Published in *Steel Construction – Design and Research*, 10(3), 216-221.
http://doi.org/10.1002/stco.201710026
Also presented at EUROSTEEL 2017, September 13–15, 2017, Copenhagen, Denmark.

O. Kunle Fajuyitan [1]
Adam J. Sadowski [2]
M. Ahmer Wadee [3]

Buckling of very short elastic cylinders with weld imperfections under uniform bending

The length-dependent behaviour domains of thin elastic cylindrical shells under uniform bending have recently received significant research attention. Ovalization is known to affect very long cylinders that undergo significant cross-sectional flattening before failing by local buckling. This effect is restrained by the end boundary conditions in shorter cylinders, which instead fail by local buckling at moments close to the classical analytical prediction. In very short cylinders, however, even this local buckling is restrained by the end boundary, and failure occurs instead through the development of a destabilizing meridional fold on the compressed side. Although this is a limit point instability under bending, ovalization does not play any role at all. This 'very short' length domain has only recently been explored for the first time with the aid of finite element modelling.
A brief overview of the non-linear buckling behaviour of very short elastic cylinders under uniform bending is presented in this paper. Two types of edge rotational restraint are used to illustrate the influence of a varying support condition on the stability in this short length range. It is shown that short cylinders under bending do not suffer at all from local short-wave buckling. Additionally, when the meridional dimension of such cylinders becomes particularly short, the resulting numerical models may predict indefinite stiffening without a limit point, even when the shell is modelled using more complete 3D solid continuum finite elements. Idealized weld depressions, which are realistic representations of a systemic manufacturing defect, are used to demonstrate only a very mild sensitivity to geometric imperfections at such short lengths owing to a pre-buckling stress state dominated by local compatibility bending. The topic should be of interest to researchers studying shell problems dominated by local bending with computational tools and designers of multi-segment shells with very close segment spacing.

Keywords: rotational restraint; very short cylinders; uniform bending; limit point buckling.

[1] PhD student [2] Lecturer [3] Professor
Imperial College London, Department of Civil and Environmental Engineering

1 Introduction

It is known that the length of a cylindrical shell plays an important role in governing its elastic stability behaviour under the fundamental load cases of uniform axial compression, external pressure and torsion. In design, such shells are categorized into length domains defined as 'short', 'medium' or 'long' according to the dominant buckling behaviour [1]–[4]. Although each of the above load cases exhibits the same three domains qualitatively, the length boundaries defining each domain occur at different numerical values for each one [5]. In a 'short' cylindrical shell under uniform axial compression, the end boundaries are known to restrain local buckling severely and failure only occurs through global snap-through buckling at a stress significantly higher than the classical algebraic prediction σ_{cl} (Eq. (1)). This classical elastic critical stress σ_{cl} is derived by assuming a uniform pre-buckling meridional membrane stress state:

$$\sigma_{cl} = \frac{E}{\sqrt{3\left(1-\nu^2\right)}} \cdot \frac{t}{r} \approx 0.605E\frac{t}{r} \text{ for } \nu = 0.3 \tag{1}$$

where E is the Young's modulus, ν is the Poisson's ratio with r and t being the radius to middle surface and thickness of the cylinder wall respectively. In cylinders of 'medium' length, the influence of the end boundary condition is localized near the edges and local buckling occurs at a stress slightly below σ_{cl}, the reduction being caused by geometric nonlinearity dependent on the edge support conditions [2], [6]. 'Long' cylinders fail by Euler column buckling at a stress far below σ_{cl}.

According to the European design standard for metal shells, EN-1993-1-6 [5], the length domains for cylinders under fundamental loads are categorized in terms of a dimensionless length parameter ω [2]:

$$\omega = \frac{L}{\sqrt{rt}} \tag{2}$$

which varies linearly with the length of the cylinder L and allows the effect of a varying slenderness ratio, i.e. radius-to-thickness ratio r/t, to be isolated within each length domain. For cylindrical shells under uniform axial compression, the range of lengths is defined as $\omega \leq 1.7$ for 'short', $1.7 < \omega \leq 0.5(r/t)$ for 'medium' and $\omega > 0.5(r/t)$ for 'long' cylinders. Other fundamental load cases are characterized in a similar manner.

For cylinders subjected to the closely-related load case of uniform bending, the complete effect of cylinder length on the non-linear elastic stability was characterized only recently in a computational study by Rotter et al. [7]. The study established that, unlike the three fundamental load cases, cylinders under uniform bending exhibit four length domains, with an additional 'transitional' domain lying between the 'medium' and 'long' ones. This 'transitional' domain is a consequence of the well-known Brazier [8] cross-section ovalization phenomenon, and represents the point at which the buckling strength of the cylinder begins to fall significantly below the classical elastic critical buckling moment M_{cl}, which describes local buckling reasonably accurately for 'medium' length cylinders:

$$M_{cl} = \pi r^2 t \sigma_{cl} \approx 1.901 E r t^2 \text{ for } \nu = 0.3 \tag{3}$$

The above expression for M_{cl} was derived based on a 'local buckling hypothesis' [9], [10], which assumes that local short-wave meridional buckles form once the stress in the most compressed fibre reaches the same σ_{cl} for uniform axial compression. Beyond the 'transitional' domain, 'long' elastic cylinders experience fully-developed ovalization and fail at a moment approximately 5 % below the predicted Brazier moment M_{Braz} defined by:

$$M_{Braz} = \frac{2\sqrt{2}}{9} \cdot \frac{E \pi r t^2}{\sqrt{1-\nu^2}} \approx 1.035 E r t^2 \approx 0.544 M_{cl} \text{ for } \nu = 0.3 \tag{4}$$

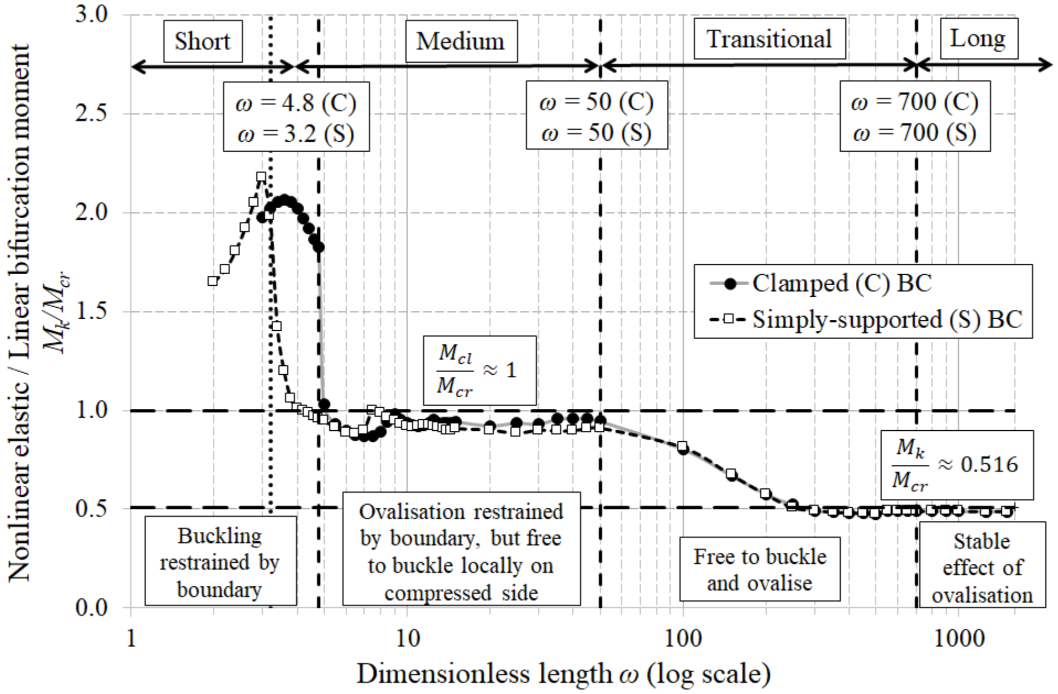


Fig. 1. Normalized buckling moment M_k/M_{cr} vs. ω of perfect cylindrical shells (r/t = 100) at different lengths under uniform bending after Fajuyitan et al. [11]

The study by Rotter et al. [7] also appears to be the first to report the existence of a corresponding 'short' domain in cylinders under uniform bending, and suggests that the domain boundary between ‘short’ and ‘medium’ occurs at ω = 4.8 for perfect cylinders with clamped end support conditions (C). For perfect cylinders with simply supported end conditions (S), a follow-up study by Fajuyitan et al. [11] suggested that this boundary occurs instead at a lower value of ω = 3.2 owing to the weaker rotational restraint offered by this boundary condition (Fig. 1). However, more recent studies [11]–[13] have demonstrated that when geometric imperfections are introduced, the length range defining each domain for a perfect shell potentially changes, and that the domain boundaries should perhaps be established at ω = 4 and 3 for clamped and simply supported conditions respectively [13]. A brief selection of the authors’ recent investigations into the behaviour of very short cylinders under uniform bending is presented in this paper.

2 Numerical model

Extensive computational analyses were performed on very short cylindrical shells using the commercial finite element program ABAQUS v6.14.2 [14]. Details of the numerical model employed are illustrated in Fig. 2. Two planes of symmetry were exploited for computational efficiency, while the end displacements and rotations of the cylinder were controlled by a reference node placed at the centroid of the cross-section and connected to the edges of the cylinder through a rigid body kinematic coupling. This treatment is acceptable where the shell does not undergo torsional deformations [7], [15]. The reference node also served as the point of application of the bending moment.

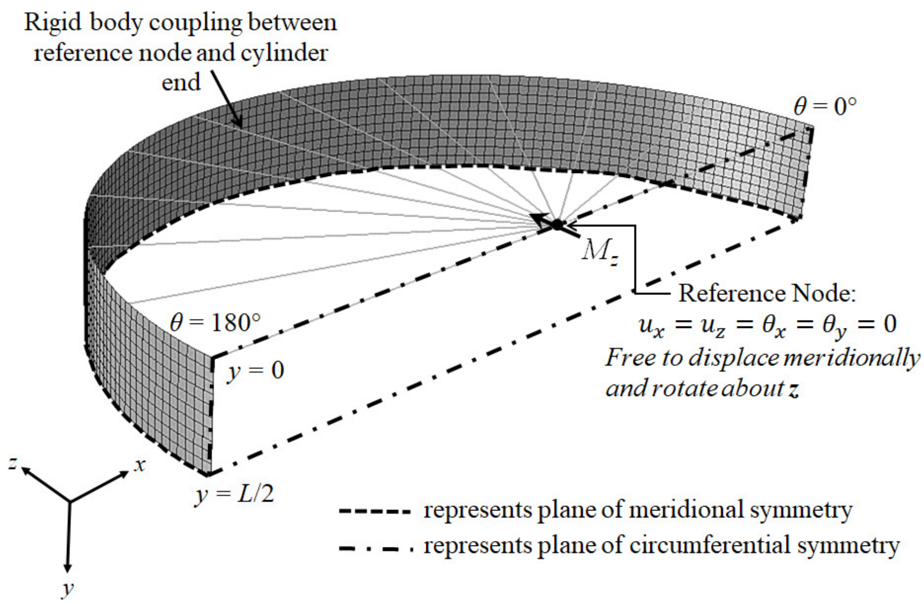


Fig. 2. Illustration of the numerical model, taken from Fajuyitan et al. [13]

The cylinder was assumed to be homogeneous and isotropic with the properties of steel (Young's modulus E = 200 GPa, Poisson's ratio ν = 0.3) and a radius-to-thickness ratio r/t = 100, assuming a unit wall thickness. The lengths of the cylinder investigated were carefully selected to capture the full extent of non-linear elastic buckling behaviour of the 'short' domain and its transition to the 'medium' domain but without being affected by ovalization.

The general-purpose linear shell element with reduced integration S4R was used in the majority of the analyses. However, to verify the numerical models for such 'very short' cylinders and increase confidence in the predictions, many of which fall outside the range of any known computational shell buckling studies to date, specific validation studies were conducted using a further three types of finite element. These included the 8-node quadratic thick-shell element with reduced integration S8R, the 8-node linear solid continuum brick element with incompatible modes C3D8I and the 20-node quadratic solid continuum brick element with reduced integration C3D20R. Following an initial mesh sensitivity study, the S4R numerical model was discretized with an approximately square mesh having an element density of 30 elements per linear bending half-wavelength λ, defined as:

$$\lambda = \frac{\pi\sqrt{rt}}{\left[3\left(1-\nu^2\right)\right]^{0.25}} \approx 2.444\sqrt{rt} \text{ for } \nu = 0.3 \tag{5}$$

A justification for this higher element density is that since the lengths of the very short cylinders are comparable with λ, the pre-buckling stress state is dominated by bending actions owing to the kinematic compatibility requirement with the end boundary conditions [10], [16]. Hence, a significantly higher mesh resolution is needed to capture the high local curvatures associated with this effect. Two different end boundary conditions were employed relating directly to the rotations of the shell middle surface in the meridional direction about the circumferential edge. One was the clamped (BC1r) condition in which these rotations are restrained, while the other was the simply supported condition (BC2f) where the rotations are free. The Rotter and Teng [17] 'Type A' weld depression was employed as an imperfection form placed at mid-span:

$$\delta = \delta_0 \exp\left(-\frac{\pi\left|y-L/2\right|}{\lambda}\right)\left\{\cos\left(\frac{\pi\left|y-L/2\right|}{\lambda}\right)+\sin\left(\frac{\pi\left|y-L/2\right|}{\lambda}\right)\right\} \tag{6}$$

where δ_o is the nominal amplitude of the imperfection, y is the meridional coordinate, $L/2$ is the current mid-span position of the depression and λ is the linear bending half-wavelength (Eq. (5)). This form of imperfection is considered to be representative of true manufacturing defects in cylindrical shells and has been widely used in recent years in numerical investigations of this nature

[18], [19], albeit for significantly longer shells. Normalized imperfection amplitudes δ_o/t = 0.1, 0.25, 0.5, 1.0 and 2.0 were investigated.

The non-linear elastic buckling strength of a perfect cylinder may be computed by a geometrically non-linear analysis (GNA), while the imperfection sensitivity of the cylinder may be investigated through geometrically non-linear analyses with imperfections (GNIA). Both analyses were performed using the modified Riks [20] algorithm in ABAQUS, which automatically checks for structural instability at every load increment and reports a negative eigenvalue in the global tangent stiffness matrix if such an instability is detected [21]. With one exception, the buckling strength was taken as the maximum moment attained immediately before a negative eigenvalue was detected in the global stiffness matrix. The non-linear equilibrium path of the cylinder was represented by the moment–curvature relationship. The curvature was computed as the mean value over the full length of the cylinder as follows:

$$\varphi = \frac{2\theta_z}{L} \tag{7}$$

where θ_z is the rotation of the cross-section at each edge. This mean curvature φ was normalized by the curvature predicted at buckling φ_{cl} [7], obtained using beam bending theory:

$$\varphi_{cl} = \frac{t}{r^2\sqrt{3\left(1-\nu^2\right)}} \approx 0.605\frac{t}{r^2} \text{ for } \nu = 0.3 \tag{8}$$

3 Buckling of short perfect cylinders under bending

The response of perfect short cylinders under uniform bending follows an initially linear path up to the classical elastic critical buckling moment M_{cl} but continues thereafter to exceed this value significantly without exhibiting any local short-wave buckling on the compressed side. However, geometric softening initiates as soon as M_{cl} is exceeded, although the level of softening varies with the degree of rotational restraint offered by the cylinder wall (Fig. 3). This softening is due to the gradual development of a destabilizing meridional fold on the compressed side of the cylinder, essentially an amplified boundary layer of local compatibility bending, which grows deeper as the moment is increased until a limit point instability is attained. Although both clamped and simply supported short cylinders exhibit limit point buckling, it should be understood that this is not at all due to Brazier ovalization, which instead would predict circumferential flattening of the cross-section [10], [22], [23].

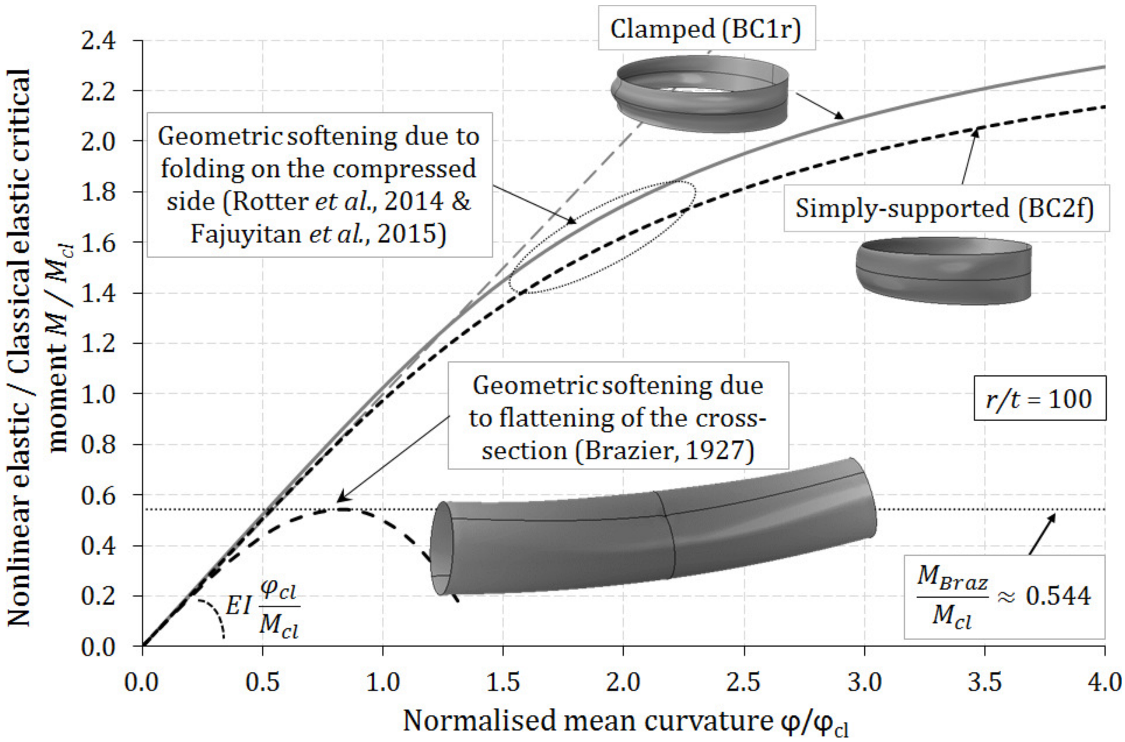
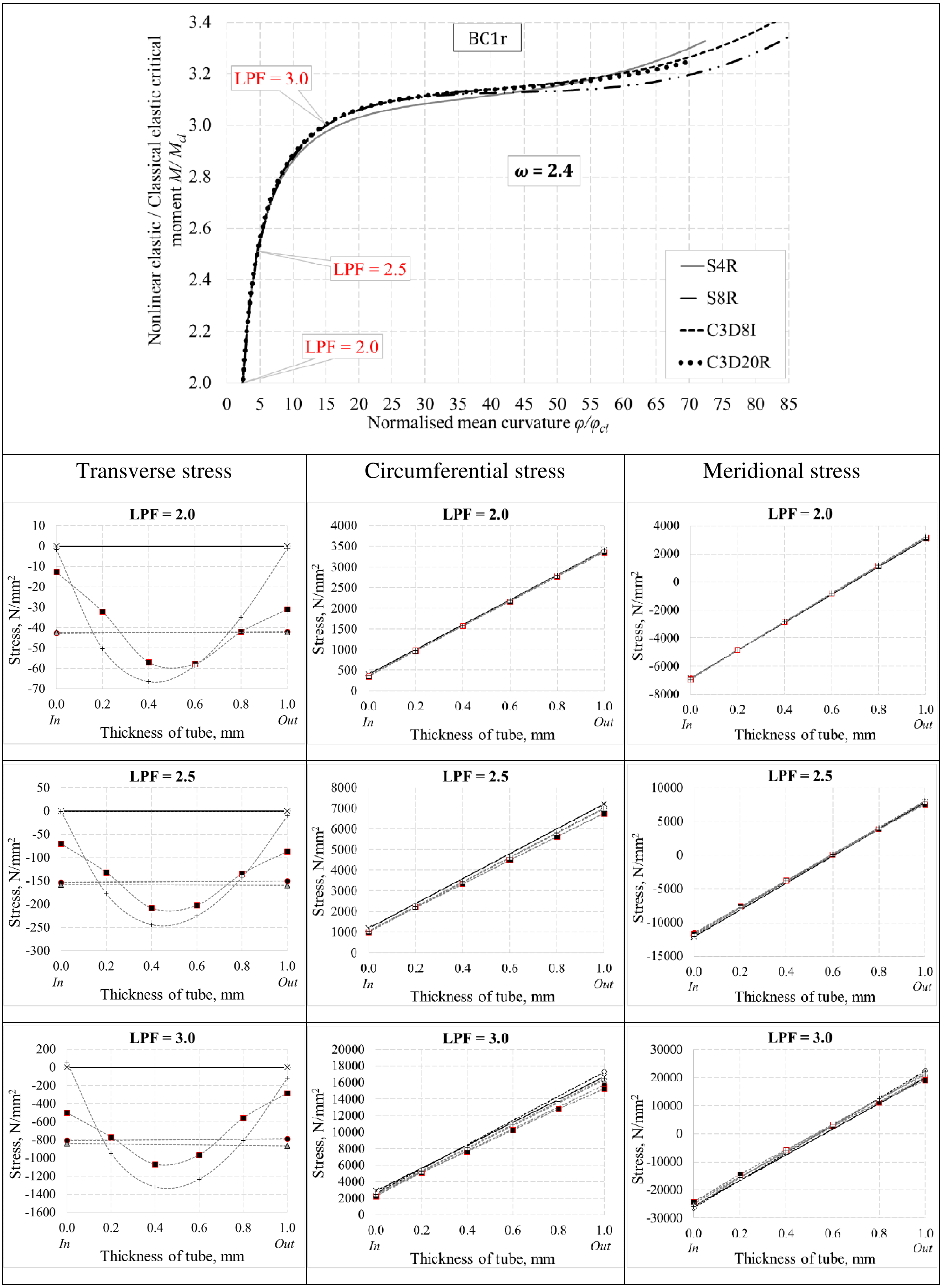


Fig. 3. Non-linear equilibrium paths of very short and very long cylinders under uniform bending indicating the difference between geometric softening due to circumferential flattening or ovalization (long) and due to meridional folding (short), taken from Fajuyitan et al. [13]

When the length of the cylinder is made to be exceptionally short (i.e. $\omega < 2.6$ and 1.4 for BC1r and BC2f respectively), the elastic behaviour changes from a limit point instability with a distinct peak as described above to a type in which there is a smooth transition from geometric softening to apparently indefinite geometric hardening at very large curvatures (Fig. 4). The likely authenticity of this non-linear elastic behaviour was established by performing further analyses at selected very short lengths using three additional finite element types, S8R, C3D8I and C3D20R, as mentioned earlier. For the 3D solid continuum elements (C3D8I and C3D20R), five elements were used through the thickness of the cylinder wall to enable an accurate assessment of the through-thickness variation in bending stresses. The predictions of the full 3D stress distribution were compared across all elements, with results extracted at mid-span of the most compressed meridian and load levels of 2, 2.5 and 3 times M_{cl} (Fig. 4).


BC1r
LPF = 3.0
LPF = 2.5
LPF = 2.0
ω = 2.4
S4R
S8R
C3D8I
C3D20R
Nonlinear elastic / Classical elastic critical moment M/ M_cl
Normalised mean curvature φ/φ_cl
Transverse stress
Circumferential stress
Meridional stress
LPF = 2.0
LPF = 2.5
LPF = 3.0
Stress, N/mm²
Thickness of tube, mm
In
Out

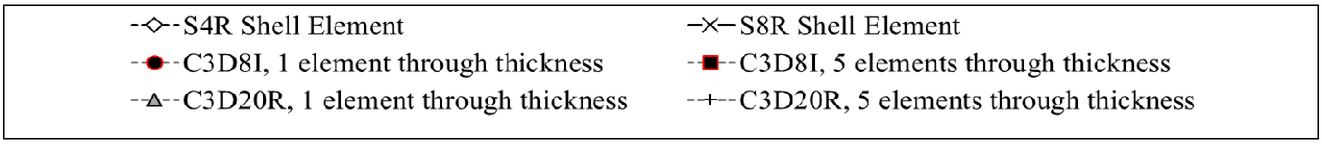


Fig. 4. Non-linear equilibrium paths of very short elastic cylinders (ω = 2.4) under uniform bending showing indefinite geometric hardening and a through-thickness stress distribution comparison of three stress components extracted at mid-span and at the most compressed meridian (i.e. $y = L/2$, $\theta = 0°$)

The comparison primarily suggests that late on the equilibrium path, the solid continuum elements develop a significant transverse stress component owing to the severe deformations encountered through the thickness. The shell element formulation naturally assumes a negligible transverse normal stress component at all times [10], [24], [25]. Despite this growing discrepancy, the shell elements (S4R and S8R) appear capable of modelling much the same highly non-linear response nearly as accurately as their solid continuum counterparts, and at a significantly lower computational cost.

4 Buckling of short imperfect cylinders under bending

The highly detrimental nature of the imperfection sensitivity of cylindrical shells under uniform axial compression is widely known and forms an important consideration when designing metal shells. It arises owing to a pre-buckling stress state dominated by uniform membrane action [26]–[28]. Cylinders subjected to the closely-related load case of uniform bending may be expected to exhibit a similar imperfection sensitivity to axially compressed cylinders. However, a preliminary study by Fajuyitan et al. [11] suggested that the sensitivity is milder and strongly length-dependent when considering the critical buckling eigenmode imperfection form. This work was extended recently by Fajuyitan et al. [13] to consider the weld depression imperfection form of Rotter and Teng [17].

Individual imperfection sensitivity relationships for ‘short’ cylinders (e.g. ω = 3 in Fig. 5) exhibit only a very mildly detrimental effect that does not necessarily become more detrimental with increasing amplitude. The waviness of the imperfection sensitivity curve suggests only a modest ~13 % and ~37 % drop in buckling strength at δ_o/t = 2.0 for BC1r and BC2f cylinders respectively . At such deep imperfection amplitudes, the equilibrium path in fact may exhibit a sudden change of slope at which there is no reported numerical instability, indicating a smooth transition from the pre-buckling to the post-buckling regions. In such cases, the buckling moment was conservatively reported corresponding to the change of slope. For ‘medium’ length cylinders with ω = 15 that are still quite short, a significant reduction in the buckling moment was found to occur as δ_o/t grows to 1, after which the buckling strength also begins to rise as the boundary condition begins to influence the growing buckling mode of more imperfect cylinders. The percentage drop in buckling strength is always less severe than that for the case of uniform axial compression [17], [26].

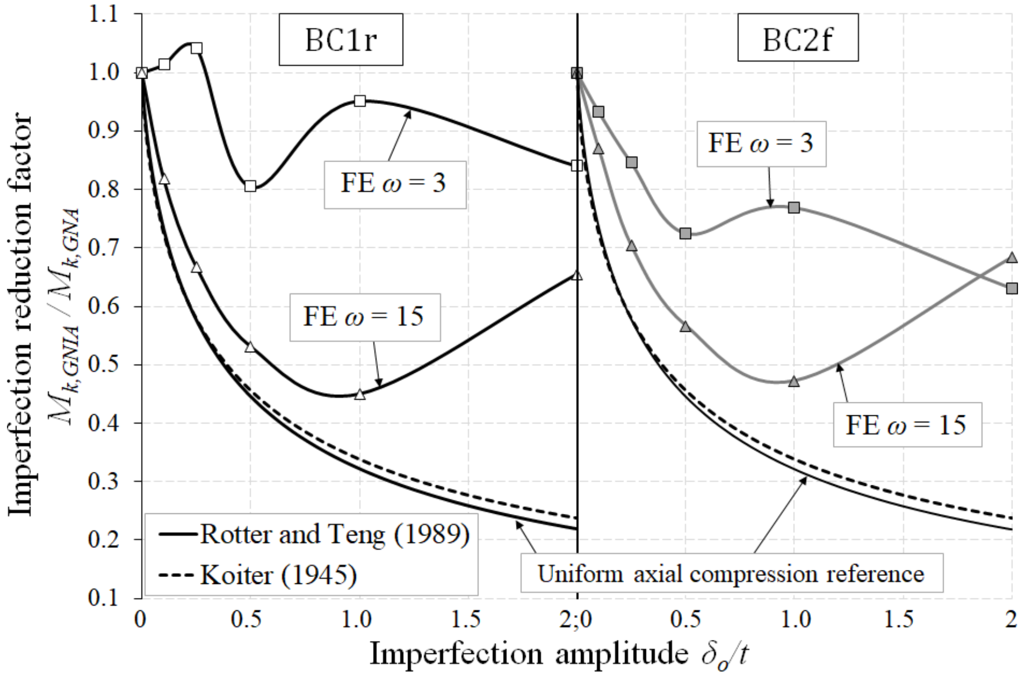


Fig. 5. Imperfection sensitivity of representative 'short' and 'medium' cylinders under bending using the weld depression imperfection form

Computed relationships between the normalized buckling moment M_k/M_{cr} and length ω for both boundary conditions are presented in Figs. 6 and 7. For clamped cylinders (Fig. 6), even a slight imperfection amplitude (δ_o/t = 0.1) appears to decrease the short domain boundary to $\omega \approx 4$, down from 4.8 for a perfect cylinder, although by δ_o/t = 0.25, the boundary oddly shifts back to $\omega \approx 4.8$. The boundary appears more stable for BC2f cylinders (Fig. 7), although this is more evident in the cylinders with higher imperfection amplitudes. This suggests that the length of the onset of the 'short' domain should perhaps be established on the basis of predictions for an imperfect shell, and should be set conservatively low to enable the more detrimental imperfection sensitivity for 'medium' cylinders to persist over a wider range of lengths. A far more extensive discussion is presented in [13].

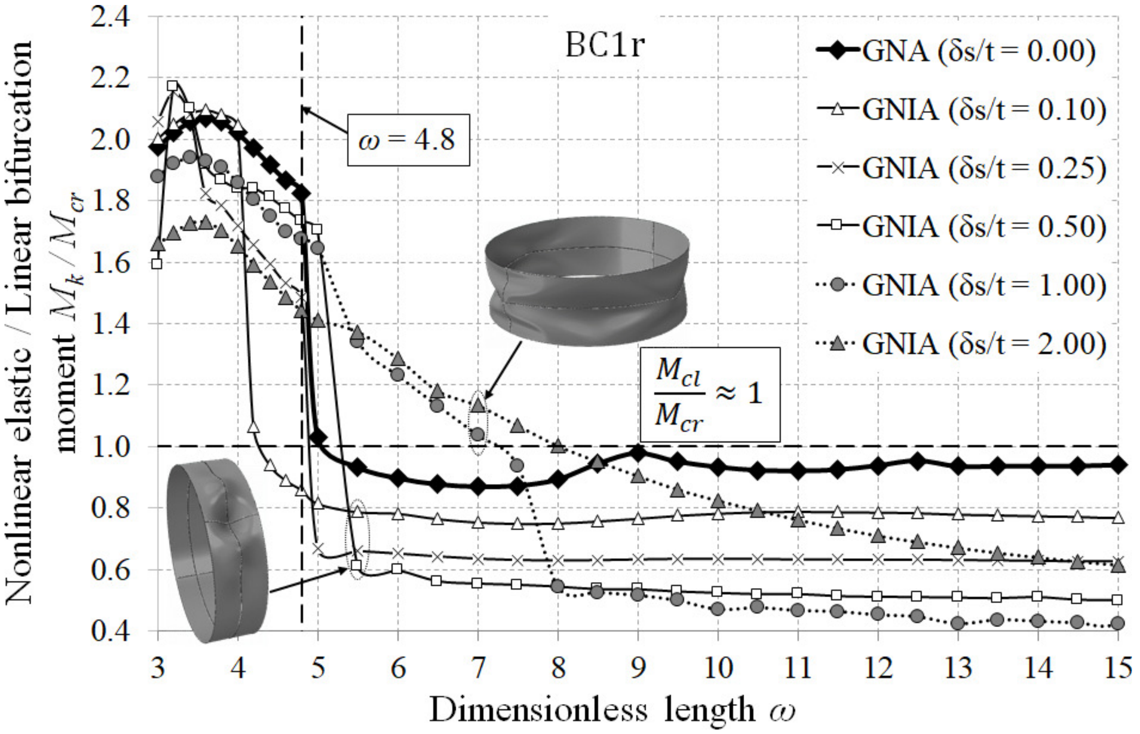


Fig. 6. Computed M_k/M_{cr} vs. ω relationship for perfect and imperfect BC1r cylinders with the weld depression imperfection form

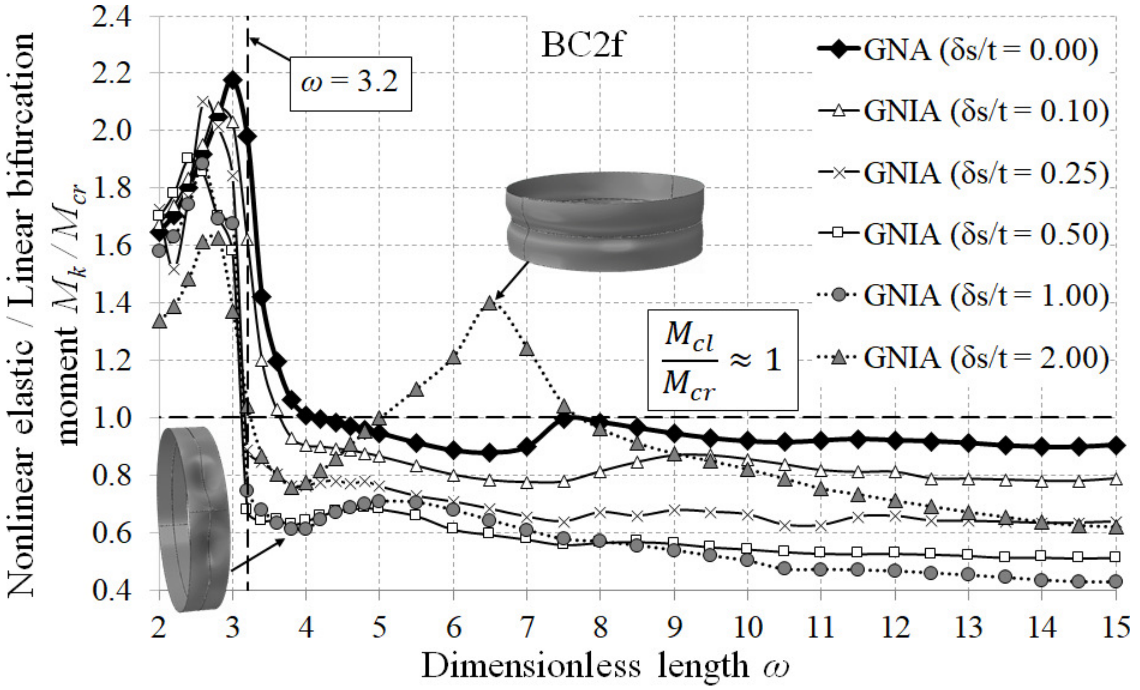


Fig. 7. Computed M_k/M_{cr} vs. ω relationship for perfect and imperfect BC2f cylinders with the weld depression imperfection form

5 Conclusions

This short paper presented a brief overview of recent developments in characterizing the non-linear buckling behaviour of short elastic cylinders under uniform bending, considering two different types of end rotational restraint and the circumferential weld depression imperfection form. It was shown that under geometrically non-linear conditions, the formation of a local buckle can be completely eliminated in short cylinders, which instead develop unfavourable geometric changes in the form of unstable meridional folding on the compressed side. They proceed to fail through limit point buckling at significantly higher moments and curvatures than expected based on analytical predictions. Particularly short cylinders may instead stiffen indefinitely and develop extremely high curvatures. Lastly, the influence of the weld depression geometric imperfection is shown to be not nearly as severe in the 'short' and 'medium' length domains under uniform bending as it is for the corresponding length domains under uniform compression. Further research to characterize definitively the full length-dependent imperfection sensitivity of cylinders under uniform bending is currently ongoing.

Acknowledgements

This project is funded by the Petroleum Technology Development Fund (PTDF) of Nigeria. The authors are grateful to Prof. Michael Rotter for joint discussions on the topic.

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