# Existence of Mori fibre spaces for 3-folds in char $p$

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Abstract. We prove the following results for projective klt pairs of dimension 3 over an algebraically closed field of characteristic $p > 5$: the cone theorem, the base point free theorem, the contraction theorem, finiteness of minimal models, termination with scaling, existence of Mori fibre spaces, etc.

## Contents



*Date*: March 27, 2017.
2010 MSC: 14E30.

## 1. Introduction

We work over an algebraically closed field $k$ (mostly of char $p > 5$). Boundary divisors are always assumed to be with real coefficients unless otherwise stated.

After [12], many results concerning the log minimal model program (LMMP) for 3-folds over $k$ of characteristic $p > 5$ were settled in [3] including the existence of log flips and log minimal models, special cases of the base point free and contraction theorems, special cases of Kollár-Shokurov connectedness principle, existence of $\mathbb{Q}$-factorial dlt models, ACC for log canonical thresholds, etc. One of the main problems not treated in [3] is the existence of Mori fibre spaces. Their existence is proved in this paper. We also settle various other problems that are discussed below.

**Cone theorem.** In characteristic 0, the base point free theorem and the cone and contraction theorems are among the first results of the LMMP to be proved. They are derived from the Kawamata-Viehweg vanishing theorem. But the story in positive characteristic is quite different because such vanishing theorems fail. Unlike in characteristic 0, existence of flips, minimal models is a fundamental ingredient of the proof of the cone and contraction and base point free theorems.

**Theorem 1.1.** *Let $(X, B)$ be a projective $\mathbb{Q}$-factorial dlt pair of dimension 3 over $k$ of characteristic $p > 5$. Then there is a countable number of rational curves $\Gamma_i$ such that*

*(i)* $\overline{NE}(X) = \overline{NE}(X)_{K_X+B\geq 0} + \sum_i \mathbb{R}_{\geq 0}[\Gamma_i]$,
*(ii)* $-6 \leq (K_X + B) \cdot \Gamma_i < 0$,
*(iii) for each ample $\mathbb{R}$-divisor $A$, only finitely many of the rays $\mathbb{R}_{\geq 0}[\Gamma_i]$ are contained in $\overline{NE}(X)_{K_X+B+A<0}$, and*
*(iv) the rays $\mathbb{R}_{\geq 0}[\Gamma_i]$ do not accumulate in $\overline{NE}(X)_{K_X+B<0}$.*

The theorem is proved in Section 3 where we also prove some other results concerning extremal rays (see Proposition 3.8). Special cases of the theorem were proved in [14, Proposition 0.6][7, Theorem 1.7].

**Base point freeness.** The proof of the next result, given in Section 9, relies on the results in Sections 3 to 8. The whole proof occupies a big chunk of this paper and it contains many of our key ideas and technical results.

**Theorem 1.2.** *Let $(X, B)$ be a projective klt pair of dimension 3 over $k$ of characteristic $p > 5$ and $X \to Z$ a projective contraction. Assume that $D$ is an $\mathbb{R}$-divisor such that $D$ is nef/$Z$ and $D - (K_X + B)$ is nef and big/$Z$. Then $D$ is semi-ample/$Z$.*

The theorem was proved in [3][22] when $D$ is a big $\mathbb{Q}$-divisor using existence of minimal models and Keel's semi-ampleness techniques. When $D$ is not big, Keel's methods do not apply, at least not directly. To deal with this issue, in

[7], a canonical bundle formula is used to reduce the problem to surfaces when $D$ is a $\mathbb{Q}$-divisor with numerical dimension $\nu(D) = 2$ and assuming that $B + A$ is a $\mathbb{Q}$-boundary with coefficients $> \frac{2}{p}$ for some $0 \leq A \sim_{\mathbb{R}} D - (K_X + B)$. The canonical bundle formula is derived from the theory of moduli of pointed curves. Our proof is very different and it does not involve canonical bundle formulas.

**Contraction theorem.** The next result is a consequence of the base point free theorem and the cone theorem above.

**Theorem 1.3.** *Let $(X, B)$ be a projective $\mathbb{Q}$-factorial dlt pair of dimension 3 over $k$ of characteristic $p > 5$ and $X \to Z$ a projective contraction. Then any $K_X + B$-negative extremal ray/$Z$ can be contracted by a projective contraction.*

The proof is given in Section 9. The theorem was proved in [3, Theorem 1.5]and [22] for extremal rays of flipping or divisorial type.

**Finiteness of minimal models.** We prove finiteness of minimal models under suitable assumptions and derive termination with scaling. This is similar to the characteristic 0 case [6].

First we introduce some notation. Let $X \to Z$ be a projective contraction of normal projective varieties over $k$ of characteristic $p > 5$ where $X$ is $\mathbb{Q}$-factorial of dimension 3. Let $A \geq 0$ be a $\mathbb{Q}$-divisor on $X$, and $V$ a rational finite dimensional affine subspace of the space of $\mathbb{R}$-Weil divisors on $X$. Define

$$\mathcal{L}_A(V) = \{\Delta \mid 0 \leq (\Delta - A) \in V, \text{ and } (X, \Delta) \text{ is lc}\}.$$

As in [20, 1.3.2], one can show that $\mathcal{L}_A(V)$ is a rational polytope inside the rational affine space $A+V$, that is, it is the convex hull of finitely many rational points in $A + V$: this follows from existence of log resolutions.

**Theorem 1.4.** *Under the above setting, assume in addition that $A$ is big/$Z$. Let $\mathcal{C} \subseteq \mathcal{L}_A(V)$ be a rational polytope such that $(X, \Delta)$ is klt for every $\Delta \in \mathcal{C}$. Then there are finitely many birational maps $\phi_i \colon X \dashrightarrow Y_i/Z$ such that for any $\Delta \in \mathcal{C}$ with $K_X + \Delta$ pseudo-effective/$Z$, there is an $i$ such that $(Y_i, \Delta_{Y_i})$ is a log minimal model of $(X, \Delta)$ over $Z$.*

As usual $\Delta_{Y_i}$ means the pushdown $(\phi_i)_*\Delta$. A conditional proof of the theorem is given in Section 4. At the end in Section 9 the extra assumptions are removed.

**Termination with scaling.** Minimal models were constructed in [3] by a rather indirect approach. It is useful in many situations to know that running an LMMP ends up with a minimal model. It is even more important for constructing Mori fibre spaces.

**Theorem 1.5.** *Let $(X, B+C)$ be a $\mathbb{Q}$-factorial projective klt pair of dimension 3 over $k$ of characteristic $p > 5$ and $X \to Z$ a projective contraction. Assume that $B \geq 0$ is big/$Z$, $C \geq 0$ is $\mathbb{R}$-Cartier, and $K_X + B + C$ is nef/$Z$. Then we can run the LMMP/$Z$ on $K_X + B$ with scaling of $C$ and it terminates.*

**Theorem 1.6.** *Let $(X, B+C)$ be a $\mathbb{Q}$-factorial projective klt pair of dimension 3 over $k$ of characteristic $p > 5$ and $X \to Z$ a projective contraction. Assume $C \geq 0$ is ample, and $K_X + B + C$ is nef/$Z$. Then we can run the LMMP/$Z$ on $K_X + B$ with scaling of $C$ and it terminates.*

The proofs are given in Section 4 under certain assumptions. Unconditional proofs are in Section 9.

**Mori fibre spaces.** Finally we come to the result which is the title of this paper.

**Theorem 1.7.** *Let $(X, B)$ be a dlt pair of dimension 3 over $k$ of characteristic $p > 5$ and $X \to Z$ a projective contraction. Assume $K_X + B$ is not pseudo-effective/$Z$. Then $(X, B)$ has a Mori fibre space/$Z$. If $X$ is $\mathbb{Q}$-factorial, then we can run an LMMP/$Z$ on $K_X + B$ which ends with a Mori fibre space/$Z$.*

The proof is given at the very end of the paper in Section 9. This theorem combined with [3, Theorem 1.2] says that any klt pair $(X, B)$ of dimension 3 over an algebraically closed field $k$ of characteristic $p > 5$, projective over some base $Z$, either has a log minimal model or a Mori fibre space over $Z$.

**Acknowledgements.** This work was partially supported by a grant of the Leverhulme Trust. Part of this work was done when the first author visited National Taiwan University in August-September 2014 with the support of the Mathematics Division (Taipei Office) of the National Center for Theoretical Sciences, and the visit was arranged by Jungkai A. Chen. He wishes to thank them for their hospitality. We would like to thank Omprokash Das and Hiromu Tanaka for helpful comments. Finally many thanks to the referee for the valuable comments and corrections.

## 2. Preliminaries

All the varieties and algebraic spaces in this paper are defined over an algebraically closed field $k$ unless otherwise stated.

2.1. **Contractions and divisors endowed with a map.** A *contraction* $f\colon X \to Z$ of algebraic spaces over $k$ is a proper morphism such that $f_*\mathcal{O}_X = \mathcal{O}_Z$. When $X, Z$ are quasi-projective varieties over $k$ and $f$ is projective, we refer to $f$ as a *projective contraction* to avoid confusion. In this case, by a *fibre* of $f$ we always mean a scheme-theoretic fibre unless stated otherwise.

For a nef $\mathbb{Q}$-divisor $L$ on a projective scheme $X$ over $k$, the *exceptional locus* $\mathbb{E}(L)$ is the union of those positive-dimensional integral subschemes $Y \subseteq X$ such that $L|_Y$ is not big, i.e. $(L|_Y)^{\dim Y} = 0$. We say $L$ is *endowed with a map* $f\colon X \to V$, where $V$ is an algebraic space over $k$ and $f$ is a proper morphism, if an integral subscheme $Y$ is contracted by $f$ (i.e. $\dim Y > \dim f(Y)$) if and only if $L|_Y$ is not big.

2.2. **Rational maps.**

**Lemma 2.3.** *Let $\phi\colon X \dashrightarrow Y$ be a birational map between normal projective varieties over $k$. Assume $\phi^{-1}$ does not contract divisors. Let $D$ be a nef $\mathbb{R}$-divisor on $X$ such that $D_Y = \phi_* D$ is $\mathbb{R}$-Cartier. Let $f\colon W \to X$ and $g\colon W \to Y$ be a common resolution. Then, $E := g^*D_Y - f^*D$ is effective and exceptional/$Y$.*

*Proof.* It is obvious that $E$ is exceptional/$Y$. The effectivity is a consequence of the negativity lemma.

□

**Lemma 2.4.** *Let $\phi\colon X \dashrightarrow Y$ be a birational map between normal projective varieties over $k$. Assume $\phi^{-1}$ does not contract divisors. Then there is an open subset $U \subseteq X$ such that $\phi|_U$ is an isomorphism and codimension of $Y \setminus \phi(U)$ is at least 2.*

*Proof.* Let $U \subseteq X$ be the largest open set such that $\phi|_U$ is an isomorphism. Assume that codimension of $Y \setminus \phi(U)$ is one and let $S$ be one of its components of codimension one. Since $\phi^{-1}$ does not contract divisors, $\phi^{-1}$ is an isomorphism near the generic point of $S$. This means that there is an open set $V \subseteq X$ intersecting the birational transform of $S$ such that $\phi|_V$ is an isomorphism. But then $\phi|_{U\cup V}$ is an isomorphism which contradicts the maximality of $U$.

□

2.5. **Resolution of singularities.** Let $X$ be a quasi-projective variety of dimension at most 3 over an algebraically closed field $k$ of characteristic $p > 0$. Let $P \subset X$ be a closed subset. Assume that there is an open set $U \subset X$ such that $P \cap U$ is a divisor with simple normal crossing (snc) singularities. Then there is a log resolution of $(X, P)$ which is an isomorphism over $U$, that is, there is a projective birational morphism $f : Y \to X$ such that the union of the exceptional locus of $f$ and the birational transform of $P$ is an snc divisor, and $f$ is an isomorphism over $U$. This follows from [11] when $k$ has characteristic $p > 5$, and from [9] and [10] in general.

2.6. **Pairs.** A *pair* $(X, B)$ consists of a normal quasi-projective variety $X$ over $k$ and an $\mathbb{R}$-*boundary* $B$, that is an $\mathbb{R}$-divisor $B$ on $X$ with coefficients in $[0, 1]$, such that $K_X + B$ is $\mathbb{R}$-Cartier. When $B$ has rational coefficients we say $B$ is a $\mathbb{Q}$-*boundary.* We say that $(X, B)$ is *log smooth* if $X$ is smooth and $\operatorname{Supp} B$ has simple normal crossing singularities.

Let $(X, B)$ be a pair. For a prime divisor $D$ on some birational model of $X$ with a nonempty centre on $X$, $a(D, X, B)$ denotes the *log discrepancy.* This is defined by taking a projective birational morphism $f\colon Y \to X$ from a normal variety containing $D$ as a prime divisor and putting $a(D, X, B) = 1 - b$ where $b$ is the coefficient of $D$ in $B_Y$ and $K_Y + B_Y = f^*(K_X + B)$.

As in characteristic 0, we can define various types of singularities using log discrepancies. Let $(X, B)$ be a pair. We say that the pair is *log canonical* or lc for short (resp. *Kawamata log terminal* or klt for short) if $a(D, X, B) \geq 0$ (resp. $a(D, X, B) > 0$) for any prime divisor $D$ over $X$, that is, on birational models of $X$. An *lc centre* of $(X,B)$ is the image in $X$ of a $D$ with $a(D,X,B)=0$. On the other hand, we say that $(X,B)$ is *dlt* if there is a closed subset $P\subset X$ such that $(X,B)$ is log smooth outside $P$ and no lc centre of $(X,B)$ is contained in $P$. In particular, the lc centres of $(X,B)$ are exactly the components of $S_1\cap\cdots\cap S_r$ where $S_i$ are among the components of $\lfloor B\rfloor$. Moreover, there is a log resolution $f\colon Y\to X$ of $(X,B)$ such that $a(D,X,B)>0$ for any prime divisor $D$ on $Y$ which is exceptional$/X$, e.g. take a log resolution $f$ which is an isomorphism over $X\setminus P$. Finally, we say that $(X,B)$ is *plt* if it is dlt and each connected component of $\lfloor B\rfloor$ is irreducible. In particular, the only lc centres of $(X,B)$ are the components of $\lfloor B\rfloor$.

2.7. **Minimal models and Mori fibre spaces.** Let $(X,B)$ be a pair and $X\to Z$ a projective contraction over $k$. A pair $(Y,B_Y)$ with a projective contraction $Y\to Z$ and a birational map $\phi\colon X\dashrightarrow Y/Z$ is a *log birational model* of $(X,B)$ if $B_Y$ is the sum of the birational transform of $B$ and the reduced exceptional divisor of $\phi^{-1}$. We say that $(Y,B_Y)$ is a *weak lc model* of $(X,B)$ over $Z$ if in addition:

(1) $K_Y+B_Y$ is nef$/Z$, and
(2) for any prime divisor $D$ on $X$ which is exceptional$/Y$, we have
$$a(D,X,B)\le a(D,Y,B_Y).$$

And we call $(Y,B_Y)$ a *log minimal model* of $(X,B)$ over $Z$ if in addition:

(3) $(Y,B_Y)$ is $\mathbb{Q}$-factorial dlt, and
(4) the inequality in (2) is strict.

A weak lc model or log minimal model $(Y,B_Y)$ is said to be *good* if $K_Y+B_Y$ is semi-ample$/Z$. When $K_X+B$ is big$/Z$, the *lc model* of $(X,B)$ over $Z$ is a weak lc model $(Y,B_Y)$ of $(X,B)$ over $Z$ with $K_Y+B_Y$ ample$/Z$.

On the other hand, a log birational model $(Y,B_Y)$ of $(X,B)$ is called a *Mori fibre space* of $(X,B)$ over $Z$ if there is a $K_Y+B_Y$-negative extremal projective contraction $Y\to T/Z$ with $\dim Y>\dim T$, and if for any prime divisor $D$ over $X$ we have
$$a(D,X,B)\le a(D,Y,B_Y)$$
with strict inequality if $D$ is a divisor on $X$ which is exceptional over $Y$.

Note that the above definitions are the same as in [3] but slightly different from the traditional definitions in that we allow $\phi^{-1}$ to contract divisors. However, if $(X,B)$ is plt (hence also klt) the definitions coincide. Actually in this paper we usually deal with models such that $\phi^{-1}$ does not contract divisors.

Let $(X,B)$ be an lc pair over $k$. A $\mathbb{Q}$-factorial dlt pair $(Y,B_Y)$ is a *$\mathbb{Q}$-factorial dlt model* of $(X,B)$ if there is a projective birational morphism $f\colon Y\to X$ such that $K_Y+B_Y=f^*(K_X+B)$ and such that every exceptional prime divisor of $f$ has coefficient 1 in $B_Y$.

One of the fundamental ingredients of this paper is the following result which was proved after the developments in [12].

**Theorem 2.8** ([3])**.** *Let $(X,B)$ be a klt pair of dimension 3 over $k$ of characteristic $p>5$ and $X\to Z$ a projective contraction. If $K_X+B$ is pseudo-effective over $Z$, then $(X,B)$ has a log minimal model over $Z$.*

2.9. **LMMP with scaling.** Let $(X,B+C)$ be an lc pair and $X\to Z$ a projective contraction over $k$ such that $K_X+B+C$ is nef$/Z$, $B\geq 0$, and $C\geq 0$ is $\mathbb{R}$-Cartier. Suppose that either $K_X+B$ is nef$/Z$ or there is an extremal ray $R/Z$ such that $(K_X+B)\cdot R<0$ and $(K_X+B+\lambda_1 C)\cdot R=0$ where

$$\lambda_1:=\inf\{t\geq 0\mid K_X+B+tC \text{ is nef}/Z\}.$$

If $R$ defines a Mori fibre structure, we stop. Otherwise assume that $R$ gives a divisorial contraction or a log flip $X\dashrightarrow X'$. We can now consider $(X',B'+\lambda_1 C')$ where $B'+\lambda_1 C'$ is the birational transform of $B+\lambda_1 C$ and proceed similarly. That is, suppose that either $K_{X'}+B'$ is nef$/Z$ or there is an extremal ray $R'/Z$ such that $(K_{X'}+B')\cdot R'<0$ and $(K_{X'}+B'+\lambda_2 C')\cdot R'=0$ where

$$\lambda_2:=\inf\{t\geq 0\mid K_{X'}+B'+tC' \text{ is nef}/Z\}$$

and so on. By continuing this process, we obtain a special kind of LMMP$/Z$ which is called the *LMMP$/Z$ on $K_X+B$ with scaling of $C$*; note that it is not unique. When we speak of running such an LMMP we make sure that all the necessary ingredients exist, e.g. the extremal rays, the contractions of the rays, etc.

2.10. **Nef reduction maps.**

**Theorem 2.11** ([1][7])**.** *Let $X$ be a normal projective variety over an uncountable field $k$, and $L$ a nef $\mathbb{R}$-divisor on $X$. Then there is a rational map $f\colon X\dashrightarrow Z$ and a nonempty open subset $V\subseteq Z$ such that*
- *$f$ is proper over $V$,*
- *$L|_F\equiv 0$ for the very general fibres $F$ of $f$ over $V$, and*
- *if $x\in X$ is a very general point and $C$ a curve passing through $x$, then $L\cdot C=0$ iff $C$ is contained in the fibre of $f$ containing $x$.*

We call $f$ a *nef reduction map* of $L$ and call $\dim Z$ the *nef dimension* of $L$ and denote it by $n(L)$. If $k$ is countable we can define $n(L)$ to be the nef dimension of $L$ after extending $k$ to an uncountable algebraically closed field.

The theorem was proved in [1] for $k$ of characteristic 0 and $L$ a $\mathbb{Q}$-divisor. It was remarked in [7, 2.4] that the proof in [1] also works in characteristic $p>0$ and for $L$ an $\mathbb{R}$-divisor.

In general nef divisors with maximal nef dimension are far from being big or even having non-negative Kodaira dimension. However, nef log divisors behave much better in this sense as the following statement shows.

**Theorem 2.12** ([7][3])**.** *Let $(X,B)$ be a projective pair over $k$ such that $B$ is big and $K_X+B$ is nef. If the nef dimension $n(K_X+B)=\dim X$, then $K_X+B$ is big.*

The theorem was proved in [7, Theorem 1.4] in characteristic $p > 0$. A short proof was given in [3, Theorem 1.11] in any characteristic. The theorem actually also holds if $B$ is not a boundary, i.e. has arbitrary coefficients.

2.13. **The $p > 5$ assumption.** The restriction on the characteristic arises due to the coincidence of klt and strongly $F$-regular surface singularities in characteristic $p > 5$, which is used in [12] to construct flips. It is unknown whether the LMMP works for 3-folds in lower characteristics, or whether this should be expected. There are several phenomena which occur only in characteristics 2,3 and 5. For instance Maddock [17] constructed a smooth 5-fold Mori fibre space of relative dimension 2 in characteristic 2 such that the generic fibre has nonzero irregularity. He then proved that this cannot happen for a threefold over a curve when the characteristic is greater than 3 in [18].

We recall another unusual behaviour in low characteristic. Remember that a smooth sextic double solid is a double cover of $\mathbb{P}^3$ ramified over a smooth sextic surface. In [8, Example 1.5], Cheltsov and Park discuss an example of such a variety which is birational to an elliptic fibration where the characteristic $p = 5$. They remark just before that example that a smooth sextic double solid cannot be birational to an elliptic fibration if $p > 5$.

## 3. Extremal rays and the cone theorem

3.1. **The cone theorem.** In this subsection we will prove Theorem 1.1. First we make some preparations.

We first recall the definition of extremal curves. Let $X$ be a normal projective variety and $H$ a fixed ample Cartier divisor (in practice we do not mention $H$ and assume that it is already fixed). Let $R$ be a ray of $\overline{NE}(X)$. An *extremal curve* for $R$ is a curve $\Gamma$ generating $R$ such that $H \cdot \Gamma \leq H \cdot C$ for any other curve $C$ generating $R$. Let $D$ be an $\mathbb{R}$-Cartier divisor with $D \cdot R < 0$. Then $D \cdot \Gamma \geq D \cdot C$ for any other curve $C$ generating $R$ [3, 3.1].

**Lemma 3.2.** *Let $(X, B)$ be a $\mathbb{Q}$-factorial projective dlt pair of dimension 3 over $k$ of characteristic $p > 5$. Suppose that $R$ is a $K_X + B$-negative extremal ray such that $N \cdot R = 0$ for some nef and big $\mathbb{Q}$-Cartier divisor $N$. Then $R$ is generated by some rational curve $\Gamma$ with $(K_X + B) \cdot \Gamma \geq -3$.*

*Proof.* Perturbing the coefficients of $B$ and using the above property of extremal curves, we can assume $(X, B)$ is klt and $B$ is a $\mathbb{Q}$-divisor. Since $N \cdot R = 0$ for some nef and big $\mathbb{Q}$-Cartier divisor $N$, by [3, 3.3], there is an ample $\mathbb{Q}$-divisor $A$ such that $L = K_X + B + A$ is nef and big and $L^{\perp} = R$. Moreover, by [3, 1.4 and 1.5], $L$ is semi-ample and $R$ can be contracted via a projective birational contraction $X \to Z$ which is either a flipping or divisorial contraction.

First suppose $X \to Z$ is a divisorial contraction and let $S$ be the contracted divisor. Let $b$ be the coefficient of $S$ in $B$ and let $\Delta = B + (1 - b)S$. By adjunction we can write

$$K_{S^\nu} + \Delta_{S^\nu} \sim_{\mathbb{Q}} (K_X + \Delta)|_{S^\nu}$$

where $S^\nu$ is the normalization of $S$ and $\Delta_{S^\nu} \geq 0$ [14, 5.3]. Let $S' \to S^\nu$ be the minimal resolution and $S' \to V$ the contraction determined by the Stein factorization of $S' \to Z$. Write the pullback of $K_{S^\nu} + \Delta_{S^\nu}$ to $S'$ as $K_{S'} + \Delta_{S'}$. Since $-(K_X + \Delta)$ is ample$/Z$, we can see that $-(K_{S'} + \Delta_{S'})$ is nef and big$/V$. So running an LMMP$/V$ on $K_{S'}$ ends with a Mori fibre space over $V$ which is either $\mathbb{P}^2$ or a $\mathbb{P}^1$-bundle. Either way there is a covering family of rational curves on $S'$ over $V$ such that for the general member $\Gamma_{S'}$ we have $-3 \leq K_{S'} \cdot \Gamma_{S'} < 0$, hence $-3 \leq (K_{S'} + \Delta_{S'}) \cdot \Gamma_{S'} < 0$. Taking the image of the family on $S$ gives a covering family of curves on $S$ over $Z$ such that for a general member $\Gamma$ we have

$$-3 \leq (K_{S'} + \Delta_{S'}) \cdot \Gamma_{S'} = (K_X + \Delta) \cdot \Gamma < (K_X + B) \cdot \Gamma < 0$$

so we are done in the divisorial case.

Now assume that $X \to Z$ is a flipping contraction and let $X \dashrightarrow X^+/Z$ be its log flip which exists by [3, Theorem 1.1]. Let $P^+$ be a sufficiently ample divisor on $X^+$. Let $\phi\colon W \to X$ be a log resolution of $(X, B)$ such that the induced map $\psi\colon W \dashrightarrow X^+$ is a morphism. Let $B_W$ be the sum of the birational transform of $B$ and the reduced exceptional divisor of $\phi$, let $A_W$ be the pullback of $A$, and let $P_W$ be the pullback of $P^+$. Since $(X, B)$ is klt,

$$K_W + B_W + A_W = \phi^*(K_X + B + A) + E_W \equiv E_W/Z$$

where $E_W$ is effective and its support is equal to the support of the reduced exceptional divisor of $\phi$.

We run an LMMP$/X$ on $K_W + B_W + A_W + P_W$ as follows. Assume $R_W$ is a $K_W + B_W + A_W + P_W$-negative extremal ray$/Z$. Then $(K_W + B_W + A_W) \cdot R_W < 0$ as $P_W$ is nef, hence $E_W \cdot R_W < 0$. Similarly, $(K_W + B_W) \cdot R_W < 0$ as $A_W$ is the pullback of $A$. By [3, Theorem 1.5], $R_W$ can be contracted via a projective morphism $\theta\colon W \to V$ and all the curves generating $R_W$ are contained in $\operatorname{Supp} E_W$. Therefore, by restricting to a suitable component $S$ of $\lfloor B_W \rfloor$ and applying the cone theorem on surfaces relative to the morphism $\theta|_S$, we can find a curve $\Gamma_W$ generating $R_W$ such that

$$-3 \leq (K_W + B_W + A_W) \cdot \Gamma_W < 0.$$

In particular, $P_W \cdot \Gamma_W \leq 3$. Since $P_W$ is the pullback of a sufficiently ample divisor, it is necessary to have $P_W \cdot \Gamma_W = 0$. Therefore, $R_W$ is contracted over $X^+$. We can apply similar arguments after taking the divisorial contraction or log flip of $R_W$, hence repeating the process we get an LMMP such that $P_W$ intersects every extremal ray in the process trivially. Moreover, by special termination [3, Proposition 5.5], the LMMP terminates with a model $Y/X$ and the induced map $Y \dashrightarrow X^+$ is a morphism. Denote $Y \to X$ and $Y \to X^+$ by $\alpha$ and $\beta$ respectively.

By construction,

$$K_Y + B_Y + A_Y \equiv E_Y/Z$$

where $E_Y$ is effective and its support is equal to the support of the reduced exceptional divisor of $\alpha$, which is equal to the reduced exceptional divisor of $\beta$. Obviously, $\beta$ is not an isomorphism, hence it contracts some divisor as $X^+$ is

$\mathbb{Q}$-factorial. Thus $E_Y \neq 0$, and by the negativity lemma, $K_Y + B_Y + A_Y + P_Y$ is not nef$/X^+$. On the other hand, if $a \gg 0$, then $K_Y + B_Y + aA_Y + P_Y$ is nef$/Z$ because arguing as in the last paragraph we know that any $K_Y + B_Y + A_Y + P_Y$-negative extremal ray$/Z$ is generated by a curve $C$ with

$$-3 \leq (K_Y + B_Y + A_Y + P_Y) \cdot C.$$

By construction of $Y$, $C$ cannot be contracted over $X$ and so $A_Y \cdot C > 0$. So we have shown that $K_Y + B_Y + A_Y$ is not nef$/X^+$ but $K_Y + B_Y + aA_Y$ is nef$/X^+$ for any $a \gg 0$.

Let $\lambda$ be the smallest number such that $K_Y + B_Y + \lambda A_Y$ is nef$/X^+$. Note that $\lambda > 1$ since $K_Y + B_Y + A_Y$ is not nef$/X^+$. Now by [3, 3.4] there is an extremal ray $R_Y/X^+$ such that $(K_Y + B_Y) \cdot R_Y < 0$ but $(K_Y + B_Y + \lambda A_Y) \cdot R_Y = 0$. By construction, $E_Y \cdot R_Y < 0$, so there is a rational curve $\Gamma_Y$ generating $R_Y$ such that $-3 \leq (K_Y + B_Y) \cdot \Gamma_Y$. Thus $A_Y \cdot \Gamma_Y \leq 3$ as $(K_Y + B_Y + A_Y) \cdot \Gamma_Y < 0$. Since $\Gamma_Y$ is contracted over $X^+$, $P_Y \cdot \Gamma_Y = 0$. Since $K_Y + B_Y + A_Y + P_Y$ is nef$/X$, $\Gamma_Y$ is not contracted over $X$. Let $\Gamma \subset X$ be the image of $\Gamma_Y$. Then $\Gamma$ generates $R$ and $A \cdot \Gamma \leq A_Y \cdot \Gamma_Y \leq 3$. Therefore, $-3 \leq (K_X + B) \cdot \Gamma$.

□

**Lemma 3.3.** *Let $(X, B)$ be a $\mathbb{Q}$-factorial projective dlt pair of dimension 3 over $k$ of characteristic $p > 5$ such that $B$ is a $\mathbb{Q}$-boundary and $K_X + B$ is not nef. Then there is a natural number $n$ depending only on $(X, B)$ such that if $H$ is an ample Cartier divisor and*

$$\lambda = \min\{t \mid K_X + B + tH \text{ is nef}\}$$

*then $\lambda = \frac{n}{m}$ for some natural number $m$. Moreover, there is a rational curve $\Gamma$ such that*

$$-6 \leq (K_X + B) \cdot \Gamma \leq 0 \text{ and } (K_X + B + \lambda H) \cdot \Gamma = 0.$$

*Proof.* Let $I$ be a natural number so that $I(K_X + B)$ is Cartier. Fix an ample Cartier divisor $H$ and let $\lambda$ be as in the statement of the lemma. It is enough to show that there is a rational curve $\Gamma$ satisfying the last claim of the proposition because then $\lambda = \frac{-I(K_X+B)\cdot\Gamma}{IH\cdot\Gamma}$ so taking $n = (6I)!$ we can write $\lambda = \frac{n}{m}$ for some natural number $m$.

Now assume that $L = K_X + B + \lambda H$ is big. Then by [3, 3.4], there is an extremal ray $R$ such that $L \cdot R = 0$. Moreover, by [3, 3.3], $R$ is generated by some curve, hence $\lambda$ is rational and $L$ is $\mathbb{Q}$-Cartier. Since $L$ is nef and big, we can apply Lemma 3.2.

From now on we assume $L$ is not big. By extending $k$ we can assume it is uncountable. By Theorem 2.12, the nef dimension of $L$ is at most 2 and there is a nef reduction map $f\colon X \dashrightarrow Z$ for $L$. Recall that the map $f$ is regular and proper over some open subset $V \subseteq Z$. For the moment assume that $\dim Z > 0$. Let $\phi\colon W \to X$ be a resolution so that $h\colon W \dashrightarrow Z$ is a morphism. Let $P$ be a general effective Cartier divisor on $Z$ intersecting $V$ and let $G$ be its pullback to $X$, that is, $G = \phi_* h^* P$. Let $S$ be a component of $G$ whose generic point maps into $V$ and such that $S$ is not a component of $\lfloor B \rfloor$, and let $Q$ be the image of $S$

on $Z$. There is a rational number $s > 0$ such that the coefficient of $S$ in $B+sG$ is 1. Let $\Theta = B + sG$ and let $S^\nu$ be the normalization of $S$ (note that $\Theta$ is not necessarily a boundary). By adjunction, we can write

$$K_{S^\nu} + \Theta_{S^\nu} \sim_{\mathbb{Q}} (K_X + \Theta)|_{S^\nu}$$

for some $\Theta_{S^\nu} \geq 0$. Let $H_{S^\nu} = H|_{S^\nu}$. Then $K_{S^\nu} + \Theta_{S^\nu} + \lambda H_{S^\nu}$ is numerically trivial over the generic point of $V_Q := V \cap Q$. Let $T \to S^\nu$ be the minimal resolution of $S^\nu$. Then the pullback of $K_{S^\nu} + \Theta_{S^\nu} + \lambda H_{S^\nu}$ to $T$ can be written as $K_T + \Theta_T + \lambda H_T$ where $H_T$ is the pullback of $H_{S^\nu}$. Let $T^0$ be the open subset of $T$ which is the pre-image of $S \cap f^{-1}(V)$. Then $T^0 \to V_Q$ is a projective morphism. Run a $K_{T_0}$-MMP$/V_Q$, which ends with a Mori fibre space because $K_{T_0} \equiv -\Theta_{T_0} - \lambda H_{T_0}$ over the generic point of $V_Q$ and because $H_T$ is big over $V_Q$. The latter follows from the fact that $H|_S$ is ample and $T \to S$ is birational. The Mori fibre space is either $\mathbb{P}^2$ (implying that $V_Q$ was a point), or a $\mathbb{P}^1$-bundle (which can have base $V_Q$, or some curve over $V_Q$ if $V_Q$ is a point). Therefore, there is a covering family of rational curves on $T^0/V_Q$ such that $-3 \leq K_T{\cdot}\Gamma_T < 0$ for the general members $\Gamma_T$ of the family. Since $\Theta_T \cdot \Gamma_T \geq 0$ and $H_T \cdot \Gamma_T > 0$, we get $-3 \leq (K_T + \Theta_T) \cdot \Gamma_T < 0$ for the very general members $\Gamma_T$. Taking the image of the family on $X$ we get a covering family of rational curves of $S/V_Q$ such that

$$-3 \leq (K_T + \Theta_T) \cdot \Gamma_T = (K_X + \Theta) \cdot \Gamma = (K_X + B) \cdot \Gamma < 0$$

for the very general members $\Gamma$ of the family. Note that by construction, $L \cdot \Gamma = 0$.

Finally we treat the case $\dim Z = 0$, that is, when $L \equiv 0$. In this case $-(K_X + B)$ is ample. Let $C$ be a smooth projective curve inside the smooth locus of $X$ such that $B \cdot C \geq 0$. We can obtain such $C$ by cutting $X$ by hypersurface sections. Note that $K_X \cdot C < 0$. Fix a closed point $c \in C$. Now by [15, Chapter II, Theorem 5.8], there is a rational curve $\Gamma$ passing through $c$ such that

$$\lambda H \cdot \Gamma \leq 6 \frac{\lambda H \cdot C}{-K_X \cdot C}$$

Since $B \cdot C \geq 0$, we have $-K_X \cdot C \geq -(K_X + B) \cdot C$, hence

$$\lambda H \cdot \Gamma \leq 6 \frac{\lambda H \cdot C}{-(K_X + B) \cdot C} = 6.$$

But then $-(K_X + B) \cdot \Gamma = \lambda H \cdot \Gamma \leq 6$ as required.

□

*Proof.* (of Theorem 1.1) First we prove that the $K_X + B$-negative extremal rays do not accumulate in $\overline{NE}(X)_{K_X+B<0}$. Assume that there is a sequence $R_i$ of $K_X + B$-negative extremal rays which accumulate to some $K_X + B$-negative ray (not necessarily extremal). Replacing the sequence and perturbing the coefficients of $B$ we can assume $B$ is a $\mathbb{Q}$-boundary. By Lemma 3.3 and [16, Theorem 3.15], there is a collection of rays $\tilde{R}_j$ of $\overline{NE}(X)$ such that

$$\overline{NE}(X) = \overline{NE}(X)_{K_X+B\geq 0} + \sum_j \tilde{R}_j$$

and such that the $\tilde{R}_j$ do not accumulate in $\overline{NE}(X)_{K_X+B<0}$. For each $i$, since $R_i$ is extremal, there is some $j$ such that $R_i = \tilde{R}_j$. Therefore, the $R_i$ cannot accumulate in $\overline{NE}(X)_{K_X+B<0}$, a contradiction.

Next we prove that there are only finitely many $K_X+B+A$-negative extremal rays for any ample $\mathbb{R}$-divisor $A$. Assume that there is an infinite sequence $R_i$ of $K_X+B+A$-negative extremal rays. Replacing the sequence we can assume the limit of $R_i$ exists as a ray, say $R$. By the last paragraph, $(K_X+B+A)\cdot R = 0$. But then $(K_X+B)\cdot R < 0$, hence the $R_i$ are an accumulating sequence of $K_X+B$-negative extremal rays which contradicts the last paragraph.

Now let $R$ be a $K_X+B$-negative extremal ray. We will show that $R$ is generated by a rational curve $\Gamma$ such that $-6 \leq (K_X+B)\cdot\Gamma$. Let $A$ be an ample $\mathbb{R}$-divisor. Pick $\epsilon, \delta > 0$ so that $(K_X+B+(\epsilon+\delta)A)\cdot R < 0$. Since there are only finitely many $K_X+B+\epsilon A$-negative extremal rays, we can find a nef $\mathbb{R}$-divisor $N$ such that $N^\perp = R$. In particular, $R$ is the only $K_X+B+nN+\epsilon A$-negative extremal ray if $n$ is large enough. Now there is an ample $\mathbb{R}$-divisor $A'$ with sufficiently small coefficients and supported on $\mathrm{Supp}(B+nN+\epsilon A)$ so that there is a $\mathbb{Q}$-boundary $B' \sim_{\mathbb{Q}} B+nN+\epsilon A+A'$ with $(X,B')$ dlt and $(K_X+B'+\delta A)\cdot R < 0$. Therefore, by Lemma 3.3, we can find an ample $\mathbb{Q}$-divisor $H'$ so that $L' = K_X+B'+H'$ is nef and $L'^\perp = R$. Moreover, by Lemma 3.2 (if $L'$ is big) and by Lemma 3.3 (if $L'$ is not big), there is a rational curve $\Gamma$ generating $R$ such that $-6 \leq (K_X+B')\cdot\Gamma$. From $(K_X+B'+\delta A)\cdot\Gamma < 0$, we deduce that $\delta A\cdot\Gamma \leq 6$, hence $\Gamma$ belongs to a bounded family of curves, so there are only finitely many possibilities for $(K_X+B)\cdot\Gamma$. Moreover, by choosing $\epsilon$ and the coefficients of $A'$ to be small enough, we can assume $(A'+\epsilon A)\cdot\Gamma$ is sufficiently small. On the other hand, $-6-(A'+\epsilon A)\cdot\Gamma \leq (K_X+B)\cdot\Gamma$. Therefore, $-6 \leq (K_X+B)\cdot\Gamma$. This completes the proof of the theorem. □

3.4. **Lifting curves birationally.** Here we prove some results which we will need in the next subsection.

**Lemma 3.5.** *Let $(X,B)$ be a $\mathbb{Q}$-factorial dlt pair of dimension 3 over $k$ of characteristic $p>5$. Assume $f\colon X\to Z$ is a $K_X+B$-negative extremal birational contraction such that $-S$ is ample/$Z$ for some component $S$ of $\lfloor B\rfloor$. Let $C$ be a curve on $Z$. Then there is a curve $D$ on $X$ such that the induced morphism $D\to Z$ maps $D$ birationally onto $C$.*

*Proof.* If $C$ is not contained in the image of the exceptional locus of $f$, which is always the case for a flipping contraction, then the statement is clear. So we can assume $S$ is contracted and mapped onto $C$. By [12][3, Lemma 5.2], $S$ is normal. Since $f$ has connected fibres and since all the positive dimensional fibres are contained in $S$, the fibres of $S\to C$ are also connected. Let $S\to C'$

be the contraction given by the Stein factorization of $S \to C$. Then $C' \to C$ is the normalization of $C$ and it is birational.

By adjunction, we can write $K_S + B_S \sim_{\mathbb{Q}} (K_X + B)|_S$ where $(S, B_S)$ is dlt. Moreover, $-(K_S + B_S)$ is ample$/C'$, hence $-K_S$ is big$/C'$. Let $U$ be an open subset of $S$ such that $U$ is smooth and $U \to C'$ is proper over its image, say $V$. Running an LMMP on $K_U$ ends with a $\mathbb{P}^1$-bundle $T \to V$. In particular, there is a curve $D_T$ on $T$ which maps birationally onto $V$. Now let $D \subset S$ be the birational transform of $D_T$. Then $D$ maps birationally onto $C$.

□

**Lemma 3.6.** *Let $(X, B)$ be a klt pair of dimension 3 over $k$ of characteristic $p > 5$, and $C$ a curve on $X$. Let $\phi \colon W \to X$ be a log resolution of $(X, B)$. Then there is a curve $D$ on $W$ such that the induced map $D \to X$ maps $D$ birationally onto $C$.*

*Proof.* Let $B_W$ be the sum of the birational transform of $B$ and the reduced exceptional divisor of $\phi$. Then $K_W + B_W = \phi^*(K_X + B) + E$ where $E$ is effective and its support is equal to the reduced exceptional divisor of $\phi$. Run an LMMP$/X$ on $K_W + B_W$ which is also an LMMP on $E$ whose support is contained in $\lfloor B_W \rfloor$. By special termination [3, Proposition 5.5], the LMMP ends with a model $Y/X$, and by the negativity lemma $E$ is contracted, hence $Y \to X$ is a small contraction. There is a curve $D_Y$ on $Y$ mapping birationally onto $C$. Let $W_i \dashrightarrow W_{i+1}/Z_i$ be a step of the LMMP which is either a flip or a divisorial contraction with $W_{i+1} = Z_i$. Assume we have already found a curve $D_{i+1}$ on $W_{i+1}$ mapping birationally onto $C$. In the divisorial contraction case, apply Lemma 3.5 to find a curve $D_i$ on $W_i$ mapping birationally onto $D_{i+1}$, hence mapping birationally onto $D$. In the flip case, apply Lemma 3.5 to find a curve $D_i$ on $W_i$ mapping birationally onto the image of $D_{i+1}$ on $Z_i$, so it also maps birationally onto $C$. So inductively we can find the required $D$ on $W$.

□

3.7. **Polytopes of boundary divisors.** In this subsection, we study polytopes of divisors. This is similar to the characteristic 0 case as treated in [4, Section 3] but for convenience we reproduce the details.

Let $X$ be a $\mathbb{Q}$-factorial projective klt variety of dimension 3 over $k$ of characteristic $p > 5$. Let $V$ be a finite-dimensional rational affine subspace of the space of Weil $\mathbb{R}$-divisors on $X$. As mentioned in the introduction,

$$\mathcal{L} = \{\Delta \in V \mid (X, \Delta) \text{ is lc}\}$$

is a rational polytope in $V$. By Theorem 1.1, for any $\Delta \in \mathcal{L}$ and any extremal curve $\Gamma$ of an extremal ray $R$ we have $(K_X + \Delta) \cdot \Gamma \geq -6$; note that although $(X, \Delta)$ may not be dlt, we can use the fact that $(X, a\Delta)$ is klt for any $a \in [0, 1)$ and then take the limit over $a$.

Let $B_1, \dots, B_r$ be the vertices of $\mathcal{L}$, and let $m \in \mathbb{N}$ such that $m(K_X + B_j)$ are Cartier. For any $B \in \mathcal{L}$, there are non-negative real numbers $a_1, \dots, a_r$ such that $B = \sum a_j B_j$ and $\sum a_j = 1$. Moreover, for any curve $\Gamma$ on $X$ the intersection number

$$(K_X + B) \cdot \Gamma = \sum a_j (K_X + B_j) \cdot \Gamma$$

is of the form $\sum a_j \frac{n_j}{m}$ for certain $n_1, \dots, n_r \in \mathbb{Z}$. If $\Gamma$ is an extremal curve of an extremal ray, then the $n_j$ satisfy $n_j \geq -6m$.

For an $\mathbb{R}$-divisor $D = \sum d_i D_i$ where the $D_i$ are the irreducible components of $D$, we define $||D|| := \max\{|d_i|\}$.

**Proposition 3.8.** *Let $X$, $V$, and $\mathcal{L}$ be as above, and fix $B \in \mathcal{L}$. Then there are real numbers $\alpha, \delta > 0$, depending only on $(X, B)$ and $V$, such that*

1. *if $\Gamma$ is any extremal curve and if $(K_X+B)\cdot\Gamma > 0$, then $(K_X+B)\cdot\Gamma > \alpha$;*
2. *if $\Delta \in \mathcal{L}$, $||\Delta - B|| < \delta$ and $(K_X + \Delta) \cdot R \leq 0$ for an extremal ray $R$, then $(K_X + B) \cdot R \leq 0$;*
3. *let $\{R_t\}_{t \in T}$ be a family of extremal rays of $\overline{NE}(X)$. Then the set*
$$\mathcal{N}_T = \{\Delta \in \mathcal{L} \mid (K_X + \Delta) \cdot R_t \geq 0 \text{ for any } t \in T\}$$
*is a rational polytope;*
4. *assume $K_X + B$ is nef, $\Delta \in \mathcal{L}$, and that $X_i \dashrightarrow X_{i+1}/Z_i$ is a sequence of $K_X + \Delta$-flips which are $K_X + B$-trivial and $X = X_1$; then for any curve $\Gamma$ on any $X_i$, we have $(K_{X_i} + B_i) \cdot \Gamma > \alpha$ if $(K_{X_i} + B_i) \cdot \Gamma > 0$ where $B_i$ is the birational transform of $B$;*
5. *in addition to the assumptions of (4) suppose that $||\Delta - B|| < \delta$; if $(K_{X_i}+\Delta_i)\cdot R \leq 0$ for an extremal ray $R$ on some $X_i$, then $(K_{X_i}+B_i)\cdot R = 0$ where $\Delta_i$ is the birational transform of $\Delta$.*

*Proof.* Let $B_1, \dots, B_r$ be the vertices of $\mathcal{L}$ and let $m$ be a natural number so that $m(K_X+B_j)$ are all Cartier. Write $B = \sum a_j B_j$ where $a_j \geq 0$ and $\sum a_j = 1$.

(1) If $B$ is a $\mathbb{Q}$-divisor, then the statement is trivially true even if $\Gamma$ is not extremal. If $B$ is not a $\mathbb{Q}$-divisor, then

$$(K_X + B) \cdot \Gamma = \sum a_j (K_X + B_j) \cdot \Gamma$$

and if $(K_X + B) \cdot \Gamma < 1$, then there are only finitely many possibilities for the intersection numbers $(K_X+B_j)\cdot\Gamma$ because $(K_X+B_j)\cdot\Gamma \geq -6$ as $\Gamma$ is extremal, and this in turn implies that there are only finitely many possibilities for the intersection number $(K_X + B) \cdot \Gamma$. So the existence of $\alpha$ is clear.

(2) If the statement is not true then there is an infinite sequence of $\Delta_t \in \mathcal{L}$ and extremal rays $R_t$ such that for each $t$ we have

$$(K_X + \Delta_t) \cdot R_t \leq 0 \text{ but } (K_X + B) \cdot R_t > 0$$

and $||\Delta_t - B||$ converges to 0. There are non-negative real numbers $a_{1,t}, \dots, a_{r,t}$ such that $\Delta_t = \sum a_{j,t} B_j$ and $\sum a_{j,t} = 1$. Since $||\Delta_t - B||$ converges to 0, after replacing the sequence we can choose the $a_j$ and the $a_{j,t}$ so that $a_j = \lim_{t\to\infty} a_{j,t}$ (note that $\mathcal{L}$ is not necessarily a simplex so the numbers $a_j, a_{j,t}$ are not necessarily unique). Moreover, we can assume that the sign of $(K_X+B_j)\cdot R_t$ is independent of $t$ for each $j$. On the other hand, we can assume that for each

$t$ there is $\Delta'_t \in \mathcal{L}$ such that $(K_X + \Delta'_t) \cdot R_t < 0$, hence we have an extremal curve $\Gamma_t$ for $R_t$, by Theorem 1.1.

Now, if $(K_X + B_j) \cdot \Gamma_t \leq 0$, then $-6 \leq (K_X + B_j) \cdot \Gamma_t \leq 0$ by Theorem 1.1, hence there are only finitely many possibilities for $(K_X + B_j) \cdot \Gamma_t$, so we can assume that it is independent of $t$. On the other hand, if $(K_X + B_j) \cdot \Gamma_t > 0$ and if $a_j \neq 0$, then $(K_X + B_j) \cdot \Gamma_t$ is bounded from below and above because

$$(K_X + \Delta_t) \cdot \Gamma_t = \sum a_{j,t}(K_X + B_j) \cdot \Gamma_t \leq 0$$

and because for $t \gg 0$, $a_{j,t}$ is bounded from below as it is sufficiently close to $a_j$. Therefore, if $a_j \neq 0$, then there are only finitely many possibilities for $(K_X + B_j) \cdot \Gamma_t$ and we could assume that it is independent of $t$.

Rearranging the indexes we can assume that $a_j \neq 0$ for $1 \leq j \leq l$ but $a_j = 0$ for $j > l$. Then by (1) the number

$$(K_X + \Delta_t) \cdot \Gamma_t = \sum a_{j,t}(K_X + B_j) \cdot \Gamma_t =$$

$$(K_X + B) \cdot \Gamma_t + \sum_{j \leq l} (a_{j,t} - a_j)(K_X + B_j) \cdot \Gamma_t + \sum_{j > l} a_{j,t}(K_X + B_j) \cdot \Gamma_t$$

is positive if $t \gg 0$ because $(K_X + B) \cdot \Gamma_t > \alpha$, and if $j \leq l$, then $|(a_{j,t} - a_j)(K_X + B_j) \cdot \Gamma_t|$ is sufficiently small, and if $j > l$, then $a_{j,t}(K_X + B_j) \cdot \Gamma_t$ is either positive or $|a_{j,t}(K_X + B_j) \cdot \Gamma_t|$ is sufficiently small. This is a contradiction.

(3) We may assume that for each $t \in T$ there is some $\Delta \in \mathcal{L}$ such that $(K_X + \Delta) \cdot R_t < 0$, in particular, $(K_X + B_j) \cdot R_t < 0$ for some vertex $B_j$ of $\mathcal{L}$ and that $R_t$ is generated by some extremal curve. Since by Theorem 1.1 the set of such extremal rays is discrete, we may assume that $T \subseteq \mathbb{N}$.

Obviously, $\mathcal{N}_T = \bigcap_{t \in T} \mathcal{N}_{\{t\}}$ is a convex compact subset of $\mathcal{L}$ since each $\mathcal{N}_{\{t\}}$ is a convex closed subset. If $T$ is finite, the claim is trivial because $\mathcal{N}_T$ is then cut out of $\mathcal{L}$ by finitely many inequalities with rational coefficients. So we may assume that $T = \mathbb{N}$. By (2) and by the compactness of $\mathcal{N}_T$, there are $\Delta_1, \ldots, \Delta_n \in \mathcal{N}_T$ and $\delta_1, \ldots, \delta_n > 0$ such that $\mathcal{N}_T$ is covered by the balls $\mathcal{B}_i = \{\Delta \in \mathcal{L} \mid ||\Delta - \Delta_i|| < \delta_i\}$ and such that if $\Delta \in \mathcal{B}_i$ with $(K_X + \Delta) \cdot R_t < 0$ for some $t$, then $(K_X + \Delta_i) \cdot R_t = 0$.

Let

$$T_i = \{t \in T \mid (K_X + \Delta) \cdot R_t < 0 \text{ for some } \Delta \in \mathcal{B}_i\}$$

Then by construction $(K_X + \Delta_i) \cdot R_t = 0$ for any $t \in T_i$. We claim that

$$\mathcal{N}_T = \bigcap_{1 \leq i \leq n} \mathcal{N}_{T_i}$$

Let $T' = \bigcup_{1 \leq i \leq n} T_i$ and let $S = T \setminus T'$. Pick $s \in S$. Since $s \notin T_i$ for each $i$, $(K_X + \Delta) \cdot R_s \geq 0$ for every $\Delta \in \bigcup_{1 \leq i \leq n} \mathcal{B}_i$. Thus $\mathcal{N}_T \subseteq \bigcup_{1 \leq i \leq n} \mathcal{B}_i \subseteq \mathcal{N}_S$ which in turn implies that

$$\mathcal{N}_T = \mathcal{N}_{T'} = \bigcap_{1 \leq i \leq n} \mathcal{N}_{T_i}$$

because the $\mathcal{B}_i$ give an open cover of $\mathcal{N}_T$ and $\mathcal{N}_{T'}$ is a convex closed set containing $\mathcal{N}_T$.

By the last paragraph, it is enough to prove that each $\mathcal{N}_{T_i}$ is a rational polytope and by replacing $T$ with $T_i$, we could assume from the beginning that there is some $\Delta \in \mathcal{N}_T$ such that $(K_X+\Delta)\cdot R_t = 0$ for every $t \in T$. If $\dim \mathcal{L} = 1$, this already proves the proposition.

Assume $\dim \mathcal{L} > 1$ and let $\mathcal{L}^1, \dots, \mathcal{L}^p$ be the proper faces of $\mathcal{L}$. Then each $\mathcal{N}_T^i := \mathcal{N}_T \cap \mathcal{L}^i$ is a rational polytope by induction. Moreover, for each $\Delta'' \in \mathcal{N}_T$ which is not equal to $\Delta$, there is $\Delta'$ on some face $\mathcal{L}^i$ such that $\Delta''$ is on the line segment determined by $\Delta$ and $\Delta'$. Since $(K_X+\Delta)\cdot R_t = 0$ for every $t \in T$, $\Delta' \in \mathcal{N}_T^i$. Hence $\mathcal{N}_T$ is the convex hull of $\Delta$ and all the $\mathcal{N}_T^i$. Now, there is a finite subset $V \subset T$ such that for each $i$ we have $\mathcal{N}_T^i = \mathcal{N}_V \bigcap \mathcal{L}^i$. But then the convex hull of $\Delta$ and all the $\mathcal{N}_T^i$ is nothing but $\mathcal{N}_V$, hence $\mathcal{N}_T = \mathcal{N}_V$ and we are done.

(4) Note that the sequence being $K_X+B$-trivial means that $K_{X_i}+B_i$ is numerically trivial over $Z_i$ for each $i$. This in particular implies that $K_{X_i}+B_i$ is nef for every $i$. Since $K_X+B$ is nef, $B \in \mathcal{N}_T$ where we take $\{R_t\}_{t\in T}$ to be the family of all the extremal rays of $\overline{NE}(X)$. Since $\mathcal{N}_T$ is a rational polytope by (3), there are positive real numbers $a'_1, \dots, a'_{r'}$, and $m' \in \mathbb{N}$ so that $\sum a'_j = 1$, $B = \sum a'_j B'_j$, and each $m'(K_X+B'_j)$ is Cartier where $B'_j$ are among the vertices of $\mathcal{N}_T$. Therefore, since $K_X+B = \sum a'_j(K_X+B'_j)$ and since each $K_X+B'_j$ is nef, the sequence $X_i \dashrightarrow X_{i+1}/Z_i$ is also $K_X+B'_j$-trivial for each $j$. Thus $K_{X_i}+B'_{j,i}$ is nef and $(X_i, B'_{j,i})$ is log canonical for every $i$.

Now fix $i$ and let $\phi\colon W \to X$ and $\psi\colon W \to X_i$ be a common log resolution. Since $X = X_1$ is klt, $(X_i, \Theta)$ is also klt for some $\Theta$. Then by Lemma 3.6, there is a curve $D$ on $W$ which maps birationally onto $\Gamma \subset X_i$. This implies that if $(K_{X_i}+B'_{j,i})\cdot\Gamma > 0$, then

$$(K_{X_i}+B_i)\cdot\Gamma \geq a'_j(K_{X_i}+B'_{j,i})\cdot\Gamma = a'_j\psi^*(K_{X_i}+B'_{j,i})\cdot D = a'_j\phi^*(K_X+B'_j)\cdot D \geq \frac{a'_j}{m'}$$

Therefore, perhaps after replacing $\alpha$ of (1) with a smaller one, we have $(K_{X_i}+B_i)\cdot\Gamma > \alpha$ if $(K_{X_i}+B_i)\cdot\Gamma > 0$.

(5) Let $\Delta'$ be on the boundary of $\mathcal{L}$ so that $\Delta$ belongs to the line segment determined by $B$ and $\Delta'$. There are non-negative real numbers $r, s$ such that $s > 0$, $r+s = 1$ and $\Delta = rB + s\Delta'$. In particular, the sequence $X_i \dashrightarrow X_{i+1}/Z_i$ is also a sequence of $K_X+\Delta'$-flips and $(X_i, \Delta'_i)$ is lc for each $i$. Suppose that there is an extremal ray $R$ on some $X_i$ such that $(K_{X_i}+\Delta_i)\cdot R \leq 0$ but $(K_{X_i}+B_i)\cdot R > 0$. By Theorem 1.1, $(K_{X_i}+\Delta'_i)\cdot\Gamma \geq -6$ for some curve $\Gamma$ generating $R$ (note that $X_i$ is $\mathbb{Q}$-factorial klt so we can apply 1.1 to $(X_i, \Delta'_i)$ although it may not be dlt). On the other hand, by (4), $(K_{X_i}+B_i)\cdot\Gamma > \alpha$. Now

$$(K_{X_i}+\Delta_i)\cdot\Gamma = r(K_{X_i}+B_i)\cdot\Gamma + s(K_{X_i}+\Delta'_i)\cdot\Gamma > r\alpha - 6s$$

and it is obvious that this is positive if $r > \frac{6s}{\alpha}$. In other words, if $\Delta$ is sufficiently close to $B$, then we get a contradiction. Therefore, it is enough to replace the $\delta$ of (2) by one sufficiently smaller.

□

3.9. **Big log divisors.** We can derive base point freeness in the big case from results we have proved so far.

**Proposition 3.10.** *Let $(X, B)$ be a projective klt pair of dimension 3 and $X \to Z$ a projective contraction over $k$ of characteristic $p > 5$. If $K_X + B$ is nef/$Z$ and big/$Z$, then it is semi-ample/$Z$.*

*Proof.* Let $P$ be the pullback of a sufficiently ample divisor on $Z$. By Theorem 1.1, $K_X + B + P$ is globally nef and big. Since $P$ is nef and $K_X + B + P$ is nef and big, there exist $\epsilon > 0$ and

$$\Delta \sim_{\mathbb{R}} B + P + \epsilon(K_X + B + P)$$

such that $(X, \Delta)$ is klt. It is enough to show $K_X + \Delta$ is semi-ample. Replacing $B$ with $\Delta$ we can then assume $Z$ is a point. We may also replace $(X, B)$ by a crepant $\mathbb{Q}$-factorialisation in order to assume $X$ is $\mathbb{Q}$-factorial.

By Proposition 3.8 (3), there are $\mathbb{Q}$-boundaries $B_j$ and non-negative real numbers $a_j$ such that $\sum a_j = 1$, $K_X + B = \sum a_j(K_X + B_j)$, and such that $K_X + B_j$ is nef for each $j$. Since $(X, B)$ is klt and $K_X + B$ is big, we can choose the $B_j$ so that $(X, B_j)$ is klt and $K_X + B_j$ is big for each $j$. By [3, Theorem 1.4], each $K_X + B_j$ is semi-ample, hence $K_X + B$ is also semi-ample.

□

## 4. Finiteness of minimal models and termination

In this section we prove finiteness of minimal models and derive termination with scaling under certain assumptions.

**Remark 4.1** In the setting of Theorem 1.4, let $B \in \mathcal{L}_A(V)$ be such that $(X, B)$ is klt. We can write $A \sim_{\mathbb{R}} A' + G/Z$ where $A' \geq 0$ is an ample $\mathbb{Q}$-divisor and $G \geq 0$ is also a $\mathbb{Q}$-divisor. Then there is a sufficiently small rational number $\epsilon > 0$ such that

$$(X, \Delta_B := B - \epsilon A + \epsilon A' + \epsilon G)$$

is klt. Note that

$$K_X + \Delta_B \sim_{\mathbb{R}} K_X + B/Z$$

Moreover, there is an open neighborhood of $B$ in $\mathcal{L}_A(V)$ such that for any $B'$ in that neighborhood

$$(X, \Delta_{B'} := B' - \epsilon A + \epsilon A' + \epsilon G)$$

is also klt. In particular, if $\mathcal{C} \subseteq \mathcal{L}_A(V)$ is a rational polytope containing $B$, then perhaps after shrinking $\mathcal{C}$ (but preserving its dimension) we can assume that $\mathcal{D} := \{\Delta_{B'} \mid B' \in \mathcal{C}\}$ is a rational polytope of klt boundaries in $\mathcal{L}_{\epsilon A'}(W)$ where $W$ is the rational affine space $V + (1 - \epsilon)A + \epsilon G$. The point is that we can change $A$ and get an ample part $\epsilon A'$ in the boundary. So, when we are concerned with a problem locally around $B$ we feel free to assume that $A$ is actually ample by replacing it with $\epsilon A'$.

**Proposition 4.2.** *Theorem 1.4 holds if either:*
(1) $K_X + \Delta$ *is big over* $Z$ *for every* $\Delta \in \mathcal{C}$, *or*
(2) *Theorem 1.2 holds.*

*Proof.* We may assume that the dimension of $\mathcal{C}$ is positive. We can also assume that the proposition holds for polytopes with smaller dimension. Since $\mathcal{C}$ is compact, it is enough to prove the statement locally near a fixed $B \in \mathcal{C}$. If $K_X + B$ is not pseudo-effective/$Z$, then the same holds in a neighborhood of $B$ inside $\mathcal{C}$. So we may assume that $K_X + B$ is pseudo-effective/$Z$. Then $(X, B)$ has a log minimal model $(Y, B_Y)$ over $Z$ by Theorem 2.8. Moreover, the polytope $\mathcal{C}$ determines a rational polytope $\mathcal{C}_Y$ of $\mathbb{R}$-divisors on $Y$ by taking birational transforms of elements of $\mathcal{C}$. If we shrink $\mathcal{C}$ around $B$ we can assume that for every $\Delta \in \mathcal{C}$ the log discrepancies satisfy

$$a(D, X, \Delta) < a(D, Y, \Delta_Y)$$

for any prime divisor $D$ on $X$ which is contracted by $X \dashrightarrow Y$. So for each $\Delta \in \mathcal{C}$, a log minimal model of $(Y, \Delta_Y)$ over $Z$ is also a log minimal model of $(X, \Delta)$ over $Z$. Therefore, we can replace $(X, B)$ by $(Y, B_Y)$ and replace $\mathcal{C}$ by $\mathcal{C}_Y$, hence from now on assume that $K_X + B$ is nef/$Z$. Then $K_X + B$ is semi-ample/$Z$: in case (1) we use Proposition 3.10 and in case (2) we use Theorem 1.2. So $K_X + B$ defines a contraction $f \colon X \to S/Z$.

Now by our assumption at the beginning of this proof, we may assume that there are finitely many birational maps $\psi_j \colon X \dashrightarrow Y_j/S$ such that for any $\Delta'$ on the boundary of $\mathcal{C}$ with $K_X + \Delta'$ pseudo-effective/$S$, there is $j$ such that $(Y_j, \Delta'_{Y_j})$ is a log minimal model of $(X, \Delta')$ over $S$. On the other hand, by Proposition 3.8, there is a sufficiently small $\epsilon > 0$ such that for any $\Delta \in \mathcal{C}$ with $||B - \Delta|| < \epsilon$ and any $K_{Y_j} + \Delta_{Y_j}$-negative extremal ray $R/Z$ we have the equality $(K_{Y_j} + B_{Y_j}) \cdot R = 0$ for all $j$. Note that all the pairs $(Y_j, B_{Y_j})$ are klt, $K_{Y_j} + B_{Y_j} \equiv 0/S$ and $K_{Y_j} + B_{Y_j}$ is nef/$Z$.

Pick $\Delta \in \mathcal{C}$ with $0 < ||B - \Delta|| < \epsilon$ such that $K_X + \Delta$ is pseudo-effective/$Z$, and let $\Delta'$ be the unique point on the boundary of $\mathcal{C}$ such that $\Delta$ belongs to the line segment given by $B$ and $\Delta'$. Since $K_X + B \equiv 0/S$, there is some $t > 0$ such that

$$K_X + \Delta' = K_X + B + \Delta' - B \equiv \Delta' - B = t(\Delta - B) \equiv t(K_X + \Delta)/S$$

hence $K_X + \Delta'$ is pseudo-effective/$S$, and $(Y_j, \Delta'_{Y_j})$ is a log minimal model of $(X, \Delta')$ over $S$ for some $j$. Moreover, $(Y_j, \Delta_{Y_j})$ is a log minimal model of $(X, \Delta)$ over $S$ for the same $j$. Now assuming $\epsilon$ is sufficiently small, $(Y_j, \Delta_{Y_j})$ is a log minimal model of $(X, \Delta)$ over $Z$ because any $K_{Y_j} + \Delta_{Y_j}$-negative extremal ray $R/Z$ would be over $S$ by the last paragraph.

□

**Proposition 4.3.** *Theorem 1.5 holds if either*
(1) $K_X + B$ *is big or*

(2) *Theorems 1.2 and 1.3 hold.*

*Proof.* Since $B$ is big/$Z$, we can assume $B \geq A$ for some (globally) ample $\mathbb{R}$-divisor $A$. Thus there are only finitely many $K_X+B$-negative extremal rays by Theorem 1.1 (iii). Moreover, by adding to $A$ the pullback of a sufficiently ample divisor on $Z$, changing $A$ up to $\mathbb{R}$-linear equivalence and applying Theorem 1.1, we can assume that all the $K_X+B$-negative extremal rays are over $Z$ and that $K_X+B+C$ is globally nef. Therefore, we can run the LMMP globally which would automatically be over $Z$.

Let $\lambda \geq 0$ be the smallest number such that $K_X+B+\lambda C$ is nef. We can assume $\lambda > 0$. So there is an extremal ray $R$ such that $(K_X+B)\cdot R < 0$ but $(K_X+B+\lambda C)\cdot R = 0$. The ray $R$ can be contracted: in case (1) we can use [3, Theorem 1.5] but in case (2) we use Theorem 1.3. If this contraction is not birational, then we have a Mori fibre space and we stop. Otherwise, we let $X \dashrightarrow X'$ be the corresponding flip or divisorial contraction, and we continue with $X'$ and so on. This shows that we can run the LMMP with scaling.

We will show that the LMMP terminates. Assume to the contrary that there is an infinite sequence $X_i \dashrightarrow X_{i+1}/Z_i$ of log flips. We may assume that $X = X_1$. Let $\lambda_i$ be the numbers appearing in the LMMP with scaling, and put $\lambda = \lim \lambda_i$. By definition, $K_{X_i}+B_i+\lambda_i C_i$ is nef but numerically zero over $Z_i$ where $B_i$ and $C_i$ are the birational transforms of $B$ and $C$ respectively. Replacing $B$ with $B+\lambda C$, we can assume $\lambda = 0$.

Let $H_1, \dots, H_m$ be general effective ample Cartier divisors on $X$ which generate the space $N^1(X)$. Since $B$ is big, we may assume that $B - \epsilon(H_1 + \cdots + H_m) \geq 0$ for some rational number $\epsilon > 0$. Replacing $A$ we can assume $A = \frac{\epsilon}{2}(H_1+\cdots+H_m)$. Let $V$ be the $\mathbb{R}$-vector space generated by the components of $B+C$, and let $\mathcal{C} \subset \mathcal{L}_A(V)$ be a rational polytope of maximal dimension containing an open neighborhood of $B$ and such that $(X,\Delta)$ is klt for every $\Delta \in \mathcal{C}$. In case (1) we can choose $\mathcal{C}$ so that $K_X+\Delta$ is big for every $\Delta \in \mathcal{C}$.

We can choose $\mathcal{C}$ such that for each $i$ there is an ample $\mathbb{Q}$-divisor $G_i = \sum g_{i,j} H_{i,j}$ on $X_i$ with sufficiently small coefficients, where $H_{i,j}$ on $X_i$ is the birational transform of $H_j$, such that $\Delta^i$ the birational transform of $B_i+G_i+\lambda_i C_i$ on $X$ belongs to $\mathcal{C}$. In particular, $K_{X_i}+B_i+G_i+\lambda_i C_i$ is ample and $(X_i, B_i+G_i+\lambda_i C_i)$ is the lc model of $(X,\Delta^i)$.

Now, by Proposition 4.2, there are finitely many birational maps $\phi_l \colon X \dashrightarrow Y_l$ such that for any $\Delta \in \mathcal{C}$ with $K_X+\Delta$ pseudo-effective, there is $l$ such that $(Y_l, \Delta_{Y_l})$ is a log minimal model of $(X,\Delta)$. Since $K_{X_i}+B_i+G_i+\lambda_i C_i$ is ample and since the lc model is unique, for each $i$, there is some $l$ such that $\phi_{1,i}\phi_l^{-1}$ is an isomorphism where $\phi_{i,j}$ is the birational map $X_i \dashrightarrow X_j$. Therefore, there exist $l$ and an infinite set $I \subseteq \mathbb{N}$ such that $\phi_{1,i}\phi_l^{-1}$ is an isomorphism for any $i \in I$. This in turn implies that $\phi_{i,j}$ is an isomorphism for any $i,j \in I$. This is not possible as any log flip increases some log discrepancy.

$\square$

**Lemma 4.4.** *Let $(X,B)$ be a projective klt pair and suppose that $(Y,B_Y)$ is a log minimal model for $(X,B)$. Also assume that $K_X+B$ is numerically a limit of movable $\mathbb{R}$-divisors. Then $X$ and $Y$ are isomorphic in codimension 1.*

*Proof.* We can assume $\dim X \geq 2$. Let $\phi\colon W \to X$ and $\psi\colon W \to Y$ be a common log resolution for $(X,B)$ and $(Y,B_Y)$. We can find a boundary $B_W$ such that $(W,B_W)$ is klt and

$$K_W+B_W=\phi^*(K_X+B)+G$$

where $G$ is effective and exceptional over $X$. Moreover,

$$\phi^*(K_X+B)=\psi^*(K_Y+B_Y)+E$$

where $E$ is effective, exceptional over $Y$, and its support contains all the prime exceptional divisors of $X \dashrightarrow Y$. Thus

$$K_W+B_W=\psi^*(K_Y+B_Y)+G+E.$$

Assume $S$ is the birational transform of a prime divisor contracted by $X \dashrightarrow Y$. Let $A_W$ be the pullback of some ample divisor on $X$. By assumption, for each $\epsilon>0$, there is $0\leq D_W \sim_{\mathbb{R}} K_W+B_W+\epsilon A_W$ such that $S$ is not a component of $D_W$. On the other hand, by Proposition 4.3, we can run an LMMP$/Y$ on $K_W+B_W$ which contracts all the components of $G+E$. Thus in some step we arrive at a model $W'$ on which $K_{W'}+B_{W'}$ negatively intersects a covering family of curves of the birational transform of $S$. This is a contradiction by the existence of the $D_W$ mentioned above and by taking $\epsilon$ to be small enough.

□

**Proposition 4.5.** *Theorem 1.6 holds if $K_X+B$ is pseudo-effective$/Z$.*

*Proof.* As in the proof of Proposition 4.3, we can reduce the problem to the global case, hence ignore $Z$. The fact that we can run the LMMP is also proved there. Let $X_i \dashrightarrow X_{i+1}/Z_i$ be the steps of the LMMP which are either flips or divisorial contractions with $X_{i+1}=Z_i$, and $X=X_1$. Let $\lambda_i$ be the numbers that appear in the LMMP and let $\lambda=\lim\lambda_i$. Assuming that the LMMP does not terminate, we will derive a contradiction. If $\lambda>0$, then the LMMP is also an LMMP on $K_X+B+\frac{\lambda}{2}C$ with scaling of $(1-\frac{\lambda}{2})C$, so the theorem follows from Proposition 4.3 in this case. Thus we can assume $\lambda=0$. In particular, this means $K_{X_i}+B_i$ is (numerically) a limit of movable divisors for any $i\gg 0$. Moreover, since $C$ is ample, we can assume that its components generate $N^1(X)$.

By Theorem 2.8, $(X,B)$ has a log minimal model $(Y,B_Y)$ which is also a log minimal model of $(X_i,B_i)$ for each $i$. Since $K_{X_i}+B_i$ is (numerically) a limit of movable $\mathbb{R}$-divisors for $i\gg 0$, the induced maps $X_i \dashrightarrow Y$ are all isomorphisms in codimension one when $i\gg 0$, by Lemma 4.4. Fix $i \gg 0$. Since the components of $C_i$ generate $N^1(X_i)$, there is an ample divisor $H_i$ on $X_i$ supported on $\operatorname{Supp} C_i$. Let $H_Y$ be the birational transform of $H_i$ on $Y$. Pick a sufficiently small number $\epsilon>0$. Then $(Y,B_Y+\epsilon H_Y+\lambda_i C_Y)$ is klt. Run an LMMP on $K_Y+B_Y+\epsilon H_Y+\lambda_i C_Y$ with scaling of some ample divisor. This LMMP terminates by Proposition 4.3. As $K_{X_i}+B_i+\epsilon H_i+\lambda_i C_i$ is ample and

$X_i$ is $\mathbb{Q}$-factorial, the LMMP ends with $X_i$. This implies that the LMMP does not contract a divisor as $Y$ and $X_i$ are isomorphic in codimension 1. Moreover, since $\epsilon$ and $\lambda_i$ are sufficiently small and $K_Y + B_Y$ is nef, by Proposition 3.8 (5), the LMMP is $K_Y + B_Y$-trivial (note that although $H_Y$ depends on $i$ but $\mathrm{Supp}(\epsilon H_Y + \lambda_i C_Y)$ is independent of $i$ and its coefficients are sufficiently small). Therefore, $K_{X_i} + B_i$ is also nef. This is a contradiction since $X_i \dashrightarrow X_{i+1}/Z_i$ is a $K_{X_i} + B_i$-flip.

$\square$

## 5. Relatively numerically trivial divisors

**Lemma 5.1.** *Let $X$ be a normal projective $\mathbb{Q}$-factorial threefold over $k$ with ample divisors $H_1, \dots, H_n$ generating $N^1(X)$. Then the set of 1-cycles $\{H_i \cdot H_j\}_{1\le i,j\le n}$ generates $N_1(X)$.*

*Proof.* Let $V$ be the vector subspace of $N_1(X)$ generated by $\{H_i \cdot H_j\}_{1\le i,j\le n}$. Let $L$ be an $\mathbb{R}$-divisor with trivial intersection with every element of $V$. As the $H_i$ generate $N^1(X)$ we can write $L \equiv \sum_1^n a_i H_i$. Pick a curve $C$ on $X$ and let $S$ be a prime divisor on $X$ containing $C$. Let $S^\nu$ be the normalization of $S$. We can write $S \equiv \sum_1^n b_i H_i$. Now

$$L|_S \cdot L|_S = L \cdot L \cdot S = L \cdot (\sum a_i H_i) \cdot (\sum b_i H_i) = 0.$$

Similarly $L|_S \cdot H_i|_S = 0$. Take a resolution $\phi\colon S' \to S^\nu$. Apply the Hodge index theorem to $\phi^*(L|_{S^\nu})$: we can find an ample divisor $H'$ on $S'$ with $\phi_* H' = H_i|_{S^\nu}$ for some $i$, so

$$\phi^*(L|_{S^\nu}) \cdot H' = \phi^*(L|_{S^\nu}) \cdot \phi^*(H_i|_{S^\nu}) = L|_S \cdot H_i|_S = 0$$

and also $\phi^*(L|_{S^\nu}) \cdot \phi^*(L|_{S^\nu}) = L|_S \cdot L|_S = 0$. Therefore, $\phi^*(L|_{S^\nu}) \equiv 0$, hence $L|_S \equiv 0$ and $L \cdot C = 0$. This shows that $L \equiv 0$. Since $L$ was chosen arbitrarily from among all divisors which intersect trivially with 1-cycles in $V$, $V = N_1(X)$.

$\square$

**Lemma 5.2.** *Let $f\colon X \to Z$ be a projective contraction from a normal projective $\mathbb{Q}$-factorial threefold onto a smooth curve over $k$, and let $L$ be a nef $\mathbb{Q}$-divisor on $X$. If $L \equiv 0/Z$, then $L \equiv f^*D$ for some $\mathbb{Q}$-divisor $D$ on $Z$.*

*Proof.* Let $P$ be a point on $Z$. Let $H_1, \dots, H_n$ be very ample divisors generating $N^1(X)$. We may assume each $H_i$ is normal and irreducible [21]. By Lemma 5.1, the cycles $H_i \cdot H_j$ generate $N_1(X)$. For each $i$, choose an ample divisor $A_i$ on $H_i$ and let $n_i$ be the value for which the divisor $D_i = n_i P$ on $Z$ satisfies $(L|_{H_i} - f|_{H_i}^* D_i) \cdot A_i = 0$. Then

$$(L|_{H_i} - f|_{H_i}^* D_i) \cdot (L|_{H_i} - f|_{H_i}^* D_i) \ge 0$$

because $L$ is nef, $(f|_{H_i}^* D_i) \cdot L|_{H_i} = 0$, and $(f|_{H_i}^* D_i) \cdot (f|_{H_i}^* D_i) = 0$. Therefore by the Hodge index theorem applied on a resolution, $L|_{H_i} - f|_{H_i}^* D_i \equiv 0$. In particular, for each $i, j$ we have

$$(L - f^* D_i) \cdot H_i \cdot H_j = 0$$

hence

$$f^*D_i \cdot H_i \cdot H_j = f^*D_j \cdot H_i \cdot H_j$$

which implies that $n_i = n_j$, so $D_i = D_j$. As the pairwise intersections $H_i \cdot H_j$ generate $N_1(X)$, we have $L \equiv f^*D$ where $D = D_i$.

□

**Lemma 5.3.** *Let $f\colon X \to Z$ be a projective contraction from a normal quasi-projective variety onto a smooth curve over $k$. Assume $L$ is a nef/$Z$ $\mathbb{R}$-divisor on $X$ such that $L|_F \equiv 0$ where $F$ is the generic fibre of $f$. Then $L \equiv 0/Z$.*

*Proof.* Pick a curve $C$ contracted by $f$. We will show that $L \cdot C = 0$. Choose a surface $S$ containing $C$ such that $S \to Z$ is surjective. It is enough to show that $L|_S \cdot C = 0$. So by replacing $X$ with the normalization $S^\nu$ and $X \to Z$ with the Stein factorization of $S^\nu \to Z$, we can assume $\dim X = 2$.

By extending $k$ we can assume it is uncountable. We will use some of the notation and results of [2]. If $G$ is a very general fibre of $f$ and $F$ is the generic fibre of $f$, then $h^0(\langle mL\rangle|_F) = h^0(\langle mL\rangle|_G)$ for every natural number $m$. Since $L|_F$ is numerically trivial,

$$\limsup_{m\to+\infty} \frac{h^0(\langle mL\rangle|_F)}{m} = 0$$

by [2, Proposition 4.3]. Thus

$$\limsup_{m\to+\infty} \frac{h^0(\langle mL\rangle|_G)}{m} = 0$$

hence $L|_G \equiv 0$ referring to the same result.

Now let $G$ be a very general fibre over a closed point and $H$ be any fibre over a closed point. Since $G \sim H$, we have $L \cdot H = L \cdot G = 0$ which means $L|_H \equiv 0$. Therefore, $L \equiv 0/Z$ as claimed.

□

**Lemma 5.4.** *Let $f\colon X \to Z$ be a flat projective morphism between quasi-projective schemes over $k$. Assume $L$ is a nef/$Z$ $\mathbb{R}$-divisor on $X$ such that $L|_F \equiv 0$ for the fibres $F$ of $f$ over some dense open subset of $Z$. Then $L \equiv 0/Z$.*

*Proof.* Let $C$ be a curve contracted by $f$ to a point $z$. We want to show $L \cdot C = 0$. We can assume there is a component $T$ of $Z$ of positive dimension containing $z$. Let $V$ be the normalization of a curve in $T$ passing through $z$. Let $Y = V \times_Z X$ and $g\colon Y \to V$ the induced morphism. Then $g$ is flat and we can choose $V$ so that $L|_Y$ is numerically trivial over some nonempty open subset of $V$. Replacing $f$ with $g$ we can assume $Z$ is a smooth curve. Now the flatness means that every associated point of $X$ maps to the generic point of $Z$, hence every irreducible component of $X$ maps onto $Z$. Replacing $X$ with one of its irreducible components (with reduced structure) containing $C$, we can assume $X$ is integral, that is, it is a variety. Now apply Lemma 5.3 to the pullback of $L$ to the normalization of $X$.

□

**Definition 5.5** Suppose $f : X \to Z$ is a dominant morphism of normal varieties over $k$. A divisor $L$ on $X$ is *exceptional* if every component $L_i$ of $L$ is contracted by $f$, that is, $\dim f(L_i) < \dim(L_i)$, and $f(L_i) \neq Z$. On the other hand, $L$ is *very exceptional* if it is exceptional and for any prime divisor $P$ on $Z$, some divisorial component $Q$ of $f^{-1}(P)$ with $f(Q) = P$ is not contained in $\operatorname{Supp} L$.

The following is an adaptation of a result of Kawamata [13, Proposition 2.1].

**Lemma 5.6.** *Let $f\colon X \to Z$ be a projection contraction between normal quasi-projective varieties over $k$ and $L$ a nef/$Z$ $\mathbb{R}$-divisor on $X$ such that $L|_F \sim_{\mathbb{R}} 0$ where $F$ is the generic fibre of $f$. Assume $\dim Z \leq 3$ if $k$ has characteristic $p > 0$. Then there exist a diagram*

$$\begin{tikzcd} X' \arrow[r, "\phi"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Z' \arrow[r, "\psi"] & Z \end{tikzcd}$$

*with $\phi, \psi$ projective birational, and an $\mathbb{R}$-Cartier divisor $D$ on $Z'$ such that $\phi^*L \sim_{\mathbb{R}} f'^*D$.*

*Proof.* Since $L|_F \sim_{\mathbb{R}} 0$, we can replace $L$ up to $\mathbb{R}$-linear equivalence and hence assume that $L|_F = 0$. So $L|_G = 0$ for the general fibres $G$ of $f$. On the other hand, by flattening, there exist a diagram

$$\begin{tikzcd} X'' \arrow[r, "\pi"] \arrow[d, "f''"] & X \arrow[d, "f"] \\ Z'' \arrow[r, "\mu"] & Z \end{tikzcd}$$

with $\pi, \mu$ projective birational and $f''$ flat (but $X''$ and $Z''$ may not be normal). Applying Lemma 5.4 to $\pi^*L$, we deduce that $\pi^*L \equiv 0/Z''$. Now we can extend the diagram as

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"] & X'' \arrow[r] \arrow[d, "f''"] & X \arrow[d, "f"] \\ Z' \arrow[r] & Z'' \arrow[r] & Z \end{tikzcd}$$

with $X' \to X''$ and $Z' \to Z''$ projective birational, $X'$ normal and $Z'$ smooth. Denote the induced maps $X' \to X$ and $Z' \to Z$ by $\phi$ and $\psi$ respectively. Then $\phi^*L \equiv 0/Z'$.

Since $\phi^*L$ is vertical/$Z'$, there is an $\mathbb{R}$-divisor $E$ on $X'$ vertical/$Z'$ such that $\phi^*L + E \sim_{\mathbb{R}} 0/Z'$. For each divisorial component $P$ of $f(E)$, we can add $af^*P$ to $E$ for some appropriate number $a$ to assume that every component of $E$ mapping onto $P$ has non-negative coefficient, and at least one of them has coefficient zero. But then since $E \equiv 0/Z'$, we arrive at the situation in which $f(E)$ has codimension at least two. Now if $E \neq 0$, we get a contradiction with the negativity lemma: indeed by cutting by hypersurface sections, we can find a normal subvariety $Y'$ of $X'$ such that $Y' \to Z'$ is generically finite; now apply the negativity lemma to $E|_{Y'}$ and $-E|_{Y'}$ which are exceptional/$Z'$ to deduce that $E|_{Y'} = 0$. So $E = 0$ and $\phi^* L \sim_{\mathbb{R}} f'^* D$ for some $\mathbb{R}$-Cartier divisor $D$ on $Z'$. □

## 6. Kodaira dimension of log divisors with big boundary

6.1. **Nef reduction maps of log divisors.** Theorem 2.12 demonstrates that nef reduction maps of log divisors are special. In dimension 3 we can say much more (also see Proposition 6.6 below).

**Proposition 6.2.** *Let $(X, B)$ be a $\mathbb{Q}$-factorial projective klt pair of dimension 3 over an uncountable field $k$ of characteristic $p > 0$. Assume that $B$ is a big $\mathbb{Q}$-boundary and that $K_X + B$ is nef. Then there exist a projective birational morphism $\phi\colon W \to X$ from a normal variety and a projective contraction $h\colon W \to Z$ to a normal variety, and a $\mathbb{Q}$-Cartier divisor $D$ on $Z$ such that*
- $\dim Z = n(K_X + B)$,
- *$D$ is nef, and*
- $\phi^*(K_X + B) \sim_{\mathbb{Q}} h^* D$.

*Proof.* If $n(K_X + B) = 3$ the statement is trivial. Let $f\colon X \dashrightarrow Z$ be a nef reduction map of $K_X + B$.

First assume $n(K_X + B) = 2$. Then $Z$ is a surface and the singular locus of $X$ is vertical/$Z$, hence $X$ is smooth near the general fibres of $f$. Let $F$ be the generic fibre. Since $B$ is big, $B|_F$ is ample, and since $(K_X + B)|_F \equiv 0$, $K_F = K_X|_F$ is anti-ample. This implies that $\overline{F}$ is isomorphic to $\mathbb{P}^1$ by [7, Lemma 6.5] where $\overline{F}$ is the geometric generic fibre. Thus there is a natural number $m$ such that the pullback of $m(K_X + B)$ to $\overline{F}$ is linearly equivalent to 0. So by base change $h^0(m(K_X + B)|_F) > 0$ which implies that $m(K_X + B)|_F$ is linearly equivalent to 0 because $F$ is an integral scheme and $K_X + B \equiv 0$. Let $\phi\colon W \to X$ be a projective birational morphism from a normal variety such that the induced map $h\colon W \dashrightarrow Z$ is a morphism. Now applying Lemma 5.6 to $h$ and $\phi^*(K_X + B)$, we can replace $W$ and $Z$ so that $\phi^*(K_X + B) \sim_{\mathbb{Q}} h^* D$ for some $\mathbb{Q}$-Cartier divisor $D$ on $Z$ which is necessarily nef.

We can then assume $n(K_X + B) = 1$. Then $f$ is a morphism as it is regular over the generic point of the smooth curve $Z$. By Lemma 5.3, $K_X + B \equiv 0/Z$, and by Lemma 5.2, $K_X + B \equiv f^* D'$ for some $\mathbb{Q}$-divisor $D'$. Let $P = K_X + B - f^* D'$. We use an argument similar to [7, proof of Theorem 1.9] to continue. Let $a\colon X \to \mathrm{Alb}$ be the Albanese morphism where Alb is the dual abelian variety of $\mathrm{Pic}^0(X)_{\mathrm{red}}$. Then for some sufficiently divisible natural number $m$, the divisor $mP$ belongs to $\mathrm{Pic}^0(X)_{\mathrm{red}}$ and $mP$ is the pullback of a numerically trivial divisor on Alb. Now let $g\colon X \to S$ be the Stein factorization of $(f, a)\colon X \to Z \times \mathrm{Alb}$. Assume for now that $\dim S = 1$. Then the induced morphism $e\colon S \to Z$ is an isomorphism which in turn implies that $a$ factors through $f$ and that $P$ is the pullback of some numerically trivial $\mathbb{Q}$-divisor $Q$ on $Z$, hence $K_X + B \sim_{\mathbb{Q}} f^* D$ where $D = D' + Q$. So we can take $W = X$ and $h = f$.

We show that in fact $\dim S = 1$. Assume otherwise, so $\dim S = 2$ or 3, hence the projection $S \to Z$ contracts curves. Let $A$ be an ample divisor on $S$. Since

$\dim S \geq 2$ and $\dim Z = 1$, $g^*A$ is not numerically trivial along the general fibres of $f$. As $B$ is big, we can assume that $B' := B - \epsilon g^*A \geq 0$ for some $\epsilon > 0$. Now from

$$K_X + B' \equiv -\epsilon g^*A/Z$$

we deduce that there is a $K_X + B'$-negative extremal ray $R$ over $Z$ because by assumption there is some curve contracted by $S \to Z$. By Theorem 1.1, $R$ is generated by some rational curve $\Gamma$ which is contracted over $Z$. By construction, $\Gamma$ is not contracted over $S$ since $K_X + B' \equiv 0/S$. On the other hand, $\Gamma$ is contracted by $a$ as $\Gamma$ is a rational curve. Therefore, $\Gamma$ should be contracted over $S$ too, a contradiction.

Lastly suppose $n(K_X + B) = 0$, so $K_X + B \equiv 0$. We use a similar argument to the previous paragraph. Let $a\colon X \to \mathrm{Alb}$ be the Albanese map and let $g\colon X \to T$ be the contraction given by the Stein factorization of $a$.

We will show that $T$ is a point. Assume not. Let $A$ be an ample divisor on $T$, and again we may assume $B' := B - \epsilon g^*A \geq 0$. Pick a $K_X + B'$-negative extremal ray $R$, which exists since we assumed $T$ is not a point. By Theorem 1.1, $R$ is generated by some rational curve $\Gamma$. Since $g^*A \cdot R > 0$, $\Gamma$ is not contracted by $g$, a contradiction. Therefore the Albanese map is trivial. In this case both Alb and $\mathrm{Pic}^0(X)_{\mathrm{red}}$ are single points. This implies that any numerically trivial line bundle on $X$ is torsion, so some positive multiple $m(K_X + B)$ satisfies $m(K_X + B) \sim 0$.

□

6.3. **ACC for horizontal coefficients.** The following result is of independent interest but also crucial for the proof of Proposition 6.6 below.

**Proposition 6.4.** *Let $\Lambda \subset [0, 1]$ be a DCC set of real numbers. Then there is a finite subset $\Lambda^0 \subset \Lambda$ with the following property: let $(X, B)$ be a pair and $f\colon X \to Z$ a projective contraction such that*

- *$(X, B)$ is $\mathbb{Q}$-factorial dlt of dimension 3 over $k$ of characteristic $p > 5$,*
- *$K_X + B$ is nef$/Z$ but not big$/Z$,*
- *$B = \lambda H + B'$ where $H, B' \geq 0$, $H$ is big$/Z$, and $\lambda \in \Lambda$,*
- *the horizontal$/Z$ coefficients of $H$ and $B'$ are in $\Lambda$, and*
- $\dim X > \dim Z \geq 1$.

*Then $\lambda$ belongs to $\Lambda^0$.*

*Proof.* Assume $\dim Z = 2$, let $F$ be the generic fibre of $f$, and $\overline{F}$ the geometric generic fibre. Since $K_X + B$ is not big$/Z$, $(K_X + B)|_F \equiv 0$. As in the proof of Proposition 6.2, one shows that $\overline{F}$ is isomorphic to $\mathbb{P}^1$ and that $K_{\overline{F}}$ is the pullback of $K_X$. Let $B_{\overline{F}}$ be the pullback of $B$ to $\overline{F}$. Since $X$ is smooth near $F$, each coefficient of $B_{\overline{F}}$ is of the form $nb$ for some $n \in \mathbb{N}$ and some coefficient $b$ of $B$. In particular, the coefficients of $B_{\overline{F}}$ belong to some DCC set depending only on $\Lambda$. Therefore, these coefficients belong to some finite set depending only on $\Lambda$ because $\deg B_{\overline{F}} = 2$. This in turn implies that the horizontal$/Z$ coefficients of $B$ belong to some finite set depending only on $\Lambda$, hence $\lambda$ belongs to some finite set depending only on $\Lambda$.

We can then assume $\dim Z = 1$. Fix a fibre $F$ over a closed point $z$ such that $F$ and $B$ have no common components. Let $\phi\colon W \to X$ be a log resolution of $(X, B+F)$ so that over $U := X \setminus \operatorname{Supp} F$, $\phi$ does not contract divisors with log discrepancy 0 with respect to $(X,B)$. Such $\phi$ exist by the dlt assumption of $(X,B)$. Let $\Delta_W$ be the sum of the birational transform of $B$, plus the sum of the birational transform of the irreducible components of $F$ (each with coefficient 1), and the sum of the exceptional prime divisors of $\phi$. Then $(W, \Delta_W)$ is dlt and $\operatorname{Supp} \phi^* F \subset \lfloor \Delta_W \rfloor$. Moreover, if we write

$$K_W + \Delta_W = \phi^*(K_X + B) + E$$

then we argue that $\operatorname{Supp} E \subseteq \lfloor \Delta_W \rfloor$: the exceptional/$X$ components of $E$ are obviously components of $\lfloor \Delta_W \rfloor$; on the other hand,

$$\phi_*\Delta_W - B = \phi_* E \text{ and } \operatorname{Supp}(\phi_*\Delta_W - B) = \operatorname{Supp} F$$

hence $\operatorname{Supp} \phi_* E = \operatorname{Supp} F$ which shows that the non-exceptional/$X$ components of $E$ are also components of $\lfloor \Delta_W \rfloor$. In addition, over $U$ the divisor $E$ is effective and its support is the reduced exceptional divisor of $\phi$ because $(X,B)$ is dlt, because over $U$ the boundary $\Delta_W$ is the sum of the birational transform of $B$ and the reduced exceptional divisor of $W \to X$, and because of our choice of $\phi$.

Run an LMMP/$X$ on $K_W + \Delta_W$ with scaling of some ample divisor as in [3, 3.5]. By special termination [3, Proposition 5.5], the LMMP terminates with a model $Y/X$ because it is an LMMP on $E$ and $\operatorname{Supp} E \subseteq \lfloor \Delta_W \rfloor$. In particular, since over $U$ the divisor $E$ is effective with support equal to the reduced exceptional divisor of $\phi$, the LMMP contracts any component of $E$ whose generic point maps into $U$. Thus $Y \to X$ is an isomorphism over $U$ because $X$ is $\mathbb{Q}$-factorial, and $E_Y$ maps into $\operatorname{Supp} F$. In particular, this means that $E_Y$ is supported on the fibre of the induced morphism $g\colon Y \to Z$ over the point $z = f(F)$.

Now run an LMMP/$Z$ on $K_Y + \Delta_Y$ with scaling of some ample divisor. By special termination, the LMMP terminates near $\lfloor \Delta_Y \rfloor$.

Since $K_Y + \Delta_Y$ is nef over $Z \setminus \{z\}$, the LMMP contracts only extremal rays all of whose curves are contained in the fibre over $z$, hence the LMMP terminates globally since the support of the fibre is contained in $\lfloor \Delta_Y \rfloor$. Denote the resulting model by $Y'$, and denote the birational transform of $H$ by $H_{Y'}$. Then we can write $\Delta_{Y'} = \lambda H_{Y'} + \Delta'_{Y'}$, and by replacing $X$ with $Y'$, $H$ with $H_{Y'}$, and $B'$ with $\Delta'_{Y'}$, we can assume that $\operatorname{Supp} F \subseteq \lfloor B' \rfloor$ (we also need to add 1 to $\Lambda$ since some of the coefficients of the new $B'$ are equal to 1).

Let $S$ be a component of $F$ which intersects $H$. Note that $S$ is automatically normal as $(X,B)$ is $\mathbb{Q}$-factorial dlt [12, Proposition 4.1](see also [3, Lemma 5.2]). By adjunction, we can write

$$K_S + B'_S = (K_X + B')|_S \text{ and } K_S + B_S = (K_X + B)|_S$$

where $B_S := \lambda H|_S + B'_S$ and the coefficients of $B_S$ and $B'_S$ belong to a DCC set determined by $\Lambda$ [3, Proposition 4.2]. More precisely, the coefficient of each prime divisor $V \subset S$ in $B_S$ is of the form

$$\frac{l-1}{l} + \sum \frac{b'_i m_i}{l} + \lambda \sum \frac{h_j n_j}{l}$$

for some $l \in \mathbb{N}$ and $m_i, n_j \in \mathbb{N} \cup \{0\}$ where $b'_i$ and $h_j$ are the coefficients of $B' - S$ and $H$ respectively ($m_i > 0$ if the corresponding component of $B' - S$ contains $V$; similarly $n_j > 0$ if the corresponding component of $H$ contains $V$).

We show that we can choose $S$ so that $H|_S$ is big. Since $H$ is big/$Z$, we can write $H \sim_{\mathbb{R}} A + N/Z$ where $A$ is ample and $N \geq 0$. Let $t$ be the smallest real number such that $N + tF \geq 0$. Then there is a component $S$ of $F$ which is not a component of $N + tF$. But then $H|_S \sim_{\mathbb{R}} A|_S + (N + tF)|_S$ is big.

Now since $K_X + B$ is not big/$Z$, $K_X + B$ restricted to any fibre of $f$ is not big, so $(K_X + B)|_F$ is not big. This in turn implies that $K_S + B_S$ is not big. However, $K_S + B_S$ is semi-ample and it defines a contraction $S \to T$. Since $H|_S$ is big, $H|_S$ is horizontal/$T$. Applying [3, Proposition 11.7], the horizontal/$T$ coefficients of $B_S$ belong to a finite set depending only on $\Lambda$. In particular, $\lambda$ belongs to a finite set depending only on $\Lambda$.

□

6.5. **Kodaira dimension.** We come to the main result of this section.

**Proposition 6.6.** *Let $(X, B)$ be a $\mathbb{Q}$-factorial projective klt pair of dimension 3 over an uncountable field $k$ of characteristic $p > 5$. Assume that $B$ is a big $\mathbb{Q}$-boundary and that $K_X + B$ is nef. Then there exist a projective birational morphism $\phi\colon W \to X$ from a normal variety, a projective contraction $h\colon W \to Z$ to a normal variety, and a $\mathbb{Q}$-Cartier divisor $D$ on $Z$ such that*
- *$D$ is nef and big, and*
- *$\phi^*(K_X + B) \sim_{\mathbb{Q}} h^* D$.*

*In particular,*

$$\kappa(K_X + B) = \nu(K_X + B) = n(K_X + B) = \dim Z.$$

*Proof. Step 1.* Assume there is a birational map $X \dashrightarrow X'$ whose inverse does not contract any divisor and such that $(X', B')$ is $\mathbb{Q}$-factorial klt and the pullbacks of $K_X + B$ and $K_{X'} + B'$ are equal on some common resolution of $X$ and $X'$. If $X \dashrightarrow X'$ contracts divisors, we replace $(X, B)$ with $(X', B')$. Repeating this process, we can assume that any map $X \dashrightarrow X'$ as above is an isomorphism in codimension one.

Now if $K_X + B$ is big, then the proposition is trivial. Moreover, by Theorem 2.12, we can assume the nef dimension $n(K_X + B)$ is at most 2. By Proposition 6.2, there exist a projective birational morphism $\phi\colon W \to X$ from a normal variety, a projective contraction $h\colon W \to Z$ to a normal variety, and a $\mathbb{Q}$-Cartier divisor $D$ on $Z$ such that $\dim Z = n(K_X + B)$, $D$ is nef, and $\phi^*(K_X + B) \sim_{\mathbb{Q}} h^* D$. Moreover, the induced map $f\colon X \dashrightarrow Z$ is a nef reduction map of $K_X + B$, so in particular, it is regular and proper over some nonempty open subset of $Z$.

If $n(K_X + B) = 0$, then $D$ is torsion, and if $n(K_X + B) = 1$, then $D$ is ample, so in these cases we are done. Thus from now on we may assume $n(K_X + B) = 2$, hence $\dim Z = 2$. It remains to show that $D$ is big. Suppose that $D$ is not big. We will derive a contradiction in several steps.

*Step 2.* Let $H_Z \geq 0$ be an ample $\mathbb{Q}$-divisor on $Z$, and let $H = \phi_* h^* H_Z$. Since $B$ is big, perhaps after replacing $B$ and $H$, we can assume $B \geq H + A$ where $A \geq 0$ is ample. Now let $\epsilon > 0$ be a sufficiently small rational number. Then $K_X + B - \epsilon H$ is not pseudo-effective because by Lemma 2.3,

$$\phi^*(K_X + B - \epsilon H) \leq \phi^*(K_X + B) - \epsilon h^* H_Z \sim_{\mathbb{Q}} h^*(D - \epsilon H_Z)$$

and because $D - \epsilon H_Z$ is not pseudo-effective. Let $\delta$ be the smallest number so that $K_X + B - \epsilon H + \delta A$ is pseudo-effective. Note as $\epsilon$ is sufficiently small, $\delta$ is sufficiently small too. We can assume that $(X, B - \epsilon H + \delta A)$ is klt. By Theorem 2.8, the pair has a log minimal model $(Y, B_Y - \epsilon H_Y + \delta A_Y)$, and by Theorem 2.12, the nef dimension of $K_Y + B_Y - \epsilon H_Y + \delta A_Y$ is at most 2.

We want to show that $X \dashrightarrow Y$ is an isomorphism in codimension one. Run an LMMP on $K_X + B - \epsilon H + \delta A$ with scaling of some large multiple of $A$. Denote the steps of the LMMP by $X_i \dashrightarrow X_{i+1}/Z_i$ which is either a flip or a divisorial contraction with $X_{i+1} = Z_i$. Assume that for each $i < l$, $X_i \dashrightarrow X_{i+1}/Z_i$ is a flip and that it is $K_X + B$-trivial, i.e. $K_{X_i} + B_i$ is numerically trivial over $Z_i$. By Proposition 3.8 (5), $K_{X_l} + B_l$ is numerically trivial over $Z_l$, and by the first paragraph of Step 1, $X_l \to Z_l$ is a flipping contraction. Therefore, the LMMP is $K_X + B$-trivial and it does not contract any divisor. Moreover, the LMMP terminates with a log minimal model by Proposition 4.5. Since different log minimal models are isomorphic in codimension one, we deduce $X \dashrightarrow Y$ is also an isomorphism in codimension one.

*Step 3.* Let $g\colon Y \dashrightarrow V$ be a nef reduction map of $K_Y + B_Y - \epsilon H_Y + \delta A_Y$. By Step 2, we can assume $\dim V > 0$ otherwise $K_Y + B_Y - \epsilon H_Y + \delta A_Y \equiv 0$, so $K_X + B - \epsilon H + \delta A \equiv 0$, and taking the limit when $\epsilon$ approaches 0 we end up with $K_X + B \equiv 0$, a contradiction.

Let $\tau$ be the largest number such that $K_Y + B_Y - \tau H_Y$ is pseudo-effective over the generic point of $V$. Since $K_X + B$ is pseudo-effective, $K_Y + B_Y$ is pseudo-effective, so $\tau \geq 0$. On the other hand, since $A_Y$ is big, $K_Y + B_Y - \epsilon H_Y$ is not pseudo-effective over the generic point of $V$, hence $\tau < \epsilon$. We want to show that $\tau = 0$. Assume not. By construction, $(Y, B_Y - \epsilon H_Y)$ is klt. Moreover, by ACC for lc thresholds [3, Theorem 1.10], $(Y, B_Y)$ is lc because $\epsilon$ is sufficiently small, and because we can write $B_Y - \epsilon H_Y = B'_Y + (1-\epsilon)H_Y$ where the coefficients of $B'_Y$ and $H_Y$ belong to a fixed finite set depending only on $(X, B)$ and $H$. Thus $(Y, B_Y - \tau H_Y)$ is klt because $\tau > 0$ and $(Y, B_Y - \epsilon H_Y)$ is klt.

Let $(Y', B_{Y'} - \tau H_{Y'})$ be a log minimal model of $(Y, B_Y - \tau H_Y)$ over some nonempty open subset $V'$ of $V$. By definition of $\tau$, $K_{Y'} + B_{Y'} - \tau H_{Y'}$ is not big$/V'$. On the other hand, $K_Y + B_Y + \delta A_Y \equiv \epsilon H_Y$ over the generic point of $V'$, hence $H_Y$ is big over $V'$ which in turn implies that $H_{Y'}$ is big over $V'$. So Proposition 6.4 implies that $1 - \tau = 1$ or that $1 - \tau$ is bounded away from 1. In our case $1 - \tau = 1$ is the only possibility because $\tau$ is sufficiently small, hence

$\tau = 0$, a contradiction.

*Step 4.* Assume that $\dim V = 2$. By the last step, $K_Y + B_Y$ is pseudo-effective but not big over the generic point of $V$. So $K_Y + B_Y$ restricted to the geometric generic fibre of $g\colon Y \dashrightarrow V$ is torsion as the geometric generic fibre is isomorphic to $\mathbb{P}^1$ (cf. proof of Proposition 6.2). Thus $K_Y + B_Y$ restricted to the generic fibre of $g$ is torsion too. This in turn implies $K_X + B \sim_{\mathbb{Q}} M$ for some $M$ whose support does not intersect the generic fibre of $g$. Therefore, $(K_Y + B_Y) \cdot G = 0$ for the general fibres $G$ of $g$.

Let $\alpha$ denote the map $X \dashrightarrow Y$. By Lemma 2.4, there is an open subset $U$ of $X$ such that $\alpha|_U$ is an isomorphism and such that the complement of $U_Y := \alpha(U)$ in $Y$ has dimension at most 1. Since $\dim V = 2$, $Y \setminus U_Y$ is vertical$/V$, hence $G \subset U_Y$. Therefore, if $G^\sim$ on $X$ is the birational transform of $G$, then $(K_X + B) \cdot G^\sim = 0$. So if $G$ is very general, then $G^\sim$ is contained in some very general fibre of $f\colon X \dashrightarrow Z$. In particular, $H \cdot G^\sim = 0$ because $H$ is vertical$/Z$. But then $H_Y \cdot G = 0$, hence

$$0 = (K_Y + B_Y - \epsilon H_Y + \delta A_Y) \cdot G = \delta A_Y \cdot G = \delta A \cdot G^\sim$$

which contradicts the assumption that $A$ is ample.

*Step 5.* Now assume $\dim V = 1$. Then $g\colon Y \dashrightarrow V$ is a morphism and $K_Y + B_Y - \epsilon H_Y + \delta A_Y \equiv 0/V$ by Lemma 5.3. By Step 3, $(Y, B_Y)$ is lc and $K_Y + B_Y$ is pseudo-effective but not big$/V$. We want to argue that $(Y, B_Y)$ has a weak lc model over $V$. Indeed since $(Y, B_Y)$ is lc and $(Y, B_Y - \epsilon H_Y + \delta A_Y)$ is klt and $K_Y + B_Y - \epsilon H_Y + \delta A_Y \equiv 0/V$, we have

$$K_Y + B_Y \equiv 2(K_Y + B_Y - \tilde{\epsilon} H_Y + \tilde{\delta} A_Y)/V$$

where $\tilde{\epsilon} = \frac{\epsilon}{2}, \tilde{\delta} = \frac{\delta}{2}$ and $(Y, B_Y - \tilde{\epsilon} H_Y + \tilde{\delta} A_Y)$ is klt. Thus $(Y, B_Y)$ has a $\mathbb{Q}$-factorial weak lc model $(Y', B_{Y'})$ over $V$ such that $Y' \dashrightarrow Y$ does not contract divisors.

Now by Theorem 1.1, $K_{Y'} + B_{Y'} + P_{Y'}$ is globally nef where $P_{Y'}$ is the pullback of a sufficiently ample divisor on $V$. Let $Y' \dashrightarrow S$ be a nef reduction map of $K_{Y'} + B_{Y'} + P_{Y'}$. If $C$ is a very general fibre of $Y' \dashrightarrow S$, then $(K_{Y'} + B_{Y'} + P_{Y'})|_C \equiv 0$ and by our choice of $P_{Y'}$ we can actually assume that $(K_{Y'} + B_{Y'})|_C \equiv 0$. Since $Y' \dashrightarrow X$ does not contract divisors and since $K_X + B$ is nef, by Lemma 2.3, $(K_X + B)|_{C^\sim} \equiv 0$ where $C^\sim$ is the birational transform of $C$.

Assume $\dim S = 1$ and let $C_Z$ on $Z$ be the image of $C^\sim$. Then $D|_{C_Z} \equiv 0$ where $D$ is as in Step 1. This shows that the nef dimension $n(D) < 2$, a contradiction.

Now assume $\dim S = 2$. Denote $X \dashrightarrow Y'$ by $\alpha'$. Since $\alpha'^{-1}$ does not contract divisors, by Lemma 2.4, there is an open subset $U' \subseteq X$ such that $\alpha'|_{U'}$ is an isomorphism and $Y' \setminus U'_{Y'}$ is of dimension at most 1 where $U'_{Y'} = \alpha'(U')$. Since $\dim S = 2$, $Y' \setminus U'_{Y'}$ is vertical$/S$, hence the very general fibres $C$ of $Y' \dashrightarrow S$ are contained in $U'_{Y'}$. Thus $(K_X + B) \cdot C^\sim = 0$, from which we deduce that $C^\sim$ is contained in the very general fibres of $X \dashrightarrow Z$. In particular, $H \cdot C^\sim = 0$, hence $H_{Y'} \cdot C = 0$. But then

$$0 = (K_{Y'} + B_{Y'} - \epsilon H_{Y'} + \delta A_{Y'}) \cdot C = \delta A_{Y'} \cdot C = \delta A \cdot C^{\sim}$$

which contradicts the assumption that $A$ is ample.

□

## 7. Some semi-ampleness criteria

**Lemma 7.1.** *Let $f\colon X \to Z$ be a surjective morphism from a normal projective variety to a normal projective surface, over $k$ of characteristic $p > 0$. Assume that $L$ is a nef $\mathbb{Q}$-Cartier divisor on $X$ such that*

- *$L \sim_{\mathbb{Q}} f^*D$ for some nef and big $\mathbb{Q}$-divisor $D$ on $Z$, and*
- *$L|_{f^{-1}\mathbb{E}(D)}$ is semi-ample.*

*Then $L$ is semi-ample.*

*Proof.* It is enough to show that $D$ is semi-ample. By [21, Theorem 7'], a general hyperplane section of a normal variety over an infinite field is normal. Therefore by taking hyperplane sections we can find a normal closed subvariety $Y \subset X$ such that the induced map $Y \to Z$ is generically finite. By replacing $X$ with $Y$ and $L$ with $L|_Y$, we can assume that $f$ is generically finite. Take the Stein factorization $g \circ h\colon X \to T \to Z$ of $f$ with $h$ birational and $g$ finite. As $g$ is finite, $D$ is semi-ample if and only if $g^*D$ is semi-ample and $g^{-1}\mathbb{E}(D) = \mathbb{E}(g^*D)$. So by replacing $Z$ with $T$ and $D$ with $g^*D$, we can assume $f$ is birational. Since $X$ and $Z$ are surfaces, $\mathbb{E}(L) = f^{-1}\mathbb{E}(D) \cup S$ where $S$ is disjoint from $f^{-1}\mathbb{E}(D)$ and $S$ is contracted by $f$. Thus $L|_{\mathbb{E}(L)}$ is semi-ample because $L|_{f^{-1}\mathbb{E}(D)}$ is semi-ample by assumption and $L|_S \sim_{\mathbb{Q}} 0$. Now by Keel's semi-ampleness criterion [14], $L$ is semi-ample and so as $f$ is birational, $D$ is semi-ample too.

□

**Lemma 7.2.** *Let $X$ be a normal projective variety over an uncountable $k$ of characteristic $p > 0$. Suppose $L$ is a nef $\mathbb{Q}$-divisor on $X$ with equal Kodaira and nef dimensions $\kappa(L) = n(L) \leq 2$. Then $L$ is endowed with a map $X \to V$ to a proper algebraic space $V$ of dimension equal to $\kappa(L)$.*

*Proof.* Since $\kappa(L) \geq 0$, we can assume $L \geq 0$. There is a nef reduction map $f\colon X \dashrightarrow Z$ to some normal projective variety $Z$, where $n(L) = \dim Z$. Replacing $X$ we may assume $f$ is a morphism. Since $L \geq 0$ and $L|_F \equiv 0$, $L|_F = 0$ where $F$ is the generic fibre of $f$. Moreover, by Lemma 5.6, perhaps after replacing $X$ and $Z$, we can assume $L \sim_{\mathbb{Q}} f^*D$ for some nef $\mathbb{Q}$-divisor $D$ on $Z$, in particular,

$$\kappa(D) = \kappa(L) = n(L) = n(D) = \dim Z.$$

It is enough to show that $D$ is endowed with a map.

Suppose first that $\kappa(D) = 2$. Then $Z$ is a surface and $\mathbb{E}(D)$ is a finite union of curves with $D|_{\mathbb{E}(D)} \equiv 0$, so $D|_{\mathbb{E}(D)}$ is endowed with the constant map to a point, hence by [14, Theorem 1.9], $D$ is endowed with a map $Z \to V$. Now suppose $\kappa(D) = 1$. Then $Z$ is a smooth curve and $D$ is a big divisor on it, which is therefore ample, hence we take $V = Z$. Finally, if $\kappa(D) = 0$, then $D \equiv 0$ is endowed with the constant map $Z \to V$ to a point.

Note that the proof shows that when $\kappa(L) = 0$ or 1, then $L$ is actually semi-ample and $X \to V$ is the projective contraction associated to $L$.

□

**Lemma 7.3.** *Let $X$ be a normal projective variety over $k$ of characteristic $p > 0$, and $L$ a nef $\mathbb{Q}$-divisor on $X$ with Kodaira dimension $0 \le \kappa(L) \le 2$. Assume that $L$ is endowed with a map $f: X \to V$ onto a proper algebraic space of dimension $\kappa(L)$. Moreover assume that $L|_F \sim_{\mathbb{Q}} 0$ for every fibre $F$ of $f$. Then $L$ is semi-ample.*

*Proof.* Taking the Stein factorization of $f$ we can assume $f$ is a contraction. Since $\kappa(L) \ge 0$, we can assume $L \ge 0$. After applying Chow's lemma and replacing $X$ we can assume $f$ factors as $X \to Z \to V$ where $h: X \to Z$ is a projective contraction and $Z \to V$ is birational. In the same way we may also assume $Z$ is smooth. Since $L \equiv 0/Z$ and $L \ge 0$, Lemma 5.6 gives us a nef $\mathbb{Q}$-divisor $D$ on $Z$ such that $L \sim_{\mathbb{Q}} h^*D$. Note that since $D$ is endowed with the birational map $Z \to V$, we deduce that $D$ is nef and big and that $\kappa(D) = \kappa(L) = \dim Z = \dim V$.

Assume first that $\kappa(L) = 2$. Then $\dim Z = 2$, $\mathbb{E}(D)$ is contracted to a point by $Z \to V$, and $L|_{h^{-1}\mathbb{E}(D)}$ is semi-ample as $L$ is torsion on the fibres of $X \to V$ and each connected component of $h^{-1}\mathbb{E}(D)$ is contained in such a fibre. Now apply Lemma 7.1. On the other hand suppose $\kappa(L) = 1$. Then $\dim Z = 1$ which implies that $D$ is ample and that $L$ is semi-ample. Finally if $\kappa(L) = 0$, then $Z$ is a point and $L$ is torsion.

□

## 8. Good log minimal models

**Remark 8.1** Let $(X, B)$ be a projective klt pair of dimension 3 over $k$. Assume that $B = B' + A$ where $B', A \ge 0$ are $\mathbb{Q}$-divisors and $A$ is ample. Let $\phi: W \to X$ be a log resolution of $(X, B)$. We can write

$$K_W + B'_W = \phi^*(K_X + B') + E'$$

where $B'_W$ is a $\mathbb{Q}$-boundary, $\phi_* B'_W = B'$, $(W, B'_W)$ is klt, $E' \ge 0$ is exceptional$/X$ and its support contains all the exceptional divisors of $\phi$. Since $\phi$ is obtained by a sequence of blow-ups with smooth centres, there is an exceptional$/X$ divisor $G$ which is ample$/X$. By the negativity lemma, $G \le 0$. Now $\phi^*A + \epsilon G$ is ample for some small $\epsilon > 0$. Pick a general $A_W \sim_{\mathbb{Q}} \phi^*A + \epsilon G$. Then we have

$$K_W + B'_W + A_W \sim_{\mathbb{Q}} \phi^*(K_X + B' + A) + E$$

where $E := E' + \epsilon G \ge 0$ is exceptional$/X$ and its support contains all the exceptional divisors of $\phi$.

It is easy to see from the definitions and using the negativity lemma that any log minimal model (resp. weak lc model) of $(W, B'_W + A_W)$ is also a log minimal model (resp. weak lc model) of $(X, B' + A)$.

**Lemma 8.2.** *Let $(X, B = B' + A)$ be a projective $\mathbb{Q}$-factorial dlt pair of dimension 3 over $k$ of characteristic $p > 5$ such that $B', A \geq 0$ are $\mathbb{Q}$-divisors, $A$ is ample, and $\lfloor B \rfloor = \lfloor B' \rfloor$. Assume $(Y, B_Y)$ is a weak lc model of $(X, B)$ such that $(Y, B_Y)$ is $\mathbb{Q}$-factorial dlt and $Y \dashrightarrow X$ does not contract divisors. Then $(K_Y + B_Y)|_{\lfloor B_Y \rfloor}$ is semi-ample.*

*Proof.* Since $(Y, B_Y)$ is dlt, its lc centres are the lc centres of $(Y, \lfloor B_Y \rfloor)$, in particular, as $\lfloor B_Y \rfloor = \lfloor B'_Y \rfloor$, $\operatorname{Supp} A_Y$ does not contain any lc centre. On the other hand, if $U \subseteq X$ is the largest open subset over which $\alpha\colon X \dashrightarrow Y$ is an isomorphism, then $\operatorname{Supp} A_Y$ contains $Y \setminus \alpha(U)$ [3, proof of Theorem 9.5]. So $Y \setminus \alpha(U)$ does not contain any lc centre of $(Y, B_Y)$.

Let $H_Y$ be a general ample $\mathbb{Q}$-divisor on $Y$ sufficiently small that $A - H$ is also ample where $H$ is the birational transform of $H_Y$. Let $A' \sim_{\mathbb{Q}} A - H$ be general. Pick a small rational number $\epsilon > 0$, and let

$$\Delta := B' + (1 - \epsilon)A + \epsilon A' + \epsilon H.$$

Then $(X, \Delta)$ is dlt on $U$, hence $(Y, \Delta_Y)$ is dlt on $\alpha(U)$. Moreover, since $(Y, B_Y)$ has no lc centre inside $Y \setminus \alpha(U)$, $\operatorname{Supp}(A'_Y + H_Y)$ does not contain any lc centre of $(Y, B_Y)$, so we see that $(Y, \Delta_Y)$ is dlt everywhere.

Now apply [3, Theorem 1.9] to deduce that $(K_Y + \Delta_Y)|_{\lfloor \Delta_Y \rfloor}$ is semi-ample which in turn implies that $(K_Y + B_Y)|_{\lfloor B_Y \rfloor}$ is semi-ample because $K_Y + \Delta_Y \sim_{\mathbb{Q}} K_Y + B_Y$ and $\lfloor \Delta_Y \rfloor = \lfloor B'_Y \rfloor$.

$\square$

**Proposition 8.3.** *Let $(X, B)$ be a projective klt pair of dimension 3 over $k$ of characteristic $p > 5$. Assume that $B$ is a big $\mathbb{Q}$-boundary and that $K_X + B$ is pseudo-effective. Then $(X, B)$ has a good log minimal model.*

*Proof.* We mimic the proof of [3, Theorem 1.3]. By extending $k$ we can assume it is uncountable. Applying Proposition 6.6 to a log minimal model of $(X, B)$, we deduce that $\kappa(K_X + B) \geq 0$. If $K_X + B$ is big, the result is already known [3, Theorems 1.2 and 1.4], so we can assume $\kappa(K_X + B) \leq 2$. Applying [3, Lemma 7.7] we can assume $X$ is $\mathbb{Q}$-factorial.

*Step 1.* By assumption $B$ is a big $\mathbb{Q}$-divisor, so by [22][3, Lemma 9.2], we can assume $B = B' + A$ where $B'$ is an effective $\mathbb{Q}$-divisor and $A \geq 0$ is an ample $\mathbb{Q}$-divisor. Take $M$ such that $K_X + B \sim_{\mathbb{Q}} M \geq 0$. By Remark 8.1, we can assume $(X, B + M)$ is log smooth. Replacing $A$ up to $\mathbb{Q}$-linear equivalence we may assume that $A$ and $B' + M$ share no components. Since

$$K_X + B' + \epsilon M + A \sim_{\mathbb{Q}} (1 + \epsilon)(K_X + B)$$

we may also replace $B'$ with $B' + \epsilon M$ for then we can assume $\operatorname{Supp} M \subseteq \operatorname{Supp} B'$.

We want to prove a slightly more general statement: we consider triples $(X, B, M)$ such that the following hold:

(1) $(X, B)$ is a dlt threefold pair over $k$ of characteristic $p > 5$, and $B$ is a $\mathbb{Q}$-boundary,
(2) $K_X + B \sim_{\mathbb{Q}} M \geq 0$ for some $\mathbb{Q}$-divisor $M$,

(3) $(X, B+M)$ is log smooth,
(4) $0 \leq \kappa(K_X+B) \leq 2$,
(5) $B = B' + A$ where $B', A \geq 0$ are $\mathbb{Q}$-divisors, and $A$ is ample with no common components with $B' + M$, and
(6) $\operatorname{Supp} \lfloor B \rfloor \subseteq \operatorname{Supp} M \subseteq \operatorname{Supp} B'$.

Pick $(X, B, M)$ satisfying the above conditions. We will show that $(X, B)$ has a good log minimal model $(Y, B_Y)$ such that $Y \dashrightarrow X$ does not contract divisors. Define $\theta(X, B, M)$ to be the number of components of $M$ which are not components of $\lfloor B \rfloor = \lfloor B' \rfloor$.

*Step 2.* First assume $\theta(X, B, M) = 0$, which is equivalent to $\operatorname{Supp} M \subseteq \lfloor B \rfloor$. We can run an LMMP on $K_X + B$ which terminates with a log minimal model $(Y, B_Y)$ by special termination [3, Proposition 5.5]. By assumption, $\kappa(K_Y + B_Y) \geq 0$. If $K_Y + B_Y$ is numerically trivial, then by (2) it is torsion and we are done. If $K_Y + B_Y$ is not numerically trivial, then $\kappa(K_Y + B_Y) = n(K_Y + B_Y)$ by Proposition 6.6, hence $K_Y + B_Y$ is endowed with a map $Y \to V$ by Lemma 7.2. Moreover, $(K_Y + B_Y)|_{\lfloor B_Y \rfloor}$ is semi-ample by Lemma 8.2.

Since $Y \to V$ is the map associated to $K_Y + B_Y$, on any of its fibres $F$ we have $M_Y|_F \equiv 0$. So either $F \cap \operatorname{Supp} M_Y = \emptyset$ or $F_{\mathrm{red}} \subseteq \operatorname{Supp} M_Y$. In the former case we have $M_Y|_F = 0$. In the latter case we show $M_Y|_F \sim_{\mathbb{Q}} 0$: from $F_{\mathrm{red}} \subseteq \operatorname{Supp} M_Y \subseteq \lfloor B_Y \rfloor$ and $(K_Y + B_Y)|_{\lfloor B_Y \rfloor} \sim_{\mathbb{Q}} M_Y|_{\lfloor B_Y \rfloor}$ being semi-ample we deduce $M_Y|_{F_{\mathrm{red}}}$ is semi-ample which in turn implies that $M_Y|_F$ is semi-ample as characteristic $k > 0$; since $M_Y|_F \equiv 0$ we must have $M_Y|_F \sim_{\mathbb{Q}} 0$. Now by Lemma 7.3, $M_Y$ is semi-ample. So from now on we can assume $\theta(X, B, M) > 0$.

*Step 3.* For an $\mathbb{R}$-divisor $D = \sum r_i D_i$ we let $D^{=1} = \sum r_i' D_i$ where $r_i' = 1$ if $r_i = 1$ but $r_i' = 0$ otherwise. Similarly let $D^{\leq 1} = \sum r_i'' D_i$ where $r_i'' = \min\{r_i, 1\}$. Define

$$\alpha := \min\{t > 0 \mid (B + tM)^{=1} \neq \lfloor B \rfloor\}.$$

Define $C$ and $N$ by the equalities $(B + \alpha M)^{\leq 1} = B + C$ and $\alpha M = C + N$. By construction $C, N \geq 0$ and $\operatorname{Supp} N = \lfloor B \rfloor$ by (6). By definition of $\alpha$ there must be some components of $\lfloor B + C \rfloor$ which are not in $\lfloor B \rfloor$, and these components are in $\operatorname{Supp} M$. The construction ensures that

$$\theta(X, B + C, M + C) < \theta(X, B, M).$$

It is easy to show that $(X, B+C, M+C)$ satisfies the properties (1)-(6) of Step 1. Indeed properties (1) and (3) follow from the assumption that $(X, B+M)$ is log smooth, and that $C$ is supported on $\operatorname{Supp} M$ and $B + C$ has coefficients at most 1. Properties (2),(5) and (6) are obvious. Property (4) is a consequence of

$$K_X + B \leq K_X + B + tC \leq K_X + B + \alpha M \sim_{\mathbb{Q}} (1 + \alpha)(K_X + B)$$

as it implies

$$\kappa(K_X + B) = \kappa(K_X + B + tC)$$

for all rational numbers $t \in [0, 1]$.

Let $\mathcal{T}$ be the set of those $t \in [0,1]$ such that $(X, B+tC)$ has a good log minimal model $(Y, B_Y + tC_Y)$ such that $Y \dashrightarrow X$ does not contract divisors. Arguing by induction on $\theta(X,B,M)$ and taking into account the previous paragraph, we can assume $1 \in \mathcal{T}$ (note that the case $\theta = 0$ of the induction was settled in Step 2).

*Step 4.* Choose $0 < t \in \mathcal{T}$. We want to show that there is an $\epsilon > 0$ such that $[t-\epsilon, t] \subset \mathcal{T}$. Let $(Y, B_Y + tC_Y)$ be a good log minimal model of $(X, B+tC)$ such that $Y \dashrightarrow X$ does not contract divisors. As $K_Y + B_Y + tC_Y$ is semi-ample, it defines a contraction $f\colon Y \to T$. Choose a sufficiently small $\epsilon > 0$ and run a $K_Y + B_Y + (t-\epsilon)C_Y$-LMMP over $T$ with scaling of $\epsilon C_Y$ as in [3, 3.5]. This is an LMMP on

$$M_Y + (t-\epsilon)C_Y = \frac{1}{\alpha}N_Y + (\frac{1}{\alpha} + t - \epsilon)C_Y$$

and as $C_Y$ is positive on each extremal ray in the process, the LMMP is also an LMMP on $N_Y$. The LMMP terminates on some model $Y'$ by special termination [3, Proposition 5.5] because $\operatorname{Supp} N_Y \subseteq \lfloor B_Y \rfloor$. Note that the LMMP is also an LMMP$/T$ on $K_Y + B_Y + (t-\epsilon')C_Y$ for any $\epsilon' \in (0,\epsilon)$. So we can replace $\epsilon$ with a smaller number if necessary. In particular, we can assume $t-\epsilon$ is rational and that $K_{Y'} + B_{Y'} + (t-\epsilon)C_{Y'}$ is globally nef by Theorem 1.1 or Proposition 3.8 (2).

If $K_{Y'} + B_{Y'} + (t-\epsilon)C_{Y'}$ is numerically trivial, then it is torsion as it has non-negative Kodaira dimension. If it is not numerically trivial, then its Kodaira dimension and nef dimension are equal by Proposition 6.6, hence $K_{Y'} + B_{Y'} + (t-\epsilon)C_{Y'}$ is endowed with a map $Y' \to V$ by Lemma 7.2. By construction, $K_{Y'} + B_{Y'} + tC_{Y'}$ is semi-ample and $\mathbb{R}$-linearly trivial over $T$. Moreover, $Y' \to T$ factors through $Y' \to V$ as $\epsilon$ is sufficiently small: indeed we can assume $K_{Y'} + B_{Y'} + (t-\delta)C_{Y'}$ is nef for some $\delta > \epsilon$, hence for any curve $\Gamma$ contracted by $Y' \to V$ we have $(K_{Y'} + B_{Y'} + (t-\epsilon)C_{Y'}) \cdot \Gamma = 0$ which implies that $(K_{Y'} + B_{Y'} + tC_{Y'}) \cdot \Gamma = 0$ and $(K_{Y'} + B_{Y'} + (t-\delta)C_{Y'}) \cdot \Gamma = 0$; thus any curve contracted by $Y' \to V$ is also contracted by $Y' \to T$. We conclude that

$$K_{Y'} + B_{Y'} + tC_{Y'} \sim_{\mathbb{R}} 0/V.$$

In particular, as $K_{Y'} + B_{Y'} + (t-\epsilon)C_{Y'} \equiv 0/V$, we get $N_{Y'} \equiv 0/V$ and $C_{Y'} \equiv 0/V$.

Put $t' := t-\epsilon$. Let $F$ be a fibre of $Y' \to V$. Since $N_{Y'} \equiv 0/V$, either $F \cap \operatorname{Supp} N_{Y'} = \emptyset$ or $F_{\mathrm{red}} \subseteq \operatorname{Supp} N_{Y'}$. In the first situation $N_{Y'}|_F = 0$, so from $(K_{Y'} + B_{Y'} + tC_{Y'})|_F \sim_{\mathbb{R}} 0$ we deduce $C_{Y'}|_F \sim_{\mathbb{Q}} 0$ and $(K_{Y'} + B_{Y'} + t'C_{Y'})|_F \sim_{\mathbb{Q}} 0$. In the second situation, by Lemma 8.2, $(K_{Y'} + B_{Y'} + t'C_{Y'})|_{\lfloor B_{Y'} \rfloor}$ is semi-ample which implies that $(K_{Y'} + B_{Y'} + t'C_{Y'})|_F$ is semi-ample as $F_{\mathrm{red}} \subseteq \operatorname{Supp} N_{Y'} \subseteq \lfloor B_{Y'} \rfloor$. So $(K_{Y'} + B_{Y'} + t'C_{Y'})|_F \sim_{\mathbb{Q}} 0$. Thus in any case the conditions of Lemma 7.3 are satisfied and so $K_{Y'} + B_{Y'} + t'C_{Y'}$ is semi-ample. Therefore, $K_{Y'} + B_{Y'} + t''C_{Y'}$ is semi-ample and $(Y', B_{Y'} + t''C_{Y'})$ is a good log minimal model of $(X, B+t''C)$ for any $t'' \in [t-\epsilon, t]$, hence $[t-\epsilon, t] \subseteq \mathcal{T}$ as claimed.

*Step 5.* Let $\tau := \inf \mathcal{T}$. Assuming $\tau \notin \mathcal{T}$, we derive a contradiction. Take a strictly decreasing sequence of rational numbers $t_i \in \mathcal{T}$ approaching $\tau$. For each $i$, there is a good log minimal model $(Y_i, B_{Y_i} + t_iC_{Y_i})$ of $(X, B + t_iC)$ such that $Y_i \dashrightarrow X$ does not contract divisors. By taking a subsequence, we can assume that all the $Y_i$ are isomorphic in codimension one. In particular, $K_{Y_1}+B_{Y_1}+\tau C_{Y_1}$ is (numerically) a limit of movable divisors. Run the LMMP on $K_{Y_1}+B_{Y_1}+\tau C_{Y_1}$ with scaling of $(t_1-\tau)C_{Y_1}$. Reasoning as in the first paragraph of Step 4, the LMMP terminates with a model $Y$ on which $K_Y+B_Y+\tau C_Y$ is nef. Note that the LMMP does not contract any divisor by the above movability property. Moreover, $K_Y + B_Y + (\tau + \delta)C_Y$ is nef for some $\delta > 0$. Now, by replacing the sequence, we can assume that $K_Y + B_Y + t_iC_Y$ is nef for every $i$ and by replacing each $Y_i$ with $Y$ we can assume that $Y_i = Y$ for every $i$. Taking limits of log discrepancies (cf. [5, Claim 3.5]) shows that for any prime divisor $D$ on birational models of $Y$ we have

$$a(D, X, B + \tau C) \leq a(D, Y, B_Y + \tau C_Y).$$

Thus $(Y, B_Y + \tau C_Y)$ is a $\mathbb{Q}$-factorial dlt weak lc model of $(X, B + \tau C)$. If we show that $K_Y + B_Y + \tau C_Y$ is semi-ample, then $\tau \in \mathcal{T}$ by Proposition 4.5, a contradiction.

*Step 6.* It remains to show that $K_Y + B_Y + \tau C_Y$ is semi-ample. Let $Y \to T_i$ be the contraction defined by $K_Y + B_Y + t_iC_Y$. For each $i$, the map $T_{i+1} \dashrightarrow T_i$ is a morphism because any curve contracted by $Y \to T_{i+1}$ is also contracted by $Y \to T_i$: to see this, note that each of $K_Y + B_Y + t_iC_Y$, $K_Y + B_Y + t_{i+1}C_Y$ and $K_Y + B_Y + \tau C_Y$ are nef, and $t_i > t_{i+1} > \tau$, so if a curve $\Gamma$ satisfies

$$(K_Y + B_Y + t_{i+1}C_Y) \cdot \Gamma = 0,$$

then we get

$$(K_Y + B_Y + t_iC_Y) \cdot \Gamma = 0 \text{ and } (K_Y + B_Y + \tau C_Y) \cdot \Gamma = 0$$

implying the claim. Perhaps after replacing the sequence, we can assume that $T_i$ is independent of $i$ so we can drop the subscript and simply use $T$. Note that $C_Y \sim_{\mathbb{Q}} 0/T$.

Assume that $\tau$ is irrational. If $K_Y + B_Y + (\tau - \epsilon)C_Y$ is nef for some $\epsilon > 0$, then $K_Y + B_Y + \tau C_Y$ is semi-ample because in this case $K_Y + B_Y + (\tau - \epsilon)C_Y$ is the pullback of a nef divisor on $T$ and $K_Y + B_Y + t_iC_Y$ is the pullback of an ample divisor on $T$. If there is no $\epsilon$ as above, then there is a curve $\Gamma$ generating some extremal ray such that $(K_Y + B_Y + \tau C_Y) \cdot \Gamma = 0$ and $C_Y \cdot \Gamma > 0$ by [3, 3.4] and Theorem 1.1. This is not possible since $\tau$ is assumed to be irrational. So from now on we assume that $\tau$ is rational.

By Proposition 6.6 and Lemma 7.2, $K_Y + B_Y + \tau C_Y$ is endowed with a map $f\colon Y \to V$. Any curve contracted by $Y \to T$ is also contracted by $Y \to V$. So $Y \to V$ factors through $Y \to T$. Now $K_Y + B_Y + \tau C_Y \equiv 0/V$, so $C_Y$ is nef$/V$ but $N_Y$ is anti-nef$/V$. Let $F$ be a fibre of $Y \to V$. Then $F$ is disjoint from $\operatorname{Supp} N_Y$ or $F_{\mathrm{red}}$ is contained in $\operatorname{Supp} N_Y$. Suppose $F \cap \operatorname{Supp} N_Y = \emptyset$. Then near $F$, $K_Y + B_Y + \tau C_Y$ is a positive multiple of $K_Y + B_Y + t_iC_Y$, hence $(K_Y + B_Y + \tau C_Y)|_F$ is semi-ample as $(K_Y + B_Y + t_iC_Y)|_F$ is semi-ample. Thus $(K_Y + B_Y + \tau C_Y)|_F \sim_{\mathbb{Q}} 0$. On the other hand, if $F_{\mathrm{red}} \subseteq \operatorname{Supp} N_Y$, then again

$(K_Y + B_Y + \tau C_Y)|_F \sim_{\mathbb{Q}} 0$ because $(K_Y + B_Y + \tau C_Y)|_{\lfloor B_Y \rfloor}$ is semi-ample by Lemma 8.2 and $F_{\mathrm{red}} \subseteq \operatorname{Supp} N_Y \subseteq \lfloor B_Y \rfloor$. The conditions of Lemma 7.3 are now satisfied, hence $K_Y + B_Y + \tau C_Y$ is semi-ample.

□

## 9. Proof of main results

Theorem 1.1 was already proved in Section 3. We will give the proofs of the other main results.

*Proof.* (of Theorem 1.2) We extend $k$ so that it is uncountable. By taking a $\mathbb{Q}$-factorialization [3, Lemma 6.7] we may assume that $X$ is $\mathbb{Q}$-factorial. Let $A = D - (K_X + B)$. By changing $A$ and $B$ we can assume $(X, \Delta := B + A)$ is klt and that $A$ is an ample $\mathbb{Q}$-divisor. Moreover, if $P$ is the pullback of a sufficiently ample divisor on $Z$, then $K_X + \Delta + P$ is globally nef by Theorem 1.1, and semi-ampleness of $K_X + \Delta + P$ implies semi-ampleness of $K_X + \Delta$ over $Z$. So replacing $\Delta$ with a boundary $\mathbb{R}$-linearly equivalent to $\Delta + P$, we can assume $Z$ is a point.

By Proposition 3.8, there exist real numbers $r_j > 0$ and $\mathbb{Q}$-boundaries $\Delta_j$ such that $\Delta = \sum r_j \Delta_j$, $||\Delta - \Delta_j||$ are sufficiently small, $\Delta_j \geq A$, $(X, \Delta_j)$ are klt, and $K_X + \Delta_j$ are all nef. By Proposition 8.3, $K_X + \Delta_j$ is semi-ample for each $j$. Therefore, $K_X + \Delta$ is semi-ample too.

□

*Proof.* (of Theorem 1.3) Let $R$ be a $K_X + B$-negative extremal ray/$Z$. Note that $R$ being over $Z$ means that $P \cdot R = 0$ where $P$ is the pullback of some ample divisor on $Z$. By adding a small ample divisor to $B$ and by perturbing the coefficients, we can assume $B$ is a big $\mathbb{Q}$-boundary. In particular, there are only finitely many negative extremal rays of $K_X + B$ and they are all generated by extremal curves with bounded intersection with $K_X + B$, by Theorem 1.1. Thus there is an ample $\mathbb{Q}$-divisor $H$ such that $L = K_X + B + H$ is globally nef and $L^{\perp} = R$. By Theorem 1.2, $L$ is semi-ample so it defines a projective contraction $X \to T$. The morphism $X \to T$ is nothing but the contraction of $R$. Since $R$ is an extremal ray over $Z$, the morphism $X \to Z$ factors through $X \to T$.

□

*Proof.* (of Theorem 1.4) This follows from Theorem 1.2 and Proposition 4.2. □

*Proof.* (of Theorem 1.5) This follows from Theorems 1.2 and 1.3 and Proposition 4.3. □

*Proof.* (of Theorem 1.6) If $K_X + B$ is pseudo-effective/$Z$, this is already proved in Proposition 4.5. If $K_X + B$ is not pseudo-effective/$Z$, then the LMMP is also an LMMP on $K_X + B + \epsilon C$ with scaling of $(1 - \epsilon)C$, for some $\epsilon > 0$, hence it terminates by Theorem 1.5. □

*Proof.* (of Theorem 1.7) We can find a projective contraction $\overline{f}\colon \overline{X} \to \overline{Z}$ of normal projective varieties such that $X$ is an open subset of $\overline{X}$ and $\overline{f}$ restricted to $X$ is $f$. Let $\phi\colon \overline{W} \to \overline{X}$ be a log resolution such that any prime exceptional divisor of $\phi$ whose generic point maps into $X$, has positive log discrepancy with respect to $(X, B)$. Let $B_{\overline{W}}$ be the sum of the birational transform of $B$ and the reduced exceptional divisor of $\phi$. Run an LMMP$/\overline{X}$ on $K_{\overline{W}} + B_{\overline{W}}$ with scaling of some ample divisor. By our choice of $\phi$ we reach a model $\overline{Y}$ such that $\overline{Y} \to \overline{X}$ is a small morphism over $X$. So we can replace $(X, B)$ with $(\overline{Y}, B_{\overline{Y}})$, hence assume $X$ is projective and $\mathbb{Q}$-factorial.

Pick a general sufficiently ample divisor $A$ so that $K_X + B + A$ is nef$/Z$ and $(X, B + A)$ is dlt. Let $\epsilon > 0$ be small enough so that $K_X + B + \epsilon A$ is not pseudo-effective$/Z$. We can find a boundary $\Delta \sim_{\mathbb{R}} B + \epsilon A/Z$ such that $(X, \Delta + (1 - \epsilon)A)$ is klt. Now run an LMMP$/Z$ on $K_X + \Delta$ with scaling of $(1 - \epsilon)A$. By Theorem 1.5, the LMMP terminates with a Mori fibre space of $(X, \Delta)$ over $Z$. The LMMP is also an LMMP on $K_X + B$ with scaling of $A$, hence the Mori fibre space is also a Mori fibre space of $(X, B)$ over $Z$.

$\square$

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