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Representations of Analytic Functions and Weihrauch Degrees

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Abstract. This paper considers several representations of the analytic functions on the unit disk and their mutual translations. All translations that are not already computable are shown to be Weihrauch equivalent to closed choice on the natural numbers. Subsequently some similar considerations are carried out for representations of polynomials. In this case in addition to closed choice the Weihrauch degree LPO^* shows up as the difficulty of finding the degree or the zeros.

Keywords: computable analysis, analytic function, Weihrauch reduction, polynomials, closed choice, LPO^*

1 Introduction

In order to make sense of computability questions in analysis, the spaces of objects involved have to be equipped with representations: A representation determines the kind of information that is provided (or has to be provided) when computing on these objects. When restricting from a more general to more restrictive setting, there are two options: Either to merely restrict the scope to the special objects and retain the representation, or to actually introduce a new representation containing more information.

As a first example of this, consider the closed subsets of $[0, 1]^2$ and the closed convex subsets of $[0, 1]^2$ (following [8]). The former are represented by an enumeration of open balls exhausting their complement. The latter are represented as the intersection of a decreasing sequence of rational polygons. Thus, prima facie the notion of <u>closed set which happens to be convex</u> and <u>convex closed set</u> are different. In this case it turns out they are computably equivalent after all (the proof, however, uses the compactness of $[0, 1]^2$).

This paper focuses on a different example of the same phenomenon: The difference between an <u>analytic function</u> and a <u>continuous function that happens to be analytic</u>. It is known that these actually are different notions. Sections 3.1 and 3.2 quantify how different they are using the framework of Weihrauch reducibility. As a further example Sections 3.3 and 3.4 consider <u>continuous functions that happen to be polynomials</u> versus <u>analytic functions that happen</u>

to be polynomials versus polynomials. All translations turn out to be either computable, or Weihrauch equivalent to one of the two well-studied principles $C_{\mathbb{N}}$ and LPO^*. The results are summarized in Figure 3 on Page 11 and Figure 5 on Page 13.

The additional information one needs about an analytic function over a continuous function can be expressed by a single natural number – the same holds for the other examples studied. Thus, this can be considered as an instance of computation with discrete advice as introduced in [20]. That finding this number is Weihrauch equivalent to $C_{\mathbb{N}}$ essentially means that while the number can be chosen to be verifiable (i.e. wrong values can be detected eventually), this is the only computationally relevant restriction on how complicated the relationship between object and associated number can be.

Before ending this introduction, we shall briefly mention two alternative perspectives on the phenomenon: Firstly, recall that in intuitionistic logic a negative translated statement behaves like a classical one, and that double negations generally do not cancel. In this setting the difference boils down to considering either analytic functions or continuous functions that are not not analytic. Secondly, from a topological perspective, Weihrauch equivalence of a translation to $C_{\mathbb{N}}$ implies that the topologies induced by the representations differ. Indeed, the suitable topology on the space of analytic functions is not just the subspace topology inherited from the space of continuous functions but in fact obtained as a direct limit.

A version of this paper containing all the proofs can be found on the arXiv [17].

2 Background

This section provides a very brief introduction to the required concepts from computable analysis, Weihrauch reducibility, and then in more detailed introduction of the representations of analytic functions that are considered. For a more in depth introduction into computable analysis and further information, the reader is pointed to the standard textbook in computable analysis [19], and to [14]. Also, [18] should be mentioned as an excellent source, even though the approach differs considerably from the one taken here. The research programme of Weihrauch reducibility was formulated in [2], a more up-to-date introduction to Weihrauch reducibility can be found in the introduction of [3].

2.1 Represented Spaces

Recall that a **represented space** $\mathbf{X} = (X, \delta_{\mathbf{X}})$ is given by a set X and a partial surjection $\delta_{\mathbf{X}} :\subseteq \mathbb{N}^{\mathbb{N}} \to X$ from Baire space onto it. The elements of $\delta_{\mathbf{X}}^{-1}(x)$ should be understood as encodings of x and are called the **X-names** of x. Since Baire space inherits a topology, each represented space can be equipped with a topology: The final topology of the chosen representation. We usually refrain from mentioning the representation of a represented space in the same way as the topology of a topological space is usually not mentioned. For instance the set of natural numbers is regarded as a represented space with the representation $\delta_{\mathbb{N}}(p) := p(0)$. Therefore, from now on denote by $\mathbb{N}$ not only the set or the topological space, but the **represented space of natural numbers**. If the set that is to be represented already inherits a topology, we always choose the representation such that it fits the topology. This can be checked easily for the case $\mathbb{N}$ above, where the final topology of the representation is the discrete topology.

If $\mathbf{X}$ is a represented space and Y is a subset of $\mathbf{X}$, then Y can be turned into a represented space by considering the range restriction of the representation of $\mathbf{X}$ on it. We denote the represented space arising in this way by $\mathbf{X}|_Y$. Note that here only set inclusion is considered. The set Y may be a subset of many different represented spaces and the restrictions need not coincide. They often turn out to be inappropriate. We use the same notation $\mathbf{X}|_{\mathbf{Y}}$ if $\mathbf{Y}$ is a represented space already. In this case, however, no information about the representation of $\mathbf{Y}$ is carried over to $\mathbf{X}|_{\mathbf{Y}}$.

The remainder of this section introduces the represented spaces that are needed for the content of the paper.

Sets of Natural Numbers. Let $\mathcal{O}(\mathbb{N})$ resp. $\mathcal{A}(\mathbb{N})$ denote the **represented spaces of open** resp. **closed subsets of** $\mathbb{N}$. The underlying set of both $\mathcal{O}(\mathbb{N})$ and $\mathcal{A}(\mathbb{N})$ is the power set of $\mathbb{N}$. The representation of $\mathcal{O}(\mathbb{N})$ is defined by

$$\delta_{\mathcal{O}(\mathbb{N})}(p) = O \quad \Leftrightarrow \quad O = \{p(n) - 1 \mid p(n) > 0\}.$$

That is: A name of an open set is an enumeration of that set, however, to include the empty set, the enumeration is allowed to not return an element of the set in each step. The closed sets $\mathcal{A}(\mathbb{N})$ are represented as complements of open sets:

$$\delta_{\mathcal{A}(\mathbb{N})}(p) = A \quad \Leftrightarrow \quad \delta_{\mathcal{O}(\mathbb{N})}(p) = A^c.$$

Normed Spaces, $\mathbb{R}$, $\mathbb{C}$, $\boldsymbol{\mathcal{C}(D)}$. Given a triple $\mathcal{M} = (M, d, (x_n)_{n\in\mathbb{N}})$ such that (M, d) is a separable metric space and x_n is a dense sequence, $\mathcal{M}$ can be turned into a represented space by equipping it with the representation

$$\delta_{\mathcal{M}}(p) = x \quad \Leftrightarrow \quad \forall n \in \mathbb{N} : d(x, x_{p(n)}) < 2^{-n}.$$

In this way $\mathbb{R}$, $\mathbb{R}^d$, $\mathbb{C}$ (where the dense sequences are standard enumerations of the rational elements) and $C([0,1])$, $\mathcal{C}(D)$ (where D is a compact subset of $\mathbb{R}^d$ and the dense sequences are standard enumerations of the polynomials with rational coefficients) can be turned into represented spaces.

Sequences in a Represented Space. For a represented space $\mathbf{X}$ there is a canonical way to turn the set of sequences in $\mathbf{X}$ into a **represented space** $\mathbf{X}^{\mathbb{N}}$:

Let $\langle \cdot, \cdot \rangle : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a standard paring function (i.e. bijective, recursive with recursive projections). Define a function $\langle \cdot \rangle : \left(\mathbb{N}^{\mathbb{N}}\right)^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ by

$$\langle (p_k)_{k\in\mathbb{N}} \rangle (\langle m, n \rangle) := p_m(n).$$

For a represented space $\mathbf{X}$ define a representation of the set $X^{\mathbb{N}}$ of the sequences in the set X underlying $\mathbf{X}$ by

$$\delta_{\mathbf{X}^{\mathbb{N}}}(\langle (p_k)_{k\in\mathbb{N}} \rangle) = (x_k)_{k\in\mathbb{N}} \quad \Leftrightarrow \quad \forall m \in \mathbb{N} : \delta_{\mathbf{X}}(p_m) = x_m.$$

In particular the spaces $\mathbb{R}^{\mathbb{N}}$ and $\mathbb{C}^{\mathbb{N}}$ of real and complex sequences are considered represented spaces in this way. Also $\mathcal{C}(D)^{\mathbb{N}}$ briefly shows up in Section 3.2.

2.2 Weihrauch Reducibility

Recall that a multivalued function f from X to Y (or $\mathbf{X}$ to $\mathbf{Y}$) is an assignment that assigns to each element x of its domain a set $f(x)$ of acceptable return values. Multivaluedness of a function is indicated by $f : \mathbf{X} \rightrightarrows \mathbf{Y}$. The domain of a multivalued function is the set of elements such that the image is not empty. Furthermore, recall that we write $f :\subseteq \mathbf{X} \to \mathbf{Y}$ if the function f is allowed to be partial, that is if its domain can be a proper subset of $\mathbf{X}$.

Definition 1. *A partial function $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is a* ***realizer*** *of a multivalued function $f :\subseteq \mathbf{X} \rightrightarrows \mathbf{Y}$ if $\delta_{\mathbf{Y}}(F(p)) \in f(\delta_{\mathbf{X}}(p))$ for all $p \in \delta_{\mathbf{X}}^{-1}(\mathrm{dom}(f))$ (compare Figure 1).*

A function between represented spaces is called computable if it has a computable realizer, where computability on Baire space is defined via oracle Turing machines (as in e.g. [6]) or via Type-2 Turing machines (as in e.g. [19]). The computable Weierstraß approximation theorem can be interpreted to state that an element of $\mathcal{C}([0,1])$ is computable if and only if it has a computable realizer as function on the represented space $\mathbb{R}$.

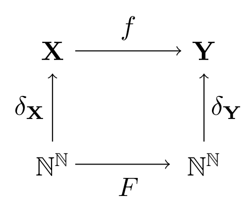

Fig. 1. Realizer

Every multivalued function $f :\subseteq \mathbf{X} \rightrightarrows \mathbf{Y}$ corresponds to a computational task. Namely: 'given information about x and the additional assumption $x \in \mathrm{dom}(f)$ find suitable information about some $y \in f(x)$'. What information about x resp. $f(x)$ is provided resp. asked for is reflected in the choice of the representations for $\mathbf{X}$ and $\mathbf{Y}$. The following example of this is very relevant for the content of this paper:

Definition 2. *Let* ***closed choice on the integers*** *be the multivalued function $C_{\mathbb{N}} :\subseteq \mathcal{A}(\mathbb{N}) \rightrightarrows \mathbb{N}$ defined on nonempty sets by*

$$y \in C_{\mathbb{N}}(A) \Leftrightarrow y \in A.$$

The corresponding task is 'given an enumeration of the complement of a set of natural numbers and provided that it is not empty, return an element of the set'. $C_{\mathbb{N}}$ does not permit a computable realizer: Whenever a machine decides that the name of the element of the set should begin with n, it has only read a finite beginning segment of the enumeration. The next value might as well be n.

From the point of view of multi-valued functions as computational tasks, it makes sense to compare their difficulty by comparing the corresponding multivalued functions. This paper uses Weihrauch reductions as a formalization of such a comparison. Weihrauch reductions define a rather fine pre-order on multivalued functions between represented spaces.

Definition 3. *Let f and g be partial, multivalued functions between represented spaces. Say that f is* ***Weihrauch reducible*** *to g, in symbols $f \leq_W g$, if there are computable functions $K :\subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ and $H :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that whenever G is a realizer of g, the function $F := (p \mapsto K(p, G(H(p))))$ is a realizer for f.*

H is called the **pre-processor** and K the **post-processor** of the Weihrauch reduction. This definition and the nomenclature is illustrated in Figure 2. The relation $\leq_{\mathrm{W}}$ is reflexive and transitive. We use $\equiv_{\mathrm{W}}$ to denote that reductions in both directions exist and $<_{\mathrm{W}}$ the other reduction does not exist. The equivalence class of a multivalued function with respect to the equivalence relation $\equiv_{\mathrm{W}}$ is called the **Weihrauch degree** of the function. A Weihrauch degree is called non-computable if it contains no computable function.

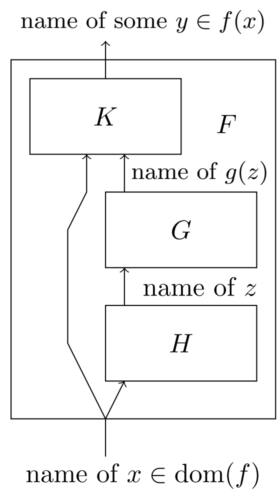

Fig. 2. Weihrauch reduction

The Weihrauch degree corresponding to $C_{\mathbb{N}}$ has received significant attention (see for instance [1,2,3],[10,11,12,13]). In particular, as shown in [15], a function between computable Polish spaces is Weihrauch reducible to $C_{\mathbb{N}}$ if and only if it is piecewise computable or equivalently is effectively Δ^0_2-measurable.

For the purposes of this paper, the following representatives of this degree are also relevant:

Lemma 1 ([16]). *The following are Weihrauch equivalent:*

- *$C_{\mathbb{N}}$, that is closed choice on the natural numbers.*
- $\max :\subseteq \mathcal{O}(\mathbb{N}) \to \mathbb{N}$ *defined on the bounded sets in the obvious way.*
- $\mathrm{Bound} :\subseteq \mathcal{O}(\mathbb{N}) \rightrightarrows \mathbb{N}$*, where $n \in \mathrm{Bound}(U)$ iff $\forall m \in U : n \geq m$.*

In the later chapters of this paper another non-computable Weihrauch degree is encountered: LPO^*. Here, LPO is short for 'limited principle of omniscience'. We refrain from stating LPO^* explicitly as it would need more machinery than we introduced. Instead we characterize it by specifying the representative that is used in the proofs: Consider the function

$$\min_{\mathrm{B}} : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}, \quad p \mapsto \min\{p(n) \mid n \in \mathbb{N}\}.$$

Here, the index B is for Baire space and to distinguish the function from the integer minimum function used on the right hand side of the definition.

Proposition 1. $\min_B$ *is a representative of the Weihrauch degree* LPO^**.*

LPO^* is also called the Weihrauch degree of finitely many mind changes: To obtain the minimum of an element of Baire space you may guess that it is the smallest value assumed on arguments up to n, and you will only be wrong a finite number of times.

To give a little more intuition as to why this Weihrauch degree shows up in this paper, note the following: LPO^* is derived from the maybe simplest non-computable Weihrauch degree $\mathrm{LPO} : \mathbb{N}^\mathbb{N} \to \{0, 1\}$ defined via

$$\mathrm{LPO}(p) := \begin{cases} 1 & \text{if } p \text{ is the zero function, i.e. } \forall n : p(n) = 0. \\ 0 & \text{otherwise.} \end{cases}$$

In computable analysis LPO shows up as the Weihrauch degree of the equality test for real (or complex) numbers $\neq : \mathbb{R} \times \mathbb{R}$. Now, LPO^* corresponds to carrying out a fixed finite but arbitrary high number of equality tests on the real or complex numbers. It is known that $\mathrm{LPO} <_\mathrm{W} \mathrm{LPO}^* <_\mathrm{W} \mathrm{C}_\mathbb{N}$.

2.3 Representations of Analytic Functions

Recall that a function is analytic if it is locally given by a power series:

Definition 4. *Let $D \subseteq \mathbb{C}$ be a set. A function $f : D \to \mathbb{C}$ is called* ***analytic****, if for every $x_0 \in D$ there is a neighborhood U of x_0 and a sequence $(a_k)_{k\in\mathbb{N}} \in \mathbb{C}^\mathbb{N}$ such that for each $x \in U \cap D$*

$$f(x) = \sum_{k\in\mathbb{N}} a_k x^k.$$

The set of analytic functions is denoted by $\mathcal{C}^\omega(D)$. Each analytic function is continuous, that is $\mathcal{C}^\omega(D) \subseteq \mathcal{C}(D)$. If D is open, the analytic functions on D are smooth, i.e. infinitely often differentiable. An analytic function can be analytically extended to an open superset of its domain.

Definition 5. *A pair $(x, (a_k)_{k\in\mathbb{N}})$ is called* ***germ*** *of $f \in \mathcal{C}^\omega(D)$ if x is an element of D and $(a_k)_{k\in\mathbb{N}} \in \mathbb{C}^\mathbb{N}$ is a series expansion of f around x.*

As long as the domain is connected, an analytic function is uniquely determined by each of its germs. The one to one correspondence of germs and analytic functions only partially carries over to the computability and complexity realm: It is well known that an analytic function on the unit disk is computable if and only if the germ around any computable point of the domain is computable [5]. However, the proofs of these statements are inherently non-uniform. The operations of obtaining a germ from a function and a function from a germ are discontinuous and therefore not computable [9]. This paper classifies them to be Weihrauch equivalent to closed choice on the naturals in Theorems 3 and 4.

There is a more suitable representation for the analytic functions than the restriction of the representation of continuous functions. This representation has been investigated by different authors for instance in [4],[7],[9]. For simplicity we restrict to the case of analytic functions on the unit disk. Thus, let D denote the closed unit disk from now on. And let U_m denote the open ball $B_{r_m}(0)$ of radius $r_m := 2^{\frac{1}{m+1}}$ around zero. Recall from the introduction that the space $\mathcal{C}(D)$ of continuous functions is represented as a metric space (where $\mathbb{C}$ is identified with $\mathbb{R}^2$).

Definition 6. *Let $\mathcal{C}^\omega(D)$ denote the* ***represented space of analytic functions on*** D*, where the representation is defined as follows: A $q \in \mathbb{N}^\mathbb{N}$ is a name of an analytic function f on D, if and only if f extends analytically to the closure of $U_{q(0)}$, the extension is bounded by $q(0)$ and $n \mapsto q(n+1)$ is a name of $f \in \mathcal{C}(D)$.*

Note that the representation of $\mathcal{C}^\omega(D)$ arises from the restriction of the representation of continuous functions by adding discrete additional information. This information is quantified by the advice function $\mathrm{Adv}_{\mathrm{C}^\omega} :\subseteq \mathcal{C}(D) \to \mathbb{N}$ whose domain are the analytic functions and that on those is defined by

$$\begin{aligned} \mathrm{Adv}_{\mathrm{C}^\omega}(f) &:= \{q(0) \mid q \text{ is a } \mathcal{C}^\omega(D)\text{-name of } f)\} \\ &= \{m \in \mathbb{N} \mid f \text{ has an analytic cont. to } U_m \text{ bounded by } m\}. \end{aligned} \tag{1}$$

This function turns up in the results of this paper. In the terminology of [4], one would say that $\mathcal{C}^\omega(D)$ arises from the restriction $\mathcal{C}(D)|_{\mathcal{C}^\omega(D)}$ by enriching with the discrete advice $\mathrm{Adv}_{\mathrm{C}^\omega}$.

The topology induced by the representation of $\mathcal{C}^\omega(D)$ is well known and used in analysis: It can be constructed as a direct limit topology and makes $\mathcal{C}^\omega(D)$ a so called Silva-Space. For more information on this topology and its relation to computability and complexity theory also compare [7].

Consider the set of germs around zero, i.e. of power series with radius of convergence strictly larger than 1. Since the base point 0 is fixed, it is often omitted and the germ identified with a sequence. This set may be represented as follows:

Definition 7. *Let $\mathcal{O}$ denote the represented space of germs around zero, where the representation is defined as follows: A $q \in \mathbb{N}^\mathbb{N}$ is a name of a germ $(0, (a_k)_{k\in\mathbb{N}})$, if and only if*

$$\forall k \in \mathbb{N} : |a_k| \leq 2^{-\frac{k}{q(0)+1}} q(0)$$

and $n \mapsto q(n+1)$ is a name of the sequence $(a_k)_{k\in\mathbb{N}}$ as element of $\mathbb{C}^\mathbb{N}$.

As above, this representation is related to the restriction of the representation of $\mathbb{C}^\mathbb{N}$ by means of the advice function $\mathrm{Adv}_\mathcal{O} :\subseteq \mathbb{C}^\mathbb{N} \rightrightarrows \mathbb{N}$ whose domain are the sequences with radius of convergence strictly larger than one and that is defined on those by

$$\begin{aligned}\mathrm{Adv}_{\mathcal{O}}((a_k)_{k\in\mathbb{N}}) &:= \{q(0) \mid q \text{ is a } \mathcal{O}\text{-name of } (a_k)_{k\in\mathbb{N}}\} \\ &= \{n \in \mathbb{N} \mid \forall k \in \mathbb{N} : |a_k| \le 2^{-\frac{k}{n+1}} \cdot n\}\end{aligned} \tag{2}$$

Again, the topology induced by this representation is well known and used in analysis: It is the standard choice of a topology on the set of germs and can be introduced as a direct limit topology.

Proofs that the following holds can be found in [4] or [9]:

Theorem 1 (computability of summation). *The assignment*

$$\mathcal{O} \to \mathcal{C}^\omega(D), \quad (a_k)_{k\in\mathbb{N}} \mapsto \left(x \mapsto \sum_k a_k x^k\right)$$

is computable.

A proof of the following can be found in [4]:

Theorem 2. *Differentiation is computable as mapping from $\mathcal{C}^\omega(D)$ to $\mathcal{C}^\omega(D)$.*

3 The Results

We open this chapter with an addition to Lemma 1. Given $p \in \mathbb{N}^\mathbb{N}$ denote the support of this function by $\mathrm{supp}(p) := \{n \in \mathbb{N} \mid p(n) > 0\}$. Furthermore, for a set A denote the number of elements of that set by $\#A$.

Lemma 2. *The function* $\mathrm{Count} :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}$*, defined via*

$$\mathrm{dom}(\mathrm{Count}) = \{p \in \mathbb{N}^\mathbb{N} \mid \mathrm{supp}(p) \text{ is finite}\} \quad \mathrm{Count}(p) = \#\mathrm{supp}(p)$$

is Weihrauch equivalent to $C_\mathbb{N}$, that is: Closed choice on the naturals.

3.1 Summing Power Series

In Section 2.3 it was mentioned that the operation of summing a power series is not computable on $\mathbb{C}^\mathbb{N}$. Recall that $\mathrm{Adv}_\mathcal{O}$ was the advice function of the representation of the represented space $\mathcal{O}$ of germs around zero of analytic functions on the unit disk. The computational task corresponding to this multivalued function is to find from a sequence that is guaranteed to have radius of convergence bigger than one a constant witnessing the exponential decay of the absolute value of the coefficients (compare eq. (2) on page 8). Theorem 1 states that summation is computable on $\mathcal{O}$. Therefore, the advice function $\mathrm{Adv}_\mathcal{O}$ cannot be computable. The following theorem classifies the difficulty of summing power series and $\mathrm{Adv}_\mathcal{O}$ in the sense of Weihrauch reductions.

Theorem 3. *The following are Weihrauch-equivalent:*

- $\boldsymbol{C}_{\mathbb{N}}$*, that is: Closed choice on the naturals.*
- Sum*, that is: The partial mapping from* $\mathbb{C}^{\mathbb{N}}$ *to* $\mathcal{C}(D)$ *defined on the sequences with radius of convergence strictly larger than one by*

$$\mathrm{Sum}((a_k)_{k\in\mathbb{N}})(x) := \sum_{k\in\mathbb{N}} a_k x^k.$$

I.e. summing a power series.
- $\mathrm{Adv}_{\mathcal{O}}$*, that is: The function from eq.* (2) *on page 8. I.e. obtaining the constant from the series.*

Proof (ideas). Build a Weihrauch reduction circle:

$\mathrm{C}_{\mathbb{N}} \leq_{\mathbf{W}}$ Sum: Lemma 2 permits to replace $\mathrm{C}_{\mathbb{N}}$ by Count. Let the pre-processor assign to $p \in \mathbb{N}^{\mathbb{N}}$ the sequence

$$a_k := \begin{cases} 1 & \text{if } p(k) > 0 \\ 0 & \text{if } p(k) = 0 \end{cases}.$$

For the post-processor use a realizer of the evaluation in 1.

Sum $\leq_{\mathbf{W}}$ $\mathrm{Adv}_{\mathcal{O}}$: Follows from Theorem 1.

$\mathrm{Adv}_{\mathcal{O}} \leq_{\mathbf{W}} \mathrm{C}_{\mathbb{N}}$: Let the pre-processor be the function that maps a given name p of $(a_k)_{k\in\mathbb{N}} \in \mathbb{C}^{\omega}$ to an $\mathcal{A}(\mathbb{N})$-name of the set $\mathrm{Adv}_{\mathcal{O}}((a_k)_{k\in\mathbb{N}})$. Note that an enumeration of the complement of this set can be extracted from p as follows: For all k and $m \in \mathbb{N}$ dovetail the test $|a_k| > 2^{-\frac{k}{m+1}} m$. If it holds for some k, return m as an element of the complement. Applying closed choice to this set will give result in a valid return value.

3.2 Differentiating Analytic Functions

In Section 2.3 it was remarked that it is not possible to compute the germ of an analytic function just from a name as continuous function. The proof that this is in general impossible from [9], however, argues about analytic functions on an interval. The first lemma of this chapter proves that for analytic functions on the unit disk it is possible to compute a germ if its base point is well inside of the domain. We only consider the case where the base point is zero, but the proof works whenever a lower bound on the distance of the base point to the boundary of the disk is known.

Lemma 3. Germ*, that is: The partial mapping from* $\mathcal{C}(D)$ *to* $\mathbb{C}^{\mathbb{N}}$ *defined on analytic functions by mapping them to their series expansion around zero, is computable.*

Proof (sketch). Use the Cauchy integral Formula.

The next theorem is very similar to Theorem 3. Both the advice function $\mathrm{Adv}_{\mathrm{C}^{\omega}}$ and computing a germ around a boundary point are shown to be Weihrauch equivalent to $\mathrm{C}_{\mathbb{N}}$. Note that the coefficients of the series expansion $(a_k)_{k\in\mathbb{N}}$ of an analytic function f around a point x_0 are related to the derivatives $f^{(k)}$ of the function via $k!a_k = f^{(k)}(x_0)$. Therefore, computing a series expansion around a point is equivalent to computing all the derivatives in that point.

Theorem 4. *The following are Weihrauch equivalent:*

- $\boldsymbol{C}_{\mathbb{N}}$*, that is closed choice on the naturals.*
- Diff*, that is the partial mapping from* $\mathcal{C}(D)$ *to* $\mathbb{C}$ *defined on analytic functions by*

$$\mathrm{Diff}(f) := f'(1).$$

  I.e. evaluating the derivative of an analytic function in 1*.*
- $\mathrm{Adv}_{\boldsymbol{C}^\omega}$*, that is the function from eq.* (1)*. I.e. obtaining the constant from the function.*

Proof (outline). By building a circle of Weihrauch reductions:

$\mathbf{C}_{\mathbb{N}} \leq_{\mathbf{W}}$ Diff: Use Lemma 2 and show Count $\leq_{\mathrm{W}}$ Diff instead. For the preprocessor fix a computable sequence of analytic functions $f_n : D \to \mathbb{C}$ such that $f_n'(1) = 1$ and $|f_n(x)| < 2^{-n}$ for all $x \in D$ (compare Figure 4). For $p \in \mathbb{N}^{\mathbb{N}}$ consider the function

$$f(x) := \sum_{n \in \mathrm{supp}(p)} f_n(x).$$

Note that applying Diff to the function f results in

$$\mathrm{Diff}(f) = f'(1) = \sum_{n \in \mathrm{supp}(p)} f_n'(1) = \#\mathrm{supp}(p).$$

Therefore, the post-processor $K(p, q) := q$ results in a Weihrauch reduction.

Diff $\leq_{\mathbf{W}}$ $\mathrm{Adv}_{\mathbf{C}^\omega}$: Use Theorem 2.

$\mathrm{Adv}_{\mathbf{C}^\omega} \leq_{\mathbf{W}} \mathbf{C}_{\mathbb{N}}$: Theorem 3 proved that $\mathrm{Adv}_{\mathcal{O}} \equiv_{\mathrm{W}} \mathrm{C}_{\mathbb{N}}$.
Let the pre-processor be a realizer of the function Germ from Lemma 3. Applying $\mathrm{Adv}_{\mathcal{O}}$ will return a constant n for the sequence. Set $m := 4(n+1)^2$, then for $|x| \leq 2^{\frac{1}{m+1}} \leq 2^{\frac{1}{2(n+1)}}$

$$\left| \sum_{k \in \mathbb{N}} a_k x^k \right| \leq \sum_{k \in \mathbb{N}} 2^{-\frac{k}{2(n+1)}} n = \frac{1}{1 - 2^{-\frac{1}{2(n+1)}}} n \leq 4(n+1)^2 = m.$$

Therefore, the sum can be evaluated to an analytic function bounded by m on $B_{2^{\frac{1}{m+1}}}(0)$ and m is a valid value for the post-processor.

Recall from the introduction that $\mathcal{C}(D)|_{\mathcal{C}^\omega(D)}$ resp. $\mathbb{C}^{\mathbb{N}}|_{\mathcal{O}}$ denote the represented spaces obtained by restricting the representation of $\mathcal{C}(D)$ resp. $\mathbb{C}^{\mathbb{N}}$ to $\mathcal{C}^\omega(D)$, resp. $\mathcal{O}$. Theorems 1, 3 and 4 and Lemma 3 are illustrated in fig. 3.

3.3 Polynomials as Finite Sequences

Consider the set $\mathbb{C}[X]$ of polynomials with complex coefficients in one variable X. There are several straightforward ways to represent polynomials. The first one that comes to mind is to represent a polynomial by a finite list of complex numbers. One can either demand the length of the list to equal the degree of the polynomial or just to be big enough to contain all of the non-zero coefficients. The first option fails to make operations like addition of polynomials computable.

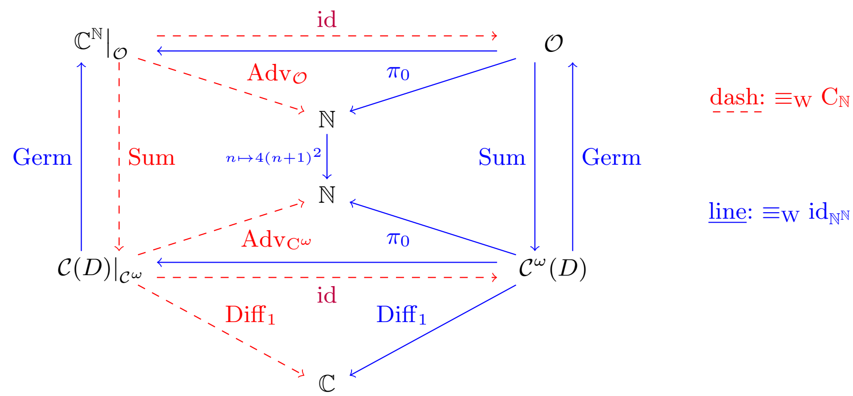

Fig. 3. The results of Theorems 1, 3 and 4 and Lemma 3.

Definition 8. *Let* $\mathbb{C}[X]$ *denote the* ***represented space of polynomials****, where* $p \in \mathbb{N}^{\mathbb{N}}$ *is a* $\mathbb{C}[X]$*-name of* P *if* $p(0) \geq \deg(P)$ *and* $n \mapsto p(n+1)$ *is a* $\mathbb{C}^{p(0)}$*-name of the first* $p(0)$ *coefficients of* P*.*

Let $\mathbb{C}_m[X]$ denote the set of monic polynomials over $\mathbb{C}$, i.e. the polynomials with leading coefficient equal to one. Make $\mathbb{C}_m[X]$ a represented space by restricting the representation of $\mathbb{C}[X]$. Monic polynomials are important because it is possible to compute their roots – albeit in an unordered way. To formalize this define a representation of the disjoint union $\mathbb{C}^\times := \coprod_{n\in\mathbb{N}} \mathbb{C}^n$ as follows: A $p \in \mathbb{N}^{\mathbb{N}}$ is a name of $x \in \mathbb{C}^\times$ if and only if $x \in \mathbb{C}^{p(0)}$ and $n \mapsto p(n+1)$ is a $\mathbb{C}^{p(0)}$ name of x. Note that the construction of the representation of $\mathbb{C}[X]$ is very similar. The only difference being that vectors with leading zeros are not identified with shorter vectors.

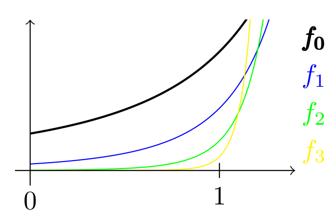

Fig. 4. $f_n(x) := (x - x_n)^{-2^{n+1}}$ for appropriate x_n.

Now, the task of finding the zeros in an unordered way can be formalized by computing the multivalued function that maps a polynomial to the set of lists of its zeros, each appearing according to its multiplicities:

$$\text{Zeros} : \mathbb{C}[X] \rightrightarrows \mathbb{C}^\times, P \mapsto \left\{ (a_1, \ldots, a_{\deg(P)}) \mid \exists \lambda : P = \lambda \prod_{k=1}^{\deg(P)} (X - a_k) \right\} \quad (3)$$

The importance of $\mathbb{C}_m[X]$ is reflected in the following well known lemma:

Lemma 4. *Restricted to* $\mathbb{C}_m[X]$ *the mapping* Zeros *is computable.*

The main difficulty in computing the zeros of an arbitrary polynomial is to find its degree. A polynomial of known degree can be converted to a monic polynomial with the same zeros by scaling. On $\mathbb{C}[X]$ consider the following functions:

- deg: The function assigning to a polynomial its degree.
- Dbnd: The multivalued function where an integer is a valid return value if and only if it is an upper bound of the degree of the polynomial.

Dbnd is computable by definition of the representation of $\mathbb{C}[X]$. The mapping deg, in contrast, is not computable on the polynomials, however, the proof of Lemma 4 includes a proof of the following:

Lemma 5. *On $\mathbb{C}_m[X]$ the degree mapping is computable.*

The next result classifies finding the degree, turning a polynomial into a monic polynomial and finding the zeros to be Weihrauch equivalent to LPO^*.

Proposition 2. *The following are Weihrauch-equivalent to LPO^*:*

- deg*, that is the mapping from $\mathbb{C}[X]$ to $\mathbb{N}$ defined in the obvious way.*
- Monic*, that is the mapping from $\mathbb{C}[X]$ to $\mathbb{C}_m[X]$ defined on the non-zero polynomials by*

$$P = \sum_{k=0}^{\deg(P)} a_k X^k \mapsto \sum_{k=0}^{\deg(P)} \frac{a_k}{a_{\deg(P)}} X^k.$$

- Zeros $:\subseteq \mathbb{C}[X] \rightrightarrows \mathbb{C}^\times$*, mapping a non-zero polynomial to the set of its zeros, each appearing according to its multiplicity (compare eq.* (3)*).*

3.4 Polynomials as Functions

As polynomials induce analytic functions on the unit disk, the representations of $\mathcal{C}^\omega(D)$ and $\mathcal{C}(D)$ can be restricted to the polynomials. The represented spaces that result from this are $\mathcal{C}^\omega(D)|_{\mathbb{C}[X]}$, resp. $\mathcal{C}(D)|_{\mathbb{C}[X]}$. Here, the choice of the unit disk D as domain seems arbitrary: A polynomial defines a continuous resp. analytic function on the whole space. The following proposition can easily be checked to hold whenever the domain contains an open neighborhood of zero and, since translations are computable with respect to all the representations we consider, if it contains any open set.

Denote the versions of the degree resp. degree bound functions that take continuous resp. analytic functions by $\deg_{\mathcal{C}(D)}$, $\mathrm{Dbnd}_{\mathcal{C}(D)}$ resp. $\deg_{\mathcal{C}^\omega(D)}$, $\mathrm{Dbnd}_{\mathcal{C}^\omega(D)}$. When polynomials are regarded as functions, resp. analytic functions, these maps become harder to compute.

Theorem 5. *The following are Weihrauch-equivalent:*

- $C_\mathbb{N}$*, that is: Closed choice on the naturals.*
- $\mathrm{Dbnd}_{\mathcal{C}^\omega(D)}$*, that is: Given an analytic function which is a polynomial, find an upper bound of its degree.*

– $\deg_{\mathcal{C}^\omega(D)}$: *Given an analytic function which is a polynomial, find its degree.*

Proof. **$\mathbf{C}_\mathbb{N} \leq_\mathbf{W} \mathrm{Dbnd}_{\mathcal{C}^\omega(D)}$:** Use Lemma 1 and reduce to Bound instead. For an enumeration p of a bounded set consider $P(X) := \sum 2^{-\max\{n,p(n)\}} X^{p(n)}$. A $\mathcal{C}^\omega(D)$-name of the function f corresponding to P can be computed from p. Let the pre-processor H be a realizer of this assignment. Set $K(p,q) := q$.

$\mathrm{Dbnd}_{\mathcal{C}^\omega(D)} \leq_\mathbf{W} \deg_{\mathcal{C}^\omega(D)}$: Is trivial.

$\deg_{\mathcal{C}^\omega(D)} \leq_\mathbf{W} \mathbf{C}_\mathbb{N}$: By Lemma 1 replace $\mathrm{C}_\mathbb{N}$ with max. Let p be a $\mathcal{C}^\omega(D)$-name of the function corresponding to some polynomial P. Use Lemma 3 to extract a $\mathbb{C}^\mathbb{N}$-name q of the series of coefficients. Define the pre-processor by $H(p)(\langle m,n\rangle) := n+1$ if the dyadic number encoded by $q(\langle m,n\rangle)$ is bigger than 2^{-m} and 0 otherwise. Set $K(p,q) := q$.

From the proof of the previous theorem it can be seen, that stepping down from analytic to continuous functions is not an issue. For sake of completeness we add a slight tightening of the third item of Theorem 4 and state this as theorem:

Theorem 6. *The following are Weihrauch-equivalent to $\mathrm{C}_\mathbb{N}$:*

– $\deg_{\mathcal{C}(D)}$: *Given a continuous function which is a polynomial, find its degree.*
– $\mathrm{Dbnd}_{\mathcal{C}(D)}$: *Given an analytic function which is a polynomial, find an upper bound of its degree.*
– $\mathrm{Adv}_{\mathcal{C}^\omega}|_{\mathbb{C}[X]}$: *Given a continuous function which happens to be a polynomial, find the constant needed to represent it as analytic function.*

$\mathrm{Dbnd}_{\mathcal{C}^\omega(D)}$ may be regarded as the advice function of $\mathbb{C}[X]$ over $\mathcal{C}^\omega(D)$: The representation where p is a name of a polynomial P if and only if $p(0) = \mathrm{Dbnd}_{\mathcal{C}^\omega(D)}$ and $n \mapsto p(n+1)$ is a $\mathcal{C}^\omega(D)$-name of P is computationally equivalent to the representation of $\mathbb{C}[X]$. The same way, $\mathrm{Dbnd}_{\mathcal{C}(D)}$ can be considered an advice function of $\mathbb{C}[X]$ over $\mathcal{C}(D)$.

Figure 5 illustrates Lemma 4, Proposition 1 and Theorems 5 and 6.

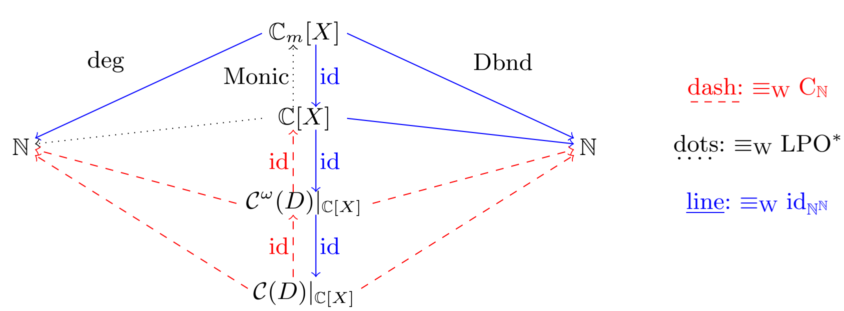

Fig. 5. The result of Lemma 4, Proposition 1 and Theorems 5 and 6.

4 Conclusion

Many of the results proved in Section 3 work for more general domains: Lemma 3 generalizes to any computable point of the interior of an arbitrary domain. It can be made a uniform statement by including the base point of a germ. In this case for the proof to go through computability of the distance function of the complement of the domain of the analytic function is needed.

Another example is the part of Theorem 4 that says finding a germ on the boundary is difficult. In this case a disc of finite radius touching the boundary in a computable point is needed. Alternatively, a simply connected bounded Lipshitz domain with a computable point in the boundary can be used. Also in this case it seems reasonable to assume that a uniform statement can be proven.

Furthermore, after considering polynomials and analytic functions [7] also investigates representations for the set of distributions with compact support. In the same vain as in this paper one could compare these representation and the representation of distributions as functions on the spaces of test functions.

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