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Decomposing tournaments into paths

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Abstract

In this work we consider a generalisation of Kelly's conjecture which is due Alspach, Mason, and Pullman from 1976. Kelly's conjecture states that every regular tournament has an edge decomposition into Hamilton cycles, and this was proved by Kühn and Osthus for large tournaments. The conjecture of Alspach, Mason, and Pullman concerns general tournaments and asks for the minimum number of paths needed in an edge decomposition of each tournament into paths. There is a natural lower bound for this number in terms of the degree sequence of the tournament and they conjecture this bound is correct for tournaments of even order. Almost all cases of the conjecture are open and we prove many of them.

1 Introduction

There has been a great deal of recent activity in the study of decompositions of graphs and hypergraphs. The general prototypical question is this area asks whether, for some given class $\mathcal{C}$ of graphs, hypergraphs or directed graphs, the edge set of each $H \in \mathcal{C}$ can be decomposed into parts satisfying some given property. A striking development in the area is the proof of the existence of designs due to Keevash [7] (and proved later by a different method by Glock, Kühn, Lo, and Osthus [5]) resolving a 150 year old problem. The special case of this problem where one wishes to establish the existence of Steiner systems asks for a decomposition of the edge set of the complete r-uniform hypergraph into r-uniform cliques of a fixed given size. In a different direction, the development of the robust expanders technique by Kühn and Osthus [8] is a second major breakthrough allowing the resolution of several conjectures relating to the decomposition of (directed) graphs into spanning structures such as matchings and Hamilton cycles; see e.g. [4, 9].

The problem we address in this paper is that of decomposing tournaments into paths. A tournament is an orientation of the complete graph, that is, one obtains a tournament by assigning a direction to each edge of the (undirected) complete graph. Let us begin however in the more general setting of directed graphs.

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Let D be a directed graph with vertex set V(D) and edge set $E(D)$. A path decomposition of D is a collection of paths $P_1, \ldots, P_k$ of D whose edge sets $E(P_1), \ldots, E(P_t)$ partition $E(D)$. Given any directed graph D, it is natural to ask what the minimum number of paths is in a path decomposition of D. This is called the path number of D and is denoted $\mathrm{pn}(D)$. A natural lower bound on $\mathrm{pn}(D)$ is obtained by examining the degree sequence of D. For each vertex $v \in V(D)$, write $d_D^+(v)$ (resp. $d_D^-(v)$) for the number of edges exiting (resp. entering) v. The excess at vertex v is defined to be $\mathrm{ex}_D(v) := \max\{d_D^+(v) - d_D^-(v), 0\}$. We note that in any path decomposition of D, at least $\mathrm{ex}(v)$ paths must start at v and therefore we have

$$\mathrm{pn}(D) \geq \mathrm{ex}(D) := \sum_{v \in V(D)} \mathrm{ex}(v),$$

where $\mathrm{ex}(D)$ is called the excess of D. Any digraph for which equality holds above is called consistent. Clearly not every digraph is consistent; in particular any nonempty digraph D of excess 0 cannot be consistent. However, Alspach, Pullman, and Mason [1] conjectured that every even tournament is consistent.

Conjecture 1.1. *Every tournament T with an even number of vertices satisfies* $\mathrm{pn}(T) = \mathrm{ex}(T)$.

It is almost immediate to see that this conjecture is a considerable generalisation of Kelly's conjecture stated below.

Conjecture 1.2. *(Kelly; see e.g. [3]) The edge set of every regular tournament can be decomposed into Hamilton cycles.*

Kühn and Osthus [8] proved Kelly's conjecture for large tournaments using their powerful robust expanders technique, which was subsequently used to prove several other conjectures on edge decompositions of (directed) graphs [9, 4].

Theorem 1.3. *Every sufficiently large regular tournament has a Hamilton decomposition.*

To see that Conjecture 1.1 implies Conjecture 1.2, take any regular $(n+1)$-vertex tournament T and any $v \in V(T)$, and note that $\mathrm{ex}(T - v) = n/2$. If Conjecture 1.1 holds, then $T - v$ can be decomposed into $n/2$ paths, so they must be Hamilton paths. Adding v back to $T - v$, it is easy to see that each path can be completed to a Hamilton cycle, giving a Hamilton decomposition of T. The converse is also easy to see. Thus the special case of Conjecture 1.1 in which $\mathrm{ex}(T) = n/2$ is equivalent to Kelly's Conjecture. In general, however, $\mathrm{ex}(T)$ can take a large range of values.

Proposition 1.4. *If T is an n-vertex tournament with n even, then $n/2 \leq \mathrm{ex}(T) \leq n^2/4$. Furthermore each value in the range occurs.*

As we saw, the lower bound occurs for any almost-regular tournament and it is easy to verify that the upper bound occurs for the transitive tournament (in fact for any tournament with a vertex partition into two equal parts A and B where all edges are directed from A to B). Alspach and Pullman [2] showed that for any tournament T, $\mathrm{pn}(T) \leq n^2/4$ thus verifying Conjecture 1.1 for the special case $ex(T) = n^2/4$ (and this was generalised to digraphs [11]). Thus the conjecture has been solved for the two extreme values of excess, namely $n/2$ and $n^2/4$: for every other value of $\mathrm{ex}(T)$ between $n/2$ and $n^2/4$ the conjecture remains open. Our main contribution is to solve many more cases of the conjecture.

Theorem 1.5. *There exists $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that if T is a tournament on $n > n_0$ vertices with n even and* $\mathrm{ex}(T) \geq n^{2-\varepsilon}$*, then* $\mathrm{pn}(T) = \mathrm{ex}(T)$.

The proof of this result is self-contained relying on a novel application of the absorption technique due to Rödl, Ruciński, and Szemerédi [12] (with special forms appearing in earlier work e.g. [10]). However, we believe that a refinement of the ideas used for Theorem 1.5 combined with Theorem 1.3 will allow us to prove the following result, which is work in progress.

Theorem 1.6. *There exists $C > 0$ such that if T is an n-vertex tournament with n even and* $\mathrm{ex}(T) > Cn$ *then* $\mathrm{pn}(T) = \mathrm{ex}(T)$.

In the next section we discuss the ideas behind the proof of Theorem 1.5.

2 Sketch proof of Theorem 1.5

Below we state some easy observations about oriented graphs in the form of a proposition. These turn out to be useful in the proof of Theorem 1.5.

Proposition 2.1. *(a) If G is an acyclic oriented graph then* $\mathrm{pn}(G) = \mathrm{ex}(G)$.

(b) For every oriented graph G, we can find edge-disjoint subgraphs G_A and G_E of G such that G_A is acyclic, G_E is Eulerian, and $G = G_A \cup G_E$.

Proof. Part (a) can be shown e.g. by repeatedly removing a path of maximum length, and part (b) by repeatedly removing cycles from G and adding them to G_E until no cycles remain. □

The key step in our proof of Theorem 1.5 is to show that for every tournament T with $\mathrm{ex}(T) \geq n^{2-\varepsilon}$, we can find a oriented subgraph H of T which has the following properties:

(i) $\mathrm{pn}(H) = \mathrm{ex}(H)$;

(ii) writing $H' := T - E(H)$, we have $\mathrm{ex}(T) = \mathrm{ex}(H) + \mathrm{ex}(H')$;

(iii) for any Eulerian graph F on $V(T)$ that is edge-disjoint from H, we have $\mathrm{pn}(H \cup F) = \mathrm{ex}(H \cup F) = ex(H)$.

We call a subgraph H of T that satisfies (i), (ii), and (iii) an absorber. If we can find an absorber H in T then it follows that the tournament is consistent. Indeed let $H' = T - E(H)$ and using the proposition, decompose H' as $H' = H'_A \cup H'_E$ where H'_A is acyclic and H'_E is Eulerian. Now $T = (H \cup H'_E) \cup H_A$ and we have

$$\mathrm{ex}(T) \leq \mathrm{pn}(T) \leq \mathrm{pn}(H \cup H'_E) + \mathrm{pn}(H'_A) = \mathrm{ex}(H) + \mathrm{ex}(H'_A) = \mathrm{ex}(H) + \mathrm{ex}(H') = ex(T).$$

The absorber H can in fact be easily described. We define a (k, ℓ)-path system of an n-vertex tournament $T = (V, E)$ to be a collection of nk paths P_i^v of T indexed by $v \in V$ and $i = 1, \ldots, k$, where

- for each fixed v, $P_1^v, \ldots, P_k^v$ are vertex disjoint paths except that they all pass through v;

- all paths are edge disjoint;
- every path has length at most ℓ;
- the oriented graph H formed by taking the union of the paths and the oriented graph $H' = T - E(H)$ satisfy $\mathrm{pn}(H) = \mathrm{ex}(H) = nk$ and $\mathrm{ex}(H) + \mathrm{ex}(H') = \mathrm{ex}(T)$.

For suitable values of k, ℓ, the union of paths in any (k, ℓ)-path-system of T gives an absorber. The reason for this is roughly as follows. Suppose $F \subseteq T$ is any Eulerian subgraph of T that is edge disjoint from H. Then by a result of Huang, Ma, Shapira, Sudakov, and Yuster [6], one can decompose any Eulerian graph into at most $n^{3/2}$ cycles. We can use the path decomposition of H to absorb the cycles of F one at a time into the path system as follows. Given a cycle C, we carefully pick vertices $v_1, \ldots, v_t$ on C and paths $Q_1, \ldots, Q_t$ from the path system such that Q_j is one of the paths that passes through v_j, i.e. one of the paths $P_i^{v_j}$. Assume each path Q_i is from vertex a_i to b_i. If we make our choices carefully, then we can ensure that Q_j is vertex-disjoint from $v_{j-1}Cv_{j+1}$ the segment of the cycle between v_{j-1} and v_{j+1} (where indices are understood to be modulo t). Now each path Q_i can be replaced by the path $a_iQ_iv_iCv_{i+1}Q_{i+1}b_{i+1}$. These new paths contain all the edges of the original paths and the edges of C. Showing that every cycle in the cycle decomposition of F can be absorbed in this way shows that H has the property of an absorber.

Finally, provided the tournament T has sufficiently high excess $\mathrm{ex}(T) > n^{2-\varepsilon}$, a careful iterated application of Menger's Theorem allows us to construct a (k, ℓ)-path system in T.

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