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Relationship between the methods of bounding time averages

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Abstract

The problem of finding bounds of time-averaged characteristics of dynamical systems, such as for example the bound on the mean energy dissipation rate in a turbulent flow governed by incompressible Navier-Stokes equations, is considered. It is shown that both the well-known background flow method of Doering and Constantin and the direct method proposed by Seis in 2015 correspond to the same quadratic storage functional in the framework of the indefinite storage functional method. In particular, a background flow can be found corresponding to the linear functional used in the direct method and vice versa. It is shown that any bound obtained with the background flow method can also be obtained by the direct method. The reverse is true subject to an additional constraint. The relative advantages of the three methods are discussed.

1 Introduction

The problem of bounding time-averaged characteristics of fluid flows is one of the well-known problems in fluid dynamics. For the following the key references are [1], [2], and [3]. Doering and Constantin [1] introduced the widely-used background flow method. Seis [2] proposed an alternative method. This note aims at revealing the connection between these two methods by recasting them in the framework of the method proposed in [3]. An extensive introduction can be found in [2].

2 General formulation of the three methods and their relationship

2.1 Indefinite storage functional method

The method of [3] can be named an indefinite storage functional method, since it utilizes a functional which satisfies all the requirements for a storage functional except the requirement of being non-negative, or an indefinite Lyapunov functional method, due to the relation between Lyapunov and storage functionals.

In what follows we will use generic formulations or specific ones whichever is easier to understand. The main statements in this subsection also apply in a finite-dimensional case with functionals and operators replaced with functions, variational derivatives replaced with gradients, and inner products replaced with dot products. This might further simplify understanding.

Some specific applications to Navier-Stokes equations will be considered in the following sections. We assume the existence of the relevant limits and solutions.

Consider an infinite-dimensional dynamical system, the state of which is described by a vector $\mathbf{u}$. The dynamical system then determines the time derivative of $\mathbf{u}$:

$$\frac{d\mathbf{u}}{dt} = \mathbf{f}(\mathbf{u}). \tag{1}$$

Let $V[\mathbf{u}]$ be a differentiable functional of $\mathbf{u}$. Then the time derivative of $V[\mathbf{u}(t)]$ can be found from (1) and is a functional of $\mathbf{u}$:

$$D[\mathbf{u}] := \frac{dV}{dt} = \left(\mathbf{f}, \frac{\delta V}{\delta \mathbf{u}}\right).$$

Here, $\delta V/\delta\mathbf{u}$ denotes the gradient of V in the vector space of $\mathbf{u}$, and $(\cdot,\cdot)$ denotes an inner product in this space. In problems involving time averages $\mathbf{u}(t)$ is usually bounded, and, hence, so is $V[\mathbf{u}(t)]$. This proves the following

Lemma 1 *For any bounded differentiable $V[\mathbf{u}]$, if*

$$D[\mathbf{u}] = \left(\mathbf{f}, \frac{\delta V}{\delta \mathbf{u}}\right)$$

then the infinite-time average of $D[\mathbf{u}(t)]$ is zero:

$$\overline{D[\mathbf{u}(t)]} = 0.$$

Proof

$$\overline{D[\mathbf{u}(t)]} = \lim_{T\to\infty} \frac{1}{T}\int_0^T D[\mathbf{u}(t)]\,dt = \lim_{T\to\infty} \frac{V[\mathbf{u}(T)] - V[\mathbf{u}(0)]}{T} = 0.$$

■

This in turn proves the following

Theorem 1 *Let $F[\mathbf{u}]$ be a functional the time-average of which is of interest. Then if there exists a bounded differentiable $V[\mathbf{u}]$ and a constant B such that*

$$F[\mathbf{u}] + D[\mathbf{u}] \le B \qquad \forall \mathbf{u} \tag{2}$$

then $\overline{F[\mathbf{u}(t)]} \le B$, that is B is an upper bound for the time average of $F[\mathbf{u}(t)]$.

A lower bound can be obtained similarly.

Note that if $V[\mathbf{u}] \geq 0$ for all $\mathbf{u}$, and $\mathbf{u}$ is finite-dimensional then by definition $V[\mathbf{u}]$ is a storage function with supply rate $B - F[\mathbf{u}]$ [4]. Storage functions are obviously related to Lyapunov functions encountered in stability theory. For finding bounds the requirement $V[\mathbf{u}] \geq 0$ is not needed.

The optimisation problem of finding the best possible upper bound is

$$B_{\rm opt} = \min_{\substack{B,V \\ \text{s.t. (2)}}} B. \tag{3}$$

The indefinite storage functional method [3] of finding bounds for time averages consists in solving (3), or a relaxation of it. Note that the optimisation problem (3) is convex.

2.2 Quadratic storage functional

In certain cases, and in particular in the case of incompressible fluid flow governed by the Navier-Stokes equations, the right-hand side of (1) is quadratic and can, therefore, be represented as a sum of a homogeneous quadratic operator $\mathbf{n}(\mathbf{u})$ and a linear operator $\mathbf{l}(\mathbf{u})$:

$$\mathbf{f}(\mathbf{u}) = \mathbf{n}(\mathbf{u}) + \mathbf{l}(\mathbf{u}).$$

The most usual quantity of interest, the energy dissipation rate, is also quadratic.

Theorem 2 *If* $\mathbf{f}(\mathbf{u})$, $F[\mathbf{u}]$, *and* $V[\mathbf{u}]$ *are quadratic and satisfy* (2) *then* $V[\mathbf{u}]$ *has the form*

$$V[\mathbf{u}] = Q[\mathbf{u}] + L[\mathbf{u}], \tag{4}$$

where $L[\mathbf{u}]$ *is a linear functional and* $Q[\mathbf{u}]$ *is a quadratic functional such that*

$$\left(\mathbf{n}(\mathbf{u}), \frac{\delta Q}{\delta \mathbf{u}}\right) = 0 \quad \forall \mathbf{u}. \tag{5}$$

Proof Since $V[\mathbf{u}]$ is quadratic $\delta V/\delta \mathbf{u}$ is linear in $\mathbf{u}$. Hence, $(\mathbf{f}, \delta V/\delta \mathbf{u})$ is cubic and, therefore, (2) cannot be satisfied in the general case [5]. The cubic terms in $(\mathbf{f}, \delta V/\delta \mathbf{u})$ can be identically zero if and only if (5) is satisfied. ■

Under the assumptions of Theorem 2,

$$D[\mathbf{u}] = \left(\mathbf{l}(\mathbf{u}), \frac{\delta Q[\mathbf{u}]}{\delta \mathbf{u}}\right) + \left(\mathbf{f}(\mathbf{u}), \frac{\delta L(\mathbf{u})}{\delta \mathbf{u}}\right). \tag{6}$$

In fluid dynamics it is typical that the nonlinear part of the governing equations conserves the kinetic energy, and then often $Q[\mathbf{u}] = E[\mathbf{u}] = ||\mathbf{u}||^2/2 = (\mathbf{u}, \mathbf{u})/2$ is the kinetic energy of the flow.

2.3 Background flow method for energy dissipation rate

The background flow method [1] can be interpreted [3] as an indefinite storage functional method with the storage functional of the form

$$V[\mathbf{u}] = \alpha E[\mathbf{u} - \mathbf{U}] = \alpha \left|\left|\mathbf{u} - \mathbf{U}\right|\right|^2 /2 = \alpha \left|\left|\mathbf{u}\right|\right|^2 /2 - \alpha \left(\mathbf{u}, \mathbf{U}\right) + \alpha \left|\left|\mathbf{U}\right|\right|^2 /2, \quad (7)$$

where $\mathbf{U}$ is a tunable parameter called the background flow, and α is a tunable constant. The name is justified since $\mathbf{U}$ belongs to the same functional space as $\mathbf{u}$ and stems from the particular applications in which $\mathbf{u}$ is the velocity field of the flow in question. By taking a variational derivative we obtain

$$D[\mathbf{u}] = \alpha \left(\mathbf{l}(\mathbf{u}), \frac{\delta \left|\left|\mathbf{u}\right|\right|^2 /2}{\delta \mathbf{u}} \right) - \alpha \left(\mathbf{f}(\mathbf{u}), \frac{\delta \left(\mathbf{u}, \mathbf{U}\right)}{\delta \mathbf{u}} \right). \quad (8)$$

Then the bound is obtained by using Theorem 1.

2.4 Direct method for energy dissipation rate

The recently proposed direct method [2] is not formulated in a general form. Instead, a number of examples are given, and the general description has to be derived from these examples. In the cases considered in [2] an identity of the form

$$\overline{D[\mathbf{u}(t)]} = 0 \quad (9)$$

with a certain functional $D[\mathbf{u}]$ is derived from the governing equations and boundary conditions.

Then, using various integral inequalities and algebraic transformations, *but not using the governing equations,* an inequality of the form

$$\overline{D[\mathbf{u}(t)]} \le \phi \left(\overline{F[\mathbf{u}(t)]} \right) \quad \forall \mathbf{u}(t) \quad (10)$$

is derived.

Finally, the bound for $\overline{F[\mathbf{u}(t)]}$ is obtained from the inequality $0 \le \phi \left(\overline{F[\mathbf{u}(t)]} \right)$.

The particular derivations and the form of the functional $D[\mathbf{u}]$ and the function $\phi(F)$ vary from example to example.

The following observations can be made. In [2],

1. The method of proving (9) is equivalent to finding a functional $V[\mathbf{u}]$ such that
$$D[\mathbf{u}] = \left(\mathbf{f}, \frac{\delta V}{\delta \mathbf{u}} \right)$$
and using Lemma 1. (This will be illustrated in Section 3.)

2. The functional $V[\mathbf{u}]$ is quadratic and has the form (4) satisfying (5).

3. The derivation of (10) also applies for instantaneous values of the functionals, leading to a similar inequality for instantaneous values:

$$D[\mathbf{u}] \le \phi\left(F[\mathbf{u}]\right) \quad \forall \mathbf{u}. \tag{11}$$

   This is because in [2] all such derivations use only the specific form of $D[\mathbf{u}]$ but not the governing equations and are thus valid for any $\mathbf{u}(t)$, including $\mathbf{u}$ independent of time.

2.5 Relationship between the background flow method and the direct method

While recognizing that our observations concerning the direct method might be not general, in this section we will assume them to be a part of how the direct method is defined. Then the links between the background flow method and the direct method are

1. The storage functional used in the background flow method can also be used in the direct method and vice versa. For this it is necessary that

$$L[\mathbf{u}] = \frac{\delta\left(\mathbf{u}, \mathbf{U}\right)_2}{\delta \mathbf{u}}. \tag{12}$$

   Proof For a given $\mathbf{u}$ the corresponding L is given by (12). For a given L the corresponding $\mathbf{U}$ exists according to the Riesz-Markov-Kakutani representation theorem, even though the background flow can turn out to be a generalised function. ■

2. If a certain bound is found by the background flow method, the same bound can be obtained by the direct method.

   Proof If B is the bound obtained by the background flow method then averaging (2) over time gives

$$\overline{D[\mathbf{u}(t)]} \le B - \overline{F[\mathbf{u}(t)]} \quad \text{for any} \quad \mathbf{u} = \mathbf{u}(t).$$

   This proves the inequality (10) of the direct method with $\phi(\overline{F}) = B - \overline{F}$. Then from $\phi(\overline{F}) \ge 0$, the same bound follows. ■

3. If a bound B is found by the direct method, the same bound can be obtained by the background flow method provided that the function ϕ satisfies the condition

$$\exists\, \alpha \ge 0 : x + \alpha\phi(x) \le B \;\forall x. \tag{13}$$

   Proof From (13) it follows that

$$F[\mathbf{u}] + \alpha D[\mathbf{u}] \le F[\mathbf{u}] + \alpha\phi(F[\mathbf{u}]) \le B \qquad \forall \mathbf{u},$$

   and hence B is the upper bound for $\overline{F}$ obtained by the background flow method. ■

   Remark Recollecting that $\phi(\overline{F}) \le 0$ for all $\overline{F} \ge B$ it is easy to see that condition (13) is satisfied for any concave function ϕ.

3 Plain Couette flow

We will now illustrate our observations with an analysis of the plane Couette flow.

3.1 Problem formulation

We will insignificantly generalize the reasoning in [2] of obtaining the bound for energy dissipation rate in plane Couette flow, staying as close to [2] as possible for our purposes. We will consider the solutions of the Navier-Stokes equations

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \nabla^2 \mathbf{u} \tag{14}$$

and the continuity equation

$$\nabla \cdot \mathbf{u} = 0 \tag{15}$$

in the domain $0 \leq x \leq L$, $0 \leq y \leq L$, $0 \leq z \leq 1$, where x, y, and z are Cartesian coordinates, and $\mathbf{u} = (u, v, w)$ is the velocity field. The lower wall is located at $z = 0$, where the no-slip condition $\mathbf{u} = 0$ is imposed. The upper wall at $y = 1$ is moving at a constant speed Re, so that the boundary condition is $u = Re$, $v = w = 0$. The solution is assumed to be periodic in x and y with the periods L.

We will distinguish the horizontal average defined as

$$\langle f(t,x,y,z) \rangle_h = \frac{1}{L^2} \iint_{x,y=0}^{x,y=L} f(t,x,y,z)\, dx\, dy$$

and the time and horizontal average denoted and defined as in [2]:

$$\langle f(t,x,y,z) \rangle = \overline{\langle f(t,x,y,z) \rangle_h}.$$

The spatial average of the energy dissipation rate is

$$\epsilon[\mathbf{u}] = \int_0^1 \left\langle |\nabla \mathbf{u}|^2 \right\rangle_h dz,$$

and the time-averaged dissipation rate is

$$\overline{\varepsilon} = \overline{\epsilon[\mathbf{u}(t)]}.$$

The problem consists in deriving an upper bound for $\overline{\varepsilon}$.

3.2 Storage functionals

To obtain the bound for $\overline{\varepsilon}$ we will loosely follow [2], but without time averaging. Multiplying (14) by $\mathbf{u}$, taking horizontal average, integrating by parts with the periodic boundary conditions used, integrating with respect to z and using the boundary conditions gives the energy equation

$$\frac{d}{dt}\int_0^1 \frac{\left\langle \mathbf{u}^2\right\rangle_h}{2}\,dz = Re\,\frac{\partial \langle u\rangle_h}{\partial z}\bigg|_{z=1} - \epsilon[\mathbf{u}], \tag{16}$$

which corresponds to (3.5) in [2] in the sense that it becomes (3.5) in [2] after time averaging.

Taking the horizontal average of the x-component of (14) gives[1]

$$\frac{d}{dt}\langle u\rangle_h = -\frac{\partial \langle uv\rangle_h}{\partial z} + \frac{\partial^2 \langle u\rangle_h}{\partial z^2} \quad \text{for all } z, \tag{17}$$

which corresponds to (3.6) in [2]. Integrating (17) with respect to z and using the boundary conditions gives

$$\frac{d}{dt}\int_1^z \langle u\rangle_h\big|_{z=\zeta}\,d\zeta = -\langle uv\rangle_h + \frac{\partial \langle u\rangle_h}{\partial z} - \frac{\partial \langle u\rangle_h}{\partial z}\bigg|_{z=1} \quad \text{for all } z. \tag{18}$$

Adding (18) multiplied by Re to (16) gives

$$\frac{d}{dt}\left(\int_0^1 \frac{\left\langle \mathbf{u}^2\right\rangle_h}{2}\,dz - Re\int_z^1 \langle u\rangle_h\big|_{z=\zeta}\,d\zeta\right) = -\epsilon[\mathbf{u}] + Re\left\langle \frac{\partial u}{\partial z} - uw\right\rangle_h \quad \text{for all } z, \tag{19}$$

which corresponds to (3.7) in [2].

We will now introduce the weight $\rho(z)$ such that

$$\int_0^1 \rho\,dz = 1,$$

and take a weighted z-average of (19), arriving at

$$\frac{d}{dt}\int_0^1\left(\frac{\left\langle \mathbf{u}^2\right\rangle_h}{2} - Re\rho(z)\int_z^1 \langle u\rangle_h\big|_{z=\zeta}\,d\zeta\right)dz = -\epsilon[\mathbf{u}] + Re\int_0^1 \rho(z)\left\langle \frac{\partial u}{\partial z} - uw\right\rangle_h dz, \tag{20}$$

which corresponds to (3.8) in [2] if $\rho(z)$ is taken to be

$$\rho(z) = \begin{cases} 1/l, & 0 \le z \le l \\ 0, & l < z \le 1. \end{cases} \tag{21}$$

Hence, the indefinite storage functional corresponding to the analysis of [2] is

$$V_D[\mathbf{u}] = \int_0^1\left(\frac{\left\langle \mathbf{u}^2\right\rangle_h}{2} - Re\rho(z)\int_z^1 \langle u\rangle_h\big|_{z=\zeta}\,d\zeta\right)dz, \tag{22}$$

[1]This is the well-known Reynolds-averaged Navier-Stokes equation for a unidirectional mean flow depending on one coordinate.

and (20) can be written as

$$\frac{dV_D[\mathbf{u}]}{dt} = D_D[\mathbf{u}] = -\epsilon[\mathbf{u}] + Re\int_0^1 \rho(z)\left\langle \frac{\partial u}{\partial z} - uw \right\rangle_h dz. \tag{23}$$

For the same problem, the indefinite storage functional corresponding to the background flow method can be taken to be

$$V_B[\mathbf{u}] = \alpha E[\mathbf{u} - \mathbf{U}] + \text{const} = \alpha V_D[\mathbf{u}] \tag{24}$$

with $\mathbf{U} = (U(z), 0, 0)$ and

$$\frac{dU}{dz} = \rho Re, \quad U(0) = 0. \tag{25}$$

This can be verified by substituting (25) into (22) and integrating by parts. For the weight (21) used in [2] the corresponding background flow is

$$U(z) = \begin{cases} Re z/l, & 0 \le z \le l \\ Re, & l < z \le 1. \end{cases}$$

3.3 Bounds equivalence

We can derive the equation (3.8) from [2] by time-averaging (20), which gives

$$0 = -\overline{\varepsilon} + Re\int_0^1 \rho(z)\left\langle \frac{\partial u}{\partial z} - uw \right\rangle dz, \tag{26}$$

In [2] using several integral inequalities and a suitable choice of l the integral term of (26) is bounded in terms of $\overline{\varepsilon}$, giving (compare with (3.16) and (3.17) in [2])

$$0 \le -\overline{\varepsilon} + A\overline{\varepsilon}^{2/3} =: \phi(\overline{\varepsilon})$$

where

$$A = \left(\frac{81\sqrt{6}}{128\pi^2}\right)^{1/3} Re.$$

This immediately leads to the bound [2]

$$\overline{\varepsilon} \le A^3.$$

Note that $-\overline{\varepsilon} + A\overline{\varepsilon}^{2/3}$ is a concave function of $\overline{\varepsilon}$.

In the background flow method, choosing the storage functional as in (24), a value B will be an upper bound of $\overline{\varepsilon}$ provided that there exists a suitable value α such that for any $\mathbf{u}$

$$\epsilon[\mathbf{u}] + \alpha D_D[\mathbf{u}] \le B. \tag{27}$$

Applying the same integral inequalities and the same choice of l to the integral in (26) that were applied to the integral term of (26) leads to

$$D_D[\mathbf{u}] \le -\epsilon[\mathbf{u}] + A(\epsilon[\mathbf{u}])^{2/3}.$$

It is then trivial to verify that with $\alpha = 3$ and $B = A^3$ the condition (27) is satisfied for all $\mathbf{u}$. This means that the background flow method produces the same bound as the method of [2].

Instead of using (3.16) of [2], we can use (3.12) of [2], but the resulting bound again turns out to be the same.

4 Concluding remarks

It has already been known [3] that the background flow method [1] is a special case of the indefinite storage functional method, in which the storage functional is quadratic, and we have now demonstrated that the direct method [2] also corresponds to a quadratic storage functional. For dynamical systems with quadratic nonlinearity the quadratic part of the storage functional has to be conserved by the quadratic part of the equation governing the dynamical system, which leaves very little freedom in the choice of the storage functional. This, in essence, is the principal reason for the close connection between the background flow method and the direct method, making them equivalent in a wide class of problems.

The direct method has an advantage of relative simplicity as compared to the other two methods, thanks to eliminating the time derivatives of the storage functional early in the analysis. It is also possible that while always being able to give the same bound as the background flow method, it might sometimes give a better bound.

The background flow method, being equivalent to the indefinite storage functional method with a quadratic storage functional, has an advantage when the functional to be bounded is also quadratic, as for example it is the case when the bound for the energy dissipation rate is sought for. In such cases the constraint in the bound optimization problem becomes quadratic, reducing this problem to a linear eigenvalue problem, which is easier to solve.

The main advantage of the indefinite storage functional method is its conceptual simplicity. It is this simplicity that allowed to establish the connection between the background flow method and the direct method in the present work. The simplicity of this method was also crucial in extending it to systems with noise [3], thus allowing to obtain bounds insensitive to unstable solutions, with the first example given in [6], and to relaxing the bound optimization problem to a finite-dimensional semi-definite optimization problem. The method of such reduction was first proposed in [5] and then implemented in [7] in the context of Lyapunov stability of a particular fluid flow. Another advantage of the indefinite storage functional method is the possibility of using polynomial storage functionals of higher-degree than quadratic. This can give better bounds, as it was demonstrated on finite-dimensional examples in [3, 6, 8] and is also implied by the Lyapunov stability results of [7] for the full Navier-Stokes equations case. At least some of these bounds are better than the best bounds obtainable by the background flow method.

References

[1] C.R. Doering and P. Constantin. Variational bounds on energy dissipation in incompressible flows: Shear flow. *Phys. Rev. E*, 49:4087–4099, 1994.

[2] C. Seis. Scaling bounds on dissipation in turbulent flows. *J. of Fluid Mech.*, 777:591603, 2015.

[3] S. I. Chernyshenko, P. Goulart, D. Huang, and A. Papachristodoulou. Polynomial sum of squares in fluid dynamics: a review with a look ahead. *Phil. Trans. R. Soc. B.*, 372(2020), 2014.

[4] J.C. Willems. Dissipative dynamical systems part I: General theory. *Archive for Rational Mechanics and Analysis*, 45(5):321–351, 1972.

[5] P. J. Goulart and S. Chernyshenko. Global stability analysis of fluid flows using sum-of-squares. *Physica D*, 241(6):692–704, March 2012.

[6] G. Fantuzzi, D. Goluskin, D. Huang, and S.I. Chernyshenko. Bounds for deterministic and stochastic dynamical systems using sum-of-squares optimization. *SIAM Journal on Applied Dynamical Systems*, 15:1962–1988, 2016.

[7] D. Huang, S. Chernyshenko, P. Goulart, D. Lasagna, O. Tutty, and F. Fuentes. Sum-of-squares of polynomials approach to nonlinear stability of fluid flows: an example of application. *Proc. Roy. Soc. A: Mathematical, Physical and Engineering Sciences*, 471, 2015.

[8] D. Goluskin. Bounding averages rigorously using semidefinite programming: mean moments of the Lorenz system. *arXiv:1610.05335 [math.DS]*, 2016.