Computability on the countable ordinals and the Hausdorff-Kuratowski theorem (Extended Abstract*)

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Abstract. While there is a well-established notion of what a computable ordinal is, the question which functions on the countable ordinals ought to be computable has received less attention so far. In order to remedy this, we explore various potential representations of the set of countable ordinals. An equivalence class of representations is then suggested as a standard, as it offers the desired closure properties. With a decent notion of computability on the space of countable ordinals in place, we can then state and prove a computable uniform version of the Hausdorff-Kuratowski theorem.

1 Introduction

In TURING's seminal paper [35], he suggested to call a real number *computable* iff its decimal expansion is. However, in the corrections [36], he pointed out that it is better to use the definition that a real number is computable, iff there is a computable sequence of rational intervals collapsing to it (an idea by BROUWER). Both definitions yield the same class of real numbers – but the natural notions of what a computable function on the real numbers that come along with them differ. For example, $x \mapsto 3x$ is only computable regarding the latter, but not the former notion.

We shall show that there is a similar phenomenon regarding the notion of a computable ordinal: While there is a very well-established notion of what a computable ordinal is, various equivalent definitions do yield different notions of what a computable function on the countable ordinal is. Like multiplication with 3 for the real numbers, some simple functions such as the maximum of two ordinals fail to be computable w.r.t. several common representations of the ordinals; whereas others do yield nice effective closure properties. We will investigate some candidates, and suggest one equivalence class of representations as the standard to be adopted.

As an application, we continue a research programme to investigate concepts from descriptive set theory in the very general setting of represented spaces, and in a fashion that produces both classical and effective results simultaneously. A

* A full version is available as [28].

survey of this approach is given in [27]. One of the first theorems studied in this way is the Jayne-Rogers theorem [16] (simplified proof in [22]); a computable version holding also in some non-Hausdorff spaces was proven by the author and de Brecht in [31] using results about Weihrauch reducibility in [1]. Our goal here is to state and prove a corresponding version of the Hausdorff-Kuratowski theorem.

1.1 Represented spaces

We shall briefly introduce the notion of a *represented space*, which underlies computable analysis [38]. For a more detailed presentation we refer to [26]. A represented space is a pair $\mathbf{X} = (X, \delta_X)$ of a set X and a partial surjection $\delta_X :\subseteq \mathbb{N}^\mathbb{N} \to X$ (the representation). A represented space is called *complete*, iff its representation is a total function.

A multi-valued function between represented spaces is a multi-valued function between the underlying sets. For $f :\subseteq \mathbf{X} \rightrightarrows \mathbf{Y}$ and $F :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$, we call F a realizer of f (notation $F \vdash f$), iff $\delta_Y(F(p)) \in f(\delta_X(p))$ for all $p \in \operatorname{dom}(f\delta_X)$.

$$\begin{array}{ccc} \mathbb{N}^\mathbb{N} & \xrightarrow{F} & \mathbb{N}^\mathbb{N} \\ \downarrow{\scriptstyle \delta_\mathbf{X}} & & \downarrow{\scriptstyle \delta_\mathbf{Y}} \\ \mathbf{X} & \xrightarrow{f} & \mathbf{Y} \end{array}$$

A map between represented spaces is called computable (continuous), iff it has a computable (continuous) realizer. Similarly, we call a point $x \in \mathbf{X}$ computable, iff there is some computable $p \in \mathbb{N}^\mathbb{N}$ with $\delta_\mathbf{X}(p) = x$. We write $\mathbf{X} \cong \mathbf{Y}$ to denote that $\mathbf{X}$ and $\mathbf{Y}$ are computably isomorphic.

Given two represented spaces $\mathbf{X}$, $\mathbf{Y}$ we obtain a third represented space $\mathcal{C}(\mathbf{X}, \mathbf{Y})$ of functions from X to Y by letting 0^n1p be a $[\delta_X \to \delta_Y]$-name for f, if the n-th Turing machine equipped with the oracle p computes a realizer for f. As a consequence of the UTM theorem, $\mathcal{C}(-,-)$ is the exponential in the category of continuous maps between represented spaces, and the evaluation map is even computable (as are the other canonic maps, e.g. currying).

Based on the function space construction, we can obtain the hyperspaces of open $\mathcal{O}$, closed $\mathcal{A}$, overt $\mathcal{V}$ and compact $\mathcal{K}$ subsets of a given represented space using the ideas of synthetic topology [8].

Let $\Delta :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ be defined on the sequences containing only finitely many 0s, and let it map those to their tail starting immediately after the last 0, with each entry reduced by 1. This is a surjection. Given a represented space $\mathbf{X} = (X, \delta_\mathbf{X})$, we define the represented space $\mathbf{X}^\nabla := (X, \delta_\mathbf{X} \circ \Delta)$. Informally, in this space, finitely many mindchanges are allowed. The operation $^\nabla$ even extends to an endofunctor on the category of represented spaces [30, 40].

1.2 Weihrauch reducibility

Several of our results are negative, i.e. show that certain operations are not computable. We prefer to be more precise, and not to merely state failure of computability. Instead, we give lower bounds for Weihrauch reducibility. The reader not interested in distinguishing degrees of non-computability may skip the remainder of the subsection, and in the rest of the paper, read any statement involving Weihrauch reducibility ($\leq_{\mathrm{W}}$, $\equiv_{\mathrm{W}}$, $<_{\mathrm{W}}$) as merely indicating the non-computability of the maps involved.

Definition 1 (Weihrauch reducibility). *Let f, g be multi-valued functions on represented spaces. Then f is said to be* Weihrauch reducible *to g, in symbols $f \leq_W g$, if there are computable functions $K, H :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that $K\langle \mathrm{id}, GH\rangle \vdash f$ for all $G \vdash g$.*

The relation $\leq_{\mathrm{W}}$ is reflexive and transitive. We use $\equiv_{\mathrm{W}}$ to denote equivalence regarding $\leq_{\mathrm{W}}$, and by $<_{\mathrm{W}}$ we denote strict reducibility. By $\mathfrak{W}$ we refer to the partially ordered set of equivalence classes. As shown in [3,25], $\mathfrak{W}$ is a distributive lattice. The algebraic structure on $\mathfrak{W}$ has been investigated in further detail in [5, 15].

A prototypic non-computable function is $\mathrm{LPO} : \mathbb{N}^{\mathbb{N}} \to \{0, 1\}$ defined via $\mathrm{LPO}(0^{\mathbb{N}}) = 1$ and $\mathrm{LPO}(p) = 0$ for $p \neq 0^{\mathbb{N}}$. The degree of this function was already studied by Weihrauch [37].

A few years ago several authors (Gherardi and Marcone [9], P. [24, 25], Brattka and Gherardi [2]) noticed that Weihrauch reducibility would provide a very interesting setting for a metamathematical inquiry into the computational content of mathematical theorems. The fundamental research programme was outlined in [2], and the introduction in [4] may serve as a recent survey.

2 Representations of the space of countable ordinals

We shall investigate several representations of the set of all countable ordinals (to be denoted by COrd), and identity their equivalence classes up to computable translations. Along the way, we shall see how the representations of the countable ordinals restrict to the finite ordinals, and compare to established representations of the natural numbers. Theorem 3 will establish a number of candidates as equivalent, and we shall tentatively propose to consider these the standard representations of COrd. An investigation of which operations on the countable ordinals are computable is postponed until Section 3.

Our first candidate is a straightforward adaption of Kleene's notation [18] of the recursive ordinals to a representation of the countable ordinals. Here and below we use a countable standard pairing function $\langle\ ,\ \rangle : \left(\mathbb{N}^{\mathbb{N}}\right)^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$.

Definition 2. *We define $\delta_K :\subseteq \mathbb{N}^{\mathbb{N}} \to$ COrd inductively via:*

1. $\delta_K(0p) = 0$
2. $\delta_K(1p) = \delta_K(p) + 1$
3. $\delta_K(2\langle p_0, p_1, p_2, \ldots\rangle) = \sup_{i\in\mathbb{N}} \delta_K(p_i)$*, provided that* $\forall i \in \mathbb{N}\ \delta_K(p_i) < \delta_K(p_{i+1})$.

A potential modification of the preceding definition that immediately comes to mind would be to drop the restriction of sup's to increasing sequences. We thus arrive at:

Definition 3. *We define* $\delta_{nK} :\subseteq \mathbb{N}^\mathbb{N} \to COrd$ *inductively via:*

1. $\delta_{nK}(0p) = 0$
2. $\delta_{nK}(1p) = \delta_{nK}(p) + 1$
3. $\delta_{nK}(2\langle p_0, p_1, p_2, \ldots\rangle) = \sup_{i\in\mathbb{N}} \delta_{nK}(p_i)$.

A third definition proceeding along similar lines can be extracted from Moschovakis' definition of the Borel codes in [21]:

Definition 4. *We define* $\delta_M :\subseteq \mathbb{N}^\mathbb{N} \to COrd$ *inductively via:*

1. $\delta_M(0p) = 0$
2. $\delta_M(1\langle p_0, p_1, p_2, \ldots\rangle) = \sup_{i\in\mathbb{N}} (\delta_M(p_i) + 1)$.

Another scheme to obtain representations of the countable ordinals starts with the view of countable ordinals as the heights of countable wellfounded relations. A countable relation is given by two sets $A \subseteq \mathbb{N}$ and $R \subseteq \mathbb{N}\times\mathbb{N}$, where A denotes which points are present, and then R provides the order relation. There are three common spaces of subsets of $\mathbb{N}$, the open subsets $\mathcal{O}(\mathbb{N})$, the closed subsets $\mathcal{A}(\mathbb{N})$ or the clopens $\mathcal{O}(\mathbb{N})\wedge\mathcal{A}(\mathbb{N})$. The computable points in these spaces are the recursively enumerable, the co-recursively enumerable and the decidable subsets of $\mathbb{N}$ respectively. Thus, we arrive at a number of representations:

Definition 5. *Let* $X, Y \in \{\mathcal{O}(\mathbb{N}), \mathcal{A}(\mathbb{N}), \mathcal{O}(\mathbb{N}) \wedge \mathcal{A}(\mathbb{N})\}$*. We define a representation* $\delta_R^{X,Y} :\subseteq \mathbb{N}^\mathbb{N} \to COrd$ *by* $\delta_R^{X,Y}(\langle p, q\rangle) = \alpha$*, iff* α *is the height of the poset* $(A, \prec)$*, where* p *is an* X*-name for* A*,* q *an* Y*-name for* R*, and* $\forall i, j \in A \ (i \prec j \Leftrightarrow \langle i, j\rangle \in R)$*.*

Potentially, it would appear to be more appropriate to consider countable ordinals as order types of countable wellorders, rather than just heights of well-founded orders. This is the approach taken by Hamkins and Li [20].

Definition 6. *Let* $X, Y \in \{\mathcal{O}(\mathbb{N}), \mathcal{A}(\mathbb{N}), \mathcal{O}(\mathbb{N})\wedge\mathcal{A}(\mathbb{N})\}$*. Let* $\delta_{wR}^{X,Y} :\subseteq \mathbb{N}^\mathbb{N} \to COrd$ *be the restriction of* $\delta_R^{X,Y}$ *to those* $\langle p, q\rangle$ *where* q *encodes a wellorder.*

Finally, we introduce a representation tailor-made for the formulation and proof of a computable Hausdorff-Kuratowski theorem below. Let a *nice relation* be a well-founded quasi-order $\preceq$ on $\mathbb{N}$, such that $\forall n,\ n \preceq 0$, and whenever $n \prec m$, then $n > m$.

Definition 7. *We define a representation* $\delta_{nR} :\subseteq \{0, 1\}^\mathbb{N} \to COrd$ *by* $\delta_{nR}(p) = \alpha$*, iff the relation* $\preceq_p$ *defined via* $n \preceq_p m$ *iff* $p(\langle n, m\rangle) = 1$ *is a nice relation of height* $\alpha+1$ *(the height of any nice relation is a countable successor ordinal, and every countable successor ordinal arises as the height of some nice relation).*

To obtain some initial understanding of how the various representations work, we shall consider what happens to the finite ordinals. Besides the usual natural numbers $\mathbb{N}$, also the spaces $\mathbb{N}_<$, $\mathbb{N}_>$ and $\mathbb{N}^\nabla$, where a number n is represented by a non-decreasing, respectively non-increasing, respectively arbitrary sequence of integers which eventually converge to n.

Observation 1. $\big(\mathrm{id} : \mathbb{N}^\nabla \to \mathbb{N}\big) \equiv_W (\mathrm{id} : \mathbb{N}_< \to \mathbb{N}) \equiv_W C_\mathbb{N}$*;* $(\mathrm{id} : \mathbb{N}_> \to \mathbb{N}) \equiv_W$ *LPO* and* $LPO \leq_W (\mathrm{id} : \mathbb{N}_> \to \mathbb{N}_<)$*.*

Proposition 1. *1.* $(COrd, \delta_K)\,|_\mathbb{N} \cong \mathbb{N}$
2. $(COrd, \delta_{nK})\,|_\mathbb{N} \cong \mathbb{N}_<$
3. $(COrd, \delta_M)\,|_{\{n\in\mathbb{N} | n>0\}} \cong (\mathbb{N}_<)\,|_{\{n\in\mathbb{N} | n>0\}}$
4. $\left(COrd, \delta_{wR}^{\mathcal{A}(\mathbb{N}),Y}\right)|_\mathbb{N} \cong \left(COrd, \delta_{R}^{\mathcal{A}(\mathbb{N}),Y}\right)|_\mathbb{N} \cong \mathbb{N}^\nabla$ *(regardless of the choice of* $Y \in \{\mathcal{O}(\mathbb{N}), \mathcal{A}(\mathbb{N}), \mathcal{O}(\mathbb{N}) \wedge \mathcal{A}(\mathbb{N})\}$*)*
5. $\left(COrd, \delta_{wR}^{\mathcal{O}(\mathbb{N}),\mathcal{O}(\mathbb{N})}\right)|_\mathbb{N} \cong \mathbb{N}_<$

We can extend Proposition 1 (3) to:

Lemma 1. $(COrd, \delta_M)\,|_{\{\alpha>0\}} \cong (COrd, \delta_{nK})\,|_{\{\alpha>0\}}$

Merely requiring the domain of the structure to be enumerable, rather than decidable, does not impact the representation at all though. For not necessarily wellordered relations, the same applies to the relation itself.

Lemma 2. $\left(COrd, \delta_R^{\mathcal{O}(\mathbb{N}),Y}\right) \cong \left(COrd, \delta_R^{\mathcal{O}(\mathbb{N})\wedge\mathcal{A}(\mathbb{N}),Y}\right)$ *and* $\left(COrd, \delta_{wR}^{\mathcal{O}(\mathbb{N}),Y}\right) \cong \left(COrd, \delta_{wR}^{\mathcal{O}(\mathbb{N})\wedge\mathcal{A}(\mathbb{N}),Y}\right)$

Lemma 3. $\left(COrd, \delta_R^{X,\mathcal{O}(\mathbb{N})}\right) \cong \left(COrd, \delta_R^{X,\mathcal{O}(\mathbb{N})\wedge\mathcal{A}(\mathbb{N})}\right)$

Proof. Essentially, whenever new information about the relationship between two already settled points occurs, one can create a fresh copy of everything encountered so far. As smaller relations (w.r.t subset inclusion) have smaller height, the extra copy does not impact the ordinal represented thus. □

Theorem 2. *Let* δ *be a representation of COrd such that*

1. $0 \in (COrd, \delta)$
2. $+1 : (COrd, \delta) \to (COrd, \delta)$
3. $\sup : \mathcal{C}(\mathbb{N}, (COrd, \delta)) \to (COrd, \delta)$

are all computable. Then $\mathrm{id} : (COrd, \delta_{nK}) \to (COrd, \delta)$ *is computable.*

Proof. Induction along the definition of δ_{nK}. □

Theorem 3. *The following representations are equivalent:*

$$\delta_{nK} \qquad \delta_{nR} \qquad \delta_R^{\mathcal{O}(\mathbb{N})\wedge\mathcal{A}(\mathbb{N}),\mathcal{O}(\mathbb{N})\wedge\mathcal{A}(\mathbb{N})}$$

$$\delta_R^{\mathcal{O}(\mathbb{N}),\mathcal{O}(\mathbb{N})\wedge\mathcal{A}(\mathbb{N})} \qquad \delta_R^{\mathcal{O}(\mathbb{N})\wedge\mathcal{A}(\mathbb{N}),\mathcal{O}(\mathbb{N})} \qquad \delta_R^{\mathcal{O}(\mathbb{N}),\mathcal{O}(\mathbb{N})}$$

Theorem 4. *1.* $(\mathrm{id} : (COrd, \delta_K) \to (COrd, \delta_{nK}))$ *is computable,*
but $C_\mathbb{N} \leq_W (\mathrm{id} : (COrd, \delta_{nK}) \to (COrd, \delta_K))$
2. $(\mathrm{id} : (COrd, \delta_M) \to (COrd, \delta_{nK}))$ *is computable,*
but $LPO \equiv_W (\mathrm{id} : (COrd, \delta_{nK}) \to (COrd, \delta_M))$

3. $\left(\mathrm{id} : (COrd, \delta_{nK}) \to (COrd, \delta_R^{X,\mathcal{A}(\mathbb{N})})\right)$ *is computable, but* $LPO^* \leq_W \left(\mathrm{id} : (COrd, \delta_R^{X,\mathcal{A}(\mathbb{N})}) \to (COrd, \delta_K)\right)$
4. $\left(\mathrm{id} : (COrd, \delta_{nK}) \to (COrd, \delta_R^{\mathcal{A}(\mathbb{N}),Y})\right)$ *is computable, but* $LPO^* \leq_W \left(\mathrm{id} : (COrd, \delta_R^{\mathcal{A}(\mathbb{N}),Y}) \to (COrd, \delta_K)\right)$
5. $\left(\mathrm{id} : (COrd, \delta_{wR}^{\mathcal{O},\mathcal{O}}) \to (COrd, \delta_{nK})\right)$ *is computable, but* $\left(\mathrm{id} : (COrd, \delta_{nK}) \to (COrd, \delta_{wR}^{\mathcal{A}\wedge\mathcal{O},\mathcal{A}\wedge\mathcal{O}})\right)$ *is not computable.*
6. $(\mathrm{id} : (COrd, \delta_K) \to (COrd, \delta_M))$ *is computable, but* $C_{\mathbb{N}} \leq_W (\mathrm{id} : (COrd, \delta_M) \to (COrd, \delta_K))$

Definition 8. *We will consider the equivalence class of* δ_{nK} *identified in Theorem 3 as the standard representation of COrd, and thus abbreviate* $\mathbf{COrd} := (COrd, \delta_{nK})$.

Besides **COrd**, we will also consider $\mathbf{COrd}_{\mathrm{M}} := (\mathrm{COrd}, \delta_{\mathrm{M}})$, $\mathbf{COrd}_{\mathrm{K}} := (\mathrm{COrd}, \delta_{\mathrm{K}})$ and $\mathbf{COrd}_{\mathrm{HL}} := (\mathrm{COrd}, \delta_{\mathrm{wR}}^{\mathcal{A}\wedge\mathcal{O},\mathcal{A}\wedge\mathcal{O}})$. Their mutual relations are demonstrated in Figure 1. The representations using well-founded structures given as closed sets would seem to be too weak to be of much interest, and thus will no longer be considered.

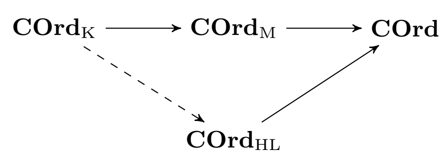

Fig. 1. Translatability between the representations. The dashed arrow refers to the Open Question 2

3 Computability on COrd

In order to justify the stance that the represented space **COrd** really is *the* space of countable ordinals, we shall investigate the computable operations on it and related properties.

Theorem 5. *The following operations are computable:*

1. $+ : \mathbf{COrd} \times \mathbf{COrd} \to \mathbf{COrd}$
2. $\times : \mathbf{COrd} \times \mathbf{COrd} \to \mathbf{COrd}$
3. $\sup : \mathbf{COrd}^{\mathbb{N}} \to \mathbf{COrd}$

4. $(-1) : \mathbf{COrd} \to \mathbf{COrd}$, *where* $(-1)(\alpha+1) = \alpha$ *and for limit ordinals* γ, $(-1)(\gamma) = \gamma$
5. $\mathrm{Smaller} : \mathbf{COrd} \rightrightarrows \mathbf{COrd}^{\mathbb{N}}$ *where* $(\alpha_i)_{i\in\mathbb{N}} \in \mathrm{Smaller}(\alpha)$ *iff* $\{0\} \cup \{\beta \in COrd \mid \beta < \alpha\} = \{\alpha_i \mid i \in \mathbb{N}\}$
6. $(\alpha, \beta) \mapsto \alpha^\beta : \mathbf{COrd} \times \mathbf{COrd} \to \mathbf{COrd}$

Proposition 2. $LPO^* \leq_W (- : \mathbf{COrd} \times \mathbf{COrd} \to \mathbf{COrd})$

4 Computability on $\mathbf{COrd}_\mathbf{K}$

In order to define the concept of a *computable ordinal*, Kleene's definition resulting in the space $\mathbf{COrd}_\mathrm{K}$ seems to be the typical choice. A strong reason to reject $\mathbf{COrd}_\mathrm{K}$ as the natural candidate for computability on the countable ordinals nonetheless, lies in the following result:

Proposition 3. $LPO \leq_W (\max : \mathbf{COrd}_K \times \mathbf{COrd}_K \to \mathbf{COrd}_K)$

The reason that calling the computable elements in $\mathbf{COrd}_\mathrm{K}$ the computable ordinals is justified regardless of $\mathbf{COrd}_\mathrm{K}$ not being the right space lies in the fact that both $\mathbf{COrd}_\mathrm{K}$ and $\mathbf{COrd}$ have the same computable points. This situation is somewhat reminiscent of Turing's transient mistake of defining the computable real numbers via the decimal expansion at first [35] before correcting himself [36].

Proposition 4. *The map* $\mathrm{UpperBound} : \mathbf{COrd} \rightrightarrows \mathbf{COrd}_K$ *defined by* $\beta \in \mathrm{UpperBound}(\alpha)$ *iff* $\beta \geq \alpha$ *is computable.*

Proof. The computation proceeds by induction, using the representations δ_{nK} and δ_K. For 0 and successor, both representations agree anyway. Given a supremum $\alpha = \sup_{n\in\mathbb{N}} \alpha_n$, we apply UpperBound to each α_n to obtain an upper bound β_n. Now $\beta = \sup_{n\in\mathbb{N}} (\beta_0 + \ldots \beta_n)$ is a valid output for $\mathrm{UpperBound}(\alpha)$ (note that addition is computable on $\mathbf{COrd}_\mathrm{K}$). □

Corollary 1. *The computable elements of* $\mathbf{COrd}_K$, $\mathbf{COrd}_M$ *and* $\mathbf{COrd}$ *are the same.*

Proof. From Proposition 4 in conjunction with Theorem 5 (5). □

5 $\mathbf{COrd}_\mathbf{M}$ and boundedness

Given that $\mathbf{COrd}_\mathrm{M}$ is very similar to $\mathbf{COrd}$, only differing in the properties of 0, and that $\mathbf{COrd}$ has the better closure properties (as sup is not computable on $\mathbf{COrd}_\mathrm{M}$[1]), one may wonder what the point of this space is. The special treatment of 0 in $\mathbf{COrd}_\mathrm{M}$ allows us to obtain a very useful extension of the $\leq$-relation on $\mathbf{COrd}_\mathrm{M}$, which ultimately can be used to prove that all continuous functions from Baire space into the countable ordinals are bounded:

[1] Any algorithm attempting to compute sup on $\mathbf{COrd}_\mathrm{M}$ needs to decide whether or not the result is 0 after finitely many steps – and this questions essentially is LPO.

Theorem 6 (Gregoriades, Kispéter and P. [10][2]). *For every continuous (even: every Borel-measurable) function $f : \mathbb{N}^\mathbb{N} \to \mathbf{COrd}_M$ there is some $\alpha \in COrd$ such that $\forall p \in \mathbb{N}^\mathbb{N} \quad f(p) \leq \alpha$.*

Corollary 2. *For every continuous (even: every Borel-measurable) function $f : \mathbb{N}^\mathbb{N} \to \mathbf{COrd}$ there is some $\alpha \in COrd$ such that $\forall p \in \mathbb{N}^\mathbb{N} \quad f(p) \leq \alpha$.*

Proof. Using Theorem 6 together with Proposition 4. □

Corollary 3. *There is no total representation $\delta : \mathbb{N}^\mathbb{N} \to COrd$ such that* id : $(COrd, \delta) \to \mathbf{COrd}$ *could be Borel measurable.*

Unfortunately, the proof of Theorem 6 is entirely non-constructive and does not offer a way to extract a bound from a description of the function. As a result of Spector establishes the corresponding version in the computable discrete realm, there seems to be hope for a positive answer to at least the weak version of the following:

Question 1. Is the function $\sup : \mathcal{C}(\mathbb{N}^\mathbb{N}, \mathbf{COrd}) \to \mathbf{COrd}$ computable? Is the multifunction $\mathrm{UpperBound} : \mathcal{C}(\mathbb{N}^\mathbb{N}, \mathbf{COrd}) \rightrightarrows \mathbf{COrd}$ computable?

6 Computability on $\mathbf{COrd}_{\mathbf{HL}}$

Computability on the space $\mathbf{COrd}_{\mathrm{HL}}$ was studied by Joel Hamkins and Zhenhao Li in [20]. We briefly survey some of their results:

Theorem 7 (Hamkins & Li [20]). *The following operations are computable:*

1. $+ : \mathbf{COrd}_{HL} \times \mathbf{COrd}_{HL} \to \mathbf{COrd}_{HL}$
2. $\times : \mathbf{COrd}_{HL} \times \mathbf{COrd}_{HL} \to \mathbf{COrd}_{HL}$
3. $(\alpha, \beta) \mapsto \alpha^\beta : \mathbf{COrd}_{HL} \times \mathbf{COrd}_{HL} \to \mathbf{COrd}_{HL}$
4. $\alpha + 1 \mapsto \alpha :\subseteq \mathbf{COrd}_{HL} \to \mathbf{COrd}_{HL}$
5. $\omega^{CK} + \omega \mapsto \omega^{CK} :\subseteq \mathbf{COrd}_{HL} \to \mathbf{COrd}_{HL}$

As with Proposition 3 for $\mathbf{COrd}_{\mathrm{K}}$, the first item of the following justifies our rejection of $\mathbf{COrd}_{\mathrm{HL}}$ as proposed *standard computability structure* on the countable ordinals. We point out that the technique introduced in [20, Theorem 16] essentially is a Wadge game relative to the representation, similar to the generalizations of the classical Wadge hierarchy on $\mathbb{N}^\mathbb{N}$ to represented spaces in [32] by Pequignot and [7] by Duparc and Fournier.

Theorem 8 (Hamkins & Li [20]). *The following operations are not computable:*

1. $\max : \mathbf{COrd}_{HL} \times \mathbf{COrd}_{HL} \to \mathbf{COrd}_{HL}$
2. $\alpha \mapsto \max\{\alpha, \omega + 1\} : \mathbf{COrd}_{HL} \to \mathbf{COrd}_{HL}$
3. $\omega \times \alpha \mapsto \alpha :\subseteq \mathbf{COrd}_{HL} \to \mathbf{COrd}_{HL}$

[2] This result essentially is folklore.

4. *Reduce*$_n$ $:\subseteq$ $\mathbf{COrd}_{HL} \to \mathbf{COrd}_{HL}$ *where* *Reduce*$_n(\omega) = n$ *and* *Reduce*$_n(\omega + \omega) = \omega$
5. $D :\subseteq \mathbf{COrd}_{HL} \to \{0, 1\}$ *where* $D(\omega) = 0$ *and* $D(\omega + 1) = 1$

Corollary 4. $\mathrm{id} : \mathbf{COrd}_{HL} \to \mathbf{COrd}_K$ *is not computable.*

An open question raised in [20] is whether the supremum of strictly increasing sequences of ordinals can be computed. This boils down to the following:

Question 2 (Hamkins & Li [20]). Is $\mathrm{id} : \mathbf{COrd}_{\mathrm{K}} \to \mathbf{COrd}_{\mathrm{HL}}$ computable?

Finally, we point out that the investigations in [20, Section 5] concern the *point degree spectrum* of $\mathbf{COrd}_{\mathrm{HL}}$ (without using this terminology, though). Point degree spectra of represented spaces were introduced by Kihara and P. in [17].

7 A non-deceiving representation of COrd?

The trusted recipe of identifying suitable representations of some structure is to pick an admissible representation whose final topology coincides with some natural topology on the structure[3]. However, the usual topology on COrd would be the order topology, which is not separable – and every represented space is separable. In this section, we shall explore whether a weaker topological requirement could be imposed on a representation.

Inspired by a property studied in the context of winning conditions for infinite sequential games in [19] by Le Roux and P., we shall call a function $f :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathrm{COrd}$ *non-deceiving*, iff whenever $(p_n)_{n\in\mathbb{N}}$ is a sequence converging to p in $\mathrm{dom}(f)$ such that $\forall n \in \mathbb{N}\ f(p_n) < f(p_{n+1})$, then $\forall i \in \mathbb{N}\ f(p_i) < f(p)$.

Theorem 9 (Gregoriades[4]). *Any non-deceiving function* $f :\subseteq \mathbb{N}^{\mathbb{N}} \to$ *COrd is bounded by some countable ordinal.*

Corollary 5. *There is no non-deceiving representation of COrd.*

The preceding corollary presumably destroys any hope to find a suitable represention of COrd that is admissible w.r.t. some weak limit space structure in the sense of Schröder [33, 34].

8 The computable Hausdorff-Kuratowski theorem

We shall now prepare the formulation of the Hausdorff-Kuratowski theorem in the framework of computable endofunctors on the category of represented spaces as introduced by de Brecht and P. in [6, 29, 31]. The setting closely follows the

[3] In fact, it is sometimes claimed that it *has* to be done like that – the present work ought to disprove this.

[4] This theorem is based on a personal communication by Vassilios Gregoriades.

corresponding section in [6] by de Brecht, where a weaker (and non-effective) version of our desired result was proven.

For any sequence of countable ordinals $(\alpha_i)_{i\in\mathbb{N}}$, we define a function $L_{(\alpha_i)_{i\in\mathbb{N}}} :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$. The sequence only impacts the domain, but whenever $L_{(\alpha_i)_{i\in\mathbb{N}}}(p)$ is defined, then $2L_{(\alpha_i)_{i\in\mathbb{N}}}(p)(n) = p(\max\{i \in \mathbb{N} \mid p(i) \text{ is odd}\} + n + 1)$; i.e. $L_{(\alpha_i)_{i\in\mathbb{N}}}$ takes the maximal tail of its input consisting of only even values, and returns the result of pointwise division by 2. Obviously any sequence in the domain of $L_{(\alpha_i)_{i\in\mathbb{N}}}$ has to contain only finitely many odd entries; and we additionally demand that for $p \in \operatorname{dom}(L_{(\alpha_i)_{i\in\mathbb{N}}})$, if $n < m$, and $p(n) = 2k+1$ and $p(m) = 2j+1$, then $\alpha_k > \alpha_j$.

Definition 9. *We define a computable endofunctor* $\mathfrak{L}_{(\alpha_n)_{n\in\mathbb{N}}}$ *by* $\mathfrak{L}_{(\alpha_n)_{n\in\mathbb{N}}}(X,\delta) = (X, \delta \circ L_{(\alpha_i)_{i\in\mathbb{N}}})$ *and the straightforward extension to functions.*

Each endofunctor $\mathfrak{L}_{(\alpha_n)_{n\in\mathbb{N}}}$ captures a version of computability with finitely many mindchanges (e.g. [39,40]): The regular outputs are encoded as even numbers. Finitely many times, the output can be reset by using an odd number, however, when doing so, one has to count down within the list of ordinals parameterizing the function (which in particular ensures that it happens only finitely many times). We thus find it connected to the *level* introduced by Hertling [12], and further studied by him and others in [6,11,13,14,23,25].

Definition 10. *Given a function* $f :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$*, we define the sets* $\mathcal{L}_\alpha(f) \subseteq \mathbb{N}^\mathbb{N}$ *inductively via:*

1. $\mathcal{L}_0(f) = \operatorname{dom}(f)$
2. $\mathcal{L}_{\alpha+1}(f) = \overline{\{x \in \mathcal{L}_\alpha(f) \mid f|_{\mathcal{L}_\alpha} \text{ is discontinuous at } x\}}$
3. $\mathcal{L}_\gamma(f) = \bigcap_{\beta<\gamma} \mathcal{L}_\beta(f)$ *for limit ordinals* γ.

Then we say $\operatorname{Lev}(f) := \min\{\alpha \mid \mathcal{L}_\alpha(f) = \emptyset\}$.

Theorem 10. *If* $f : \mathbb{N}^\mathbb{N} \to \mathfrak{L}_{(\alpha_i)_{i\in\mathbb{N}}}\mathbb{N}^\mathbb{N}$ *is continuous, then* $\operatorname{Lev}(f) \leq (\sup_{i\in\mathbb{N}} \alpha_i)+1$.

Proposition 5. *Let* $(\alpha_i)_{i\in\mathbb{N}}$ *be such that* $\exists \alpha \in COrd$ *with* $\{\alpha_i \mid i \in \mathbb{N}\} = \{\beta \in COrd \mid \beta < \alpha\}$. *Then* $\operatorname{Lev}(L_{(\alpha_i)_{i\in\mathbb{N}}}) = \alpha + 1$.

The computable Hausdorff-Kuratowski theorem has at its heart a dependent sum type; namely the construction $\sum_{(\alpha_i)_{i\in\mathbb{N}}\in\mathbf{COrd}^\mathbb{N}} \left(\mathcal{C}(\mathbf{X}, \mathfrak{L}_{(\alpha_i)_{i\in\mathbb{N}}}\mathbf{Y})\right)$ for some represented spaces $\mathbf{X}$, $\mathbf{Y}$. A point in this space is a pair, consisting of a sequence of countable ordinals and a function $f : \mathbf{X} \to \mathbf{Y}$, the latter given only in a $\mathfrak{L}_{(\alpha_i)_{i\in\mathbb{N}}}$-continuous way.

Theorem 11 (Computable Hausdorff-Kuratowski theorem). *Let* $\mathbf{X}$, $\mathbf{Y}$ *be represented spaces, and* $\mathbf{X}$ *be complete. Then the map* $HK : \mathcal{C}(\mathbf{X}, \mathbf{Y}^\nabla) \rightrightarrows \sum_{(\alpha_i)_{i\in\mathbb{N}}\in\mathbf{COrd}^\mathbb{N}} \left(\mathcal{C}(\mathbf{X}, \mathfrak{L}_{(\alpha_i)_{i\in\mathbb{N}}}\mathbf{Y})\right)$ *where* $((\alpha_i)_{i\in\mathbb{N}}, g) \in HK(f)$ *iff* $f = g$*, is computable.*

Corollary 6. *Let* $f : \mathbf{X} \to \mathbf{Y}$ *be computable with finitely many mindchanges, and* $\mathbf{X}$ *be complete. Then Lev(f) exists and is a computable ordinal.*

The result of the preceding corollary was also announced by Selivanov at CCA 2014.

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