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Packing degenerate graphs greedily

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Abstract

We prove that if $\mathcal{G}$ is a family of graphs with at most n vertices each, with constant degeneracy, with maximum degree at most $O(n/\log n)$, and with total number of edges at most $(1-o(1))\binom{n}{2}$, then $\mathcal{G}$ packs into the complete graph K_n.

Keywords: tree packing conjecture, graph packing, graph processes

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1 Introduction

A *packing* of a family $\mathcal{G} = \{G_1, \dots, G_k\}$ of graphs into a host graph H is a colouring of the edges of H with the colours $0, 1, \dots, k$ such that the edges of colour i form an isomorphic copy of G_i for each $1 \le i \le k$. Graph packing problems can be considered as a common generalisation of a number of important directions in Extremal Graph Theory. Here we focus on packings of large connected graphs that either exhaust all (*perfect packings*) or almost all the edges of the host graph H (*near-perfect packings*). The first and still the most famous problems are the *Tree Packing Conjectures*. In 1963 Ringel conjectured that if T is any $n+1$-vertex tree, then $2n+1$ copies of T pack into K_n, and in 1976 Gyárfás conjectured that if T_i is an i-vertex tree for each $1 \le i \le n$ then $\{T_1, \dots, T_n\}$ packs into K_n. Since we have $(2n+1)e(T) = \binom{n}{2}$ and $\sum e(T_i) = \binom{n}{2}$, both conjectures ask for perfect packings. Despite many partial results both these problems were wide open until recently.

The first near-perfect packing result in this direction was obtained in [1], where it was shown that one can pack into K_n any family of trees whose maximum degree is at most Δ, whose order is at most $(1-\delta)n$, and whose total number of edges is at most $(1-\delta)\binom{n}{2}$, provided that n is sufficiently large given the constants $\Delta \in \mathbb{N}$ and $\delta > 0$. This approximately answers the Tree Packing Conjectures for bounded degree trees. Various generalisations were obtained in quick succession. The paper [6] shows that one can replace trees with graphs from any nontrivial minor-closed family. This was improved in [2] by allowing the graphs to be packed to be spanning. The paper [5] proves a near-perfect packing result for families of graphs with bounded maximum degree which are otherwise unrestricted. Both Tree Packing Conjectures for trees of bounded maximum degree were solved in [4]. The paper [3] gives near-perfect packing results for spanning trees, and for almost spanning trees, allowing the maximum degrees to be as big $O\big(n^{1/6}/\log^6 n\big)$, and $O\big(n/\log n\big)$, respectively.

Our result is a near-perfect packing theorem for spanning graphs with bounded degeneracy and maximum degrees up to $O\big(n/\log n\big)$, extending the mentioned packing results. [5]

Theorem 1.1 *For each $\gamma > 0$ and each $D \in \mathbb{N}$ there exist $c > 0$ and $n_0 \in \mathbb{N}$ such that the following holds for each $n > n_0$. Suppose that $(G_t)_{t\in[t^*]}$ is a family*

[5] While our result extends these results in the setting of complete host graphs, the focus of [3] is on packing into random graphs, and [5] provides a general packing result in the setting of the Regularity lemma.

of D-degenerate graphs, each of which has at most n vertices and maximum degree at most $\frac{cn}{\log n}$. Suppose further that the total number of edges of $(G_t)_{t\in[t^*]}$ is at most $(1-\gamma)\binom{n}{2}$. Then $(G_t)_{t\in[t^*]}$ packs into K_n.

To make the presentation of the sketch of the proof of Theorem 1.1 more accessible we shall simplify the description. In particular, the auxiliary results required to prove Theorem 1.1 are more involved than the ones stated here.

2 Outline of the main idea

We call the host graph H; it turns out that it is convenient not to restrict ourselves to $H = K_n$ but to a more general setting of quasirandom graphs. An n-vertex graph with $p\binom{n}{2}$ edges is (ϵ, Δ)-*quasirandom* if the common neighbourhood $\mathsf{N}(S)$ of any set S of at most Δ vertices has size $(1 \pm \epsilon)p^{|S|}n$.

The proof of Theorem 1.1 breaks into three steps. We first select almost spanning subgraphs G'_t of the graphs G_t in the given family and a sparse quasirandom graph $H^* \subseteq K_n$. Then we embed the family $(G_t)_t$ into the almost-complete graph $H := K_n - H^*$. This main step is stated in Theorem 2.1 (in a simplified version). Finally, we complete the packing of $(G'_t)_t$ to a packing of $(G_t)_t$ by a matching argument.

Theorem 2.1 *For each $\nu > 0$ and each $D \in \mathbb{N}$ there exist $c, \epsilon > 0$ and $n_0 \in \mathbb{N}$ such that the following holds for each $n > n_0$. Suppose that $(G'_t)_{t\in[t^*]}$ is a family of $t^* \leq 2n$ many D-degenerate graphs, each of which has at most $(1-\nu)n$ vertices and maximum degree at most $\frac{cn}{\log n}$. Suppose that H is a $(\epsilon, 2D+3)$-quasirandom graph of order n. Suppose further that the total number of edges of $(G'_t)_{t\in[t^*]}$ is at most $e(H) - \nu\binom{n}{2}$. Then $(G'_t)_{t\in[t^*]}$ packs into H.*

We outline the proof of Theorem 2.1 in Section 3. Let us now explain how to reduce Theorem 1.1 to Theorem 2.1. Suppose that $(G_t)_{t\in[t^*]}$ is a family as in Theorem 1.1. We first deal with the possibility $t^* > 2n$. We remove any isolated vertices from all graphs in the family, and so we obtain $v(G_t) \leq 2e(G_t)$ for each $t \in [t^*]$. Now, given G_{t_1} and G_{t_2} both of which have at most $n/4$ edges and hence at most $n/2$ vertices, we merge them into a single graph $G_{t_1} \sqcup G_{t_2}$ with at most n vertices. Repeating this procedure until no further merging is possible, we end up with t^* graphs each having at least $n/4$ edges; since the total number of edges in the family is at most $\binom{n}{2}$ we have $t^* \leq 2n$, as is required in Theorem 2.1. Any packing of the modified family (which we still call $(G_t)_{t\in[t^*]}$) gives a packing of the original family.

Next, we want to find subgraphs $G'_t \subseteq G_t$ of order at most $(1-\nu)n$. Since Theorem 2.1 gives us a packing of (G'_t) we want to choose G'_t in order to make it easy to extend this packing to a packing of (G_t). To this end we find an independent set I_t in G_t of size $\Theta(n)$ in which vertices have the same degrees $d \leq 2D$. To show such an independent set exists, we make use of degeneracy of G_t. We set $G'_t = G_t - I_t$. Now Theorem 2.1 gives a packing of the (G'_t) into any n-vertex sufficiently quasirandom graph H with nearly $\binom{n}{2}$ edges. To complete the derivation of Theorem 1.1 we explain how we choose H inside K_n and how we complete the packing of (G'_t) to a packing of (G_t).

We choose H by taking away a random subgraph from K_n, and we let $H^* = K_n - H$. We choose the number of edges in H large enough that Theorem 2.1 applies, but small enough that H^* contains much more than $\sum_{t\in[t^*]} e(G_t) - e(G'_t)$ edges. We apply Theorem 2.1 to pack (G'_t) into H, and then for each $t \in [t^*]$ we find a way to complete the copy of G'_t in H to a copy of G_t in K_n using edges of H^*. The vertices of G_t remaining to embed are an independent set I_t. Each vertex $x \in I_t$ has $d \leq 2D$ neighbours $y_1, \ldots, y_d$ in G_t, which are all in G'_t and hence already embedded to vertices $v_1, \ldots, v_d$ of K_n. Now we complete the embeddings of the G_t, starting with $t = 1$. For $t = 1$, we only allow embedding x to vertices in the candidate set

$$C(x) := \left\{u \in K_n \colon u \notin \mathrm{im}(G'_t), uv_1, \ldots, uv_d \in H^*\right\},$$

and we simply need to match the vertices of I_t to the vertices of K_n such that each x is matched to a vertex of $C(x)$. To see that this matching exists, we need to verify Hall's condition. Part of the unstated strengthening of Theorem 2.1 that we need to do this roughly states that the sets $C(x)$ are distributed in a random-like fashion. It is straightforward to argue from this that Hall's condition holds.

For $t \geq 2$, of course when we want to complete the embedding of G_t we should not use edges of H^* which were used to complete any of $G_1, \ldots, G_{t-1}$, and the definition of $C(x)$ must change accordingly. The other unstated strengthening of Theorem 2.1 that we require is that the vertices adjacent to those in I_t are embedded to sets distributed in a random-like fashion. This means that during the entire packing process we will use only a few edges of H^* at each vertex, and Hall's condition is robust enough to allow for such a change.

3 Proof of Theorem 2.1

The vertices of the graphs G'_t will be always the first $v(G'_t)$ natural numbers, in a degeneracy order. We proceed by packing the graphs $G'_1, \ldots, G'_{t^*}$ one by one in this order and call the randomised algorithm which embeds the graph G'_t *RandomEmbedding*. The graphs $H =: H_0 \supset \ldots \supset H_{t^*}$ record the host graph edges remaining throughout the process. At a given stage $t = 1, \ldots, t^*$, we embed the graph G'_t into H_{t-1} as follows. We embed the vertex 1 into H_{t-1} uniformly at random. Having embedded vertices $1, \ldots, j-1$ of G'_t to H_{t-1}, we need to embed the vertex j. We simply pick a valid choice uniformly at random. In other words, we choose uniformly an image for j from the set of vertices $x \in V(H_{t-1})$ to which we have not embedded any vertex $1, \ldots, j$ of G'_t, and which are adjacent to all of the embedded left-neighbours of j. If this set is ever empty then *RandomEmbedding* fails; if for each stage $t \in [t^*]$ and $j \in V(G'_t)$ it is not empty, then the sequence of *RandomEmbedding*s gives an embedding of each G'_t into H_{t-1}, hence a packing of the (G'_t) into H. Therefore we need to analyse the evolution of (H_t) and the run of *RandomEmbedding* at each stage t. In order to analyse the run of *RandomEmbedding* at stage t, we need H_{t-1} to be very quasirandom; on the other hand, the graph H_t will be a little less quasirandom than H_{t-1}. We set $\alpha_x = C^{-1} \exp(\frac{C(x-2n)}{n})$ for some large constant C. The required quasirandomness for $H = H_0$ is α_0; note that this quantity does not depend on n. Our strategy is to prove that with high probability the sequence of *RandomEmbedding*s does not fail and each of the graphs H_i is $(\alpha_i, 2D+3)$-quasirandom. The following two lemmas are key.

Lemma 3.1 *The probability that* RandomEmbedding *fails when embedding a D-degenerate graph G of order at most $(1-\nu)n$ into an n-vertex $(\alpha, 2D+3)$-quasirandom graph H with $p\binom{n}{2}$ edges, $\alpha \ll p$, is $o(1/n)$.*

Lemma 3.2 *Suppose that we are in the setting described above Lemma 3.1. As the graphs $G'_1, G'_2, \ldots$ are embedded one by one, for each t the following holds. Provided that H_i is $(\alpha_i, 2D+3)$-quasirandom for each $0 \leq i < t$ and that* RandomEmbedding *does not fail before the end of stage t, the probability that H_{t-1} fails to be $(\alpha_t, 2D+3)$-quasirandom is $o(1/n)$.*

These two lemmas imply Theorem 2.1. Indeed, if some *RandomEmbedding* fails, then there must be a first time t when either *RandomEmbedding* fails and H_{t-1} is quasirandom, or *RandomEmbedding* succeeds but the resulting H_t is not quasirandom. Lemmas 3.1 and 3.2 respectively state that these two events have probability $o(1/n)$. Theorem 2.1 follows by a union bound over t.

3.1 Sketch of the proof of Lemma 3.1

Since H is quasirandom, if j has d left-neighbours then about $p^d n$ vertices in H are adjacent to all these embedded left-neighbours, so failure of *Ran-*

domEmbedding can only occur if these vertices have been eaten up by the previous embeddings. We show that this is unlikely; in fact, we show that the following stronger *diet condition* for each $t \in V(H)$ is likely to hold:

For each $S \subseteq V(H)$, $|S| \leq 2D+3$, we have $|\mathsf{N}(S) \setminus \mathrm{im}(G[t])| \approx p^{|S|}(n-t)$.

We fix S and aim to show that S is very unlikely to be a set which witnesses the diet condition failing at the first time. In other words, assuming the diet condition holds up to time $t-1$, we want to show that the sum $\sum_{i=1}^{t} \mathbb{1}\big(i \text{ is embedded to } \mathsf{N}(S)\big)$ is likely to be about $p^{|S|}t$. If these Bernoulli random variables were independent, Hoeffding's inequality would tell us that the sum is very likely to be close to its expectation. They are not independent, but nevertheless a martingale version of Hoeffding's inequality shows that the sum is likely to be close to the sum of conditional expectations

$$\sum_{i=1}^{t} \sum_{w \in \mathsf{N}(S)} \mathbb{P}\big(i \text{ embedded to } w \big| \mathscr{H}_{i-1}\big) \tag{1}$$

where $\mathscr{H}_{i-1}$ denotes the history, that is, the choices for embedding vertices $1, \ldots, i-1$. This sum is itself a random variable, but it turns out to be easier to control. To avoid a technical complication, let us pretend that each i has exactly d left-neighbours. Letting κ be a very small constant, for i in the interval $j+1, \ldots, j+\kappa n$ there is a chance to embed i to w each time w is in the candidate set of i; that is, each time that w is adjacent to all the embedded neighbours of i. The following *cover condition* states that this happens about as often as one would expect:

For each $j \in V(G)$ and $w \in V(H)$, there are about $p^d \kappa n$ vertices among $[j+1, j+\kappa n] \subseteq V(G)$ which contain w in their candidate set. (2)

For each i in the interval $j+1, \ldots, j+\kappa n$ whose candidate set contains $w \notin \mathrm{im}(G[1, \ldots, i-1])$, the vertex i is embedded to a set of size about $p^d(n-i+1) \approx p^d(n-j)$ by the diet condition, where the approximation is since κ is very small. So the probability of embedding i to w is about $p^{-d}(n-j)^{-1}$. On the other hand, by the diet condition we have $\big|\mathsf{N}(S) \setminus \mathrm{im}(G[1, \ldots, i-1])\big| \approx p^{|S|}(n-j)$, which gives the number of vertices w contributing to the sum (1). Summing up, if the cover condition holds then the interval $j+1, \ldots, j+\kappa n$ contributes about $p^{|S|}\kappa n$ to the sum (1). So the cover condition holding implies that the whole sum (1) comes to about $p^{|S|}t$, as desired. This means that, provided the cover condition did not yet fail, the diet condition is unlikely to fail.

We sketch why the cover condition is likely to hold provided the diet condition has not yet failed. When we embed a vertex i, provided the diet condition did not yet fail, the probability of embedding it to a neighbour of w is about p. Now a similar application of a martingale Hoeffding inequality shows that the probability of a given w and j witnessing the failure of the cover condition, given that the diet condition did not yet fail, is very small.

Consider the first time at which one of the cover and diet conditions fails. Before this time both hold, so the probability that t is that first time is by the above argument very small. Taking the union bound over t we conclude that with high probability no such first time exists, and therefore *RandomEmbedding* succeeds, as desired.

3.2 Idea of the proof of Lemma 3.2

We use similar ideas of showing that sums of dependent random variables concentrate using martingale inequalities. The interesting feature is that, because we allow the G'_t to have vertices of very high degree but (because the G'_t are D-degenerate) these must be very few, this time the random variables we are summing (such as the number of edges removed at a vertex of H_{t-1} to form H_t) have maximum values vastly larger than the expected value. In this situation Hoeffding-type inequalities perform very poorly. However, we can use Freedman's martingale inequality to obtain the desired concentration.

References

[1] Böttcher, J., J. Hladký, D. Piguet, and A. Taraz. An approximate version of the tree packing conjecture. *Israel J. Math.*, 211(1):391–446, 2016.

[2] Ferber, A., C. Lee, and F. Mousset. Packing spanning graphs from separable families. arXiv:1512.08701, to appear in Israel J. Math.

[3] Ferber, A. and W. Samotij. Packing trees of unbounded degrees in random graphs. arXiv:1607.07342.

[4] Joos, F., J. Kim, D. Kühn, and D. Osthus. Optimal packings of bounded degree trees. arXiv:1606.03953.

[5] Kim, J., D. Kühn, D. Osthus, and M. Tyomkyn. A blow-up lemma for approximate decompositions. arXiv:1604.07282.

[6] Messuti, S., V. Rödl, and M. Schacht. Packing minor-closed families of graphs into complete graphs. *J. Combin. Theory Ser. B*, 119:245–265, 2016.