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RESEARCH ARTICLE

An ergodic theorem of a parabolic Anderson model driven by Lévy noise

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Abstract In this paper, we study an ergodic theorem of a parabolic Andersen model driven by Lévy noise. Under the assumption that $A = (a(i,j))_{i,j\in S}$ is symmetric with respect to a σ-finite measure π, we obtain the long-time convergence to an invariant probability measure ν_h starting from a bounded nonnegative A-harmonic function h based on self-duality property. Furthermore, under some mild conditions, we obtain the one to one correspondence between the bounded nonnegative A-harmonic functions and the extremal invariant probability measures with finite second moment of the nonnegative solution of the parabolic Anderson model driven by Lévy noise, which is an extension of the result of Y. Liu and F. X. Yang.

Keywords Parabolic Anderson model, ergodic theorem, invariant measure, Lévy noise, self-duality
MSC 60H15, 60B10, 60K35

1 Introduction

The parabolic Anderson model has long been of interest to physicists and mathematicians. It describes the entrapment of electrons of crystals with impurities originally introduced by physicist Anderson at 1958 [2]. It also presents the relevant models for chemical kinetics and population dynamics (see [3,12] for more background and applications). On the other hand, the rigorous analysis to some real world phenomenon, for example, intermittency effect has provided mathematical challenges, and often requires some new mathematical ideas and techniques. References [13,14,17] and the survey [12] provided the recent interesting progress on the mathematical aspects of parabolic Anderson

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model.

As a mathematical object, the parabolic Anderson model has led to substantial research, which relates to some other interesting topics, such as the spectrum of random Schrödinger operator [18], the Lyapunov exponent for infinite dimensional random dynamical system [7–9,11], and the Feynman-Kac formula [15]. It can be regarded as a linear interacting particle system in the sense of Liggett's book [21, Chapter IX] also. The ergodic theory of stochastic interacting systems, for instance, characterizing all invariant probability measures, is one of the important themes of the research on this field. It is hoped that this would present a better understanding of the phenomenon of phase transition. The solutions of related problems has led in turn to the new probabilistic methods and tools, such as duality theory and coupling method.

In this paper, inspired by Liggett, Spitzer, and Shiga et al. (see [19,20,22,29, 30]), we are concerned with characterizing the invariant probability measures of a parabolic Anderson model driven by Lévy noise under some mild conditions. The specific model is described as the following, of which the sample Lyapunov exponent was researched by Furuoya and Shiga in [11].

Let S be a countable set, and let $A = (a(i,j))_{i,j\in S}$ be an $S\times S$ real matrix satisfying

$$a(i,j) \geqslant 0 \ \text{ for } i \neq j, \ \sum_{j\in S} a(i,j) = 0, \ \text{and} \ \sup_{i\in S} |a(i,i)| < \infty, \tag{1.1}$$

i.e., A is an infinitesimal generator of a continuous time Markov chain on S. Let $\pi = (\pi_i)_{i\in S}$ be a σ-finite measure on S, A is symmetric with respect to π, i.e.,

$$\pi_i a(i,j) = \pi_j a(j,i), \quad \forall\ i,j\in S. \tag{1.2}$$

Let $(Y(t))_{t\geqslant 0}$ be a 1-dimensional Lévy process with a characteristic exponent $\psi(z)$,

$$\psi(z) = -\frac{\alpha^2}{2} z^2 + \mathrm{i}\beta z + \int_{\mathbb{R}\setminus\{0\}} (\mathrm{e}^{\mathrm{i}zu} - 1 - \mathrm{i}zu I_{\{|u|<1/2\}})\rho(\mathrm{d}u), \tag{1.3}$$

where $\rho(\mathrm{d}u)$ is a Radon measure on $\mathbb{R}\setminus\{0\}$ such that

$$\int_{\mathbb{R}\setminus\{0\}} \min(u^2,1)\rho(\mathrm{d}u) < \infty. \tag{1.4}$$

Now, let us consider the following linear interacting diffusion on S, which is a parabolic Andersen model driven by Lévy processes:

$$\mathrm{d}X_i(t) = \kappa \sum_{j\in S} a(i,j) X_j(t)\mathrm{d}t + X_i(t-)\mathrm{d}Y_i(t), \quad i\in S, \tag{1.5}$$

where $(Y_i(t))_{t\geqslant 0}$ $(i\in S)$ are independent copies of $(Y(t))_{t\geqslant 0}$. The integral

$$\int_0^{t+} X_i(s-)\mathrm{d}Y_i(s)$$

is understood as follows (see [16]). Consider the Lévy-Itô decomposition of $(Y_i(t))_{t\geqslant 0},\ i\in S$,

$$Y_i(t)=\alpha B_i(t)+\beta t+\int_0^{t+}\int_{\{|u|<1/2\}}u\widetilde{N}_i(\mathrm{d}s\mathrm{d}u)+\int_0^{t+}\int_{\{|u|\geqslant 1/2\}}uN_i(\mathrm{d}s\mathrm{d}u), \tag{1.6}$$

where $\{B_i(t)_{t\geqslant 0},\, i\in S\}$ are independent standard Brownian motions, $\{N_i(\mathrm{d}s\mathrm{d}u),\ i\in S\}$ are independent Poisson random measures on $[0,\infty)\times\mathbb{R}\backslash\{0\}$ with intensity measure $\mathrm{d}s\rho(\mathrm{d}u)$, and

$$\widetilde{N}_i(\mathrm{d}s\mathrm{d}u)=N(\mathrm{d}s\mathrm{d}u)-\mathrm{d}s\rho(\mathrm{d}u),\quad i\in S,$$

are the martingale measures. Then,

$$\begin{aligned}\int_0^{t+}X_i(s)\mathrm{d}Y_i(s)=\alpha\int_0^{t+}X_i(s)\mathrm{d}B_i(\mathrm{d}s)+\beta\int_0^{t}X_i(s)\mathrm{d}s\\ +\int_0^{t+}\int_{\{|u|<1/2\}}X_i(s-)u\widetilde{N}_i(\mathrm{d}s\mathrm{d}u)\\ +\int_0^{t+}\int_{\{|u|\geqslant 1/2\}}X_i(s-)uN_i(\mathrm{d}s\mathrm{d}u).\end{aligned} \tag{1.7}$$

Especially, we investigate a special form of (1.5) in this paper:

$$\begin{aligned}\mathrm{d}X_i(t)&=\kappa\sum_{j\in S}a(i,j)X_j(t)\mathrm{d}t+\alpha X_i(t)\mathrm{d}B_i(t)+\int_{\{|u|<1/2\}}uX_i(t-)\widetilde{N}_i(\mathrm{d}t\mathrm{d}u)\\ &\quad+\int_{\{|u|\geqslant 1/2\}}uX_i(t-)N_i(\mathrm{d}t\mathrm{d}u)-\int_{\{|u|\geqslant 1/2\}}uX_i(t-)\rho(\mathrm{d}u)\mathrm{d}t\\ &=\kappa\sum_{j\in S}a(i,j)X_j(t)\mathrm{d}t+\alpha X_i(t)\mathrm{d}B_i(t)+\int_{\mathbb{R}\backslash\{0\}}uX_i(t-)\widetilde{N}_i(\mathrm{d}t\mathrm{d}u).\end{aligned} \tag{1.8}$$

Our main purposes in this paper are constructing the one-to-one correspondence between the A-bounded nonnegative harmonic functions and the extremal invariant probability measures of the nonnegative solution of (1.8) (see Theorem 2.9 in Section 2).

In [6], the parabolic Anderson model driven by Brownian motions on S (A was assumed to be doubly stochastic) and a subclass of linear interacting system constructed through Poisson point processes on $\mathbb{Z}^d$ (A was assumed to be translation invariant) were studied by Cox, Klenke and Perkins. They showed the long-time convergence to an invariant measure ν_θ starting from a constant initial state θ by the self-duality of linear system introduced in [21]. Moreover, they presented that the convergence to the invariant probability measure ν_θ held for a broad class of initial distributions denoted by $\mathscr{M}_\theta$. In [25] or [33], the authors constructed the one-to-one correspondence between the A-bounded nonnegative harmonic functions and the extremal invariant probability measures of the nonnegative solution of the parabolic Anderson model driven by Brownian motions on S by using the self-duality property, the second moment estimates, and the truncation technique, if A was symmetric and transient and the diffusion parameter was less than a threshold ([25, Theorem 2.5]). Furthermore, if A was doubly stochastic and satisfied the so-called Case I (see the definition of Case I in [29]), they proved that the system locally died out independent of the diffusion parameter through a comparison theorem in [4] ([25, Theorem 2.9]).

Now, we will generalize the approaches in [25] or [33] to (1.8), the parabolic Anderson model driven by Lévy noise. Here, we extend slightly that A is symmetric with respect to a σ-finite measure π (see (1.2)). Actually, in [25] or [33], A is only assumed to be symmetric, i.e., $a(i,j)=a(j,i)$. The self-duality property is the heart of our proofs. Although this property of linear systems in sense of Liggett's book ([21]) was shown, and Cox et al. only claimed it held for the parabolic Anderson model driven by Brownian motions if A was doubly stochastic in [6], we are unable to get a quick conclusion from their results, which is suitable for our case. Hence, for the completeness of this paper, we will verify the self-duality property in Appendix briefly. On the other hand, it seems difficult to extend [25, Theorem 2.9] to (1.8), because it is unclear for us whether the comparison theorem in [4] is available for Lévy noise or not.

The remainder of this paper is organized as follows. The main results (Theorems 2.8 and 2.9) are given in Section 2. At the same time, some preliminary propositions, necessary notations, assumptions, and conditions are presented in Section 2, too. In Section 3, the proofs of the main theorems and lemmas are shown. To be self-contained, we give some lemmas of the proofs of Propositions 2.4 and 2.6 in Appendix. The proof of the self-duality property is shown in Appendix, too.

2 Setup and main results

We set

$$L^p(\gamma)=\Big\{x\in[0,\infty)^S\;\Big|\;\|x\|_{\gamma,p}^p=\sum_{i\in S}\gamma_i x_i^p<\infty\Big\},$$

where $p\geqslant 1$, $\gamma=\{\gamma_i\}_{i\in S}\in[0,\infty)^S$ is a strictly positive, summable reference sequence (i.e., $\sum_{i\in S}\gamma_i<\infty$) satisfying for some constant $\Gamma>0$,

$$\sum_{i\in S}\gamma_i|a(i,j)|\leqslant\Gamma\gamma_j,\quad j\in S. \tag{2.1}$$

Moreover, we need the following conditions on Lévy's measure $\rho(\mathrm{d}u)$.

Condition 2.1 $\rho((-\infty,0))=0$.

Condition 2.2

$$\int_{(1,\infty)}u^2\rho(\mathrm{d}u)<\infty.$$

Condition 2.3 For some constant K, $\rho\{u \mid u > K\} = 0$.

Similar to [11, Condition [A]], Conditions 2.1 and 2.2 guarantee that there exists a unique nonnegative solution in $L^2(\gamma)$ with $X_i(0) \geqslant 0,\ i \in S$.

Remark 2.1 In fact, in [11], Furouya and Shiga studied the sample Lyapunov exponent in the sense of $L^1(\gamma)$ solution. However, because we consider the ergodic theory in the $L^2(\gamma)$ framework in the present paper, Condition 2.2 is stronger than the corresponding one (1.6) in [11].

Proposition 2.4 *Assume that inequality* (2.1), *Conditions* 2.1 *and* 2.2 *hold. Let* $X(0) = \{X_i(0),\ i \in S\}$ *be an* $\mathscr{F}_0$*-measurable random vector in* $L^2(\gamma)$ *a.s. Then* (1.5) *has a unique strong solution in* $L^2(\gamma)$, *and satisfies that*

1) *if* $E(\|X(0)\|_{\gamma,2}^2) < \infty$, *then* $E(\|X(t)\|_{\gamma,2}^2) < \infty,\ \forall\ t > 0$;

2) $(X(t))_{t\geqslant 0}$ *is a Feller process*;

3) (1.5) *has a unique nonnegative* $L^2(\gamma)$ *solution, if* $X(0) \in L^2(\gamma)$ *and* $X_i(0) \geqslant 0$ *for all* $i \in S$;

4) *for any* $i \in S,\ X_i(\cdot) \in D[0,\infty;\mathbb{R}]$ *a.s., denote by* $D[0,\infty;\mathbb{R}]$ *the space of functions* $f\colon [0,\infty) \to \mathbb{R}$ *which are right-continuous and admit left-hand limits for every* $t > 0$.

Proof Since Condition 2.2 implies that

$$\int_{(1,\infty)} u\rho(\mathrm{d}u) < \infty,$$

and $E(\|X(0)\|_{\gamma,2}^2) < \infty$ implies $E(\|X(0)\|_{\gamma,1}) < \infty$, this means that if

$$E(\|X(0)\|_{\gamma,2}^2) < \infty,$$

then there is a unique nonnegative $L^1(\gamma)$ pathwise solution of (1.5), which is a Feller process (or see Lemma A5 in Appendix), by [11, Theorem 2.1, Corollary 2.1]. Furthermore, by Lemma A4 in Appendix, if $E(\|X(0)\|_{\gamma,2}^2) < \infty$, then $E(\|X(t)\|_{\gamma,2}^2) < \infty,\ \forall\ t > 0$. And by Lemma A6 in Appendix, we have 4). Hence, the proposition holds. □

Let Ξ_F be the set of $x \in [0,\infty)^S$ such that $x_j = 0$ for all but finitely many $j \in S$, and let $L^{\infty,+}(S)$ be the set of all bounded nonnegative functions on S. Let $X(t,x)$ be the solution of (1.5) with deterministic initial data $x \in L^2(\gamma)$.

Proposition 2.5 (Self-duality) *For any* $x \in L^{\infty,+}(S)$ *and* $\widetilde{x} \in \Xi_F$, *we have*

$$\langle X(t,x), \widetilde{x}\rangle_\pi \stackrel{d}{=} \langle X(t,\widetilde{x}), x\rangle_\pi, \tag{2.2}$$

where $\stackrel{d}{=}$ *denotes the equality in distribution and* $\langle x,y\rangle_\pi = \sum_{i\in S} x_i y_i \pi_i$.

Remark 2.2 The proof of the self-duality property is provided in Appendix. Through this proof, we know that $X(t) \in L^2(\gamma)$ guarantees (2.2) make sense.

Denote by $\{T_t\}_{t\geqslant 0}$ the Feller semigroup of $\{X(t)\}_{t\geqslant 0}$, where T_t satisfies $T_t f(x) = E^x f(X(t))$ for any bounded Borel-measurable function f on $L^2(\gamma)$. Define the continuous time kernel a_t by

$$a_t(i,j) = \sum_{n=0}^{\infty} \frac{t^n}{n!}\, a^{(n)}(i,j),$$

where $a^{(n)}$ is the n-th iterate of a, i.e.,

$$a^{(n)} = \underbrace{a \times a \times \cdots \times a}_{n}.$$

For $\phi\colon\ S \to [0,\infty)$, define

$$a_t\phi(i) = \sum_{j\in S} a_t(i,j)\phi_j, \quad \phi a_t(j) = \sum_{i\in S} \phi_i a_t(i,j),$$

and for $\phi,\varphi\colon\ S \to [0,\infty)$, define

$$\langle \phi, \varphi\rangle = \sum_{i\in S} \phi_i \varphi_i.$$

Let $\mathscr{P}$ be the set of probability measures on $L^2(\gamma)$, and let $\mathscr{I}$ be the set of probability measures which are invariant for $\{X(t)\}_{t\geqslant 0}$, i.e., $\mathscr{I} = \{\mu \in \mathscr{P} \mid T_t^*\mu = \mu \text{ for } t \geqslant 0\}$, where $T_t^*\mu$ satisfies

$$\int f(x)\ T_t^*\mu(\mathrm{d}x) = \int T_t f(x)\ \mu(\mathrm{d}x)$$

for any bounded Borel-measurable function f on $L^2(\gamma)$. $\mathscr{I}_{\rm ex}$ denotes the set of the extreme point of $\mathscr{I}$. We set $\mathscr{H} = \{h(\cdot) \mid h \text{ is a bounded function on } S,\ h(i) \geqslant 0,\ \sum_{j\in S} a(i,j)h(j) = 0\}$. For $\nu,\mu \in \mathscr{P}$, let $\nu\otimes\mu$ be the product measure of ν and μ.

Without loss of generality, we set $\kappa = 1$.

Suppose that $h \in \mathscr{H}$ is fixed, and $\{X(t)\}_{t\geqslant 0}$ is the solution of (1.8) with initial state $X(0) = h$. For any $\widetilde{x} \in \Xi_F$, Proposition 2.5 implies that

$$\langle X(t,h), \widetilde{x}\rangle_\pi \overset{d}{=} \langle h, X(t,\widetilde{x})\rangle_\pi. \tag{2.3}$$

Then we have the following result.

Proposition 2.6 *For $h \in \mathscr{H}$, $\langle h, X(t,\widetilde{x})\rangle_\pi$ is a càdlàg square-integrable martingale.*

The proof of Proposition 2.6 is given in Section 3. From Proposition 2.6, the right-hand side of (2.3) is a nonnegative martingale, and hence converges almost surely as $t \to \infty$. Therefore, the left-hand side of (2.3) must converge as $t \to \infty$. It follows that there is a probability measure ν_h on $L^2(\gamma)$ such that

$$\mathscr{L}[X(t)] \Rightarrow \nu_h, \quad t \to \infty, \tag{2.4}$$

where $\mathscr{L}$ denotes the law of random variable and $\Rightarrow$ denotes the weak convergence of probability measure. By (2.3) and (2.4), we have

$$E^{\widetilde{x}}(\mathrm{e}^{-\langle h, X(t,\widetilde{x})\rangle_\pi}) = E^h(\mathrm{e}^{-\langle X(t,h),\widetilde{x}\rangle_\pi}) \to \int \mathrm{e}^{-\langle x,\widetilde{x}\rangle_\pi} \nu_h(\mathrm{d}x), \quad t \to \infty. \tag{2.5}$$

For $h \in \mathscr{H}$, we define $\mathscr{M}_h$ to be the set of probability measures ν on $L^2(\gamma)$ such that

$$\sup_k \int x_k^2 \nu(\mathrm{d}x) < \infty, \tag{2.6}$$

$$\lim_{t\to\infty} \int (a_t x(k) - h(k))^2 \nu(\mathrm{d}x) = 0, \quad k \in S. \tag{2.7}$$

Let

$$\mathscr{J}_2 = \Big\{ \nu \in \mathscr{P} \Big| \sup_k \int x_k^2 \nu(\mathrm{d}x) < \infty \Big\}.$$

Then we have the following result.

Proposition 2.7 *Let $h \in \mathscr{H}$ and $\nu \in \mathscr{M}_h$. If $\mathscr{L}[X(0)] = \nu$ and $\widetilde{x} \in \Xi_F$, then*

$$\langle X(0) - h, X(t,\widetilde{x})\rangle_\pi \to 0 \tag{2.8}$$

in $\nu \otimes P^{\widetilde{x}}$-probability as $t \to \infty$.

Theorem 2.8 *If $\mathscr{L}[X(0)] \in \mathscr{M}_h$, then $\mathscr{L}[X(t)] \Rightarrow \nu_h$ as $t \to \infty$.*

Proposition 2.7 is a main technical result to show Theorem 2.8. Similar to [6, condition (5.42)], Condition 2.3 is a key assumption to show Proposition 2.7 through [6, Lemma 3.2]. The proofs of Proposition 2.7 and Theorem 2.8 are similar to those of [6, Proposition 2.1, Theorem 2.3], respectively.

Theorem 2.9 *Assume that $(a_t)_{t\geqslant 0}$ is transient and satisfies*

$$G := \sup_{i\in S} \int_0^\infty \sum_k a_t(i,k) a_t(i,k) \mathrm{d}t < \infty. \tag{2.9}$$

Moreover, suppose that $\pi_i > 0,\ i \in S$, and $\sup_{i\in S} \pi_i < \infty$. If

$$\Big(\alpha^2 + \int u^2 \rho(\mathrm{d}u)\Big) G < 1,$$

then

$$\{\mathscr{I} \cap \mathscr{J}_2\}_{\mathrm{ex}} = \{\nu_h \mid h \in \mathscr{H}\}.$$

Remark 2.3 It seems that the assumption of symmetry of A (see (1.2)) is a key technical condition in our proofs. In [6], for the parabolic Andersen model driven by Brownian motions on S, if A is doubly stochastic, i.e., $\sum_{i\in S} a(i,j) = 0$, Cox et al. obtained the partial result of Theorem 2.8. They only showed that $\mathscr{L}[X(t)] \Rightarrow \nu_\theta$, if θ is a constant function on S. However, we are not able to prove Theorems 2.8 and 2.9 under the same assumption as theirs by now.

Remark 2.4 Assumption (2.9) in Theorem 2.9 is also a technical condition, but it implies some interesting cases. For example, suppose that $S = \mathbb{Z}^d$ (d-dimensional cubic lattice space) and $(a(i,j))_{i,j\in S}$ is transient and satisfies

$$a(i,j) = a(j,i), \quad a(i,j) = a(0, j-i), \quad i,j \in \mathbb{Z}^d,$$

then

$$\begin{aligned}\sup_i \int_0^\infty \sum_k a_t(i,k)a_t(i,k)\mathrm{d}t &= \sup_i \int_0^\infty a_{2t}(i,i)\mathrm{d}t \\ &= \sup_i \int_0^\infty a_{2t}(0,0)\mathrm{d}t \\ &= \int_0^\infty a_{2t}(0,0)\mathrm{d}t \\ &< \infty.\end{aligned}$$

(2.9) is satisfied naturally.

Moreover, if $0 < \inf_{i\in S}\pi_i \leqslant \sup_{i\in S}\pi_i < \infty$ and $(a(i,j))_{i,j\in S}$ is transient, then

$$\begin{aligned}(\inf_i \pi_i)\int_0^\infty \sum_k a_t(i,k)a_t(i,k)\mathrm{d}t &\leqslant \pi_i \int_0^\infty \sum_k a_t(i,k)a_t(i,k)\mathrm{d}t \\ &= \int_0^\infty \sum_k a_t(i,k)\pi_k a_t(k,i)\mathrm{d}t \\ &\leqslant (\sup_k \pi_k)\int_0^\infty a_{2t}(i,i)\mathrm{d}t,\end{aligned}$$

and hence,

$$\sup_i \int_0^\infty \sum_k a_t(i,k)a_t(i,k)\mathrm{d}t \leqslant \frac{\sup_k \pi_k}{\inf_i \pi_i} \sup_i \int_0^\infty a_{2t}(i,i)\mathrm{d}t. \tag{2.10}$$

Similarly, we get

$$\sup_i \int_0^\infty \sum_k a_t(i,k)a_t(i,k)\mathrm{d}t \geqslant \frac{\inf_k \pi_k}{\sup_i \pi_i} \sup_i \int_0^\infty a_{2t}(i,i)\mathrm{d}t. \tag{2.11}$$

Hence, (2.10) and (2.11) yield that

$$\sup_{i\in S} \int_0^\infty a_t(i,i)\mathrm{d}t < \infty \Longleftrightarrow \sup_{i\in S} \int_0^\infty \sum_k a_t(i,k)a_t(i,k)\mathrm{d}t < \infty. \tag{2.12}$$

Remark 2.5 Since S is a countable set, A is the generator of a Markov chain, in general, there exist non-constant bounded nonnegative harmonic functions of A, for example, the isotropic random walk on a homogeneous tree (see [28]).

Remark 2.6 In [15], for the parabolic Anderson model driven by Brownian motions on $\mathbb{Z}^d$, if the symmetrized transition kernel of A is transient, Greven and den Hollander pointed out that there exists a $b^* > 0$, such that if the diffusion parameter $\alpha > b^*$, then the system locally dies out. We believe that the similar result holds for (1.8) also, i.e., there is a $b^* > 0$, if

$$\alpha^2 + \int u^2 \rho(\mathrm{d}u) > b^*,$$

then the system locally dies out. However, we have not got a proof till now.

3 Proofs

3.1 Proofs of Propositions 2.6, 2.7, and Theorem 2.8

Proof of Proposition 2.6 It is easy to check that

$$\begin{aligned}\langle h, X(t,\widetilde{x})\rangle_\pi = \langle h, \widetilde{x}\rangle_\pi + \alpha \sum_{i\in S} \int_0^t \pi_i h(i) X_i(s,\widetilde{x}) \mathrm{d}B_i(s) \\ + \sum_{i\in S} \int_0^t \int_{\mathbb{R}\backslash\{0\}} u\pi_i h(i) X_i(s-,\widetilde{x}) \widetilde{N}_i(\mathrm{d}s\mathrm{d}u).\end{aligned} \tag{3.1}$$

By Lemma A7 in Appendix, we have

$$E^{\widetilde{x}}(\langle h, X(t,\widetilde{x})\rangle_\pi^2) \leqslant \langle h, \widetilde{x}\rangle_\pi^2 \mathrm{e}^{ct} < \infty,$$

where $c = \alpha^2 + \int u^2 \rho(\mathrm{d}u)$.

On the other hand, By Lemma A8 in Appendix, $\langle h, X(t,\widetilde{x})\rangle_\pi$ is a càdlàg process. Therefore, it follows from (3.1) that for $\widetilde{x} \in \Xi_F$, $\langle h, X(t,\widetilde{x})\rangle_\pi$ is a càdlàg square-integrable martingale. □

Proof of Proposition 2.7 We note that $\langle X(t,\widetilde{x}), 1\rangle_\pi$ is a nonnegative càdlàg square-integrable martingale. By (3.1), we have

$$\langle X(t,\widetilde{x}), 1\rangle_\pi = \langle \widetilde{x}, 1\rangle_\pi + \sum_i M_i(t),$$

where

$$M_i(t) = \alpha \int_0^t \pi_i X_i(s,\widetilde{x}) \mathrm{d}B_i(s) + \int_0^t \int_{\mathbb{R}\backslash\{0\}} u\pi_i X_i(s-,\widetilde{x}) \widetilde{N}_i(\mathrm{d}s\mathrm{d}u),$$

and the sum converges in $L^2(\gamma)$. $\{M_i(t)\}_{i\in S}$ are square integrable orthogonal martingales, with

$$\langle M_i\rangle_t := \langle M_i, M_i\rangle_t = \left(\alpha^2 + \int u^2 \rho(\mathrm{d}u)\right) \int_0^t \pi_i^2 X_i^2(s,\widetilde{x}) \mathrm{d}s.$$

$\langle X(t,\widetilde{x}),1\rangle_\pi$ has the predictable quadratic variation process $A_t=\sum_i\langle M_i\rangle_t$. Let $T_n=\inf\{t\mid\langle X(t,\widetilde{x}),1\rangle_\pi\geqslant n\}$. We set $Z_t=\langle X(t,\widetilde{x}),1\rangle_\pi$. By Condition 2.3, it can be shown that for each positive integer n,

$$E[(Z_{T_n}-Z_{T_n-})1_{\{T_n<\infty\}}]<\infty.$$

Then by [6, Lemma 3.2], we have

$$\begin{aligned}A_\infty&=\sum_i\langle M_i\rangle_\infty\\&=\left(\alpha^2+\int u^2\rho(\mathrm{d}u)\right)\int_0^\infty\sum_i\pi_i^2X_i^2(s,\widetilde{x})\mathrm{d}s<\infty\quad\text{a.s. }P^{\widetilde{x}}.\end{aligned}\tag{3.2}$$

Applying Itô's formula to $a_{t-s}X(s,\widetilde{x})(i)$, we have

$$X_i(t,\widetilde{x})=a_t\widetilde{x}(i)+\int_0^t\sum_j a_{t-s}(i,j)d\overline{M}_j(s),\tag{3.3}$$

where

$$\overline{M}_i(t)=\alpha\int_0^tX_i(s,\widetilde{x})\mathrm{d}B_i(s)+\int_0^t\int_{\mathbb{R}\setminus\{0\}}uX_i(s-,\widetilde{x})\widetilde{N}_i(\mathrm{d}s\mathrm{d}u).$$

Let

$$\Lambda=X(0)-h,\quad \mathbf{P}=\nu\otimes P^{\widetilde{x}},$$

and let $\mathbf{E}$ denote expectation with respect to $\mathbf{P}$. Noting that $\nu\in\mathscr{M}_h$, we have

$$\begin{aligned}C:=&\sup_i\int(\Lambda(i))^2\nu(\mathrm{d}x)\\=&\sup_i\int(x_i-h(i))^2\nu(\mathrm{d}x)\\\leqslant&\,2\sup_i\int x_i^2\nu(\mathrm{d}x)+2\sup_i\int h^2(i)\nu(\mathrm{d}x)\\<&\,\infty.\end{aligned}\tag{3.4}$$

Hence,

$$\begin{aligned}\sup_{i,t}\int(a_t\Lambda(i))^2\nu(\mathrm{d}x)=&\sup_{i,t}\int\left(\sum_k a_t(i,k)\Lambda_k\right)^2\nu(\mathrm{d}x)\\=&\sup_{i,t}\int\left(\sum_k a_t(i,k)(x_k-h(k))\right)^2\nu(\mathrm{d}x)\\\leqslant&\sup_{i,t}\int\sum_k a_t(i,k)(x_k-h(k))^2\nu(\mathrm{d}x)\\\leqslant&\sup_{i,t}\sum_k a_t(i,k)\sup_k\int(x_k-h(k))^2\nu(\mathrm{d}x)\\=&\,C,\end{aligned}\tag{3.5}$$

and

$$\mathbf{E}\bigg(\sum_i \int_0^t (a_{t-s}\Lambda(i))^2\pi_i^2 X_i^2(s,\widetilde{x})\mathrm{d}s\bigg) \leqslant C\int_0^t E^{\widetilde{x}}(\langle X(s),1\rangle_\pi^2)\mathrm{d}s < \infty. \tag{3.6}$$

Set

$$N_s^t = \sum_i \int_0^s a_{t-r}\Lambda(i)\mathrm{d}M_i(r), \quad s \leqslant t. \tag{3.7}$$

Then

$$\langle \Lambda, X(t,\widetilde{x})\rangle_\pi = \langle \Lambda, a_t\widetilde{x}\rangle_\pi + N_t^t. \tag{3.8}$$

Here, $\{N_s^t,\ s\leqslant t\}$ is a square-integrable martingale under $\mathbf{P}$, and

$$\langle N^t\rangle_s := \langle N^t, N^t\rangle_s = \int_0^s \sum_i (a_{t-r}\Lambda(i))^2\mathrm{d}\langle M_i\rangle_r < \infty \quad \text{a.s. } P^{\widetilde{x}}.$$

First, we will show that

$$\langle \Lambda, a_t\widetilde{x}\rangle_\pi \to 0 \quad \text{in } \nu\otimes P^{\widetilde{x}}\text{-probability as } t\to\infty. \tag{3.9}$$

It is straightforward to check that

$$\begin{aligned}\int \langle \Lambda, a_t\widetilde{x}\rangle_\pi^2\nu(\mathrm{d}x) &= \int \langle a_t\Lambda, \widetilde{x}\rangle_\pi^2\nu(\mathrm{d}x)\\ &= \sum_{j,k}\widetilde{x}_j\widetilde{x}_k\pi_j\pi_k\int (a_t\Lambda(j))(a_t\Lambda(k))\nu(\mathrm{d}x)\\ &\leqslant \bigg(\sum_j \widetilde{x}_j\pi_j^2\bigg[\int (a_t\Lambda(j))^2\nu(\mathrm{d}x)\bigg]^{1/2}\bigg)^2.\end{aligned} \tag{3.10}$$

Using (2.7), (3.5), and $\widetilde{x}\in\Xi_F$, we can obtain

$$\int \langle \Lambda, a_t\widetilde{x}\rangle_\pi^2\nu(\mathrm{d}x) \to 0, \quad t\to\infty. \tag{3.11}$$

Hence, (3.9) holds.

Next, let us show that

$$N_t^t \to 0 \quad \text{in } \nu\otimes P^{\widetilde{x}}\text{-probability as } t\to\infty. \tag{3.12}$$

To this end, we consider

$$\begin{aligned}\int \langle N^t\rangle_t\nu(\mathrm{d}x) &= \int_0^t \sum_i \int (a_{t-r}\Lambda(i))^2\nu(\mathrm{d}x)\mathrm{d}\langle M_i\rangle_r\\ &= \sum_i \int_0^\infty \bigg(I_{\{r<t\}}\int (a_{t-r}\Lambda(i))^2\nu(\mathrm{d}x)\bigg)\mathrm{d}\langle M_i\rangle_r.\end{aligned} \tag{3.13}$$

Note that

$$I_{\{r<t\}}\int(a_{t-r}\Lambda(i))^2\nu(\mathrm{d}x)\to 0,\quad t\to\infty,$$

and is bounded by C. By (3.2) and the bounded convergence theorem, we have

$$P^{\widetilde{x}}\Big(\lim_{t\to\infty}\int\langle N^t\rangle_t\nu(\mathrm{d}x)=0\Big)=1. \tag{3.14}$$

Thus,

$$\mathbf{P}(\langle N^t\rangle_t>\varepsilon')=E^{\widetilde{x}}(\nu(\langle N^t\rangle_t>\varepsilon'))\to 0,\quad t\to\infty. \tag{3.15}$$

Let

$$\tau_\varepsilon=\inf\{s\geqslant 0\mid \langle N^t\rangle_s>\varepsilon^3\}.$$

Note that $\{|N_s^t|^2-\langle N^t\rangle_s,\ s\leqslant t\}$ is a martingale under $\mathbf{P}$. By the stopping time theorem, we have

$$\mathbf{E}(|N^t_{t\wedge\tau_\varepsilon}|^2)=\mathbf{E}(\langle N^t\rangle_{t\wedge\tau_\varepsilon}).$$

Then,

$$\begin{aligned}\mathbf{P}(|N_t^t|>\varepsilon)&\leqslant\mathbf{P}(\sup_{0\leqslant s\leqslant t}|N_s^t|>\varepsilon)\\&\leqslant\mathbf{P}(\sup_{0\leqslant s\leqslant t}|N_s^t-N_s^tI_{\{s<\tau_\varepsilon\}}|>0)+\mathbf{P}(\sup_{0\leqslant s\leqslant t}|N_s^tI_{\{s<\tau_\varepsilon\}}|>\varepsilon)\\&\leqslant\mathbf{P}(\langle N^t\rangle_t>\varepsilon^3)+\frac{\mathbf{E}(|N^t_{t\wedge\tau_\varepsilon}|^2)}{\varepsilon^2}\\&\leqslant\mathbf{P}(\langle N^t\rangle_t>\varepsilon^3)+\frac{\mathbf{E}(\langle N^t\rangle_{t\wedge\tau_\varepsilon})}{\varepsilon^2}\\&\leqslant\mathbf{P}(\langle N^t\rangle_t>\varepsilon^3)+\frac{\varepsilon^3}{\varepsilon^2}\\&=\mathbf{P}(\langle N^t\rangle_t>\varepsilon^3)+\varepsilon.\end{aligned} \tag{3.16}$$

Clearly, the claim (3.12) follows from (3.15) and (3.16). Therefore, it follows from (3.9) and (3.12) that

$$\langle X(0)-h,X(t,\widetilde{x})\rangle_\pi\to 0\quad\text{in }\nu\otimes P^{\widetilde{x}}\text{-probability as }t\to\infty.$$ □

Proof of Theorem 2.8 For any $\widetilde{x}\in\Xi_F$, let $X(0,\widetilde{x})\equiv\widetilde{x}$. Then we have

$$\begin{aligned}\mathbf{E}(\mathrm{e}^{-\langle X(t),\widetilde{x}\rangle_\pi})&=E^\nu\otimes E^{\widetilde{x}}(\mathrm{e}^{-\langle X(0),X(t,\widetilde{x})\rangle_\pi})\\&=E^\nu\otimes E^{\widetilde{x}}(\mathrm{e}^{-\langle X(0)-h,X(t,\widetilde{x})\rangle_\pi}\mathrm{e}^{-\langle h,X(t,\widetilde{x})\rangle_\pi}).\end{aligned} \tag{3.17}$$

From Proposition 2.7, it is easy to see that

$$\mathrm{e}^{-\langle X(0)-h,X(t,\widetilde{x})\rangle_\pi}\to 1\quad\text{in }\nu\otimes P^{\widetilde{x}}\text{-probability as }t\to\infty. \tag{3.18}$$

(3.17) and (3.18) imply that

$$\lim_{t\to\infty}E^\nu(\mathrm{e}^{-\langle X(t),\widetilde{x}\rangle_\pi})=\lim_{t\to\infty}E^{\widetilde{x}}(\mathrm{e}^{-\langle h,X(t,\widetilde{x})\rangle_\pi}). \tag{3.19}$$

It follows from (2.5) that

$$\lim_{t\to\infty} E^{\nu}(\mathrm{e}^{-\langle X(t),\widetilde{x}\rangle_\pi}) = \int \mathrm{e}^{-\langle x,\widetilde{x}\rangle_\pi}\nu_h(\mathrm{d}x). \tag{3.20}$$

Therefore, $\mathscr{L}[X(t)] \Rightarrow \nu_h$ as $t\to\infty$. □

3.2 Proof of Theorem 2.9

We need the following lemmas to prove Theorem 2.9.

Lemma 3.1 *If $h\in\mathscr{H}$, then $\nu_h\in\mathscr{M}_h$.*

Lemma 3.2 *If $\mu\in(\mathscr{I}\cap\mathscr{J}_2)_{\mathrm{ex}}$, then*

$$\lim_{t\to\infty}\sum_{k,l} a_t(i,k)a_t(j,l)\int x_kx_l\mu(\mathrm{d}x) = h(i)h(j),\quad i,j\in S,$$

where $h(i)=\int x_i\mu(\mathrm{d}x)$, $i\in S$.

The proofs of Lemmas 3.1 and 3.2 will be given later.

Proof of Theorem 2.9 The proof will be done in two steps.

Step 1 We will show that $\forall\ h\in\mathscr{H}$, $\nu_h\in(\mathscr{I}\cap\mathscr{J}_2)_{\mathrm{ex}}$. From Lemma 3.1 and Theorem 2.8, we know that $\nu_h\in\mathscr{I}\cap\mathscr{J}_2$. Suppose $\nu_h=\lambda\mu+(1-\lambda)\nu$, where $\mu,\nu\in\mathscr{I}\cap\mathscr{J}_2$ and $0<\lambda<1$. Then

$$\begin{aligned}\int\Big(\sum_k a_s(i,k)x_k-h(i)\Big)^2\nu_h(\mathrm{d}x) &= \lambda\int\Big(\sum_k a_s(i,k)x_k-h(i)\Big)^2\mu(\mathrm{d}x)+(1-\lambda)\\ &\quad\cdot\int\Big(\sum_k a_s(i,k)x_k-h(i)\Big)^2\nu(\mathrm{d}x).\end{aligned} \tag{3.21}$$

From Lemma 3.1, the left-hand side of (3.21) converges to 0 as $t\to\infty$, and then each term on the right-hand side of (3.21) must converge to 0 as $t\to\infty$. Noting that $\mu,\nu\in\mathscr{J}_2$, we get $\mu,\nu\in\mathscr{M}_h$. It follows from Theorem 2.8 that $T_t^*\mu\Rightarrow\nu_h$ and $T_t^*\nu\Rightarrow\nu_h$ as $t\to\infty$. Since $\mu,\nu\in\mathscr{I}$, we obtain $\mu=\nu=\nu_h$. This shows that $\nu_h\in(\mathscr{I}\cap\mathscr{J}_2)_{\mathrm{ex}}$.

Step 2 We will show that $(\mathscr{I}\cap\mathscr{J}_2)_{\mathrm{ex}}\subset\{\nu_h\mid h\in\mathscr{H}\}$. $\forall\ \mu\in(\mathscr{I}\cap\mathscr{J}_2)_{\mathrm{ex}}$, we need to show $\mu=\nu_h$ for some $h\in\mathscr{H}$. Let $h(i)=\int x_i\mu(\mathrm{d}x)$, $i\in S$. Using the same method as the proof of [5, Lemma 1], we have

$$E^{\mu}X_i(t)=\sum_k a_t(i,k)\int x_k\mu(\mathrm{d}x)=\sum_k a_t(i,k)h(k). \tag{3.22}$$

Since μ is invariant, (3.22) implies that $h\in\mathscr{H}$.

In order to show $\mu=\nu_h$, from Theorem 2.8 and $\mu\in\mathscr{I}$, it suffices to show $\mu\in\mathscr{M}_h$. Since $\mu\in\mathscr{J}_2$, it remains to show

$$\lim_{t\to\infty}\int\Big(\sum_k a_t(i,k)x_k-h(i)\Big)^2\mu(\mathrm{d}x)=0,\quad i\in S. \tag{3.23}$$

It follows from Lemma 3.2 and (3.22) that

$$\begin{aligned}
&\lim_{t\to\infty}\int\Big(\sum_k a_t(i,k)x_k-h(i)\Big)^2\mu(\mathrm{d}x)\\
&=\lim_{t\to\infty}\Big(\sum_{k,l}a_t(i,k)a_t(i,l)\int x_kx_l\mu(\mathrm{d}x)-2h(i)\sum_k a_t(i,k)\int x_k\mu(\mathrm{d}x)+h^2(i)\Big)\\
&=\lim_{t\to\infty}\sum_{k,l}a_t(i,k)a_t(i,l)\int x_kx_l\mu(\mathrm{d}x)-h^2(i)\\
&=0. \qquad (3.24)
\end{aligned}$$

Hence, (3.23) holds. □

Proof of Lemma 3.1 In order to show $\nu_h\in\mathscr{M}_h$, we need to show

$$\sup_i\int x_i^2\nu_h(\mathrm{d}x)<\infty, \qquad (3.25)$$

$$\lim_{s\to\infty}\int\Big(\sum_k a_s(i,k)x_k-h(i)\Big)^2\nu_h(\mathrm{d}x)=0,\quad i\in S. \qquad (3.26)$$

First, we use the similar method to the proof of [31, Lemma 2.1] to prove (3.25). Let $f_{ij}(t)=E^{\delta_h}(X_i(t)X_j(t))$. Applying Itô's formula to $f_{ij}(t)$, we get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}f_{ij}(t)=&\sum_k a(i,k)E^{\delta_h}(X_k(t)X_j(t))+\sum_l a(j,l)E^{\delta_h}(X_i(t)X_l(t))\\
&+\Big(\alpha^2+\int u^2\rho(\mathrm{d}u)\Big)\delta(i,j)E^{\delta_h}(X_i^2(t))\\
=&(A_1+A_2)f_{ij}(t)+\Big(\alpha^2+\int u^2\rho(\mathrm{d}u)\Big)\delta(i,j)f_{ij}(t), \qquad (3.27)
\end{aligned}$$

where

$$(A_1+A_2)f_{ij}=\sum_k a(i,k)f_{kj}+\sum_l a(j,l)f_{il}.$$

Denote by $(\xi_t,P_{(i,j)})$ the continuous time Markov chain on $S\times S$, which has the generator A_1+A_2 and the transition probability $a_t\otimes a_t$, where

$$a_t\otimes a_t((i,j),(k,l))=a_t(i,k)a_t(j,l).$$

We set $b=\alpha^2+\int u^2\rho(\mathrm{d}u)$. Applying Feynman-Kac's formula, we have

$$f_{ij}(t)=E_{(i,j)}\Big(f_{\xi_t}(0)\exp\Big(\int_0^t bI_\triangle(\xi_s)\mathrm{d}s\Big)\Big), \qquad (3.28)$$

where $\triangle = \{(i,j) \in S \times S \mid i = j\}$. Let $\overline{C} = \max_{i \in S} h(i)$. Then $f_{ij}(0) \leqslant \overline{C}^2$ for all $i, j \in S$. Hence,

$$\begin{aligned} E^{\delta_h}(X_i^2(t)) &= f_{ii}(t) \\ &= E_{(i,i)}\Big(f_{\xi_t}(0) \exp\Big(\int_0^t bI_\triangle(\xi_s)\mathrm{d}s \Big)\Big) \\ &\leqslant \overline{C}^2 E_{(i,i)}\Big(\exp\Big(\int_0^t bI_\triangle(\xi_s)\mathrm{d}s \Big)\Big). \end{aligned} \tag{3.29}$$

Using Taylor's expansion for the exponential function and combing it with (2.9), we have

$$\begin{aligned} &E_{(i,i)}\Big(\exp\Big(\int_0^t bI_\triangle(\xi_s)\mathrm{d}s \Big)\Big) \\ &= E_{(i,i)}\Big(\sum_{n=0}^\infty \frac{\big(\int_0^t bI_\triangle(\xi_s)\mathrm{d}s\big)^n}{n!} \Big) \\ &= E_{(i,i)}\Big(\sum_{n=0}^\infty b^n \int\cdots\int_{0<t_1<\cdots<t_n<t} I_\triangle(\xi_{t_1})\cdots I_\triangle(\xi_{t_n})\mathrm{d}t_1\mathrm{d}t_2\cdots\mathrm{d}t_n \Big) \\ &= \sum_{n=0}^\infty b^n \int\cdots\int_{0<t_1<\cdots<t_n<t} P_{(i,i)}\big(\xi_{t_1}\in\triangle,\cdots,\xi_{t_n}\in\triangle\big)\mathrm{d}t_1\mathrm{d}t_2\cdots\mathrm{d}t_n \\ &= \sum_{n=0}^\infty b^n \int\cdots\int_{0<t_1<\cdots<t_n<t} \sum_{k_1\cdots k_n} a_{t_1}(i,k_1)a_{t_1}(i,k_1)\cdots a_{t_n-t_{n-1}}(k_{n-1},k_n) \\ &\quad \cdot a_{t_n-t_{n-1}}(k_{n-1},k_n)\mathrm{d}t_1\mathrm{d}t_2\cdots\mathrm{d}t_n \\ &\leqslant G\sum_{n=0}^\infty b^n \int\cdots\int_{0<t_1<\cdots<t_{n-1}<t} \sum_{k_1\cdots k_{n-1}} a_{t_1}(i,k_1)a_{t_1}(i,k_1)\cdots a_{t_{n-1}-t_{n-2}}(k_{n-2},k_{n-1}) \\ &\quad \cdot a_{t_{n-1}-t_{n-2}}(k_{n-2},k_{n-1})\mathrm{d}t_1\mathrm{d}t_2\cdots\mathrm{d}t_{n-1} \\ &\leqslant \sum_{n=0}^\infty (bG)^n \\ &< \infty. \end{aligned} \tag{3.30}$$

Let

$$L = \overline{C}^2 \sum_{n=0}^\infty (bG)^n.$$

Then L is a constant independent of i. Hence, (3.29) and (3.30) imply that for all $i \in S$,

$$E^{\delta_h}(X_i^2(t)) = f_{ii}(t) \leqslant L < \infty. \tag{3.31}$$

Since $T_t^*\delta_h \Rightarrow \nu_h$, we have

$$\int (x_i^2 \wedge N)\nu_h(\mathrm{d}x) = \lim_{t\to\infty} E^{\delta_h}(X_i^2(t) \wedge N) \leqslant \liminf_{t\to\infty} E^{\delta_h}(X_i^2(t)) \leqslant L. \tag{3.32}$$

It follows from (3.32) that (3.25) holds.

Next, let us prove (3.26). It is straightforward to check that

$$\begin{aligned}
&\int \Big(\sum_k a_s(i,k)x_k - h(i)\Big)^2 \nu_h(\mathrm{d}x) \\
&\leqslant \liminf_{t\to\infty} \int \Big(\sum_k a_s(i,k)x_k - h(i)\Big)^2 T_t^*\delta_h(\mathrm{d}x) \\
&= \liminf_{t\to\infty} \Big[\sum_{k,l} a_s(i,k)a_s(i,l)E^{\delta_h}(X_k(t)X_l(t)) \\
&\quad -2h(i)\sum_k a_s(i,k)E^{\delta_h}X_k(t)\Big] + h^2(i).
\end{aligned} \tag{3.33}$$

Similar to (3.22) and using the fact that $h \in \mathscr{H}$, we get

$$E^{\delta_h}X_k(t) = \sum_l a_t(k,l)h(l) = h(k), \quad k \in S, \tag{3.34}$$

$$E^{\delta_h}(X_k(t)X_l(t)) = h(k)h(l) + b\int_0^t \sum_m a_{t-r}(k,m)a_{t-r}(l,m)E^{\delta_h}(X_m^2(r))\mathrm{d}r, \tag{3.35}$$

where $b = \alpha^2 + \int u^2\rho(\mathrm{d}u)$. Inserting (3.34) and (3.35) into (3.33), and combining with (3.31) and (2.9), we have

$$\begin{aligned}
&\int \Big(\sum_k a_s(i,k)x_k - h(i)\Big)^2 \nu_h(\mathrm{d}x) \\
&\leqslant b\liminf_{t\to\infty} \sum_{k,l} a_s(i,k)a_s(i,l)\int_0^t \sum_m a_{t-r}(k,m)a_{t-r}(l,m)E^{\delta_h}(X_m^2(r))\mathrm{d}r \\
&\leqslant bL\liminf_{t\to\infty}\int_0^t \sum_m a_{t+s-r}(i,m)a_{t+s-r}(i,m)\mathrm{d}r \\
&\leqslant bL\int_s^\infty \sum_m a_r(i,m)a_r(i,m)\mathrm{d}r \\
&\to 0, \quad s \to \infty.
\end{aligned} \tag{3.36}$$

Hence, (3.26) holds. □

Proof of Lemma 3.2 **Step 1** We will show that

$$\lim_{t\to\infty} \sum_{k,l} a_t(i,k)a_t(j,l)\int x_k x_l \mu(\mathrm{d}x)$$

exists. Similar to (3.35), for any $t > 0$, we have

$$E^{\mu}(X_i(t)X_j(t)) = \sum_{k,l} a_t(i,k)a_t(j,l)\int x_k x_l \mu(\mathrm{d}x) + b\int_0^t \sum_k a_{t-r}(i,k)a_{t-r}(j,k)E^{\mu}(X_k^2(r))\mathrm{d}r. \tag{3.37}$$

Since μ is invariant, (3.37) can be rewritten as

$$\int x_i x_j \mu(\mathrm{d}x) = \sum_{k,l} a_t(i,k)a_t(j,l)\int x_k x_l \mu(\mathrm{d}x) + b\int_0^t \sum_k a_r(i,k)a_r(j,k)\int x_k^2 \mu(\mathrm{d}x)\mathrm{d}r. \tag{3.38}$$

Noting that $\mu \in \mathscr{I}_2$, we have

$$\begin{aligned}\int_0^\infty \sum_k a_r(i,k)a_r(j,k)\int x_k^2\mu(\mathrm{d}x)\mathrm{d}r &\leqslant \frac{\widetilde{C}}{\pi_i}\int_0^\infty \sum_k \pi_k a_r(k,i)a_r(j,k)\mathrm{d}r \\ &\leqslant \frac{\widetilde{C}}{\pi_i}\max_k \pi_k \int_0^\infty a_{2r}(j,i)\mathrm{d}r \\ &< \infty,\end{aligned} \tag{3.39}$$

where

$$\widetilde{C} = \sup_k \int x_k^2 \mu(\mathrm{d}x).$$

(3.39) means that

$$\int_0^t \sum_k a_r(i,k)a_r(j,k)\int x_k^2\mu(\mathrm{d}x)\mathrm{d}r$$

has limit as $t \to \infty$. It follows from (3.38) that

$$\lim_{t\to\infty} \sum_{k,l} a_t(i,k)a_t(j,l)\int x_k x_l \mu(\mathrm{d}x)$$

exists.

Step 2 We will show that for some increasing sequence $\{t_n\}$ tending to ∞,

$$h(i)h(j) = \lim_{n\to\infty}\frac{1}{t_n}\int_0^{t_n}\int (T_s x_j)x_i\mu(\mathrm{d}x)\mathrm{d}s. \tag{3.40}$$

Fix $i \in S$. Noting that

$$\int x_i\mu(\mathrm{d}x) = h(i), \quad \int x_i^2\mu(\mathrm{d}x) < \infty,$$

we have

$$\lim_{m\to\infty}\int(x_i\wedge m)\mu(\mathrm{d}x)=h(i),\quad \lim_{m\to\infty}\int x_i^2 I_{\{x_i>m\}}\mu(\mathrm{d}x)=0.$$

Then $\forall\ \varepsilon>0,\ \exists\ N$ large enough such that

$$\int(x_i\wedge N)\mu(\mathrm{d}x)>h(i)-\varepsilon, \tag{3.41}$$

$$\int x_i^2 I_{\{x_i>N\}}\mu(\mathrm{d}x)<\varepsilon^2. \tag{3.42}$$

For arbitrary $\delta\in(0,1/2)$, we define two probability measures μ_1 and μ_2 by

$$\langle\mu_1,f\rangle=\frac{\int f(x)(\frac{x_i\wedge N}{2N}+\delta)\mu(\mathrm{d}x)}{\int(\frac{x_i\wedge N}{2N}+\delta)\mu(\mathrm{d}x)},\quad \forall\ f\in\mathscr{B}_b(L^2(\gamma)),$$

$$\langle\mu_2,f\rangle=\frac{\int f(x)(1-\frac{x_i\wedge N}{2N}-\delta)\mu(\mathrm{d}x)}{\int(1-\frac{x_i\wedge N}{2N}-\delta)\mu(\mathrm{d}x)},\quad \forall\ f\in\mathscr{B}_b(L^2(\gamma)),$$

where $\mathscr{B}_b(L^2(\gamma))$ is the set of all bounded Borel-measurable functions on $L^2(\gamma)$. Then it holds that $\mu=\lambda\mu_1+(1-\lambda)\mu_2$, where

$$\lambda=\int\Big(\frac{x_i\wedge N}{2N}+\delta\Big)\mu(\mathrm{d}x)\in(0,1).$$

Since $\mu\in\mathscr{J}_2,\ \mu_1,\mu_2\in\mathscr{J}_2$. Then, it can be shown that both

$$\frac{1}{t}\int_0^t\int T_s x_i^2\mu_1(\mathrm{d}x)\mathrm{d}s\quad\text{and}\quad\frac{1}{t}\int_0^t\int T_s x_i^2\ \mu_2(\mathrm{d}x)\mathrm{d}s$$

are uniformly bounded in i, t. Thus,

$$\frac{1}{t}\int_0^t T_s^*\mu_1\mathrm{d}s,\quad \frac{1}{t}\int_0^t T_s^*\mu_2\mathrm{d}s$$

are tight, and therefore, for some increasing sequence $\{t_n\}$ tending to ∞,

$$\lim_{n\to\infty}\frac{1}{t_n}\int_0^{t_n}T_s^*\mu_1\mathrm{d}s=\overline{\mu}_1\in\mathscr{I},\quad \lim_{n\to\infty}\frac{1}{t_n}\int_0^{t_n}T_s^*\mu_2\mathrm{d}s=\overline{\mu}_2\in\mathscr{I}.$$

Then $\mu=\lambda\overline{\mu}_1+(1-\lambda)\overline{\mu}_2$. It follows from $\mu\in\mathscr{I}_{\rm ex}$ that $\mu=\overline{\mu}_1=\overline{\mu}_2$.

Noting that

$$\frac{1}{t_n}\int_0^{t_n}T_s^*\mu_1\mathrm{d}s\Rightarrow\overline{\mu}_1=\mu$$

and $\mu\in\mathscr{J}_2$, we have

$$\lim_{n\to\infty}\frac{1}{t_n}\int_0^{t_n}\langle T_s^*\mu_1,x_j\rangle\mathrm{d}s=\langle\mu,x_j\rangle=\int x_j\mu(\mathrm{d}x). \tag{3.43}$$

By the definition of μ_1, we have

$$\int\Big(\frac{x_i\wedge N}{2N}+\delta\Big)\mu(\mathrm{d}x)\int x_j\mu(\mathrm{d}x)$$
$$=\lim_{n\to\infty}\frac{1}{t_n}\int_0^{t_n}\int(T_sx_j)\Big(\frac{x_i\wedge N}{2N}+\delta\Big)\mu(\mathrm{d}x)\mathrm{d}s. \tag{3.44}$$

Then, (3.44) yields

$$\Big(\frac{1}{2N}\int(x_i\wedge N)\mu(\mathrm{d}x)+\delta\Big)h(j)$$
$$=\lim_{n\to\infty}\frac{1}{t_n}\int_0^{t_n}\Big(\int(T_sx_j)\frac{x_i\wedge N}{2N}\mu(\mathrm{d}x)+\delta\int(T_sx_j)\mu(\mathrm{d}x)\Big)\mathrm{d}s$$
$$=\lim_{n\to\infty}\frac{1}{t_n}\int_0^{t_n}\Big(\int(T_sx_j)\frac{x_i\wedge N}{2N}\mu(\mathrm{d}x)+\delta h(j)\Big)\mathrm{d}s$$
$$=\frac{1}{2N}\lim_{n\to\infty}\frac{1}{t_n}\int_0^{t_n}\int(T_sx_j)(x_i\wedge N)\mu(\mathrm{d}x)\mathrm{d}s+\delta h(j). \tag{3.45}$$

Hence, it follows from (3.45) that

$$\int(x_i\wedge N)\mu(\mathrm{d}x)h(j)=\lim_{n\to\infty}\frac{1}{t_n}\int_0^{t_n}\int(T_sx_j)(x_i\wedge N)\mu(\mathrm{d}x)\mathrm{d}s. \tag{3.46}$$

On the one hand, by the Hölder inequality and (3.42), we have

$$\lim_{n\to\infty}\frac{1}{t_n}\int_0^{t_n}\int(T_sx_j)(x_i\wedge N)\mu(\mathrm{d}x)\mathrm{d}s$$
$$=\lim_{n\to\infty}\frac{1}{t_n}\int_0^{t_n}\Big(\int(T_sx_j)x_i\mu(\mathrm{d}x)-\int(T_sx_j)(x_i-N)I_{\{x_i>N\}}\mu(\mathrm{d}x)\Big)\mathrm{d}s$$
$$=\lim_{n\to\infty}\frac{1}{t_n}\int_0^{t_n}\Big(\int(T_sx_j)x_i\mu(\mathrm{d}x)$$
$$-\int\sum_k a_s(j,k)x_k(x_i-N)I_{\{x_i>N\}}\mu(\mathrm{d}x)\Big)\mathrm{d}s$$
$$\geqslant\lim_{n\to\infty}\frac{1}{t_n}\int_0^{t_n}\Big(\int(T_sx_j)x_i\mu(\mathrm{d}x)-\sum_k a_s(j,k)\int x_kx_iI_{\{x_i>N\}}\mu(\mathrm{d}x)\Big)\mathrm{d}s$$
$$\geqslant\lim_{n\to\infty}\frac{1}{t_n}\int_0^{t_n}\Big(\int(T_sx_j)x_i\mu(\mathrm{d}x)$$
$$-\sum_k a_s(j,k)\Big(\int x_k^2\mu(\mathrm{d}x)\Big)^{1/2}\Big(\int x_i^2I_{\{x_i>N\}}\mu(\mathrm{d}x)\Big)^{1/2}\Big)\mathrm{d}s$$
$$\geqslant\lim_{n\to\infty}\frac{1}{t_n}\int_0^{t_n}\Big(\int(T_sx_j)x_i\mu(\mathrm{d}x)-\sum_k a_s(j,k)\widetilde{C}^{1/2}\varepsilon\Big)\mathrm{d}s$$

$$= \lim_{n\to\infty} \frac{1}{t_n} \int_0^{t_n} \int (T_s x_j) x_i \mu(\mathrm{d}x) \mathrm{d}s - \widetilde{C}^{1/2}\varepsilon. \tag{3.47}$$

On the other hand,

$$\lim_{n\to\infty} \frac{1}{t_n} \int_0^{t_n} \int (T_s x_j)(x_i \wedge N) \mu(\mathrm{d}x) \mathrm{d}s \leqslant \lim_{n\to\infty} \frac{1}{t_n} \int_0^{t_n} \int (T_s x_j) x_i \mu(\mathrm{d}x) \mathrm{d}s. \tag{3.48}$$

Hence, (3.47) and (3.48) yield that

$$\begin{aligned} &\Big| \lim_{n\to\infty} \frac{1}{t_n} \int_0^{t_n} \int (T_s x_j)(x_i \wedge N) \mu(\mathrm{d}x) \mathrm{d}s - \lim_{n\to\infty} \frac{1}{t_n} \int_0^{t_n} \int (T_s x_j) x_i \mu(\mathrm{d}x) \mathrm{d}s \Big| \\ &\quad \leqslant \widetilde{C}^{1/2}\varepsilon. \end{aligned} \tag{3.49}$$

(3.41) implies

$$\Big| \int (x_i \wedge N) \mu(\mathrm{d}x) h(j) - h(i)h(j) \Big| \leqslant \varepsilon h(j). \tag{3.50}$$

By (3.46), (3.49), and (3.50), we have

$$\begin{aligned} &\Big| h(i)h(j) - \lim_{n\to\infty} \frac{1}{t_n} \int_0^{t_n} \int (T_s x_j) x_i \mu(\mathrm{d}x) \mathrm{d}s \Big| \\ &\leqslant \Big| \int (x_i \wedge N) \mu(\mathrm{d}x) h(j) - h(i)h(j) \Big| \\ &\quad + \Big| \int (x_i \wedge N) \mu(\mathrm{d}x) h(j) - \lim_{n\to\infty} \frac{1}{t_n} \int_0^{t_n} \int (T_s x_j)(x_i \wedge N) \mu(\mathrm{d}x) \mathrm{d}s \Big| \\ &\quad + \Big| \lim_{n\to\infty} \frac{1}{t_n} \int_0^{t_n} \int (T_s x_j)(x_i \wedge N) \mu(\mathrm{d}x) \mathrm{d}s - \lim_{n\to\infty} \frac{1}{t_n} \int_0^{t_n} \int (T_s x_j) x_i \mu(\mathrm{d}x) \mathrm{d}s \Big| \\ &\leqslant \varepsilon h(j) + \widetilde{C}^{1/2}\varepsilon. \end{aligned} \tag{3.51}$$

Since ε is arbitrary, (3.40) holds.

Step 3 Let

$$h(i,j) = \lim_{t\to\infty} \sum_{k,l} a_t(i,k) a_t(j,l) \int x_k x_l \mu(\mathrm{d}x).$$

In order to show

$$\lim_{t\to\infty} \sum_{k,l} a_t(i,k) a_t(j,l) \int x_k x_l \mu(\mathrm{d}x) = h(i)h(j),$$

by (3.40), we need to show that

$$h(i,j) = \lim_{n\to\infty} \frac{1}{t_n} \int_0^{t_n} \int (T_s x_j) x_i \mu(\mathrm{d}x) \mathrm{d}s. \tag{3.52}$$

From Remark of [29, Lemma 4.7], we know that for arbitrarily fixed $i \in S$, $h(i,\cdot)$ is harmonic with respect to A. By (3.38), we have

$$\begin{aligned}
&\pi_j \lim_{n\to\infty} \frac{1}{t_n}\int_0^{t_n}\int (T_s x_j) x_i \mu(\mathrm{d}x)\mathrm{d}s \\
&= \pi_j \lim_{n\to\infty} \frac{1}{t_n}\int_0^{t_n}\Big(\int \sum_k a_s(j,k) x_k x_i \mu(\mathrm{d}x)\Big)\mathrm{d}s \\
&= \pi_j \lim_{n\to\infty} \frac{1}{t_n}\int_0^{t_n}\Big(\sum_k a_s(j,k) \int x_k x_i \mu(\mathrm{d}x)\Big)\mathrm{d}s \\
&= \pi_j \lim_{n\to\infty} \frac{1}{t_n}\int_0^{t_n}\sum_k a_s(j,k)\Big(h(i,k) \\
&\quad + b\int_0^\infty \sum_m a_r(k,m)a_r(i,m)\int x_k^2\mu(\mathrm{d}x)\mathrm{d}r\Big)\mathrm{d}s \\
&= \pi_j \lim_{n\to\infty} \frac{1}{t_n}\int_0^{t_n}\Big(h(i,j) + b\int_0^\infty \sum_{k,m} a_s(j,k)a_r(k,m)a_r(i,m)\int x_k^2\mu(\mathrm{d}x)\mathrm{d}r\Big)\mathrm{d}s \\
&= \pi_j h(i,j) + b\pi_j \lim_{n\to\infty} \frac{1}{t_n}\int_0^{t_n}\int_0^\infty \Big(\sum_{k,m} a_s(j,k)a_r(k,m)a_r(i,m) \\
&\quad \cdot \int x_k^2\mu(\mathrm{d}x)\Big)\mathrm{d}r\mathrm{d}s. \qquad (3.53)
\end{aligned}$$

Noting that

$$\begin{aligned}
0 &\leqslant \pi_j \lim_{n\to\infty} \frac{1}{t_n}\int_0^{t_n}\int_0^\infty \Big(\sum_{k,m} a_s(j,k)a_r(k,m)a_r(i,m)\int x_k^2\mu(\mathrm{d}x)\Big)\mathrm{d}r\mathrm{d}s \\
&\leqslant \widetilde{C}\pi_j \lim_{n\to\infty} \frac{1}{t_n}\int_0^{t_n}\int_0^\infty \sum_m a_{s+r}(j,m)a_r(i,m)\mathrm{d}r\mathrm{d}s \\
&= \widetilde{C} \lim_{n\to\infty} \frac{1}{t_n}\int_0^{t_n}\int_0^\infty \sum_m \pi_m a_{s+r}(m,j)a_r(i,m)\mathrm{d}r\mathrm{d}s \\
&\leqslant \frac{1}{2}\widetilde{C}\max_m \pi_m \lim_{n\to\infty} \frac{1}{t_n}\int_0^{t_n}\int_s^\infty a_r(i,j)\mathrm{d}r\mathrm{d}s \\
&= 0 \quad (\text{by the transience of } a_t), \qquad (3.54)
\end{aligned}$$

we have

$$\pi_j \lim_{n\to\infty} \frac{1}{t_n}\int_0^{t_n}\int_0^\infty \Big(\sum_{k,m} a_s(j,k)a_r(k,m)a_r(i,m)\int x_k^2\mu(\mathrm{d}x)\Big)\mathrm{d}r\mathrm{d}s = 0. \qquad (3.55)$$

Hence, (3.53) and (3.55) imply that for all $i,j \in S$,

$$\pi_j h(i,j) = \pi_j \lim_{n\to\infty} \frac{1}{t_n}\int_0^{t_n}\int (T_s x_j) x_i \mu(\mathrm{d}x)\mathrm{d}s.$$

Therefore, (3.52) holds. □

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Appendix

A1 Some lemmas of proof of Proposition 2.4

In order to prove Propositions 2.4 and 2.6, we need the following lemmas. For any $E(\|X(0)\|^2_{\gamma,2}) < \infty$, we consider (1.5),

$$\begin{aligned} X_i(t) = X_i(0) &+ \kappa \int_0^t \sum_{j\in S} a(i,j)X_j(s)\mathrm{d}s + \beta \int_0^t X_i(s)\mathrm{d}s \\ &+ \int_0^t \alpha X_i(s)\mathrm{d}B_i(s) + \int_0^{t+}\int_{|u|<\delta} uX_i(s-)\widetilde{N}_i(\mathrm{d}s\mathrm{d}u) \\ &+ \int_0^{t+}\int_{|u|\geqslant\delta} uX_i(s-)N_i(\mathrm{d}s\mathrm{d}u), \quad i \in S. \end{aligned} \tag{A1}$$

Let $\{S_n\}_{n\in\mathbb{N}}$ be a sequence of finite sets which increase to S as $n \to \infty$. Consider the following finite-dimensional equation:

$$X_i^{(n)}(t) = \begin{cases} X_i^{(n)}(0) + \kappa \int_0^t \sum_{j\in S} a(i,j)X_j^{(n)}(s)\mathrm{d}s + \beta \int_0^t X_i^{(n)}(s)\mathrm{d}s \\ \quad + \int_0^t \alpha X_i^{(n)}(s)\mathrm{d}B_i(s) + \int_0^{t+}\int_{|u|<\delta} uX_i^{(n)}(s-)\widetilde{N}_i(\mathrm{d}s\mathrm{d}u) \\ \quad + \int_0^{t+}\int_{|u|\geqslant\delta} uX_i^{(n)}(s-)N_i(\mathrm{d}s\mathrm{d}u), \quad i \in S_n, \\ X_i(0), \quad i \in S\backslash S_n. \end{cases} \tag{A2}$$

Since (A2) is a finite-dimensional linear equation, using classical method, it is easy to prove that (A2) has a pathwise unique solution $\{X_i^{(n)}(t)\}$ and $E(\sup_{0\leqslant s\leqslant T}|X_i^{(n)}(s)|^2) < \infty$. Moreover, we have the following lemma, which implies that $E(\sup_{0\leqslant s\leqslant T}|X_i^{(n)}(s)|^2) < C$ independent of n.

Lemma A1 *In the case $i \in S_n$,*

$$E(\sup_{0\leqslant s\leqslant T}|X_i^{(n)}(s)|^2) \leqslant 5\sum_{j\in S}(\mathrm{e}^{TM_T})_{ij}E(|X_j^{(n)}(0)|^2), \tag{A3}$$

where

$$M_T(i,j) = 10\kappa^2 T\sup_{i\in S}|a(i,i)| + \delta_{ij}5\Big[C_2\int_{\mathbb{R}\backslash\{0\}} u^2\rho(\mathrm{d}u) + \widetilde{C}_2\alpha^2 + \varpi^2 T\Big], \quad i,j \in S,$$

and $\varpi = \int_{|u|\geqslant\delta} u\rho(\mathrm{d}u)+\beta$. *Note that* A *is a* Q*-matrix satisfying* $\sup_{i\in S}|a(i,i)| < \infty$. *Thus,* e^{TM_T} *makes sense for any* $T > 0$.

Proof By (A2), we have

$$X_i^{(n)}(t) = \begin{cases} X_i^{(n)}(0) + \kappa\displaystyle\int_0^t \sum_{j\in S} a(i,j)X_j^{(n)}(s)\mathrm{d}s + \varpi\int_0^t X_i^{(n)}(s)\mathrm{d}s \\ \quad + \displaystyle\int_0^t \alpha X_i^{(n)}(s)\mathrm{d}B_i(s) + \int_0^{t+}\int_{\mathbb{R}\backslash\{0\}} uX_i^{(n)}(s-)\widetilde{N}_i(\mathrm{d}s\mathrm{d}u), \quad i\in S_n, \\ X_i(0), \quad i\in S\backslash S_n. \end{cases}$$

For any $M>0$, denote $\sigma_M = \inf\{t\geqslant 0 \mid \sup_{i\in S_n}|X_i^{(n)}(t)|\geqslant M\}$ and

$$\widetilde{X}_i^{(n)}(t) = \begin{cases} X_i^{(n)}(t), & t<\sigma_M, \\ \displaystyle\lim_{t\to\sigma_M-} X_i^{(n)}(t), & t\geqslant\sigma_M. \end{cases}$$

For $i\in S_n$ and $t<\sigma_M$, we get

$$\begin{aligned} |\widetilde{X}_i^{(n)}(t)|^2 \leqslant 5\bigg\{ & |X_i^{(n)}(0)|^2 + \bigg|\kappa\int_0^t \sum_{j\in S} a(i,j)\widetilde{X}_j^{(n)}(s)\mathrm{d}s\bigg|^2 + \bigg|\varpi\int_0^t \widetilde{X}_i^{(n)}(s)\mathrm{d}s\bigg|^2 \\ & + \bigg|\int_0^t \alpha\widetilde{X}_i^{(n)}(s)\mathrm{d}B_i(s)\bigg|^2 + \bigg|\int_0^{t+}\int_{\mathbb{R}\backslash\{0\}} u\widetilde{X}_i^{(n)}(s-)\widetilde{N}_i(\mathrm{d}s\mathrm{d}u)\bigg|^2\bigg\} \\ =: {} & 5\{\mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3+\mathrm{I}_4+\mathrm{I}_5\}. \end{aligned} \tag{A4}$$

By (A4), if $\sigma_M > T$, then

$$\sup_{0\leqslant t\leqslant T} |\widetilde{X}_i^{(n)}(t)|^2 \leqslant \sup_{0\leqslant t\leqslant T} 5\{\mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3+\mathrm{I}_4+\mathrm{I}_5\}; \tag{A5}$$

if $\sigma_M \leqslant T$, then

$$\begin{aligned} \sup_{0\leqslant t\leqslant T} |\widetilde{X}_i^{(n)}(t)|^2 &= \sup_{0\leqslant t<\sigma_M} |\widetilde{X}_i^{(n)}(t)|^2 \\ &\leqslant \sup_{0\leqslant t<\sigma_M} 5\{\mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3+\mathrm{I}_4+\mathrm{I}_5\} \\ &\leqslant \sup_{0\leqslant t\leqslant T} 5\{\mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3+\mathrm{I}_4+\mathrm{I}_5\}. \end{aligned} \tag{A6}$$

It follows from (1.1) that

$$\begin{aligned} E(\sup_{0\leqslant t\leqslant T} \mathrm{I}_2) &\leqslant \kappa^2 E\bigg(\sup_{0\leqslant t\leqslant T}\int_0^t \sum_{j\in S}|a(i,j)|\mathrm{d}s\cdot\int_0^t\sum_{j\in S}|a(i,j)|\,|\widetilde{X}_j^{(n)}|^2\mathrm{d}s\bigg) \\ &\leqslant 2\kappa^2 T\sup_{i\in S}|a(i,i)|E\bigg(\sup_{0\leqslant t\leqslant T}\int_0^t\sum_{j\in S}|a(i,j)|\,|\widetilde{X}_j^{(n)}|^2\mathrm{d}s\bigg) \end{aligned}$$

$$\leqslant 2\kappa^2 T \sup_{i\in S}|a(i,i)| \int_0^T \sum_{j\in S}|a(i,j)|E(\sup_{0\leqslant s'\leqslant s}|\widetilde{X}_j^{(n)}(s')|^2)\mathrm{d}s. \tag{A7}$$

For I_3, by Hölder's inequality, we have

$$E(\sup_{0\leqslant t\leqslant T}\mathrm{I}_3) \leqslant \varpi^2 T \int_0^T E\big(\sup_{0\leqslant s'\leqslant s}|\widetilde{X}_i^{(n)}(s')|^2\big)\mathrm{d}s. \tag{A8}$$

By the Burkholder-Davis-Gundy inequality (see [26]), we have

$$\begin{aligned} E(\sup_{0\leqslant t\leqslant T}\mathrm{I}_4) &\leqslant \alpha^2\widetilde{C}_2 E\bigg(\int_0^T [\widetilde{X}_i^{(n)}(s)]^2\mathrm{d}s\bigg) \\ &\leqslant \alpha^2\widetilde{C}_2\int_0^T E(\sup_{0\leqslant s'\leqslant s}|\widetilde{X}_i^{(n)}(s')|^2)\mathrm{d}s, \end{aligned} \tag{A9}$$

$$\begin{aligned} E(\sup_{0\leqslant t\leqslant T}\mathrm{I}_5) &\leqslant C_2 E\bigg(\int_0^{T+}\int_{\mathbb{R}\backslash\{0\}}[u\widetilde{X}_i^{(n)}(s-)]^2 N_i(\mathrm{d}s\mathrm{d}u)\bigg) \\ &\leqslant C_2 E\bigg(\int_0^T\int_{\mathbb{R}\backslash\{0\}}[u\widetilde{X}_i^{(n)}(s)]^2\rho(\mathrm{d}u)\mathrm{d}s\bigg) \\ &\leqslant C_2\int_{\mathbb{R}\backslash\{0\}}u^2\rho(\mathrm{d}u)\cdot\int_0^T E(\sup_{0\leqslant s'\leqslant s}|\widetilde{X}_i^{(n)}(s')|^2)\mathrm{d}s. \end{aligned} \tag{A10}$$

Therefore, by (A5)–(A10), we have

$$\begin{aligned} &E(\sup_{0\leqslant s\leqslant T}|\widetilde{X}_i^{(n)}(s)|^2) \\ &\leqslant 5E(|X_i^{(n)}(0)|^2) + 10\kappa^2 T\sup_{i\in S}|a(i,i)|\int_0^T\sum_{j\in S}|a(i,j)|E(\sup_{0\leqslant s'\leqslant s}|\widetilde{X}_j^{(n)}(s')|^2)\mathrm{d}s \\ &\quad + 5\bigg[C_2\int_{\mathbb{R}\backslash\{0\}}u^2\rho(\mathrm{d}u) + \widetilde{C}_2\alpha^2 + \varpi^2 T\bigg]\int_0^T E(\sup_{0\leqslant s'\leqslant s}|\widetilde{X}_i^{(n)}(s')|^2)\mathrm{d}s. \end{aligned} \tag{A11}$$

Due to Gronwall's inequality and $E(\sup_{0\leqslant s\leqslant T}|X_i^{(n)}(s)|^2) < \infty$, we have

$$E(\sup_{0\leqslant s\leqslant T}|\widetilde{X}_i^{(n)}(s)|^2) \leqslant 5\sum_{j\in S}(\mathrm{e}^{TM_T})_{ij}E(|X_j^{(n)}(0)|^2).$$

Referring to [27, Eq. (1.10)], we have

$$\begin{aligned} &P(\omega \mid X_i^{(n)}(T) \neq X_i^{(n)}(T-)) \\ &\quad\leqslant P\bigg(\omega\,\bigg|\int_0^{T+}\int_{\mathbb{R}\backslash\{0\}}u\widetilde{N}_i(\mathrm{d}s\mathrm{d}u) \neq \lim_{t\to T-}\int_0^{t+}\int_{\mathbb{R}\backslash\{0\}}u\widetilde{N}_i(\mathrm{d}s\mathrm{d}u)\bigg) \\ &\quad= 0. \end{aligned} \tag{A12}$$

By Fatou's lemma and (A12), (A3) holds. □

Lemma A2 *For any $t \in [0,T]$, we have*

$$E(\sup_{0\leqslant s\leqslant t}\|X^{(n)}(s)\|_{\gamma,2}^2) \leqslant 5E(\|X^{(n)}(0)\|_{\gamma,2}^2)\cdot \exp\bigg\{5t\bigg[2\kappa^2 T\sup_{i\in S}|a(i,i)|\Gamma\sup_{j\in S}\gamma_j + C_2\int_{\mathbb{R}\setminus\{0\}}u^2\rho(\mathrm{d}u) + \widetilde{C}_2\alpha^2 + \varpi^2 T\bigg]\bigg\}. \tag{A13}$$

Proof It follows from (A11) that

$$\begin{aligned}
&E\bigg(\sum_{i\in S}\gamma_i\sup_{0\leqslant s\leqslant t}|X_i^{(n)}(s)|^2\bigg)\\
&\leqslant 5E\bigg(\sum_{i\in S}\gamma_i|X_i^{(n)}(0)|^2\bigg)\\
&\quad + 10\kappa^2 T\sup_{i\in S}|a(i,i)|\int_0^t\sum_{i\in S}\sum_{j\in S}\gamma_i|a(i,j)|E(\gamma_j\sup_{0\leqslant s'\leqslant s}|X_j^{(n)}(s')|^2)\mathrm{d}s\\
&\quad + 5\bigg[C_2\int_{\mathbb{R}\setminus\{0\}}u^2\rho(\mathrm{d}u) + \widetilde{C}_2\alpha^2 + \varpi^2 T\bigg]\int_0^t E\bigg(\sum_{i\in S}\gamma_i\sup_{0\leqslant s'\leqslant s}|X_i^{(n)}(s')|^2\bigg)\mathrm{d}s.
\end{aligned}$$

Noting that

$$\begin{aligned}
&\sum_{i\in S}\sum_{j\in S}\gamma_i|a(i,j)|E(\gamma_j\sup_{0\leqslant s'\leqslant s}|X_j^{(n)}(s')|^2)\\
&\quad = \sum_{j\in S}E(\gamma_j\sup_{0\leqslant s'\leqslant s}|X_j^{(n)}(s')|^2)\sum_{i\in S}\gamma_i|a(i,j)|\\
&\quad \leqslant \sum_{j\in S}E(\gamma_j\sup_{0\leqslant s'\leqslant s}|X_j^{(n)}(s')|^2)\cdot\Gamma\gamma_i\\
&\quad \leqslant \Gamma\sup_{j\in S}\gamma_j E\bigg(\sum_{i\in S}\gamma_i\sup_{0\leqslant s'\leqslant s}|X_i^{(n)}(s')|^2\bigg),
\end{aligned} \tag{A14}$$

we have

$$\begin{aligned}
&E\bigg(\sum_{i\in S}\gamma_i\sup_{0\leqslant s\leqslant t}|X_i^{(n)}(s)|^2\bigg)\\
&\leqslant 5E\bigg(\sum_{i\in S}\gamma_i|X_i^{(n)}(0)|^2\bigg) + 5\bigg[2\kappa^2 T\sup_{i\in S}|a(i,i)|\Gamma\sup_{j\in S}\gamma_j\\
&\quad + C_2\int_{\mathbb{R}\setminus\{0\}}u^2\rho(\mathrm{d}u) + \widetilde{C}_2\alpha^2 + \varpi^2 T\bigg]\cdot\int_0^t E\bigg(\sum_{i\in S}\gamma_i\sup_{0\leqslant s'\leqslant s}|X_i^{(n)}(s')|^2\bigg)\mathrm{d}s.
\end{aligned}$$

Note that $X^{(n)}(0)\in L^2(\gamma)$ and $X^{(n)}$ is a finite-dimensional equation essentially. Then

$$E\bigg(\sum_{i\in S}\gamma_i\sup_{0\leqslant s\leqslant t}|X_i^{(n)}(s)|^2\bigg) < \infty.$$

By Gronwall's inequality,

$$
\begin{aligned}
&E\bigg(\sum_{i\in S}\gamma_i\sup_{0\leqslant s\leqslant t}|X_i^{(n)}(s)|^2\bigg)\\
&\quad\leqslant 5E\bigg(\sum_{i\in S}\gamma_i|X_i^{(n)}(0)|^2\bigg)\cdot\exp\bigg\{5\bigg[2\kappa^2T\sup_{i\in S}|a(i,i)|\Gamma\sup_{j\in S}\gamma_j\\
&\qquad+C_2\int_{\mathbb{R}\backslash\{0\}}u^2\rho(\mathrm{d}u)+\widetilde{C}_2\alpha^2+\varpi^2T\bigg]t\bigg\}. \qquad \text{(A15)}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
E(\sup_{0\leqslant s\leqslant t}\|X^{(n)}(s)\|_{\gamma,2}^2)\leqslant 5E(\|X^{(n)}(0)\|_{\gamma,2}^2)\cdot\exp\bigg\{5t\bigg[2\kappa^2T\sup_{i\in S}|a(i,i)|\Gamma\sup_{j\in S}\gamma_j\\
+C_2\int_{\mathbb{R}\backslash\{0\}}u^2\rho(\mathrm{d}u)+\widetilde{C}_2\alpha^2+\varpi^2T\bigg]\bigg\}.
\end{aligned}
$$

The proof is complete. □

Lemma A3 $\lim_{n,m\to\infty}E(\sup_{0\leqslant s\leqslant T}\|X^{(n)}(s)-X^{(m)}(s)\|_{\gamma,2}^2)=0$.

Proof For $n\geqslant m$, we have

$$
X_i^{(n)}(s)-X_i^{(m)}(s)
=\begin{cases}
\kappa\displaystyle\int_0^t\sum_{j\in S}a(i,j)(X_j^{(n)}(s)-X_j^{(m)}(s))\mathrm{d}s+\varpi\int_0^t(X_i^{(n)}(s)-X_i^{(m)}(s))\mathrm{d}s\\
\quad+\displaystyle\int_0^t\alpha(X_i^{(n)}(s)-X_i^{(m)}(s))\mathrm{d}B_i(s)\\
\quad+\displaystyle\int_0^{t+}\int_{\mathbb{R}\backslash\{0\}}u(X_i^{(n)}(s-)-X_i^{(m)}(s-))\widetilde{N}_i(\mathrm{d}s,\mathrm{d}u),\quad i\in S_m,\\
\kappa\displaystyle\int_0^t\sum_{j\in S}a(i,j)X_j^{(n)}(s)\mathrm{d}s+\varpi\int_0^tX_i^{(n)}(s)\mathrm{d}s+\int_0^t\alpha X_i^{(n)}(s)\mathrm{d}B_i(s)\\
\quad+\displaystyle\int_0^{t+}\int_{\mathbb{R}\backslash\{0\}}uX_i^{(n)}(s-)\widetilde{N}_i(\mathrm{d}s,\mathrm{d}u)-X_i(0),\quad i\in S_n\backslash S_m,\\
0,\quad i\in S\backslash S_n.
\end{cases}
$$

Applying the same argument as in Lemma A1, for $i\in S_m$, we get

$$
\begin{aligned}
&E(\sup_{0\leqslant s\leqslant T}|X_i^{(n)}(s)-X_i^{(m)}(s)|^2)\\
&\leqslant 10\kappa^2T\sup_{i\in S}|a(i,i)|\int_0^T\sum_{j\in S}|a(i,j)|E(\sup_{0\leqslant s'\leqslant s}|X_j^{(n)}(s')-X_j^{(m)}(s')|^2)\mathrm{d}s\\
&\quad+5\bigg[C_2\int_{\mathbb{R}\backslash\{0\}}u^2\rho(\mathrm{d}u)+\widetilde{C}_2\alpha^2+\varpi^2T\bigg]
\end{aligned}
$$

$$\cdot \int_0^T E(\sup_{0\leqslant s'\leqslant s} |X_i^{(n)}(s') - X_i^{(m)}(s')|^2)\mathrm{d}s.$$

It is easy to see

$$E(\sup_{0\leqslant s\leqslant T} |X_i^{(n)}(s) - X_i^{(m)}(s)|^2) \begin{cases} \leqslant 2E(\sup_{0\leqslant s\leqslant T} |X_i^{(n)}(s)|^2) + 2E(|X_i(0)|^2), & i \in S_n \backslash S_m, \\ = 0, & i \in S\backslash S_n, \end{cases}$$

By the above inequalities, we have

$$\begin{aligned} &E\bigg(\sum_{i\in S}\gamma_i \sup_{0\leqslant s\leqslant T} |X_i^{(n)}(s) - X_i^{(m)}(s)|^2\bigg) \\ &\leqslant 10\kappa^2 T \sup_{i\in S}|a(i,i)| \int_0^T \sum_{i\in S}\gamma_i \sum_{j\in S}|a(i,j)| E(\sup_{0\leqslant s'\leqslant s} |X_j^{(n)}(s') - X_j^{(m)}(s')|^2)\mathrm{d}s \\ &\quad + 5\bigg[C_2\int_{\mathbb{R}\backslash\{0\}} u^2\rho(\mathrm{d}u) + \widetilde{C}_2\alpha^2 + \varpi^2 T\bigg] \\ &\quad \cdot \int_0^T E\bigg(\sum_{i\in S}\gamma_i \sup_{0\leqslant s'\leqslant s}|X_i^{(n)}(s') - X_i^{(m)}(s')|^2\bigg)\mathrm{d}s \\ &\quad + \sum_{i\in S_n\backslash S_m}\gamma_i(2E(\sup_{0\leqslant s\leqslant T}|X_i^{(n)}(s)|^2) + 2E(|X_i(0)|^2)). \end{aligned}$$

Together with (A14), we have

$$\begin{aligned} &E(\sup_{0\leqslant s\leqslant T} \|X^{(n)}(s) - X^{(m)}(s)\|_{\gamma,2}^2) \\ &\leqslant \sum_{i\in S_n\backslash S_m}\gamma_i(2E(\sup_{0\leqslant s\leqslant T}|X_i^{(n)}(s)|^2) + 2E(|X_i(0)|^2)) \\ &\quad \cdot \exp\bigg\{5\bigg[2\kappa^2 T\sup_{i\in S}|a(i,i)|\Gamma\sup_{j\in S}\gamma_j + C_2\int_{\mathbb{R}\backslash\{0\}}u^2\rho(\mathrm{d}u) + \widetilde{C}_2\alpha^2 + \varpi^2 T\bigg]T\bigg\}. \end{aligned}$$

Therefore, Lemma A3 holds by Lemmas A1 and A2. □

Lemma A3 implies that there exists an $L^2(\gamma)$-valued stochastic process $(X(t))_{t\geqslant 0}$ such that for $t > 0$,

$$\lim_{n\to\infty} E(\sup_{0\leqslant s\leqslant t}\|X^{(n)}(s) - X(s)\|_{\gamma,2}^2) = 0. \tag{A16}$$

In fact, $(X(t))_{t\geqslant 0}$ is the solution of (1.5). Moreover, by Lemma A2, it is easy to prove the following lemma.

Lemma A4 *For any $t \in [0,T]$, we have*

$$E(\sup_{0\leqslant s\leqslant t}\|X(s)\|_{\gamma,2}^2) \leqslant 5E(\|X(0)\|_{\gamma,2}^2)\exp\bigg\{5t\bigg[2\kappa^2 T\sup_{i\in S}|a(i,i)|\Gamma\sup_{j\in S}\gamma_j$$

$$+ C_2 \int_{\mathbb{R}\setminus\{0\}} u^2 \rho(\mathrm{d}u) + \widetilde{C}_2 \alpha^2 + \varpi^2 T \Big] \Big\}. \tag{A17}$$

Lemma A5 $(X(t))_{t \geqslant 0}$ *has the Feller property.*

Proof Let $C_b(L^2(\gamma))$ be the space of bounded continuous real-valued functions, and let $\mathrm{Lip}_b(L^2(\gamma))$ be the set of bounded Lipschitz real-valued functions.

Note that $\mathrm{Lip}_b(L^2(\gamma))$ is dense in $C_b(L^2(\gamma))$ under sup norm. Therefore, we need only to prove that for any $f \in \mathrm{Lip}_b(L^2(\gamma))$,

$$\lim_{y \to x} E(f(X(t,y))) = E(f(X(t,x))), \quad x, y \in L^2(\gamma). \tag{A18}$$

By Lemmas A1–A3, for $\varepsilon > 0$, there exist $\delta > 0$ and $N > 0$, such that for $z \in B_\delta(x)$ and $n > N$,

$$E\big(\sup_{0 \leqslant t \leqslant T} \|X(t,z) - X^{(n)}(t,z)\|_{\gamma,2}^2\big) < \varepsilon.$$

Using the same argument in Lemma A2, we have

$$\begin{aligned} & E\big(\sup_{0 \leqslant s \leqslant T} \|X^{(n)}(s,x) - X^{(n)}(s,y)\|_{\gamma,2}^2\big) \\ & \leqslant 5\|x-y\|_{\gamma,2}^2 \exp\Big\{5T\Big[2\kappa^2 T \sup_{i \in S} |a(i,i)| \Gamma \sup_{j \in S} \gamma_j \\ & \quad + C_2 \int_{\mathbb{R}\setminus\{0\}} u^2 \rho(\mathrm{d}u) + \widetilde{C}_2 \alpha^2 + \varpi^2 T \Big] \Big\}. \end{aligned}$$

Thus,

$$\lim_{y \to x} E(\|X(t,x) - X(t,y)\|_{\gamma,2}^2) = 0.$$

Therefore,

$$\begin{aligned} \lim_{y \to x} |E(f(X(t,y))) - E(f(X(t,x)))| & \leqslant \lim_{y \to x} C(E(\|X(t,x) - X(t,y)\|_{\gamma,2}^2))^{1/2} \\ & = 0. \end{aligned}$$

□

Lemma A6 $X_i(\cdot) \in D[0,\infty;\mathbb{R}]$ a.s.

Proof From [1], we need only to show that Aldous's tightness criterion holds.

By (A3), we have for any $t > 0$, $\{X_i^{(n)}(t)\}_{n \in \mathbb{N}}$ is tight in $\mathbb{R}$. Let $\{\tau_n, \delta_n\}$ satisfy the following conditions (see [1]).

(i) For each n, τ_n is a stopping time on the process $\{X_i^{(n)}(t), 0 \leqslant t \leqslant T\}$ with respect to the natural σ-fields, and τ_n takes only finitely many values.

(ii) For each n, δ_n is a constant, $0 \leqslant \delta_n \leqslant 1$, $\lim_{n \to \infty} \delta_n = 0$.

Similar to Lemma A2 and noting (A15), we have

$$
\begin{aligned}
&E(\|X^{(n)}(\tau_n+\delta_n)-X^{(n)}(\tau_n)\|_{\gamma,2}^2)\\
&\leqslant 4\sum_{i\in S}\gamma_i E\bigg[\bigg|\kappa\int_{\tau_n}^{\tau_n+\delta_n}\sum_{j\in S}a(i,j)X_j^{(n)}(s)\mathrm{d}s\bigg|^2+\bigg|\varpi\int_{\tau_n}^{\tau_n+\delta_n}X_i^{(n)}(s)\mathrm{d}s\bigg|^2\\
&\quad+\bigg|\int_{\tau_n}^{\tau_n+\delta_n}\alpha X_i^{(n)}(s)\mathrm{d}B_i(s)\bigg|^2+\bigg|\int_{\tau_n}^{\tau_n+\delta_n}\alpha X_i^{(n)}(s)\mathrm{d}B_i(s)\bigg|^2\\
&\quad+\bigg|\int_{\tau_n}^{(\tau_n+\delta_n)+}\int_{\mathbb{R}\backslash\{0\}}uX_i^{(n)}(s-)\widetilde{N}_i(\mathrm{d}s,\mathrm{d}u)\bigg|^2\bigg],\\
&\leqslant CE\bigg(\sum_{i\in S}\gamma_i\sup_{t\in[0,T]}|X_i^{(n)}(t)|^2\mathrm{d}s\bigg)\delta_n\\
&\leqslant C\delta_n\cdot 5E(\|X^{(n)}(0)\|_{\gamma,2}^2)\cdot \mathrm{e}^{5TC/4},
\end{aligned}
$$

where

$$
C=4\bigg[2\kappa^2T\sup_{i\in S}|a(i,i)|\Gamma\sup_{j\in S}\gamma_j+C_2\int_{\mathbb{R}\backslash\{0\}}u^2\rho(\mathrm{d}u)+\widetilde{C}_2\alpha^2+\varpi^2T\bigg].
$$

Thus,

$$
\begin{aligned}
E(|X_i^{(n)}(\tau_n+\delta_n)-X_i^{(n)}(\tau_n)|^2)&\leqslant\frac{1}{\gamma_i^2}E(\|X^{(n)}(\tau_n+\delta_n)-X^{(n)}(\tau_n)\|_{\gamma,2}^2)\\
&\leqslant\frac{\widetilde{C}}{\gamma_i^2}\delta_n\\
&\to 0,\quad \delta_n\to 0, \qquad\qquad (\text{A19})
\end{aligned}
$$

where

$$
\widetilde{C}=4CE(\|X^{(n)}(0)\|_{\gamma,2}^2)\cdot\mathrm{e}^{5T[2\kappa^2T\sup_{i\in S}|a(i,i)|\Gamma\sup_{j\in S}\gamma_j+\int_{\mathbb{R}\backslash\{0\}}u^2\rho(\mathrm{d}u)+\alpha^2+\varpi^2T]}.
$$

It follows from (A19) that

$$
X_i^{(n)}(\tau_n+\delta_n)-X_i^{(n)}(\tau_n)\to 0\quad\text{in probability.}
$$

Hence, $\{X_i^{(n)}(t),\, t\geqslant 0\}_{n\in\mathbb{N}}$ satisfies Aldous's tightness criterion, which implies that $\{X_i^{(n)}(t),\, t\geqslant 0\}_{n\in\mathbb{N}}$ is tight in $D[0,\infty;\mathbb{R}]$. Therefore, we get the lemma.□

Lemma A7 *For any $\widetilde{x}\in\Xi_F$, we have*

$$
E^{\widetilde{x}}(\langle h,X(t,\widetilde{x})\rangle_\pi^2)\leqslant\langle h,\widetilde{x}\rangle_\pi^2\mathrm{e}^{ct}<\infty,
$$

where $c:=\alpha^2+\int u^2\rho(\mathrm{d}u)$.

Proof Note that

$$\frac{\mathrm{d}}{\mathrm{d}t}E^{\widetilde{x}}(X_i(t)X_j(t)) = \sum_k a(i,k)E^{\widetilde{x}}(X_k(t)X_j(t)) + \sum_l a(j,l)E^{\widetilde{x}}(X_i(t)X_l(t)) + \Big(\alpha^2 + \int u^2\rho(\mathrm{d}u)\Big)\delta(i,j)E^{\widetilde{x}}(X_i^2(t)).$$

Using the same method as the proof of [5, Lemma 1], we have

$$E^{\widetilde{x}}(X_i(t)X_j(t)) = \sum_{k,l} a_t(i,k)a_t(j,l)\widetilde{x}_k\widetilde{x}_l + \Big(\alpha^2 + \int u^2\rho(\mathrm{d}u)\Big) \cdot \int_0^t \sum_k a_{t-r}(i,k)a_{t-r}(j,k)E^{\widetilde{x}}(X_k^2(r))\mathrm{d}r. \tag{A20}$$

For the sake of convenience, we write

$$E^{\widetilde{x}}(\langle h, X(t)\rangle_\pi^2) = E^{\widetilde{x}}(\langle h, X(t,\widetilde{x})\rangle_\pi^2).$$

From (A20), we get that for $h \in \mathscr{H}$,

$$\begin{aligned}E^{\widetilde{x}}(\langle h, X(t)\rangle_\pi^2) &= \sum_{i,j}\pi_i\pi_j h(i)h(j)E^{\widetilde{x}}(X_i(t)X_j(t)) \\ &= \langle h,\widetilde{x}\rangle_\pi^2 + \Big(\alpha^2 + \int u^2\rho(\mathrm{d}u)\Big)\int_0^t \sum_k \pi_k^2 h^2(k)E^{\widetilde{x}}(X_k^2(r))\mathrm{d}r \\ &\leqslant \langle h,\widetilde{x}\rangle_\pi^2 + \Big(\alpha^2 + \int u^2\rho(\mathrm{d}u)\Big)\int_0^t E^{\widetilde{x}}(\langle h, X(r)\rangle_\pi^2)\mathrm{d}r. \end{aligned}\tag{A21}$$

Similar to (A21), we have

$$E^{\widetilde{x}}(\langle h, X^{(n)}(t)\rangle_\pi^2) \leqslant \langle h,\widetilde{x}\rangle_\pi^2 + \Big(\alpha^2 + \int u^2\rho(\mathrm{d}u)\Big)\int_0^t E^{\widetilde{x}}(\langle h, X^{(n)}(r)\rangle_\pi^2)\mathrm{d}r. \tag{A22}$$

Note that

$$\begin{aligned}E^{\widetilde{x}}(\langle h, X^{(n)}(t)\rangle_\pi^2) &= E^{\widetilde{x}}\Big(\sum_{i\in S_n} h(i)\pi_i X_i^{(n)}(t) + \sum_{i\in S\setminus S_n} h(i)\pi_i X_i(0)\Big)^2 \\ &\leqslant 2E^{\widetilde{x}}\Big(\sum_{i\in S_n} h(i)\pi_i X_i^{(n)}(t)\Big)^2 + 2E^{\widetilde{x}}\Big(\sum_{i\in S\setminus S_n} h(i)\pi_i\widetilde{x}_i\Big)^2 \\ &\leqslant \sum_{i,j\in S_n} h(i)h(j)\pi_i\pi_j[E^{\widetilde{x}}(X_i^{(n)}(t))^2 + E^{\widetilde{x}}(X_j^{(n)}(t))^2] \\ &\quad + 2\Big(\sum_{i\in S\setminus S_n} h(i)\pi_i\widetilde{x}_i\Big)^2.\end{aligned}$$

By Lemma A1 and using the fact that $\widetilde{x} \in \Xi_F$, for $t \geqslant 0$, we have

$$E^{\widetilde{x}}(\langle h, X^{(n)}(t)\rangle_\pi^2) < \infty.$$

Thus, by Gronwall's inequality, (A22) implies that for all $\widetilde{x} \in \Xi_F$ and $t \geqslant 0$,

$$E^{\widetilde{x}}(\langle h, X^{(n)}(t)\rangle_\pi^2) \leqslant \langle h, \widetilde{x}\rangle_\pi^2 \mathrm{e}^{ct}, \tag{A23}$$

where $c = \alpha^2 + \int u^2 \rho(\mathrm{d}u)$.

Note that

$$\lim_{n\to\infty} E^{\widetilde{x}} \|X^{(n)}(t) - X(t)\|_{L^2(\gamma)}^2 = 0.$$

Then there exists an increasing sequence $\{n_k\}$ tending to ∞, such that

$$X^{(n_k)}(t) \to X(t) \text{ a.s.}, \quad n_k \to \infty,$$

and

$$X_i^{(n_k)}(t) \to X_i(t) \text{ a.s.}, \quad n_k \to \infty.$$

By Fatou's lemma, and combining with $\langle h, X(t)\rangle_\pi \geqslant 0$, we have

$$\begin{aligned}
E^{\widetilde{x}}(\langle h, X(t)\rangle_\pi^2) &= E^{\widetilde{x}}\bigg(\sum_{i\in S} h(i)\pi_i X_i(t)\bigg)^2 \\
&= E^{\widetilde{x}}\bigg(\sum_{i\in S} h(i)\pi_i \liminf_{n_k\to\infty} X_i^{(n_k)}(t)\bigg)^2 \\
&\leqslant E^{\widetilde{x}}(\liminf_{n_k\to\infty} \langle h, X^{(n_k)}(t)\rangle_\pi^2) \\
&\leqslant \liminf_{n_k\to\infty} E^{\widetilde{x}}(\langle h, X^{(n_k)}(t)\rangle_\pi^2) \\
&\leqslant \langle h, \widetilde{x}\rangle_\pi^2 \mathrm{e}^{ct} \\
&< \infty.
\end{aligned} \tag{A24}$$

□

Lemma A8 $\langle h, X(t, \widetilde{x})\rangle_\pi$ *is a càdlàg process.*

Proof For any $n \in \mathbb{N}$, we have

$$\begin{aligned}
\langle h, X^{(n)}(t, \widetilde{x})\rangle_\pi = \langle h, \widetilde{x}\rangle_\pi &+ \alpha \sum_{i\in S} \int_0^t \pi_i h(i) X_i^{(n)}(s, \widetilde{x}) \mathrm{d}B_i(s) \\
&+ \sum_{i\in S} \int_0^t \int_{\mathbb{R}\setminus\{0\}} u\pi_i h(i) X_i^{(n)}(s-, \widetilde{x}) \widetilde{N}_i(\mathrm{d}s\mathrm{d}u).
\end{aligned} \tag{A25}$$

By the Burkholder-Davis-Gundy inequality, we have

$$
\begin{aligned}
&E^{\widetilde{x}}[\sup_{t\in[0,T]}\langle h, X^{(n)}(t,\widetilde{x})\rangle_\pi^2]\\
&\leqslant 3\langle h,\widetilde{x}\rangle_\pi^2 + 3\alpha^2 E^{\widetilde{x}}\bigg[\sup_{t\in[0,T]}\Big|\sum_{i\in S}\int_0^t \pi_i h(i)X_i^{(n)}(s,\widetilde{x})\mathrm{d}B_i(s)\Big|^2\bigg]\\
&\quad + 3E^{\widetilde{x}}\bigg[\sup_{t\in[0,T]}\Big|\sum_{i\in S}\int_0^t\int_{\mathbb{R}\backslash\{0\}} u\pi_i h(i)X_i^{(n)}(s-,\widetilde{x})\widetilde{N}_i(\mathrm{d}s\mathrm{d}u)\Big|^2\bigg]\\
&\leqslant 3\langle h,\widetilde{x}\rangle_\pi^2 + 3\alpha^2 E^{\widetilde{x}}\bigg[\sum_{i\in S}\int_0^T \pi_i^2 h^2(i)(X_i^{(n)}(s,\widetilde{x}))^2\mathrm{d}s\bigg]\\
&\quad + 3E^{\widetilde{x}}\bigg[\sum_{i\in S}\int_0^T\int_{\mathbb{R}\backslash\{0\}} u^2\pi_i^2 h(i)(X_i^{(n)}(s,\widetilde{x}))^2\mathrm{d}s\rho(\mathrm{d}u)\bigg]\\
&\leqslant 3\langle h,\widetilde{x}\rangle_\pi^2 + \bigg(3\alpha^2 + 3\int u^2\rho(\mathrm{d}u)\bigg)E^{\widetilde{x}}\bigg[\sum_{i\in S}\int_0^T \pi_i^2 h^2(i)(X_i^{(n)}(s,\widetilde{x}))^2\mathrm{d}s\bigg]\\
&\leqslant 3\langle h,\widetilde{x}\rangle_\pi^2 + \bigg(3\alpha^2 + 3\int u^2\rho(\mathrm{d}u)\bigg)E^{\widetilde{x}}\bigg[\int_0^T \langle h, X^{(n)}(t,\widetilde{x})\rangle_\pi^2\mathrm{d}t\bigg]\\
&\leqslant 3\langle h,\widetilde{x}\rangle_\pi^2 + \bigg(3\alpha^2 + 3\int u^2\rho(\mathrm{d}u)\bigg)\int_0^T E^{\widetilde{x}}[\sup_{s\in[0,t]}\langle h, X^{(n)}(s,\widetilde{x})\rangle_\pi^2]\mathrm{d}t.
\end{aligned}
$$

By Gronwall's inequality, we have

$$
E^{\widetilde{x}}[\sup_{t\in[0,T]}\langle h, X^{(n)}(t,\widetilde{x})\rangle_\pi^2] \leqslant 3\langle h,\widetilde{x}\rangle_\pi^2 \mathrm{e}^{c_1 T}, \tag{A26}
$$

here $c_1 = 3\alpha^2 + 3\int u^2\rho(\mathrm{d}u)$.

For any $\{\tau_n,\delta_n\}$ satisfying (i) and (ii) in Lemma A6 (here using $\langle h, X^{(n)}(t,\widetilde{x})\rangle_\pi$ instead of $X_i^{(n)}(t)$), by (A25) and (A26), we have

$$
\begin{aligned}
&E^{\widetilde{x}}(\langle h, X^{(n)}(\tau_n+\delta_n) - X^{(n)}(\tau_n)\rangle_\pi^2)\\
&\leqslant 2\alpha^2 E^{\widetilde{x}}\bigg[\Big|\sum_{i\in S}\int_{\tau_n}^{\tau_n+\delta_n} \pi_i h(i)X_i^{(n)}(s,\widetilde{x})\mathrm{d}B_i(s)\Big|^2\bigg]\\
&\quad + 2E^{\widetilde{x}}\bigg[\Big|\sum_{i\in S}\int_{\tau_n}^{\tau_n+\delta_n}\int_{\mathbb{R}\backslash\{0\}} u\pi_i h(i)X_i^{(n)}(s-,\widetilde{x})\widetilde{N}_i(\mathrm{d}s\mathrm{d}u)\Big|^2\bigg]\\
&\leqslant \bigg(2\alpha^2 + 2\int u^2\rho(\mathrm{d}u)\bigg)E^{\widetilde{x}}\bigg[\sum_{i\in S}\int_{\tau_n}^{\tau_n+\delta_n} \pi_i^2 h^2(i)(X_i^{(n)}(s,\widetilde{x}))^2\mathrm{d}s\bigg]\\
&\leqslant \bigg(2\alpha^2 + 2\int u^2\rho(\mathrm{d}u)\bigg)E^{\widetilde{x}}(\sup_{r\in[0,T]}\langle h, X^{(n)}(r)\rangle_\pi^2)\cdot\delta_n\\
&\leqslant 6\bigg(\alpha^2 + \int u^2\rho(\mathrm{d}u)\bigg)\langle h,\widetilde{x}\rangle_\pi^2 \mathrm{e}^{c_1 T}\cdot\delta_n.
\end{aligned}
\tag{A27}
$$

Similar to Lemma A6, using Aldous's tightness criterion in $D[0,\infty;\mathbb{R}]$ (see [1]), we can get that $\{\langle h, X^{(n)}(t)\rangle_\pi,\ t\geqslant 0\}_{n\in\mathbb{N}}$ is tight in $D[0,\infty;\mathbb{R}]$ by (A22) and (A27). Thus, $\langle h, X(t,\widetilde{x})\rangle_\pi$ is a càdlàg process. □

A2 Proof of Proposition 2.5

Let $\{S_n\}$ be an increasing sequence of finite subset of S satisfying $\cup_{n\geqslant 1}S_n = S$. We set $\{\Phi_i,\ i\in S_n\}$ be a family of independent and identically distributed compound Poisson processes with characteristic measure of $\varrho(\mathrm{d}u)$ on $\mathbb{R}$. We consider the following stochastic differential equation (SDE):

$$X_i^{(n)}(t) = \begin{cases} X_i(0) + \displaystyle\int_0^t \kappa \sum_{j\in S} a(i,j) X_j^{(n)}(s)\mathrm{d}s \\ \quad + \displaystyle\int_0^{t+}\int_{\mathbb{R}} X_i^{(n)}(s-)u\Phi_i(\mathrm{d}s,\mathrm{d}u), \quad i\in S_n, \\ X_i(0), \quad i\notin S_n. \end{cases} \tag{A28}$$

$$\Xi_F \equiv \{x\in[0,\infty)^S \mid x_j = 0 \text{ for all but finitely many } j\in S\}, \tag{A29}$$

$$L^{\infty,+}(S) \equiv \{x\in[0,\infty)^S \mid \sup_{j\in S}|x_j| < \infty\}, \tag{A30}$$

Lemma A9 *Let* $(X^{(n)}(t,x))_{t\geqslant 0}$ *be the solution of* (A28) *with initial state* $x\in L^\infty(S)$, *and let* $(\widetilde{X}^{(n)}(t,\widetilde{x}))_{t\geqslant 0}$ *be the solution of* (A28) *with initial state* $\widetilde{x}\in\Xi_F$. *Then*

$$\langle X^{(n)}(t,x), \widetilde{x}\rangle_\pi \overset{d}{=} \langle \widetilde{X}^{(n)}(t,\widetilde{x}), x\rangle_\pi. \tag{A31}$$

Proof Let $\Phi^{S_n} \equiv (\Phi_i,\ i\in S_n)$. Then Φ^{S_n} is also a compound Poisson process with characteristic measure of

$$\underbrace{\varrho(\mathrm{d}u)\times\cdots\times\varrho(\mathrm{d}u)}_{\mathrm{card}(S_n)}$$

on $\mathbb{R}^{\mathrm{card}(S_n)}$. Let $\tau_1,\tau_2,\ldots$ be the jump times of the compound Poisson process Φ^{S_n}. Then we can construct $X_t^{(n)}$ on event $\{\tau_k\leqslant t<\tau_{k+1}\}$ by

$$\mathrm{e}^{(t-\tau_k)(\kappa A)}B_{k-1}\mathrm{e}^{(\tau_k-\tau_{k-1})(\kappa A)}B_{k-2}\cdots B_1\mathrm{e}^{\tau_1(\kappa A)}x, \tag{A32}$$

where

$$\mathrm{e}^{t(\kappa A)} = \sum_{k=0}^\infty \frac{(t(\kappa A))^n}{n!},$$

the semigroup generated by κA on $L^\infty(S)$, and B_k is finite-rank operator from $L^\infty(S)$ to $L^\infty(S)$ satisfying

$$(B_k x)_i = \begin{cases} (\Phi_i(\tau_{k+1}) - \Phi_i(\tau_k))x_i, & i\in S_n, \\ 0, & i\notin S. \end{cases}$$

Because $\sup_{i\in S}|a(i,i)|<\infty$, formula (A32) is well-posed. Since $\widetilde{x}\in\Xi_F$, A is symmetric with respect to π, and B_k is a diagonal matrix, we have

$$\begin{aligned}&\langle \mathrm{e}^{(t-\tau_k)(\kappa A)}B_{k-1}\mathrm{e}^{(\tau_k-\tau_{k-1})(\kappa A)}B_{k-2}\cdots B_1\mathrm{e}^{\tau_1(\kappa A)}x,\widetilde{x}\rangle_\pi\\&\quad=\langle x,\mathrm{e}^{\tau_1(\kappa A)}xB_1\cdots \mathrm{e}^{(\tau_k-\tau_{k-1})(\kappa A)}B_{k-1}\mathrm{e}^{(t-\tau_k)(\kappa A)}\widetilde{x}\rangle_\pi.\end{aligned}$$

Note that

$$(\tau_1,B_1,\tau_2-\tau_1,B_2,\dots,B_{k-1},t-\tau_k),\quad (t-\tau_k,B_{k-1},\dots,B_2,\tau_2-\tau_1,B_1,\tau_1) \tag{A33}$$

have the same distribution conditioned on event $\{\tau_k\leqslant t<\tau_{k+1}\}$. We get the lemma. □

Let $\{\Lambda_i^{(m)},\ i\in S_n\}$ be a family of independent and identically distributed Poisson processes with intensity m on the same probability space $(\Omega,\mathscr{F},(\mathscr{F})_t,P)$ as $(Y_i,\ i\in S)$, which are independent of $\{Y_i,\ i\in S_n\}$.

We set

$$\Phi_i^{(m)}(t)=Y_i\Big(\frac{\Lambda_i^{(m)}(t)}{m}\Big),\ i\in S_n,\ t\geqslant 0;\quad \Phi^{S_n,m}=(\Phi_i^{(m)},\ i\in S_n).$$

$\Phi_i^{(m)}$ is a compound Poisson process with characteristic measure of $m\cdot\varrho_m(\mathrm{d}x)$, where

$$\varrho_m(\mathrm{d}x)=P\Big(Y_i\Big(\frac{1}{m}\Big)\in \mathrm{d}x\Big).$$

Now, for any $x\in L^2(\gamma)$, we consider the following SDE:

$$\overline{X}_i^{(n,m)}(t)=\begin{cases} x_i+\displaystyle\int_0^t\kappa\sum_{j\in S}a(i,j)\overline{X}_j^{(n,m)}(s)\mathrm{d}s\\ \qquad+\displaystyle\int_0^{t+}\int_{\mathbb{R}}\overline{X}_i^{(n,m)}(s-)u\Phi_i^{(m)}(\mathrm{d}s,\mathrm{d}u),\quad i\in S_n,\\ x_i,\quad i\notin S_n.\end{cases} \tag{A34}$$

Let $C^2_{\rm ub}(\mathbb{R}^n)$ be the collection of bounded, uniformly continuous, and twice smooth functions on $\mathbb{R}^n$, and let $C_{\rm ub}(L^2(\gamma))$ be the collection of bounded uniformly continuous functions on $L^2(\gamma)$. Without loss of generality, we set $\mathrm{card}(S_n)=n$.

Lemma A10 *For any $f\in C_{\rm ub}(L^2(\gamma))$, $x\in L^2(\gamma)$, and $t\geqslant 0$, we have*

$$\lim_{m\to\infty}E(f(\overline{X}^{(n,m)}(t)))=E(f(X^{(n)}(t)), \tag{A35}$$

where $(\overline{X}_i,\ i\in S)$ is the solution of (A2) *with initial state x.*

Proof Denote by $\mathscr{L}^n$ and $\mathscr{L}^{n,m}$ the generators of $X^{(n)}$ and $X^{(n,m)}$ on $C^2_{\mathrm{ub}}(\mathbb{R}^n)$, respectively. For $f \in C^2_{\mathrm{ub}}(\mathbb{R}^n)$, we have

$$\begin{aligned}\mathscr{L}^n f(x_i, i\in S_n) = &\sum_{i\in S_n}\left(\sum_{j\in S}\kappa a(i,j)x_j + \beta x_i\right)\frac{\partial f}{\partial x_i} + \frac{\alpha^2}{2}\sum_{i\in S_n} x_i^2\frac{\partial^2 f}{\partial x_i^2}\\ &+\sum_{i\in S_n}\int_{\{|u|\geqslant\delta\}}(f(x_i+x_iu)-f(x_i))x_iu\rho(\mathrm{d}u)\\ &+\sum_{i\in S_n}\int_{\{|u|<\delta\}}\Big(f(x_i+x_iu)-f(x_i)-x_iu\frac{\partial f}{\partial x_i}\Big)\rho(\mathrm{d}u),\end{aligned}$$

$$\begin{aligned}&\mathscr{L}^{n,m} f(x_i, i\in S_n)\\ &=\sum_{i\in S_n}\left(\sum_{j\in S}\kappa a(i,j)x_j+\beta x_i\right)\frac{\partial f}{\partial x_i}+\sum_{i\in S_n}\int_{\mathbb{R}}(f(x_i+x_iu)-f(x))m\varrho_m(\mathrm{d}u)\\ &=\sum_{i\in S_n}\left(\sum_{j\in S}\kappa a(i,j)x_j+\beta x_i\right)\frac{\partial f}{\partial x_i}+m\sum_{i\in S_n}E\Big[f\Big(x_i+x_iY_i\Big(\frac{1}{m}\Big)\Big)-f(x)\Big]\\ &=\sum_{i\in S_n}\left(\sum_{j\in S}\kappa a(i,j)x_j+\beta x_i\right)\frac{\partial f}{\partial x_i}+\frac{m\alpha^2}{2}\sum_{i\in S_n}x_i^2\int_0^{1/m}\frac{\partial^2 f}{\partial x_i^2}(x_i+x_iY_i(s))\mathrm{d}s\\ &\quad+m\sum_{i\in S_n}\int_0^{1/m}\int_{\{|u|\geqslant\delta\}}(f(x_i+x_iY_i(s-)+u)-f(x_i+x_iY_i(s-)))\mathrm{d}s\rho(\mathrm{d}u)\\ &\quad+m\sum_{i\in S_n}\int_0^{1/m}\int_{\{|u|<\delta\}}\Big(f(x_i+x_iY_i(s-)+u)-f(x_i+x_iY_i(s-))\\ &\quad-x_iu\frac{\partial f}{\partial x_i}(x_i+x_iY_i(s-))\Big)\mathrm{d}s\rho(\mathrm{d}u).\end{aligned}\tag{A36}$$

These imply that for any compact set $A\subset\mathbb{R}^n$,

$$\lim_{m\to\infty}\sup_{(x_i,i\in S_n)\in A}|\mathscr{L}^{(n,m)}f(x)-\mathscr{L}^{(n)}f(x)|=0.\tag{A37}$$

By the standard argument of approximations of martingale problem for jump diffusions (see [10,23,24,32]), we can get $(\overline{X}^{(n,m)}(t))_{t\geqslant0}$ converges weakly to $(\overline{X}^{(n)}(t))_{t\geqslant0}$ as $m\to\infty$. Therefore, it is not hard to show (A35). □

Lemma A11 *Let* $(X^{(n)}(t))_{t\geqslant0}$ *be the solution of* (A2) *with initial state* $x\in L^{\infty,+}(S)$, *and let* $(\widetilde{X}^{(n)}(t))_{t\geqslant0}$ *be the solution of* (A2) *with initial state* $\widetilde{x}\in\Xi_F$. *Then*

$$\langle X^{(n)}(t),\widetilde{x}\rangle_\pi\overset{d}{=}\langle\widetilde{X}^{(n)}(t),x\rangle_\pi.\tag{A38}$$

Proof For any $z\in\Xi_F$ and $f\in C_{\mathrm{ub}}(\mathbb{R})$, $f(\langle x,z\rangle_\pi)\in C_{\mathrm{ub}}(L^2(\gamma))$. Due to Lemma A10, we have

$$\lim_{m\to\infty}E(f(\langle\overline{X}^{n,m}(t),\widetilde{x}\rangle_\pi))=E(f(\langle X^n(t),\widetilde{x}\rangle_\pi)),\tag{A39}$$

$$\lim_{m\to\infty} E(f(\langle \overline{\widetilde{X}}^{n,m}(t), x\rangle_\pi)) = E(f(\langle \widetilde{X}^n(t), x\rangle_\pi)). \tag{A40}$$

Thus, (A38) follows from Lemma A9. □

Proof of Proposition 2.5 Since (A16) and $(X(t))_{t\geqslant 0}$ is the solution of (1.5), we obtain that $E\|X^{(n)}(t) - X(t)\|^2_{\gamma,2} \to 0$. Therefore, using a same argument as Lemma A11, for any $x, \widetilde{x} \in \Xi_F$, we can show

$$\langle X(t), \widetilde{x}\rangle_\pi \stackrel{d}{=} \langle \widetilde{X}(t), x\rangle_\pi. \tag{A41}$$

For $x \in L^{\infty,+}(S)$, denote

$$x_i^{(n)} = \begin{cases} x_i, & i \in S_n, \\ 0, & i \notin S_n, \end{cases}$$

and let $X^{(n)}$ be the solution of (A2) with initial state $x^{(n)}$. On the one hand, since $E\|X^{(n)}(t) - X(t)\|^2_{\gamma,2} \to 0$, for any $f \in C_{\rm ub}(\mathbb{R})$, we have

$$\lim_{n\to\infty} E(f(\langle X^{(n)}(t), \widetilde{x}\rangle_\pi)) = E(f(\langle X(t), \widetilde{x}\rangle_\pi)). \tag{A42}$$

On the other hand, there is a constant $0 < M(x, \widetilde{x}, \pi) < \infty$ dependent on x, $\widetilde{x}$, π only, such that $E(\langle X^{(n)}(t), \widetilde{x}\rangle^2_\pi) \leqslant M$. Since

$$0 \leqslant \langle \widetilde{X}(t), x^{(n)}\rangle_\pi \uparrow \langle \widetilde{X}(t), x\rangle_\pi, \quad \text{a.s. } dP,$$

this shows

$$E(\langle \widetilde{X}(t), x\rangle^2_\pi) = \lim_{n\to\infty} E(\langle X^{(n)}(t), \widetilde{x}\rangle^2_\pi) \leqslant M. \tag{A43}$$

Therefore, $P(\langle \widetilde{X}(t), x\rangle_\pi < \infty) = 1$. Thus,

$$\lim_{n\to\infty} E(f(\langle \widetilde{X}(t), x^{(n)}\rangle_\pi)) = E(f(\langle \widetilde{X}(t), x\rangle_\pi)). \tag{A44}$$

Combining (A41), (A42), and (A44), we obtain the self-duality of (1.5). □

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