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Positive Herz-Schur multipliers and approximation properties of crossed products

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Abstract

For a C^*-algebra A and a set X we give a Stinespring-type characterisation of the completely positive Schur A-multipliers on $\mathcal{K}(\ell^2(X)) \otimes A$. We then relate them to completely positive Herz-Schur multipliers on C^*-algebraic crossed products of the form $A \rtimes_{\alpha,r} G$, with G a discrete group, whose various versions were considered earlier by Anantharaman-Delaroche, Bédos and Conti, and Dong and Ruan. The latter maps are shown to implement approximation properties, such as nuclearity or the Haagerup property, for $A \rtimes_{\alpha,r} G$.

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1. Introduction

Schur multipliers, a class of maps generalising the operators of entrywise (Schur) multiplication on finite matrices, were first abstractly studied by Grothendieck in his famous

'Résumé' [**Gro**]. Since then they have played a very important role in operator theory, the theory of Banach spaces, and operator space theory (for a longer discussion we refer to the introduction of [**MTT**], and to Section 5 of [**Pis**]). In the simplest situation they arise in the following manner: to a (discrete) set X and a function $\phi : X \times X \to \mathbb{C}$, one associates an operator S_ϕ on the space of compact operators on the Hilbert space $\ell^2(X)$; if the resulting map is (completely) bounded, we call S_ϕ a Schur multiplier with symbol ϕ. A related class of maps is that of the Herz-Schur multipliers associated to a discrete (or, more generally, locally compact) group G. On one hand, these can be viewed as the special class of Schur multipliers whose symbol $\phi : G \times G \to \mathbb{C}$ is 'invariant'. On the other hand, they form a natural extension of the Fourier multipliers from classical harmonic analysis; in particular, in this case, the operator S_ϕ can be viewed as acting on the reduced group C^*-algebra $C_r^*(G)$ of G.

Since the pioneering work of Haagerup, it has been known that the existence of Herz-Schur multipliers of a particular type encodes various approximation properties of $C_r^*(G)$, see Chapter 12 of [**BrO**] (it is worth mentioning that the same fact persists for discrete unimodular *quantum groups* — see the recent survey [**Bra**]). An important ingredient of the proof of such results, which features again in this paper, can be summarised as follows: if $(\phi_i)_{i \in I}$ is a net of Herz-Schur multipliers on G with certain properties then the associated operators S_{ϕ_i} implement an approximation property of $C_r^*(G)$; conversely any family of approximating maps on $C_r^*(G)$ can be 'averaged' into Herz-Schur multipliers. An early example of this technique is Lance's proof [**Lan**] that a discrete group is amenable if and only if its reduced group C^*-algebra is nuclear.

Recently both classes of maps discussed above have been generalised to the C^*-algebra-valued case. In the paper [**MTT**], written by three authors of the current article, a class of *Schur A-multipliers* is identified, where A is a C^*-algebra faithfully represented on a Hilbert space H. In this 'operator-valued' case, the starting point is a function ϕ, defined on the direct product $X \times X$, and taking values in the space $CB(A, \mathcal{B}(H))$ of all completely bounded maps from A into the C^*-algebra $\mathcal{B}(H)$ of all bounded linear operators on H. The associated operator S_ϕ acts from $\mathcal{K}(\ell^2(X)) \otimes A$ to $\mathcal{K}(\ell^2(X)) \otimes B(H)$; the function ϕ is called a Schur A-multiplier if the map S_ϕ is completely bounded. Again, in the case where X is in fact a discrete group G, the C^*-algebra A is equipped with an action of G, and the function ϕ satisfies a natural invariance property with respect to the action, we are led to a generalisation of Herz-Schur multipliers, this time acting on the C^*-algebraic crossed product $A \rtimes_{\alpha, r} G$. Such maps have also been studied in a series of papers by Bédos and Conti (see [**BéC**] and references therein), where they are viewed as generalisations of Fourier multipliers. In [**MTT**] several general properties and examples of Schur A-multipliers were established; later, in [**McK**], it was shown that the completely bounded approximation property of reduced crossed products can be characterised *via* the existence of Herz-Schur A-multipliers of a particular type.

The descriptions above imply that the operator-space-theoretic concept of complete boundedness plays an important role in the theory of Schur and Herz-Schur multipliers. In the study of approximation properties of C^*-algebras it is well known that a special role is played by *completely positive* maps. These are the subject of our paper. We first characterise, in Theorem 2·6, those functions $\phi : X \times X \to CB(A, \mathcal{B}(H))$ for which S_ϕ is a completely positive Schur A-multiplier, calling such ϕ *functions of positive type*. Then we prove Theorem 2·8, a transference-type result: in the case where X is a discrete group G acting on A by automorphisms, there is a one-to-one correspondence between

certain 'invariant' functions $\phi : G \times G \to CB(A)$ of positive type, and functions $F : G \to CB(A)$ leading to completely positive Herz-Schur multipliers on the reduced crossed product $A \rtimes_{\alpha,r} G$. The latter class of maps is then compared to those studied earlier by Anantharaman-Delaroche [**A-D**], Bédos and Conti [**BéC**], and Dong and Ruan [**DoR**]. These general results are later applied to show that approximation properties of C^*-algebraic crossed products can be realised using Herz-Schur A-multipliers, generalising the corresponding statements for reduced group C^*-algebras (which can be viewed as 'crossed products with trivial coefficients'), mentioned earlier in this introduction.

The plan of the paper is as follows: after finishing this section by introducing some general notational conventions, in Section 2 we discuss functions of positive type and associated completely positive Schur A-multipliers, establishing a Stinespring-type representation for the latter. We also introduce the related completely positive Herz-Schur A-multipliers, which act on reduced crossed products, and compare the resulting class with those considered earlier by other authors. Section 3 presents a characterisation of the Haagerup property for the reduced crossed product in terms of the existence of a certain class of completely positive Herz-Schur A-multipliers; Section 4 solves the analogous problem for nuclearity.

For C^*-algebras A and B, we denote by $A \otimes B$ their minimal/spatial tensor product; $\mathcal{Z}(A)$ stands for the centre of A, while A^+ designates the cone of positive elements of A. Scalar products will be always linear on the left. For Hilbert spaces H and K, we denote by $H \otimes K$ their Hilbertian tensor product. We let $\mathcal{B}(H, K)$ be the space of all bounded linear operators from H into K and set $\mathcal{B}(H) = \mathcal{B}(H, H)$. For a vector space V, we denote, as usual, by $M_n(V)$ the space of all n by n matrices with entries in V. For a map $\phi : V \to W$ between vector spaces V and W, we let $\phi^{(n)} : M_n(V) \to M_n(W)$ be the map given by $\phi^{(n)}((x_{i,j})_{i,j}) = (\phi(x_{i,j}))_{i,j}$. For operator spaces $\mathcal{X}$ and $\mathcal{Y}$, we denote by $CB(\mathcal{X}, \mathcal{Y})$ the (operator) space of all completely bounded maps from $\mathcal{X}$ into $\mathcal{Y}$. The paper will rely on acquaintance with basic operator space theory; we refer the reader to [**EfR**] and [**Pau**] for necessary background.

2. *Completely positive Herz-Schur multipliers*

We recall some notions and results from [**MTT**]. Let X be a set, H a Hilbert space, and $A \subseteq B(H)$ a (nondegenerate) C^*-algebra. We write $\mathcal{K} = \mathcal{K}(\ell^2(X))$ for the algebra of all compact operators on $\ell^2(X)$. Set $\mathcal{H} = \ell^2(X) \otimes H$; we identify $\mathcal{H}$ with the Hilbert space $\ell^2(X, H)$ of all square summable sequences in H. It was noted in [**MTT**] that, for $k \in \ell^2(X \times X, A)$, the formula

$$(T_k \xi)(x) = \sum_{y \in X} k(x, y)\big(\xi(y)\big), \quad x \in X, \ \xi \in \mathcal{H},$$

defines a bounded operator on $\mathcal{H}$ with $\|T_k\| \leq \|k\|_2$. The functions $k : X \times X \to A$ will often be referred to as *kernels*. We let

$$\mathcal{S}_2(X \times X, A) = \{T_k : k \in \ell^2(X \times X, A)\}$$

and note that $\mathcal{S}_2(X \times X, A)$ is a dense subspace of the minimal tensor product $\mathcal{K} \otimes A$. Given a bounded function $\varphi : X \times X \to CB(A, \mathcal{B}(H))$ and $k \in \ell^2(X \times X, A)$, we write $\varphi \cdot k$ for the function in $\ell^2(X \times X, A)$ given by $(\varphi \cdot k)(x, y) = \varphi(x, y)(k(x, y))$, $x, y \in X$. Note we are using here a different convention to that of [**MTT**]; see also the comment before Definition 2·7. We let $S_\varphi : \mathcal{S}_2(X \times X, A) \to \mathcal{S}_2(X \times X, \mathcal{B}(H))$ be the map given

by $S_\varphi(T_k) = T_{\varphi \cdot k}$; it is clear that S_φ is linear and bounded with respect to the norm $\| \cdot \|_2$. If S_φ is completely bounded, when $\mathcal{S}_2(X \times X, A)$ is equipped with the operator space structure arising from its inclusion into $\mathcal{K} \otimes A$, the function φ is called a *Schur A-multiplier*. We denote the algebra of all Schur A-multipliers on $X \times X$ by $\mathfrak{S}(X, A)$. For $\varphi \in \mathfrak{S}(X, A)$, we write $\|\varphi\|_{\mathrm{m}} = \|S_\varphi\|_{\mathrm{cb}}$.

We note that if in the above paragraph the map $\varphi(x, y) : A \to \mathcal{B}(H)$, for $x, y \in X$, is only assumed to be bounded (as opposed to completely bounded), then the complete boundedness of the map $S_\varphi : \mathcal{S}_2(X \times X, A) \to \mathcal{S}_2(X \times X, A)$ implies that $\varphi(x, y)$ is in fact completely bounded.

Let $\varphi \in \mathfrak{S}(X, A)$. Then the map S_φ has a (unique) completely bounded extension to a map on $\mathcal{K} \otimes A$, which we denote in the same way. The following result, with the additional assumptions that the set X be countable and the C^*-algebra A be separable, was established in [**MTT**] as a special case of Theorem 2.6 therein. An inspection of the proof shows that these assumptions are not needed when X is a discrete set endowed with counting measure.

THEOREM 2·1. *Let X be a set, H a Hilbert space, $A \subseteq \mathcal{B}(H)$ a C^*-algebra, and $\varphi : X \times X \to CB(A, \mathcal{B}(H))$ a bounded function. The following are equivalent:*

(i) *φ is a Schur A-multiplier;*

(ii) *there exist a Hilbert space K, a non-degenerate $*$-representation $\rho : A \to \mathcal{B}(K)$, and bounded functions $V, W : X \to \mathcal{B}(H, K)$ such that*

$$\varphi(x, y)(a) = V(x)^* \rho(a) W(y), \quad x, y \in X, a \in A.$$

Moreover, if (i) holds true then the functions V and W in (ii) can be chosen so that $\|\varphi\|_{\mathrm{m}} = \sup_{x \in X} \|V(x)\| \sup_{y \in X} \|W(y)\|$.

Suppose that $\varphi \in \mathfrak{S}(X, A)$. The map $S_\varphi : \mathcal{K} \otimes A \to \mathcal{K} \otimes \mathcal{B}(H)$ has a canonical extension to a map from $\mathcal{B}(\ell^2(X)) \otimes A$ into the weak* spatial tensor product $\mathcal{B}(\ell^2(X)) \bar\otimes \mathcal{B}(H)$, which we now describe. Let ρ, V and W be as in Theorem 2·1 (ii). Fix $T \in \mathcal{B}(\ell^2(X)) \otimes A$ and write $T = (a_{x,y})_{x,y}$, where $a_{x,y} \in A$, $x, y \in X$. Since ρ is completely bounded, we have that $\rho^{(\infty)}(T) \stackrel{\mathrm{def}}{=} (\rho(a_{x,y}))_{x,y}$ defines a bounded operator on $\ell^2(X) \otimes H$. Letting V (resp. W) be the diagonal operator from $\ell^2(X) \otimes H$ into $\ell^2(X) \otimes K$ with entries $V(x)$ (resp. $W(y)$), we set

$$\Phi(T) = V^* \rho^{(\infty)}(T) W. \tag{2·1}$$

We have that $\Phi(T) = (V(x)^* \rho(a_{x,y}) W(y))_{x,y}$ and thus Φ agrees with the map S_φ on $\mathcal{K} \otimes A$. In the sequel, if needed we will use the same symbol, S_φ, to denote the mapping Φ. Note that the mapping Φ coincides with the restriction to $\mathcal{B}(\ell^2(X)) \otimes A$ of the map $\mathcal{E} \circ S_\varphi^{**}$, where $S_\varphi^{**} : \mathcal{B}(\ell^2(X)) \bar\otimes A^{**} \to \mathcal{B}(\ell^2(X)) \bar\otimes \mathcal{B}(H)^{**}$ is the second dual of S_φ and $\mathcal{E} : \mathcal{B}(\ell^2(X)) \bar\otimes \mathcal{B}(H)^{**} \to \mathcal{B}(\ell^2(X)) \bar\otimes \mathcal{B}(H)$ is the canonical projection, whose existence follows from the fact that $\mathcal{B}(H)$ is a dual Banach space.

Recall that a kernel $k : X \times X \to A$ is called *hermitian* if $k(x, y)^* = k(y, x)$ for all $x, y \in X$, and that k is called *positive definite* if

$$\big(k(x_i, x_j)\big)_{i,j} \in M_n(A)^+, \quad \text{for all } x_1, \ldots, x_n \in X \text{ and all } n \in \mathbb{N}.$$

PROPOSITION 2·2. *Let $k \in \ell^2(X \times X, A)$ be a hermitian kernel. The operator T_k is positive if and only if k is positive definite.*

Proof. Suppose that k is positive definite and $\xi \in \ell^2(X, H)$ is finitely supported, say $\mathrm{supp}\,(\xi) = \{x_1, \ldots, x_n\}$. Then

$$\langle T_k \xi, \xi \rangle = \sum_{x \in X} \left\langle \sum_{y \in X} k(x, y)\xi(y), \xi(x) \right\rangle = \sum_{i,j=1}^{n} \langle k(x_i, x_j)\xi(x_j), \xi(x_i) \rangle \geq 0.$$

Since the H-valued functions with finite support are dense in $\ell^2(X, H)$, we conclude that the operator T_k is positive.

Now suppose that T_k is positive. Let $\xi_1, \ldots, \xi_n \in H$ and $x_1, \ldots, x_n \in X$. Define $\xi \in \ell^2(X, H)$ by letting

$$\xi(x) = \begin{cases} \xi_i & x = x_i; \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\sum_{i,j} \langle k(x_i, x_j)\xi_j, \xi_i \rangle = \langle T_k \xi, \xi \rangle \geq 0,$$

so k is positive definite. $\quad\square$

DEFINITION 2·3. *A bounded function* $\varphi : X \times X \to CB(A, \mathcal{B}(H))$ *will be called of positive type if, for any* $x_1, \ldots, x_m \in X$ *and any* $a_{p,q} \in A$, $p, q = 1, \ldots, m$, *such that* $(a_{p,q})_{p,q} \in M_m(A)^+$, *we have that*

$$\left(\varphi(x_p, x_q)(a_{p,q})\right)_{p,q} \in M_m(\mathcal{B}(H))^+.$$

REMARK 2·4. *If we only assume that each* $\varphi(x, y)$ *is bounded then being of positive type implies the complete boundedness of* $\varphi(x, y)$ *for all* $x, y \in X$. *The latter statement follows by arguments similar to those in the proof of Theorem 2·6 below.*

LEMMA 2·5. *Suppose that a bounded function* $\varphi : X \times X \to CB(A, \mathcal{B}(H))$ *is of positive type.*
(i) *If* $x_1, \ldots, x_m \in X$ *and* $C_{p,q} \in M_n(A)$, $p, q = 1, \ldots, m$, *are such that* $(C_{p,q})_{p,q} \in M_{mn}(A)^+$, *then*

$$\left(\varphi(x_p, x_q)^{(n)}(C_{p,q})\right)_{p,q} \in M_{mn}(\mathcal{B}(H))^+; \tag{2·2}$$

(ii) *If* $k \in \ell^2(X \times X, A)$ *is a positive definite kernel then* $\varphi \cdot k$ *is such, too.*

Proof. (i) Letting $C_{p,q} = (a_{i,j}^{p,q})_{i,j=1}^{n}$, we have that $(a_{i,j}^{p,q})_{p,i,q,j} \in M_{mn}(A)^+$. Set $y_{p,i} = x_p$, $p = 1, \ldots, m$, $i = 1, \ldots, n$. Then

$$\left(\varphi(x_p, x_q)^{(n)}(C_{p,q})\right)_{p,q} = \left(\varphi(y_{p,i}, y_{q,j})(a_{i,j}^{p,q})\right)_{p,i,q,j} \in M_{mn}(A)^+.$$

(ii) is straightforward from the definitions. $\quad\square$

The following theorem provides a characterisation of those Schur A-multipliers φ for which S_φ is completely positive.

THEOREM 2·6. *Let* X *be a set,* H *a Hilbert space,* $A \subseteq \mathcal{B}(H)$ *a non-degenerate* C^*-*algebra, and* $\varphi : X \times X \to CB(A, \mathcal{B}(H))$ *a bounded function. The following are equivalent:*
(i) φ *is of positive type;*
(ii) φ *is a Schur A-multiplier,* S_φ *is completely positive as a map from* $\mathcal{K} \otimes A$ *to* $\mathcal{K} \otimes \mathcal{B}(H)$, *and* $\|S_\varphi\|_{\mathrm{cb}} = \sup_{x \in X} \|\varphi(x, x)\|_{\mathrm{cb}}$;

(iii) there exists a Hilbert space K, a non-degenerate $*$-representation $\rho : A \to \mathcal{B}(K)$, and a bounded function $V : X \to \mathcal{B}(H, K)$, such that

$$\varphi(x, y)(a) = V(x)^* \rho(a) V(y), \quad x, y \in X, \ a \in A.$$

Further, if these conditions hold then the canonical extension of S_φ to $\mathcal{B}(\ell^2(X)) \bar{\otimes} \mathcal{B}(H)$ is also completely positive.

Proof. (i)$\Rightarrow$(ii) Let $\varphi : X \times X \to CB(A, \mathcal{B}(H))$ be of positive type and assume that $k_{i,j} \in \ell^2(X \times X, A)$, $i, j = 1 \ldots, n$, are such that $T = (T_{k_{i,j}})_{i,j}$ is a positive operator. Let $[n] = \{1, \ldots, n\}$, $Y = [n] \times X$ and $k : Y \times Y \to A$ be given by $k(i, x, j, y) = k_{i,j}(x, y)$. After identifying $\ell^2(Y, H)$ with $\mathbb{C}^n \otimes \ell^2(X, H)$, and $\mathcal{B}(\ell^2(Y, H))$ with $M_n(\mathcal{B}(\ell^2(X, H)))$, we have that $T = T_k$. By Proposition 2·2, k is positive definite. Let $\psi : Y \times Y \to CB(A, \mathcal{B}(H))$ be given by $\psi(i, x, j, y) = \varphi(x, y)$ for all $i, j \in [n]$ and all $x, y \in X$. We claim that ψ is of positive type. Indeed, let $m \in \mathbb{N}$, $x_1, \ldots, x_m \in X$, and $(a_{i,p,j,q})_{i,p,j,q} \in M_{nm}(A)^+$. Set $C_{p,q} = (a_{i,p,j,q})_{i,j} \in M_n(A)$. By Lemma 2·5 (i),

$$(\psi(i, x_p, j, x_q)(a_{i,p,j,q}))_{i,p,j,q} = (\varphi^{(n)}(x_p, x_q)(C_{p,q})) \in M_{mn}(\mathcal{B}(H))^+.$$

By Lemma 2·5 (ii), $\psi \cdot k$ is positive definite. In addition, $(T_{\varphi \cdot k_{i,j}})_{i,j} = T_{\psi \cdot k}$, and another application of Proposition 2·2 shows that $(T_{\varphi \cdot k_{i,j}})_{i,j} \in M_n(\mathcal{B}(H))^+$. Thus, S_φ is completely positive on $\mathcal{S}_2(X \times X, A)$.

Set $C = \sup_{x \in X} \|\varphi(x, x)\|_{\mathrm{cb}}$. Let $E \subseteq X$ be a finite set, say $|E| = m$. We view $\ell^2(E) \otimes H$ as a subspace of $\ell^2(X) \otimes H$, and $M_m \otimes A \equiv \mathcal{S}_2(E \times E, A) \subseteq \mathcal{S}_2(X \times X, A)$. The map S_φ leaves $\mathcal{S}_2(E \times E, A)$ invariant and, by the previous paragraph, its restriction $S_{\varphi, E}$ to the latter space is completely positive. Let $(e_i)_{i \in \mathcal{I}}$ be an approximate unit in A. It is easy to check that the family $(\tilde{e}_i)_{i \in \mathcal{I}}$, where $\tilde{e}_i$ is the 'diagonal function' constantly equal to e_i, is an approximate unit in the C^*-algebra $\mathcal{S}_2(E \times E, A)$. Since each of the maps $\varphi(x, x)$ is completely positive, and for a completely positive map on a C*-algebra its completely bounded norm can be expressed as the limit of the norms of the images of an approximate unit (as follows from Stinespring theorem for non-unital completely positive maps, see for example Appendix A of [**NeS**]), we have

$$\|S_{\varphi, E}\|_{\mathrm{cb}} = \lim_{i \in \mathcal{I}} \|S_{\varphi, E}(\tilde{e}_i)\| = \lim_{i \in \mathcal{I}} \max_{x \in E} \|\varphi(x, x)(e_i)\| = \max_{x \in E} \|\varphi(x, x)\|_{\mathrm{cb}} \leq C.$$

Since the spaces $(\mathcal{S}_2(E \times E, A))_{E \subseteq X}$ form an upwards directed net dense in $\mathcal{S}_2(X \times X, A)$, where the indexing set of all finite subsets of X is given the inclusion order, we conclude that S_φ is completely bounded on $\mathcal{S}_2(X \times X, A)$ and $\|S_\varphi\|_{\mathrm{cb}} \leq C$. Since the map $\varphi(x, x)$ is a compression of S_φ, we have that $\|\varphi(x, x)\|_{\mathrm{cb}} \leq \|S_\varphi\|_{\mathrm{cb}}$, $x \in X$, and hence $\|S_\varphi\|_{\mathrm{cb}} = C$. On the other hand, the density of $\mathcal{S}_2(X \times X, A)$ in $\mathcal{K} \otimes A$ implies that the matricial cones of $\mathcal{K} \otimes A$ are closures of the corresponding cones of $\mathcal{S}_2(X \times X, A)$. It follows that $S_\varphi : \mathcal{K} \otimes A \to \mathcal{K} \otimes A$ is completely positive.

(ii)$\Rightarrow$(iii) follows the steps of the proof of [**MTT**, Theorem 2.6], using the Stinespring theorem instead of the Haagerup-Paulsen-Wittstock theorem. We leave the detailed verification to the interested reader.

(iii)$\Rightarrow$(i) Let $x_1, \ldots, x_m \in X$ and $a_{p,q} \in A$ be such that $C = (a_{p,q})_{p,q} \in M_m(A)^+$. Letting $V = \oplus_{p=1}^m V(x_p)$, we have that

$$\begin{aligned}(\varphi(x_p, x_q)(a_{p,q}))_{p,q} &= ((V(x_p)^*(\rho(a_{p,q}))V(x_q))_{p,q} \\ &= V^* \rho^{(m)}(C) V \in M_m(\mathcal{B}(H))^+.\end{aligned}$$

The last statement follows from the representation (2·1), taking into account that in the case S_φ is completely positive we may choose $V = W$. $\square$

We fix a discrete group G, a Hilbert space H, a non-degenerate C^*-algebra $A \subseteq \mathcal{B}(H)$, and a homomorphism $\alpha : G \to \mathrm{Aut}(A)$; we thus have that (A, G, α) is a C^*-dynamical system. We let $\ell^1(G, A)$ be the $*$-algebra of all summable functions $f : G \to A$. We write $\mathcal{H} = \ell^2(G) \otimes H$ and identify $\mathcal{H}$ with $\ell^2(G, H)$. We denote by $\lambda : G \to \mathcal{B}(\mathcal{H})$, $t \to \lambda_t$, the unitary representation of G given by $\lambda_t \xi(s) = \xi(t^{-1}s)$, $s \in G$, $\xi \in \mathcal{H}$, and write λ_t^0 for the corresponding left regular unitary acting on $\ell^2(G)$. Thus, $\lambda_t = \lambda_t^0 \otimes I$. We also let $\pi : A \to \mathcal{B}(\mathcal{H})$ be the $*$-representation given by $\pi(a)\xi(s) = \alpha_{s^{-1}}(a)(\xi(s))$, $s \in G$, $a \in A$. We note the covariance relation

$$\pi(\alpha_t(a)) = \lambda_t \pi(a) \lambda_t^*, \quad a \in A, t \in G. \tag{2·3}$$

The pair (π, λ) gives rise to a $*$-representation $\tilde{\pi} : \ell^1(G, A) \to \mathcal{B}(\mathcal{H})$ such that

$$\tilde{\pi}(f) = \sum_{s \in G} \pi(f(s))\lambda_s, \quad f \in \ell^1(G, A). \tag{2·4}$$

(Note that the series on the right hand side of (2·4) converges in norm for every $f \in \ell^1(G, A)$.) The reduced crossed product $A \rtimes_{\alpha, r} G$ is defined by letting

$$A \rtimes_{\alpha, r} G = \overline{\tilde{\pi}(\ell^1(G, A))},$$

where the closure is taken in the operator norm of $\mathcal{B}(\mathcal{H})$. Note that, after identifying A with $\pi(A)$, we may consider A as a C^*-subalgebra of $A \rtimes_{\alpha, r} G$.

Identifying $\mathcal{H}$ with $\oplus_{s \in G} H$, we associate to every operator $x \in \mathcal{B}(\mathcal{H})$ a corresponding matrix $(x_{s,t})_{s,t \in G}$, where $x_{s,t} \in \mathcal{B}(H)$ (we use the standard identification: for $\xi, \eta \in H$ and $s, t \in G$ we have $\langle x(\delta_t \otimes \xi), \delta_s \otimes \eta \rangle = \langle x_{s,t}\xi, \eta \rangle$). Note that if $x \in A \rtimes_{\alpha, r} G$ then the diagonal of its matrix coincides with $(\alpha_{t^{-1}}(a_x))_{t \in G}$ for some $a_x \in A$. This in particular implies that the transformation $\mathcal{E}$ that maps x to a_x is a conditional expectation from $A \rtimes_{\alpha, r} G$ onto A. We also consider the maps $\mathcal{E}_t : A \rtimes_{\alpha, r} G \to A$ given by $\mathcal{E}_t(x) = \mathcal{E}(x\lambda_t^*)$; note that $\mathcal{E}_e = \mathcal{E}$. Thus, to every operator $x \in A \rtimes_{\alpha, r} G$, one can associate the family $(a_t)_{t \in G} \subseteq A$, where $a_t = \mathcal{E}_t(x)$; we write $x \sim \sum_{t \in G} \pi(a_t)\lambda_t$, although the series is formal and no convergence is assumed. We note that

$$\alpha_t(\mathcal{E}(x)) = \mathcal{E}(\lambda_t x \lambda_t^*), \quad t \in G, x \in A \rtimes_{\alpha, r} G. \tag{2·5}$$

The latter equality is straightforward in the case $x = \sum_{s \in F} \pi(a_s)\lambda_s$ for a finite subset $F \subseteq G$, and follows by continuity for a general $x \in A \rtimes_{\alpha, r} G$. Similarly, we can check that for any $x \in A \rtimes_{\alpha, r} G$ and $p, q \in G$, we have $x_{p,q} \in A$ and

$$x_{p,q} = \alpha_{p^{-1}}(x_{e,qp^{-1}}). \tag{2·6}$$

Finally note that, as is well-known, the construction of the reduced crossed product does not depend (up to a $*$-isomorphism) on the choice of the initial faithful embedding $A \subseteq \mathcal{B}(H)$.

If $F : G \to CB(A)$ is a bounded map and $f \in \ell^1(G, A)$, let $F \cdot f \in \ell^1(G, A)$ be the function given by

$$(F \cdot f)(t) = F(t)(f(t)), \quad t \in G.$$

Recall [**MTT**] that F is called a Herz-Schur (A, G, α)-multiplier (or simply a Herz-Schur

multiplier if the dynamical system is clear from the context) if the map S_F, given by

$$S_F(\tilde{\pi}(f)) = \tilde{\pi}(F \cdot f), \quad f \in \ell^1(G, A),$$

is completely bounded. If F is a Herz-Schur multiplier, then the map S_F has a (unique) extension to a completely bounded map on $A \rtimes_{\alpha,r} G$, which will be denoted in the same way.

For a bounded function $F : G \to CB(A)$, let $\mathcal{N}(F) : G \times G \to CB(A)$ be the function given by

$$\mathcal{N}(F)(s,t)(a) = \alpha_{s^{-1}}(F(st^{-1})(\alpha_s(a))), \quad s,t \in G, a \in A.$$

It was shown in [**MTT**] that $\mathcal{N}$ is an isometric injection from the algebra of all Herz-Schur (A, G, α)-multipliers into the algebra of all Schur A-multipliers. We note that in [**MTT**] a different (but similar) convention was used for defining $\mathcal{N}(F)$; however we found that for the purposes of positivity the definition given above is more natural.

DEFINITION 2·7. *Let (A, G, α) be a C^*-dynamical system. A Herz-Schur (A, G, α)-multiplier F will be called* completely positive *if the map $S_F : A \rtimes_{\alpha,r} G \to A \rtimes_{\alpha,r} G$ is completely positive.*

THEOREM 2·8. *Let $F : G \to CB(A)$ is a bounded function. The following are equivalent:*

(i) F is a completely positive Herz-Schur (A, G, α)-multiplier;
(ii) the function $\mathcal{N}(F)$ is of positive type;
(iii) $\mathcal{N}(F)$ is a Schur A-multiplier and $S_{\mathcal{N}(F)}$ is completely positive.
Moreover, if (i) holds then $\|S_F\|_{\mathrm{cb}} = \|S_{\mathcal{N}(F)}\|_{\mathrm{cb}} = \|F(e)\|_{\mathrm{cb}}$.

Proof. (i)$\Rightarrow$(ii) Let $n \in \mathbb{N}$, $a_i \in A$ and $s_i \in G$, $i = 1, \ldots, n$. Using (2·3), we obtain

$$(\pi(\alpha_{s_i}(a_i^* a_j))\lambda_{s_i s_j^{-1}})_{i,j=1}^n = (\lambda_{s_i} \pi(a_i^* a_j)\lambda_{s_j}^*)_{i,j=1}^n \in M_n(A \rtimes_{\alpha,r} G)^+,$$

and hence

$$\begin{aligned} X &\overset{\text{def}}{=} (\lambda_{s_i} \pi(\alpha_{s_i^{-1}}(F(s_i s_j^{-1})(\alpha_{s_i}(a_i^* a_j))))\lambda_{s_j^{-1}})_{i,j=1}^n \\ &= (\pi(F(s_i s_j^{-1})(\alpha_{s_i}(a_i^* a_j)))\lambda_{s_i s_j^{-1}})_{i,j=1}^n \in M_n(A \rtimes_{\alpha,r} G)^+. \end{aligned}$$

Letting $\Lambda \in M_n(A \rtimes_{\alpha,r} G)$ be the diagonal matrix with diagonal entries (in this order) $\lambda_{s_1^{-1}}, \ldots, \lambda_{s_n^{-1}}$, we have that

$$(\pi(\alpha_{s_i^{-1}}(F(s_i s_j^{-1})(\alpha_{s_i}(a_i^* a_j)))))_{i,j=1}^n = \Lambda X \Lambda^* \in M_n(A \rtimes_{\alpha,r} G)^+.$$

Since every positive matrix in $M_n(A)$ is the sum of matrices of the form $(a_i^* a_j)_{i,j}$, we have that $\mathcal{N}(F)$ is of positive type.

(ii)$\Rightarrow$(iii) follows from Theorem 2·6.

(iii)$\Rightarrow$(i) It suffices to show that $S_{\mathcal{N}(F)}(\pi(a)\lambda_s) = \pi(F(s)(a))\lambda_s$ for all $a \in A$ and $s \in G$. Let $a \in A$ and $s \in G$. Writing $(b_{p,q})_{p,q \in G}$ for the A-valued matrix of $\pi(a)\lambda_s$, we have that

$$b_{p,q} = \begin{cases} \alpha_{p^{-1}}(a), & \text{if } pq^{-1} = s; \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $S_{\mathcal{N}(F)}(\pi(a)\lambda_s) = (c_{p,q})_{p,q\in G}$, where

$$c_{p,q} = \begin{cases} \mathcal{N}(F)(p,q)(\alpha_{p^{-1}}(a)), & \text{if } pq^{-1} = s; \\ 0 & \text{otherwise;} \end{cases}$$

that is, $S_{\mathcal{N}(F)}(\pi(a)\lambda_s) = \pi(F(s)(a))\lambda_s$.

Finally, the equalities involving the norms follow from [**MTT**, Theorem 3.8] and Theorem 2·6. $\square$

COROLLARY 2·9. *Let $\varphi : G \times G \to CB(A)$ be a Schur A-multiplier. The following are equivalent:*

(i) *S_φ is completely positive and leaves $A \rtimes_{\alpha,r} G$ invariant;*

(ii) *$\varphi = \mathcal{N}(F)$ for some completely positive Herz-Schur (A, G, α)-multiplier $F : G \to CB(A)$.*

Proof. (i)$\Rightarrow$(ii) The fact that S_φ leaves $A \rtimes_{\alpha,r} G$ invariant shows that if $s \in G$ then for every $a \in A$ there exists $b \in A$ such that

$$\varphi(p,q)(\alpha_{p^{-1}}(a)) = \alpha_{p^{-1}}(b), \quad \text{whenever } pq^{-1} = s.$$

Thus, the function $(p,q) \to \alpha_p(\varphi(p,q)(\alpha_{p^{-1}}(a)))$ depends only on pq^{-1}. Set $F(s)(a) = \alpha_p(\varphi(p,q)(\alpha_{p^{-1}}(a)))$, where $p, q \in G$ are such that $pq^{-1} = s$. Then $F : G \to CB(A)$ is well-defined and $\mathcal{N}(F) = \varphi$. Since S_φ is completely positive, F is a completely positive Herz-Schur (A, G, α)-multiplier.

(ii)$\Rightarrow$(i) follows from Theorem 2·8. $\square$

REMARKS 2·10. **(i)** It follows from Theorem 2·6 that if $F : G \to CB(A)$ is a bounded function such that the function $\mathcal{N}(F)$ is of positive type then F is automatically a Herz-Schur (A, G, α)-multiplier.

(ii) Suppose that $F : G \to CB(A)$ is a Herz-Schur (A, G, α)-multiplier. The positivity of the map $S_F : A \rtimes_{\alpha,r} G \to A \rtimes_{\alpha,r} G$ does not imply its complete positivity. This becomes evident if one considers the case where $G = \{e\}$ and A is any C^*-algebra that admits positive maps that are not completely positive (*e.g.* $A = M_n$ for $n \geq 2$).

We next compare the notion of completely positive Herz–Schur (A, G, α)-multipliers to other similar notions that can be found in the literature. Let (A, G, α) be a C^*-dynamical system.

- Let us call a function $T : G \times A \to A$, which is linear in the second variable, *positive definite in the sense of Bédos-Conti*, or *BC positive definite*, if for any $n \in \mathbb{N}$, any $s_1, \ldots, s_n \in G$, and any $a_1, \ldots, a_n \in A$, the matrix

$$\left(\alpha_{s_i}\left(T(s_i^{-1}s_j, \alpha_{s_i^{-1}}(a_i^* a_j))\right) \right)_{i,j=1}^n$$

  is a positive element of $M_n(A)$. This definition was given by Bédos and Conti [**BéC**, Definition 4.7] in the more general case of a twisted C^*-dynamical system; here we consider only the trivial twist and have simplified the definition accordingly.
- Let us call a function $h : G \to \mathcal{Z}(A)$ *positive definite in the sense of Dong-Ruan*, or *DR positive definite*, if, for any $n \in \mathbb{N}$ and any $s_1, \ldots, s_n \in G$, the matrix

$$\left(\alpha_{s_j}\left(h(s_i^{-1}s_j)\right) \right)_{i,j=1}^n$$

is a positive element of $M_n(A)$. This definition was given by Dong and Ruan [**DoR**, p. 436]; only centre-valued functions are considered because this is a necessary condition for such a map to be a 'multiplier' of the reduced crossed product in the sense of [**DoR**].

- Let us call a function $\varphi : G \to A$ α-*positive definite*, if for any $n \in \mathbb{N}$ and any $s_1, \ldots, s_n \in G$ the matrix

$$\left(\alpha_{s_i} \big(\varphi(s_i^{-1} s_j) \big) \right)_{i,j=1}^n$$

is a positive element of $M_n(A)$. This definition was given in [**A-D**, Définition 2.1], and used in [**BéC**, p. 3].

We comment on how the above notions compare to Definition 2·7; we will not consider DR positive definiteness since it is similar to α-positive definiteness.

One can easily show that a function $T : G \times A \to A$, which is linear in the second variable, is a completely bounded multiplier of (A, G, α) in the sense of Bédos–Conti [**BéC**] if and only if the function

$$F_T : G \to CB(A); \ \ F_T(t)(a) \overset{\text{def}}{=} T(t, a), \quad t \in G, \ a \in A,$$

is a Herz–Schur (A, G, α)-multiplier. Let $n \in \mathbb{N}$, $s_1, \ldots, s_n \in G$, and $a_1, \ldots, a_n \in A$; then

$$\left(\alpha_{s_i^{-1}} \big(T(s_i s_j^{-1}, \alpha_{s_i}(a_i^* a_j)) \big) \right)_{i,j=1}^n = \left(\alpha_{s_i^{-1}} \big(F_T(s_i s_j^{-1}) \big(\alpha_{s_i}(a_i^* a_j) \big) \big) \right)_{i,j=1}^n$$
$$= \big(\mathcal{N}(F_T)(s_i, s_j)(a_i^* a_j) \big)_{i,j=1}^n.$$

This implies that T is BC positive definite if and only if F_T is a Herz–Schur (A, G, α)-multiplier of positive type, since any positive matrix in $M_n(A)$ is a sum of matrices of the form $(a_i^* a_j)_{i,j=1}^n$.

Now suppose that $\varphi : G \to \mathcal{Z}(A)$ is α-positive definite. Let $F_\varphi : G \to CB(A)$ be given by

$$F_\varphi(t)(a) = \varphi(t)a, \quad t \in G, \ a \in A.$$

Let $n \in \mathbb{N}$, $s_1, \ldots, s_n \in G$, and $(a_{i,j})$ a positive matrix in $M_n(A)$; then

$$\big(\mathcal{N}(F_\varphi)(s_i, s_j)(a_{i,j}) \big)_{i,j=1}^n = \left(\alpha_{s_i^{-1}} \big(F_\varphi(s_i s_j^{-1}) \big(\alpha_{s_i}(a_{i,j}) \big) \big) \right)_{i,j=1}^n$$
$$= \left(\alpha_{s_i} \big(\varphi(s_i s_j^{-1}) \big) a_{i,j} \right)_{i,j=1}^n,$$

which is positive as the Schur product of a positive matrix in $M_n(\mathcal{Z}(A))$ and a positive matrix in $M_n(A)$ (as follows from an elementary calculation; note that the argument would not be valid for general A-valued maps). Conversely, if $\mathcal{N}(F_\varphi)$ is a Herz-Schur (A, G, α)-multiplier of positive type then, since the matrix in $M_n(A)$ with all entries equal to 1_A is positive, we have for any $n \in \mathbb{N}$ and any $s_1, \ldots, s_n \in G$,

$$0 \le \left(\mathcal{N}(F_\varphi)(s_i, s_j)(1_A) \right)_{i,j=1}^n = \left(\alpha_{s_i^{-1}} \big(\varphi(s_i s_j^{-1}) \big) \right)_{i,j=1}^n,$$

which shows that φ is an α-positive definite function.

We finish this section by exhibiting a class of examples of positive Herz-Schur (A, G, α)-multipliers.

PROPOSITION 2·11. *Let F be a finite subset of G and $\Phi \in CB(A)$. The map $h_F : G \to \mathcal{B}(A)$, given by*

$$h_F(s)(a) = \sum_{p \in F \cap sF} \alpha_p \circ \Phi \circ \alpha_p^{-1}(a), \quad a \in A,$$

is a Herz-Schur (A, G, α)-multiplier with $\|S_{h_F}\|_{\mathrm{cb}} \leq |F| \|\Phi\|_{\mathrm{cb}}$. Moreover, h_F is a completely positive Herz-Schur (A, G, α)-multiplier if the map Φ is completely positive.

Proof. We have

$$\mathcal{N}(h_F)(s,t)(a) = \alpha_{s^{-1}}(h_F(st^{-1})(\alpha_s(a))) = \sum_{p \in F \cap st^{-1}F} \alpha_{s^{-1}p}(\Phi(\alpha_{p^{-1}s}(a)))$$

$$= \sum_{p \in (s^{-1}F) \cap (t^{-1}F)} \alpha_p(\Phi(\alpha_p^{-1}(a)))$$

$$= \sum_{p \in G} \chi_{s^{-1}F}(p)\alpha_p(\Phi(\alpha_p^{-1}(a)))\chi_{t^{-1}F}(p)$$

(here we denote by χ_E the characteristic function of a set E). As $\alpha_p \circ \Phi \circ \alpha_p^{-1}$ is a completely bounded map on A for each $p \in G$, there exist a Hilbert space H_p, $*$-representation $\pi_p : A \to \mathcal{B}(H_p)$ and bounded operators $V_p, W_p \in B(H, H_p)$ such that

$$(\alpha_p \circ \Phi \circ \alpha_p^{-1})(a) = V_p^* \pi_p(a) W_p, \quad a \in A,$$

and $\|\alpha_p \circ \Phi \circ \alpha_p^{-1}\|_{\mathrm{cb}} = \|\Phi\|_{\mathrm{cb}} = \|V_p\|^2 = \|W_p\|^2$.

Let $\rho = \oplus_{p \in G} \pi_p : A \to \mathcal{B}(\oplus_{p \in G} H_p)$, and $V(s), W(s) : H \to \oplus_{p \in G} H_p$ be the column operators given by $V(s) = (\chi_{s^{-1}F}(p)V_p)_{p \in G}$ and $W(s) = (\chi_{s^{-1}F}(p)W_p)_{p \in G}$. The latter are bounded operators with norms

$$\|V(s)\| = \left\| \sum_{p \in s^{-1}F} V_p^* V_p \right\|^{1/2} \leq |F|^{1/2} \|\Phi\|_{\mathrm{cb}}^{1/2}$$

and

$$\|W(t)\| = \left\| \sum_{p \in t^{-1}F} W_p^* W_p \right\|^{1/2} \leq |F|^{1/2} \|\Phi\|_{\mathrm{cb}}^{1/2}.$$

Moreover,

$$\mathcal{N}(h_F)(s,t)(a) = V(s)^* \rho(a) W(t), \quad s, t \in G.$$

Hence, by Theorem 2·1, $\mathcal{N}(h_F)$ is a Schur A-multiplier with $\|\mathcal{N}(h_F)\|_{\mathrm{m}} \leq |F| \|\Phi\|_{\mathrm{cb}}$. By [**MTT**, Theorem 3.8], h_F is a Herz-Schur (A, G, α)-multiplier. If Φ is completely positive then we can choose $V_p = W_p$ and hence $V(s) = W(s)$ for every $s \in G$. In this case, by Theorem 2·6 and Theorem 2·8, h_F is a completely positive Herz-Schur (A, G, α)-multiplier. $\square$

3. *The Haagerup property*

In this section we let A be a *unital C^*-algebra*, whose identity will be denoted by 1_A, equipped with a faithful tracial state τ. We denote by $L^2(A, \tau)$ the completion of A with respect to the norm $\|a\|_{2,\tau} := (\tau(a^*a))^{1/2}$. We say that a map $\Phi : A \to A$ is *L^2-bounded* if there exists a constant $C > 0$ such that $\|\Phi(a)\|_{2,\tau} \leq C\|a\|_{2,\tau}$ for every $a \in A$. If this

happens, there exists a (unique) bounded operator $T_\Phi : L^2(A, \tau) \to L^2(A, \tau)$ with the property that $T_\Phi(a) = \Phi(a)$ whenever $a \in A$. An L^2-bounded map Φ will be called L^2-compact if T_Φ is compact. An application of the Cauchy-Schwarz inequality shows that a unital completely positive map Φ satisfying the condition $\tau \circ \Phi \leq \tau$ is L^2-bounded and that T_Φ is a contraction.

The following definition is due to Dong [**Don**].

DEFINITION 3·1. *The pair (A, τ), where A is a unital C^*-algebra and τ is a faithful tracial state on A, is said to possess the C^*-algebra Haagerup property if there exists a net $(\Phi_i)_{i \in I}$ of unital completely positive maps on A such that*
  (i) $\tau \circ \Phi_i \leq \tau$ for all $i \in I$;
  (ii) Φ_i is L^2-compact for all $i \in I$;
 (iii) $\|\Phi_i(x) - x\|_{2,\tau} \to 0$, for all $x \in A$.

Let (A, G, α) be a C^*-dynamical system, where the group G is discrete, A has a faithful tracial state τ, and α is τ-preserving. Following Dong [**Don**], we will consider the reduced crossed product $A \rtimes_{\alpha,r} G$ endowed with the induced trace τ', given by

$$\tau'(x) = \tau(\mathcal{E}(x)), \quad x \in A \rtimes_{\alpha,r} G.$$

LEMMA 3·2. *Let $F : G \to CB(A)$ be a completely positive Herz-Schur (A, G, α)-multiplier such that $\tau \circ F(e) \leq \tau$ and $F(e)(1_A) = 1_A$. Then $F(t)$ is L^2-bounded and $T_{F(t)}$ is a contraction, for each $t \in G$.*

Proof. Set $\varphi = \mathcal{N}(F)$; for $s, t \in G$, $a \in A$ we have $\varphi(s,t)(a) = \alpha_{s^{-1}}(F(st^{-1})(\alpha_s(a)))$. Since φ is of positive type, it is hermitian. Thus, if $s, t \in G$ and $b \in A$, we have $\varphi(t,s)(b^*) = \varphi(s,t)(b)^*$, that is,

$$\alpha_{t^{-1}}(F(ts^{-1})(\alpha_t(b^*))) = \alpha_{s^{-1}}(F(st^{-1})(\alpha_s(b)))^*, \quad s, t \in G, b \in A.$$

Letting $a = \alpha_t(b)$ and using the fact that α is a homomorphism, we obtain

$$F(ts^{-1})(a^*) = \alpha_{ts^{-1}}(F(st^{-1})(\alpha_{st^{-1}}(a)))^*,$$

and so

$$F(r)(a^*) = \alpha_r(F(r^{-1})(\alpha_{r^{-1}}(a)))^*, \quad r \in G, a \in A;$$

the latter identity can also be rewritten as

$$F(r)(a)^* = \alpha_r(F(r^{-1})(\alpha_{r^{-1}}(a)^*)), \quad r \in G, a \in A. \tag{3·1}$$

Fix $a \in A$ and let $a_1 = 1_A$ and $a_2 = a^*$. In (2·2), set $m = 2$, $n = 1$, $x_1 = e$, $x_2 = t$, $a^{p,q} = a_p a_q^*$, $p, q = 1, 2$. Since φ is of positive type, we have

$$\begin{pmatrix} F(e)(1_A) & F(t^{-1})(a) \\ \alpha_{t^{-1}}(F(t)(\alpha_t(a^*))) & \alpha_{t^{-1}}(F(e)(\alpha_t(a^*a))) \end{pmatrix} \in M_2(A)^+.$$

Since $F(e)(1_A) = 1_A$, and α_t is unital and completely positive,

$$\begin{pmatrix} 1_A & \alpha_t(F(t^{-1})(a)) \\ F(t)(\alpha_t(a^*)) & F(e)(\alpha_t(a^*a)) \end{pmatrix} \in M_2(A)^+;$$

writing $b = \alpha_t(a^*)$, we have

$$\begin{pmatrix} 1_A & \alpha_t(F(t^{-1})(\alpha_{t^{-1}}(b^*))) \\ F(t)(b) & F(e)(bb^*) \end{pmatrix} \in M_2(A)^+.$$

In view of (3·1),

$$\begin{pmatrix} 1_A & F(t)(b)^* \\ F(t)(b) & F(e)(bb^*) \end{pmatrix} \in M_2(A)^+$$

and hence

$$F(t)(b)F(t)(b)^* \le F(e)(bb^*).$$

It follows that

$$\|F(t)(b)\|_{2,\tau}^2 = \tau(F(t)(b)^*F(t)(b)) \le \tau(F(e)(bb^*))$$
$$\le \tau(bb^*) = \tau(b^*b) = \|b\|_{2,\tau}^2,$$

for every $b \in A$; thus, $F(t)$ is L^2-bounded and $T_{F(t)}$ is a contraction. $\qquad\square$

In view of Lemma 3·2, condition (i) in the following definition implies that $F_i(s)$ is L^2-bounded, $s \in G$, $i \in I$, and hence it makes sense to formulate conditions (ii) and (iii) in the subsequent definition.

DEFINITION 3·3. *Let (A, G, α) be a C^*-dynamical system, where the group G is discrete, A has a faithful tracial state τ, and α is τ-preserving. We say that (A, G, α, τ) has the* Haagerup property *if there is a net $(F_i)_{i \in I}$ of completely positive Herz-Schur (A, G, α)-multipliers such that*
  (i) *$F_i(e)$ is unital and $\tau \circ F_i(e) \le \tau$, $i \in I$;*
  (ii) *$F_i(t)$ is L^2-compact, $t \in G$;*
  (iii) *the function $s \to \|T_{F_i(s)}\|$ vanishes at infinity, $i \in I$;*
  (iv) *$\|F_i(t)(a) - a\|_{2,\tau} \to 0$ for all $t \in G$ and all $a \in A$.*

We note that, in the case $A = \mathbb{C}$, Definition 3·3 reduces to the definition of the Haagerup property for the group G.

Let Φ be a bounded linear map on $A \rtimes_{\alpha,r} G$, and $h_\Phi : G \to \mathcal{B}(A)$ be the function given by

$$h_\Phi(s)(a) = \mathcal{E}(\Phi(\pi(a)\lambda_s)\lambda_s^*). \tag{3·2}$$

The next proposition can be found in [**BéC**]; for convenience of the reader we include a proof.

PROPOSITION 3·4. *If Φ is a completely positive map on $A \rtimes_{\alpha,r} G$ then h_Φ is a completely positive Herz-Schur (A, G, α)-multiplier.*

Proof. By Stinespring's Theorem, there exist a Hilbert space K, an operator $V : \mathcal{H} \to K$ and a $*$-representation $\rho : A \rtimes_{\alpha,r} G \to \mathcal{B}(K)$ such that

$$\Phi(x) = V^*\rho(x)V, \quad x \in A \rtimes_{\alpha,r} G.$$

Using (2·5), we have

$$\mathcal{N}(h_\Phi)(s,t) = \alpha_{s^{-1}}(h_\Phi(st^{-1})(\alpha_s(a))) = \alpha_{s^{-1}}(\mathcal{E}(\Phi(\pi(\alpha_s(a))\lambda_{st^{-1}})\lambda_{st^{-1}}^*))$$
$$= \alpha_{s^{-1}}(\mathcal{E}(\Phi(\lambda_s\pi(a)\lambda_{s^{-1}}\lambda_{st^{-1}})\lambda_{st^{-1}}^*))$$
$$= \mathcal{E}(\lambda_{s^{-1}}\Phi(\lambda_s\pi(a)\lambda_{t^{-1}})\lambda_t\lambda_{s^{-1}})\lambda_s))$$
$$= \mathcal{E}(\lambda_{s^{-1}}\Phi(\lambda_s\pi(a)\lambda_{t^{-1}})\lambda_t))$$
$$= \mathcal{E}(\lambda_s^*V^*\rho(\lambda_s)\rho(\pi(a))\rho(\lambda_t)^*V\lambda_t)).$$

Letting $V(s) = \rho(\lambda_s)^* V \lambda_s$, $s \in G$, we obtain

$$\mathcal{N}(h_\Phi)(s,t) = \mathcal{E}(V(s)^* \rho(\pi(a)) V(t))), \quad a \in A, s, t \in G.$$

It follows that h_Φ is a function of positive type. $\quad\square$

We will denote by $L^2(\tau)$ (resp. $L^2(\tau')$) the Hilbert space $L^2(A, \tau)$ (resp. $L^2(A \rtimes_{\alpha,r} G, \tau')$). For $t \in G$, let $L^2_t(\tau')$ be the closure, in the norm $\|\cdot\|_{2,\tau'}$, of the subspace

$$\mathcal{L}_t := \{\pi(a)\lambda_t : a \in A\} \tag{3.3}$$

of $L^2(\tau')$.

Lemma 3·5. *The following statements hold true.*
(i)We have an orthogonal decomposition

$$L^2(\tau') = \oplus_{t \in G} L^2_t(\tau'). \tag{3.4}$$

(ii)For each $t \in G$, the map $a \to \pi(a)\lambda_t$ extends to an isometry V_t from $L^2(\tau)$ to $L^2(\tau')$, such that $V_t(L^2(\tau)) = L^2_t(\tau')$.
(iii)Let P_t the orthogonal projection of $L^2(\tau')$ onto $L^2_t(\tau')$. Then the map $V_t^* P_t : L^2(\tau') \to L^2(\tau)$ satisfies

$$V_t^* P_t(z) = \mathcal{E}(z\lambda_t^*), \quad z \in A \rtimes_{\alpha,r} G.$$

Proof. (i) and (ii). If $a, b \in A$ and $s, t \in G$ then

$$\langle \pi(a)\lambda_s, \pi(b)\lambda_t \rangle = \tau'(\lambda_t^* \pi(b^* a)\lambda_s) = \tau'(\pi(b^* a)\lambda_{st^{-1}}) = \tau(\mathcal{E}(\pi(b^* a)\lambda_{st^{-1}})).$$

It follows that $\mathcal{L}_s \perp \mathcal{L}_t$ whenever $s \neq t$, and that the map $a \to \pi(a)\lambda_t$ extends to a unitary from $L^2(\tau)$ onto $L^2_t(\tau')$. The claim follows from the fact that $\cup_{t \in G} \mathcal{L}_t$ is total in $L^2(\tau')$.

(iii) follows after a simple check for $z = \pi(a)\lambda_s$, combined with the continuity of the involved maps. $\quad\square$

In the next lemma, we use Lemma 3·5 to identify $L^2(\tau')$ with $\oplus_{t \in G} L^2(\tau)$.

Lemma 3·6. *Let $F : G \to CB(A)$ be a completely positive Herz-Schur (A, G, α)-multiplier such that $\tau \circ F(e) \leq \tau$ and $F(e)(1_A) = 1_A$. Then S_F is L^2-bounded and $T_{S_F} = \oplus_{t \in G} T_{F(t)}$. In particular, T_{S_F} is a contraction.*

Proof. Clearly, S_F leaves the space $\mathcal{L}_t$, defined in (3·3), invariant. After identifying $L^2_t(\tau')$ with $L^2(\tau)$ by Lemma 3·5, we have that the restriction of S_F to $\mathcal{L}_t$ coincides with $F(t)$. By Lemma 3·2, $\oplus_{t \in G} T_{F(t)}$ is a well-defined contraction. Since T_F coincides with the latter operator on a dense subspace, we conclude that S_F is L^2-bounded and $T_{S_F} = \oplus_{t \in G} T_{F(t)}$. $\quad\square$

Lemma 3·7. *Let $(F_i)_{i \in I}$ be a net of completely positive Herz-Schur (A, G, α)-multipliers such that $F_i(e)$ is unital and $\tau \circ F_i \leq \tau$ for each i. The following are equivalent:*
(i) $\|S_{F_i}(x) - x\|_{2,\tau'} \to 0$, $x \in A \rtimes_{\alpha,r} G$;
(ii) $\|F_i(t)(a) - a\|_{2,\tau} \to 0$ for all $t \in G$ and all $a \in A$.

Proof. For $s \in G$ and $a \in A$ we have, by Lemma 3·5 (ii),

$$\|\pi(F_i(s)(a))\lambda_s - \pi(a)\lambda_s\|_{2,\tau'} = \|F_i(s)(a) - a\|_{2,\tau}, \quad i \in I.$$

The equivalence now follows easily from Lemma 3·6 and the fact that, by Lemma 3·6, $\|T_{S_{F_i}}\| \leq 1$ for all $i \in I$. $\quad\square$

Now we can characterise the Haagerup property for $A \rtimes_{\alpha,r} G$.

THEOREM 3·8. *Let (A, G, α) be a C^*-dynamical system, where the group G is discrete, A has a faithful tracial state τ and α is τ-preserving. The following are equivalent:*
(i) *(A, G, α, τ) has the Haagerup property;*
(ii) *$(A \rtimes_{\alpha,r} G, \tau')$ has the C^*-algebra Haagerup property.*

Proof. (i)$\Rightarrow$(ii) Let $(F_i)_{i \in I}$ be a net of completely positive Herz-Schur (A, G, α)-multipliers satisfying conditions (i)–(iv) of Definition 3·3. Since $F_i(e)(1_A) = 1_A$, the map S_{F_i} is unital. By definition, S_{F_i} is completely positive. By Lemma 3·6, $T_{S_{F_i}}$ is a compact contraction for all $i \in I$. By Lemma 3·7, $(T_{S_{F_i}})_{i \in I}$ converges to the identity operator in the strong operator topology.

(ii)$\Rightarrow$(i) Let $(\Phi_i)_{i \in I}$ be a net associated with the C^*-algebra Haagerup property of $A \rtimes_{\alpha,r} G$ as in Definition 3·1. Write $F_i = h_{\Phi_i}$, $i \in I$. By Proposition 3·4, F_i is a completely positive Herz-Schur (A, G, α)-multiplier. Since Φ_i is unital, $F_i(e)(1_A) = 1_A$. Moreover, if $a \in A^+$ then

$$(\tau \circ F_i(e))(a) = \tau(\mathcal{E}(\Phi_i(\pi(a)))) = \tau'(\Phi_i(\pi(a))) \le \tau'(\pi(a)) = \tau(a);$$

that is, $\tau \circ F_i(e) \le \tau$, $i \in I$.

Fix $t \in G$ and $a \in A$. By the Cauchy-Schwarz inequality (see *e.g.* [**Pau**, Proposition 3.3]),

$$\begin{aligned}
\|F_i(t)(a)\|_{2,\tau}^2 &= \tau\Big(\mathcal{E}(\Phi_i(\pi(a)\lambda_t)\lambda_t^*)^* \big(\mathcal{E}(\Phi_i(\pi(a)\lambda_t)\lambda_t^*))\big)\Big) \\
&\le \tau\Big(\mathcal{E}\big((\Phi_i(\pi(a)\lambda_t)\lambda_t^*)^*(\Phi_i(\pi(a)\lambda_t)\lambda_t^*)\big)\Big) \\
&= \tau'\Big((\Phi_i(\pi(a)\lambda_t)\lambda_t^*)^*(\Phi_i(\pi(a)\lambda_t)\lambda_t^*)\Big) = \|\Phi_i(\pi(a)\lambda_t)\|_{2,\tau'}^2 \\
&= \|T_{\Phi_i}(\pi(a)\lambda_t)\|_{2,\tau'}^2.
\end{aligned}$$

Since T_{Φ_i} is compact,

$$\{\Phi_i(\pi(a)\lambda_t) : \|\pi(a)\|_{2,\tau'} \le 1\}$$

is a relatively compact set. Since $\|\pi(a)\|_{2,\tau'} = \|a\|_{2,\tau}$, $a \in A$, it follows that the subset $\{F_i(t)(a) : \|a\|_{2,\tau} \le 1\}$ of $L^2(\tau')$ is relatively compact, and hence $F_i(t)$ is L^2-compact, $i \in I$.

Letting $P_t : L^2(\tau') \to L_t^2(\tau')$ be the orthogonal projection, we have that

$$T_{F_i(t)} = V_t^* P_t T_{\Phi_i} P_t V_t; \tag{3·5}$$

indeed, for $a \in A$, by Lemma 3·5 (iii), we have

$$\begin{aligned}
T_{F_i(t)}(a) = F_i(t)(a) = h_{\Phi_i}(t)(a) &= \mathcal{E}(\Phi_i(\pi(a)\lambda_t)\lambda_t^*) \\
&= V_t^* P_t \Phi_i(\pi(a)\lambda_t) = V_t^* P_t T_{\Phi_i}(\pi(a)\lambda_t) = V_t^* P_t T_{\Phi_i} P_t V_t(a).
\end{aligned}$$

Since V_t is an isometry, T_{Φ_i} is a compact operator, and $P_s \to_{s \to \infty} 0$ in the strong operator topology, we have that $\|T_{F_i(t)}\| \to_{t \to \infty} 0$. On the other hand, identity (3·5) and the uniform boundedness of the net $(T_{F_i(t)})_{i \in I}$ show that $T_{F_i(t)} \to_i$ id in the strong operator topology, and the proof is complete. $\square$

The following result is well-known (see for example [**Men**]); we next show how one can deduce it quickly from our Theorem 3·8.

COROLLARY 3·9. *Let (A, G, α) be a C^*-dynamical system, where the group G is discrete, A has a faithful tracial state τ and α is τ-preserving. Assume that $(A \rtimes_{\alpha, r} G, \tau')$ has the C^*-algebra Haagerup property. Then G has the Haagerup property and (A, τ) has the C^*-algebra Haagerup property.*

Proof. By Theorem 3·8, there exists a net $(F_i)_{i \in I}$ of positive Herz–Schur (A, G, α)-multipliers satisfying the conditions of Definition 3·3. Set

$$T_i = F_i(e), \quad i \in I,$$
$$\varphi_i(s) = \tau(F_i(s)(1_A)), \quad i \in I, s \in G.$$

The fact that the net $(T_i)_{i \in I}$ yields the C^*-algebra Haagerup property for A is a direct consequence of Definitions 3·1 and 3·3. It thus suffices to check that $(\varphi_i)_{i \in I}$ is a net of normalised positive definite functions on G vanishing at infinity and converging pointwise to 1. The fact that φ_i is normalised is immediate from the unitality of $F_i(e)$, $i \in I$. If $s \in G$ then

$$|\varphi_i(s)| = |\langle F_i(s)(1_A), 1_A \rangle_{2, \tau}| \leq \|1_A\|_{2, \tau} \|T_{F_i(s)}\| \|1_A\|_{2, \tau} = \|T_{F_i(s)}\|,$$

and property (iii) in Definition 3·3 shows that $\varphi_i(s) \to 0$ as $s \to \infty$. Further,

$$|\varphi_i(s) - 1| = |\langle F_i(s)(1_A) - 1_A, 1_A \rangle_{2, \tau}| \leq \|1_A\|_{2, \tau} \|F_i(s)(1_A) - 1_A\|_{2, \tau}$$

and so, by condition (iv) in Definition 3·3, $\varphi_i(s) \to 1$. Finally, the positive definiteness of φ_i can be checked by a standard matrix computation: given $n \in \mathbb{N}$, $s_1, \ldots, s_n \in G$ and $\mu_1, \ldots, \mu_n \in \mathbb{C}$, we have

$$\sum_{k, l=1}^{n} \overline{\mu_k} \varphi(s_k^{-1} s_l) \mu_l = \tau' \left(\sum_{k, l=1}^{n} \overline{\mu_k} S_{F_i}(\lambda_{s_k^{-1} s_l}) \mu_l \lambda_{s_l^{-1} s_k} \right)$$
$$= \tau' \left(\sum_{k, l=1}^{n} \overline{\mu_k} \lambda_{s_k} S_{F_i}(\lambda_{s_k^{-1} s_l}) \mu_l \lambda_{s_l^{-1}} \right),$$

where we used the trace property of τ'. Now note that

$$\sum_{k, l=1}^{n} \overline{\mu_k} \lambda_{s_k} S_{F_i}(\lambda_{s_k^{-1} s_l}) \mu_l \lambda_{s_l^{-1}}$$

coincides with the operator

$$\begin{bmatrix} \overline{\mu_1} \lambda_{s_1} & \cdots & \overline{\mu_n} \lambda_{s_n} \end{bmatrix} S_{F_i}^{(n)} \left([\lambda_{s_k^{-1} s_l}]_{k, l=1}^{n} \right) \begin{bmatrix} \overline{\mu_1} \lambda_{s_1} & \cdots & \overline{\mu_n} \lambda_{s_n} \end{bmatrix}^*,$$

which is positive since $S_{F_i}^{(n)}$ is so. $\square$

REMARKS 3·10. **(i)** Note that the converse of Corollary 3·9, *i.e.* the statement that if (A, τ) has the C^*-algebra Haagerup property and G has the Haagerup property then $(A \rtimes_{\alpha, r} G, \tau')$ has the C^*-algebra Haagerup property, which was claimed to hold in [**Men**], is false: for example, both $\mathrm{SL}(2, \mathbb{Z})$ and $\mathbb{Z}^2$ have the Haagerup property but $\mathbb{Z}^2 \rtimes \mathrm{SL}(2, \mathbb{Z})$ does not (see the references given in [**BrO**, pg. 359, 374]). The error originates in the earlier article [**You**]: the maps Φ_n considered at the beginning of Section 2 of [**You**] need not be completely positive.

(ii) Dong and Ruan [**DoR**] study a Hilbert module version of the Haagerup approximation property for crossed products and show that the relevant approximating maps

— which are required to be module maps — can always be constructed *via* multipliers associated to maps of the form $F(g)(a) = h(g)(a)$ ($g \in G, a \in A$), where $h : G \to \mathcal{Z}(A)$. It is clear that the A-modularity condition on the maps S_F, together with the positivity requirement, force F to be of the type described above, making the potential class of approximants much narrower compared to the one considered here. On the other hand, as the approximation property of Dong and Ruan requires only the compactness of the maps S_F with respect to A, and not as Hilbert space operators, the conditions on the multipliers they study do not involve any trace on A, which we need to impose in order to obtain the genuine C^*-algebra Haagerup property.

4. *Nuclearity*

In this section, we provide a characterisation of the nuclearity of $A \rtimes_{\alpha,r} G$ in terms of Herz-Schur (A, G, α)-multipliers. Recall that a C^*-algebra B is called *nuclear* if, for any other C^*-algebra A, the algebraic tensor product $A \odot B$ admits a unique C^*-norm. It is well-known (and due to Choi, Effros, and Kirchberg, see Theorem 3.8.7 in [**BrO**]) that B is nuclear if and only if there exist a net $(k_i)_{i \in I}$ of natural numbers and completely positive contractions $\varphi_i : B \to M_{k_i}(\mathbb{C})$ and $\psi_i : M_{k_i}(\mathbb{C}) \to B$ (which moreover can be assumed to be unital) such that $\|(\psi_i \circ \varphi_i)(x) - x\| \to 0$ for every $x \in B$. Let $\mathcal{F}(B)$ denote the set of linear maps on B of finite rank; note that [**ChE**] shows that B is nuclear if and only if there exists a net of completely positive contractions in $\mathcal{F}(B)$ approximating the identity map on B pointwise in norm.

DEFINITION 4·1. *A C^*-dynamical system (A, G, α), where G is a discrete group and A is a C^*-algebra, will be called* nuclear *if there exists a net $(F_i)_{i \in I}$ of completely positive, finitely supported, Herz-Schur (A, G, α)-multipliers, such that*
(i) $\|F_i(e)\|_{\mathrm{cb}} \le 1$ for all $i \in I$,
(ii) $F_i(s) \in \mathcal{F}(A)$ for all $s \in G$ and all $i \in I$, and
(iii) $\|F_i(s)(a) - a\| \to_{i \in I} 0$ for all $s \in G$ and all $a \in A$.

For the next lemma recall the definition of the maps h_Φ from (3·2).

LEMMA 4·2. *(i) Let $(F_i)_{i \in I}$ be a net of completely positive Herz-Schur multipliers of (A, G, α) such that $\sup_i \|F_i(e)\| < \infty$. Then $\|F_i(s)(a) - a\| \to_{i \in I} 0$ for all $s \in G$ and $a \in A$ if and only if $\|S_{F_i}(x) - x\| \to_{i \in I} 0$ for all $x \in A \rtimes_{\alpha,r} G$.*

(ii) If $(\Phi_i)_{i \in I}$ is a net of bounded linear maps on $A \rtimes_{\alpha,r} G$ such that $\|\Phi_i(x) - x\| \to_{i \in I} 0$ for every $x \in A \rtimes_{\alpha,r} G$ then $\|h_{\Phi_i}(s)(a) - a\| \to_{i \in I} 0$ for every $a \in A$ and every $s \in G$.

Proof. (i) Assume that $\|F_i(s)(a) - a\| \to_{i \in I} 0$ for all $s \in G$ and $a \in A$. Since

$$\|S_{F_i}\|_{\mathrm{cb}} = \|S_{F_i}(1)\| = \|F_i(e)(1_A)\|$$

and $\sup_i \|F_i(e)\| < \infty$, it suffices to show that $\|S_{F_i}(x) - x\| \to_{i \in I} 0$ whenever $x = \sum_{s \in E} \pi(a_s) \lambda_s$ for a finite set $E \subseteq G$. But, in this case,

$$\|S_{F_i}(x) - x\| = \left\| \sum_{s \in E} \pi(F_i(s)(a) - a) \lambda_s \right\| \le \sum_{s \in E} \|F_i(s)(a) - a\| \to_{i \in I} 0.$$

Conversely, suppose that $\|S_{F_i}(x) - x\| \to_{i \in I} 0$ for all $x \in A \rtimes_{\alpha,r} G$. Then for fixed $s \in G$ and $a \in A$, we have

$$\|F_i(s)(a) - a\| = \|S_{F_i}(\pi(a)\lambda_s) - \pi(a)\lambda_s\| \to_{i \in I} 0.$$

(ii) Assume that $\|\Phi_i(x) - x\| \to_i 0$ for all $x \in A \rtimes_{\alpha,r} G$. Then

$$\|h_{\Phi_i}(s)(a) - a\| = \|\mathcal{E}(\Phi_i(\pi(a)\lambda_s)\lambda_s^*) - \mathcal{E}((\pi(a)\lambda_s)\lambda_s^*)\|$$
$$\leq \|\Phi_i(\pi(a)\lambda_s) - \pi(a)\lambda_s\| \to_{i \in I} 0.$$

$\square$

We are now ready to state and prove the main result of this section.

THEOREM 4·3. *Let G be a discrete group and (A, G, α) be a C^*-dynamical system. The following are equivalent:*

(i) (A, G, α) *is nuclear;*
(ii) $A \rtimes_{\alpha,r} G$ *is nuclear.*

Proof. (ii)$\Rightarrow$(i) Assume that the C^*-algebra $B := A \rtimes_{\alpha,r} G$ is nuclear. Then there exist a net $(k_i)_{i \in I} \subseteq \mathbb{N}$ and unital completely positive maps $\varphi_i : B \to M_{k_i}(\mathbb{C})$ and $\psi_i : M_{k_i}(\mathbb{C}) \to B$ such that $\psi_i \circ \varphi_i \to_{i \in I} \mathrm{id}_B$ in point-norm topology. Let

$$B_0 = \left\{ \sum_{s \in F} \pi(a_s)\lambda_s : F \subseteq G \text{ is finite}, a_s \in A, s \in F \right\}.$$

We first show that there exists a net $\tilde{\psi}_i : M_{k_i}(\mathbb{C}) \to B$ of completely positive contractions such that the range of $\tilde{\psi}_i$ is in B_0 and $\tilde{\psi}_i \circ \varphi_i \to \mathrm{id}_B$ in the point-norm topology. Since ψ_i is completely positive, and the matrix $(E_{p,q})_{p,q=1}^{k_i} \in M_{k_i}(M_{k_i}(\mathbb{C}))$ is positive, where $\{E_{p,q} : p, q = 1, \ldots, k_i\}$ is the canonical matrix unit system of $M_{k_i}(\mathbb{C})$, we have $(\psi_i(E_{p,q}))_{p,q=1}^{k_i} \in M_{k_i}(B)^+$. As $M_{k_i}(B_0)^+$ is dense in $M_{k_i}(B)^+$, given $\varepsilon_i > 0$, $i \in I$, $\varepsilon_i \to_{i \in I} 0$, there exists a matrix $(b_{p,q}^i)_{p,q=1}^{k_i} \in M_{k_i}(B_0)^+$ such that

$$\|(\psi_i(E_{p,q}))_{p,q=1}^{k_i} - (b_{p,q}^i)_{p,q=1}^{k_i}\| < \varepsilon_i/k_i^3.$$

Let $\psi_i' : M_{k_i}(\mathbb{C}) \to B$ be the map given by

$$\psi_i'((\lambda_{p,q})_{p,q=1}^{k_i}) = \sum_{p,q=1}^{k_i} \lambda_{p,q} b_{p,q}^i.$$

By Choi's Theorem [**Pau**, Theorem 3.14], ψ_i' is completely positive; moreover,

$$\|\|\psi_i'\|_{\mathrm{cb}} - 1\| = \|\|\psi_i'(1)\| - \|\psi_i(1)\|\| \leq \|\psi_i(1) - \psi_i'(1)\|$$
$$\leq \sum_{p=1}^{k_i} \|\psi_i(E_{p,p}) - b_{p,p}^i\| \leq \varepsilon_i/k_i^2.$$

Thus, $\|\psi_i'\|_{\mathrm{cb}} \to_{i \in I} 1$.

Let $\tilde{\psi}_i = \psi_i'/\|\psi_i'\|_{\mathrm{cb}}$. Then $\tilde{\psi}_i$ is a completely positive contraction. Moreover, for each $p, q = 1, \ldots, k_i$, we have

$$\|\psi_i(E_{p,q}) - \tilde{\psi}_i(E_{p,q})\| \quad \leq \frac{\|\psi_i(E_{p,q}) - \psi_i'(E_{p,q})\| + |1 - \|\psi_i'\|_{\mathrm{cb}}|\|\psi_i(E_{p,q})\|}{\|\psi_i'\|_{\mathrm{cb}}}$$
$$\leq \left(\frac{\varepsilon_i}{k_i^3} + \frac{\varepsilon_i}{k_i^2} \right) / \|\psi_i'\|_{\mathrm{cb}},$$

and estimating very roughly we obtain for every matrix $x = (\lambda_{p,q})_{p,q=1}^{k_i} \in M_{k_i}(\mathbb{C})$

$$\|(\psi_i - \tilde{\psi}_i)((\lambda_{p,q})_{p,q})\| = \left\| \sum_{p,q=1}^{k_i} \lambda_{p,q}(\psi_i - \tilde{\psi}_i)(E_{p,q}) \right\|$$

$$\leq \sum_{p,q=1}^{k_i} |\lambda_{p,q}| \|(\psi_i - \tilde{\psi}_i)(E_{p,q})\|$$

$$\leq \left(\frac{\varepsilon_i}{k_i^3} + \frac{\varepsilon_i}{k_i^2} \right) \sum_{p,q=1}^{k_i} \frac{|\lambda_{p,q}|}{\|\psi_i'\|_{\mathrm{cb}}}$$

$$\leq \left(\frac{\varepsilon_i}{k_i} + \varepsilon_i \right) \frac{\|(\lambda_{p,q})_{p,q}\|}{\|\psi_i'\|_{\mathrm{cb}}},$$

giving

$$\|\psi_i - \tilde{\psi}_i\| \leq \frac{2\varepsilon_i}{\|\psi_i'\|_{\mathrm{cb}}}.$$

Hence, for each $x \in B$,

$$\|(\tilde{\psi}_i \circ \varphi_i)(x) - x\| \leq \|(\tilde{\psi}_i - \psi_i) \circ \varphi_i(x)\| + \|\psi_i \circ \varphi_i(x) - x\|$$

$$\leq \|\tilde{\psi}_i - \psi_i\| \|x\| + \|(\psi_i \circ \varphi_i)(x) - x\| \to_{i \in I} 0.$$

Let $\Phi_i = \tilde{\psi}_i \circ \varphi_i$ and $h_i = h_{\Phi_i}$, $i \in I$. Note that, as the rank of Φ_i is finite, there exists a finite set $F_i \subseteq G$ such that, for each $s \in G$ and $a \in A$, we have $\Phi_i(\pi(a)\lambda_s) = \sum_{t \in F_i} \pi(a_t)\lambda_t$. By Proposition 3·4, h_i is a completely positive Herz-Schur (A, G, α)-multiplier such that $\|h_i(e)\|_{\mathrm{cb}} \leq 1$. As $h_i(s)(a) = \mathcal{E}(\Phi_i(\pi(a)\lambda_s)\lambda_s^*)$, and the range of Φ_i is finite dimensional, so is the range of $h_i(s)$ for each $s \in G, i \in I$. Further, as for any $s, t \in G$ we have $\mathcal{E}(\pi(a_t)\lambda_t\lambda_s^*) = a_t$ for $s = t$ and zero otherwise, we obtain that h_i is finitely supported (on the set F_i). Moreover,

$$\|h_i(s)(a) - a\| = \|\mathcal{E}((\Phi_i(\pi(a)\lambda_s)\lambda_s^*)) - \mathcal{E}(\pi(a)\lambda_s\lambda_s^*)))\| \to_{i \in I} 0$$

for all $a \in A$ and all $s \in G$.

(i)$\Rightarrow$(ii) Let $(F_i)_{i \in I}$ be a net satisfying the conditions of Definition 4·1. By Theorem 2·8, the map Φ_i on $A \rtimes_{\alpha,r} G$, given by

$$\Phi_i \left(\sum_{s \in E} \pi(a_s)\lambda_s \right) = \sum_{s \in E} \pi(F_i(s)(a_s))\lambda_s,$$

where $E \subseteq G$ is finite, is completely positive and $\|\Phi_i\|_{\mathrm{cb}} = \|F_i(e)\| \leq 1$. As F_i is finitely supported and $F_i(s) \in \mathcal{F}(A)$ for each $s \in G$, the range of Φ_i is finite dimensional. Since $\|\Phi_i(x) - x\| \to_{i \in I} 0$ for $x = \sum_{s \in E} \pi(a_s)\lambda_s$ with E a finite set, the uniform boundedness of the net $(\Phi_i)_{i \in I}$ shows that $\|\Phi_i(x) - x\| \to_{i \in I} 0$ for every $x \in A \rtimes_{\alpha,r} G$. Thus, $A \rtimes_{\alpha,r} G$ is nuclear. $\square$

REMARK 4·4. It follows from the above theorem that if (A, G, α) is a nuclear C*-dynamical system then A is nuclear. This can be seen directly: the net $(F_i(e))_{i \in I}$ is a sequence of completely positive, contractive, finite rank maps on A converging pointwise to the identity.

We now indicate how to express the proof of the result given by Brown and Ozawa [**BrO**,

Theorem 4.3.4] on amenable actions and nuclearity of the reduced crossed product in our language. Assume for simplicity that A is a unital C^*-algebra with identity 1_A.

DEFINITION 4·5. *We say that an action α of a discrete group G on a unital C^*-algebra A is* amenable *if there exists a net $(T_i)_{i \in I}$ of finitely supported functions $T_i : G \to \mathcal{Z}(A)^+$ such that $\sum_{i \in I} T_i(t)^2 = 1_A$ and*

$$\left\| \sum_{s \in G} (T_i(s) - \alpha_t(T_i(t^{-1}s)))^*(T_i(s) - \alpha_t(T_i(t^{-1}s))) \right\| \to_{i \in I} 0,$$

for every $t \in G$.

COROLLARY 4·6. *Assume that G is a discrete group acting amenably on a unital, C^*-algebra A. Then $A \rtimes_{\alpha,r} G$ is nuclear if and only if A is nuclear.*

Proof. The forward implication follows from the existence of conditional expectation from $A \rtimes_{\alpha,r} G$ onto A, and does not require amenability of the action (see Remark 4·4).

Assume that A is nuclear and the action of G on A is amenable. We will show that if $(\Phi_j)_{j \in J}$ is an approximating net of unital completely positive maps for A and $(T_i)_{i \in I}$ is a net as in Definition 4·5 (where T_i is supported on $\mathcal{F}_i$), then the maps $F_{i,j}(s)$ $(i \in I, j \in J, s \in G)$, given by

$$F_{i,j}(s)(a) = \sum_{p \in G} T_i(p)\alpha_p(\Phi_j(\alpha_p^{-1}(a)))\alpha_s(T_i(s^{-1}p)), \quad a \in A,$$

yield Herz-Schur multipliers providing the completely positive finite rank approximations for $A \rtimes_{\alpha,r} G$. Note that the summation above is in fact over the finite set $\mathcal{F}_i \cap s\mathcal{F}_i$.

We first show that $F_{i,j}$ is a completely positive Herz-Schur (A, G, α)-multiplier. For s, $t \in G$ and $a \in A$, we have

$$\mathcal{N}(F_{i,j})(s,t)(a) = \sum_{p \in G} \alpha_{s^{-1}}(T_i(p)\alpha_p(\Phi_j(\alpha_p^{-1}(\alpha_s(a))))\alpha_{st^{-1}}(T_i(ts^{-1}p)))$$

$$= \sum_{p \in G} \alpha_{s^{-1}}(T_i(p))\alpha_{s^{-1}p}(\Phi_j(\alpha_{s^{-1}p}^{-1}(a)))\alpha_{t^{-1}}(T_i(ts^{-1}p)))$$

$$= \sum_{p \in G} \alpha_{s^{-1}}(T_i(sp))\alpha_p(\Phi_j(\alpha_p^{-1}(a)))\alpha_{t^{-1}}(T_i(tp))).$$

As each Φ_j is a unital completely positive map on $A \subseteq B(H)$, there exist Hilbert spaces $H_{p,j}$, $*$-representations $\pi_{p,j} : A \to \mathcal{B}(H_{p,j})$, and bounded operators $V_{p,j} \in \mathcal{B}(H, H_{p,j})$, such that

$$\alpha_p(\Phi_j(\alpha_p^{-1}(a))) = V_{p,j}^* \pi_{p,j}(a) V_{p,j}, \quad p \in G, j \in J, a \in A.$$

Moreover,

$$\|\alpha_p \circ \Phi_j \circ \alpha_p^{-1}\|_{\mathrm{cb}} = \|\Phi_j\|_{\mathrm{cb}} = \|V_{p,j}\|^2 = 1.$$

Let $\rho_j := \oplus_{p \in G} \pi_{p,j}$ and let $V_{i,j}(s) : H \to \oplus_{p \in G} H_{p,j}$ be the column operator given by $(V_{p,j}\alpha_{s^{-1}}(T_i(sp)))_{p \in G}$. We then get

$$\mathcal{N}(F_{i,j})(s,t)(a) = V_{i,j}(s)^* \rho_j(a) V_{i,j}(s)$$

with

$$\|V_{i,j}(s)\|^2 = \left\| \sum_{p \in G} \alpha_{s^{-1}}(T_i(sp))^* V_{p,j}^* V_{p,j} \alpha_{s^{-1}}(T_i(sp)) \right\|$$

$$\leq \left\| \sum_{p \in G} \alpha_{s^{-1}}(T_i(sp))^2 \right\| = \left\| \sum_{p \in G} T_i(p)^2 \right\| = 1.$$

By Theorems 2·6 and 2·8, $F_{i,j}$ is a completely postive Herz-Schur (A, G, α)-multiplier. Furthermore,

$$F_{i,j}(e)(a) = \sum_{p \in \mathcal{F}_i} T_i(p) \alpha_p(\Phi_j(\alpha_p^{-1}(a))) T_i(p)$$

and, since each Φ_j is a unital completely positive map,

$$\|F_{i,j}(e)\|_{\mathrm{cb}} = \|F_{i,j}(e)(1_A)\| \leq \left\| \sum_{p \in G} T_i(p)^2 \right\| = 1.$$

Since $\Phi_j \in \mathcal{F}(A)$, we have that $F_{i,j}(s) \in \mathcal{F}(A)$ for all $s \in G$, $i \in I$ and $j \in J$. Finally,

$$\|F_{i,j}(s)(a) - a\| = \left\| \sum_{p \in \mathcal{F}_i \cap s\mathcal{F}_i} T_i(p) \alpha_p(\Phi_j(\alpha_p^{-1}(a))) \alpha_s(T_i(s^{-1}p)) - a \right\|$$

$$\leq \left\| \sum_{p \in \mathcal{F}_i \cap s\mathcal{F}_i} T_i(p) \alpha_s(T_i(s^{-1}p)) \right\| \left\| \alpha_p(\Phi_j(\alpha_p^{-1}(a)) - \alpha_p^{-1}(a)) \right\|$$

$$+ \left\| \sum_{p \in \mathcal{F}_i \cap s\mathcal{F}_i} T_i(p) \alpha_s(T_i(s^{-1}p)) - 1_A \right\| \|a\|.$$

The latter converges to zero for any $s \in G$ by [**BrO**, Lemma 4.3.2]. $\square$

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