of c_1, c_2, c_3 would be required in the latter case as pointed out in [2]. Conversely, the performance improves for smaller T and smaller N while the difference between (44) and (49) vanishes. Since the friction forces are additive in (36), the adaptation law (43) attempts to compensate their cumulative effect and consequently the individual estimates might differ from the real values during the transient. Differently from the un-delayed case, while the Coulomb friction coefficient $\hat{\theta}_2$ converges to the correct value F_a, the estimate of the stiction $\hat{\theta}_3$ can differ from F_m and become larger in the presence of input delay. This effect is a direct consequence of the adaptation law (43) for the parameter $\hat{\theta}_3$, which consists of an integrative term that is incremented until the piston is set in motion.

7. Conclusions

This paper presented a new adaptive algorithm for a class of nonlinear discrete-time system in strict feedback form characterised by disturbances and input delay. The control law was constructed using the I&I approach which is employed to compensate the disturbances and to correct the prediction error, resulting in a recursive algorithm. Additionally, a compact form of the control law particularly suitable for the implementation on digital microcontrollers was derived. The stability of the closed-loop system was analysed with Lyapunov functions and guidelines for tuning the design parameters were outlined. Incidentally, the traditional predictive algorithm is recovered as a special case of the proposed control scheme, which conversely provides additional flexibility in shaping the dynamics of the estimation errors. The control scheme was applied to the tracking problem of a pneumatic drive with multiple disturbances consisting of viscous friction, Coulomb friction, and stiction. The design of the adaptive algorithm was outlined and the effectiveness of the proposed approach was demonstrated with simulations. In particular, two versions of the control law were considered for comparison purposes. Future work will extend the current results to systems with both input delay and state delays and to systems for which only a subset of states are measurable. In parallel, the control of continuous-time systems with input delay and disturbances will be investigated. Finally, the results will be validated experimentally with a prototype representative of the chosen application.

8. Appendix

Derivations of the adaptation law for system (36)

Computing (42) for the next time step in a similar fashion to (4) we obtain:

$$
\begin{aligned}
z_1^+ &= \beta_1 T\big(\tilde{y}_1 - z_4 - (\hat{\theta}_1 + \beta_1^- x_2 - z_1)x_2 - (\hat{\theta}_2 + \beta_2^- x_2 - z_2)\mathrm{sign}(x_2) \\
&\qquad - (\hat{\theta}_3 + \beta_3^- x_2 - z_3)\delta(x_2) - p_0\big) + \hat{\theta}_1^+ + \beta_1 x_2 - \theta_1 \\
z_2^+ &= \beta_2 T\big(\tilde{y}_1 - z_4 - (\hat{\theta}_1 + \beta_1^- x_2 - z_1)x_2 - (\hat{\theta}_2 + \beta_2^- x_2 - z_2)\mathrm{sign}(x_2) \\
&\qquad - (\hat{\theta}_3 + \beta_3^- x_2 - z_3)\delta(x_2) - p_0\big) + \hat{\theta}_2^+ + \beta_2 x_2 - \theta_2 \\
z_3^+ &= \beta_3 T\big(\tilde{y}_1 - z_4 - (\hat{\theta}_1 + \beta_1^- x_2 - z_1)x_2 - (\hat{\theta}_2 + \beta_2^- x_2 - z_2)\mathrm{sign}(x_2) \\
&\qquad - (\hat{\theta}_3 + \beta_3^- x_2 - z_3)\delta(x_2) - p_0\big) + \hat{\theta}_3^+ + \beta_3 x_2 - \theta_3 \\
z_4^+ &= \tilde{y}_1^+ - u
\end{aligned}
\tag{A.1}
$$

where $\beta_1, \beta_2, \beta_3$ are nonlinear functions of the state to be defined. Extending (5) to the case of multiple disturbances, the update law $\hat{\theta}_1^+, \hat{\theta}_2^+, \hat{\theta}_3^+$ and the control law u are chosen as:

$$
\begin{aligned}
\hat{\theta}_1^+ &= \hat{\theta}_1 - \beta_1 T\big(\tilde{y}_1 - (\hat{\theta}_1 + \beta_1^- x_2)x_2 - (\hat{\theta}_2 + \beta_2^- x_2)\mathrm{sign}(x_2) - (\hat{\theta}_3 + \beta_3^- x_2)\delta(x_2) - p_0\big) \\
&\qquad + (\beta_1^- - \beta_1)x_2 \\
\hat{\theta}_2^+ &= \hat{\theta}_1 - \beta_2 T\big(\tilde{y}_1 - (\hat{\theta}_1 + \beta_1^- x_2)x_2 - (\hat{\theta}_2 + \beta_2^- x_2)\mathrm{sign}(x_2) - (\hat{\theta}_3 + \beta_3^- x_2)\delta(x_2) - p_0\big) \\
&\qquad + (\beta_2^- - \beta_2)x_2 \\
\hat{\theta}_3^+ &= \hat{\theta}_3 - \beta_3 T\big(\tilde{y}_1 - (\hat{\theta}_1 + \beta_1^- x_2)x_2 - (\hat{\theta}_2 + \beta_2^- x_2)\mathrm{sign}(x_2) - (\hat{\theta}_3 + \beta_3^- x_2)\delta(x_2) - p_0\big) \\
&\qquad + (\beta_3^- - \beta_3)x_2 \\
u &= \tilde{y}_1^+ - \beta_4 z_4
\end{aligned}
\tag{A.2}
$$

In order to prove the convergence of (A.1) to zero, the following Lyapunov function candidate is chosen:

$$V_2 = z_1^2 + z_2^2 + z_3^2 + z_4^2 \geq 0 \tag{A.3}$$

Computing (A.3) for the next time step and substituting (A.1),(A.2) gives:

$$
\begin{aligned}
V_2^+ &= (z_1(1 + \beta_1 T x_2) + z_2 \beta_1 T \mathrm{sign}(x_2) + z_3 \beta_1 T \delta(x_2) - z_4 \beta_1 T)^2 \\
&\quad + \big(z_1 \beta_2 T x_2 + z_2(1 + \beta_2 T \mathrm{sign}(x_2)) + z_3 \beta_2 T \delta(x_2) - z_4 \beta_2 T\big)^2 \\
&\quad + \big(z_1 \beta_3 T x_2 + z_2 \beta_3 T \mathrm{sign}(x_2) + z_3(1 + \beta_3 T \delta(x_2)) - z_4 \beta_3 T\big)^2 + \beta_4^2 z_4^2 \geq 0
\end{aligned}
\tag{A.4}
$$

Exploiting the structure of (A.4) and introducing the constant parameter $c_3 > 0$, the nonlinear functions $\beta_1, \beta_2, \beta_3$ are chosen as:

$$
\begin{aligned}
\beta_1 &= -c_3 x_2 \\
\beta_2 &= -c_3 \mathrm{sign}(x_2) \\
\beta_3 &= -c_3 \delta(x_2)
\end{aligned}
\tag{A.5}
$$

Substituting (A.5) back into (A.4), and since $\delta(x_2)x_2 = 0$; $\delta(x_2)\text{sign}(x_2) = 0, \forall x_2 \in \Re$, we obtain:

$$\begin{aligned} V_2^+ = &(z_1(1 - c_3Tx_2^2) - z_2c_3T|x_2| + z_4c_3Tx_2)^2 \\ &+ (-z_1c_3T|x_2| + z_2(1 - c_3T\text{sign}(x_2)^2) + z_4c_3T\text{sign}(x_2))^2 \\ &+ (z_3(1 - c_3T\delta(x_2)^2) + z_4c_3T\delta(x_2))^2 + \beta_4^2z_4^2 \geq 0 \end{aligned} \tag{A.6}$$

In accordance with (37), we can split (A.6) in two cases:

$$\begin{aligned} &x_2 = 0 \quad V_2^+ = (z_1)^2 + (z_2)^2 + (z_3(1 - c_3T) + z_4c_3T)^2 + \beta_4^2z_4^2 \\ &x_2 \neq 0 \quad V_2^+ = (z_1(1 - c_3Tx_2^2) - z_2c_3T|x_2| + z_4c_3Tx_2)^2 + (z_3)^2 \\ &\qquad +\left(-z_1c_3T|x_2| + z_2(1 - c_3T) + z_4c_3T\text{sign}(x_2)\right)^2 + \beta_4^2z_4^2 \end{aligned} \tag{A.7}$$

Computing the squares in (A.7), regrouping the terms and simplifying common factors we obtain:

$$\begin{aligned} &x_2 = 0 \quad V_2^+ \leq {z_1}^2 + {z_2}^2 + z_3^2(1 - c_3T)^2(1 + \varepsilon_0) + z_4^2(\beta_4^2 + {c_3}^2T^2(1 + 1/\varepsilon_0)) \\ &x_2 \neq 0 \quad V_2^+ \leq z_1^2((1 - c_3Tx_2^2)^2(1 + \varepsilon_1 + \varepsilon_2) + {c_3}^2T^2x_2^2(1 + 1/\varepsilon_4 + 1/\varepsilon_5)) + \\ &\qquad +z_2^2((1 - c_3T)^2(1 + \varepsilon_4 + \varepsilon_6) + {c_3}^2T^2x_2^2(1 + 1/\varepsilon_1 + \varepsilon_3)) + z_3^2 + \\ &\qquad +z_4^2(\beta_4^2 + {c_3}^2T^2x_2^2(1 + 1/\varepsilon_3 + 1/\varepsilon_2) + {c_3}^2T^2(1 + \varepsilon_5 + 1/\varepsilon_6)) \end{aligned} \tag{A.8}$$

which holds $\forall \varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6 > 0, \forall (z_1, z_2, z_3, z_4) \in \Re^4$. Considering that $T \ll 1$, (A.8) can be further simplified neglecting all terms in T^2:

$$\begin{aligned} &x_2 = 0 \quad V_2^+ \leq {z_1}^2 + {z_2}^2 + z_3^2(1 - c_3T)^2(1 + \varepsilon_0) + z_4^2(\beta_4^2) \\ &x_2 \neq 0 \quad V_2^+ \leq z_1^2(1 - c_3Tx_2^2)^2(1 + \varepsilon_1 + \varepsilon_2) + z_2^2(1 - c_3T)^2(1 + \varepsilon_4 + \varepsilon_6) + z_3^2 + z_4^2(\beta_4^2) \end{aligned} \tag{A.9}$$

Comparing the corresponding terms in (A.3) and (A.9) gives:

$$\begin{array}{cc} x_2 \neq 0 & z_1^2(1 - c_3Tx_2^2)^2(1 + \varepsilon_1 + \varepsilon_2) < z_1^2 \\ x_2 \neq 0 & z_2^2(1 - c_3T)^2(1 + \varepsilon_4 + \varepsilon_6) < z_2^2 \\ x_2 = 0 & z_3^2(1 - c_3T)^2(1 + \varepsilon_0) < z_3^2 \\ \forall x_2 & z_4^2(\beta_4^2) < z_4^2 \end{array} \tag{A.10}$$

which is satisfied for some $\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_4, \varepsilon_6 > 0$ and for $|1 - c_3T| < 1, |1 - c_3Tx_2^2| < 1, |\beta_4| < 1$, and proves the convergence of (A.1) to zero. Given that the speed of the pneumatic drive for the application considered is typically low $|x_2| \ll 1$ (i.e. $|x_2| < 0.05$ m/s), the above limits for the case $N = 1$ can be further simplified to $c_3 > 0, c_3 < 1/T, |\beta_4| < 1$.

9. References

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