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Stability for vertex cycle covers

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Abstract

In 1996 Kouider and Lonc proved the following natural generalization of Dirac's Theorem: for any integer $k \geqslant 2$, if G is an n-vertex graph with minimum degree at least n/k, then there are $k-1$ cycles in G that together cover all the vertices.

This is tight in the sense that there are n-vertex graphs that have minimum degree $n/k-1$ and that do not contain $k-1$ cycles with this property. A concrete example is given by $I_{n,k} = K_n \setminus K_{(k-1)n/k+1}$ (an edge-maximal graph on n vertices with an independent set of size $(k-1)n/k+1$). This graph has minimum degree $n/k-1$ and cannot be covered with fewer than k cycles. More generally, given positive integers $k_1, \dots, k_r$ summing to k, the disjoint union $I_{k_1 n/k, k_1} + \dots + I_{k_r n/k, k_r}$ is an n-vertex graph with the same properties.

In this paper, we show that there are no extremal examples that differ substantially from the ones given by this construction. More precisely, we obtain the following stability result: if a graph G has n vertices and minimum degree *nearly* n/k, then it either contains $k-1$ cycles covering all vertices, or else it must be close (in 'edit distance') to a subgraph of $I_{k_1 n/k, k_1} + \dots + I_{k_r n/k, k_r}$, for some sequence $k_1, \dots, k_r$ of positive integers that sum to k.

Our proof uses Szemerédi's Regularity Lemma and the related machinery.

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1 Introduction

The theorem of Dirac [10] saying that any graph G on $n \geqslant 3$ vertices with minimum degree at least $n/2$ contains a Hamilton cycle is one of the classical results of graph theory. There is a rich collection of extensions of this theorem in various directions. One possibility to replace the Hamilton cycle with another spanning subgraph and ask what minimum degree guarantees its existence.

For example, Bollobás [6] conjectured that for $c > 1/2$ and $\Delta > 0$, every sufficiently large n-vertex graph G with minimum degree at least cn contains every spanning tree of maximum degree at most Δ. The proof of this conjecture was given by Komlós, Sárközy and Szemerédi [20], using the regularity method. Another example is the famous Hajnal-Szemerédi theorem [18], saying that every graph on kn vertices with minimum degree $(k-1)n$ contains n vertex-disjoint copies of K_k. Yet another well-known example is the conjecture of Pósa [15] and Seymour [32] that any n-vertex graph with minimum degree at least $kn/(k+1)$ contains the k-th power of a Hamilton cycle. If true, this would imply both Dirac's theorem and the Hajnal-Szemerédi theorem. The Pósa-Seymour conjecture was proved for large n by Komlós, Sárközy, and Szemerédi [22], [23]. Later, Levitt, Sárközy, Szemerédi [28] and Chau, DeBiasio, and Kierstead [9] proved the same result with different methods, for smaller values of n. When we consider the square of a Hamilton path instead of the square of a Hamilton cycle, Fan and Kierstead [16] proved that $(2n-1)/3$ is the optimal minimum degree for every n.

All of these results are about graphs with minimum degree larger than $n/2$. Indeed, as soon as the minimum degree can be below $n/2$, one loses a lot of global structure: for example, the graph may no longer be connected. Here we will explore the direction where the minimum degree can be smaller than $n/2$. Already Dirac observed that every 2-connected graph G contains a cycle of length at least $\min\{v(G), 2\delta(G)\}$ [10]. The connectivity assumption in this result might seem artificial, and indeed several researchers have looked at the case without this assumption. Alon [2] proved that any n-vertex graph G with minimum degree at least n/k must contain a cycle of length at least $\lfloor n/(k-1)\rfloor$. Later Bollobás and Häggkvist [7] proved that such a graph must in fact contain a cycle of length $\lceil n/(k-1)\rceil$, which is optimal. An Ore-type condition for the same problem is considered in [12]. More recently, Nikiforov and Schelp [30] and Allen [1] have considered the problem of finding cycles of a specified length in graphs of minimum degree at least n/k.

In 1987 Enomoto, Kaneko and Tuza [14] conjectured that any graph G on n vertices with $\delta(G) \geqslant n/k$ contains a collection of at most $k-1$ cycles that cover all vertices of G. Note that in the case $k = 2$ this reduces to Dirac's theorem. Moreover, since at least one of these cycles would need to have length at least $\lceil n/(k-1)\rceil$, the conjecture implies the result of Bollobás and Häggkvist mentioned above. The case $k = 3$ was already shown by Enomoto, Kaneko and Tuza [14]. For the case of 2-connected graphs G, the conjecture was shown by Kouider in [25]. An Ore-type condition for $k = 3$ was given in [13]. Finally, Kouider and Lonc solved the conjecture (even in the stronger Ore-version) in [26]. Thus:

Theorem 1 (Kouider and Lonc [26]). *Let $k \geqslant 2$ be an integer and let G be an n-vertex* *graph with minimum degree $\delta(G) \geqslant n/k$. Then the vertex set of G can be covered with $k-1$ cycles*[1].

This leaves open the problem of determining the structure of the extremal examples. The minimum degree condition in Theorem 1 is tight by a family of examples of graphs with n vertices and minimum degree $n/k-1$ that cannot be covered with $k-1$ cycles.

First, we can consider the disjoint union of k copies of $K_{n/k}$. This graph has n vertices and minimum degree $n/k-1$ and yet clearly cannot be covered with $k-1$ cycles, because every cycle must be confined to a single copy of $K_{n/k}$. On the other extreme, we can imagine a graph on n vertices with minimum degree $n/k-1$ that contains an independent set of size $(k-1)n/k+1$. A concrete example is given by the graph $I_{n,k} := K_n \setminus K_{(k-1)n/k+1}$, although there are also sparser examples that still have minimum degree $n/k-1$. Note that every cycle in such a graph can cover at most $n/k-1$ vertices of the independent set, so at least k cycles are needed to cover all vertices. Finally, the we can interpolate between these two types of examples as follows. For any sequence $k_1, \dots, k_r$ of positive integers such that $k_1 + \cdots = k_r = k$, we may consider the disjoint union $G = I_{k_1 n/k, k_1} + \cdots + I_{k_r n/k, k_r}$. Then G is an n-vertex graph with minimum degree $n/k-1$ that cannot be covered with $k-1$ cycles. Note here that $I_{n/k,1} = K_{n/k}$, so this construction includes the disjoint union of cliques.

It is natural to ask whether there are families of examples that are substantially different. The main result of this paper is that this is not the case: if the minimum degree is close to n/k then either the graph can be covered by $k-1$ cycles, or it is close to a subgraph of $I_{k_1 n/k, k_1} + \cdots + I_{k_r n/k, k_r}$ for a sequence $k_1, \dots, k_r$ of positive integers summing to k. To make this precise, we need the following definitions:

Definition 1 (separable partition)**.** A partition of the vertices of a graph G into sets $X_1, \dots, X_r$ is *separable* if for all $i \neq j$ there exists a single-vertex X_i-X_j-cut in G.

Definition 2 (**(k', k, β)**-stable)**.** Let G be a graph of order n. Given a positive integer k and real numbers $k' \in [1, k]$ and $\beta \in (0,1)$, we say that a subset $X \subseteq V(G)$ is *(k', k, β)-stable* if there exists a subset $I \subseteq X$ such that:

(S1) $|X| = k'n/k \pm \beta n$ and $|I| = (k'-1)n/k \pm \beta n$;

(S2) we have $\delta(G[X]) \geqslant n/k^4 - \beta n$ and all but at most βn vertices in X have degree at least $n/k - \beta n$ in $G[X]$;

(S3) $e(G[I]) \leqslant \beta n^2$.

With these definitions, our main result reads as follows:

Theorem 2. *Given an integer $k \geqslant 2$ and $\beta > 0$, there is $\alpha > 0$ such that the following holds for sufficiently large n. Assume that G is a graph with n vertices and minimum degree at least $(1-\alpha)n/k$ whose vertices cannot be covered by $k-1$ cycles. Then there is a separable partition $X_1, \dots, X_r$ of the vertices of G and positive integers $k_1, \dots, k_r$ such that*

[1]Edges and vertices count as cycles.

- *each X_i is (k_i, k, β)-stable in G, and*
- $k_1 + \cdots + k_r = k$.

Note that if X_i is (k_i, k, β)-stable, then $G[X_i]$ is 'β-close' to a subgraph of $I_{k_i n/k, k_i}$ with minimum degree n/k, with the set I in Definition 2 playing the role of the (nearly) independent set. Note also that if $X_1, \ldots, X_r$ is a separable partition, then G can only contain very few (say, at most n) edges going between different parts X_i and X_j. Thus every graph G as in the theorem can be turned into a subgraph of $I_{k_1 n/k, k_1} + \cdots + I_{k_r n/k, k_r}$ by changing at most $C\beta n^2$ edges, for a constant $C > 0$ independent of β.

Results of this type are usually referred to as *stability theorems*, the most famous example being the Erdős-Simonovits stability theorem for the Turán problem [33]. It would also be interesting to find the characterization of extremal families when $k = k(n) \to \infty$ as $n \to \infty$.

For the proof of Theorem 2, we use the method of connected matchings, invented by Łuczak [29], which is based on an application of Szemerédi's Regularity Lemma [34]. This method seems to be widely applicable; for more applications (especially in Ramsey theory) see [3, 4, 8, 17, 31, 27].

2 Preliminaries

The following lemma states several useful properties of stable sets. The proof of the lemma is not very interesting but quite technical, so we postpone it to Section 4 at the end of this paper.

Lemma 3 (Properties of stable sets)**.** *For every integer $k \geqslant 2$, there is some $\beta > 0$ such that the following holds for all sufficiently large n. Let G be a graph of order n and assume that X is an (k', k, β)-stable subset of $V(G)$, for some $k' \in [1, k]$. Then*

(a) the vertices of $G[X]$ can be covered with $\lceil k' \rceil$ cycles;

(b) given any two vertices $x, y \in X$, the vertices of $G[X]$ can be covered by $\lceil k' \rceil - 1$ cycles and a single path with endpoints x and y;

(c) if, additionally, all but at most n/k^3 vertices $x \in X$ satisfy $d(x, X) \geqslant |X|/k'$, then the vertices of $G[X]$ can be covered with $\lceil k' \rceil - 1$ cycles.

We will also have occasion to use Szemerédi's Regularity Lemma [34] and some related results. Given $\varepsilon \in (0, 1)$, we say that a pair (A, B) of disjoint sets of vertices of a graph G is *ε-regular* if, for all subsets $X \subseteq A$ and $Y \subseteq B$ such that $|X| \geqslant \varepsilon|A|$ and $|Y| \geqslant \varepsilon|B|$, we have

$$\left| d_G(X, Y) - d_G(A, B) \right| \leqslant \varepsilon.$$

A pair (A, B) is called *(ε, δ)-super-regular* if it is ε-regular and

$$d_G(a, B) \geqslant \delta|B| \text{ for all } a \in A \qquad \text{and} \qquad d_G(b, A) \geqslant \delta|A| \text{ for all } b \in B.$$

Theorem 4 (Degree form of the Regularity Lemma [24])**.** *For every $\varepsilon > 0$ and every positive integer t_0, there is an $M = M(\varepsilon, t_0)$ such that the following holds for every graph $G = (V, E)$ of order $n \geqslant M$ and every real number $d \in [0, 1]$. There exists an integer $t \in [t_0, M]$, a partition $(V_i)_{i=0}^t$ of the vertex set V into $t + 1$ sets (called* clusters*), and a subgraph $G' \subseteq G$ with the following properties:*

(R1) $|V_0| \leqslant \varepsilon|V|$,

(R2) all clusters V_i are of the same size $m \in ((1 - \varepsilon)n/t, n/t)$,

(R3) $d_{G'}(v) > d_G(v) - (d + \varepsilon)n$ for all $v \in V \setminus V_0$,

(R4) $e(G'[V_i]) = 0$ for all $1 \leqslant i \leqslant t$,

(R5) for all $1 \leqslant i < j \leqslant t$, the pair (V_i, V_j) is ε-regular, with a density either 0 or greater than d.

A partition as in Theorem 4 is usually called an *ε-regular partition* with *exceptional set* V_0. Given a partition $(V_i)_{i=0}^t$ of the vertex set V and a subgraph $G' \subseteq G$ satisfying conditions (R1)–(R5), we define the *(ε, δ)-reduced graph* as the graph R with vertex set $[t]$ and edges corresponding to those pairs ij for which (V_i, V_j) is ε-regular and with density at least δ.

We shall also use the following special case of the famous Blow-up Lemma of Komlós, Sárközy, and Szemerédi [21] (see also Lemma 24 and the first remark after Lemma 25 in [19]).

Lemma 5 (Blow-up Lemma)**.** *For every $\delta > 0$ there exists an $\varepsilon > 0$ such that the following holds. Assume that a graph G contains an (ε, δ)-super-regular pair (A, B) with $|A| = |B|$ and let $x \in A, y \in B$. Then $G[A, B]$ contains a Hamilton path with endpoints x and y.*

We will also need the Chernoff bounds for binomial and hypergeometric random variables. Recall that a random variable X is binomially distributed if it is a sum of a fixed number of i.i.d. $\{0, 1\}$-valued random variables, while it is hypergeometrically distributed with parameters N, K, n if it counts the number of successes in a subset of size n drawn uniformly at random from a population of N elements that contains K successes.

Lemma 6 (Chernoff bounds [11, Theorem 1.17])**.** *Assume that X is either binomially or hypergeometrically distributed. Then for all $\varepsilon \in (0, 1)$*

$$\Pr[|X - \mathbf{E}[X]| \geqslant \varepsilon \mathbf{E}[X]] \leqslant 2 \exp(-\varepsilon^2 \mathbf{E}[X]/3).$$

3 Proof of Theorem 2

Let $k \geqslant 2$ be a fixed integer. Without loss of generality, we may assume that the given $\beta > 0$ is sufficiently small. Let us choose constants $\varepsilon, d, \alpha \in (0, 1)$ and $t_0 \in \mathbb{N}$ such that

$$\frac{1}{t_0} \prec \alpha \prec \varepsilon \prec d \prec \beta,$$

where by $a \prec b$ we mean that a is chosen to be sufficiently smaller than b.

Let G be a graph of order n with $\delta(G) \geqslant (1-\alpha)n/k$, where n is sufficiently large. Let $\hat{\varepsilon} := \varepsilon/(4k)$ and $\hat{d} := d + (k+1)\hat{\varepsilon}$. We apply the Regularity Lemma (Theorem 4) to G with parameters $\hat{\varepsilon}$, t_0, and $\hat{d}$ to obtain a partition $(V_i)_{i=0}^t$ and a subgraph $G' \subseteq G$ satisfying (R1–5), for some integer $t_0 \leqslant t \leqslant M$ and with $\hat{\varepsilon}, \hat{d}$ instead of ε, δ. We denote by R the $(\hat{\varepsilon}, \hat{d})$-reduced graph corresponding to this partition.

Structure of the reduced graph.

The reduced graph R is has t vertices and it satisfies

$$\delta(R) \geqslant \frac{(1-2dk)t}{k} > \frac{t}{k+1}. \tag{1}$$

To see this, simply observe that the vertices of every cluster in R with degree less than $(1-2dk)t/k$ would have degree at most $(1-2dk)(t/k)\cdot(n/t) + \hat{\varepsilon}n < (1-\alpha)n/k - (\hat{d}+\hat{\varepsilon})n$ in G', contradicting property (R3) of Theorem 4.

Let us denote by r the number of components of R and by $R_1, \dots, R_r$ the components themselves. Since each component has size at least $\delta(R) > t/(k+1)$, there can be at most k components altogether, i.e., $r \leqslant k$.

For each component R_i, define a real number

$$s_i := \frac{kv(R_i)}{(1-2dk)t} \in (1, k+1), \tag{2}$$

where the given bounds follow from $\delta(R_i) < v(R_i) \leqslant t$ and (1). Note that by combining (1) and (2) we have

$$\delta(R_i) \geqslant v(R_i)/s_i. \tag{3}$$

Finally, since $v(R_1) + \ldots + v(R_r) = t$, we have

$$\sum_{i=1}^{r} s_i = \frac{k}{1-2dk} \leqslant (1+3dk)k, \tag{4}$$

a fact that will be important later.

The components where $s_i < 2 + 4dk^2$ have such a large minimum degree that we can treat them by a special argument. For the others, we have the following structural lemma, whose proof we postpone to a later point.

Lemma 7. *Let $i \in [r]$ and assume that $s_i \geqslant 2 + 4dk^2$. Let $m_i = \lfloor s_i - 4dk^2 \rfloor$ and $t_i = v(R_i)$. Then at least one of the following is the case:*

(i) the graph R_i contains a subset $I \subseteq V(R_i)$ of size $(s_i - 1)t_i/s_i - 6dk^2 s_i t_i$ that is almost independent in the sense that $e(I) \leqslant 4dk^2 s_i t_i^2$, or

(ii) the graph R_i contains matchings $M_1, \dots, M_{m_i}$ and disjoint subsets of vertices D_1, D_2 with the following properties:

(a) *$D_1 \cap V(M_1) = \emptyset$, and for $j > 1$, $D_2 \cap V(M_j) = \emptyset$;*

(b) *each vertex of R_i has at least $dt_i/(3s_i)$ neighbours in each set D_1, D_2;*

(c) *the matchings $M_1, \ldots, M_{m_i}$ cover the vertex set of R_i.*

We apply Lemma 7 to each component R_i where $s_i \geqslant 2 + 4dk^2$. From all such components that are in case (ii) of the lemma, we obtain a collection $\mathcal{M}$ of matchings in R. Since each component R_i contributes at most m_i matchings, and using (4), we see that $\mathcal{M}$ contains at most k matchings.

The significance of the matchings and the sets D_1 and D_2 will become clear at a later point. Essentially, we will see that for each matching $M \in \mathcal{M}$, we can cover the vertices in the subgraph of G induced by the clusters participating in the matching using a single cycle (covering also a certain number of exceptional vertices in V_0 that are 'assigned' to the matching M). This will be an application of the Blow-up Lemma. Here the sets D_1 and D_2 are used on the one hand to balance the sizes of the clusters in M and also to absorb the exceptional vertices assigned to the matching. However, to be able to apply the Blow-up Lemma, we need to modify the regular partition a little, which we will do next.

Modifying the regular partition.

We need to modify the initial regular partition $V_0, \ldots, V_t$ in such a way that each edge in each matching in $\mathcal{M}$ corresponds not just to an $\hat{\varepsilon}$-regular pair of density at least $\hat{d}$, but in fact to an (ε, d)-*super-regular* pair (this is the reason why we applied the Regularity Lemma with the slightly stronger parameters $\hat{\varepsilon}, \hat{d}$ instead of ε, d). For this, we proceed as follows. For each edge (V_i, V_j) of a given matching $M \in \mathcal{M}$, we observe that by regularity, at most $\hat{\varepsilon}|V_i|$ vertices of V_i have fewer then $\hat{d}|V_j| - \hat{\varepsilon}|V_j| = d|V_j| + k\hat{\varepsilon}|V_j|$ neighbours in V_j and vice-versa. We move all vertices in these sets to the exceptional set. Since $\mathcal{M}$ contains at most k matchings, we remove at most $k\hat{\varepsilon}|V_i|$ vertices from each cluster V_i. By removing some additional vertices, we can make sure that after this, all clusters are still of the same size. Then one can check that the properties (R1–5) of Theorem 4 hold for the new partition with ε, d instead of $\hat{\varepsilon}, \hat{d}$ (and with the same G' and t). Moreover, we have gained the property that the edges of the matchings in $\mathcal{M}$ correspond to (ε, d)-super-regular pairs in G.

Partitioning into stable sets.

For each $i \in [r]$, let us define G_i as the subgraph of G induced by the vertices in clusters in R_i. Note that

$$\bigcup_{i=1}^{r} V(G_i) = V(G) \setminus V_0.$$

Using (R2), we have

$$\frac{(1-\varepsilon)nv(R_i)}{t} \leqslant v(G_i) \leqslant \frac{nv(R_i)}{t},$$

so by plugging in the definition of s_i, we get

$$\frac{(1-\varepsilon)(1-2dk)ns_i}{k} \leqslant v(G_i) \leqslant \frac{(1-2dk)ns_i}{k}. \tag{5}$$

Using the inequalities

$$(1+dk)(1-2dk) = 1 - dk - 2d^2k^2 \leqslant 1 - \alpha - (d+\varepsilon)k$$

and $\delta(G) \geqslant (1-\alpha)n/k$ and properties (R3)–(R5), we obtain that

$$\delta(G_i) \geqslant \frac{(1-\alpha)n}{k} - (d+\varepsilon)n \geqslant (1+dk)\frac{v(G_i)}{s_i}. \tag{6}$$

Since $1 \leqslant r \leqslant k$ and since $|V(G_1) \cup \cdots \cup V(G_r)| \geqslant n - \varepsilon n$, it is clear that for every vertex $v \in V(G)$, there exists at least one $i \in [r]$ such that the degree of v into $V(G_i)$ is at least $(\delta(G) - \varepsilon n)/k$. Thus, we can partition the exceptional set V_0 into sets $U_1, \ldots, U_r$, where U_i contains only vertices with at least

$$(1-\alpha-k\varepsilon)n/k^2 \geqslant n/k^3$$

neighbours in $V(G_i)$. The sets $Y_i := V(G_i) \cup U_i$ form a partition of the vertex set of G. This is *almost* the partition $X_1, \ldots, X_r$ that Theorem 2 asks for; as we will see, the sets Y_i for which $s_i \in [2, 2+4dk^2)$ might have to be partitioned further.

To complete the proof of Theorem 2, we need to distinguish three cases. Each case is handled by one of the following lemmas.

Lemma 8. *If $1 \leqslant s_i < 2$, then $G[Y_i]$ is Hamiltonian. Moreover, if $s_i < 1 + 4dk^2$, then Y_i is actually $(1, k, \beta)$-stable in G.*

Lemma 9. *If $2 \leqslant s_i < 2 + 4dk^2$, then at least one of the following holds:*

(i) $G[Y_i]$ is Hamiltonian,

(ii) Y_i is $(2, k, \beta)$-stable in G, or

(iii) there is a partition of Y_i into two $(1, k, \beta)$-stable sets in G.

Lemma 10. *If $2 + 4dk^2 \leqslant s_i$, then either Y_i can be covered with $\lfloor s_i - 4dk^2 \rfloor$ cycles in $G[Y_i]$, or Y_i is $(\lfloor s_i \rfloor, k, \beta)$-stable in G.*

The proofs of the first two lemmas are elementary. However, for the last lemma, we will need to use the structure given by Lemma 7 and the Blow-up Lemma (Lemma 5). Given these three lemmas, we can complete the proof of Theorem 2.

Proof of Theorem 2. By combining Lemmas 3, 8, 9 and 10, we see that each graph $G[Y_i]$ can be covered with at most $\lfloor s_i \rfloor$ cycles. By (4), we have $\lfloor s_1 \rfloor + \cdots + \lfloor s_r \rfloor \leqslant k$. Since we assume that G cannot be covered with $k-1$ cycles, this inequality is really an equality, i.e., $\lfloor s_1 \rfloor + \cdots + \lfloor s_r \rfloor = k$.

This implies that for every $i \in [r]$, we have $\lfloor s_i \rfloor \leqslant s_i < \lfloor s_i \rfloor + 4dk^2$, as otherwise (4) yields the contradiction

$$k = \sum_{i=1}^{r} \lfloor s_i \rfloor = \sum_{i=1}^{r} s_i - \sum_{i=1}^{r} (s_i - \lfloor s_i \rfloor) \leqslant (1 + 3dk)k - 4dk^2 < k.$$

But then, using again that we cannot cover the vertices of G with $k-1$ cycles, Lemmas 8, 9 and 10 tell us that the situation is as follows:

- if $1 \leqslant s_i < 2$, then Y_i is $(1, k, \beta)$-stable;
- if $2 \leqslant s_i < 2 + 4dk^2$, then Y_i is either $(2, k, \beta)$-stable or the union of two disjoint $(1, k, \beta)$-stable sets;
- if $s_i \geqslant 2 + 4dk^2$, then Y_i is $(\lfloor s_i \rfloor, k, \beta)$-stable.

By splitting some of the sets Y_i into two stable sets (if they are in the second case and not stable already), we obtain a partition of the vertices into $r' \geqslant r$ sets $X_1, \ldots, X_{r'}$ such that each X_i is (k_i, k, β)-stable for some integer k_i, where moreover $k_1 + \cdots + k_{r'} = k$.

To complete the proof, we show that $X_1, \ldots, X_{r'}$ is a separable partition. For this, let $i \neq j$ and assume for a contradiction that there is no single-vertex X_i-X_j-cut in G. Then by Menger's theorem there are two vertex-disjoint X_i-X_j-paths in G, and so using Lemma 3 (b) it is possible to cover $X_i \cup X_j$ by a $k_i + k_j - 1$ cycles in G. Moreover, by Lemma 3 (a) it is possible to cover all other sets X_ℓ by k_ℓ cycles. Hence, there is a cover of the vertices of G by $k_1 + \cdots + k_{r'} - 1 = k - 1$ cycles, which we assumed is not the case. □

It remains to give the proofs of Lemmas 7, 8, 9, and 10.

Proof of Lemma 7. Since by (1), we have $v(R_i) \geqslant t/(k+1) \geqslant t_0/(k+1)$, we can assume that $t_i = v(R_i)$ is very large compared to $1/d$. The only other property of R_i that we will need is that $\delta(R_i) \geqslant v(R_i)/s_i$, by (3).

First, we show that it is possible to choose disjoint subsets $D_1, D_2 \subseteq V(R_i)$, each of size at most $2dt_i$, in such a way that every vertex in $V(R_i)$ has at least $dt_i/(3s_i)$ neighbours in D_j, for $j \in \{1, 2\}$. For this, let D be a random subset of $V(R_i)$ in which every cluster is included independently with probability d. Then let $D_1 \cup D_2 = D$ be a partition of D into two sets chosen uniformly at random. The expected size of D_1 and D_2 is $dt_i/2$. Thus, by Markov's inequality, with probability at least $1/2$, we have $|D_1|, |D_2| \leqslant 2dt_i$. Fix some vertex $v \in V(R_i)$. The expected number of neighbours of v that are in D_1 is at least $d\delta(R_i)/2 \geqslant dt_i/(2s_i)$. Using the Chernoff bounds, the probability that the neighborhood of v does not contain at least $dt_i/(3s_i)$ elements of D_1 is smaller than $1/(4t_i)$, provided that t_i is large enough. Similarly, the probability that the neighborhood of v does not contain at least $\beta t_i/(3s_i)$ elements of D_2 is smaller than $1/(4t_i)$. The union bound shows that there exists a good choice for D_1 and D_2. From now on, fix such a choice.

Let $m_i := \lfloor s_i - 4dk^2 \rfloor$ and observe that by assumption, we have $m_i \geqslant 2$. We want to cover the set $V(R_i)$ by m_i matchings $M_1, \ldots, M_{m_i}$, so that M_1 is disjoint from D_1 and $M_2, \ldots, M_{m_i}$ are disjoint from D_2. To do this, we first let M_1 be a maximal matching in

R_i that covers D_2 and is disjoint from D_1. Note that there certainly exists such a matching because of the minimum degree condition (3) and because D_1 and D_2 are very small. Now, to choose the matchings M_j for $j \geqslant 2$, we partition the set $V(R_i) \setminus V(M_1)$ equitably into sets $A_2, \ldots, A_{m_i}$. Then we let M_j be a matching that is disjoint from D_2 and that covers the maximum number of vertices of A_j (among all matchings that are disjoint from D_2); moreover, we assume that M_j has maximum size among all such matchings. There are now two cases.

Non-extremal case.

If $|M_1| \geqslant t_i/s_i + 2dk^2 s_i t_i$, then we claim that we are in case (ii) of the lemma. The only thing to check is whether the matchings cover R_i. The set $V(R_i) \setminus V(M_1)$ has size

$$\begin{aligned} t_i - 2|M_1| \leqslant t_i - \frac{2t_i}{s_i} - 4dk^2 s_i t_i &= \frac{t_i(s_i - 2 - 4dk^2 s_i^2)}{s_i} \\ &\leqslant \frac{t_i(s_i - 4dk^2 - 2)(1 - 2ds_i)}{s_i} \leqslant \frac{t_i(m_i - 1)(1 - 2ds_i)}{s_i}, \end{aligned}$$

so for each $2 \leqslant j \leqslant m_i$, we have

$$|A_j| \leqslant \left\lceil \frac{t_i - 2|M_1|}{m_i - 1} \right\rceil \leqslant \lceil t_i/s_i - 2dt_i \rceil \leqslant \delta(R_i) - |D_2|.$$

Thus, there exists a matching disjoint from D_2 that covers A_j completely, and since M_j was chosen to cover the most vertices of A_j among all matchings disjoint from D_2, the matchings cover every vertex of R_i.

Extremal case.

If $|M_1| < t_i/s_i + 2dk^2 s_i t_i$, then we will see that the graph must have a special structure.

We will first show that $|M_1| \geqslant t_i/s_i - 2dt_i$. Write U for the set $V(R_i) \setminus (D_1 \cup V(M_1))$ of *uncovered* vertices that are not in D_1. Note that U is an independent set in R_i (or the matching M_1 would not be maximal). If $|U| \leqslant 1$, then, since $s_i \geqslant 2+4dk^2 \geqslant 2/(1-2d-1/t_i)$ and $|D_1| \leqslant 2dt_i$, we have

$$2|M_1| \geqslant t_i - |D_1| - 1 \geqslant t_i - 2dt_i - 1 \geqslant 2t_i/s_i,$$

and we are done. Otherwise, there are at least two vertices $u, v \in U$. Since M_1 is maximal, we know that every neighbor of u is either in D_1 or is covered by an edge of M_1, and similarly for v. Moreover, there are *no* edges of M_1 between a neighbor of u and a neighbor of v. Therefore

$$2t_i/s_i \leqslant 2\delta(R_i) \leqslant d(u) + d(v) \leqslant 2|D_1| + 2|M_1|,$$

which implies that

$$|M_1| \geqslant t_i/s_i - |D_1| \geqslant t_i/s_i - 2dt_i. \tag{7}$$

Now, since $|M_1| < t_i/s_i + 2dk^2 s_i t_i$, we have

$$|U| = t_i - |D_1| - 2|M_1| \geqslant (s_i - 2)t_i/s_i - 5dk^2 s_i t_i. \tag{8}$$

To complete the proof of the lemma, we will show that there exists a set of size $|U| + |M_1|$ which contains very few edges. For this, observe that by the maximality of M_1, for every edge $xy \in M_1$ at least one of the vertices x, y has at most one neighbor in U. Thus, we may split $V(M_1)$ into two disjoint sets A and B of size $|M_1|$ by placing, for each edge of M_1, an endpoint with at most one neighbor in U into A, and the other endpoint into B. Then we have $e(U, A) \leqslant |A|$; the 'nearly independent set' that we are looking for will be $U \cup A$.

To show that $U \cup A$ contains few edges, we will first show that most vertices in B have at least two neighbours in U. Indeed, let $X := \{v \in B \mid d(v, U) < 2\}$. Since U is an independent set and since $V(R_i) = A \cup B \cup U \cup D_2$, we have

$$\begin{aligned}|X| + |U|(|B| - |X|) &\geqslant e(B, U) \geqslant |U|\delta(R_i) - e(U, V(R_i) \setminus B)\\ &\geqslant |U|\delta(R_i) - e(U, A) - e(U, D_2)) \geqslant |U|\delta(R_i) - |B| - |U||D_2|.\end{aligned}$$

Rearranging this inequality, and using that $|B| - \delta(R_i) \leqslant 2dk^2 s_i t_i$ and $|D_2| \leqslant 2dt_i$, as well as the fact that $|U| = \Omega(t_i)$ is sufficiently large, we get

$$|X| \leqslant \frac{|B| + |U||B| - |U|\delta(R_i) + |U||D_2|}{|U| - 1} \leqslant 3dk^2 s_i t_i.$$

Let us now estimate the number of edges inside of $U \cup A$. We know that $e(U) = 0$ and $e(U, A) \leqslant |A|$. To bound $e(A)$, consider some edge $xy \in E(A)$ and denote by x' and y' the vertices matched to x and y in M_1, respectively. Then, by the maximality of M_1, we can see that at least one of x' and y' has at most one neighbor in U. It follows that $e(A) \leqslant |A||X|$. Thus, using $e(U, A) \leqslant |A|$, $|X| \leqslant 3dk^2 s_i t_i$, $|A| \leqslant t_i$ and the fact that U is an independent set, we get

$$e(U \cup A) \leqslant e(A) + e(U, A) \leqslant |A||X| + |A| \leqslant 4dk^2 s_i t_i^2$$

and, using (7) and (8),

$$|U \cup A| = |U| + |M_1| \geqslant (s_i - 1)t_i/s_i - 6dk^2 s_i t_i.$$

So we are in case (i) of the lemma. □

Proof of Lemma 8. We start with the first part. Recall that $Y_i = V(G_i) \cup U_i$, where U_i is a set of at most εn vertices that have degree at least n/k^3 into $V(G_i)$. Also recall that by (6), we have $\delta(G_i) \geqslant (1 + dk)v(G_i)/s_i \geqslant (1 + dk)v(G_i)/2$. In particular, any two vertices $u, v \in V(G_i)$ are connected by at least $dkv(G_i) \geqslant 3|U_i|$ disjoint paths of length two. Then we can greedily construct a path P of length $4|U_i| - 2$ in $G[Y_i]$ such that P starts and ends in vertices of G_i and contains all vertices of U_i. More precisely, for each vertex $u \in U_i$, we can find two neighbours in $V(G_i)$ such that all neighbours are distinct; then we can connect these into a path by using the fact that any two neighbours have more than $3|U_i|$ common neighbours in G_i.

The graph $G_i - V(P)$ still satisfies Dirac's condition. Let C be a Hamilton cycle in $G_i - V(P)$ and let $u, v \in G_i$ be the endpoints of P. Then, by the minimum degree condition, there are vertices u', v' that are adjacent on C and such that $uu', vv' \in E(G_i)$. By opening the cycle C on the edge $u'v'$ and connecting u' to u and v' to v, we obtain a Hamilton cycle in $G[Y_i]$.

To see the second statement of the lemma, just let $X = Y_i$ and $I = \emptyset$. Since d is very small compared to β, the conditions of Definition 2 are easily verified. Specifically, (S1) follows from (5), (S2) follows from (6) and the definition of U_i, and (S3) is trivially true. $\square$

Proof of Lemma 9. Assume that $2 \leqslant s_i < 2 + 4dk^2$. By (6) we have

$$\delta(G_i) \geqslant (1 + dk)v(G_i)/s_i \geqslant (1 - 3dk^2)v(G_i)/2.$$

Moreover, recall that $Y_i = V(G_i) \cup U_i$, where U_i is a set of at most εn vertices that each have at least n/k^3 neighbours in $V(G_i)$.

We will show that at least one of the following holds:

(i) $G[Y_i]$ is Hamiltonian,

(ii) $G[Y_i]$ contains an independent set of size at least $(1 - 10dk^2)|Y_i|/2$, or

(iii) Y_i contains two disjoint sets A, B of size at least $(1-5dk^2)|Y_i|/2$ such that $e(A, B) = 0$.

It is straightforward to verify that if we are in case (ii), then Y_i is $(2, k, \beta)$-stable (let I be the independent set of size $(1 - 10dk^2)|Y_i|/2$ and let $X := Y_i \setminus I$). Similarly, if we are in case (iii), then one easily checks that $Y_i = A \cup B$ is a partition into two $(1, k, \beta)$-stable sets.

Thus, from now on, we shall assume that neither (ii) nor (iii) holds. Then for any two vertices $u, v \in V(G_i)$ and every subset $A \subseteq V(G_i)$ of size at least $v(G_i) - dn$, the graph $G[A \cup \{u, v\}]$ contains a path of length at most three that goes from u to v. To see this, observe that both u and v have at least

$$(1 - 3dk^2)v(G_i)/2 - dn - 1 \geqslant (1 - 5dk^2)v(G_i)/2$$

neigbors in A. If they have a common neighbor in A, or if there is an edge from a neighbor of u in A to a neighbor of v in A, then we are done. Otherwise, the neighborhoods of u and v are disjoint subsets of size at least $(1 - 5dk^2)v(G_i)/2$ with no edges between them, and we are in case (iii).

From this observation, it is now easy to see that $G[Y_i]$ must contain a path P of length $5|U_i| - 3$ that contains all vertices of U_i and whose endpoints are in Y_i. The construction is the same as in the proof of Lemma 8: for each vertex of U_i we find two neighbours in $V(G_i)$ such that all neighbours are distinct, and then we connect these neighbours using the observation to build the path P. Let us write a_P and b_P for the endpoints of P.

Let G_i' be the subgraph of G_i induced by $\{a_P, b_P\} \cup V(G_i - P)$. Note that $v(G_i') \geqslant v(G_i) - 5\varepsilon n$, and that, consequently, G_i' has minimum degree at least $(1 - 4dk^2)|Y_i|/2$. We may also assume that $G_i' - \{a_P, b_P\}$ is at least two-connected, since otherwise, by the minimum degree of G_i', the graph would contain two sets X, Y of size at least

$$\delta(G_i') - 2 \geqslant (1 - 4dk^2)|Y_i|/2 - 2$$

that intersect only in an articulation point. But then, we would be in case (iii), contradicting our assumption.

We will show that G_i' contains a Hamilton path joining a_P to b_P. Clearly, this path will combine with P to yield a Hamilton cycle in $G[Y_i]$. Our strategy is the following. First, we will prove that $G_i' - \{a_P, b_P\}$ must contain a nearly spanning cycle. Then, we will connect a_P and b_P with this cycle to form a nearly spanning path from a_P to b_P in G_i'. Finally, we will absorb the few remaining vertices of G_i' into the path to get a Hamilton path.

To obtain the first part, we use the well-known fact (also due to Dirac [10]) that every two-connected graph with minimum degree δ contains a cycle of length at least 2δ. In our case, this means that $G_i' - \{a_P, b_P\}$ contains a cycle C of length

$$|C| \geqslant 2\delta(G_i' - \{a_P, b_P\}) \geqslant (1 - 5dk^2)|Y_i|.$$

For the second step, as both a_P and b_P have degree larger than $|C|/3$ into C, there must be a neighbor of a_P on C that is within distance at most two to a neighbor of b_P on C, the distance being measured along the cycle C (and making sure that the neighbours are distinct). Therefore, if we are generous, there is a path P' in G_i' with endpoints a_P and b_P that has length at least $(1 - 6dk^2)|Y_i|$.

To complete the proof, we show how to handle the at most $6dk^2|Y_i|$ vertices of G_i' that do not belong to P'. Consider any such vertex $v \in V(G_i')$ and let X be the set of all neighbours of v on P' that are not within distance less than two of either a_P or b_P (again, the distance being measured on P'). There must be at least $(1 - 10dk^2)|Y_i|/2$ such vertices. If any two neighbours u and w of v are neighbours on P', then we can absorb v to P' by following P' from a_P to u, using uv and vw, and following P' from w to b_P. So, assume this is not the case.

Orient P' from a_P to b_P, and let Y be the set of the immediate successors of vertices in X on the path. Since this is a set of size at least $(1 - 10dk^2)|Y_i|/2$, it must contain at least one edge uw, or else we would be in case (ii). However, using this edge, one can rotate the path P' to obtain a path going from a_P to b_P that contains all vertices of P', as well as the additional vertex v. Indeed: let $u' \in X$ be the predecessor of u and $w' \in X$ be the predecessor of w on P'. We absorb v to P' by following P' from a_P to u', using $u'v$ and vw', following P' from w' to u, using uw, and following P' from w to b_P.

In this way, it is possible to absorb all left-over vertices until the path spans the whole of G_i'. □

Proof of Lemma 10. If $s_i \geqslant 2 + 4dk^2$, then by Lemma 7, we know that the corresponding component R_i of the reduced graph has a certain structure: either

(i) there is a subset $I \subseteq V(R_i)$ of size $(s_i - 1)/s_i - 6dk^2 s_i t_i$ with the property that $e(I) \leqslant 4dk^2 s_i t_i^2$, or

(ii) R_i contains matchings $M_1, \ldots, M_{m_i}$, where $m_i = \lfloor s_i - 4dk^2 \rfloor$, and subsets $D_1, D_2 \subseteq V(R_i)$ such that

  (a) $D_1 \cap V(M_1) = \emptyset$, and for $j > 1$, $D_2 \cap V(M_j) = \emptyset$;

  (b) each vertex of R_i has at least $dt_i/(3s_i)$ neighbours in each set D_1, D_2;

  (c) the matchings $M_1, \ldots, M_{m_i}$ cover the vertex set of R_i.

Recall that U_i is a set of at most εn vertices that have degree at least n/k^3 into $V(G_i)$. If (i) is the case, then it follows easily from (6) and the properties of regularity that $Y_i = V(G_i) \cup U_i$ is a (s_i, k, β)-stable set in G (note that ε, d are tiny compared to β). Since by (6), all but $|U_i| \leqslant \varepsilon n$ vertices of Y_i have degree at least $|Y_i|/s_i$ in $G[Y_i]$, it follows from Lemma 3 (c) that $G[Y_i]$ can be covered with $\lceil s_i \rceil - 1$ cycles. Therefore, either we can cover $G[Y_i]$ with $\lfloor s_i - 4dk^2 \rfloor$ cycles, or $\lfloor s_i - 4dk^2 \rfloor < \lceil s_i \rceil - 1 \leqslant s_i$ and so $s_i \leqslant \lfloor s_i \rfloor + 4dk^2$. In the former case, we are done, and in the latter case, it is again easy to verify that Y_i must actually be $(\lfloor s_i \rfloor, k, \beta)$-stable (again, the extra $4dk^2$ gets lost in the much larger β). Thus if we are in case (i), then the lemma holds.

From now on, assume that we are in case (ii). Recall that each edge of each matching M_j is (ε, d)-super-regular. We may assume that ε is so small that Lemma 5 applies with $\delta = d$. The general idea is to use Lemma 5 to cover the preimage of each matching (meaning: the vertices of G participating in clusters of the matching) by a single cycle in $G[Y_i]$, and to do this in such a way that all the vertices in U_i are absorbed. Of course, if we manage to do so, then we are done. We start by assigning the exceptional vertices of U_i to clusters C into which they have large degree.

Assigning the exceptional vertices.

For each matching M_j, let us write $V_{M_j} \subseteq V(G_i)$ for the union of all clusters in M_j. As the matchings M_j cover the vertices of R_i, we have $\bigcup_{j=1}^{m_i} V(M_j) = V(G_i)$. Since each vertex of U_i has degree at least n/k^3 into $V(G_i)$, and since $m_i \leqslant k$, we see that for every $u \in U_i$ there exists a $j_u \in [m]$ such that u has n/k^4 neighbours in $V_{M_{j_u}}$. Let us write $U_i^{(j)} := \{u \in U_i \mid j_u = j\}$ for the exceptional vertices assigned to the matching M_j in this way.

Since $|V(M_j)| \leqslant |V(R)| \leqslant t$ and since each cluster has size at most $2n/t$, it follows that for each vertex $u \in U_i^{(j)}$, there are at least $t/(4k^4)$ clusters $C \in V(M_j)$ such that $d(u, C) \geqslant n/(2k^4 t)$. Indeed, if this were not true, then the degree of u into $V(M_j)$ would be strictly below

$$t \cdot \frac{n}{2k^4 t} + \frac{t}{4k^4} \cdot \frac{2n}{t} = \frac{n}{k^4},$$

a contradiction with the definition of $U_i^{(j)}$.

We now assign the vertices of $U_i^{(j)}$ to clusters in $V(M_j)$ in such a way that

(i) if u is assigned to the cluster C, then $d(u,C) \geqslant n/(2k^4t)$, and

(ii) at most $4k^4\varepsilon n/t$ vertices are assigned to each cluster.

Since $|U_i^{(j)}| \leqslant \varepsilon n$ and since each vertex has $t/(4k^4)$ candidates, such an assignment exists. Take any such assignment and write U_C for the exceptional vertices assigned to the cluster $C \in V(M_j)$.

Covering the matchings.

From the above, it is clear that the sets

$$V_{M_1} \cup U_i^{(1)}, V_{M_2} \cup U_i^{(2)}, \ldots, V_{M_{m_i}} \cup U_i^{(m_i)}$$

cover the set Y_i. In the following, we will cover each set $V_{M_j} \cup U_i^{(j)}$ by a single cycle in H_i (however, this cycle might use vertices outside of $V_{M_j} \cup U_i^{(j)}$).

Fix some $j \in [m_i]$, and assume $\ell \in \{1,2\}$ is such that D_ℓ is disjoint from the matching M_j. The embedding proceeds in two steps: first, for each cluster $C \in V(M_j)$, we connect the vertices of U_C by a short path using only vertices from D_ℓ, C, and U_C (and making sure that the paths for different C are vertex-disjoint); second, we use Lemma 5 to connect these short paths into a cycle spanning the whole of $V_{M_j} \cup U_i^{(j)}$.

In the first step, it is important to make sure that each path uses exactly the right number of vertices in the cluster C, as otherwise the second step might fail. Because we do not want to make this completely precise at this point, we assign to each cluster C an integer

$$\ell_C \in [8k^4\varepsilon n/t, 100k^4\varepsilon n/t],$$

and we will make sure that after creating the short path for C, the number of vertices of C not used by the path is exactly $|C| - \ell_C$. The bounds of ℓ_C allow us enough control over the number of remaining vertices per cluster, without hurting the super-regularity of the pairs corresponding to edges of M_j in a significant way.

Step 1: creating the small paths.

First, we assign each $C \in V(M_j)$ to a neighbor D_C of C in D_ℓ in such a way that we assign at most $3s_i/d$ clusters of $V(M_j)$ to each cluster in D_ℓ. This is possible because each vertex of R_i has at least $dt_i/(3s_i) \geqslant dt_i/(3s_i)$ neighbours in D_ℓ and because there are at most t_i clusters in $V(M_j)$.

During the construction of the paths, for every $D \in D_\ell$ and $C \in V(M_j)$, we maintain sets $A(D) \subseteq D$ and $A(C) \subseteq C$ of *available* clusters; initially $A(D) = D$ and $A(C) = C$ for all D and C, i.e., all clusters are available. The sets $A(D)$ and $A(C)$ will shrink during the construction of the paths; however, it will be true throughout that for each $C \in V(M_j)$ and $D \in D_\ell$, we have $|A(C)| \geqslant |C| - K\varepsilon|C|/d$ and $|A(D)| \geqslant |D| - K\varepsilon|D|/d$, where $K = K(k)$ is a sufficiently large constant depending only on k, but not on ε or d. Since ε is very small compared to d, this means that almost all clusters are available throughout the process.

For each cluster $C \in V(M_j)$, we shall first build a path P'_C covering the vertices of U_C. The path will have the form

$$P'_C = x_1u_1y_1\ z_2\ x_2u_2y_2\ z_3\ x_3u_3y_3\ \cdots\ z_{|U_C|}\ x_{|U_C|}u_{|U_C|}y_{|U_C|},$$

where $x_p, y_p \in C$, $u_p \in U_C$ and $z_p \in D_C$. After doing this, we will extend this path to a path P_C that uses exactly ℓ_C vertices of C, completing the first step in the outline given above.

We now describe how to construct P'_C. Recall that every vertex $u \in U_C$ has $n/(2k^4t) \geqslant 2K\varepsilon|C|/d$ neighbours in C. Order the vertices of U_C arbitrarily. For the first vertex $u_1 \in U_C$, let x_1 be an arbitrary neighbor of u_1 in $A(C)$, and let y_1 be a vertex in $A(C) \setminus \{x_1\}$ that has at least $dn/(3t)$ neighbours in $A(D_C)$. Assuming that $|A(C)| \geqslant |C| - K\varepsilon|C|/d$ and $|A(D_C)| \geqslant |D_C| - K\varepsilon|D_C|/d$, such neighbours exist by the fact that the pair (C, D_C) is ε-regular with density at least d. Remove x_1, y_1 from $A(C)$.

At every subsequent step, consider the current $u_p \in U_C$. Provided that $|A(C)| \geqslant |C| - K\varepsilon|C|/d$, there is a neighbor x_p of u_p in $A(C)$ that has a neighbor z_p in the neighborhood of y_{p-1} in $A(D_C)$, which we may assume (by induction) to be of size at least $dn/(3t) \geqslant \varepsilon|D_C|$. Similarly, there is a neighbor $y_p \in A(C)$ that has at least $dn/(3t)$ neighbours in $A(D_C) \setminus \{z_p\}$, again provided that $A(C)$ and $A(D_C)$ are large. Remove z_p from $A(D_C)$ and remove x_p, y_p from $A(C)$.

We can continue in this way as long as $A(C)$ and $A(D_C)$ are sufficiently large. For every vertex in U_C we remove at most one vertex from $A(D_C)$ and two from $A(C)$. Since (for large enough K) we have $|U_C| \leqslant 4k^4\varepsilon n/t \leqslant K\varepsilon|C|/(6s_i)$, and as only at most $3s_i/d$ clusters have chosen D_C, it follows that both $A(C)$ and $A(D)$ lose at most $K\varepsilon|C|/d$ vertices throughout this process. In other words, the process can be carried out until all vertices of U_C are covered.

Note that the path P'_C uses exactly $2|U_C| \leqslant 8k^4\varepsilon n/t$ vertices from C. However, we would like to have a path that uses exactly $\ell_C \in [8k^4\varepsilon n/t, 100k^4\varepsilon n/t]$ vertices of C. For this reason, we will extend the path in the following way.

By construction, $y_{|U_C|}$ has $dn/(3t) \geqslant \varepsilon|D_C|$ neighbours in $A(D_C)$. The typical vertex in $A(C)$ has a neighbor in this neighborhood, as well as $dn/(3t)$ additional neighbours in $A(D_C)$. Thus we may take such a vertex $x_{|U_C|+1}$ and a common neighbor $z_{|U_C|+1} \in A(D_C)$ of $x_{|U_C|+1}$ and $y_{|U_C|}$, and create a longer path $P'_C\ z_{|U_C|+1}\ x_{|U_C+1|}$. Then, we remove $z_{|U_C|+1}$ from $A(D_C)$ and $x_{|U_C|+1}$ from $A(C)$. As before, this process will not fail while $|A(C)| \geqslant |C| - K\varepsilon|C|/d$ and $|A(D_C)| \geqslant |D_C| - K\varepsilon|D_C|/d$ hold for all $C \in V(M_j)$. If K is large enough, then this means that we can continue for at least $100(k+1)k^3\varepsilon n/t$ steps, and we do so until the path contains exactly ℓ_C vertices of C.

Call the resulting path P_C. Observe that for different $C, C' \in V(M_j)$, the paths P_C and $P_{C'}$ are vertex-disjoint. Moreover, each path P_C has its endpoints in C, uses ℓ_C vertices of C (and no vertices of other clusters in $V(M_j)$), and visits all vertices in U_C.

Step 2: finishing the embedding.

Let T_j be a minimal tree in R_i containing the matching M_j as a subgraph (such a tree exists because R_i is connected), and let $m = |T_j| - 1$ be the number of edges of T_j. For each $C \in V(M_j)$, choose $\ell_C \in [8(k+1)k^3\varepsilon n/t, 100(k+1)k^3\varepsilon n/t]$ such that

$$|C| - \ell_C = \lfloor n/t \rfloor - \lfloor 20k^4\varepsilon n/t \rfloor + d_{T_j}(C).$$

This is possible since $n/t \geqslant |C| \geqslant (1-\varepsilon)n/t$ and since $d_{T_j}(C) \leqslant t$ is bounded by a constant.

By doubling the edges of T_j and considering an Euler tour in the resulting graph, one can see that there exists a surjective homomorphism $\pi\colon C_{2m} \to T_j$ that covers each edge of T_j exactly twice, i.e., for each edge $e \in T_j$, there are exactly two edges $e_1, e_2 \in E(C_{2m})$ such that $\pi(e_1) = \pi(e_2) = e$. For each edge $e \in M_j$, we (arbitrarily) color the edge e_1 red. Let us, for the moment, remove all red edges from C_{2m}, resulting in the graph C'_{2m}, which is just a system of disjoint paths. We now choose any embedding

$$\iota\colon C'_{2m} \to G$$

with the property that every $x \in V(C'_{2m})$ is mapped to a vertex in the cluster $\pi(x)$, and whose image is disjoint from the vertices of the paths P_C. Such an embedding exists by regularity: for every path $x_1, \ldots, x_r$ in C'_{2m}, we may first embed x_1 to a vertex in $\pi(x_1)$ that has at least $d|\pi(x_2)|/2$ neighbours in $\pi(x_2)$. Of these neighbours, at least half will have at least $d|\pi(x_3)|/2$ neighbours in $\pi(x_3)$, so we may embed x_2 to any such neighbor. Continuing in this way, we can completely embed $x_1, \ldots, x_r$ in G in the desired way, and we can do this for every path in C'_{2m}. Note that some vertices might be embedded into the same cluster of R_i; however, as $m \leqslant t$ is a constant and as each cluster has linear size, this does not pose any difficulty.

At this point, we have merely embedded some disjoint paths into G. For each red edge $xy \in E(C_{2m})$, we will now embed into G a path with endpoints $\iota(x)$ and $\iota(y)$ that contains the paths $P_{\pi(x)}$ and $P_{\pi(y)}$, and that, moreover, contains all vertices of $\pi(x) \cup \pi(y)$ that are not in the image of ι. Thus, we will extend ι to an embedding of a subdivision of C_{2m} into G whose image contains the set $V_{M_j} \cup U_i^{(j)}$, as required. Since for each red edge xy, the pair $(\pi(x), \pi(y))$ is $(\varepsilon, d/2)$-super-regular, this is relatively easy to achieve: first, we connect an endpoint of $P_{\pi(x)}$ to $\iota(x)$ by a path of length four (such a path exists by regularity); similarly, we connect an endpoint of $P_{\pi(y)}$ to $\iota(y)$ by a path of length four; finally, we use the Lemma 5 to connect the other endpoint of $P_{\pi(x)}$ to the other endpoint of $P_{\pi(y)}$ by a Hamilton path in the bipartite subgraph of $G[\pi(x), \pi(y)]$ induced by the remaining vertices. The only thing to check is that this subgraph is balanced. However, this follows from our choice of ℓ_C and the fact that the image of ι intersects each cluster C in exactly $d_{T_j}(C)$ vertices. $\square$

4 Proof of Lemma 3

It remains to prove Lemma 3. In this section, we will use the notations $G - e$ and $G + e$ to denote the graph obtained from G by adding or removing a given edge e. We also use the notation $G + H$ to denote the *union* of the graphs G and H, i.e., the graph $(V(G) \cup V(H), E(G) \cup E(H))$ (note that before, the same notation was used for the *disjoint* union). In the proof, we will use the following auxiliary results:

Lemma 11 (Berge [5, Chapter 10.5, Theorem 13]). *Let $G = (V, E)$ be a graph with $n \geqslant 3$ vertices such that for each $2 \leqslant j \leqslant (n+1)/2$, fewer than $j-1$ vertices have degree at most j in G. Then for any two vertices $u \neq v$, there is a Hamilton path with endpoints u and v (and in particular, G is Hamiltonian).*

Lemma 12 (Berge [5, Chapter 10.5, Theorem 15]). *Let $G = (A, B, E)$ be a bipartite graph with $|A| = |B| = n \geqslant 2$ such that for each $2 \leqslant j \leqslant (n+1)/2$, fewer than $j-1$ vertices have degree at most j in G. Then for any two vertices $a \in A$ and $b \in B$, there is a Hamilton path with endpoints a and b.*

Lemma 13. *Let $G = (A, B, E)$ be a bipartite graph. Let s, t be positive integers. Suppose that G contains a cycle C and a collection of paths $P_1, \dots, P_t$ such that $|V(P_1) \cup \cdots \cup V(P_t)| \leqslant s$ and such that the following hold:*

- *each P_i has one endpoint in $a_i \in A$ and one endpoint in $b_i \in B$;*
- *these endpoints satisfy $d(b_i, A) > (|A| + s)/2$ and $d(a_i, B) > (|B| + s)/2$, for every $i \in [t]$;*
- *$C, P_1, \dots, P_t$ are vertex-disjoint and cover all vertices of G.*

Then G is Hamiltonian.

Proof. We can define a sequence $C_0, C_1, \dots, C_t$ of cycles in G such that C_i covers exactly the vertices in $V(C) \cup V(P_1) \cup \cdots \cup V(P_i)$. For this, let $C_0 = C$. Suppose that we have defined C_{i-1}. Using the condition on the degrees of a_i and b_i, the pigeonhole principle implies that C_{i-1} contains some edge uv such that $a_i u$ and $b_i v$ are edges of G. Then we can define $C_i := C_{i-1} - uv + a_i u + b_i v + P_i$. Finally, C_t is a Hamilton cycle in G. □

To keep the proof of Lemma 3 as short as possible, we define the following notion:

Definition 3 (Good pair). Let G be a graph of order n and let an integer $r \geqslant 1$ and real numbers $\gamma, \delta \in (0, 1)$ be given. A pair (A, B) of disjoint subsets of $V(G)$ is said to be (γ, δ, r)-*good* if the following hold:

(P1) $\gamma n \leqslant |A| \leqslant |B| \leqslant r \cdot |A|$;

(P2) $d(b, A) \geqslant |A| - \delta n$ for all but at most δn vertices $b \in B$;

(P3) $d(a, B) \geqslant |B| - \delta n$ for all but at most δn vertices $a \in A$;

(P4) for all $a \in A$ and $b \in B$, we have $d(b, A) \geqslant \gamma n$ and $d(a, B) \geqslant \gamma n$.

Lemma 14. *For every integer $k \geqslant 1$ and every $\gamma > 0$, there is an integer n_0 and a real number $\delta \in (0, 1)$ such that the following holds for every $n \geqslant n_0$. Let G be a graph on n vertices and let (A, B) be a (γ, δ, r)-good pair in G, for some integer $r \in [k]$. Then $G[A, B]$ can be covered by at most r cycles.*

Proof. Let k and γ be given and let G be a graph $n \geqslant n_0$ vertices, where $n_0 = n_0(k, \gamma)$ is a sufficiently large constant. Suppose further that $\delta = \delta(k, \gamma)$ is sufficiently small. For $r \in [k]$ let us define $\gamma_r := \gamma \cdot (2k)^{r-k}$. We will show by induction on r that (γ_r, δ, r)-good pair (A, B) in G can be covered by at most r cycles, for all $1 \leqslant r \leqslant k$. Then the lemma follows by noting that $\gamma_r \leqslant \gamma$ for all $r \in [k]$.

The base case $r = 1$ follows easily from Lemma 12 applied to $G[A, B]$. Indeed, suppose (A, B) is (γ_1, δ, r)-good in G. Then by (P1) we have $|A| = |B| \geqslant 2$, and for $\gamma_1 n \leqslant j \leqslant (|A| + 1)/2$, the number of vertices with degree at most j is at most $\delta n < j - 1$, while for $j < \gamma_1 n$, the number of vertices of degree at most j is zero. Hence, $G[A, B]$ is Hamiltonian.

For the induction step, suppose that $r \geqslant 2$ and recall that $\gamma_r = \gamma \cdot (4k)^{r-k}$. Let (A, B) be a (γ_r, δ, r)-good pair in G. We claim that there exist subsets $B_1, B_2 \subseteq B$ such that $B = B_1 \cup B_2$ and such that (A, B_1) is $(\gamma_1, \delta, 1)$-good and (A, B_2) is $(\gamma_{r-1}, \delta, r - 1)$-good. If such a partition exists, then the claim follows by applying the induction hypothesis on the pairs (A, B_1) and (A, B_2).

To find the sets B_1 and B_2, we use the probabilistic method. Let B_1 be a subset of B chosen uniformly at random among all subsets of size $|A|$ (such a set exists because $|B| \geqslant |A|$). Let $B_2' := B \setminus B_1$ and let B_2'' be a subset of B_1 chosen uniformly at random among all subsets of size $\max\{0, |A| - |B_2'|\}$. Finally, let $B_2 := B_2' \cup B_2''$. Note that $|B_2| = \max\{|A|, |B| - |A|\}$. Clearly B_1 and B_2 cover B, and it is enough to show that with positive probability, (A, B_1) is $(\gamma_1, \delta, 1)$-good and (A, B_2) is $(\gamma_{r-1}, \delta, r - 1)$-good.

We first show that (A, B_2) is $(\gamma_{r-1}, \delta, r - 1)$-good with probability at least 0.6. Since (A, B) is (γ_r, δ, r)-good we have $\gamma_r n \leqslant |A| \leqslant |B| \leqslant r|A|$. Thus also

$$\gamma_{r-1} n \leqslant \gamma_r n \leqslant |A| \leqslant |B_2| = \max\{|A|, |B| - |A|\} \leqslant (r - 1)|A|,$$

verifying (P1) for the pair (A, B_2). It is easy to see that (P2) and (P3) hold for (A, B_2) automatically, using the assumption that (A, B) satisfies (P2) and (P3). As far as (P4) is concerned, it follows from the goodness of (A, B) that for all $b \in B_2$ we have $d(b, A) \geqslant \gamma_r n \geqslant \gamma_{r-1} n$. It remains to show that with probability at least 0.6 we also have $d(a, B_2) \geqslant \gamma_{r-1} n$ for all $a \in A$. Fix some $a \in A$. The degree $d(a, B_2)$ is distributed hypergeometrically with mean

$$\mathbf{E}[d(a, B_2)] \geqslant d(a, B) \cdot \frac{|B_2|}{|B|} \geqslant \gamma_r n / r \geqslant 2\gamma_{r-1} n,$$

where we used $|B_2| \geqslant |A| \geqslant |B|/r$ and the definition of γ_r. By the Chernoff bounds, we obtain

$$\mathbf{P}[d(a, B_2) < \gamma_{r-1} n] \leqslant e^{-\gamma_{r-1} n / 12} \leqslant 0.6/n,$$

if n is sufficiently large given γ and k. By the union bound, with probability 0.6, every $a \in A$ satisfies $d(a, B_2) \geqslant \gamma_{r-1}$, and so (P4) holds with probability at least 0.6. Analogously, one shows that (A, B_1) is (γ_1, δ, r)-good with probability at least 0.6, so there exists a choice of B_1 and B_2 such that (A, B_1) is $(\gamma_1, \delta, 1)$-good and (A, B_2) is $(\gamma_{r-1}, \delta, r - 1)$-good. $\square$

Before continuing, we show that for certain stable sets X (namely, those where k' is not too small) we can find a partition $X = A \cup B$ that is 'almost good'. For this, we have the following claim. Note that (P2'), (P3') and (P4') in the claim correspond exactly to (P2), (P3) and (P4) in the definition of a good pair (with $\gamma = 1/(7k^4)$). Condition (P5') tells us something about the structure of $G[B]$ in the case where $|B| > (\lceil k' \rceil - 1)|A|$.

Lemma 15. *For every $\delta > 0$ and $k \in \mathbb{N}$, there is $\beta > 0$ such that the following holds for all sufficiently large n. Let G be a graph of order n and suppose that X is (k', k, β)-stable in G where $2 - 4k\beta \leqslant k' \leqslant k$. Then there exists a partition $X = A \cup B$ with the following properties:*

(P1') $|A| = n/k \pm \delta n$ *and* $|B| = (k'-1)n/k \pm \delta n$ *and* $|B| \geqslant |A|$*;*

(P2') $d(b, A) \geqslant |A| - \delta n$ *for all but at most* δn *vertices* $b \in B$*;*

(P3') $d(a, B) \geqslant |B| - \delta n$ *for all but at most* δn *vertices* $a \in A$*;*

(P4') for all $a \in A$ *and* $b \in B$*, we have* $d(b, A) \geqslant n/(7k^4)$ *and* $d(a, B) \geqslant n/(7k^4)$*.*

(P5') we have $\Delta(G[B]) \leqslant n/(6k^4)$ *or* $|B| \leqslant (\lceil k' \rceil - 1)|A|$*.*

Proof. It is enough to show the required properties with 18δ instead of δ. Assume that β is sufficiently smaller than δ. Let $I \subseteq X$ be a nearly independent set as in Definition 2. As a first approximation, we may choose $A'' := X \setminus I$ and $B'' := I$. Then we have (say) $|A''| = n/k \pm \delta n$ and $|B''| = (k'-1)n/k \pm \delta n$, using (S1). Moreover, one can use the properties (S1–3) and an averaging argument to show that $d(b, A'') \geqslant |A''| - \delta n$ holds for all but at most δn vertices $b \in B''$ and $d(a, B'') \geqslant |B''| - \delta n$ holds for all but at most δn vertices $a \in A''$. Thus we already have (P2') and (P3') and also very nearly (P1') (but not quite, because it could be that $|B| < |A|$).

We now modify (A'', B'') to make sure that (P4') holds. Let $S \subseteq X$ be the set of at most $2\delta n$ vertices with a deficient degree, i.e., the set of vertices $x \in A''$ for which $d(x, A'') < |A''| - \delta n$ and of vertices $x \in B''$ for which $d(x, B'') < |B''| - \delta n$. Since by (S2) we have $\delta(G[X]) \geqslant n/k^4 - \beta n$, we can partition S into disjoint sets $S_A \cup S_B$ such that the vertices of S_A have at least $k^{-4}n/3$ neighbours in B'' and the vertices of S_B have at least $k^{-4}n/3$ neighbours in A''. Then we let $A' := A'' \cup S_A \setminus S_B$ and $B' := B'' \cup S_B \setminus S_A$. Since we only moved around at most $2\delta n$ vertices, we still have $|A'| = n/k \pm 3\delta n$ and $|B'| = (k'-1)n/k \pm 3\delta n$, and we have $d(a, B') \geqslant |B'| - 3\delta n$ for all but at most $3\delta n$ vertices $a \in A'$, and similarly for the vertices in B'. However, we have gained the property that every vertex in A has degree at least $k^{-4}n/4$ in B, and vice-versa.

Next, we make sure that (P1') holds. The only issue is that it might be that $|B'| < |A'|$. If so, then

$$(k'-1)n/k - 3\delta n \leqslant |B'| < |A'| \leqslant n/k + 3\delta n$$

implies $k' \leqslant 2 + 6k\delta n$. Since we further assumed that $k' \geqslant 2 - 4k\beta \geqslant 2 - 6k\delta n$, we see that in fact, we have $|A'| = (k'-1)n/k \pm 9\delta n$ and $|B'| = n/k \pm 9\delta n$, and so by switching A' with B' we obtain (P1'). Note that this swtiching cannot invalidate any of the symmetric properties (P2'), (P3'), or (P4').

Finally, to obtain (P5'), we further modify these sets A' and B' as follows: as long as both $\Delta(G[B']) > k^{-4}n/6$ and $|B'| > (\lceil k' \rceil - 1)|A'|$, we move a vertex $b \in B'$ with $d(b, B') > k^{-4}n/6$ from B' to A'. Since we have $|B'| \leqslant (k'-1)|A'| + 9\delta n$, we do no move more than $9\delta n$ vertices during this process. Call the sets resulting from these modifications A and B. Then it is easy to verify that (P1'–P5') hold with 18δ instead of δ – note in particular that it is still the case that $|A| \leqslant |B|$ since we stop the process the latest when $|B| = (\lceil k' \rceil - 1)|A| \geqslant |A|$. □

The next lemma will take care of the stable sets where $k' \leqslant 2 - 4k\beta$.

Lemma 16. *Let $k \in \mathbb{N}$ and $\beta > 0$, where $\beta > 0$ is sufficiently small. Let G be a graph of order $n \geqslant 3$ and let X be (k', k, β)-stable in G where $1 \leqslant k' \leqslant 2 - 4k\beta$. Then for any two distinct vertices $x, y \in X$, there exists a Hamilton path in $G[X]$ whose endpoints are x and y.*

Proof. This follows easily from Lemma 11. Indeed, by (S1) the graph $G[X]$ has at most

$$|X| \leqslant k'n/k + \beta n \leqslant (2 - 4k\beta)n/k + \beta n \leqslant 2n/k - 3\beta n$$

vertices. By (S2), every vertex in $G[X]$ has degree at least $n/k^4 - \beta n \geqslant n/(2k^4)$ and all but βn vertices have degree at least $n/k - \beta n > (|X|+1)/2$. Thus, for $n/(2k^4) \leqslant j \leqslant (|X|+1)/2$, there are at most $\beta n < j - 1$ vertices with degree at most j, while for $j < n/(2k^4)$, there are no vertices with degree at most j. □

Proof of Lemma 3. Let G be a graph on n vertices and assume that X is (k', k, β)-stable in G for some real number $k' \in [1, k]$, i.e., X satisfies properties (S1), (S2) and (S3) from Definition 2. Throughout the proof, we will assume that $\delta > 0$ and $\beta > 0$ are sufficiently small constants, where β is assumed to be much smaller than δ, and that n is sufficiently large.

The case $k' \leqslant 2 - 4k\beta$ of the lemma follows immediately from Lemma 16. Therefore, we will from now on assume that $k' \geqslant 2 - 4k\beta$. In particular there is a partition (A, B) of X satisfying the properties (P1'–P5') of Lemma 15. We now prove the different parts of Lemma 3.

Proof of (a).

Note that if δ is sufficiently small, which we assume, then the pair (A, B) is automatically $(k^{-4}/7, \delta, \lceil k' \rceil)$-good. Thus by Lemma 14, the graph $G[A, B]$ can be covered by $\lceil k' \rceil$ cycles, which immediately yields statement (a) of Lemma 3.

Proof of (b).

To see that (b) holds, fix two vertices $x, y \in X$. By (P2'), (P3') and (P4'), it is straightforward to find vertices $a \in A$ and $b \in B$ such that x, y, a, b are all distinct, such that $d(a, B) \geqslant |B| - \delta n$ and $d(b, A) \geqslant |A| - \delta n$, and such that there exist two vertex-disjoint paths of length at most two from x to a and from y to b, respectively. Let P_1 and P_2

be these paths and let $A' := A \setminus (V(P_1) \cup V(P_2))$ and $B' := B \setminus (V(P_1) \cup V(P_2))$. Since $k' \geqslant 2 - 4k\beta n$, one can check that one of the pairs (A', B') and (B', A') is $(k^{-4}/8, 2\delta, \lceil k' \rceil)$-good (note that (A', B') can fail to be good if $|B'| < |A'|$). By Lemma 14 this allows us to cover $G[A' \cup B']$ with at most $\lceil k' \rceil$ cycles. At least one of these cycles, call it C, has length at least $|A'|/k \geqslant n/k^3$. Then, since $d(b, A) \geqslant |A| - \delta n$ and $d(a, B) \geqslant |B| - \delta n$, the pigeonhole principle implies that we can find an edge uv in C such that au, bv are edges of G that are not used by any of the cycles used for covering $A' \cup B'$. Then we can replace C by the x-y-path $P_1 + P_2 + au + bv + (C - uv)$.

Proof of (c).

Here we assume that all but at most n/k^3 vertices $x \in X$ satisfy $d(x, X) \geqslant |X|/k'$, and we need to show that $G[X]$ can be covered with $\lceil k' \rceil - 1$ cycles. If we have $|B| \leqslant (\lceil k' \rceil - 1)|A|$ then (A, B) is a $(k^{-4}/7, \delta, \lceil k' \rceil - 1)$-good pair and using Lemma 14, we are done immediately. So assume that $|B| > (\lceil k' \rceil - 1)|A|$. Then by property (P5'), we have $\Delta(G[B]) \leqslant n/(6k^4)$.

We first claim that $G[B]$ contains a matching of size $|B| - (\lceil k' \rceil - 1)|A|$. To see this, observe that if a vertex $x \in B$ satisfies $d(x, X) \geqslant |X|/k'$ then

$$d(x, B) = d(x, X) - d(x, A) \geqslant |X|/k' - |A| = \frac{|B| - (k' - 1)|A|}{k'},$$

using $|X| = |A| + |B|$ for the last equality. Since there are at least $|B| - n/k^3 \geqslant n/(2k)$ vertices $x \in B$ such that $d(x, X) \geqslant |X|/k'$, this implies

$$e(G[B]) \geqslant \frac{|B| - (k' - 1)|A|}{4k^2} \cdot n.$$

Since the maximum degree of $G[B]$ is at most $n/(6k^4)$, Vizing's theorem implies that the edges of $G[B]$ can be properly edge-colored with $n/(5k^4)$ colors. This in turn means that $G[B]$ contains a matching of size at least

$$\frac{e(G[B])}{n/(5k^4)} \geqslant |B| - (k' - 1)|A| \geqslant |B| - (\lceil k' \rceil - 1)|A|,$$

as claimed.

Denote by M any matching of size $|B| - (\lceil k' \rceil - 1)|A|$ in $G[B]$. Note that by (P1') we have $|M| \leqslant |B| - (k' - 1)|A| \leqslant k\delta n$. By (P4') each vertex in B has at least $n/(7k^3) \geqslant 5|M|$ neighbours in A. Similarly, each vertex of A has at least $n/(7k^3) - \delta n \geqslant 5|M|$ neighbours in B. Using these properties, we can now greedily find a system of $|M|$ vertex-disjoint paths $P_1, \ldots, P_{|M|}$ of length four with the following properties: each P_i contains one edge of M and visits exactly three vertices in B and two vertices in A; moreover, the endpoints of P_i are vertices $a_i \in A$ and $b_i \in B$ such that $d(a_i, B) \geqslant |B| - \delta n$ and $d(b_i, A) \geqslant |A| - \delta n$.

Let $S := V(P_1) \cup \cdots \cup V(P_{|M|})$. We will use a probabilistic argument to show that there exists a partition of B into disjoint sets B_1 and B_2 such that the pairs (A, B_1) and $(A \setminus S, B_2 \setminus S)$ are $(k^{-5}/30, \delta, \lceil k' \rceil - 2)$-good and $(k^{-5}/30, \delta, 1)$-good, respectively. If we manage to do so, it will be easy to complete the proof. Indeed, by Lemma 14 we will be able to cover $G[A, B_1]$ with $\lceil k' \rceil - 2$ cycles and cover $G[A \setminus S, B_2 \setminus S]$ by a single cycle C. Then it follows from Lemma 13 applied to $C, P_1, \dots, P_{|M|}$ that we can in fact cover $G[A, B_2]$ by a single cycle, meaning that we can cover $G[A, B]$ with $\lceil k' \rceil - 1$ cycles, as required.

It remains for us to show how to obtain B_1 and B_2. Let $B_1 \subseteq B \setminus S$ be a subset of $B \setminus S$ chosen uniformly at random among all subsets of size $(\lceil k' \rceil - 2)|A|$ and let $B_2 := B \setminus B_1$. Note that since $|M| = |B| - (\lceil k' \rceil - 1)|A|$, we have

$$|B_2| = |B| - (\lceil k' \rceil - 2)|A| = |M| + |A|.$$

We claim that with positive probability, the pairs (A, B_1) and $(A \setminus S, B_2 \setminus S)$ are $(k^{-5}/30, \delta, \lceil k' \rceil - 2)$-good and $(k^{-5}/30, \delta, 1)$-good, respectively. First of all, both pairs satisfy (P1). For (A, B_1) this is clear since $|B_1| = (\lceil k' \rceil - 2)|A|$. For $(A \setminus S, B_2 \setminus S)$, we have

$$|B_2 \setminus S| = |B_2| - 3|M| = |A| - 2|M| = |A \setminus S|,$$

since every path P_i uses three vertices of B_2 and two vertices of A. Properties (P2) and (P3) hold because of (P2') and (P3'). Finally, using Chernoff bounds we may show that (P4) holds for both pairs with positive probability. For example, for the pair $(A \setminus S, B_2 \setminus S)$, we proceed as follows: for $a \in A$, the degree $d(a, B_2 \setminus S)$ is distributed hypergeometrically with mean

$$\mathbf{E}[d(a, B_2 \setminus S)] \geqslant \frac{n}{7k^4} \cdot \frac{|B_2 \setminus S|}{|B|} \geqslant \frac{n}{15k^5},$$

using $|B_2 \setminus S| \geqslant |A| - |S| \geqslant n/(2k)$. Then the required bound on the probability that some $a \in A$ satisfies $d(a, B_2) < n/(30k^5)$ follows from the Chernoff and union bounds. On the other hand, we have $d(b, A \setminus S) \geqslant n/(7k^4) - |S| \geqslant n/(8k^4)$ for all $b \in B_2$ deterministically. The argument for (A, B_1) is similar. $\square$

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