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Tropical Amplitudes

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Abstract. In this work, we argue that the $\alpha' \to 0$ limit of closed string theory scattering amplitudes is a tropical limit. The motivation is to develop a technology to systematize the extraction of Feynman graphs from string theory amplitudes at higher genus. An important technical input from tropical geometry is the use of tropical theta functions with characteristics to rigorously derive the worldline limit of the worldsheet propagator. This enables us to perform a non-trivial computation at two loops: we derive the tropical form of the integrand of the genus-two four-graviton type II string amplitude, which matches the direct field theory computations. At the mathematical level, this limit is an implementation of the correspondence between the moduli space of Riemann surfaces and the tropical moduli space.

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1. Introduction

It is well accepted that the field theory limit[1] of string theory scattering amplitudes reproduces the usual perturbative expansion of quantum field theory. However, a constructive general proof of that statement has not been given yet. Besides the intrinsic interest of such a proof, this problem is important for several reasons.

Firstly, string inspired methods have already proved their efficiency at one loop to compute scattering amplitudes in field theory [1–15] and to obtain more general results about amplitudes [16–22]. Secondly, it is important to better understand the mechanisms by which string theory renormalizes supergravity theories. In particular, the question of the ultraviolet (UV) divergences of maximal supergravity continues to draw much attention [23–33] and string theory provides a well-suited framework to analyze this issue [32–36].

In this paper, we revisit the $\alpha' \to 0$ limit of string theory [37] in the context of tropical geometry, a link previously unnoticed. Since tropical geometry describes—in particular—how Riemann surfaces degenerate to certain graphs called tropical graphs, it provides a framework for studying this limit. Tropical graphs are then seen as particles' worldlines.

Only at one loop, the Bern–Kosower rules [7–10] give a full-fledged method to obtain field theory amplitudes from string theory. At higher loops, such techniques are not available and this work is a step in this direction.[2]

The aim of this work is therefore computational: it is to develop methods based on tropical geometry to extract the field theory limit of higher genus closed string theory amplitudes.

The "tropicalization" of a complex variety is a particular degeneration by which the variety sees its dimension halved. Consider for instance the annulus $\Sigma = \{z, 1 < |z| < \rho\}$.

The *tropical* variety is obtained by a taking the "modulus" of the coordinate in Σ; paraphrasing [40], the tropical limit corresponds to "forgetting the phases in complex numbers". The meaning of the modulus of z is easier seen

[1] Throughout the text, we call indistinctly, "point-like", "field theory", "infinite tension" "tropical" or "$\alpha' \to 0$" this limit. We recall that the Regge slope α' of the string is a positive quantity of mass dimension -2 related to the string length ℓ_s by $\alpha' = \ell_s^2$.

[2] An alternative approach exists in the literature to study the $\alpha' \to 0$ limit of string amplitudes, based on the Schottky parametrization, see the recent works [38, 39].

by mapping the annulus to the cylinder via $z \to \exp iw$ with $w = \sigma_1 + i\sigma_2$: $|z|$ is a longitudinal coordinate along the cylinder and the tropical variety is just a segment in this case.

We will make this more precise for generic Riemann surfaces in Sect. 3. It should however already be clear that this process is similar to the point-like limit of string theory. Seeing the cylinder as the worldsheet of a closed string propagating through spacetime, the phase-dependence of the amplitude enforces the "level-matching" condition. Level-matching is a physical constraint that forces the string to be balanced and have as many left-moving as right-moving excitations. But, in the $\alpha' \to 0$ limit, one could think that the massive excitations, that have masses of order $1/\alpha'$, should decouple and make the level matching condition trivial. There is however a caveat. When the field theory amplitudes have ultraviolet (UV) divergences, the massive modes do not decouple but instead act as UV regulators. These give rise to counter-terms in the amplitudes. We shall see that these counter-terms have a natural description in tropical geometry: they correspond to certain weighted vertices.

This text begins in Sect. 2 with an introduction to tropical geometry. In Sect. 2.3.2, we prove an important lemma, on tropical theta functions with characteristics, Lemma 1. Later we make use of it to show that the $\alpha' \to 0$ limit of the string theory propagator on higher genus surfaces reduces to the worldline propagator. This tropical limit of the string propagator is one of the main contributions of this work. This step is required to extract in full rigor the form of the field theory amplitudes arising in the $\alpha' \to 0$ limit of string theory. This discussion is extended in Sect. 3 to the connection between tropical and classical geometry.

In Sect. 4, we formulate the field-theory limit of closed string theory amplitudes in the context of tropical geometry. We explain how, as $\alpha' \to 0$, a genus g, n-point string theory amplitude $\mathsf{A}^{(g,n)}_{\alpha'}$ reduces to an integral over the moduli space of tropical graphs [41,42], $\mathcal{M}^{\text{trop}}_{g,n}$

$$\lim_{\alpha' \to 0} \mathsf{A}^{(g,n)}_{\alpha'} = \int_{\mathcal{M}^{\text{trop}}_{g,n}} \mathrm{d}\mu^{\text{trop}} F_{g,n}, \tag{1.1}$$

The right-hand side of this equation is the renormalized field theory amplitude written in its "tropical representation", or in short a "tropical amplitude". The integration measure $\mathrm{d}\mu^{\text{trop}}$ is defined in terms of the Schwinger proper times of the graph—the lengths of the inner edges. The integrand $F_{g,n}$ contains the theory-dependence of the amplitude and encompasses both the numerators and denominators of the Feynman graphs [see Eq. (4.12) below]. This type of formulas are the origin of Feynman's construction of quantum field theory [43]. The novelty of our approach lies in the use of tropical geometry to extract the limit, which allows to recycle some of the string theory efficiency and compactness in field theory.

We come to practical applications in Sect. 5. We start with a review of tree-level and one-loop methods. Then we compute the tropical limit of the two-loop four-graviton type II string amplitude of D'Hoker and Phong [44–50] and find agreement with the supergravity result of [51,52]; that is another main contribution of this paper.

Besides the study of the $\alpha' \to 0$ limit of string amplitudes, our approach sheds a new light on the geometry of field theory amplitudes: they are integrals over the tropical moduli space. The components of the Feynman integrands also acquire a geometrical origin: the first Symanzik polynomial is seen to be the determinant of the period matrix of the tropical graph, while the second is written in terms of Green's functions on the graph. Similar observations were made in [52,53].

We close this introduction with a comment. String field theory constructions, Zwiebach's bosonic string field theory in particular [54], give formal representations of string field theory amplitudes in terms of certain Feynman graphs. Although massless fields (field theory fields) contributions are accounted for in these graphs, these constructions are not designed for practical implementation of the field theory limit. Their goal is rather a non-perturbative formulation of string field theory. In principle one could take formally the $\alpha' \to 0$ limit of a string field theory amplitude. This would lead us to a set of Feynman rules and a prescription to build field theory amplitudes: the exact same one as if we had started with a field theory Lagrangian.

What we want to do here is the opposite. We want to be able to take a string theory amplitude, expressed in its compact form as a single moduli space integral, and extract field theory graphs out of it, in the spirit of the Bern–Kosower rules.

Note added. *In the second version of this paper, the author added a comment on the three-loop amplitude of* [55] *at the end of Sect.* 5.

2. Tropical Geometry

Tropical geometry is a recent and active field in mathematics.[3] The basic objects, tropical varieties, can be either abstract [62] or defined as algebraic curves over certain spaces [58]. Tropical varieties also arise as the result of a degeneration of the complex structure of complex varieties called *tropicalization* [63,64].

The use of tropical geometry in physics is not new: even before the coinage of the word "tropical", the authors of [65] studied a class of embedded tropical varieties called webs, arising from the degeneration of brane systems. Also, Kontsevich and Soibelman introduced tropical geometry in the context of mirror symmetry [66], which became an active area of investigation (see the book [67]).

2.1. Tropical Graphs

An abstract tropical graph is a connected graph with labeled legs (external edges), whose inner edges have a length and whose vertices are weighted. The

[3] For introductory works, the reader is referred to [40,56–61], and to [41] for a more exhaustive bibliography.

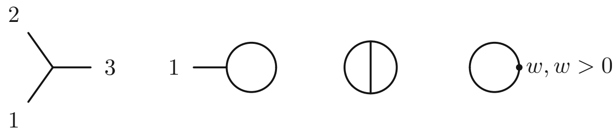

FIGURE 1. Examples of tropical graphs (*left* to *right*): a 3-point tropical tree, a once-punctured graph of genus one, a 2-loop tropical graph, a graph of genus $1+w$

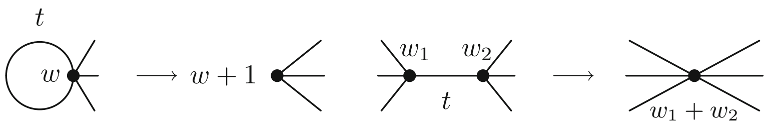

FIGURE 2. Specialization rules as $t \to 0$

external legs are called punctures or marked points, and they have infinite length. A tropical graph Γ is then a triple $\Gamma = (G, w, \ell)$ where; G is a connected graph called the combinatorial type of Γ, ℓ and w are length and weight functions on the edges and on the vertices

$$\begin{aligned} \ell &: E(G) \cup L(G) \to \mathbb{R}_+ \cup \{\infty\}, \\ w &: V(G) \to \mathbb{Z}_+. \end{aligned} \tag{2.1}$$

The quantities $E(G)$, $L(G)$ and $V(G)$ are, respectively, the sets of inner edges, legs and vertices of the graph. The total weight $|w|$ of a tropical graph Γ is the sum of all the weights of the vertices $|w| = \sum_{V(G)} w(V)$. Its genus $g(\Gamma)$ is the number of loops $g(G)$ of G plus the total weight

$$g(\Gamma) = g(G) + |w|. \tag{2.2}$$

A *pure* tropical graph is by definition a tropical graph that only has vertices of weight zero, therefore its genus of is given by the number of loops in the usual sense. In Fig. 1 we give a few examples of tropical graphs.

As for classical complex curves, a stability condition must be added to the previous definitions; we consider only genus-g tropical graphs with n punctures for which[4]

$$2g - 2 + n \geq 1. \tag{2.3}$$

This implies that every vertex of weight zero must have valency at least three and vertices of weight one should have at least one leg.

A specialization map acts on these graphs by contracting edges and adding the weights of the vertices that are brought together, as pictured in Fig. 2. This gives another interpretation of the weights; they correspond to

[4] Strictly speaking, the local valency condition should be viewed as considering *classes* of abstract tropical graphs under the equivalence relation that contracts edges connected to 1-valent vertices of weight 0, and removes weight 0 bivalent vertices. Physically, on the worldline, this equivalence relation is perfectly sensible, since no interpretation of these 1- or 2-valent vertices of weight zero seem natural in the absence of external classical sources.

degenerated loops, and it is easily checked that the genus of a graph (2.2) and the stability criterion (2.3) are stable under specialization.

Finally, a graph that can be disconnected in two components by removing a single edge is called one-particle-irreducible (1PI), otherwise it is called one-particle-reducible (1PR).

Physically, tropical graphs will be interpreted as the worldlines swept by propagating particles, just like Riemann surfaces are strings worldsheets. The lengths of the edges are Schwinger proper times, and a nonzero weight on a vertex indicates the possible insertion of a counter-term to a divergence in the graph. Since loops with very short proper times correspond to the UV region, it is intuitively clear that this should be the case. In particular, at genus g, the tropical graph corresponding to single vertex of weight g will be supporting counter-terms to the primary divergence of the amplitude.

2.2. Homology, Forms, Jacobian and Divisors

In this paragraph, following [58], we introduce the tropical analogues of some common objects of classical geometry; abelian forms, period matrices and Jacobian varieties. Some care is required because graphs of identical genus may not have the same number of inner edges. We first avoid this subtlety and start with pure graphs.

Let Γ be a pure tropical graph of genus g and $(B_1, \ldots, B_g)$ be a canonical homology basis of Γ, as in Fig. 3a. The vector space of the g independent tropical one-forms ω_I^{trop} can be canonically defined by;

$$\omega_I^{\text{trop}} = \begin{cases} 1 \text{ on } B_I, \\ 0 \text{ otherwise.} \end{cases} \tag{2.4}$$

These forms are *constant* on the edges of the graph. The period matrix $\boldsymbol{K}$ is a $g \times g$ positive definite real-valued matrix, defined by

$$\oint_{B_I} \omega_J^{\text{trop}} = K_{IJ}. \tag{2.5}$$

The Jacobian of Γ is a real torus defined by

$$J(\Gamma) = \mathbb{R}^g / \boldsymbol{K}\mathbb{Z}^g. \tag{2.6}$$

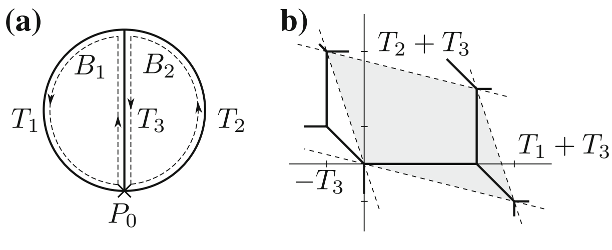

FIGURE 3. **a** Genus-two graph with edges lengths T_1, T_2, T_3. **b** Image of Γ (*thick line*) by the tropical Abel–Jacobi map in the Jacobian $J(\Gamma) = \mathbb{R}^2/K^{(2)}\mathbb{Z}^2$

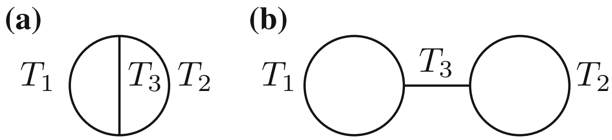

FIGURE 4. Genus-two graphs described in the examples

The tropical version of the Abel–Jacobi map μ^{trop} [58,62] is then defined by integration along a path γ between P_0 and P_1 on the graph as a map to $J(\Sigma)$;

$$\mu^{\mathrm{trop}}(P_0, P_1) = \int_\gamma (\omega_1^{\mathrm{trop}}, \dots, \omega_g^{\mathrm{trop}}) \mod \boldsymbol{K}\mathbb{Z}^g. \tag{2.7}$$

Changing γ by elements of the homology basis results in adding to the integral in the right-hand side some elements of the lattice $K\mathbb{Z}^g$. Thus μ^{trop} is well-defined as a map to the Jacobian torus. Here are two examples taken from [58].

Examble 1. Let Γ be the genus-two tropical graph depicted in Fig. 3a) with canonical homology basis as in Fig. 3. Its period matrix is

$$\boldsymbol{K}^{(2)} = \begin{pmatrix} T_1 + T_3 & -T_3 \\ -T_3 & T_2 + T_3 \end{pmatrix}. \tag{2.8}$$

Choosing P_0 as depicted, one can draw the image of Γ by the tropical Abel–Jacobi map in $J(\Gamma)$, as shown in the Fig. 3b).

Examble 2. Figure 4 depicts two inequivalent pure tropical graphs of genus two. The period matrix $K^{(2)}$ of the 1PI graph a) is given in (2.8), while that of the 1PR graph b) is given by $\mathrm{Diag}(T_1, T_2)$. This illustrates the fact that the period matrix is independent of the lengths of the separating edges.

The generalization of this discussion to the case of tropical graphs with weighted vertices depends on the approach one wants to use. A simplistic approach consists in using a homology basis of size $g(G)$ instead of $g(\Gamma)$, thereby ignoring the weights on the vertices; in this case, the definitions given before apply straightforwardly. However, in doing so, the dimension of the Jacobian drops under specialization. A more complete treatment of this question is provided in Ref. [41].

2.3. Divisors and Theta Characteristics

Now we introduce the notion of divisors and rational functions in order to define tropical theta characteristics.

2.3.1. Divisors on Graphs. A divisor D on a tropical graph is a formal sum of points, weighted by integer multiplicities;

$$D = \sum_{i=1}^{n} a_i P_i, \quad a_i \in \mathbb{Z}. \tag{2.9}$$

The degree of a divisor is given by the sum of its weights; in the previous example it is $a_1 + \cdots + a_n$.

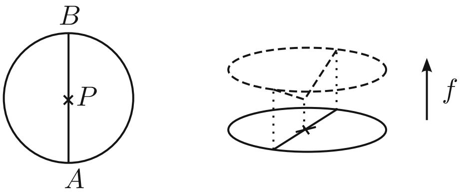

FIGURE 5. Example of rational function f on a two-loop graph

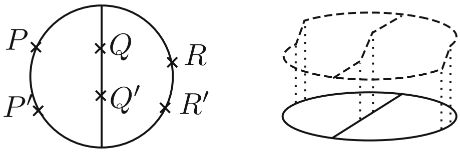

FIGURE 6. Example of linear equivalence; $P + Q + R \sim P' + Q' + R'$

A rational function on a tropical graph is a continuous, piecewise-linear function with integer slopes (see Fig. 5). The order of a rational function at a divisor P is defined by the sum of the outgoing slopes at P. A rational function is said to have a pole of order n at P if its order is $-n < 0$. It is said to have a zero of order n if its order is $n > 0$. For $n = 0$, the function is simply regular at P.[5]

The divisor $\mathrm{div}(f)$ of a rational function f is defined to be the sum of the divisors P of the graph, weighted by the order of f at P. In the example of Fig. 5, if the slopes of the f on the central edge are ± 1, then we find $\mathrm{div}(f) = 2P - A - B$.

Two divisors D and D' are said to be linearly equivalent, $D \sim D'$, if and only if there exists a rational function f whose divisor is $D - D'$, as in Fig. 6. Finally, a canonical divisor on a graph is a linear equivalence class of divisors D of which a representative K_Γ is defined by

$$K_\Gamma = \sum_{P \in \Gamma} (\mathrm{valence}(P) - 2)P. \tag{2.10}$$

For instance, if Γ is a trivalent graph, a representative canonical divisor is the sum of the points at the vertices; on the example of Fig. 5, $K = A + B$.

2.3.2. Tropical Theta Characteristics. To define tropical theta characteristics, originally introduced in [58,68], we follow [69]. A theta characteristics on a graph Γ is a class of divisors D such that $2D$ is linearly equivalent to K_Γ;

$$2D \sim K_\Gamma \tag{2.11}$$

[5] Strictly speaking, another property should be added to the definition of a rational function: it must have finitely many poles and zeros. Thus, a rational function has finitely many linear pieces.

This definition is equivalent to the following. To define a theta characteristics on a graph Γ, first define a $\mathbb{Z}_2$ flow on the graph, i.e. a cycle C on Γ (possibly disconnected) such that at each vertex the number of edges belonging to the cycle is 0 modulo 2. Then put arrows on the complement of C in G that go in the direction opposite to Γ.

Where the arrows meet, insert a divisor weighted by the numbers of edges meeting there, minus 1. Then, this divisor is a theta characteristics in the sense of Eq. (2.11), as shown in Refs. [68, Lemma 6] or [69, Lemma 3.4].

Different choices of flows produce non-equivalent tropical theta characteristics. In total, there are 2^g tropical theta characteristics [68].

While the relation between tropical and classical theta characteristics does not appear to have been discussed in the literature, we will here conjecture how to associate a g-dimensional vector to a tropical theta characteristics.

Take the flow C defined above, it is uniquely decomposed in the homology as

$$C = \cup_{i\in\mathcal{I}} B_i, \tag{2.12}$$

for some unique set $\mathcal{I}$. It is then conjectured here that the theta characteristics associated with this cycle is the vector $\boldsymbol{\beta}$ of $\frac{1}{2}(\mathbb{Z}/2\mathbb{Z})^g$ with entries β_i, $i = 1,\dots,g$ such that

$$\beta_i = \begin{cases} 1/2 & \text{if } i\in\mathcal{I}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.13}$$

An example of this construction is provided in Fig. 7.

We now have the following lemma.

Lemma 1. *Let P and Q be two points on a tropical graph Γ, let γ be a path joining them and $\mathrm{dist}_\gamma(P,Q)$ be the distance between P and Q along γ. Then, there always exist a tropical theta characteristics $\boldsymbol{\beta}\in\frac{1}{2}(\mathbb{Z}/2\mathbb{Z})^g$ such that*

$$\boldsymbol{\beta}\cdot\int_\gamma(\omega_1^{\mathrm{trop}}\dots\omega_g^{\mathrm{trop}}) = \frac{1}{2}\mathrm{dist}_\gamma(\mathrm{P},\mathrm{Q}). \tag{2.14}$$

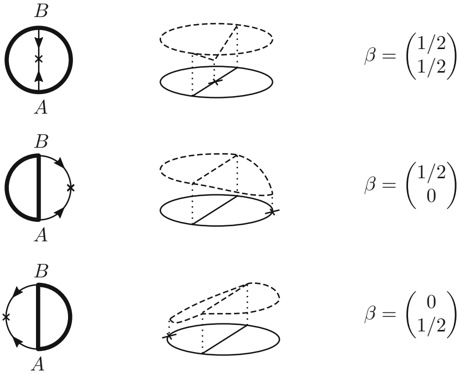

FIGURE 7. The three tropical theta characteristics at genus two

Proof. First, given two points P and Q joined by a path γ, there always exist at least one $\mathbb{Z}_2$ flow C containing γ. This cycle is decomposed uniquely as a particular union of homology cycles; this defines a corresponding set $\mathcal{I}_C$, as in (2.12).

Let $\boldsymbol{\beta}^{(C)}$ be the tropical theta characteristics associated with C as in Eq. (2.13). By definition, its only nonzero entries $\beta_J^{(C)} \neq 0$ are these for which $J \in \mathcal{I}_C$. The entries of the vector $\int_\gamma(\omega_1^{\text{trop}}, \dots, \omega_g^{\text{trop}})$, into which $\boldsymbol{\beta}^{(C)}$ is dotted, result from the integration of the tropical one-forms along γ. By definition again, the individual one-forms ω_J^{trop} integrated along γ give exactly the length of the portion of the cycle B_J that belongs to γ, which we can call γ_J. Note that if $\gamma \cap B_J = \emptyset$, then $\gamma_J = 0$. In general, several cycles share an edge $\gamma \cap B_{J_1} = \cdots = \gamma \cap B_{J_k}$ and this implies that the vector $\int_\gamma(\omega_1^{\text{trop}}, \dots, \omega_g^{\text{trop}})$ has entries that can be equal.

The scalar product with $\boldsymbol{\beta}^{(C)}$ precisely has the effect to avoid to double count these components. Indeed, among all these cycles $B_{J_1}, \dots B_{J_k}$ which would produce identical terms, the unique decomposition (2.12) picks only the one that belongs to $\mathcal{I}_C$. Therefore, the left-hand side of (2.14) is rewritten as the following sum

$$\begin{aligned} \boldsymbol{\beta}^{(C)} \cdot \int_\gamma (\omega_1^{\text{trop}} \dots \omega_g^{\text{trop}}) &= \sum_{J=1}^{g} \beta_J^{(C)} \gamma_J \\ &= \frac{1}{2} \sum_{J \in \mathcal{I}_C} \gamma_J \end{aligned} \tag{2.15}$$

where the right-hand side of the second line is one-half of the length of the path γ, as claimed. □

Figure 8 shows an illustration of this proof.

2.4. The Tropical Moduli Space

The moduli space $\mathcal{M}(\Gamma)$ associated with a particular tropical graph $\Gamma = (G, w, \ell)$ is the cone spanned by the lengths of its inner edges, modulo the discrete automorphism group of the graph;

$$\mathcal{M}(\Gamma) = \mathbb{R}_+^{|E(G)|}/\text{Aut}(G). \tag{2.16}$$

The tropical moduli space of all genus g, n-punctured graphs is defined by gluing all these cones together [41,42], we denote it $\mathcal{M}_{g,n}^{\text{trop}}$. In physical terms, this

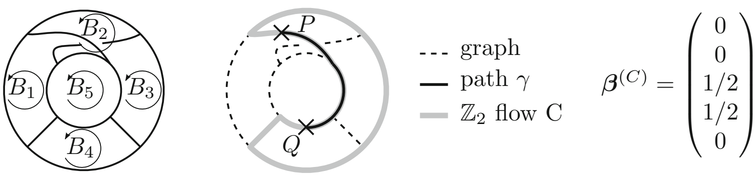

FIGURE 8. Five-loop tropical characteristics and illustration of the lemma

definition is that of the moduli space of Feynman or worldline graphs including graphs with counter-terms. We reproduce a few examples below, and start with $\mathcal{M}_{0,n}^{\mathrm{trop}}$. These latter spaces are themselves tropical varieties (actually, tropical orbifolds), of dimension $(n-3)$ [59,61]. Because of the stability condition (2.3), the smallest allowed value of n is $n=3$. The space $\mathcal{M}_{0,3}^{\mathrm{trop}}$ contains only one graph with no modulus (no inner length): the three-punctured tropical curve. The space $\mathcal{M}_{0,4}^{\mathrm{trop}}$ has more structure; it is isomorphic to the three-punctured tropical curve and contains combinatorially distinct graphs which have at most one inner length, as shown below in Fig. 9. The space $\mathcal{M}_{0,5}^{\mathrm{trop}}$ is a two-dimensional simplicial complex with an even richer structure (Fig. 10). At genus one, $\mathcal{M}_{1,1}^{\mathrm{trop}}$ is also easily described. A genus-one tropical graph with one leg is either a loop or a vertex of weight one. Hence, $\mathcal{M}_{1,1}^{\mathrm{trop}}$ is isomorphic to the half-infinite line $\{T \in \mathbb{R}_+\}$. The graph with $T=0$ is the weight-one vertex, while nonzero T's correspond to loops of length T.

For generic g and n, Euler's relation gives that a stable graph has at most $3g-3+n$ inner edges and has exactly that number if and only if the graph is pure and possess only trivalent vertices. This implies that $\mathcal{M}_{g,n}^{\mathrm{trop}}$ is

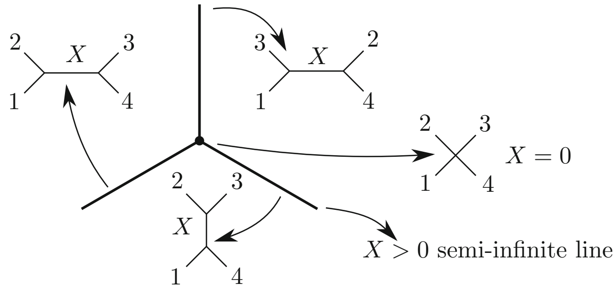

FIGURE 9. *Thick line* $\mathcal{M}_{0,4}^{\mathrm{trop}}$. The X coordinate gives the length of the inner edge of the various graphs. $X=0$ is common to the three branches

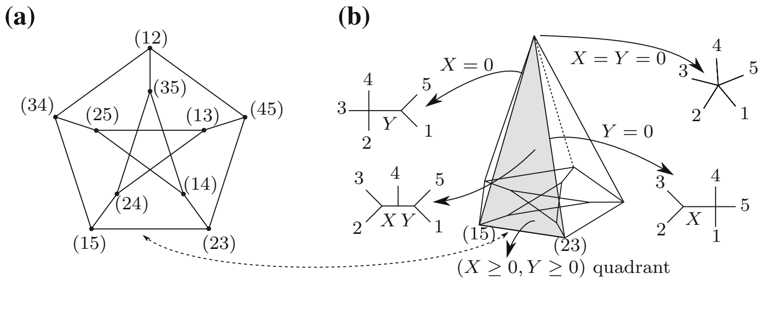

FIGURE 10. **a** A slice of the tropical moduli space $\mathcal{M}_{0,5}^{\mathrm{trop}}$. **b** $\mathcal{M}_{0,5}^{\mathrm{trop}}$, with a specific quadrant in *gray*

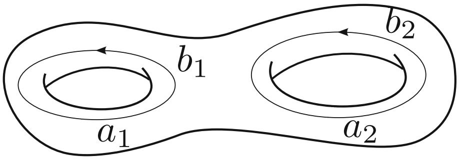

FIGURE 11. Canonical homology basis, example for $g = 2$

of dimension $3g - 3 + n$ almost everywhere, while some of its subsets (faces) are of higher codimension. Finally, note that there also exist a description of $\mathcal{M}_{g,n}^{\text{trop}}$ in terms of the category of "stacky fans", discussed in Refs. [70,71].

3. Classical Geometry and the Tropical Limit

3.1. Riemann Surfaces and Their Jacobians

Let Σ be a generic Riemann surface of genus g and let (a_I, b_J), $I, J = 1, \ldots, g$ be a canonical homology basis on Σ with intersection $a_I \cap b_J = \delta_{IJ}$ and $a_I \cap a_J = b_I \cap b_J = 0$, as in Fig. 11.

The abelian differentials ω_I, $I = 1, \ldots, g$ are holomorphic 1-forms, they can be normalized along a-cycles, so that their integral along the b-cycles defines the period matrix $\mathbf{\Omega}$ of Σ:

$$\oint_{a_I} \omega_J = \delta_{IJ}, \qquad \oint_{b_I} \omega_J = \Omega_{IJ}. \tag{3.1}$$

The modular group $\mathrm{Sp}(2g, \mathbb{Z})$ at genus g is spanned by the $2g \times 2g$ matrices of the form $\left(\begin{smallmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{smallmatrix}\right)$, where $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ and $\boldsymbol{D}$ are $g \times g$ matrices with integer coefficients satisfying $\boldsymbol{A}\boldsymbol{B}^t = \boldsymbol{B}\boldsymbol{A}^t$, $\boldsymbol{C}\boldsymbol{D}^t = \boldsymbol{D}\boldsymbol{C}^t$ and $\boldsymbol{A}\boldsymbol{D}^t - \boldsymbol{B}\boldsymbol{C}^t = \mathbf{Id}_g$, with $\mathbf{Id}_g$ the identity matrix. At $g = 1$, the modular group reduces to $\mathrm{SL}(2, \mathbb{Z})$. The Siegel upper half-plane $\mathcal{H}_g$ is the set of symmetric $g \times g$ complex matrices with positive definite imaginary part

$$\mathcal{H}_g = \{\mathbf{\Omega} \in \mathrm{Mat}(g \times g, \mathbb{C}) : \mathbf{\Omega}^t = \mathbf{\Omega}, \mathrm{Im}\,(\mathbf{\Omega}) > 0\}. \tag{3.2}$$

The modular group $\mathrm{Sp}(2g, \mathbb{Z})$ acts on $\mathcal{H}_g$ by $\mathbf{\Omega} \mapsto (\boldsymbol{A}\mathbf{\Omega} + \boldsymbol{B})(\boldsymbol{C}\mathbf{\Omega} + \boldsymbol{D})^{-1}$. Period matrices of Riemann surfaces are elements of the Siegel upper half-plane and the action of the modular group on them is produced by the so-called Dehn twists of the surface along homology cycles. The Jacobian variety $J(\Sigma)$ of Σ with period matrix $\mathbf{\Omega}$ is the complex torus

$$J(\Sigma) = \mathbb{C}^g / (\mathbb{Z}^g + \mathbf{\Omega}\mathbb{Z}^g). \tag{3.3}$$

Integration along a path C between two points p_1 and p_2 on the surface of the holomorphic one-forms defines the classical Abel–Jacobi map μ:

$$\mu(p_1, p_2) = \int_{p_1}^{p_2} (\omega_1, \ldots, \omega_g) \mod \mathbb{Z}^g + \mathbf{\Omega}\mathbb{Z}^g. \tag{3.4}$$

As in the tropical case, the right-hand side of (3.4) does not depend on the integration path. Note that, apart for the very special case of genus one where

$\mu(\Sigma_1) \cong \Sigma_1$, the image of a genus $g \geq 2$ Riemann surface Σ_g by μ is strictly contained in $J(\Sigma_g)$, $\mu(\Sigma_g) \subsetneq J(\Sigma_g)$.

3.2. Riemann Surfaces and Their Moduli Spaces $\mathcal{M}_{g,n}$, $\overline{\mathcal{M}}_{g,n}$

Smooth Riemann surfaces of genus g with n punctures span a moduli space denoted $\mathcal{M}_{g,n}$ of complex dimension $3g-3+n$ whose coordinates are called the moduli of the surface. This space is not compact, since surfaces can develop nodes when non-trivial homotopy cycles pinch off and give rise to nodal surfaces with double points. The result of adding all such nodal curves to $\mathcal{M}_{g,n}$ is the well known Deligne-Mumford compactified moduli space of curves $\overline{\mathcal{M}}_{g,n}$ [72]. The nodal curves are then boundary divisors in $\overline{\mathcal{M}}_{g,n}$. There exist two types of such degenerations, called separating and non-separating degenerations. A separating degeneration splits off the surface into a surface with two components linked by a double point, while a non-separating degeneration simply gives rise to a new surface with two points identified, whose genus is reduced by one unit (see Fig. 12). Further, no degeneration is allowed to give rise to a nodal curve that does not satisfy the stability criterion shared with tropical graphs (2.3). As a consequence, a maximally degenerated surface is composed of thrice-punctured spheres.

These degenerations induce a stratification on $\overline{\mathcal{M}}_{g,n}$. It is characterized by the so-called "dual graphs". These encore the combinatorial structure of the nodal curves and the codimension of the boundary divisors. They are defined as follow. Take a nodal curve. Draw a line that goes through each pinched cycle and turn each non-degenerated component of genus $g \geq 0$ into a vertex of weight g. Draw "legs" attached to the graph for each marked point on the surface. See examples in Fig. 13.

A surface where a node is developing locally looks like a neck whose coordinates x and y on each of its side obey the following equation

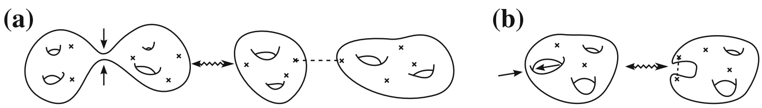

FIGURE 12. **a** A separating degeneration. **b** A non-separating degeneration. *Dashes* represent double points

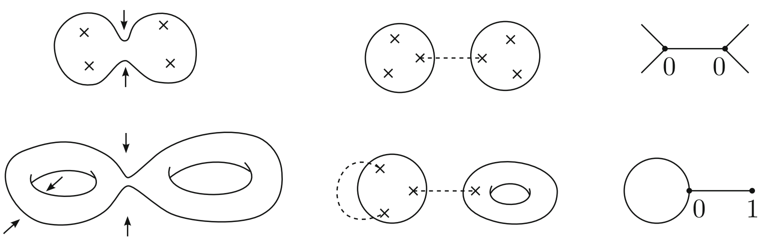

FIGURE 13. *Leftmost column* degenerating surfaces. *Centre* nodal curve. *Rightmost* dual graphs

$$xy = t, \tag{3.5}$$

where the complex number t of modulus $|t| < 1$ is a parameter measuring the deformation of the surface around the boundary divisor in $\overline{\mathcal{M}}_{g,n}$. The surface is completely pinched when $t = 0$. After a conformal transformation, this surface is alternatively described by a tube of length $-\log|t|$ and the tropicalization procedure will turn these tubes into actual lines.

3.3. Tropicalizing $\mathcal{M}_{g,n}$

The following schematic construction, not really described explicitly in the tropical geometry literature, is based on the standard physical $\alpha' \to 0$ limit of string theory amplitudes. The essential difficulty of the $\alpha' \to 0$ of string theory is that the objects that we are taking limits of are integrals over $\mathcal{M}_{g,n}$, which is not a compact space. This integrand has singularities at the various boundary divisors, and one is forced to study the integral locally to take the limit.

Decomposition of the Moduli Space. We proceed as follows: $\mathcal{M}_{g,n}$ is decomposed into a disjoint union of domains such that each of them gives rise to a combinatorially distinct set of tropical graphs;

$$\mathcal{M}_{g,n} = \bigsqcup_{G} \mathcal{D}_G \tag{3.6}$$

where $\sqcup$ symbolizes disjoint union and in the bulk of each domain $\mathcal{D}_G$ lies a nodal curve of $\overline{\mathcal{M}}_{g,n}$ with dual graph G. The existence of such a decomposition is intuitively clear from the stratum structure of the moduli space. To obtain a disjoint union as in Eq. (3.6), just ensure to redefine potentially overlapping domains so as to remove the intersections. This decomposition is not unique. The boundaries of the domains can be deformed so long as they does not start to absorb neighboring singularities. An explicit decomposition based on minimal area metrics can be found in Zwiebach's work [54], on which we come back below.

In each of these domains, we have local coordinates—like t in (3.5)—that parametrize the surfaces. Let us exclude the marked points of the following discussion, for simplicity. Close to the singularity, the surface is developing a certain number N of narrow necks or long tubes: as many as there are inner edges in G. Each of them are parametrized by a complex parameter t_j for $j = 1, \ldots, N$ whose collection form a set of local coordinates. The tropical graph is obtained by forgetting the phase on the t_j's. The lengths T_j of its edges are then given by

$$T_j = -\alpha' \log|t_j|. \tag{3.7}$$

Hence, to obtain edges of finite size, the t_i's should actually define families of curves with a particular scaling, depending on α', dictated by (3.7):

$$t_j = \exp(i(2\pi\phi + iT_j/\alpha')), \quad |t_j| \to 0, \quad \phi \in [0, 2\pi[\tag{3.8}$$

The rest of the $3g-3$ moduli describe the non-degenerating parts of the surface. The field theory limit procedure requires to integrate out these moduli to create weighted vertices. Alternatively, keeping t_j fixed in (3.7) corresponds to sending T_j to zero, which is consistent with the definition of weighted vertices as the result of specialized loops. In this paper, we do not describe the technology to handle these type of integration.[6]

Two specific kinds of domains are particularly interesting from the physical perspective that shall be called "analytic domain" and "maximally non-analytic domains", respectively. This terminology is borrowed from [14] and refers to the analyticity of the string amplitudes restricted to these domains. The analytic domain corresponds to the most superficial strata of $\overline{\mathcal{M}}_{g,n}$ which tropicalizes to the n-valent weight-g vertex. In this domain, the string theory integrand has no poles in the t_j moduli and it is possible to take the limit $\alpha' \to 0$ directly inside the integral. This gives the primary UV divergences of the field theory amplitudes, at any loop order, the most divergent parts of field theory amplitudes. The maximally non-analytic domains correspond to the deepest strata of $\overline{\mathcal{M}}_{g,n}$ and give rise to pure tropical graphs made of trivalent vertices only; this is the field theory unrenormalized amplitude.

Comment on the Relation to the Minimal Area Metrics Formalism. So far, what was described was a formal construction. Zwiebach in [54] defined an explicit decomposition of $\mathcal{M}_{g,n}$ based on a "minimal area metrics" [80, 81], which we summarize now. The idea is that for any given Riemann surface, there exists a unique metric of minimal area for which the length of any non-contractible closed loop is greater than 2π. This metric foliates the surface by closed loops of length 2π, and Feynman graphs are basically obtained by drawing on the surface a path that intersect orthogonally these curves. More precisely, if the height of a local foliation is bigger than 2π, then it corresponds to a propagator, if no foliation have height greater than 2π one is dealing with the genus-g n-point string vertex, etc. (see more details in sec. 6 of [54]). Along the time foliation, the local parameters (now real) presumably give rise to the lengths of the tropical graphs via the standard scaling (3.7) in the $\alpha' \to 0$ limit.[7] But it is not at all obvious that it is doable in practice to implement this construction in the context of the field theory limit of string theory which is the one we investigate here. In particular, when possible (i.e. when there is no "Schottky problem", so up to three loops),[8] it is more convenient to

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[6] The literature on this is too vast to be summarized here, see however recent developments at genus one [73–75], two [76–78] and higher genus [79].

[7] As is explained later in Sect. 4.2, and in the explicit computations in Sect. 5, here we actually do not need certain domains (=vertices) of the string field theory decomposition, those that correspond to graphs that contain vertices of weight 0 and valence $v \geq 4$. They contribute subleading terms in the limit. Therefore, an explicit decomposition of the kind we need here could be obtained in principle from Zwiebach's by removing the union of all of these domains from the decomposition of Eq. (3.6) and gluing them together to form an "outer" domain $\mathcal{D}_0$. The decomposition then becomes $\mathcal{M}_{g,n} = \sqcup_G \mathcal{D}_G \sqcup \mathcal{D}_0$, and the string theory integral has no support at leading order over $\mathcal{D}_0$.

[8] The Schottky problem is to identify the locus of the moduli space of Riemann surfaces (of dimension $3g-3$) inside that of Jacobian varieties, of dimension $g(g+1)/2$. These dimensions coincide up to three loops, with a subtlety at $g = 1$. At $g = 4$, the problem is solved and the locus is determined by the zero locus of a certain modular form called the Schottky-Igusa form.

parametrize the moduli space of surfaces in terms of period matrices. Below we use an such explicit decomposition.

The objective of Zwiebach's construction was to give a set of Feynman rules to construct formally full string theory amplitudes using propagators and vertices, in order to obtain a second quantized path integral formulation of string theory for instance. Therefore, the consistency of the quantization of his string field theory essentially guarantees the following. The $\alpha' \to 0$ limit of the string field theory is a well-defined quantum field theory. Moreover, it could be possible to extract field theory Feynman rules from the string field theory ones in this way.[9] This is not the goal that we are pursuing here.

In conclusion of this discussion, as far as computing string amplitudes an taking their field theory limit is concerned, first quantization appears to be the most efficient formalism. It is therefore not in the scope of this paper to investigate further the analysis of the formal field theory limit of Zwiebach's string *field* theory. Instead, we will now expose how to implement the tropical technology in order to extract field theory limits of string amplitudes in their explicit and compact first-quantized form.

Classical Versus Tropical. The definitions of previous sections lead to the following three facts:

(i) When going from surfaces to graphs, one-half of the homology disappears: the a-cycles pinch and the strings become point-like.
(ii) In particular, since the Abel–Jacobi map maps the a-cycles to the real part of the Jacobian variety, the imaginary part of the period matrices $\operatorname{Im}\boldsymbol{\Omega}$ of tropicalizing surfaces should be related to the period matrix of the tropical graph $\boldsymbol{K}$.
(iii) The classical holomorphic one-forms become one-forms that are constant on the edges.

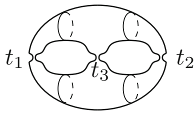
We want to interpret these in the context of the tropical limit.

Let us start with period matrices, restricting first to those of 1PI pure graphs. Consider a families of curves degenerating toward a maximal codimension singularity, with local parameters t_i, as in (3.7). Taniguchi showed in [82] that the elements of the family of period matrices are given by a certain linear combination of logarithms of the t_i's, in a rather obvious combination. An example is shown in Fig. 14, where the period matrix (2.8) of the two-loop tropical graph of Fig. 3; $\boldsymbol{\Omega}^{(2)}_{\alpha'} = i\boldsymbol{K}^{(2)}/(2\pi\alpha')+O(1)$ is immediately recovered, using the tropical scaling (3.7). This procedure generalizes straightforwardly to other cases and we obtain that, in a given domain, the tropicalizing families of curves defined by (3.7) have period matrices that approach the period matrix $\boldsymbol{K}$ of the tropical graph as

$$\operatorname{Re}\boldsymbol{\Omega}_{\alpha'} = \boldsymbol{M}_0 + O(\alpha', t_i), \qquad \operatorname{Im}\boldsymbol{\Omega}_{\alpha'} = \boldsymbol{K}/(2\pi\alpha') + \boldsymbol{M}_1 + O(\alpha', t_i), \quad (3.9)$$

where $\boldsymbol{M}_0$ and $\boldsymbol{M}_1$ are constant matrices with real coefficients. The $(1/2\pi)$ normalization is discussed shortly after Eq. (3.16). In total, at leading order

[9] Actually the bosonic closed string probably does not have a naive field theory limit anyway because of the Tachyon.



$$\mathbf{\Omega}^{(2)}_{\alpha'} = \tfrac{1}{2i\pi}\begin{pmatrix} -\log(t_1 t_3) & \log(t_3) \\ \log(t_3) & -\log(t_2 t_3) \end{pmatrix} + O(\alpha', t_i)$$

FIGURE 14. Degenerating Riemann surface parametrized by local coordinates t_1, t_2, t_3 and its period matrix. The $1/(2i\pi)$ normalization follows Taniguchi's [82] but differs from Fay's in the standard ref [83] eq. (54) because of different normalizations (recall Eq. (3.1))

and up to a rescaling by α', the tropical Jacobian is the imaginary part of the complex one.[10]

To extend this to 1PR graphs, observe that the one-forms have zero support on the separating edges. In a domain corresponding to a dual graph G where an edge e splits off G into two 1PI graphs G_1 and G_2, let t_e be a local coordinate parametrizing such a separating degeneration. The period matrix of the degenerating curve is given by;

$$\mathbf{\Omega}^{(t_e)} = \begin{pmatrix} \mathbf{\Omega}_1 & 0 \\ 0 & \mathbf{\Omega}_2 \end{pmatrix} + O(t_e), \tag{3.10}$$

which can be tropicalized further following the previous discussion and provides the same splitting for the period matrix of the corresponding tropical graphs

$$\boldsymbol{K} = \begin{pmatrix} \boldsymbol{K}_1 & 0 \\ 0 & \boldsymbol{K}_2 \end{pmatrix}. \tag{3.11}$$

The holomorphic one-forms, at a neck j parametrized by t_j, behave locally as on the cylinder:

$$\omega_I = \frac{c}{2i\pi}\frac{\mathrm{d}z}{z} + O(t_i), \tag{3.12}$$

where $c = 1$ or $c = 0$ depending on whether the cycle b_I contains the node i or not. The Abel–Jacobi map (3.4) then reduces to

$$\int^z \omega_I = \frac{c}{2i\pi}\log(z) \in J(\Gamma) \tag{3.13}$$

where it is now clear that the phase of z is mapped to real parts in $J(\Gamma)$ in the tropical limit. Moreover, consider the following tropicalizing family of points z on the tube j:

$$z_{\alpha'} = e^{i(\theta + iY/\alpha')} \tag{3.14}$$

where $\theta \in [-\pi; \pi[$ and Y is a positive real number. This yields the tropical limit of the Abel–Jacobi map

$$2\pi\alpha' \int^z \omega_i = i\int^Y \omega_I^{\mathrm{trop}} = iZ + O(\alpha') \in \alpha' \mathrm{Im}\, J(\Sigma_{\alpha'}) \equiv J(\Gamma), \tag{3.15}$$

[10] For non pure graphs, one has to be more careful with such a statement, see the remark at the end of Sect. 2.2.

where we used that $\omega_I^{\text{trop}} = 1$ on B_I. This result is in accordance with (3.9). Finally, these equations are compatible with Riemann bilinear relations

$$\int \omega_I \wedge \bar{\omega}_J = \operatorname{Im} \Omega_{IJ}, \tag{3.16}$$

which descend to a tropical version (upon multiplication by α'):

$$\alpha' \int \omega_I \wedge \bar{\omega}_J \underset{\alpha' \to 0}{\longrightarrow} \frac{\alpha'}{(2\pi)^2} \int \frac{dz \wedge \mathrm{d}\bar{z}}{|z|^2} = \frac{1}{2\pi} \int \mathrm{d}Y = \frac{K_{IJ}}{2\pi} \tag{3.17}$$

where Y is defined in Eq. (3.14). This eventually justifies the normalization in Eq. (3.9). Another explicit cross-check of the normalization is provided later at one loop (see Sect. 5.2) where one has to identify the imaginary part of the modular parameter τ with a rescaled Schwinger proper time $T/(2\pi\alpha')$. See also the discussion of [96, pp. 218].

3.4. The Tropical Prime Form

Let Σ be a Riemann surface of genus g with period matrix $\boldsymbol{\Omega}$. The classical Riemann theta function is defined on the Jacobian variety of Σ by

$$\theta(\boldsymbol{\zeta}|\boldsymbol{\Omega}) = \sum_{\boldsymbol{n} \in \mathbb{Z}^g} e^{i\pi \boldsymbol{n}\cdot\boldsymbol{\Omega}\boldsymbol{n}} e^{2i\pi \boldsymbol{n}\cdot\boldsymbol{\zeta}} \tag{3.18}$$

where $\boldsymbol{\zeta} \in J(\Sigma)$ and $\boldsymbol{\Omega} \in \mathcal{H}_g$. Here and below we call Fourier expansions these series in $e^{2i\pi\Omega_{IJ}}$. Theta functions with characteristics are defined by

$$\begin{aligned} \theta\left[\begin{smallmatrix}\boldsymbol{\beta}\\ \boldsymbol{\alpha}\end{smallmatrix}\right](\boldsymbol{\zeta}|\boldsymbol{\Omega}) &= e^{i\pi\boldsymbol{\beta}\cdot\boldsymbol{\Omega}\boldsymbol{\beta} + 2i\pi\boldsymbol{\beta}\cdot(\boldsymbol{\zeta}+\boldsymbol{\alpha})}\theta(\boldsymbol{\zeta} + \boldsymbol{\Omega}\boldsymbol{\beta} + \boldsymbol{\alpha}|\boldsymbol{\Omega}) \\ &= \sum_{\boldsymbol{n}\in\mathbb{Z}^g} e^{i\pi(\boldsymbol{n}+\boldsymbol{\beta})\cdot\boldsymbol{\Omega}(\boldsymbol{n}+\boldsymbol{\beta})} e^{2i\pi(\boldsymbol{n}+\boldsymbol{\beta})\cdot(\boldsymbol{\zeta}+\boldsymbol{\alpha})} \end{aligned} \tag{3.19}$$

where $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \frac{1}{2}(\mathbb{Z}/2\mathbb{Z})^{2g}$ are the theta characteristics. There are 2^{2g} of them and the parity of the scalar product $4\,\boldsymbol{\alpha}\cdot\boldsymbol{\beta}$ modulo 2 corresponds to the parity of both the spin structure and the theta function (in z); $\frac{1}{2}(2^{2g}+2^g)$ are even, the remaining $\frac{1}{2}(2^{2g}-2^g)$ are odd.

The prime form [83,84], is an object of central importance for string amplitudes [85,86]. It is defined by

$$E : (x,y) \in \Sigma \times \Sigma \longrightarrow E(x,y) = \frac{\theta\left[\begin{smallmatrix}\boldsymbol{\beta}\\ \boldsymbol{\alpha}\end{smallmatrix}\right](\int_x^y (\omega_1,\dots,\omega_g)|\boldsymbol{\Omega})}{h\left[\begin{smallmatrix}\boldsymbol{\beta}\\ \boldsymbol{\alpha}\end{smallmatrix}\right](x) h\left[\begin{smallmatrix}\boldsymbol{\beta}\\ \boldsymbol{\alpha}\end{smallmatrix}\right](y)} \in \mathbb{C}, \tag{3.20}$$

where $\left[\begin{smallmatrix}\boldsymbol{\beta}\\ \boldsymbol{\alpha}\end{smallmatrix}\right]$ is an odd theta characteristic and $h\left[\begin{smallmatrix}\boldsymbol{\beta}\\ \boldsymbol{\alpha}\end{smallmatrix}\right]$ are half-differentials defined on Σ by

$$h\left[\begin{smallmatrix}\boldsymbol{\beta}\\ \boldsymbol{\alpha}\end{smallmatrix}\right](z)^2 = \sum_{i=1}^{g} \omega_I(z)\partial_I \theta\left[\begin{smallmatrix}\boldsymbol{\beta}\\ \boldsymbol{\alpha}\end{smallmatrix}\right](0|\boldsymbol{\Omega}). \tag{3.21}$$

In this way, the prime form is a differential form of weight $(-1/2, 0)$ in each variables. It is also independent of the spin structure $\left[\begin{smallmatrix}\boldsymbol{\beta}\\ \boldsymbol{\alpha}\end{smallmatrix}\right]$ (this is not obvious from this definition, see for instance [86]). In a sense, it generalizes $(x-y)/\sqrt{\mathrm{d}x}\sqrt{\mathrm{d}y}$ to arbitrary Riemann surfaces and in particular it vanishes only along the diagonal $x = y$. It is multi-valued on $\Sigma \times \Sigma$ since it depends on the path of integration in the argument of the theta function. More precisely, it is invariant up to a sign if the path of integration is changed by a cycle a_I, but it picks up a multiplicative factor when changing the path of integration by a cycle b_J

$$E(x,y) \to \exp\left(-\Omega_{JJ}/2 - \int_x^y \omega_J\right) E(x,y). \tag{3.22}$$

We define the tropical prime form to be the result of the following limit:

$$E^{\text{trop}}(X,Y) := -\lim_{\alpha'\to 0}\left(\alpha' \log\left|E(x_{\alpha'}, y_{\alpha'}|\boldsymbol{\Omega}_{\alpha'})\right|\right) \tag{3.23}$$

where $\boldsymbol{\Omega}_{\alpha'}$ are the period matrices of a family of curves $\Sigma_{\alpha'}$ tropicalizing as in (3.9) to a graph Γ,

$$\boldsymbol{\Omega}_{\alpha'} = i\boldsymbol{K}/(2\pi\alpha') + \cdots \tag{3.24}$$

where the ... indicate subleading α' terms and K is the period matrix of Γ. The two families of points $x_{\alpha'}, y_{\alpha'}$ on $\Sigma_{\alpha'}$ degenerate as in (3.14) to X and Y on Γ. By the Abel–Jacobi map, we also have a family of elements in the family of Jacobian

$$\boldsymbol{\zeta}_{\alpha'} \in J(\Sigma_{\alpha'}), \tag{3.25}$$

that degenerates to an element of the tropical Jacobian

$$\boldsymbol{Z} \in J(\Gamma), \tag{3.26}$$

in such a way that

$$\boldsymbol{\zeta}_{\alpha'} = i\boldsymbol{Z}/(2\pi\alpha') + \cdots \tag{3.27}$$

where again the dots indicate subleading terms.

Now comes one of the most important results of this work, the computation of the field theory or tropical limit of the prime form.

Proposition 1. *The tropical prime form defined as above corresponds at any loop order to the graph distance $d_\gamma(X,Y)$ between X and Y along a path γ:*

$$E^{\text{trop}}(X,Y) = d_\gamma(X,Y). \tag{3.28}$$

Proof. The difficult point in this proof lies in the fact that, although the prime form does not depend on the spin structure, its various constituents do. We will actually turn this to our advantage and use Lemma 1 to pick an adequate spin structure. More precisely, having defined (fixed) the families of points $x_{\alpha'}, y_{\alpha'}$ and their limits X, Y on the graph, there will always exist a class of convenient spin structures that make the computation easier.

The first ingredient of the proof is the limit of the theta functions in the numerator of E. Below, we suppress the α' index but keep in mind that we deal with families of curves. Let us first describe the case of theta functions without characteristics defined in Eq. (3.18). Given the above scaling, in the series expansion (3.18), all terms but one are exponentially suppressed:

$$e^{i\boldsymbol{n}\cdot\boldsymbol{\Omega}\boldsymbol{n}+2i\boldsymbol{\zeta}\cdot\boldsymbol{n}} \to 0, \tag{3.29}$$

except for $\boldsymbol{n} = 0$, where we have $e^{i\boldsymbol{n}\cdot\boldsymbol{\Omega}\boldsymbol{n}+2i\boldsymbol{\zeta}\cdot\boldsymbol{n}} = 1$. The case of theta functions with (odd) characteristics is similar; generic terms in the sum read

$$e^{i\pi(\boldsymbol{n}+\boldsymbol{\beta})\cdot\boldsymbol{\Omega}(\boldsymbol{n}+\boldsymbol{\beta})}e^{2i\pi(\boldsymbol{n}+\boldsymbol{\beta})\cdot(\boldsymbol{\zeta}+\boldsymbol{\alpha})}. \tag{3.30}$$

By definition of an odd theta characteristics, $\boldsymbol{\beta} \neq \mathbf{0}$, and $\boldsymbol{\beta}+\boldsymbol{n} \neq$ for all $\mathbf{n}$ since the elements of $\boldsymbol{\beta}$ are half-integers. Therefore, all terms in the expansion (3.30) are exponentially suppressed by the positive-definiteness of $\operatorname{Im}\boldsymbol{\Omega}$. The leading order term of the theta sum is reached for two values of $\boldsymbol{n}$,

$$\boldsymbol{n} = 0 \text{ and } \boldsymbol{n} = -2\boldsymbol{\beta}, \tag{3.31}$$

and the leading order asymptotics reads

$$\theta\left[\begin{smallmatrix}\boldsymbol{\beta}\\ \boldsymbol{\alpha}\end{smallmatrix}\right](\boldsymbol{\zeta}|\boldsymbol{\Omega}) = e^{i\pi\boldsymbol{\beta}\cdot\boldsymbol{\Omega}\boldsymbol{\beta}}\left(e^{2i\pi(\boldsymbol{\zeta}+\boldsymbol{\alpha})\cdot\boldsymbol{\beta}} + e^{-2i\pi(\boldsymbol{\zeta}+\boldsymbol{\alpha})\cdot\boldsymbol{\beta}}\right) + \cdots . \tag{3.32}$$

This is rewritten

$$\theta\left[\begin{smallmatrix}\boldsymbol{\beta}\\ \boldsymbol{\alpha}\end{smallmatrix}\right](\boldsymbol{\zeta}|\boldsymbol{\Omega}) = e^{i\pi\boldsymbol{\beta}\cdot\boldsymbol{\Omega}\boldsymbol{\beta}} e^{2i\pi\boldsymbol{\beta}\cdot\boldsymbol{\alpha}} 2i\sin(2\pi\boldsymbol{\zeta}\cdot\boldsymbol{\beta}) + \cdots , \tag{3.33}$$

using that $e^{2i\pi\boldsymbol{\beta}\cdot\boldsymbol{\alpha}} = -e^{-2i\pi\boldsymbol{\beta}\cdot\boldsymbol{\alpha}}$ since $2\boldsymbol{\alpha}\cdot\boldsymbol{\beta} \equiv 1/2 \pmod 1$ for an odd theta characteristics. The prefactor $e^{i\pi\boldsymbol{\beta}\cdot\boldsymbol{\Omega}\boldsymbol{\beta}}$ renders the right-hand side of (3.33) exponentially suppressed, but the presence of the half-differentials in the prime form is going to compensate this. From their definition (3.21), we see that the computation of the limit of the $h\left[\begin{smallmatrix}\boldsymbol{\beta}\\ \boldsymbol{\alpha}\end{smallmatrix}\right]$'s is very similar to that of the theta functions; we just have to include a derivative. The extremizing values of $\boldsymbol{n}$ are still $\mathbf{0}$ and $-2\boldsymbol{\beta}$, and, as in Eq. (3.32) we have;

$$h\left[\begin{smallmatrix}\boldsymbol{\beta}\\ \boldsymbol{\alpha}\end{smallmatrix}\right](x)^2 = 2i\pi\sum_{J=1}^{g}\sum_{\boldsymbol{n}=\mathbf{0},-2\boldsymbol{\beta}} \omega_J(x)(n_J+\beta_J)e^{i\pi(\boldsymbol{n}+\boldsymbol{\beta})\cdot\boldsymbol{\Omega}(\boldsymbol{n}+\boldsymbol{\beta})}e^{2i\pi(\boldsymbol{n}+\boldsymbol{\beta})\cdot\boldsymbol{\alpha}}, \tag{3.34}$$

at leading order. Actually, only a subset of the $\omega_J(x)$'s contributes to the sum. While the one-forms $\omega_J(x)$ for which the limiting divisor X of the family $x_{\alpha'}$ belongs to the cycle tropical B_J do contribute, the other all vanish (recall (3.12)). If we call $B_{i_1}, \dots, B_{i_k}$, the set of these k cycles (there is always at least one cycle), (3.34) reduces to;

$$\begin{aligned} h\left[\begin{smallmatrix}\boldsymbol{\beta}\\ \boldsymbol{\alpha}\end{smallmatrix}\right](x)^2 &= 2i\pi\sum_{r=1}^{k}\omega_{i_r}(x)\sum_{\boldsymbol{n}=\mathbf{0},-2\boldsymbol{\beta}}(n_{i_r}+\beta_{i_r})e^{i\pi(\boldsymbol{n}+\boldsymbol{\beta})\cdot\boldsymbol{\Omega}(\boldsymbol{n}+\boldsymbol{\beta})}e^{2i\pi(\boldsymbol{n}+\boldsymbol{\beta})\cdot\boldsymbol{\alpha}} \\ &= 4i\pi e^{i\pi\boldsymbol{\beta}\cdot\boldsymbol{\Omega}\boldsymbol{\beta}}\boldsymbol{\beta}\cdot\boldsymbol{\omega}(x). \end{aligned} \tag{3.35}$$

To obtain the second line, we first used that the exponential of the quadratic form was independent of $\boldsymbol{n}$ and factored it out. Then, we simplified as above the induced cosine using $e^{2i\pi\boldsymbol{\beta}\cdot\boldsymbol{\alpha}} = -e^{-2i\pi\boldsymbol{\beta}\cdot\boldsymbol{\alpha}}$; $\cos(2\pi\boldsymbol{\alpha}\cdot\boldsymbol{\beta}) = 1$. Finally, the r summation was rewritten as a scalar product.

Collecting the previous results in (3.33) and (3.35), we obtain the explicit behavior of the prime form;

$$-\alpha'\log\left|E(x_{\alpha'}, y_{\alpha'}|\boldsymbol{\Omega}_{\alpha'})\right| = -\frac{\alpha'}{2}\log\left(\frac{\sin(2\pi\boldsymbol{\beta}\cdot\boldsymbol{\zeta}_{\alpha'})}{\sqrt{\boldsymbol{\omega}(x_{\alpha'})\cdot\boldsymbol{\beta}}\sqrt{\boldsymbol{\omega}(y_{\alpha'})\cdot\boldsymbol{\beta}}}\right) \tag{3.36}$$

where we have reintroduced the explicit index α', and where the factor of $1/2$ comes from the absolute value on the left-hand side.

Now we set the characteristics $\boldsymbol{\beta}$ as in Lemma 1. With the scaling of $\zeta_{\alpha'}$ in (3.27) and Lemma 1, the sine function in (3.36) becomes

$$\sin(2i\pi \mathrm{dist}_\gamma(X,Y)/\alpha'), \tag{3.37}$$

whose logarithm gives

$$-\frac{2\pi}{\alpha'}\mathrm{dist}_\gamma(X,Y). \tag{3.38}$$

Then we need to deal with the factors of $\boldsymbol{\beta}\cdot\boldsymbol{\omega}(x)$. With our choice of characteristics, $\boldsymbol{\beta}\cdot\boldsymbol{\omega}(x)$ produces at leading order a positive integer or half-integer, whose explicit determination is irrelevant here, as it vanishes in the logarithm in (3.23) as $\alpha'\to 0$. The only important thing is that this quantity should not vanish:[11] this is ensured by the following facts

(i) The first all entries of both vectors are positive,

(ii) Then, $\boldsymbol{\beta}$ is chosen such that its $\mathbb{Z}_2$ cycle passes through X. This implies, as we demonstrated, that at least one cycle B_J for which $X\in B_j$ has $\beta_J=1/2$.

Therefore $\boldsymbol{\beta}\cdot\boldsymbol{\omega}(x)\geq \omega_J(x)\beta_J\simeq 1/2$.

The proposition is finally proven by inserting (3.38) in (3.36). $\square$

Higher-order terms can sometimes be required to compute the tropical limit of some amplitudes in string theory. In principle, they can be extracted following the same recipe. For the amplitudes treated in this paper, only the leading order contribution described above will be needed.

4. String Theory Amplitudes, Tropical Amplitudes and the Tropical Limit

In the previous sections, we introduced tropical graphs and showed how they result from the tropicalization of Riemann surfaces. We are now ready to introduce string theory amplitudes and describe their $\alpha'\to 0$ limit.

4.1. The Tropical Limit of String Theory

Let $\mathsf{A}^{(g,n)}_{\alpha'}(X)$ denote a generic g-loop n-point string theory scattering amplitude for a scattering process X (we omit the reference to the scattering process when it is not necessary). In the Ramond-Neveu-Schwarz (RNS) formalism, the amplitudes are given by integrals over the supermoduli space of super Riemann surfaces $\mathfrak{M}_{g,n}$ [86–88]. In contrast, the pure spinor [89] and Green-Schwarz formalisms, naturally give integrals over the ordinary moduli space of Riemann surfaces, $\mathcal{M}_{g,n}$.

In this paper, we restrict ourselves to the study of the string amplitudes that can be written as integrals $\mathcal{M}_{g,n}$ only, whether they come from the pure

[11] Otherwise one should extract higher-order terms from the Fourier expansion in the half-differentials. A similar type of cancelation would occur in the argument of $\sin(2\pi\boldsymbol{Z}_\gamma\cdot\boldsymbol{\beta})$ in (3.33), and presumably the two would cancel out, but the author has not been able to show this in full generality.

spinor formalism or from a case where the RNS formalism produces such integrals.[12] Our amplitudes will therefore assume the generic form:

$$\mathsf{A}^{(g,n)}_{\alpha'} = \int_{\mathcal{M}_{g,n}} \mathrm{d}\mu_{\mathrm{bos}}\ \mathcal{F}_{g,n}. \tag{4.1}$$

In the RNS formalism, the integrand $\mathcal{F}_{g,n}$ involves a spin structure sum that accounts for the periodicity of the worldsheet fermions ψ^μ. In the cases that we deal with explicitly, the sum will already be done, so we will not be more precise about that. The bosonic measure $\mathrm{d}\mu_{\mathrm{bos}}$ is a $(3g-3+n)$-dimensional measure that can be traded for an integration over the period matrices for $g=1,2,3$, where there is no Schottky problem;

$$\mathrm{d}\mu_{bos} = \frac{|\prod_{1\le I<J\le g}\mathrm{d}\Omega_{IJ}|^2}{|\det \mathrm{Im}\,\boldsymbol{\Omega}|^{d/2}} \prod_{i=1}^{n} \mathrm{d}^2 z_i, \tag{4.2}$$

where d is the number of spacetime non-compact dimensions.[13] The integrand can be decomposed further and written as

$$\mathcal{F}_{g,n} = \mathcal{W}_{g,n} \exp(\mathcal{Q}_{g,n}). \tag{4.3}$$

The function $\mathcal{W}_{g,n}$ carries all the information about the particular scattering process. The factor $\exp(\mathcal{Q}_{g,n})$ is called the Koba–Nielsen factor. It is a universal factor present in all string theory amplitudes. Its exponent reads

$$\mathcal{Q}_{g,n} = \alpha' \sum_{1\le i<j\le n} k_i\cdot k_j\, \mathcal{G}(z_i,z_j), \tag{4.4}$$

with $\mathcal{G}$ the bosonic Green's function [85,86];

$$\mathcal{G}(z_1,z_2) = -\frac{1}{2}\log\left(|E(z_1,z_2)|^2\right) + \pi \mathrm{Im}\left(\int_{z_2}^{z_1}\omega_I\right)(\mathrm{Im}\,\boldsymbol{\Omega}^{-1})^{IJ}\mathrm{Im}\left(\int_{z_2}^{z_1}\omega_J\right). \tag{4.5}$$

Unlike the prime form, $\mathcal{G}$ is well-defined on the surface; changes in $\log|E|$ as in (3.22) are compensated by the second term in (4.5).

The procedure of Sect. 3.3 is then implemented as follows. Take the decomposition $\mathcal{M}_{g,n} = \left(\bigsqcup_{i=1}^{N}\mathcal{D}_G\right)\sqcup\mathcal{D}_0$ of Sect. 3.3. In the $\alpha'\to 0$ limit of $\mathsf{A}^{(g,n)}_{\alpha'}$, the following two points hold:

(i) Integrating over the domain $\mathcal{D}_0$ produces only subleading contributions:

$$\int_{\mathcal{D}_0} \mathrm{d}\mu_{bos}\,\mathcal{F}_{g,n} = O(\alpha'). \tag{4.6}$$

We call $\mathcal{D}_0$ the "outer" domain.

[12] We postpone to the discussion some comments on the recent works of Witten and Donagi, where it is argued that, from the supermoduli space perspective this would automatically imply a restriction to genus $g<5$.

[13] This normalization is non-standard, in the sense that the invariant measure has an inverse power of $g+1$. From the point of view of the field theory limit though, the $d/2$ is more natural; therefore, we define the measure in this way and absorb a compensating factor in the definition of the integrand. Also in all explicit examples below, we will have $d=10$.

(ii) In each domain $\mathcal{D}_G$, there exist a function $F_{g,n}$ defined over $\mathcal{M}^{\text{trop}}(\Gamma)$, the moduli space of tropical graphs $\Gamma = (G, \ell, w)$ with combinatorial type G, such that:

$$\int_{\mathcal{D}_G} \mathrm{d}\mu_{\text{bos}}\ \mathcal{F}_{g,n} = \int_{\mathcal{M}^{\text{trop}}(\Gamma)} \mathrm{d}\mu_{\text{trop}}\ F_{g,n} + O(\alpha'). \tag{4.7}$$

The measure is given by

$$\mathrm{d}\mu_{\text{trop}} := (2\pi)^{d/2-|E(G)|} \frac{\prod_{i\in E(G)} \mathrm{d}\ell(i)}{(\det \boldsymbol{K})^{d/2}}, \tag{4.8}$$

where $\boldsymbol{K}$ is the period matrix of Γ.

Compared to Zwiebach's string field theory [54], in the field theory limit, only massless modes propagate along edges of finite lengths. The contribution of massive modes stay localized on vertices with weights. We shall see this explicitly in the examples below.

As far as the explicit computations of this paper are concerned, we will build by hand these decompositions.

Physically, the right-hand side of (4.7) is the contribution of the Feynman diagrams of field theory in the tropical representation corresponding to the graph G. As above, the integrand $F_{g,n}$ can be factorized

$$F_{g,n} = W_{g,n} \exp(Q_{g,n}), \tag{4.9}$$

where $W_{g,n}$ and $Q_{g,n}$ descend from their string theory ancestors. Computing their explicit form gives the tropical representation of the integrand and is the second step of the procedure. The extraction of $W_{g,n}$ is straightforward in the cases of maximal supergravity four-graviton amplitudes discussed later for $g = 0, 1, 2$ but it is much more intricate in the general case. It requires in particular to deal with Fourier expansions in higher genus, and this will not be covered in this paper, although in principle the procedure of Sect. 3.4 gives a prescription to extract these terms. As we mentioned already, this process at genus one is fully understood since the works of Bern and Kosower [7–10].

On the other hand, $Q_{g,n}$ is a universal factor and is obtained from (4.4) by computing the tropical limit of the Green's function $\mathcal{G}$, to which we turn now. We have already studied the limits of both the prime form in (3.23) and the holomorphic differentials (3.13); therefore, all we have to do is to piece these up to obtain the tropical Green's function;

$$\begin{aligned}\lim_{\alpha'\to 0} \alpha' \mathcal{G}(z_1, z_2) &= -\frac{1}{2} E^{\text{trop}}(Z_1, Z_2) - \frac{1}{2} \left(\int_{Z_2}^{Z_1} \omega_I^{\text{trop}} \right) (\boldsymbol{K}^{-1})^{IJ} \left(\int_{Z_2}^{Z_1} \omega_J^{\text{trop}} \right) \\ &:= G^{\text{trop}}(Z_1, Z_2).\end{aligned} \tag{4.10}$$

The limit is to be understood as in Sect. 3.4 and factors of (2π) have been consistently reabsorbed in ω and $\boldsymbol{\Omega}$ to produce ω^{trop} and $\boldsymbol{K}$. This tropical Green's function coincides with the worldline Green's function computed directly in [53] (see also [15, 90–92] for earlier works). Contrary to the tropical prime form,

G^{trop} is always independent of the integration path. It follows from these definitions that the tropical representation of exponential factor in (4.3) is given by

$$Q_{g,n} = \sum k_i \cdot k_j G^{\text{trop}}(Z_i, Z_j) \tag{4.11}$$

We can now collect (4.8) and (4.11) to obtain the following formula; the tropical representation of (4.7) is

$$\int \prod_{i \in E(G)} \mathrm{d}\ell(i) \frac{W_{g,n} \exp(Q_{g,n})}{(\det \boldsymbol{K})^{d/2}}, \tag{4.12}$$

up to an overall numerical factor of the form $(2\pi)^m$. In this form, $\det(\boldsymbol{K})$ and $\exp(Q_{g,n})$ are respectively the first and second Symanzik polynomials obtained from Feynman rules in field theory,[14] and $W_{g,n}$ is the numerator of the Feynman graph integrand. This assertion is physically clear, however, a direct proof using graph theory would be of interest concerning more formal aspects of the study of Feynman diagrams.[15] Examples in genus one and two are given in Sect. 5.

We can now phrase the standard $\alpha' \to 0$ limit in the tropical language;

Conjecture 2. *The $\alpha' \to 0$ limit of the string theory integral over $\mathcal{M}_{g,n}$ is given by an integral over $\mathcal{M}_{g,n}^{\text{trop}}$*

$$\int_{\mathcal{M}_{g,n}} \mathrm{d}\mu_{\text{bos}}\ \mathcal{F}_{g,n} = \int_{\mathcal{M}_{g,n}^{trop}} \mathrm{d}\mu_{\text{trop}}\ F_{g,n} + O(\alpha'), \tag{4.13}$$

where

$$\int_{\mathcal{M}_{g,n}^{trop}} \mathrm{d}\mu_{\text{trop}} := \sum_{\Gamma} \int_{\mathcal{M}(\Gamma)} \mathrm{d}\mu^{\text{trop}}. \tag{4.14}$$

The discrete finite sum runs over all the combinatorially distinct graphs Γ of genus g with n legs. Moreover, the right-hand side of (4.13) corresponds to the field theory amplitude renormalized in the scheme induced by string theory. This scheme is defined such that

$$\mathsf{A}_{\text{trop}}^{(g,n)} = \int_{\mathcal{M}_{g,n}^{trop}} \mathrm{d}\mu_{\text{trop}}\ F_{g,n} \tag{4.15}$$

where $\mathsf{A}_{\text{trop}}^{(g,n)}$ is the field theory amplitude written in its tropical representation (in short tropical amplitude) obtained in the field theory limit.

The conjecture can be shown in the cases where one starts from a known string amplitude, mostly because an explicit $\mathcal{F}_{g,n}$ is needed. In this way, re-expressing the existing tree-level and one-loop computations in the tropical

[14] There is a slight difference of normalization compared to the usual definition given for instance in the classic reference [93] where the first and second Symanzik polynomials, denoted $\mathcal{U}$ and $\mathcal{F}$, are related to ours by: $\mathcal{U} = \det K$, $\mathcal{F} = \exp(Q_{g,n}) \det K$, and where also $\exp(Q_{g,n})$ should strictly speaking be replaced by the result of integrating out a global scale factor for the lengths of the edges of the graph to go from Schwinger proper times to Feynman parameters.

[15] Note also that in this representation, it is obvious that the first Symanzik polynomial does not depend on the positions of the punctures.

language, as we do later, can be considered as a proof of various instances of the conjecture.

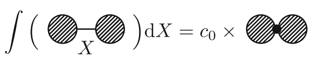
4.2. Counter-Terms, Contact Terms

Analytic and non-analytic terms. For simplicity, let us exclude the punctures of the discussion. The analytic and maximally non-analytic domains have been defined in Sect. 3.3 by the requirement that the first should correspond to the more superficial stratum of $\overline{\mathcal{M}}_g$ and the second should correspond to the deepest strata of $\overline{\mathcal{M}}_g$.

In other words, the analytic domain is defined by removing all neighborhood around the singularities of $\mathcal{M}_g$. Therefore it is a compact space. Inside that domain, the string integrand has no singularity and the limit may be safely taken directly; the factor α' present in the definition of $\mathcal{Q}_{g,n}$ simply sends $\exp(\mathcal{Q}_{g,n})$ to 1. Moreover, the dual graph of the analytic domain is a single vertex of weight g. Physically, such graphs are counter-terms to primary UV divergences, so this is consistent with the fact these correspond to the string integral over the analytic domain, as illustrated later in the one-loop example of Sect. 5.2.

The maximally non-analytic domains provide the contributions of the pure tropical graphs, the worldline graphs made of trivalent vertices only (graphs with no counter-terms). Summed over, they give the unrenormalized field theory amplitude, with all of its divergences. We present in Sect. 5.3 a computation of a tropical integrand at genus two in such a domain.

A Remark on Contact Terms. Feynman rules in non-abelian gauge theories or gravity naturally use vertices of valency higher than three to implement gauge invariance. The way that these arise in string theory is different. What is called a "contact term" in string theory is usually the vertex that results from integrating out the length dependence of a separating edge in a 1PR graph, as in (4.16) below.



(4.16)

These integrations are trivial since they are of the form $\int_0^\infty \exp(-sX)\mathrm{d}X$ where s is a kinematic invariant. However, prior to any of these trivial integrations, the locus $X = 0$ corresponds geometrically to a lower codimension face in $\mathcal{M}_{g,n}^{\mathrm{trop}}$ and does not carry any localized contribution, it is only after integration that a contact term is produced.

Maximal Simplicity of Maximally Supersymmetric Numerators. A final note in this section concerns the simplicity of the extraction of $W_{g,n}$ in the non-analytic regions. Generic string theory models exhibit chiral "tachyon poles", of the form q^{-1} or $q^{-1/2}$ at $g = 1$ and generalization thereof at higher genus (see for instance [94] at $g = 2$ in CHL models).

These poles "soak up" powers of $\partial G^{\mathrm{trop}}$ from the numerators as they extract residues of the form $\mathcal{W}_{g,n} \exp(\mathcal{Q}_{g,n})|_q$ in the Fourier expansion. This decreases the degree of the loop momentum numerator polynomials, thereby enforcing supersymmetric cancelations. The Bern–Kosower rules were a systematization of this residue extraction at one-loop, and one of the longer term goal of this tropical limit project is to extend these rules to higher loops.

In the case of maximally supersymmetric amplitudes, these tachyon poles are canceled directly at the level of the spin-structure sum and the technology presented here is usable straight away to extract the field theory numerators in the tropical or Schwinger proper-time form. We give an illustration of this at $g = 2$ in Sect. 5.3 and in the conclusion mention some work in progress at $g = 3$ based on [55].

5. Explicit Computations

In this section, we first review some examples of field theory limits at tree-level and one-loop which we formulate in the tropical framework. Then at two loops, we derive the worldline representation of the four-graviton amplitude in the non-analytic domain from the full string theory amplitude of D'Hoker and Phong. We also comment on UV divergences and counter-terms.

5.1. Tree Level (Review)

As a warm-up, we start with tree-level scattering amplitudes in string theory, as was done by Scherk in the early days of string theory [37]. We first look at the simplest example, the four-tachyon scattering in the bosonic string, then we describe the case of four-graviton scattering in the type II superstring. The general case of n-particle scattering follows from the same method as the one reviewed here.

A closed string theory tree-level n-point amplitude can be written in the general form:[16]

$$\mathsf{A}^{(0,n)}_{\alpha'} = g_c^{n-2}\frac{8\pi}{\alpha'}\int_{\mathcal{M}_{0,n}}\prod_{i=3}^{n-1}\mathrm{d}^2 z_i\,\langle (c\bar{c}V_1)(c\bar{c}V_2)V_3\ldots V_{n-1}(c\bar{c}V_n)\rangle, \tag{5.1}$$

where $\mathrm{d}^2 z := \mathrm{d}z\mathrm{d}\bar{z}$ and g_c is the string coupling constant. The vertex operators V_i insert the external scattered states at position z_i on the worldsheet. They depend on the momenta k_i and possible polarizations ϵ_i of the particles. The integration over the points $z_1,\ z_2$ and z_n is suppressed and exchanged by the insertion of $c\bar{c}$ ghosts to account for the factorization of the infinite volume of the $SL(2,\mathbb{C})$ conformal group. The integral over the set of $n-3$ distinct complex variables $z_3,\ldots,z_{n-1}$ spans the moduli space of n-punctured genus zero surfaces $\mathcal{M}_{0,n}$. The correlation function (5.1) is computed using Wick's theorem and the correlators

$$\langle X(z,\bar{z})X(w,\bar{w})\rangle = \mathcal{G}(z,w) = -\frac{\alpha'}{2}\log(|z-w|^2), \quad \langle c(z)c(w)\rangle = z-w, \tag{5.2}$$

The ghost correlator is given by

$$|\langle c(z_1)c(z_2)c(z_n)\rangle|^2 = |z_{12}z_{2n}z_{n1}|^2. \tag{5.3}$$

[16] We follow the conventions of [95].

The correlation function (5.1) can be written as in (4.1) by defining $\mathrm{d}\mu_{\mathrm{bos}} = \prod_{i=3}^{n-1} \mathrm{d}^2 z_i$ and

$$\mathcal{F}_{0,n} := g_c^{n-2} \frac{8\pi}{\alpha'} \mathcal{W}_{0,n}(z_{jk}^{-1}, \bar{z}_{lm}^{-1}) \exp(\mathcal{Q}_{0,n}), \tag{5.4}$$

$$\mathcal{Q}_{0,n} := \alpha' \sum_{3\le i<j\le n-1} k_i \cdot k_j \log |z_i - z_j|, \tag{5.5}$$

where $1 \le j, k, l, m \le n$ and $\mathcal{W}_{0,n} = 1$ for the scattering of n tachyons, while it is a rational function of the z_{jk} in the general case of massless states scattering. Its coefficients are made of factors of α', scalar products of polarization tensors and external momenta and include the color structure for gauge theory interactions.

Let us start with the scattering of four tachyon states. The vertex operator of a tachyon with momentum k_i ($k_i^2 = -m_{\mathrm{tach}}^2 := 4/\alpha'$) is a plane wave $V_j = e^{ik_j\cdot X(z_j,\bar{z}_j)}$. From (5.1) we obtain

$$\begin{aligned}\mathsf{A}_{\alpha'}^{(0,4-\mathrm{tachyons})} &= g_{\mathrm{tach}}^2 \,|z_{12}z_{24}z_{41}|^2 \\ &\quad\times \int \mathrm{d}^2 z_3\, e^{\left(\alpha' k_1\cdot k_3 \log|z_{13}z_{24}| + \alpha' k_2\cdot k_3 \log|z_{23}z_{14}| + \alpha' k_4\cdot k_3 \log|z_{12}z_{34}|\right)},\end{aligned} \tag{5.6}$$

where we have introduced the tachyon coupling constant $g_{\mathrm{tach}} = 8\pi g_c/\alpha'$ and kept z_1, z_2 and z_4 fixed but arbitrary. Momentum conservation imposes $k_1 + k_2 + k_3 + k_4 = 0$ and the Mandelstam kinematic invariants s, t, u are defined by $s = -(k_1 + k_2)^2$, $t = -(k_1 + k_4)^2$, $u = -(k_1 + k_3)^2$. Their sum is the sum of the squared masses of the particles $s + t + u = \sum_1^4 m_i^2$. The integral (5.6) can be computed explicitly and reads

$$\mathsf{A}_{\alpha'}^{(0,4-\mathrm{tachyons})} = 2\pi g_{\mathrm{tach}}^2 \frac{\Gamma(\alpha(s))\Gamma(\alpha(t))\Gamma(\alpha(u))}{\Gamma(\alpha(t)+\alpha(u))\Gamma(\alpha(u)+\alpha(s))\Gamma(\alpha(s)+\alpha(t))} \tag{5.7}$$

where $\alpha(s) := -1 - s\,\alpha'/4$. It has poles in the tachyon kinematic channels, for instance

$$\mathsf{A}_{\alpha'}^{(0,4-\mathrm{tachyons})} \underset{s\to -4/\alpha'}{\sim} g_{\mathrm{tach}}^2 \frac{1}{-s - 4/\alpha'}. \tag{5.8}$$

We want to recover these poles in the point-like limit in a tropical language. Physically, these poles originate from regions where vertex operators collide to one another. Since at tree level in field theory, there are only poles, the domains $\mathcal{D}$ of the decomposition in Eq. (3.6) precisely correspond to these regions. At four points, only one coordinate is free and the domains are just open disks of radius ℓ centered around z_1 z_2 and z_4 called $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_4$ as shown in Fig. 15 (see for instance the classic reference [96]):

$$\mathcal{M}_{0,4} = (\mathcal{D}_1 \sqcup \mathcal{D}_2 \sqcup \mathcal{D}_4) \sqcup \mathcal{D}_0. \tag{5.9}$$

We review below how the integrals over each domain provide the u, t and s channel tachyon exchanges, respectively, while the integral over $\mathcal{D}_0$ gives a subleading contribution. We start with the integral over $\mathcal{D}_1$. As the domains are

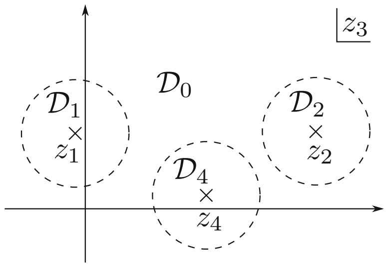


FIGURE 15. Decomposition of the moduli space $\mathcal{M}_{0,4}$

disjoint, we have $|z_{21}| > \ell$ and $|z_{41}| > \ell$. Thus, the terms $\alpha' k_2 \cdot k_3 \log |z_{32} z_{14}| + \alpha' k_4 \cdot k_3 \log |z_{34} z_{12}|$ in Eq. (5.6) behave like

$$(-\alpha' k_1 \cdot k_3 - 4) \log |z_{12} z_{14}| + O(\alpha' z_{31}, \alpha' \bar{z}_{31}) \tag{5.10}$$

which gives in the integral:

$$\int_{\mathcal{D}_1} \mathrm{d}^2 z_3 \frac{|z_{24}|^2}{|z_{12} z_{14}|^2} \, e^{\alpha' k_1 \cdot k_3 \log \left| \frac{z_{31} z_{24}}{z_{12} z_{14}} \right|} + O(\alpha'), \tag{5.11}$$

The integration over the phase of z_{31} is now trivial; hence, we may change variables to the tropical variable X as in (3.14);

$$c \times z_{31} = \exp(-X/\alpha' + i\theta), \tag{5.12}$$

where c is a conformal factor given by $c = z_{24}/(z_{12} z_{14})$ and θ is the irrelevant phase. In this variable, the closer z_3 is from z_1, the larger X is. The integration measure becomes $|c|^2 \mathrm{d}^2 z_3 = -\frac{2}{\alpha'} \, e^{-2X/\alpha'} \, dX \, d\theta$ and the radial integration domain is now $X \in [-\alpha' \log \ell, +\infty[$. We integrate out θ, drop the ℓ-dependent terms, since they are subleading, and obtain the following contribution to the amplitude

$$\mathsf{A}_{\alpha'}^{(0,4-\text{tachyons})}|_{u-\text{channel}} = g_{\text{tach}}^2 \left(\int_0^\infty dX \; e^{-((k_1+k_3)^2 + m_{\text{tach}}^2) X} + O(\alpha') \right). \tag{5.13}$$

This is simply the exponentiated the Feynman propagator of a scalar ϕ^3 theory with coupling constant g_{tach} and mass m_{tach}. In this form, the modulus X of the graph corresponds to the Schwinger proper time of the exchanged particle, as in Fig. 16. The same computation can be repeated in the other two kinematic

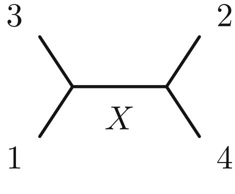


FIGURE 16. X is the modulus of the tropical graph. The larger it is, the closer z_1 from z_3

regions to obtain s- and t-channel exchanges. To conclude, one has to check that the integral over $\mathcal{D}_0$ does yield only $O(\alpha')$ contributions. In the case of tachyon scattering, this is actually not true, due to the fact that the tachyon acquires an infinite negative mass squared $m^2_{\text{tach}} = -4/\alpha'$ when $\alpha' \to 0$, which cancels the exponential damping induced by the factor α' already present in $\mathcal{Q}_{0,n}$. This is not surprising because tachyons generically lead to inconsistencies of the field theory. In the case of gravitons that we consider next, the limit will be well-defined and the integral over $\mathcal{D}_0$ will vanish.

Let us turn to graviton scattering in superstring theory. The decomposition remains unchanged. The qualitative difference with the scalar case is due to the appearance of a non-trivial $\mathcal{W}$. We will work in a representation of the integrand where all double poles have been integrated out by parts—this can always been done. [17] The tree-level four-graviton amplitude is written as

$$\mathsf{A}^{(0,4-\text{grav.})}_{\alpha'} = \frac{8\pi g_c^2}{\alpha'} \langle c\bar{c}V_{(-1,-1)}(z_1)c\bar{c}V(z_2)_{(-1,-1)}V_{(0,0)}(z_3)c\bar{c}V_{(0,0)}(z_4)\rangle. \tag{5.14}$$

The graviton vertex operators in the $(-1,-1)$ and $(0,0)$ pictures read

$$\begin{aligned} V_{(-1,-1)}(z) &= \epsilon_{\mu\nu}\psi^\mu\bar{\psi}^\nu e^{-\phi-\bar{\phi}}e^{ik\cdot X(z,\bar{z})}, \\ V_{(0,0)}(z) &= \frac{2}{\alpha'}\epsilon_{\mu\nu}\left(i\bar{\partial}X^\mu + \frac{\alpha'}{2}k\cdot\bar{\psi}\bar{\psi}^\mu\right)\left(i\partial X^\mu + \frac{\alpha'}{2}k\cdot\psi\psi^\mu\right)e^{ik\cdot X(z,\bar{z})}. \end{aligned} \tag{5.15}$$

in terms of the polarization tensors $\epsilon_{\mu\nu} := \epsilon_\mu\tilde{\epsilon}_\nu$. The bosonized superconformal ghost two-point function reads $\langle\phi(z)\phi(w)\rangle = -\log(z-w)$ while the one of the fermions reads $\psi^\mu(z)\psi^\nu(w) = \eta^{\mu\nu}/(z-w)$. In terms of these, the amplitude (5.16) can be computed explicitly (see the classic reference [101]);

$$\mathsf{A}^{(0,4-\text{grav.})}_{\alpha'} = \frac{8\pi g_c^2}{\alpha'}C(s,t,u)\,\mathcal{R}^4, \tag{5.16}$$

where $\mathcal{R}^4$ is a particular tensorial combination of four powers of the linearized Weyl tensor $R^{\mu\nu\rho\sigma} = F^{\mu\nu}\tilde{F}^{\rho\sigma}$ written in term the famous tensor t_8 as $\mathcal{R}^4 = t_8t_8R^4$. The tensors F and $\tilde{F}$ are on-shell linearized field strengths; the graviton i with polarization $\epsilon_i^{\mu\nu} = \epsilon_i^\mu\tilde{\epsilon}_i^\nu$ and momentum k_i has $F_i^{\mu\nu} = \epsilon_i^{[\mu}k_i^{\nu]}$ and $\tilde{F}_i^{\rho\sigma} = \tilde{\epsilon}_i^{[\rho}k_i^{\sigma]}$. The function C and the tensor t_8 are defined in [101], we reproduce them here:

$$C(s,t,u) = -\pi\frac{\Gamma(-\alpha's/4)\Gamma(-\alpha't/4)\Gamma(-\alpha'u/4)}{\Gamma(1+\alpha's/4)\Gamma(1+\alpha't/4)\Gamma(1+\alpha'u/4)}, \tag{5.17a}$$

$$\begin{aligned} t_8F^4 = -st(\epsilon_1\cdot\epsilon_3)(\epsilon_2\cdot\epsilon_4) + 2t(&\epsilon_2\cdot k_1\,\epsilon_4\cdot k_3\,\epsilon_3\cdot\epsilon_1 + \epsilon_3\cdot k_4\,\epsilon_1\cdot k_2\,\epsilon_2\cdot\epsilon_4 \\ &+ \epsilon_2\cdot k_4\,\epsilon_1\cdot k_3\,\epsilon_3\cdot\epsilon_4 + \epsilon_3\cdot k_1\,\epsilon_4\cdot k_2\,\epsilon_2\cdot\epsilon_1) + (2\leftrightarrow 3) + (3\leftrightarrow 4). \end{aligned} \tag{5.17b}$$

[17] See [7–10] for a one-loop proof and the more recent works [97–100] for an extensive study of the tree-level integrand representations, using integration by parts and fraction by part identities

Schematically, $t_8 F^4$ is a polynomial in the kinematic invariants with coefficients made of scalar products between polarizations and momenta

$$t_8 F^4 = C_s s + C_t t + C_u u + C_{st} st + C_{tu} tu + C_{us} us. \tag{5.18}$$

Since $C(s,t,u) \sim 1/({\alpha'}^3 stu)$, using multiple times the on-shell condition $s+t+u=0$, the amplitude (5.16) can be written as

$$\mathsf{A}^{(0,4)}_{\alpha'} \sim \frac{A_s}{s} + \frac{A_t}{t} + \frac{A_u}{u} + A_0 + O(\alpha') \tag{5.19}$$

where the A's are sums of terms like $C_s \overline{C}_t$, etc. As the tensorial structure of this object is rather complicated, we will only focus ourselves on one particular term; a contribution to A_u.

In the correlation function (5.14), such a contribution comes from the following term:

$$\begin{aligned} &-(\alpha'/2)^2 (\epsilon_2 \cdot \epsilon_4) \tfrac{1}{z_{24}^2} (\epsilon_1 \cdot k_4)(\epsilon_3 \cdot k_2) \left(\left(\tfrac{1}{z_{14}} - \tfrac{1}{z_{13}} \right) \left(\tfrac{1}{z_{32}} - \tfrac{1}{z_{31}} \right) + \tfrac{1}{z_{13}^2} \right) \\ &\quad \times (-1)(\alpha'/2)^2 \left(\tilde\epsilon_2 . \tilde\epsilon_4 \right) \tfrac{1}{\bar z_{24}^2} (\tilde\epsilon_1 \cdot k_2)(\tilde\epsilon_3 \cdot k_4) \left(\left(\tfrac{1}{\bar z_{12}} - \tfrac{1}{\bar z_{13}} \right) \left(\tfrac{1}{\bar z_{34}} - \tfrac{1}{\bar z_{31}} \right) + \tfrac{1}{\bar z_{13}^2} \right), \end{aligned} \tag{5.20}$$

where we have used the conservation of momentum $k_1 + k_2 + k_3 + k_4 = 0$, the on-shell condition $\epsilon_i \cdot k_i = 0$. It is now straightforward to check that the term corresponding to $1/|z_{31}|^2$ in the previous expression is accompanied with a factor of $|z_{12} z_{24} z_{41}|^{-2}$ which combines with the conformal factor from the $c\bar c$ ghosts integration (5.3) to give

$$-\left(\frac{\alpha'}{2}\right)^3 \int \mathrm{d}^2 z_{31} \frac{1}{|z_{31}|^2} e^{\alpha' k_1 \cdot k_3 \log |z_{31}|} + O(\alpha'). \tag{5.21}$$

The phase dependence of the integral is either pushed to $O(\alpha')$ terms or canceled due to level matching in the vicinity of z_1. Thus, we can integrate it out and recast the integral in its tropical form using the same change of variables as in (5.12) and one gets the following contribution to the amplitude of Eq. (5.14):

$$4\kappa_d^2 \left(\int_0^\infty \mathrm{d}X e^{-uX} + O(\alpha') \right), \tag{5.22}$$

where $\kappa_d = 2\pi g_c$ is the d-dimensional coupling constant that appears in the Einstein–Hilbert action. Other terms are generated in the exact same manner, by combinations of various massless poles (even A_0, despite that it has no explicit pole structure). The full amplitude is finally rewritten as an integral over $\mathcal{M}_{0,4}^{\mathrm{trop}}$ as follows;

$$\mathsf{A}^{(0,4-\mathrm{grav.})}_{\alpha'} \to \mathsf{A}^{(0,4-\mathrm{grav.})} = \int_{\mathcal{M}_{0,4}^{\mathrm{trop}}} \mathrm{d}\mu_{\mathrm{trop}}\, F_{0,4}, \tag{5.23}$$

where the measure pulls back to regular integration measure dX on each edge, and $F_{0,4} = 4\kappa_d^2 t_8 t_8 R^4 \exp\left(-X((k_i + k_3)^2)\right)$ where $i = 1,2,4$, depending on the edge of $\mathcal{M}_{0,4}^{\mathrm{trop}}$ considered.

The generalization to n points is conceptually straightforward, though combinatorially more involved. The trees with edges of finite lengths will be

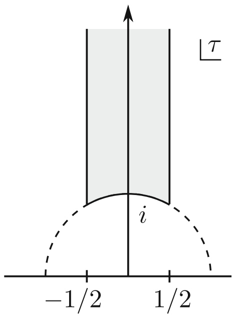


FIGURE 17. An $SL(2,\mathbb{Z})$ fundamental domain for complex tori

generated by similar regions of the moduli space where the points z_i collides toward one another. Writing out explicitly, this decomposition would not bring any new insight, so we shall turn to loops now.

5.2. One Loop (Review)

The technical aspects of the point-like limit of one-loop open and closed string theory amplitudes are well understood. In this review section, we simply recast in the tropical framework some of the older results on the subject. We first focus on the four-graviton type II superstring amplitude since we are ultimately interested in higher genus four-graviton amplitudes. That amplitude is a nice toy model to see how the tropical limiting procedure naturally generates the so-called analytic and non-analytic terms [14, 33, 35, 102] of the amplitudes together with the counter-terms. Then we discuss the n-point case. We make connection with the previous section and describe the regions of the string theory moduli space integral give rise to trees attached to the loop, recapitulating the Bern–Kosower rules.

Let us first review some elements about genus one Riemann surfaces or elliptic curves. They are complex tori $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ parametrized by a complex modulus τ in the Siegel upper half-plane $\mathcal{H}_1 = \{\tau \in \mathbb{C},\ \mathrm{Im}\,(\tau) > 0\}$.[18] Modding out by the action of the modular group $SL(2,\mathbb{Z})$ restricts τ to an $SL(2,\mathbb{Z})$ fundamental domain. The one that we use is defined by $\mathcal{F} = \{\tau \in \mathcal{H}_1,\ 1 < |\tau|,\ -1/2 \le \mathrm{Re}\,\tau < 1/2,\ \mathrm{Im}\,\tau > 0\}$, see Fig. 17. Also, recall that

$$q = \exp(2i\pi\tau).$$

If we include the three moduli associated with the four punctures at distinct positions $\zeta_i \in \mathcal{T}$, $i = 1,2,3$ where $\mathcal{T} = \{\zeta \in \mathbb{C}, -1/2 < \mathrm{Re}\,\zeta < 1/2,\ 0 \le \mathrm{Im}\,\zeta < \mathrm{Im}\,\tau\}$ and ζ_4 fixed at $\zeta_4 = \mathrm{Im}\,\tau$, we can describe completely the moduli space $\mathcal{M}_{1,4}$ over which our string theory amplitude is integrated

[18] The complex torus is actually the Jacobian variety of the surface, but at genus one both are isomorphic. This property does not hold for higher genus curves.

$$\mathsf{A}_{\alpha'}^{(1,4)} = \int_{\mathcal{M}_{1,4}} \mathrm{d}\mu_{\mathrm{bos}}\ \mathcal{F}_{1,4}. \tag{5.24}$$

We start the analysis with the four-graviton type II amplitude in 10 dimensions. Supersymmetry kills the configurations where vertex operators collide which could create poles. Thus, we will not consider regions of the moduli space $\mathcal{M}_{1,4}$ which could give rise to one-loop diagrams with trees attached to the loop. This will be justified *a posteriori*. For this amplitude $\mathcal{F}_{1,4}$ is particularly simple since it is reduced to the Koba–Nielsen factor times a constant kinematic term

$$\mathcal{F}_{1,4} = (2\pi)^8 \mathcal{R}^4 \exp\left(\alpha' \sum_{i<j} k_i \cdot k_j \mathcal{G}(\zeta_i, \bar{\zeta}_i, \zeta_j, \bar{\zeta}_j)\right), \tag{5.25}$$

where $\mathcal{R}^4$ has been defined below Eq. (5.16). The integration measure reads

$$\int_{\mathcal{M}_{1,4}} \mathrm{d}\mu_{\mathrm{bos}} = \int_{\mathcal{F}} \frac{\mathrm{d}^2\tau}{(\mathrm{Im}\,\tau)^5} \int_{\mathcal{T}} \prod_{i=1}^{3} \mathrm{d}^2\zeta_i. \tag{5.26}$$

The one-loop bosonic propagator reads

$$\mathcal{G}(\zeta_i, \bar{\zeta}_i, \zeta_j, \bar{\zeta}_j) = -\frac{1}{2}\log\left|\frac{\theta\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](\zeta_i - \zeta_j|\tau)}{\partial_\zeta \theta\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](0|\tau)}\right|^2 + \frac{2\pi(\mathrm{Im}\,(\zeta_i - \zeta_j))^2}{\mathrm{Im}\,\tau}, \tag{5.27}$$

as in (4.5). From now on we omit the dependence on the conjugate variables in $\mathcal{G}$. We start the tropicalization procedure, following Sect. 4.1. We look first at the torus alone, and include punctures later. We want to find a decomposition for $\mathcal{F}$. As q is a local coordinate on the moduli space around the nodal curve at infinity, we would want to use it as in Sect. 3.3. We saw in (3.7) that, in order to obtain a loop of finite size T, we had to set $|q| = \exp(-2\pi T/\alpha')$. This defines a family of tori parametrized by their modulus $\tau_{\alpha'}$:

$$\mathrm{Re}\ \tau_{\alpha'} = \mathrm{Re}\,\tau \in [-1/2; 1/2[, \qquad \mathrm{Im}\ \tau_{\alpha'} = T/(2\pi\alpha') \in [0; +\infty[. \tag{5.28}$$

The issue with the previous definition is that for $\mathrm{Im}\,\tau_{\alpha'} < 1$, $\mathrm{Re}\,\tau_{\alpha'}$ is not unrestricted in $\mathcal{F}$, but depends on $\mathrm{Im}\,\tau_{\alpha'}$. To build the decomposition, we follow [14] and introduce a parameter $L > 1$ to cut the fundamental domain into an upper part, the non-analytic domain $\mathcal{F}^+(L)$, and a lower part, the analytic domain $\mathcal{F}^-(L)$. They are defined by $\mathcal{F}^+(L) = \{\tau \in \mathcal{F}, \mathrm{Im}\,\tau > L\}$ and $\mathcal{F}^-(L) = \{\tau \in \mathcal{F},\ \mathrm{Im}\,\tau \leq L\}$, respectively. The decomposition then reads

$$\mathcal{F} = \mathcal{F}^+(L) \sqcup \mathcal{F}^-(L). \tag{5.29}$$

For any $T \geq 2\pi\alpha' L$ we now have the standard family of complex tori in $\mathcal{F}^+(L)$

$$\mathrm{Re}\ \tau_{\alpha'} = \mathrm{Re}\,\tau \in [-1/2; 1/2[, \qquad \mathrm{Im}\ \tau_{\alpha'} = T/2\pi\alpha' \in [L; +\infty[. \tag{5.30}$$

To complete the decomposition, we have to deal with the positions of the punctures. Firstly, note that the splitting (5.29) induces a similar decomposition of $\mathcal{M}_{1,4}$ into two domains depending on L, defined by the position of τ in $\mathcal{F}$

$$\mathcal{M}_{1,4} = \mathcal{M}_{1,4}^+(L) \sqcup \mathcal{M}_{1,4}^-(L). \tag{5.31}$$

In $\mathcal{M}_{1,4}^{-}(L)$, the positions of the punctures can be integrated out directly. In $\mathcal{M}_{1,4}^{+}(L)$ however, it is well known that to take correctly the $\alpha' \to 0$ limit, one should split the integration domain spanned the punctures into three regions, one for each inequivalent ordering of the graph [1, 103]. Hence $\mathcal{M}_{1,4}^{+}(L)$ is split further into three disjoint domains, labeled by the three permutations inequivalent under reversal symmetry $\sigma \in \mathfrak{S}_3/\mathbb{Z}_2 = \{(123),(231),(312)\}$ defined by

$$\mathcal{D}_{(ijk)} := \mathcal{F}(L)^{+} \times \{\zeta_i, \zeta_j, \zeta_k \,|\, 0 < \operatorname{Im}\zeta_i < \operatorname{Im}\zeta_j < \operatorname{Im}\zeta_k < \operatorname{Im}\tau\}. \tag{5.32}$$

In total, we have the explicit decomposition

$$\mathcal{M}_{1,4} = \left(\bigsqcup_{\sigma \in \{(123),(231),(312)\}} \mathcal{D}_{\sigma} \right) \sqcup \mathcal{M}_{1,4}^{-}(L) \tag{5.33}$$

Since the integrand vanishes by supersymmetry in the other regions of the moduli space, where a tree splits off from the torus for instance, there is no need to refine the decomposition to take into account vertex operators colliding to one another.

To determine a tropical form of the integrand, we compute the limit in the two regions $\mathcal{M}_{1,4}^{\pm}(L)$ separately. We define, following [14],

$$\begin{aligned} \mathsf{A}_{\alpha',+}^{(1,4)}(L) &= \sum_{i=(s,t),(t,u),(u,s)} \int_{\mathcal{D}_i} \mathrm{d}\mu_{\mathrm{bos}}\, \mathcal{F}_{1,4}, \\ \mathsf{A}_{\alpha',-}^{(1,4)}(L) &= \int_{\mathcal{M}_{1,4}^{-}(L)} \mathrm{d}\mu_{\mathrm{trop}}\, \mathcal{F}_{1,4}. \end{aligned} \tag{5.34}$$

Of course these partial amplitudes add up to the complete amplitude.

In $\mathcal{M}_{1,4}^{+}(L)$, we have the scaling behavior (5.30). As for the punctures, in $\mathcal{D}_{(ijk)}$ we define the following families of points:[19]

$$\zeta_{i\alpha'} = \operatorname{Re}\zeta_i + iX_i/(2\pi\alpha'), \quad \operatorname{Re}\zeta_i \in [0; 2\pi[, \quad 0 < X_i < X_j < X_k < X_4 = T. \tag{5.35}$$

Although we already derived in full rigor the field theory limit of the Green's function at any genus, it is instructive to review this standard computation at genus one. The propagator (5.27) has the following q-expansion:

$$\begin{aligned} \mathcal{G}(\zeta_i, \zeta_j) = {} & \frac{\pi(\operatorname{Im}(\zeta_i - \zeta_j))^2}{\operatorname{Im}\tau} - \frac{1}{2}\log\left|\frac{\sin(\pi(\zeta_i - \zeta_j))}{\pi}\right|^2 \\ & -2\sum_{m\geq 1}\left(\frac{q^m}{1-q^m}\frac{\sin^2(m\pi(\zeta_i - \zeta_j))}{m} + h.c.\right), \end{aligned} \tag{5.36}$$

[19] This definition is equivalent to the one defined in (3.14) at tree-level, one should just pay attention to the fact that $\zeta_{\alpha'}$ belongs to the complex torus, i.e. the Jacobian. Its inverse image via the Abel–Jacobi map, $z_{\alpha'} \simeq \exp(i\zeta_{\alpha'}) + O(q)$ does indeed satisfy (3.14).

which, in terms of $\tau_{\alpha'}$, $\zeta_{i_{\alpha'}}$ and $\zeta_{j_{\alpha'}}$, becomes

$$\alpha'\mathcal{G}(\zeta_{i_{\alpha'}},\zeta_{j_{\alpha'}}) = \frac{1}{2T}(X_i-X_j)^2 - \frac{\alpha'}{2}\log\Big(\big|e^{-(X_i-X_j)/(2\alpha')}e^{i\pi\mathrm{Re}\,(\zeta_{ij})} \\ -e^{(X_i-X_j)/(2\alpha')}e^{-i\pi\mathrm{Re}\,(\zeta_{ij})}\big|^2\Big) + O(\alpha') \tag{5.37}$$

up to $O(q)$ terms and where ζ_{ij} stands for $\zeta_i-\zeta_j$. At leading order in α', the logarithm is equal to the absolute value of X_i-X_j and one gets

$$\lim_{\alpha'\to 0}(\alpha'\mathcal{G}(\zeta_{i_{\alpha'}},\zeta_{j_{\alpha'}})) = G^{\mathrm{trop}}(X_i,X_j) = \frac{1}{2}\left(-|X_i-X_j| + \frac{(X_i-X_j)^2}{T}\right). \tag{5.38}$$

This is the well known worldline propagator on the circle derived in [90] with the exact same normalization. This expression also coincides with the one for G^{trop} given in Eq. (4.10). By plugging that result in $\mathcal{F}_{1,4}$ one obtains

$$\mathcal{F}_{1,4} \to F_{1,4} = (2\pi)^8\mathcal{R}^4\exp\left(\sum k_i\cdot k_j G^{\mathrm{trop}}(X_i,X_j)\right) + O(\alpha'), \tag{5.39}$$

where nothing depends anymore on the phases $\mathrm{Re}\,\zeta_i$ or $\mathrm{Re}\,\tau$. We can integrate them out and the measure (5.26) becomes

$$\mathrm{d}\mu_{\mathrm{bos}} \to \mathrm{d}\mu_{\mathrm{trop}} = 2\pi\alpha'\frac{\mathrm{d}T}{T^5}\prod_{i=1}^{3}\mathrm{d}X_i \tag{5.40}$$

over the integration domains

$$D_{(ijk)} = \{T\in[\alpha' L,+\infty\,[\,\}\times\{X_i,X_j,X_k\in[0;T[\;|\;0<X_i<X_j<X_k<T\}. \tag{5.41}$$

For instance in the ordering 1234, the exponential factor reduces to $Q_{1,4} = X_1(X_3-X_2)s+(X_2-X_1)(X_4-X_3)t$; this is the second Symanzik polynomial of this graph. The first Symanzik polynomial is simply T.

Collecting the various pieces, $\mathsf{A}^{(1,4)}_{\alpha',+}(L)$ is given by, at leading order;

$$\mathsf{A}^{(1,4)}_{+}(L) = \sum_\sigma\int_{D_\sigma}\mathrm{d}\mu_{\mathrm{trop}}\,F_{1,4} \\ = \alpha'(2\pi)^9\mathcal{R}^4\Big(\int_{2\pi\alpha' L}^{\infty}\frac{\mathrm{d}T}{T^2}\int_0^T\frac{\mathrm{d}X_3}{T}\int_0^{X_3}\frac{\mathrm{d}X_2}{T}\int_0^{X_2}\frac{\mathrm{d}X_1}{T}\,e^{Q_{1,4}} \\ +2\text{ other orderings}\Big), \tag{5.42}$$

This is the classic result of [1]. Now, we could in principle drop the restriction $T>2\pi\alpha' L$ and use dimensional regularization. However, in order to make the underlying tropical nature of the limit manifest, the hard UV cutoff $2\pi\alpha' L$ should be kept. Then in 10 dimensions, this integral has a power-behaved UV divergence given by

$$\mathsf{A}^{(1,4)}_{\alpha',+}\Big|_{\text{leading div.}} = \alpha'(2\pi)^9\mathcal{R}^4\left(\frac{1}{2\pi\alpha' L}\right), \tag{5.43}$$

as can be seen by a direct computation. As observed in [14], the full amplitude $\mathsf{A}^{(1,4)}_{\alpha'}$ does not depend on L, thus any non-vanishing term in $\mathsf{A}^{(1,4)}_{\alpha',+}$ that depends on L in the tropical limit should be canceled by including contributions from the analytic domain. In particular, the divergence (5.43) should be canceled by a counter-term coming from $\mathsf{A}^{(1,4)}_{\alpha',-}$.

The integrand being analytic in the compact space $\mathcal{M}^{-}_{1,4}(L)$, we can take the $\alpha' \to 0$ limit inside the integral: this sets the exponential factor to 1. The integration over the ζ_i's is now trivial and the remaining integral can be computed straight away:

$$\begin{aligned} \mathsf{A}^{(1,4)}_{\alpha',-}(L) \to \mathsf{A}^{(1,4)}_{-}(L) &= (2\pi)^8 \mathcal{R}^4 \int_{\mathcal{F}_L} \frac{\mathrm{d}^2\tau}{(\operatorname{Im}\tau)^2} + O(\alpha') \\ &= (2\pi)^9 \mathcal{R}^4 \left(\frac{1}{6} - \frac{1}{2\pi L}\right) + O(\alpha'). \end{aligned} \tag{5.44}$$

Up to the global factor, there are two physically distinct contributions; 1/6 and $-1/(2\pi L)$. The first is the so-called analytic part of the amplitude. After going from the string frame to the Einstein frame, it is solely expressed in terms of gravitational coupling constant and is the leading order contribution of higher order operators in the effective action of supergravity. The second is the counter-term required to cancel the leading UV divergence (5.43). From the tropical point of view, this integral may be thought of as being localized at the singular point $T = 0$ of the tropical moduli space which corresponds to a graph with a vertex of weight one.

We may now add up (5.42) and (5.44) to obtain the field theory amplitude written as an integral over the full tropical moduli space $\mathcal{M}^{\mathrm{trop}}_{1,4}$. This amplitude is regularized by the inclusion of a counter-term at $T = 0$. This discussion is summarized in Fig 18.

For general amplitudes, $\mathcal{W}_{1,n}$ acquires a possibly complicated structure and one often has to perform a Fourier expansion of $(\mathcal{W}_{1,n} \exp(\mathcal{Q}_{1,n}))$ in terms of q or $\sqrt{q}$ as discussed in Sect. 4.2 (see [7–10] and more recently for instance [94, 104] for heterotic string computations). At first, these terms may seem q-

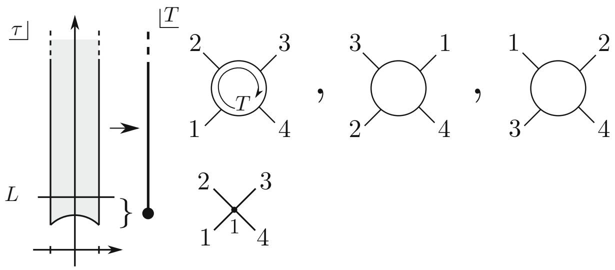


FIGURE 18. Summary of the tropicalization of the four-graviton genus one amplitude in type II string

or $\sqrt{q}$-exponentially suppressed as $\mathrm{Im}\,\tau \to \infty$. However, the worldsheet realization of generic string theory models with non-maximal supersymmetry is based on altering the spin structure sum projection: this causes the appearance of "poles" in $1/q$ and $1/\sqrt{q}$. In all consistent models, these poles are automatically either compensated by higher-order terms in the Fourier expansion or killed by real part integration via identities such as $\int_{-1/2}^{1/2} q^n \bar{q}^m \mathrm{dRe}\,\tau = 0$ if $n \neq m$. In the bosonic string, they are not, which makes the theory inconsistent at loop level.

Let us make explicit the general form of the decomposition for n-point amplitudes used in the Bern–Kosower rules, or the more recent works [16, 17, 105]. There are now $(n-1)!/2$ domains $\mathcal{D}_\sigma$ for $\sigma \in \mathfrak{S}_{n-1}/\mathbb{Z}_2$ defined exactly as in (5.32) that generate 1PI tropical graphs with orderings σ. In this previous analysis, we did not have to deal with regions in the moduli space where points collide to one another because supersymmetry prevented such configurations to contribute. In general though, they have to be included, for physical reasons—we know that there are contact terms in generic amplitudes—and for mathematical reasons—the tropical moduli space does have 1PR graphs.

Hence, we refine the previous definition of the domains $\mathcal{D}_\sigma$ and define new domains $\hat{\mathcal{D}}_\sigma$ and $\hat{\mathcal{M}}^-(L)$ by cutting out the open disks $|\zeta_i - \zeta_j| < e^{-\ell\alpha'}$ of the domains $\mathcal{D}_\sigma$.[20] The complementary set of the union of the previous domains in $\mathcal{M}^+(L)$ is made of domains of the form $\hat{\mathcal{D}}_\sigma$, where $\sigma \in \mathfrak{S}_{p-1}/\mathbb{Z}_2$ indicates the ordering of p points on the future loop, while $n-p$ points are grouped into one or more disks of radius ℓ centered around one or more of the first p points.

To finish the description of the decomposition, we have to deal with these clusters of points. Locally, such a cluster of m points on a disk of radius ℓ looks like a sphere. Thus, as in the tree-level analysis, $\mathcal{M}_{1,n}$ is decomposed into $(2m-3)!!$ domains corresponding to the $(2m-3)!!$ combinatorially distinct trees. Note the shift $m \to m+1$ compared to the tree-level case due to the fact that such trees with m external legs have one additional leg attached to the loop. At this point, one could basically conclude by invoking the Bern–Kosower rules; this would yield the desired tropical form of the one-loop amplitude. Let us then be brief and describe for simplicity, a cluster of two points, where ζ_j is treated like before (5.35) and ζ_i collides to ζ_j;

$$\zeta_{i_{\alpha'}} = \zeta_j + e^{i\theta} e^{-X/\alpha'}, \quad \theta \in [0; 2\pi[, \quad X \in [\alpha'\ell, +\infty[\tag{5.45}$$

where ζ_j is fixed, X is the tropical length of the tree connecting legs i and j to the loop as in the tree-level analysis and ℓ is an IR cutoff. In this simple example, there is no outer region $\mathcal{D}_0$ and the construction of the decomposition is complete. Concerning the tropical form of the integrand and the equation (4.12), one has to look at $\mathcal{F}_{1,n} = \mathcal{W}_{1,n} e^{\mathcal{Q}_{1,n}}$. For simplicity, we work in a representation of $\mathcal{W}_{1,n}$ where all double derivatives of the propagator have been integrated out by parts. Using the general short distance behavior of the propagator on a generic Riemann surface

[20] Note that ℓ has to be small compared to L so that $\hat{\mathcal{M}}^-(L)$ is non-empty. Typically $\ell \ll \sqrt{L/n\pi}$.

$$\mathcal{G}(z-w) = -1/2 \log |z-w|^2 + O((z-w)^3), \tag{5.46}$$

one sees that $\mathcal{Q}_{1,n}$ gives a term $-X k_i \cdot k_j$, while any term of the form $\mathcal{G}(\zeta_k, \zeta_i)$ is turned into a $\mathcal{G}(\zeta_k, \zeta_j)$ at leading order in α':

$$\sum_{k<l} (k_k \cdot k_l) \mathcal{G}(kl) = -X(k_i \cdot k_j) + \sum_{k \neq i,j} k_k \cdot (k_i + k_j) \mathcal{G}(jk) + \sum_{\substack{k<l \\ k,l \neq i,j}} (k_k \cdot k_l) \mathcal{G}(kl), \tag{5.47}$$

up to $O(\alpha')$ terms, with obvious abbreviated notation. The factor $e^{-X\, k_i \cdot k_j}$ provides a contact term via a pole in the amplitude if and only if $\mathcal{W}$ contains a factor of the form $|\partial \mathcal{G}(ij)|^2 \sim e^{2X/\alpha'}$ exactly as in the tree-level analysis. Then in $\mathcal{W}$ any ζ_i-dependent term is replaced by a ζ_j at the leading order in $O(\alpha')$. This is indeed one of the Bern–Kosower rules. A similar analysis can be performed in the region $\mathcal{M}^-(L)$ where we have to include the contributions of poles.

In this section, we have recast classic one-loop field theory limits in the tropical language. This shows a correspondence between the string theory integration over $\mathcal{M}_{1,n}$ and its field theory point-like limit, which can be expressed as an integral over the tropical moduli space $\mathcal{M}_{1,n}^{\text{trop}}$.

5.3. Two Loops

Zero- to four-point two-loop amplitudes in RNS type II and heterotic string have been worked out completely in [44–46, 48–50, 106]. The four-graviton amplitude have also been derived using the pure spinor formalism [107] and shown in [108] to be equivalent to the RNS computation.

However, the corresponding S-matrix elements in supergravity have not been extracted from these string theory amplitudes.[21] In [52], the four-graviton two-loop amplitude in maximal supergravity computed in [51] was *rewritten* in a worldline form resembling the string theory integral. In this section, our goal is to prove rigorously that the tropical limit of the string theory integrand does match this result by making use of the tropical machinery that we have developed. We also provide a decomposition of $\mathcal{M}_{2,0}$ such that each region encompasses the dual graphs corresponding to the primary and sub-divergences of the amplitude. The study of the integral restricted to the counter-term domains is left over for future work.

Let us review some facts about genus-two Riemann surfaces. At genus two (and three), there is no Schottky problem; therefore, we may parametrize $\mathcal{M}_2$ in terms of period matrices. As before, the action of the modular group $Sp(4, \mathbb{Z})$ on $\mathcal{H}_2$ restricts it to fundamental domains, of which we pick the representative $\mathcal{F}_2$ defined in [109]. This 3-dimensional complex space can be defined in terms of some inequalities that we describe below. They are similar to these defining $\mathcal{F}$ at genus one. We choose a canonical homology basis (a_I, b_J) as in Fig. 11 with normalized holomorphic one-forms (3.1). The period matrix

[21] See however [77], sec. 3.2, where a degeneration, that we call here tropical, of the so-called Kawazumi–Zhang invariant was investigated.

$\boldsymbol{\Omega}$ is parametrized by three complex moduli τ_1, τ_2 and τ_3:

$$\boldsymbol{\Omega} = \begin{pmatrix} \tau_1 + \tau_3 & -\tau_3 \\ -\tau_3 & \tau_2 + \tau_3 \end{pmatrix}. \tag{5.48}$$

In this parametrization, the inequalities of [109] can be rewritten as (see [110]);

- Conditions on $\operatorname{Re}\tau_j$ and $\operatorname{Im}\tau_j$:

$$|\operatorname{Re}\tau_3| \leq \frac{1}{2}, \ |\operatorname{Re}(\tau_j + \tau_3)| \leq \frac{1}{2}, \ \operatorname{Im}(\tau_j + \tau_3) \geq \frac{1}{2}\sqrt{3}, \ j = 1, 2, \ \operatorname{Im}\tau_3 \geq 0. \tag{5.49}$$

- Minkowski ordering:

$$\operatorname{Im}\tau_1 \geq \operatorname{Im}\tau_3, \quad \operatorname{Im}\tau_2 \geq \operatorname{Im}\tau_1, \tag{5.50}$$

- The following set of 19 inequalities:

$$|\tau_1 + \tau_3| \geq 1, \quad |\tau_2 + \tau_3| \geq 1, \quad |\tau_1 + \tau_2 + \epsilon| \geq 1, \tag{5.51}$$

  and

$$|\det(\boldsymbol{\Omega} + \boldsymbol{M})| \geq 1, \tag{5.52}$$

  for all matrices $\boldsymbol{M}$ in the set

$$\left\{\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} \epsilon & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & \epsilon \end{smallmatrix}\right), \left(\begin{smallmatrix} \epsilon & 0 \\ 0 & \epsilon \end{smallmatrix}\right), \left(\begin{smallmatrix} \epsilon & 0 \\ 0 & -\epsilon \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & \epsilon \\ \epsilon & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} \epsilon & \epsilon \\ \epsilon & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & \epsilon \\ \epsilon & \epsilon \end{smallmatrix}\right)\right\}, \ \epsilon = \pm 1. \tag{5.53}$$

Not considering punctures and ignoring the separating degeneration of the genus-two curve (we will see that it does not contribute to the field theory limit), we can define a decomposition of $\mathcal{M}_{2,0}$, as follows. We introduce by hand, in analogy with the genus one construction, a single parameter $L > 1$ and we define three domains $\mathcal{D}_i$, $i = a, b, c$ by

$$\begin{aligned} \mathcal{D}_a &= \mathcal{F}_2 \cap \{\operatorname{Im}\tau_1 \geq L\}, \\ \mathcal{D}_b &= \mathcal{F}_2 \cap \{\operatorname{Im}\tau_1 \leq L, \operatorname{Im}\tau_2 \geq L\}, \\ \mathcal{D}_c &= \mathcal{F}_2 \cap \{\operatorname{Im}\tau_1 \leq L, \operatorname{Im}\tau_2 \leq L\}. \end{aligned} \tag{5.54}$$

We checked numerically using a standard numerical minimization routine that for $L > 1$, in the domains $\mathcal{D}_a$ and $\mathcal{D}_b$ the determinant inequalities (5.52) are always satisfied, upon the constraints Eqs. (5.49), (5.51), (5.54). They turn out to be always individually greater than L^2. Of course the same procedure applied in the domain $\mathcal{D}_c$ fails for all determinant inequalities, for which the individual minimums are slightly greater than 0.7.

The three domains contain the singularities corresponding to the graphs of Fig. 19. Therefore, we identify $\mathcal{D}_a$ as the maximally non-analytic domain and $\mathcal{D}_c$ as the analytic domain. Since this decomposition is rather special (as

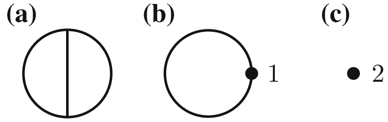


FIGURE 19. From *left* to *right*; the three master graphs entering the genus-two four-graviton amplitude

it is defined only in terms of a single parameter where one could have expected more), it is natural to wonder if the choice of L is constrained. Contrary to the one-loop case, the complexity of the definition of the fundamental domain $\mathcal{F}_2$ does not *a priori* grant us that any choice of L would give nice integrals. A good choice for L would be one that makes the real parts of the τ's in the regions $\mathcal{D}_a$ and $\mathcal{D}_b$ independent from their imaginary parts, so that they can be integrated out. Setting L big enough (of order 10 for instance) is clearly enough to ensure that the domain $\mathcal{D}_a$ is of this form, but then it is not guaranteed that $\mathcal{D}_b$ and $\mathcal{D}_c$ are suitable for easy integration. In [111] was presented a more elaborate decomposition based on two parameters, and it would be interesting to check if it is actually needed for the purpose of extracting UV divergences and sub-divergences in these amplitudes.

We leave this problem for future investigations, and from now on focus on the type II four-graviton string amplitude restricted to $\mathcal{D}_a$, in order to compute the tropical limit of the integrand.

In ten dimensions it reads [47, 108, 112, 113]

$$\mathsf{A}^{(2,4)}_{\alpha'}(\epsilon_i, k_i) = \frac{\pi}{64}\left(\frac{\kappa_{10} g_s \alpha'}{2}\right)^2 \mathcal{R}^4 \int_{\mathcal{F}_2} \frac{|\prod_{I\leq J} \mathrm{d}\Omega_{IJ}|^2}{(\det \operatorname{Im} \mathbf{\Omega})^5} \int_{\Sigma^4} |\mathcal{Y}_S|^2 \exp(\mathcal{Q}_{2,4}). \tag{5.55}$$

Here, $\int_{\Sigma^4}$ denotes integration of the four punctures over the surface Σ. The normalization in terms of the 10-dimensional gravitational coupling constant κ_{10} and the string coupling constant g_s can be found in [76] for instance.

The quantity $\mathcal{Y}_S$ arises from several contributions in the RNS computation and from fermionic zero modes in the pure spinor formalism [107, 108]. It is defined as

$$3\mathcal{Y}_S = (k_1 - k_2)\cdot(k_3 - k_4)\,\Delta(z_1, z_2)\Delta(z_3, z_4) + (13)(24) + (14)(23), \tag{5.56}$$

with

$$\Delta(z, w) = \omega_1(z)\omega_2(w) - \omega_1(w)\omega_2(z), \tag{5.57}$$

so that $|\mathcal{Y}_S|^2$ is a top form on Σ^4. Hence, we can identify a measure and an integrand as follows

$$\mathrm{d}\mu_{\mathrm{bos}} = \int_{\mathcal{F}_2} \frac{|\prod_{I\leq J} \mathrm{d}\Omega_{IJ}|^2}{(\det \operatorname{Im} \mathbf{\Omega})^5} \int_{\Sigma^4} |\mathcal{Y}_S|^2\,, \tag{5.58a}$$

$$\mathcal{F}_{2,4} = \mathcal{R}^4 \exp\Bigg(\alpha' \sum_{i<j} k_i\cdot k_j\, \mathcal{G}(z_i, z_j)\Bigg), \tag{5.58b}$$

where the numerator factor $\mathcal{W}_{2,4}$ is again trivial.

Before starting the computation, we note that it is immediate to see that the contributions coming from a separating degeneration vanish in the field theory limit. Indeed, the integrand is missing terms of the form $\partial G \bar\partial G$ that could produce $1/|z|^2$-poles, required to allow for a massless state exchange. Alternatively, this can be seen as a consequence of the "No-triangle" property of maximal supergravity, [16, 17]. This justifies why we did not have to be more precise about this region in defining the decomposition of $\mathcal{M}_2$.

The degeneration in the domain $\mathcal{D}_a$ has already been studied in details in Sect. 3.3, around Fig. 14. Here we follow a simpler approach: since we use a parametrization in terms of period matrices, we are allowed to take the tropical limit directly at this level, instead of at the level of the curve. Hence, we *define* the tropical scaling by

$$\operatorname{Im}\tau_i = -T_i/(2\pi\alpha'), \quad i = 1, 2, 3. \tag{5.59}$$

where, contrary to Eq. (3.9), no higher-order corrections enter this equation. Put differently, the q_i's, defined by

$$q_i = \exp(2i\pi\tau_i) \tag{5.60}$$

are particular local coordinates around the boundary divisor which are only equal to the t_i's at leading order, $q_i = t_i + O(q_i^2)$. On this point, see [38, eq 4.6] for an explicit relation between the Schottky representation and the q_i parameters in the case of the genus-two open string worldsheet.

We have thus defined families of curves whose period matrices tropicalize to $\boldsymbol{K}^{(2)} = \left(\begin{smallmatrix} T_1+T_3 & -T_3 \\ -T_3 & T_2+T_3 \end{smallmatrix}\right)$. Furthermore, the boundaries of $\mathcal{D}_a$ define worldline cutoff and ordering given by $\{T_1 > T_2 > 2\pi\alpha' L,\, T_3 > 0\}$.

Let us now turn to the limit of $\mathcal{Y}_S$. The tropical limit of the holomorphic one-forms (2.4) firstly gives the limit of the Δ bilinears;

$$\Delta(z_i, z_j) \sim \Delta^{\text{trop}}(ij) = \omega_1^{\text{trop}}(i)\omega_2^{\text{trop}}(j) - \omega_1^{\text{trop}}(j)\omega_2^{\text{trop}}(i) \tag{5.61}$$

up to some factor of α' that rigorously arises when combining with the anti-holomorphic part, as in Eq. (3.17). This tropical version of Δ is defined by

$$\Delta^{\text{trop}}(ij) = \begin{cases} 0 & \text{if } (i,j) \in B_1 \text{ or } (i,j) \in B_2 \\ 1 & \text{if } i \in B_1 \text{ and } j \in B_2 \\ -1 & \text{if } i \in B_2 \text{ and } j \in B_1 \end{cases} \tag{5.62}$$

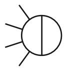
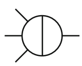
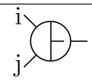
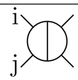
Then the tropical form of $\mathcal{Y}_S$ is immediately obtained:

$$3\mathcal{Y}_S \to 3Y_S = (k_1 - k_2)\cdot(k_3 - k_4)\,\Delta^{\text{trop}}(12)\Delta^{\text{trop}}(34) + (13)(24) + (14)(23). \tag{5.63}$$

This expression vanishes if three or four punctures lie on the same edge of the graph, while in all other cases, it is given by a kinematic invariant as in Table 1.

Let us mention that $\det \boldsymbol{K}^{(2)} = T_1T_2 + T_2T_3 + T_3T_1$ does not depend on the positions of the punctures and is easily seen to be the usual form of the first Symanzik polynomial of the sunset graph. This concludes the study of the tropicalization of the integration measure.

TABLE 1. Numerators for the two-loop four-graviton integrand.

Graph				
Y_S	0	0	$(-s_{ij})^2$	$(-s_{ij})^2$

The last thing to be done would be to compute the tropical representation of the exponential factor (5.58b). Fortunately, this was already done at any genus in (4.4), thanks to theorem (4.10). Thus we obtain our final result;

$$\begin{aligned}\mathsf{A}^{(2,4)}_{\mathrm{non-ana}}(L) = \mathcal{N}\mathcal{R}^4 \int_{T_1>T_2>2\pi\alpha' L}^{\infty} \frac{\mathrm{d}T_1\mathrm{d}T_2\mathrm{d}T_3}{(\det \boldsymbol{K})^5} \times \\ \int_{\Gamma^4} Y_S \exp\left(\sum_{i<j} k_i\cdot k_j\, G^{\mathrm{trop}}(Z_i,Z_j)\right),\end{aligned} \tag{5.64}$$

where $\mathcal{N}$ is a normalization factor and $\int_{\Gamma^4}$ stands for an integration of the positions of the four punctures on the graph. This object coincides with the one derived in [52, eq. 2.12] from the two-loop field theory computation of [51]; thus, it is the two-loop unrenormalized four-graviton amplitude.

To continue the procedure and remove the primary and sub-divergences (in dimensions when there are any), we should include the regions $\mathcal{D}_b$ and $\mathcal{D}_c$ described above in Eq. (5.54). These computations would illustrate the systematics of renormalisation in the tropicalization procedure in the presence of sub-divergences and one should match the field theory computations of [52, 114].

The computation of the $\alpha' \to 0$ limit of the genus-two Heterotic string amplitude represents a more challenging task, as we said before. It should be based, as explained in [94], on a Fourier expansion of the string integrand in the parameters q_i.

5.4. A Comment at Three Loops

An expression was proposed for a sub-sector of the four-graviton genus-three amplitude using the pure spinor formalism in [55]. Only the terms that contribute to D^6R^4 operator in the low energy limit were computed. Regardless, it would already be interesting to extract the tropical limit of this partial amplitude. Comparing the terms obtained from it to the full three-loop amplitude in supergravity would help to constrain the form of the missing terms in the string theory computation.

A quick analysis of the tropical limit of this amplitude shows the following. The integrand of this partial amplitude is a generalization of the two-loop bilinears Δ in Eq. (5.57) to trilinears of the form $\epsilon_{IJK}\omega^I\omega^J\omega^k$. This kind of terms always vanish when one B-cycle is free of punctures in the tropical limit, by antisymmetry of ϵ_{IJK}. At the level of the graphs, this implies, interestingly, that no graph with three or more particles on the same edge can appear from the 3-loop amplitude, which is consistent with supersymmetry. However, this also implies that no “ladder graphs” can be generated by these terms, since at three loops the central cycle of ladder graphs is empty. However, such graphs are definitely present in the three-loop supergravity amplitude[23, 24]. Therefore the missing terms of in the string theory amplitude will have to involve new kind of objects, different from the Δ’s.

6. Discussion

The material presented in this paper fits in the active and recent developments of the domain of string perturbation theory. These are mostly driven by the introduction of new mathematical structures, for instance in the automorphic form program [33–36,73–75] or the analysis of the structure of the supermoduli space [88,115–123] and by certain formal aspects related to genus two and higher string amplitudes [76–79,111,124–126]. These interactions between physics and mathematics have yielded significant advances in both domains and the author hopes that the present work raises some interest in both communities. **Note added.** Since this paper appeared on the arXiv, the author have become aware of the works of Bloch and collaborators [127,128]. In these works, partly inspired by the present paper, the authors describe a mathematical process very similar to the field theory limit, based on degenerating mixed Hodge structures. It would be very interesting to relate precisely the two approaches.

Let us summarize what we achieved in this paper. We formulated the old-fashioned $\alpha' \to 0$ limit of string theory amplitudes in the context of tropical geometry: the string theory integral, once split up according to the domain decomposition (3.6) provides in each domain an integral that has the exact same structure as the expected Feynman integral. By structure, we mean poles inside the integrand, or equivalently, first and second Symanzik polynomials. The proof relied on the use of tropical theta functions with characteristics and on Lemma 1 in particular. We did not prove that the result of the integration matches automatically the result obtained from field theory Feynman rules. This is a separate question, which essentially concerns string field theory. We were interested in a practical process that would make use of precomputed string theory amplitude and extract the Feynman numerators in the field theory limit. We reviewed tree and one-loop processes and performed a two-loop computation. We also commented on the field theory limit of the three-loop partial amplitude of [55]. This work can be considered as a first step toward a map between string theory and field theory numerators to all orders.

Until the recent works of Witten initiated in [88], the procedure to compute superstring amplitudes was believed to rely on the existence of a global holomorphic projection of the supermoduli space $\mathfrak{M}_{g,n}$ onto its bosonic base $\mathcal{M}_{g,n}$ [47,86]. It is now known that such a projection does not exist in general [120,121]: for $g \geq 5$, $\mathfrak{M}_{g,0}$ is not holomorphically projected. At genus two, the superstring measure (the integrand of the $n = 0$ amplitude) was computed in [47] using an explicit projection for the even spin structures of $\mathfrak{M}_{2,0}$. This result was obtained by a different method by Witten in [118]. An ansatz at genus three was proposed in [129], later extended to genus four in [130–132]. However, Witten argued [122] that the projection from the supermoduli space to its bosonic base has a pole in the bulk of the moduli space (on the hyperelliptic locus), while the ansatz of [129] is manifestly holomorphic.

Therefore, the most natural framework for the field theory limit seems to be a putative super-tropical geometry. The development of such a theory could eventually allow to treat in full generality first quantized RNS particles directly on the worldline, and generalize the seminal work [90].

Notwithstanding, there are several formulations of string theory that imply only bosonic integration. For instance the Green Schwarz and the pure spinor formalisms, but also a few other bosonic realizations of the superstring [133], like that of [134], or topological string amplitudes. Moreover, the "vertical integration" procedure recently introduced by Sen [135,136] gave a prescription to gauge fix supergravity on the worldsheet in such a way that the physical S-matrix elements are independent of this gauge choice. This procedure is fully generic and allows in principle to perform the integration over the supermoduli first, using picture changing operators [137] whose position is integrated using this vertical integration procedure.

This work was only focussed on the closed string sector. Witten's open string field theory is based on a particular decomposition of the moduli space of graphs [138,139], called the Kontsevich–Penner cell decomposition [140,141].[22] This decomposition describes the moduli space of open string field theory in terms of proper times [142]. It is different from the one we use here, and it would be interesting to relate the two. On a related note, in series of works [12, 13,38,39,143], field theory limits of open string amplitudes have been carefully studied at one and two loops, using the Schottky parametrization of Riemann surfaces. The authors of [38] also provided an analysis of the field theory limit in superstring theory based on super-Schottky parametrization, still in the open string setting. Inspiration for developing a super-tropical geometry could be sought in these works.

Another direction for developments how the Feynman $i\epsilon$ prescription fits in the field theory limit. This has been analyzed by Witten in [119] where a solution to this question in string theory was proposed and applied to the description of the field theory limit of a five-point open bosonic string amplitude restricted to a specific color ordering (12345). The moduli space of points on a disk is very similar to $\mathcal{M}_{0,5}^{\text{trop}}$, except that color ordering selects only one cone through one of the pentagons, for instance the exterior one in Fig. 10. It was shown that the correct string theory integration cycle should be a complexified version of this cone in order to account for the $i\epsilon$ prescription (see also [144]). Implementing this complexification systematically in the tropical language would lead to a sort of Lorentzian picture of tropical graphs.

Finally, to compute more general tropical limits, it is necessary to push to higher order the Fourier expansion of the prime form. In principle, the procedure explained in this paper gives a prescription for extracting such terms, by choosing the appropriate spin structure—as in Lemma 1—for each couple of points (i,j) in the factors of $\partial G(z_i, z_j)$ entering $\mathcal{W}_{g,n}$ to expand the prime form. The most suited application would be the tropical limit of the Heterotic string four-graviton two-loop amplitude of [47] studied in [94]. Also, the extraction of the leading and subleading divergences of these two-loop amplitudes should

[22] The author would like to thank Edward Witten for pointing out an erroneous use of the denomination "Kontsevich–Penner" in the first version of this draft.

be performed. An important consistency check of such a computation is to verify that overlapping and spurious divergences cancel between the different diagrams. We leave this for future work.

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