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Embedding spanning bounded degree subgraphs in randomly perturbed graphs [5]

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Abstract

We study the model of randomly perturbed dense graphs, which is the union of any graph G_α with minimum degree αn and the binomial random graph $G(n,p)$. For $p = \omega(n^{-2/(\Delta+1)})$, we show that $G_\alpha \cup G(n,p)$ contains any single spanning graph with maximum degree Δ. As in previous results concerning this model, the bound for p we use is lower by a log-term in comparison to the bound known to be needed to find the same subgraph in $G(n,p)$ alone.

Keywords: random graphs, spanning subgraphs, thresholds

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1 Introduction and Result

1.1 Thresholds in $G(n,p)$

Let $G(n,p)$ be the binomial random graph model, where among n vertices each possible edge is chosen independently with probability p.

An important part of random graph theory is the understanding of threshold behaviour with respect to certain graph properties. We say that $\hat{p}$ is a threshold for a graph property $\mathcal{F}$ if $\mathbb{P}[G(n,p) \in \mathcal{F}] \to 0$ for $p = o(\hat{p})$ and $\mathbb{P}[G(n,p) \in \mathcal{F}] \to 1$ for $p = \omega(\hat{p})$. If the latter is true, then we say that $G(n,p)$ has the property $\mathcal{F}$ with high probability (whp) and that this $\hat{p}$ is an upper bound for the threshold. Containing a graph as a (not neccesarily induced) subgraph is a monotone property and therefore admits a threshold [7].

In the following we will focus on spanning subgraphs. In their early, seminal work Erdős and Rényi [10] determined the threshold for perfect matchings in $G(n,p)$, which is $\ln n/n$. Pósa [21] and Korŝunov [15] independently showed that the property of having a Hamilton cycle has the same threshold. Recently, there has been a lot of work on the threshold for a bounded degree spanning tree, where the current best bound, by the second author [18,19], is $p \geq \Delta \ln^5 n/n$. A breakthrough result was achieved by Johannson, Kahn and Vu [13] who showed that the (sharp) threshold for a $K_{\Delta+1}$-factor, that is $n/(\Delta+1)$ vertex-disjoint copies of $K_{\Delta+1}$, is given by

$$p_\Delta := (n^{-1} \ln^{1/\Delta} n)^{\frac{2}{\Delta+1}}.$$

Turning to a much more general class of graphs, let $\mathcal{F}(n,\Delta)$ be the family of graphs on n vertices with maximum degree at most Δ. For some absolute constant C, Alon and Füredi [3] proved that, if $p \geq C(\ln n/n)^{1/\Delta}$, then $G(n,p)$ contains a fixed graph from $\mathcal{F}(n,\Delta)$ whp. This is far from optimal and since the clique-factor is widely believed to be the hardest graph in $\mathcal{F}(n,\Delta)$ to embed, it is natural to state the following conjecture.

Conjecture 1.1 *If $\Delta > 0$, $F \in \mathcal{F}(n,\Delta)$ and $p = \omega(p_\Delta)$, then whp $G(n,p)$ contains a copy of F.*

For $\Delta = 2$, this conjecture was very recently solved by Ferber, Kronenberg and Luh [11], who in fact showed a stronger so-called universality statement, is finding all graphs of the class simultaneously. larger Δ, Riordan [22] gave a general result, which requires a probability larger by a factor of $n^{\Theta(1/\Delta^2)}$ from p_Δ. The current best result in this direction is the following almost spanning version by Ferber, Luh and Nguyen [12].

Theorem 1.2 ([12]) *Let $\varepsilon > 0$ be any constant and let $\Delta \geq 5$ be an integer. Then, for every $F \in \mathcal{F}((1-\varepsilon)n, \Delta)$ and $p = \omega(p_\Delta)$, whp the random graph $G(n, p)$ contains a copy of F.*

Their approach is based on ideas from Conlon, Ferber, Nenadov and Škorić [8] who prove a stronger universality statement for the almost spanning case with probability $p \geq n^{-1/(\Delta-1)} \ln^5 n$. This also extends Theorem 1.2 to the case $\Delta = 3$, whereas $\Delta = 4$ remains open.

In the almost spanning case the ln-term in p_Δ is expected to be redundant [12], but this remains open. Essentially, we will show here that the ln-term in p_Δ is redundant if we add to $G(n, p)$ a deterministic graph with linear minimum degree.

1.2 Randomly perturbed graphs

We now change the setup in the following way, as first suggested by Bohman, Frieze and Martin [6] (though they worked with $G(n, m)$ instead of $G(n, p)$). For $\alpha \in (0, 1)$, let G_α be any graph with minimum degree at least αn and reveal more edges within the graph independently at random with probability p. That is, we study the properties of $G_\alpha \cup G(n, p)$. For a fixed G_α, containing a subgraph is a monotone property in $G_\alpha \cup G(n, p)$. Hence one can ask for upper bounds on thresholds in this model.

For $\alpha \in (0, 1/2)$ Bohman, Frieze and Martin [6] showed that if $p = \omega(1/n)$ then whp there is a Hamilton cycle in $G_\alpha \cup G(n, p)$ for any G_α. Furthermore, this is optimal, as for $p = o(1/n)$ there are graphs G_α such that $G_\alpha \cup G(n, p)$ is not Hamiltonian whp. Comparing this threshold to the threshold for Hamiltonicity in $G(n, p)$ we note an extra factor of $\ln n$ in the latter. This $\ln n$ term is necessary to guarantee minimum degree at least 2, otherwise clearly no Hamilton cycle exists. Of course if $\alpha \geq 1/2$, then G_α is itself Hamiltonian (Dirac's Theorem) and so for smaller α a few random edges can compensate for the loss in minimum degree.

Krivelevich, Kwan and Sudakov [16] studied the corresponding problem for the containment of bounded degree trees and showed that $p = \omega(1/n)$ is sufficient in this case. For $p = \omega(1/n)$ it is already possible to find any almost spanning bounded degree tree in $G(n, p)$ [4]. The addition of G_α ensures there are no isolated vertices and allows every vertex to be incorporated into the embedding.

Krivelevich, Kwan and Sudakov [17] also considered matchings and loose cycles in uniform hypergraphs. In an r-uniform hypergraph all edges have cardinality r and in a loose Hamilton cycle consecutive edges intersect in exactly one vertex. The generalized minimum degree condition in G_α is that all $(r-1)$-sets are contained in at least αn edges. Here, only a large linear number of edges is required in the random r-uniform hypergraph to ensure both properties, matchings and loose cycles, in the union with G_α. Note that for the loose Hamilton cycle the corresponding Dirac type theorem is known [14]. Comparing these bounds to the threshold for matchings and loose cycles in random hypergraphs (which are both $n^{-r+1} \ln n$ [9,13]), we again have a difference of $\ln n$.

Other monotone properties considered in this model include containing a fixed sized clique, having a small diameter, k-connectivity [5] and non-2-colorability [23].

1.3 Our Result

We analyze the model $G(n,p) \cup G_\alpha$ with respect to the containment of spanning bounded degree graphs and obtain the following.

Theorem 1.3 *Let $\alpha > 0$ be a constant, $\Delta \geq 5$ an integer and G_α a graph with minimum degree at least αn. Then, for every $F \in \mathcal{F}(n, \Delta)$ and $p = \omega\left(n^{-\frac{2}{\Delta+1}}\right)$, whp $G(n,p) \cup G_\alpha$ contains a copy of F.*

Observe that the bound on p is best possible. Indeed, in the case where F is a $K_{\Delta+1}$-factor on n vertices and $G_\alpha = K_{\alpha n,(1-\alpha)n}$, we need to find an almost spanning $K_{\Delta+1}$-factor of size $(1-\alpha\Delta)n$ in $G(n,p)$. Furthermore, compared to p_Δ this is again better by a ln-term.

2 Overview of the proof of Theorem 1.3

We give a brief outline of the steps of the proof and the tools involved.

2.1 Embedding most of the graph

Similarly as for other results in this model, we first obtain an almost spanning embedding of all but εn vertices of F, using only the edges of the random graph $G(n,p)$. For this we adapt the strategy of Ferber, Luh and Nguyen [12] to decompose the graph, and embed it using a theorem of Riordan [22] and Janson's inequality. A major difference to previous methods is that we do not choose precisely the large subgraph of F to embed, only seeking to embed an almost spanning subgraph of F which covers the sparser parts of F.

2.2 Preparing the reservoir

The key part in our proof is to obtain a so-called reservoir set. To build this we use that only random edges have been used so far and thus, the embedding is independent of G_α. The reservoir set we develop is already covered by the partial embedding. When, later in the proof, we need to use some vertices from the reservoir set, we use the deterministic graph G_α to swap out some vertices from the reservoir. Similar kind of reservoir structures were used for embedding bounded degree trees [18] and tight Hamilton cycles in hypergraphs [2], but we use the interplay of the random and deterministic graphs in a new way to create this structure.

2.3 Finishing the embedding

Using additional edges of $G(n, p)$ and G_α, we can embed the rest of F into the reservoir. The approach for the embedding again follows [12], using Janson's inequality and a Hall-type matching argument for hypergraphs [1]. It is crucial that we also use the deterministic edges of G_α here, to gain the ln-term in comparison to p_Δ. Finally, we use the properties of the reservoir to complete the embedding.

3 Concluding remarks

In fact Theorem 1.3 is valid also for $\Delta \leq 3$ and basically the same approach works. The difference is that the definition of the dense spots has to be slightly adapted to each of the cases. For $\Delta = 4$, the only dense spots for which our methods do not work are triangles with two pendant edges at each vertex extending to the rest of the graph. We do not know how to deal with many of these particular dense spots.

The multiround exposure is not crucial for our embedding, but it makes the calculations for Janson's inequality much simpler. Furthermore, we would not need the hypergraph matching theorem, if we managed to use $(\Delta + 1)/2$ many edges of G_α for the embedding of each dense spot. As there might not be th many edges leaving a dense spot, we would also need to use edges inside the dense spot, which is possible, but much harder to work with.

On the other hand, Riordan's result, which is proven by second moment calculations, is essential for our approach. Thus it seems unlikely that there is a chance to extend this to a proof of a corresponding universality statement, though we believe such a statement should hold. That is, we think that $G(n,p) \cup G_\alpha$ contains whp a copy of every graph in $\mathcal{F}(n,\Delta)$ simultaneously, with p and α as in Theorem 1.3. Similarly, it is commonly believed that p_Δ is the threshold for the property that $G(n,p)$ is universal for $\mathcal{F}(n,\Delta)$.

The third and fourth author [20] extended the result of Riordan [22] to hypergraphs. Analogous generalizations for Theorems 1.2 and 1.3 would be interesting.

Moreover, for the model $G(n,p) \cup G_\alpha$ it would be nice to know if there are any nontrivial spanning structures, for which this provides no advantage compared to $G(n,p)$, in the sense that the bound on p needed in $G_\alpha \cup G(n,p)$ is of the same order as the corresponding threshold in $G(n,p)$. For example, it might be interesting to consider the d-dimensional cube, which appears in $G(n,p)$ shortly after $p = 1/4$ [22].

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