A stress function for 3D frames

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Abstract

This paper generalises Rankine diagrams for 3D trusses to be applicable to 3D frames. Rankine diagrams are a graphical representation of a state of self-stress in a 3D truss, with the area of reciprocal polygons representing the axial force in their corresponding original bars. Rankine diagrams are a polyhedral version of the continuous Maxwell-Rankine stress function. In this paper we present a new stress function. It is piecewise linear and discontinuous and it allows the analysis of 3D frames, giving all six stress resultants of axial and shear forces and bending and torsional moments in any member. A succinct statement of the stress function is given in terms of 4D Clifford Algebra.

1. Introduction

This paper describes a generalisation of Rankine's reciprocal construction for 3D trusses to the case of 3D frames. In an associated paper (McRobie [1]) this generalisation is presented more formally using the language of 4D Clifford algebra. In this paper, the same generalisation is described more simply using standard 3D vector analysis, with the Clifford algebra description stated briefly at the end of the paper. Although this paper is thus arguably less general than [1], it nevertheless contains a number of important new results on completeness, and moreover provides a simpler introduction to the full theory which may be preferable for those who are unfamiliar with Clifford algebra.

In a Rankine reciprocal diagram [2] a polygonal face is orthogonal to its corresponding bar in the original 3D truss, with the oriented area of the face equal to the axial force in the bar. That is, a vector normal to the reciprocal polygon and of magnitude equal to the polygon area represents the bar force. In an earlier attempt to generalise this construction to be applicable to moment-carrying 3D frames, McRobie and Williams [3] presented a generalisation in which reciprocal polygons did not need to be orthogonal to their corresponding bars. In that case, the oriented area of the polygon represented the total force (axial plus shear) on the bar cross section, and the presence of shear stresses led to torsions and bending moments in each bar. This early approach thus demonstrated some potential for modelling 3D frames.

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Maxwell [4] defined two stress functions, one of which was a simple scalar field. We refer to that stress function as the Maxwell-Ranine stress function, since it has a close relationship with Rankine's reciprocal construction. The earlier McRobie and Williams approach [3] relied heavily on taking the discontinuous polyhedral limit of the continuous Maxwell-Rankine stress function. That is, the Maxwell-Rankine stress function was everywhere continuous in its values and in it first derivatives, and over each region of space (cell) the stress function was a linear function, excepting over thin transition regions between cells. The volume of these transition regions was then allowed to tend to zero, whilst maintaining continuity of slope and value. Whilst this led to some new and interesting results - such as the concept of reciprocal diagrams for grillages - a number of technical difficulties were encountered in the limiting process: it was often difficult to take limits of the Maxwell-Rankine stress function without encountering unwanted infinities.

In this paper, we therefore discard all attempts to take the discontinuous limit of some continuum stress function (such as the Maxwell-Rankine or even the Beltrami stress function), and we simply define a discontinuous one *ab initio* with the desired properties. Recall that a stress function is simply some function over a region of space, together with an operation on that function which returns a stress field. That is, the stress function is a function of position, and the stress is a function of the stress function. This paper defines such an object, and we argue that it provides a powerful new method for describing the behaviour of 3D frames

The first fundamental difference we introduce concerns position. Position will be defined within a set of polyhedral cells, and these cells give an at-least *double cover* of a region of 3D space. That is, we shall consider a region of 3D space that is covered by the projection of a 4-polytope, that being a set of conjoined polyhedra. Over each polyhedron we shall define a scalar stress function. The stress function will be linear over each cell, but there is no requirement for continuity at cell boundaries. That is, on polyhedron i we define the stress function

$$\phi_i(\mathbf{x}) = \Delta_0 + g_x x + g_y y + g_z z \tag{1}$$

This stress function has gradient $\mathbf{g} = (g_x, g_y, g_z)$, and at a point with coordinate vector $\mathbf{x}$ within the cell, the value of the stress function is $\phi_i(\mathbf{x}) = \Delta = \Delta_0 + \mathbf{g}.\mathbf{x}$

We now define the operation on the stress function which delivers the stress. Our principal departure from previous methods is that the stress function will be used to directly define the stress *resultants* in bars, rather than the stresses over a region. We first define the stress to be zero everywhere except on the edges of the polyhedra. The edges of the polyhedra are the bars, and the stress resultant in a bar is determined by the stress functions on the cells for which that bar is a common edge. Specifically, the operation that defines the stress resultants is given by the equations

$$\mathbf{F} = \frac{1}{2}\sum_c \mathbf{g}_c \times \mathbf{g}_{c+1} \tag{2}$$

$$\mathbf{M} = \frac{1}{2}\sum_c \mathbf{g}_c(\Delta_{c-1} - \Delta_{c+1}) \tag{3}$$

where $\mathbf{F}$ and $\mathbf{M}$ are the force and moment vectors respectively on the cut end of a bar.

Here $\mathbf{g}_c$ denotes the gradient of the stress function over cell c and Δ_c denotes the value of the stress function in cell c adjacent to the point under consideration. The summation is over the adjoining cells of which the bar is an edge, c being a cell and $c+1$ being the cell adjacent to cell c in the direction given by a right-hand screw rule (i.e. clockwise) when looking along the bar in question. The counting is cyclic such that for a bar that is common to n cells, then when $c = n$ we have $c + 1 = 1$.

We illustrate the stress function definition with a simple example.

2. Example: the 5-cell

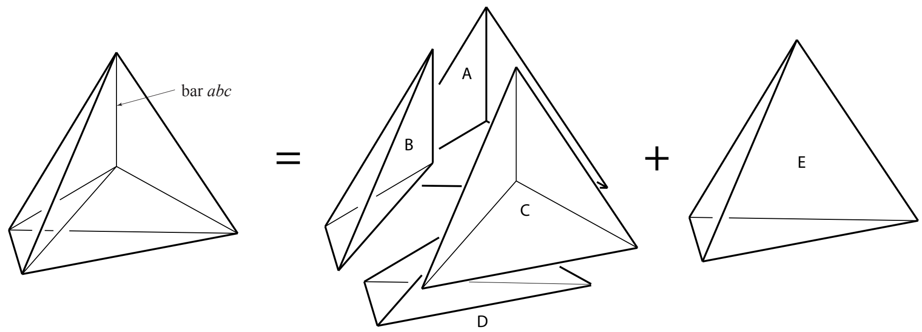

Figure 1: A frame which is the projection of a 5-cell, the 5-cell being a double cover of a region of 3D space by 5 tetrahedra.

Just as Fig.1 of Maxwell's 1864 paper [5] on reciprocal diagrams for 2D trusses begins with the simplest example - the projection of a tetrahedron - we begin here with the simplest higher dimensional generalisation; our 3D frame structure is the 3D projection of a 5-cell. This consists of 5 cells (see Fig. 1). Each cell is a tetrahedron whose four faces are each common to one of the other four tetrahedra. That is, the structure consists of 10 bars joined as shown with rigid connections, but from the perspective of this paper, we look at the spaces between the bars - that is, the tetrahedral volumes.

Look along any bar in the frame, such as the vertical bar *abc* from centre to apex. A plane transverse to the bar intersects the three cells A, B, C of which the bar is a common edge (see Fig. 2*a*). The common faces AB, BC, CA of these cells intersect the transverse plane in a set of straight lines radiating from the point representing the bar cross-section. These divide the plane into sectors, each sector being a cross-section through one of the cells (see Fig. 2*b*).

Over each cell, the gradient $\mathbf{g}$ of the stress function is denoted by a vector $\mathbf{a}$, $\mathbf{b}$, etc. These gradients are not restricted to lie within the transverse plane shown, thus they may have components parallel to the beam.

The value Δ of the stress function at a point immediately adjacent to the beam in each adjoining cell will be denoted A, B, etc.

We may thus compute the stress resultant in the bar at this point using Eqns. 2 and 3.

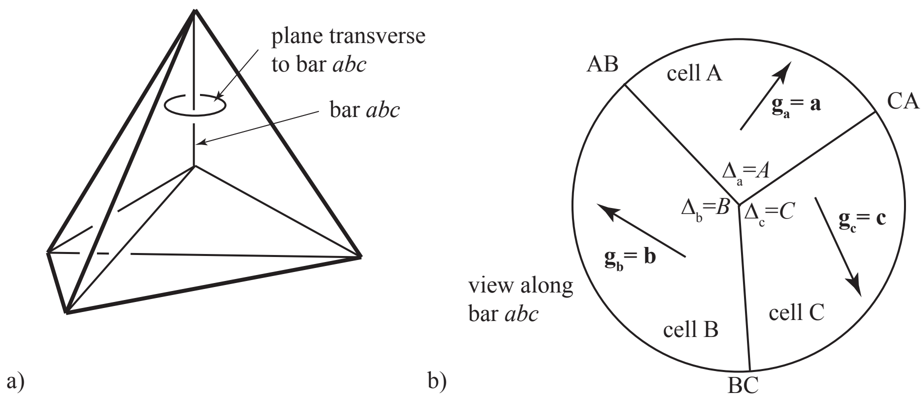

Figure 2: A plane transverse to bar *abc* which is the common edge of cells A, B and C. b) The stress function definition on that plane. Note in particular that the gradient vectors need not lie in that plane.

2.1. Force

First we compute the force in the bar. The algebraic formula (Eqn. 2) states that the force in the bar at this point is given by a sum of cross-products of the gradient vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ of the stress functions on the adjacent cells. These three vectors may be used to define a pyramid with apex at the origin, and each cross-product (such as $(1/2)\mathbf{a} \times \mathbf{b}$) is the oriented area of a triangular face of this pyramid. It follows that the sum of the cross-products is the sum of the oriented areas of the sides, which thus equals the oriented area of the base of the pyramid, the triangle with vertices $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$. We may represent that oriented area by a vector $\mathbf{F}$ perpendicular to the triangle and of magnitude equal to the area of the triangle.

This is our first generalisation of the Rankine reciprocal. In Rankine's construction, the force in a bar is purely axial, and is given by the area of a polygon which is perpendicular to the bar. In this construction, the polygon dual to a bar need not be orthogonal to the bar, and the resulting force need not be oriented along the bar. That is, the force $\mathbf{F}$ may consist of an axial component $\mathbf{P}$ and a shear force $\mathbf{S}$ (see Fig. 3).

2.2. Moment

We begin with the fundamental algebraic definition, Eqn. 3, and in the first instance we shall consider it purely as an algorithm by which one may evaluate the moment from a given stress function. We shall consider the geometric interpretation later.

Eqn. 3 states that the resultant moment at a point on the bar is a weighted sum of the adjacent stress function gradients, with the weights given by differences in stress function values at the point in question.

One motivation for this choice follows from dimensional analysis. Since a force is represented by the area of a dual polygon, which is given by a product of stress function gradients, then stress function gradients must have units of $\sqrt{\text{kN}}$. Our stress function must thus have units of $\sqrt{\text{kN}}$ m. A product of stress function value and stress function gradient (as per our fundamental definition, Eqn. 3) must thus have dimensions of kNm, as befits a moment. This argument was used in the previous attempts (see McRobie [3])

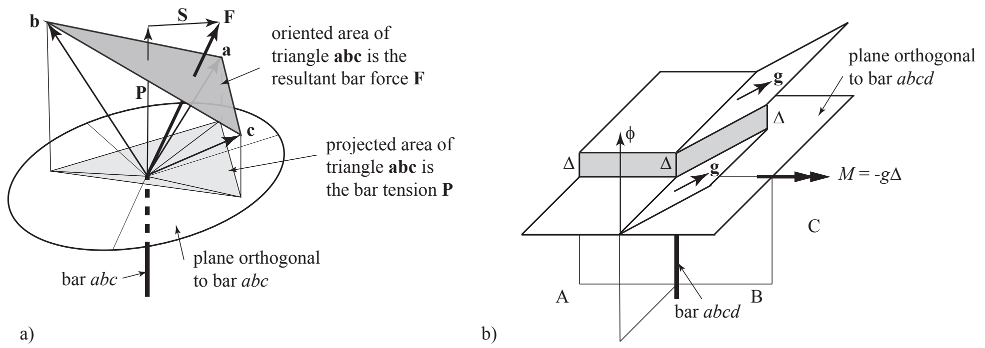

Figure 3: a) The polygon dual to a bar is defined by the gradient vectors of the stress function over the adjoining cells. The oriented area of that polygon may be represented by the vector $\mathbf{F}$ normal to the polygon, and $\mathbf{F}$ need not be parallel to the original bar *abc*. b) The discontinuous limit of a simple Maxwell-Rankine stress function, as per McRobie and Williams [3], for which the moment $\mathbf{M} = -\mathbf{g}\Delta$ is a product of changes of gradient and value.

to define a stress function for 3D frames by taking the discontinuous limit of the Maxwell-Rankine stress function. There it was possible to show that if there is a plane of stress function discontinuity Δ orthogonal to a plane across which the stress function gradient changes by $\mathbf{g}$, then the resulting moment in the bar (which lies at the intersection of the two planes) is given by the product $\mathbf{g}\Delta$. The arrangement is illustrated in Fig. 3*b*. The statement in this paper, (Eqn. 3), is the generalisation of that idea for more general, non-orthogonal intersections of cell faces, and is valid when the bar is the common edge of any number of cells.

In this paper, then, the moment at the point on the bar is given by a sum of the individual products of stress function gradients and stress function discontinuities. We describe the algorithm in detail for the bar *abc* in the tetrahedral structure. We may start the summation with cell A, which has stress function gradient $\mathbf{a}$. Taking the difference of the stress function values on the cells either side, we obtain a discontinuity $\Delta_C - \Delta_B = C - B$. The contribution from cell A is thus $\mathbf{a}(C - B)/2$. Gathering the contributions from cells B and C, we obtain the associated moment $\mathbf{M}$ in the bar as

$$\mathbf{M} = \frac{1}{2}\left[\mathbf{a}(C - B) + \mathbf{b}(A - C) + \mathbf{c}(B - A)\right] \tag{4}$$

and we note that this may be regrouped as a sum of stress function values with gradient changes:

$$\mathbf{M} = \frac{1}{2}\left[A(\mathbf{b} - \mathbf{c}) + B(\mathbf{c} - \mathbf{a}) + C(\mathbf{a} - \mathbf{b})\right] \tag{5}$$

The resultant moment $\mathbf{M}$ is a general vector. Letting $\mathbf{i}$ be the unit vector in the bar direction, then the component of magnitude $T = \mathbf{M}.\mathbf{i}$ is the torsion, and the remainder $\mathbf{B} = \mathbf{M} - T\mathbf{i}$ is the bending moment.

2.3. Variation along the bar

Let us choose coordinate axes (x, y, z) with unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ with origin at one end of the bar, and the x-axis oriented along the bar.

The force at a point x along the bar depends only on the gradients of the stress functions in the adjacent cells, and since gradients are constant within each cell, it follows that the force resultant (consisting of axial force $\mathbf{P} = P\mathbf{i}$ and shear $\mathbf{S}$) is constant along the bar.

The moment, however, may vary along the bar. If the stress function value in cell A is A_0 adjacent to the origin end of the bar, then at distance x along the bar, the stress function value is $A = A_0 + a_1 x$, where a_1 is the x-component of $\mathbf{a}$. There are similar expressions for B and C.

We may now extract the components of the moment.

For the torsion $T = \mathbf{M}.\mathbf{i}$, we have

$$\begin{aligned}
2T &= A(b_1 - c_1) + B(c_1 - a_1) + C(a_1 - b_1) \\
&= (A_0 + a_1 x)(b_1 - c_1) + (B_0 + b_1 x)(c_1 - a_1) + (C_0 + c_1 x)(a_1 - b_1) \\
&= A_0(b_1 - c_1) + B_0(c_1 - a_1) + C_0(a_1 - b_1) \qquad (6)
\end{aligned}$$

with all x-variation cancelling via the cyclic summation. That is, the torsion is constant along the bar.

The bending moment can be resolved into components $M_y = \mathbf{M}.\mathbf{j}$ and $M_z = \mathbf{M}.\mathbf{k}$ with

$$\begin{aligned}
2M_y &= (A_0 + a_1 x)(b_2 - c_2) + (B_0 + b_1 x)(c_2 - a_2) + (C_0 + c_1 x)(a_2 - b_2) \\
&= A_0(b_2 - c_2) + B_0(c_2 - a_2) + C_0(a_2 - b_2) \qquad (7) \\
&\quad + \left[(a_1 b_2 - a_2 b_1) + (b_1 c_2 - b_2 c_1) + (c_1 a_2 - c_2 a_1)\right] x \qquad (8)
\end{aligned}$$

That is, the moment about the y-axis is a constant plus a linear term. We obtain a similar result for the moment about the z-axis.

Note that the shear force component $S_z = \mathbf{k}.\mathbf{F}$, thus

$$\begin{aligned}
2S_z &= \mathbf{k}.\left[\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}\right] \\
&= (a_1 b_2 - a_2 b_1) + (b_1 c_2 - b_2 c_1) + (c_1 a_2 - c_2 a_1) \qquad (9)
\end{aligned}$$

which is the coefficient of x in the equation for M_y. We may thus express the bending moment M_y as

$$M_y = M_{0y} + S_z x \qquad (10)$$

That is, the bending moment component consists of a constant plus a linearly-varying term, and the slope of the linearly-varying term is the shear force, as usual.

Similar algebra leads to

$$M_z = M_{0z} - S_y x \qquad (11)$$

These combine to give the bending moment $\mathbf{B}$ as

$$\mathbf{B} = (M_{0y} + S_z x)\mathbf{j} + (M_{0z} - S_y x)\mathbf{k} = \mathbf{M}_0 + x\mathbf{S}_R \qquad (12)$$

this being the sum of a constant vector $\mathbf{M}_0$ and a linearly-varying vector $x\mathbf{S}_R$ (where $\mathbf{S}_R$ is a 90 degree rotation of the shear force about the beam axis). Note in particular that $\mathbf{M}_0$ and $x\mathbf{S}_R$ may be in different directions.

In summary, the proposed stress function leads to stress resultants within the beam that accord with the familiar equilibrium equations for beam members in nodally-loaded

3D frames. For this example nodal equilibrium can be readily checked term by term, showing that the forces and moments on all bars meeting at a node sum to zero. It is, however, somewhat lengthy and not particularly instructive compared to the far more general geometric proof presented later, that shows the stress function guarantees nodal equilibrium for any 3D frame that may be represented as the projection of a 4-polytope.

3. Rankine Incompleteness

Although this paper is concerned with moment-carrying 3D frames, the methods here provide a new perspective on a long-standing problem concerning 3D trusses, that of Rankine Incompleteness. It was demonstrated in Maxwell 1870 [4] that there can exist states of 3D stress that cannot be represented by the Maxwell-Rankine stress function. It follows that there can exist 3D trusses which possess states of self-stress and yet which have no Rankine reciprocal.

The question thus arises as to whether the stress function of this paper suffers from similar problems of incompleteness. For example, the 5-cell frame requires six cuts to make a tree structure and with 6 free stress resultant components at each cut, there are thus 36 possible states of self stress. However, there are only 4 coefficients of the linear stress function over each of the five cells and, without loss of generality, the stress function on one cell can be set to zero, meaning that at most only $4 \times 4 = 16$ of the 36 states of self-stress can currently be accessed.

The incompleteness arises because each bar is the common edge of only three cells. As the following theorem shows, even if each bar were the common edge of four cells, it would still not be possible to represent all states of stress in that bar.

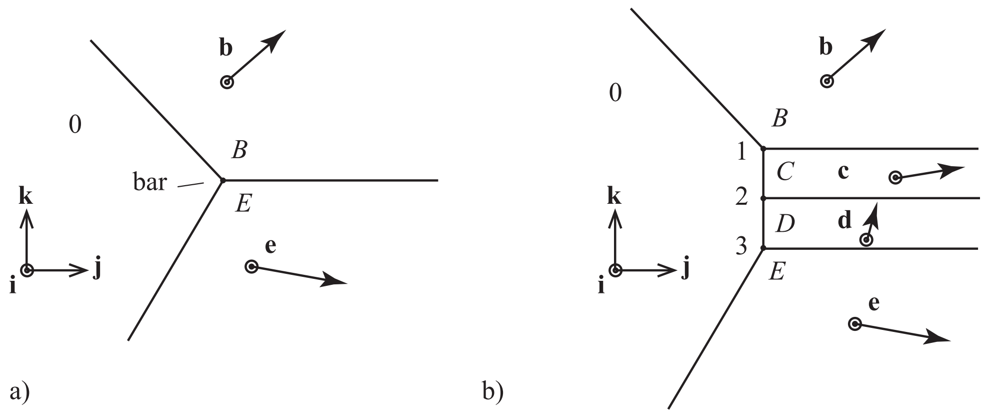

Figure 4: a) A section through a bar that is the common edge of three cells. b) Thickening the face BE to consist of two thin prismatic cells C and D over which a general stress function may be applied.

3.1. An Incompleteness Theorem

If a beam is the common edge of at most four cells then the stress function cannot represent all states of stress in that beam.

Proof: Let the beam be the common edge of cells A,B,C,D arranged cyclically. Without loss of generality, let the stress function be zero over cell A. The force and moment in the beam is then given by

$$2\mathbf{F} = \mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{d} \tag{13}$$
$$2\mathbf{M} = -C\mathbf{b}+(B-D)\mathbf{c}+C\mathbf{d} \tag{14}$$

Elementary operations show that $4\mathbf{F}.\mathbf{M} = (C-C)(\mathbf{b}\times\mathbf{c}).\mathbf{d} = 0$ thus the stress function can only represent stress resultants where $\mathbf{F}$ and $\mathbf{M}$ are orthogonal.

3.2. Five-valent bars

The foregoing proof of incompleteness does not apply to a bar which is the common edge of five cells. In that case, it is possible for $\mathbf{F}.\mathbf{M}$ to be nonzero. We thus describe a procedure which appears to be capable of representing general states of stress in a beam.

Fig. 4*a* shows a beam which is the common edge of three cells A,B,E. Adjacent to a point on the beam, the stress function values and gradients are 0,B,E and $\mathbf{0},\mathbf{b},\mathbf{e}$ respectively.

We may now artificially thicken the face that separates cells B and E into two thin prismatic cells C and D (see Fig. 4*b*). We shall refer to such cells as “face cushions”. Since there is no requirement for stress function continuity, arbitrary stress functions can be applied over the face cushion cells C and D. There are now three nodes ABC, ACD and ADE representing the beam, but these may be taken sufficiently close together that their actions may be added.

We begin by writing the general stress resultant components at some point in the beam:

$$\begin{array}{rcl} \mathbf{F} &=& P\mathbf{i} + S_y\mathbf{j} + S_z\mathbf{k} \\ \mathbf{M} &=& T\mathbf{i} + (M_{0y}+S_z x)\mathbf{j} + (M_{0z}-S_y x)\mathbf{k} \end{array} \tag{15}$$

where $\mathbf{i}$ is along the beam, and P and T are the axial force and torsion, with shears S, constant moments M_0 and linearly-varying moments of magnitude Sx.

This may be decomposed into two systems: a grillage system

$$\begin{array}{rcl} \mathbf{F} &=& S_z\mathbf{k} \\ \mathbf{M} &=& T\mathbf{i} + (M_{0y}+S_z x)\mathbf{j} \end{array} \tag{16}$$

and a 2D frame system

$$\begin{array}{rcl} \mathbf{F} &=& P\mathbf{i} + S_y\mathbf{j} \\ \mathbf{M} &=& (M_{0z}-S_y x)\mathbf{k} \end{array} \tag{17}$$

The stress functions over the face cushion cells are then chosen such as to create grillage action between cells B and C, and 2D frame action between cells C, D and E. We simply state a solution.

Let the stress functions over the cells have adjacent values and gradients as follows

$$\begin{array}{rcl} \sqrt{2}(B+\mathbf{b}) &=& (M_{0y}+S_z x)/I + (S_z/I)\mathbf{i} \\ \sqrt{2}(C+\mathbf{c}) &=& -TI/S_z + I\mathbf{j} \\ \sqrt{2}(D+\mathbf{d}) &=& 0 + I\mathbf{k} \\ \sqrt{2}(E+\mathbf{e}) &=& [-TI/S_z+(M_{0z}-S_y x)/I] + (S_y/I)\mathbf{i} + (I-P/I)\mathbf{j} \end{array} \tag{18}$$

Here I is unit gradient (with dimensions $[\sqrt{\mathrm{kN}}]$), and is incorporated merely for dimensional consistency. Also, we consider the face cushions to be so thin that any through-the-thickness change in value may be neglected. In particular, the stress function value D is zero at both nodes 2 and 3. The configuration is shown in Fig. 5 where for brevity, the $\sqrt{2}$ factors have been omitted.

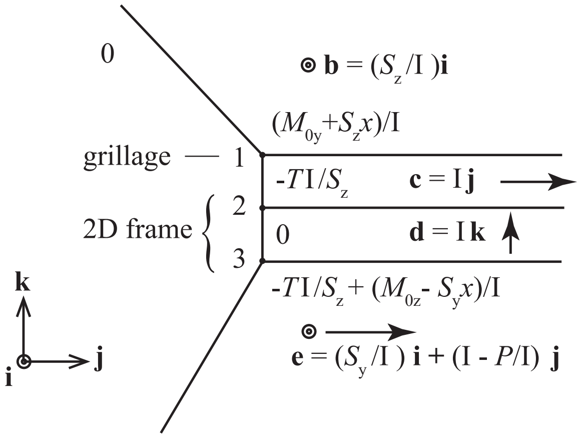

Figure 5: A stress function that creates a general stress resultant in the bar. The stress function consists of a grillage system and a 2D frame system.

For the upper node 1, we have $2\mathbf{F} = \mathbf{b} \times \mathbf{c}$ and $2\mathbf{M} = -C\mathbf{b} + B\mathbf{c}$, and elementary algebra shows this to give a grillage system (Eqn. 16). For the lower two nodes 2 and 3, the total stress resultants are $2\mathbf{F} = \mathbf{c} \times \mathbf{d} + \mathbf{d} \times \mathbf{e}$ and $2\mathbf{M} = [\mathbf{c}(0-0) + \mathbf{d}(C-0)] + [\mathbf{d}(0-E) + \mathbf{e}(0-0)] = (C-E)\mathbf{d}$, which leads to a 2D frame system (Eqn. 17). We have thus created a general state of stress in the member.

Although some effort was required here to pick the stress function coefficients in order to achieve a desired state of stress, in applications the procedure will be different. Face cushions can be introduced over any polygonal face, and given the topology of the cells, arbitrary linear stress functions can be applied and the resulting stress resultants can be trivially computed. Stress function coefficients can then be adjusted to explore possible outcomes.

4. Gauche polygons and general loops

On a number of occasions throughout the foregoing analysis, we have used sums of oriented areas, where the individual areas were not coplanar. The boundary of the total area is thus what Maxwell would call a 'gauche' polygon, a set of edges which form a closed but nonplanar loop. Such configurations have previously been problematic for approaches using polyhedral stress functions. However, the stress function of this paper has no problem with this. This is largely because the fundamental stress function definitions (Eqns. 2 and 3) have no dependence on the structural geometry. All structural dependence is topological. The linear stress function over each cell is specified, but no knowledge is required of the angles at which cell faces join at the beam. An extension of this idea would suggest that the faces of cells do not even need to be planes, but could be any surface spanning any general loop.

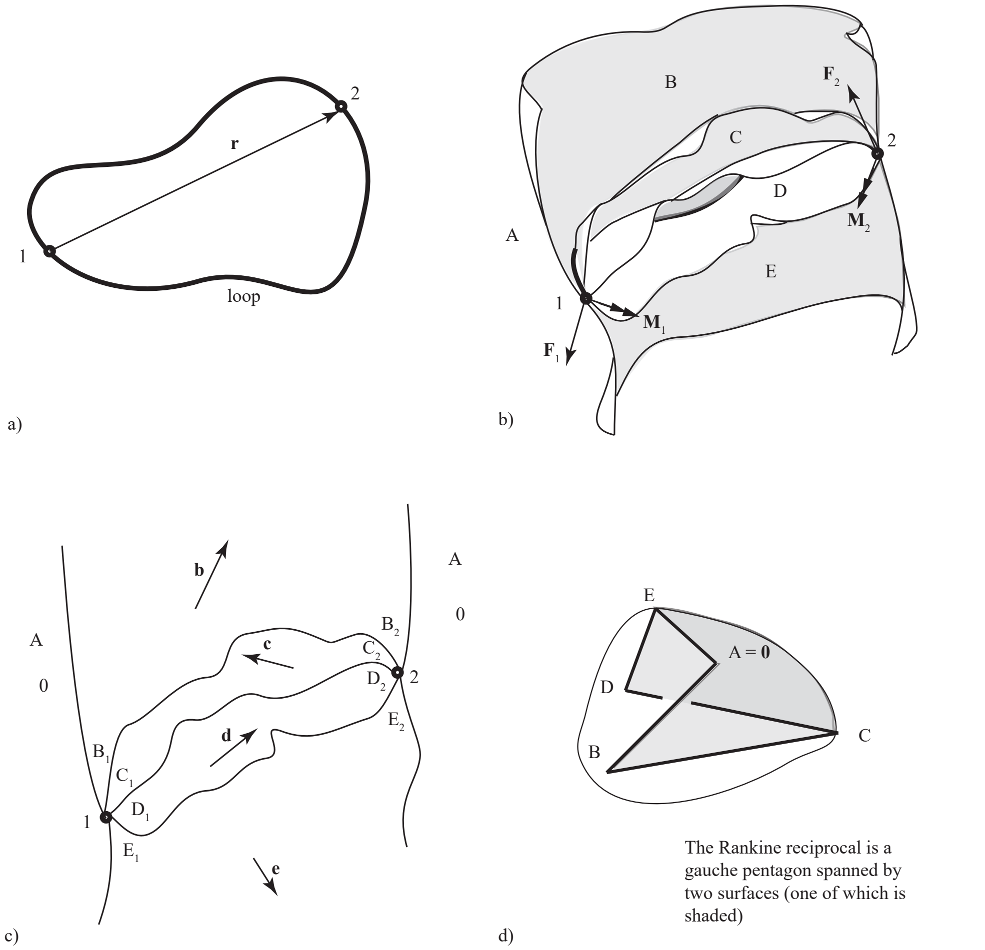

Figure 6: a) A general loop. b) A section giving two cut ends 1 and 2. c) A general stress function over 5 cells for which the loop is a common boundary. d) The Rankine reciprocal is a gauche pentagon.

Fig. 6 shows a general loop which is the common edge of five general cells A-E. Each cell shares a surface only with its two cyclically adjacent cells. Consider points 1 and 2 on the loop separated by a vector $\mathbf{r}$. Without loss of generality we let the stress function be zero on cell A. The stress resultants at the cut 1 are given by

$$2\mathbf{F}_1 = \mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{d}+\mathbf{d}\times\mathbf{e} \tag{19}$$

$$2\mathbf{M}_1 = -C_1\mathbf{b} + (B_1 - D_1)\mathbf{c} + (C_1 - E_1)\mathbf{d} + D_1\mathbf{e} \tag{20}$$

and at the cut 2 are

$$\begin{aligned} 2\mathbf{F}_2 &= -\mathbf{b}\times\mathbf{c} - \mathbf{c}\times\mathbf{d} - \mathbf{d}\times\mathbf{e} & (21)\\ 2\mathbf{M}_2 &= C_2\mathbf{b} - (B_2 - D_2)\mathbf{c} - (C_2 - E_2)\mathbf{d} - D_2\mathbf{e} & (22)\end{aligned}$$

Clearly $\mathbf{F}_1 = -\mathbf{F}_2$, thus we have force equilibrium. Taking moments about the cut 1, global moment equilibrium requires

$$\mathbf{M}_1 + \mathbf{M}_2 + \mathbf{r}\times\mathbf{F}_2 = 0 \tag{23}$$

.

Now, $B_2 - B_1 = \mathbf{r}.\mathbf{b}$, etc., thus the moment equations add to give

$$2(\mathbf{M}_1 + \mathbf{M}_2) = (\mathbf{r}.\mathbf{c})\mathbf{b} - (\mathbf{r}.\mathbf{b})\mathbf{c} + (\mathbf{r}.\mathbf{d})\mathbf{c} - (\mathbf{r}.\mathbf{c})\mathbf{d} + (\mathbf{r}.\mathbf{e})\mathbf{d} - (\mathbf{r}.\mathbf{d})\mathbf{e} \tag{24}$$

Taking the right hand terms pairwise and using the vector triple product identity that $\mathbf{r}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{r}.\mathbf{c})\mathbf{b} - (\mathbf{r}.\mathbf{b})\mathbf{c}$ we obtain

$$\begin{aligned} 2(\mathbf{M}_1 + \mathbf{M}_2) &= \mathbf{r}\times(\mathbf{b}\times\mathbf{c}) + \mathbf{r}\times(\mathbf{c}\times\mathbf{d}) + \mathbf{r}\times(\mathbf{d}\times\mathbf{e}) & (25)\\ &= \mathbf{r}\times(2\mathbf{F}_1) = -\mathbf{r}\times(2\mathbf{F}_2) & (26)\\ \text{whence} \quad 2(\mathbf{M}_1 + \mathbf{M}_2 + \mathbf{r}\times\mathbf{F}_2) &= 0 & (27)\end{aligned}$$

That is, we also have moment equilibrium. The stress function thus satisfies global equilibrium. Moreover, the previous section showed that a five-valent bar was sufficient for completeness, thus we can create any loop solution via this representation.

There being five cells, the Rankine reciprocal is a gauche pentagon (Fig. 6d), whose oriented area is given by the sum of cross products. That summation represents a triangulation of the pentagon, but any surface spanning the pentagon has the same oriented area. The latter is the more general perspective, because for example cells A and C do not share a common face, thus there is no reciprocal bar connecting reciprocal nodes A and C. Since there are two sums $\mathbf{F}_1$ and $\mathbf{F}_2$ corresponding to different faces of the cut, there are actually two oriented surfaces spanning the pentagon, whose oriented areas are of equal magnitude but opposite sign. There are thus two reciprocal cells, one interior to the two spanning surfaces and one exterior. Reciprocal to these cells are two nodes on the original diagram, these being nodes 1 and 2. The coordinates of nodes 1 and 2 thus define the gradient of a dual stress function over each cell of the reciprocal diagram. We thus have a pair of reciprocal systems.

More significantly, by abandoning any requirement for planarity of faces or linearity of edges, we appear to have arrived at an extremely general stress function formulation.

5. Clifford Algebra statement

It is evident from the form of the equations that there may exist a succinct formulation of the stress function in terms of four-dimensional Clifford Algebra. Clifford Algebra is also known as Geometric Algebra, suggesting that it may form a suitable language for a theory based heavily on geometry [6].

Consider a four-dimensional vector space spanned by orthogonal unit vectors $\mathbf{e}_0$, $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$. Let a four-vector $\mathbf{A}$ represent the stress function adjacent to a bar according to

$$\mathbf{A} = A\mathbf{e}_0 + a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 = A\mathbf{e}_0 + \mathbf{a} \tag{28}$$

with A being the value of the stress function adjacent to the bar, and $\mathbf{a}$ being the gradient.

The framework of 4D Clifford algebra consists of 1 scalar, 4 vectors, 6 bivectors, 4 trivectors and one quadrivector (the pseudoscalar or 4-volume). This is the natural language of the 4D projective geometry that underpins our generalised Rankine reciprocity. By expressing the stress function as a 4-vector, the 6D space of bivectors (oriented areas) becomes available for representing the 6 components of the stress resultants.

The wedge product of two vectors $\mathbf{A} \wedge \mathbf{B}$ is the oriented area obtained by sweeping $\mathbf{B}$ along $\mathbf{A}$, and the stress function may be succinctly stated as the cyclic sum

$$\mathbf{R} = \frac{1}{2}\sum_i \mathbf{A}_i \wedge \mathbf{A}_{i+1} \tag{29}$$

where the stress resultant $\mathbf{R} = \mathbf{F} + \mathbf{e}_0\mathbf{M}$ contains a force bivector $\mathbf{F}$ and a moment vector $\mathbf{M}$.

In this 4D setting, a number of key results emerge as simple consequences of the geometry. For example, in the Maxwell construction for 2D trusses, the reciprocal force vectors of bars meeting at a node naturally form a *closed* polygon, and this closure is the geometric expression of nodal force equilibrium. Similarly, in the Rankine construction for 3D trusses, the reciprocal force polygons of bars meeting at a node form a *closed* polyhedron. In our own description here, reciprocal to each bar is a more general polygon, which is possibly gauche, and whose vertices are points in the 4D space (and which need not be restricted to some 3D subspace). Nevertheless, the force polygons define generalised polyhedra in 4D which are also closed, and it is this simple geometric fact that ensures that the stress function of this paper guarantees force and moment equilibrium at every node.

For further information on the full 4D description, the reader is referred to McRobie [1], where a number of practical applications are given (such as the analysis of a loaded gridshell roof).

6. Summary and Conclusions

A stress function has been defined which is capable of representing the stress resultants in self-stressed 3D frames. It was presented in traditional vector terms via the defining equations 2 and 3. Issues of incompleteness were discussed, and some potential solutions were proposed (such as requiring bars to be the common edge of at least five cells). Section 8 highlighted the important feature that the stress function equations are independent of structural geometry, leading to a purely topological theory. This opens access to previously problematic areas such as gauche polygons, curved bars and cells bounded by irregular, nonplanar surfaces. Finally, the paper showed how the vector description could be succinctly encapsulated in the language of 4D Clifford Algebra.

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