

QUANTUM STOCHASTIC CONVOLUTION COCYCLES II

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ABSTRACT. Schürmann’s theory of quantum Lévy processes, and more generally the theory of quantum stochastic convolution cocycles, is extended to the topological context of compact quantum groups and operator space coalgebras. Quantum stochastic convolution cocycles on a C^* -hyperbialgebra, which are Markov-regular, completely positive and contractive, are shown to satisfy coalgebraic quantum stochastic differential equations with completely bounded coefficients, and the structure of their stochastic generators is obtained. Automatic complete boundedness of a class of derivations is established, leading to a characterisation of the stochastic generators of $*$ -homomorphic convolution cocycles on a C^* -bialgebra. Two tentative definitions of quantum Lévy process on a compact quantum group are given and, with respect to both of these, it is shown that an equivalent process on Fock space may be reconstructed from the generator of the quantum Lévy process. In the examples presented, connection to the algebraic theory is emphasised by a focus on full compact quantum groups.

INTRODUCTION

In this paper we investigate quantum stochastic evolutions with independent identically distributed increments on compact quantum groups, in other words *quantum Lévy processes*. The natural setting for this analysis is the somewhat wider one of quantum stochastic convolution cocycles. For a compact quantum group \mathbb{B} , a quantum stochastic convolution cocycle on \mathbb{B} is a family of linear maps $(l_t)_{t \geq 0}$ from \mathbb{B} to operators on the symmetric Fock space \mathcal{F} , over a Hilbert space of the form $L^2(\mathbb{R}_+; \mathfrak{k})$, satisfying

$$l_{s+t} = l_s \star (\sigma_s \circ l_t), \quad s, t \geq 0$$

and some regularity and natural adaptedness conditions. Here $(\sigma_s)_{s \geq 0}$ is the semi-group of time-shifts on $B(\mathcal{F})$ and the convolution is induced by the quantum group structure; the initial condition is specified by the counit: $l_0 = \epsilon(\cdot)I_{\mathcal{F}}$. Thus the increment of the process over the interval $[0, s+t]$ coincides with the increment over $[0, s]$ convolved with the (shifted) increment over $[0, t]$. We show that such families may be obtained as solutions of quantum stochastic differential equations with completely bounded coefficients, we analyse their positivity and multiplicativity properties, and we establish natural conditions under which all sufficiently regular cocycles arise in this way. Motivated by these results (and the purely algebraic theory), we propose two abstract definitions of quantum Lévy process on a compact quantum group and show that any process which has bounded ‘generator’ has an equivalent Fock space realisation. Precise definitions are given below.

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2000 *Mathematics Subject Classification.* Primary 46L53, 81S25; Secondary 22A30, 47L25, 16W30.

Key words and phrases. Noncommutative probability, quantum stochastic, compact quantum group, C^* -bialgebra, C^* -hyperbialgebra, operator space, stochastic cocycle, quantum Lévy process.

Stochastic cocycles on operator algebras were introduced by Accardi (under the name quantum Markovian cocycles) for Feynman-Kac type perturbation of quantum dynamical semigroups ([Acc]). Earlier work on a cocycle approach to classical Markov processes and their Itô integral representation may be found in [Pin]. Quantum stochastic differential equations ([HuP]) were quickly seen to provide examples of stochastic cocycles and in fact to characterise large classes of them in the Fock space context (see [L] and references therein).

The theory of quantum Lévy processes, developed by Schürmann and others, generalises the classical theory of Lévy processes on groups ([Hey]), and Skorohod's theory of stochastic semigroups ([Sko]), to the context of quantum groups or, more generally, $*$ -bialgebras (see [Sch], [FrScho], [Glo] and references therein). A quantum Lévy process on a quantum group \mathcal{B} is a time-indexed family of unital $*$ -homomorphisms from \mathcal{B} to some noncommutative probability space, with identically distributed and (tensor-)independent increments, satisfying the convolution increment relation given by the coproduct of \mathcal{B} , and with initial condition given by the counit of \mathcal{B} . Schürmann showed that each quantum Lévy process may be equivalently realised in a symmetric Fock space as a solution of a quantum stochastic differential equation. This led us to introduce and investigate, in this algebraic context, quantum stochastic convolution cocycles ([LS₁]). These are linear (but not necessarily unital or $*$ -homomorphic) maps from a coalgebra to a space of Fock space operator processes, satisfying the convolution increment relation and counital initial condition.

In the last twenty years there has been a growing interest in the theory of *topological* quantum groups. Starting from the fundamental paper of Woronowicz ([Wor₁]), where the concept of compact quantum groups was first introduced (under the name of compact matrix pseudogroups), it has led to a rich and well-developed theory, with a satisfactory notion of *locally compact* quantum group eventually emerging in the work of Kustermans and Vaes ([KuV]). The main object becomes a C^* -algebra, equipped with a coproduct and counit satisfying a corresponding form of coassociativity and counit relations.

In this paper we go beyond the purely algebraic context treated in [LS₁] and initiate the study of quantum Lévy processes on a compact quantum group, or more generally on a C^* -bialgebra. Heeding P.-A. Meyer's dictum once more, we again set our work in the wider context of quantum stochastic convolution cocycles on a coalgebra. The coalgebras here though are operator-space-theoretic rather than being purely algebraic. Nevertheless the stochastic cocycles in question may be obtained by solving coalgebraic quantum stochastic differential equations. In turn, every sufficiently regular completely positive and contractive quantum stochastic convolution cocycle on a C^* -hyperbialgebra is shown to satisfy a quantum stochastic differential equation of the above type. These results are obtained by, on the one hand applying techniques of operator space theory ([EfR], [Pis₂]), and on the other hand using known facts about standard quantum stochastic cocycles (see [LW₂], [L] and references therein). Here it is natural to work with processes on abstract operator spaces and C^* -bialgebras. For this we use theory developed in [LSa] and summarised in the first section. When the spaces are concrete this reduces to the existing theory. A key tool of our analysis is a convolution operation which we call the *R-map*. This transforms coalgebraic objects such as convolution cocycles and coalgebraic quantum stochastic differential equations to standard objects of quantum stochastic analysis ([L]), setting up a traffic of properties and relationships which we systematically exploit. The *R-map* gives rise to a noncommutative avatar of the transformation between convolution semigroups of measures and Markov semigroups (of operators), familiar from classical probability theory.

The structure of the stochastic generators of Markov-regular, $*$ -homomorphic convolution cocycles on a C^* -bialgebra may be characterised in terms of ϵ -structure maps, where ϵ is the counit, or *topological* Schürmann triples (cf. their purely algebraic counterparts). The complete boundedness of such generators, indeed their implementability, follows from their algebraic properties alone. We prove this by first extending well-known results of Sakai, Ringrose and Christensen, on automatic continuity and innerness properties of derivations, to the case of (π', π) -derivations. The fact that every ϵ -structure map defined on the whole C^* -bialgebra must be implemented may be viewed as a noncommutative counterpart to the fact that every classical Lévy process on a topological group which has a bounded generator must be a compound Poisson process. In this connection we note the definition of quantum Poisson process on a $*$ -bialgebra proposed in [Fra].

The axiomatisation of quantum Lévy processes on a C^* -bialgebra raises several problems connected with the fact that the product on a C^* -algebra A usually fails to extend to a continuous map from the spatial tensor product $A \otimes A$ to the algebra. We offer two different ways of overcoming this obstacle, for both of which a topological version of Schürmann's reconstruction theorem remains valid.

Our choice of examples is designed to expose the variety of connections of this work with the classical and quantum probabilistic literature. The analysis of quantum stochastic convolution cocycles in the topological context requires different methods and techniques to that of the purely algebraic and poses new nontrivial problems. However, according to our philosophy (explicitly described in the expository paper [LS₂]), purely algebraic and topological convolution cocycles may nevertheless usefully be viewed from a common vantage point. This perspective is particularly well illustrated in the last class of examples discussed here, namely that of $*$ -homomorphic quantum stochastic convolution cocycles on a full compact quantum group. We would also like to point out that conversely, due to the Fundamental Theorem on Coalgebras, one can view the purely algebraic situation as a *finite dimensional version* of the topological theory. An example of reasoning along these lines may be found in the final section of [LS₃].

The plan of the paper is as follows. In the first section we review the basic facts needed from operator space theory and quantum stochastic analysis. We work with processes on abstract operator spaces. The transition from concrete to abstract exploits a number of natural identifications and inclusions, the key ones being (1.2), (1.3) and (1.7). In Section 2 the notion of operator space coalgebra is introduced and basic properties of the R -map are established, facilitating a correspondence between mapping composition structures and convolution-type structures. Section 3 contains proofs of the existence, uniqueness and regularity of solutions of coalgebraic quantum stochastic differential equations with completely bounded coefficients. There also the ground is prepared for a traffic between standard quantum stochastic cocycles and quantum stochastic convolution cocycles. The latter are defined in Section 4 where the solutions of coalgebraic quantum stochastic differential equations are shown to lie in this class. The section concludes with a brief discussion of *opposite* convolution cocycles. In Section 5 the converse result is established for Markov-regular, completely positive, contractive quantum stochastic convolution cocycles on a C^* -hyperbialgebra: they are characterised as solutions of coalgebraic quantum stochastic differential equation with completely bounded coefficient of a particular form. Section 6 deals with $*$ -homomorphic convolution cocycles on a C^* -bialgebra. As in the purely algebraic case, their stochastic generators are characterised by structure relations involving the counit; in the topological case these amount to the generator being an ϵ -structure map where ϵ is the counit of the bialgebra. In Section 7 two candidates for the axiomatisation of

quantum Lévy processes on a C^* -bialgebra are proposed; firstly, in a weak sense of distributions, and secondly, as processes whose values are operators from a product system, in the sense of Arveson. Basic consequences of the proposed axioms are discussed, and reconstruction theorems established. Section 8 is devoted to examples, first the commutative case of classical compact groups, then the cocommutative case of the universal C^* -algebra of a discrete group, and finally the case of full compact quantum groups. In the latter case a link is established with the purely algebraic quantum stochastic convolution cocycles investigated in [LS₁]. In an appendix some results on derivations are established; these are applied to yield the automatic implementedness of ϵ -structure maps used in Section 6.

Some of the results proved here have been announced in [LS₂].

Note added in proof. It is now clear that our results extend to the context of locally compact quantum groups in the sense of Kustermans and Vaes ([LS₄]).

Notation. All vector spaces arising in this paper are complex; inner products (and all sesquilinear maps) are linear in their *second* argument. For a dense subspace E of a Hilbert space \mathfrak{h} , $\mathcal{O}(E)$ denotes the space of operators $\mathfrak{h} \rightarrow \mathfrak{h}$ with domain E and $\mathcal{O}^\ddagger(E) := \{T \in \mathcal{O}(E) : \text{Dom } T^* \supset E\}$. Thus $\mathcal{O}^\ddagger(E)$ has the natural conjugation $T \mapsto T^\ddagger := T^*|_E$. We view $B(\mathfrak{h})$ as a subspace of $\mathcal{O}^\ddagger(E)$ (via restriction/continuous linear extension). For vectors $\zeta \in E$ and $\zeta' \in \mathfrak{h}$, $\omega_{\zeta', \zeta}$ denotes the linear functional on $\mathcal{O}(E)$ given by $T \mapsto \langle \zeta', T\zeta \rangle$. We use the Dirac-inspired notations

$$|E\rangle := \{|\zeta\rangle : \zeta \in E\} \text{ and } \langle E| := \{\langle \zeta| : \zeta \in E\}$$

where $|\zeta\rangle \in |\mathfrak{h}\rangle := B(\mathbb{C}; \mathfrak{h})$ and $\langle \zeta| \in \langle \mathfrak{h}| := B(\mathfrak{h}; \mathbb{C})$ are defined by $\lambda \mapsto \lambda\zeta$ and $\eta \mapsto \langle \zeta, \eta \rangle$ respectively. A class of ampliations frequently met here is denoted as follows:

$$\iota_{\mathfrak{h}} : \mathbb{V} \rightarrow \mathbb{V} \otimes B(\mathfrak{h}), \quad x \mapsto x \otimes I_{\mathfrak{h}}, \quad (0.1)$$

where this time the operator space \mathbb{V} is determined by context (and \otimes denotes spatial tensor product).

For a vector-valued function f on \mathbb{R}_+ and subinterval I of \mathbb{R}_+ f_I denotes the function on \mathbb{R}_+ which agrees with f on I and vanishes outside I . Similarly, for a vector ξ , ξ_I is defined by viewing ξ as a constant function. This extends the standard indicator function notation. The symmetric measure space over the Lebesgue measure space \mathbb{R}_+ ([Gui]) is denoted Γ , with integration denoted $\int_{\Gamma} \cdots d\sigma$, thus $\Gamma = \{\sigma \subset \mathbb{R}_+ : \#\sigma < \infty\} = \bigcup_{n \geq 0} \Gamma^{(n)}$ where $\Gamma^{(n)} = \{\sigma \subset \mathbb{R}_+ : \#\sigma = n\}$ and \emptyset is an atom having unit measure. If \mathbb{R}_+ is replaced by a subinterval I then we write Γ_I and $\Gamma_I^{(n)}$, thus the measure of $\Gamma_I^{(n)}$ is $|I|^n/n!$ where $|I|$ denotes the length of I .

For a linear map $\psi : U \rightarrow V$ the corresponding linear map between conjugate vector spaces

$$U^\ddagger \rightarrow V^\ddagger, \quad x^\ddagger \mapsto \psi(x)^\ddagger \quad (0.2)$$

is denoted ψ^\ddagger ; $L(U^\ddagger; V^\ddagger)$ is thereby the natural conjugate space of $L(U; V)$. The collection of sesquilinear maps $\phi : U \times V \rightarrow W$ is denoted $SL(U, V; W)$; when W is a space of maps we denote values of ϕ by $\phi^{u,v}$ ($u \in U, v \in V$). The collection of bilinear maps $U \times V \rightarrow W$ is denoted $L(U, V; W)$. If \mathcal{A} is an involutive algebra and E is a dense subspace of a Hilbert space \mathfrak{h} then *weak multiplicativity* for a map $\phi : \mathcal{A} \rightarrow \mathcal{O}^\ddagger(E)$, is the property

$$\phi(a^*b) = \phi^\ddagger(a)^* \phi(b) \quad (a, b \in \mathcal{A}), \quad (0.3)$$

where $\phi^\ddagger : a \mapsto \phi(a^*)^*|_E$.

Remark. If $\phi : \mathcal{A} \rightarrow \mathcal{O}^\ddagger(E)$ is a linear map, defined on a C^* -algebra, which is *real* (that is $\phi = \phi^\ddagger$) and weakly multiplicative then ϕ is necessarily bounded-operator-valued and thus may be viewed as a $*$ -homomorphism $\mathcal{A} \rightarrow B(\mathfrak{h})$.

1. OPERATOR SPACE AND QUANTUM STOCHASTIC PRELIMINARIES

In this section we collect some relevant facts from operator space theory, recall the matrix-space construction and describe the basic properties of tensor-extended compositions. We also recall relevant results from quantum stochastic (QS) analysis.

Operator spaces ([Efr], [Pis₂]). For operator spaces V and W the Banach space of completely bounded maps from V to W is endowed with operator space structure via the linear identifications

$$M_n(CB(V; W)) = CB(V; M_n(W)) \quad (n \in \mathbb{N}),$$

where $M_n(W)$ denotes the linear space $M_n(W)$ with its natural operator space structure. When viewed as a C^* -algebra or operator space, $M_n(\mathbb{C})$ is denoted M_n .

The operator space spatial/minimal tensor product of V and W is here denoted simply $V \otimes W$. For example $M_n(W)$ may be identified with the spatial tensor product $W \otimes M_n$. When V and W are realised in $B(H)$ and $B(K)$ respectively, $V \otimes W$ is realised concretely in $B(H \otimes K) = B(H) \overline{\otimes} B(K)$ as the norm closure of the algebraic tensor product $V \odot W$. In fact $V \otimes W$ does not depend on concrete realisation of V and W ; an abstract model arises from the natural linear embedding

$$V \odot W \hookrightarrow CB(W^*; V) \tag{1.1}$$

(where W^* is defined below). For any completely bounded maps $\phi : V \rightarrow V'$ and $\psi : W \rightarrow W'$ into further operator spaces, the linear map $\phi \odot \psi$ extends uniquely to a completely bounded map $V \otimes W \rightarrow V' \otimes W'$; the extension is denoted $\phi \otimes \psi$ and satisfies $\|\phi \otimes \psi\|_{cb} = \|\phi\|_{cb} \|\psi\|_{cb}$. Each bounded operator $\phi : V \rightarrow M_n$ is automatically completely bounded and satisfies $\|\phi\|_{cb} = \|\phi^{(n)}\|$ in the notation $\phi^{(n)} : [x_{ij}] \mapsto [\phi(x_{ij})]$, in other words $\phi^{(n)} = \phi \otimes \text{id}_{M_n}$. In particular, the operator space $CB(V; \mathbb{C})$ coincides with the Banach space dual $B(V; \mathbb{C})$ and has the same norm; it is therefore denoted V^* . Note the natural completely isometric isomorphisms

$$CB(U, V; W) = CB(U; CB(V; W)) \tag{1.2}$$

for operator spaces U, V and W . We shall also exploit the natural completely isometric inclusions

$$V \otimes B(H; H') \hookrightarrow CB(\langle H' |, |H \rangle; V) \tag{1.3}$$

for operator space V and Hilbert spaces H and H' . (See below for the tensor product which delivers *isomorphism* here.)

The following short-hand notation for tensor-extended composition is useful. Let U, V, W and X be operator spaces, and let V be a vector space. If $\phi \in L(V; U \otimes V \otimes W)$ and $\psi \in CB(V; X)$ then we compose in the obvious way:

$$\psi \bullet \phi := (\text{id}_U \otimes \psi \otimes \text{id}_W) \circ \phi \in L(V; U \otimes X \otimes W). \tag{1.4}$$

Ambiguity is avoided provided that the context dictates which tensor component the second-to-be-applied map ψ should act on. This also applies to the case where $\phi \in SL(H', H; L(V; V))$ as follows: $\psi \bullet \phi \in SL(H', H; L(V; X))$ is given by

$$(\psi \bullet \phi)^{\xi', \xi} = \psi \circ \phi^{\xi', \xi}. \tag{1.5}$$

The natural inclusion $L(V; CB(\langle H' |, |H \rangle; V)) \subset SL(H', H; L(V; V))$ is relevant here.

Matrix spaces ([LW₃]). For an operator space Y in $B(\mathsf{H}; \mathsf{H}')$ and Hilbert spaces \mathfrak{h} and \mathfrak{h}' define

$$\mathsf{Y} \otimes_{\mathsf{M}} B(\mathfrak{h}; \mathfrak{h}') := \{T \in B(\mathsf{H} \otimes \mathfrak{h}; \mathsf{H}' \otimes \mathfrak{h}') = B(\mathsf{H}; \mathsf{H}') \overline{\otimes} B(\mathfrak{h}; \mathfrak{h}') : \Omega_{\zeta', \zeta}^{\mathsf{Y}}(T) \in \mathsf{Y}\} \quad (1.6)$$

where $\Omega_{\zeta', \zeta}^{\mathsf{Y}}$ denotes the slice map $\text{id}_{\mathsf{Y}} \overline{\otimes} \omega_{\zeta', \zeta}$. For us the relevant cases are $\mathsf{Y} \otimes_{\mathsf{M}} B(\mathfrak{h})$ and $\mathsf{Y} \otimes_{\mathsf{M}} |\mathfrak{h}\rangle$, referred to respectively as the \mathfrak{h} -matrix space over Y and the \mathfrak{h} -column space over Y . Matrix spaces are operator spaces which lie between the spatial tensor product $\mathsf{Y} \otimes B(\mathfrak{h}; \mathfrak{h}')$ and the ultraweak tensor product $\overline{\mathsf{Y}} \overline{\otimes} B(\mathfrak{h}; \mathfrak{h}')$, coinciding with the latter when Y is ultraweakly closed ($\overline{\mathsf{Y}}$ here denotes the ultraweak closure of Y). They arise naturally in quantum stochastic analysis where a topological state space is to be coupled with the measure-theoretic noise — if Y is a C^* -algebra then typically the inclusion $\mathsf{Y} \otimes B(\mathfrak{h}) \subset \mathsf{Y} \otimes_{\mathsf{M}} B(\mathfrak{h})$ is proper and $\mathsf{Y} \otimes_{\mathsf{M}} B(\mathfrak{h})$ is *not* a C^* -algebra. Completely bounded maps between concrete operator spaces lift to completely bounded maps between corresponding matrix spaces: if Y' is another concrete operator space, for $\phi \in CB(\mathsf{Y}; \mathsf{Y}')$ there is a unique map $\Phi : \mathsf{Y} \otimes_{\mathsf{M}} B(\mathfrak{h}; \mathfrak{h}') \rightarrow \mathsf{Y}' \otimes_{\mathsf{M}} B(\mathfrak{h}; \mathfrak{h}')$ satisfying

$$\Omega_{\zeta', \zeta}^{\mathsf{Y}'} \circ \Phi = \phi \circ \Omega_{\zeta', \zeta}^{\mathsf{Y}} \quad (\zeta \in \mathfrak{h}, \zeta' \in \mathfrak{h}');$$

it is denoted $\phi \otimes_{\mathsf{M}} \text{id}_{B(\mathfrak{h}; \mathfrak{h}')}$. Using these matrix liftings, tensor-extended compositions work in the same way for matrix spaces as for spatial tensor products. There are natural completely isometric isomorphisms

$$\mathsf{Y} \otimes_{\mathsf{M}} B(\mathfrak{h}; \mathfrak{h}') = CB(\langle \mathfrak{h}' |, |\mathfrak{h}\rangle; \mathsf{Y}) \quad (1.7)$$

(cf. (1.3)) under which $\phi \otimes_{\mathsf{M}} \text{id}_{B(\mathfrak{h}; \mathfrak{h}')}$ corresponds to $\phi \circ$, *composition with ϕ* ([LSa]). The two tensor-extended compositions are consistent.

Quantum stochastic ([Par], [Mey]; we follow [L], [LS₃], modified for abstract spaces). *Fix now, and for the rest of the paper*, a complex Hilbert space \mathfrak{k} which we refer to as the *noise dimension space*, and let $\widehat{\mathfrak{k}}$ denote the orthogonal sum $\mathbb{C} \oplus \mathfrak{k}$. Whenever $c \in \mathfrak{k}$, $\widehat{c} := \binom{1}{c} \in \widehat{\mathfrak{k}}$; for $E \subset \mathfrak{k}$, $\widehat{E} := \text{Lin}\{\widehat{c} : c \in E\}$ and when g is a function with values in \mathfrak{k} , \widehat{g} denotes the corresponding function with values in $\widehat{\mathfrak{k}}$, defined by $\widehat{g}(s) := \widehat{g(s)}$. Let \mathcal{F}_I denote the symmetric Fock space over $L^2(I; \mathfrak{k})$, dropping the subscript when the interval I is all of \mathbb{R}_+ . For any dense subspace D of \mathfrak{k} let \mathbb{S}_D denote the linear span of $\{d_{[0, t[} : d \in D, t \in \mathbb{R}_+\}$ in $L^2(\mathbb{R}_+; \mathfrak{k})$ (we always take these right-continuous versions) and let \mathcal{E}_D denote the linear span of $\{\varepsilon(g) : g \in \mathbb{S}_D\}$ in \mathcal{F} , where $\varepsilon(g)$ denotes the exponential vector $((n!)^{-\frac{1}{2}} g^{\otimes n})_{n \geq 0}$. The subscript D is dropped when $D = \mathfrak{k}$. We usually drop the tensor symbol and denote simple tensors such as $v \otimes \varepsilon(f)$ by $v\varepsilon(f)$. Also define

$$e_0 := \binom{1}{0} \in \widehat{\mathfrak{k}} \text{ and } \Delta^{QS} := P_{\{0\} \oplus \mathfrak{k}} = \begin{bmatrix} 0 & \\ & I_{\mathfrak{k}} \end{bmatrix} \in B(\widehat{\mathfrak{k}}). \quad (1.8)$$

The basic objects we consider in this paper are completely bounded quantum stochastic mapping processes on operator spaces. These are time-indexed families of completely bounded maps $\{k_t : t \geq 0\}$ from an operator space to the algebra of bounded operators on $\mathfrak{h} \otimes \mathcal{F}$, for a Hilbert space \mathfrak{h} , satisfying standard adaptedness and measurability conditions. For technical reasons we also need to consider mapping processes whose values are (at least, a priori) unbounded operators. The crucial point here is that the naturally arising operators have ‘bounded slices’: for any vectors $\varepsilon, \varepsilon' \in \mathcal{E}$ the maps

$$v \rightarrow (I_{\mathfrak{h}} \otimes \langle \varepsilon' |) k_t(v) (I_{\mathfrak{h}} \otimes |\varepsilon\rangle)$$

($t \in \mathbb{R}_+$) have values in $B(\mathfrak{h})$, and are (completely) bounded, even though the global maps k_t may not be — more precisely they have (completely) bounded columns (see

Property 2, following Theorem 1.1). This point of view, where each k_t is taken to be a family of maps indexed by pairs of exponential vectors, allows the replacement of $B(\mathfrak{h})$ by an abstract operator space and, once the somewhat technical definitions below are accepted, leads to a development of the theory which is straight-forward and effective with more transparent proofs. This said, to follow the arguments it is safe to keep in mind sesquilinear maps induced by mapping processes in the familiar sense.

Let V and W be operator spaces. In this paper we denote by $\mathbb{P}(V \rightarrow W)$ the collection of families $k = (k_t)_{t \geq 0}$ of maps in

$$L(V; L(\mathcal{E}; CB(\langle \mathcal{F} |; W))) \subset SL(\mathcal{E}, \mathcal{E}; L(V; W))$$

satisfying the following *measurability* and *adaptedness* conditions

$$s \mapsto k_s^{\varepsilon', \varepsilon} \text{ is pointwise weakly measurable, and}$$

$$k_t^{\varepsilon', \varepsilon} = \langle \varepsilon'_2, \varepsilon_2 \rangle k_t^{\varepsilon'_1, \varepsilon_1},$$

for $\varepsilon = \varepsilon(f), \varepsilon' = \varepsilon(f') \in \mathcal{E}$ and $t \in \mathbb{R}_+$, where $\varepsilon_1 = \varepsilon(f_{[0,t]})$ and $\varepsilon_2 = \varepsilon(f_{[t, \infty[})$ with ε'_1 and ε'_2 defined in the same way for f' . When $W = \mathbb{C}$ (as is the case for quantum stochastic convolution cocycles) we write $\mathbb{P}_*(V)$ instead of $\mathbb{P}(V \rightarrow \mathbb{C})$. Then $k_t \in L(V; \mathcal{O}(\mathcal{E}))$ for each $t \geq 0$ and, in terms of the exponential property of Fock space: $\mathcal{F} = \mathcal{F}_{[0,t]} \otimes \mathcal{F}_{[t, \infty[}$, adaptedness reads

$$k_t(x)\varepsilon(f) = u_t \otimes \varepsilon(f_{[t, \infty[}) \text{ where } u_t = k_t(x)\varepsilon(f_{[0,t]}) \in \mathcal{F}_{[0,t]}.$$

Here the following Banach space identifications are used:

$$CB(\langle \mathcal{F} |; \mathbb{C}) = B(\langle \mathcal{F} |; \mathbb{C}) = \mathcal{F}.$$

When k_t is viewed as a map in $L(V, \mathcal{E}; CB(\langle \mathcal{F} |; W))$ we use the notation $k_{t, |\varepsilon\rangle}(x)$. Note that if $k \in \mathbb{P}(V \rightarrow Y)$ for a concrete operator space Y then, invoking the complete isometry (1.7), $k_{t, |\varepsilon\rangle} \in L(V; Y \otimes_M |\mathcal{F}\rangle)$.

Processes k and j are identified if, for all $\varepsilon', \varepsilon \in \mathcal{E}, x \in V$ and $\varphi \in W^*$, the scalar-valued functions $t \mapsto \varphi \circ k_t^{\varepsilon', \varepsilon}(x)$ and $t \mapsto \varphi \circ j_t^{\varepsilon', \varepsilon}(x)$ agree almost everywhere. We also denote by $\mathbb{P}^\ddagger(V \rightarrow W)$ the subspace of processes k for which

$$\text{each map } \langle \varepsilon' | \mapsto k_t^{\varepsilon, \varepsilon'}(x) \text{ is completely bounded } \langle \mathcal{E} | \rightarrow W$$

($\varepsilon \in \mathcal{E}, x \in V, t \in \mathbb{R}_+$). Then, for $k \in \mathbb{P}^\ddagger(V \rightarrow W)$,

$$(k^\dagger)_t^{\varepsilon', \varepsilon} := (k_t^{\varepsilon, \varepsilon'})^\dagger$$

defines a process $k^\dagger \in \mathbb{P}^\ddagger(V^\dagger \rightarrow W^\dagger)$, where \dagger denotes conjugate operator space. When the operator space W is concrete this amounts to the usual notion of adjoint(able) process. *Complete boundedness* for a process $k \in \mathbb{P}(V \rightarrow W)$ means

$$k_t \in CB(\langle \mathcal{F} |, |\mathcal{F}\rangle; CB(V; W)) \subset L(V, \mathcal{E}; CB(\langle \mathcal{F} |; W))$$

for each $t \in \mathbb{R}_+$. Thus $\mathbb{P}_{cb}(V \rightarrow W)$, the class of such processes, is a subspace of $\mathbb{P}^\ddagger(V \rightarrow W)$. The natural inclusion

$$CB(V; W \otimes B(\mathcal{F})) \subset CB(\langle \mathcal{F} |, |\mathcal{F}\rangle; CB(V; W)) \quad (1.9)$$

and, for a concrete operator space Y , the natural identification

$$CB(\langle \mathcal{F} |, |\mathcal{F}\rangle; CB(V; Y)) = CB(V; Y \otimes_M B(\mathcal{F})) \quad (1.10)$$

([LSa]) are both worth noting here (cf. (1.3)); they explain the terminology.

We need two further properties for processes: $k \in \mathbb{P}(V \rightarrow W)$ is *weakly initial space bounded* if

$$k_t^{\varepsilon', \varepsilon} : V \rightarrow W \text{ is bounded}$$

$(\varepsilon, \varepsilon' \in \mathcal{E}, t \in \mathbb{R}_+)$ and is *weakly regular* if further

$$\sup \{ \|k_s^{\varepsilon', \varepsilon}\| : 0 \leq s \leq t \} < \infty,$$

for all $t \geq 0$. We shall be dealing with quantum stochastic differential equations of the form

$$dk_t = k_t \bullet d\Lambda_\phi(t), \quad k_0 = \iota_{\mathcal{F}} \circ \kappa, \quad (1.11)$$

where $\phi \in CB(\mathbb{V}; \mathbb{V} \otimes B(\widehat{\mathbf{k}}))$ and $\kappa \in CB(\mathbb{V}; \mathbb{W})$. Here the natural inclusion

$$CB(\mathbb{V}; \mathbb{V} \otimes B(\widehat{\mathbf{k}})) \subset CB(\langle \widehat{\mathbf{k}} |, | \widehat{\mathbf{k}} \rangle; CB(\mathbb{V})) \quad (1.12)$$

and, for a concrete operator space \mathbb{Y} in $B(\mathfrak{h})$, the natural complete isometries

$$CB(\langle \widehat{\mathbf{k}} |, | \widehat{\mathbf{k}} \rangle; CB(\mathbb{Y})) = CB(\mathbb{Y}; \mathbb{Y} \otimes_{\mathbb{M}} B(\widehat{\mathbf{k}})) \subset CB(\mathbb{V}; B(\mathfrak{h} \otimes \widehat{\mathbf{k}})) \quad (1.13)$$

are relevant (cf. (1.9) and (1.10)). A process $k \in \mathbb{P}(\mathbb{V} \rightarrow \mathbb{W})$ is a *weak* solution of the QS differential equation (1.11) if

$$\begin{aligned} s \mapsto k_s^{\varepsilon', \varepsilon} \circ \phi^{\zeta', \zeta}(x) \text{ is weakly continuous, and} \\ k_t^{\varepsilon', \varepsilon}(x) = \langle \varepsilon', \varepsilon \rangle \kappa(x) + \text{w-} \int_0^t k_s^{\varepsilon', \varepsilon} \circ \phi^{\widehat{f}'(s), \widehat{f}(s)}(x) ds \end{aligned}$$

$(\zeta, \zeta' \in \widehat{\mathbf{k}}, \varepsilon = \varepsilon(f), \varepsilon' = \varepsilon(f') \in \mathcal{E}, x \in \mathbb{V}, t \in \mathbb{R}_+)$; it is so-called due to the First Fundamental Formula of quantum stochastic calculus. The theorem we need is the following special case of Proposition 3.5 in [LS₃] which generalises [LW₃] to allow nontrivial initial conditions and abstract spaces.

Theorem 1.1. *Let $\phi \in CB(\mathbb{V}; \mathbb{V} \otimes B(\widehat{\mathbf{k}}))$ and $\kappa \in CB(\mathbb{V}; \mathbb{W})$. Then there is a unique weakly regular weak solution of the quantum stochastic differential equation (1.11).*

Notation. $k^{\kappa, \phi}$, simplifying to k^ϕ for the case where $\mathbb{V} = \mathbb{W}$ and $\kappa = \text{id}_{\mathbb{W}}$.

We list key properties of the solution processes needed in this paper next (see [LS₃]). Let $k = k^{\kappa, \phi}$ for κ and ϕ as above.

1. $k \in \mathbb{P}^\dagger(\mathbb{V} \rightarrow \mathbb{W})$ and $k^\dagger = k^{\kappa^\dagger, \phi^\dagger}$.
2. For all $\varepsilon \in \mathcal{E}$ and $t \in \mathbb{R}_+$, $k_{t, |\varepsilon} \in CB(\mathbb{V}; CB(\langle \mathcal{F} |; \mathbb{W})) = CB(\langle \mathcal{F} |; CB(\mathbb{V}; \mathbb{W}))$ (the process has completely bounded columns) and the map $s \mapsto k_{s, |\varepsilon}$ is locally Hölder-continuous with exponent 1/2. Moreover if $\phi(\mathbb{V}) \subset \mathbb{V} \otimes B(\widehat{\mathbf{k}})$, then k satisfies

$$k_{t, |\varepsilon}(\mathbb{V}) \subset \mathbb{W} \otimes |\mathcal{F}\rangle$$

(in terms of the inclusion (1.1)).

3. If $\kappa = \kappa_2 \circ \kappa_1$ where $\kappa_1 \in CB(\mathbb{V}; \mathbb{U})$ and $\kappa_2 \in CB(\mathbb{U}; \mathbb{W})$ then

$$k_{t, |\varepsilon} = \kappa_2 \bullet \widetilde{k}_{t, |\varepsilon} : x \mapsto \kappa_2 \circ \widetilde{k}_{t, |\varepsilon}(x) \text{ where } \widetilde{k} = k^{\kappa_1, \phi}$$

($t \in \mathbb{R}_+, \varepsilon \in \mathcal{E}$). If the process \widetilde{k} is completely bounded then so is k and we have the identity

$$k_t = \kappa_2 \bullet k_t^{\kappa_1, \phi}$$

($t \in \mathbb{R}_+$). In particular,

$$k_{t, |\varepsilon} = \kappa \bullet k_{t, |\varepsilon}^\phi \quad (\text{resp. } k_t = \kappa \bullet k_t^\phi \text{ when } k^\phi \in \mathbb{P}_{\text{cb}}(\mathbb{V} \rightarrow \mathbb{W})).$$

4. The following useful ‘form representation’ holds

$$k_t^{\varepsilon', \varepsilon}(x) = \langle \varepsilon', \varepsilon \rangle \text{w-} \int_{\Gamma_{[0, t]}} d\sigma \Omega_\sigma \circ \tau_{\#\sigma} \circ \kappa \bullet \phi^{\bullet \#\sigma}(x) \quad (1.14)$$

(weak integral) where $\Omega_\sigma := \omega_{\xi', \xi}$ for $\xi = \pi_{\widehat{f}}(\sigma)$ and $\xi' = \pi_{\widehat{f}'}(\sigma)$, when $\varepsilon = \varepsilon(f)$ and $\varepsilon' = \varepsilon(f')$ and, for $n \in \mathbb{N}$, τ_n is the permutation of tensor components which reverses the order of the \widehat{k} 's.

5. Let $\varepsilon, \varepsilon' \in \mathcal{E}$ and $t \in \mathbb{R}_+$. If each $k_{t, |\varepsilon)}$ is $W \otimes |\mathcal{F})$ -valued then

$$k_t^{\varepsilon', \varepsilon} = (\text{id}_W \otimes \lambda_{\varepsilon'}) \circ k_{t, |\varepsilon)},$$

and if further $k \in \mathbb{P}_{\text{cb}}(V \rightarrow W)$ and each k_t maps V into the spatial tensor product $W \otimes B(\mathcal{F})$, then

$$k_{t, |\varepsilon)} = (\text{id}_W \otimes \rho_\varepsilon) \circ k_t.$$

Here $\lambda_{\varepsilon'} : |\mathcal{F}) \rightarrow \mathbb{C}$ denotes left multiplication by $\langle \varepsilon' |$, and $\rho_\varepsilon : B(\mathcal{F}) \rightarrow |\mathcal{F}$ right multiplication by $|\varepsilon\rangle$.

6. In fact k is a *strong solution* of the QS differential equation, meaning that the integral equation

$$k_t = \iota_{\mathcal{F}} \circ \kappa + \int_0^t k_s \bullet d\Lambda_\phi(s)$$

is valid in a ‘strong sense’.

7. The process k is expressible in terms of the multiple QS integral operation:

$$k_t = \Lambda_t \circ v \text{ where } v = (v_n)_{n \geq 0} \text{ and } v_n = \tau_n \circ \kappa \bullet \phi^{\bullet n}$$

(cf. Property 4).

8. When $V = W$ and $\kappa = \text{id}_W$, $k_0 = \iota_{\mathcal{F}}$ and k enjoys the following weak cocycle property: for $s, t \in \mathbb{R}_+$, $\varepsilon = \varepsilon(f)$ and $\varepsilon' = \varepsilon(f')$ in \mathcal{E} ,

$$k_{s+t}^{\varepsilon', \varepsilon} = \langle \varepsilon'_3, \varepsilon_3 \rangle k_s^{\varepsilon'_1, \varepsilon_1} \circ k_t^{\varepsilon'_2, \varepsilon_2},$$

where

$$\varepsilon_1 = \varepsilon(f_{[0, s[}), \varepsilon_2 = \varepsilon(S_s^* f_{[s, s+t[}) \text{ and } \varepsilon_3 = \varepsilon(f_{[s+t, \infty[}), \quad (1.15)$$

$\varepsilon'_1, \varepsilon'_2$ and ε'_3 being defined similarly with f' in place of f , $(S_t)_{t \geq 0}$ is the one-parameter semigroup of right shifts on $L^2(\mathbb{R}_+; \mathfrak{k})$ and $(\sigma_t)_{t \geq 0}$ is the induced endomorphism semigroup on $B(\mathcal{F})$, amplified to $SL(\mathcal{E}, \mathcal{E}; W)$. The cocycle property simplifies to

$$k_{s+t} = k_s \bullet \sigma_s \circ k_t$$

when k is completely bounded.

Property 8, namely the fact that processes of the form k^ϕ are weak QS cocycles (also called Markovian cocycles), has a converse — subject to certain constraints, weak QS cocycles are necessarily of this form. The main results in this direction concern completely positive, contractive QS cocycles on a unital C^* -algebra and are collected next — they originate in [LPa]; a direct proof is given in [LW₅].

Theorem 1.2 ([LW₁₋₃]). *Let A be a unital C^* -algebra and let $k \in \mathbb{P}(A \rightarrow A)$. Then the following are equivalent:*

- (i) k is a Markov-regular, completely positive and contractive QS cocycle on A ;
- (ii) $k = k^\phi$ where $\phi \in CB(\langle \widehat{k} |, |\widehat{k}\rangle; CB(A)) = CB(A; CB(\langle \widehat{k} |, |\widehat{k}\rangle; A))$ satisfies $\phi(1) \leq 0$ and, in any faithful, nondegenerate representation, ϕ may be decomposed as follows:

$$\phi(x) = \Psi(x) - x \otimes \Delta^{QS} - (x \otimes |e_0\rangle)J - J^*(x \otimes \langle e_0|) \quad (1.16)$$

$$(x \in A), \text{ for some map } \Psi \in CP(A; A'' \overline{\otimes} B(\widehat{k})) \text{ and operator } J \in A'' \overline{\otimes} \widehat{k}.$$

Here, with respect to the representation, A'' denotes the double commutant of A , $\overline{\otimes}$ denotes the ultraweak tensor product and $\phi(x) \in A \otimes_M B(\widehat{k})$. The form (1.16) taken by the stochastic generator (see also [Bel]) generalises the Christensen-Evans Theorem on the generators of norm continuous completely positive contractive semigroups ([ChE]).

2. OPERATOR SPACE COALGEBRAS

In this section we adapt the basic notions of coalgebra to the category of operator spaces, and consider convolution semigroups of functionals in this context. Three structures are considered: operator space coalgebras, operator system coalgebras and C^* -bialgebras, corresponding to the three levels of question addressed in this paper, namely linear, positivity-preserving and algebraic. In fact a hybrid structure, called C^* -hyperbialgebra, plays a more prominent role than operator system coalgebras do.

Definition. An *operator space coalgebra* is an operator space \mathbb{C} equipped with complete contractions $\epsilon : \mathbb{C} \rightarrow \mathbb{C}$ and $\Delta : \mathbb{C} \rightarrow \mathbb{C} \otimes \mathbb{C}$, called the *counit* and *coproduct* respectively, satisfying

$$(OSC1) \quad (\Delta \otimes \text{id}_{\mathbb{C}}) \circ \Delta = (\text{id}_{\mathbb{C}} \otimes \Delta) \circ \Delta \quad (\text{coassociativity}),$$

$$(OSC2) \quad (\epsilon \otimes \text{id}_{\mathbb{C}}) \circ \Delta = \text{id}_{\mathbb{C}} = (\text{id}_{\mathbb{C}} \otimes \epsilon) \circ \Delta \quad (\text{counit property});$$

it is an *operator system coalgebra* if \mathbb{C} is an operator system and

$$(OSyC) \quad \epsilon \text{ and } \Delta \text{ are both unital and completely positive;}$$

a C^* -*hyperbialgebra* if \mathbb{C} is a unital C^* -algebra and

$$(C^*\text{-Hy}) \quad \epsilon \text{ is a character (i.e. it is nonzero and multiplicative) and } \Delta \text{ is unital and completely positive;}$$

and finally it is a C^* -*bialgebra* if \mathbb{C} is a (unital) C^* -algebra and

$$(C^*\text{-Bi}) \quad \epsilon \text{ and } \Delta \text{ are both unital and } * \text{-homomorphic.}$$

By (OSC1), $\Delta^{\bullet 2}$ is defined unambiguously, as is $\Delta^{\bullet n}$ for all $n \in \mathbb{N}$, and we define $\Delta^{\bullet 0} := \text{id}_{\mathbb{C}}$. Similarly (OSC2) gives $\epsilon \bullet \Delta = \text{id}_{\mathbb{C}}$, with unambiguous meaning. An operator space coalgebra is *cocommutative* if $\Delta = \Delta^{\text{op}}$ where

$$\Delta^{\text{op}} := \tau \circ \Delta,$$

τ being the tensor flip on $\mathbb{C} \otimes \mathbb{C}$. The *opposite* operator space coalgebra results from replacing Δ by Δ^{op} .

An operator space coalgebra is thus typically *not* a coalgebra in the algebraic sense ([Swe]) since the coproduct is not required to map \mathbb{C} into $\mathbb{C} \odot \mathbb{C}$. A (unital) C^* -bialgebra is a C^* -hyperbialgebra \mathbb{B} whose coproduct is also multiplicative (and thus a unital $*$ -homomorphism). Some authors (for example [Kus₂]) drop the unital condition on C^* -bialgebras, and require instead the counit to be a nondegenerate $*$ -homomorphism into the multiplier algebra $M(\mathbb{C} \otimes \mathbb{C})$. The asymmetry in the definition of C^* -hyperbialgebra — whereby ϵ is required to be multiplicative but Δ only to be completely positive — is motivated by the example of compact quantum hypergroups ([ChV]). Multiplicativity of the counit is used extensively in characterising generators of completely positive convolution cocycles (Section 5 and [S]). Finally note that the conjugate operator space of an operator space coalgebra has natural operator space coalgebra structure.

Convolution. For an operator space coalgebra \mathbb{C} and operators spaces \mathbb{V}_1 and \mathbb{V}_2 , the *convolution* of $\varphi_1 \in CB(\mathbb{C}; \mathbb{V}_1)$ and $\varphi_2 \in CB(\mathbb{C}; \mathbb{V}_2)$ is defined by

$$\varphi_1 \star \varphi_2 := (\varphi_1 \otimes \varphi_2) \circ \Delta \in CB(\mathbb{C}; \mathbb{V}_1 \otimes \mathbb{V}_2).$$

It is easily seen that convolution is associative (in the same sense as the spatial tensor product is) and enjoys submultiplicativity and unital properties:

$$(\varphi_1 \star \varphi_2) \star \varphi_3 = \varphi_1 \star (\varphi_2 \star \varphi_3) \tag{2.1}$$

$$\|\varphi_1 \star \varphi_2\|_{\text{cb}} \leq \|\varphi_1\|_{\text{cb}} \|\varphi_2\|_{\text{cb}}, \text{ and}$$

$$\epsilon \star \varphi = \varphi = \varphi \star \epsilon. \tag{2.2}$$

In particular, (\mathbb{C}^*, \star) is a unital Banach algebra. For $n \in \mathbb{N}$ and $\varphi_1, \dots, \varphi_n \in CB(\mathbb{C}; \mathbb{V})$, n -fold convolution is defined via n -fold tensor products:

$$\varphi_1 \star \dots \star \varphi_n = (\varphi_1 \otimes \dots \otimes \varphi_n) \circ \Delta^{\bullet(n-1)}. \quad (2.3)$$

We also define $\varphi^{\star 0} := \epsilon$, which is consistent with (2.2).

Given an operator space coalgebra \mathbb{C} , each operator space \mathbb{V} determines maps

$$\begin{aligned} R_{\mathbb{V}} &: CB(\mathbb{C}; \mathbb{V}) \rightarrow CB(\mathbb{C}; \mathbb{C} \otimes \mathbb{V}), & \varphi &\mapsto (\text{id}_{\mathbb{C}} \otimes \varphi) \circ \Delta; \\ E_{\mathbb{V}} &: CB(\mathbb{C}; \mathbb{C} \otimes \mathbb{V}) \rightarrow CB(\mathbb{C}; \mathbb{V}), & \phi &\mapsto (\epsilon \otimes \text{id}_{\mathbb{V}}) \circ \phi. \end{aligned}$$

Thus the action of $R_{\mathbb{V}}$ is convolve with the identity map on \mathbb{C} , putting the argument on the *right*, and that of $E_{\mathbb{V}}$ is compose in the tensor-extended sense with the counit:

$$R_{\mathbb{V}}\varphi = \text{id}_{\mathbb{C}} \star \varphi, \text{ and } E_{\mathbb{V}}\phi = \epsilon \bullet \phi.$$

In the noncocommutative case we are therefore making a choice here. We abbreviate $R_{\mathbb{C}}$ to R_{\star} .

The basic properties of these maps are collected below. They are all easily proved from the definitions, noting that under the completely isometric identification $M_n(\mathbb{C} \otimes \mathbb{V}) = \mathbb{C} \otimes M_n(\mathbb{V})$,

$$(R_{\mathbb{V}})^{(n)} = R_{M_n(\mathbb{V})}.$$

Proposition 2.1. *Let \mathbb{C} be an operator space coalgebra, and let $\mathbb{V}_1, \mathbb{V}_2$ and \mathbb{V} be operator spaces.*

(a) $R_{\mathbb{V}}$ and $E_{\mathbb{V}}$ are complete isometries satisfying

$$E_{\mathbb{V}} \circ R_{\mathbb{V}} = \text{id}_{CB(\mathbb{C}; \mathbb{V})}.$$

(b) If $\varphi_1 \in CB(\mathbb{C}; \mathbb{V}_1)$ and $\varphi_2 \in CB(\mathbb{C}; \mathbb{V}_2)$ then

$$R_{\mathbb{V}_1 \otimes \mathbb{V}_2}(\varphi_1 \star \varphi_2) = R_{\mathbb{V}_1}\varphi_1 \bullet R_{\mathbb{V}_2}\varphi_2.$$

(c) If $\varphi \in CB(\mathbb{C}; \mathbb{V})$ then

$$R_{\mathbb{V}^\dagger}(\varphi^\dagger) = (R_{\mathbb{V}}\varphi)^\dagger.$$

Remark. Noting that $\epsilon = E_{\mathbb{C}}(\text{id}_{\mathbb{C}})$ and $\Delta = R_{\mathbb{C}}(\text{id}_{\mathbb{C}})$, it is clear that operator space coalgebras could be axiomatised in terms of R - and E -maps in lieu of Δ and ϵ .

Write $CB^\Delta(\mathbb{C}; \mathbb{C} \otimes \mathbb{V})$ for $\text{Ran } R_{\mathbb{V}}$.

Corollary 2.2. *For each operator space \mathbb{V} , $R_{\mathbb{V}}$ determines a complete isometry of operator spaces*

$$CB(\mathbb{C}; \mathbb{V}) \cong CB^\Delta(\mathbb{C}; \mathbb{C} \otimes \mathbb{V}),$$

by corestriction. In case $\mathbb{V} = \mathbb{C}$ this gives an isometric isomorphism of unital Banach algebras

$$(\mathbb{C}^*, \star) \cong (CB^\Delta(\mathbb{C}), \circ).$$

A further noteworthy consequence is the following identity.

Corollary 2.3. *In $CB^\Delta(\mathbb{C}; \mathbb{C} \otimes M_n)$,*

$$\|\phi\|_{\text{cb}} = \|\phi^{(n)}\|.$$

Proof. Let $\phi \in CB^\Delta(\mathbb{C}; \mathbb{C} \otimes M_n)$, say $\phi = R_{M_n}\varphi$. Then $\varphi \in CB(\mathbb{C}; M_n)$ so

$$\|\phi\|_{\text{cb}} = \|\varphi\|_{\text{cb}} = \|\varphi^{(n)}\| = \|\epsilon \bullet \phi^{(n)}\| \leq \|\phi^{(n)}\|.$$

The result follows. \square

In particular, in $CB^\Delta(\mathbb{C})$ the completely bounded norm coincides with the bounded operator norm. As a result $CB^\Delta(\mathbb{C})$ is a closed subspace of $B(\mathbb{C})$. The next proposition collects the structure-preserving properties of $R_{\mathbb{V}}$ and its inverse, under a number of pertinent assumptions on \mathbb{C} and \mathbb{V} .

Proposition 2.4. *Let \mathbb{C} be an operator space coalgebra and \mathbb{V} an operator space, let $\varphi \in CB(\mathbb{C}; \mathbb{V})$ and $\phi = R_{\mathbb{V}}\varphi \in CB^{\Delta}(\mathbb{C}; \mathbb{C} \otimes \mathbb{V})$.*

- (a) *The map ϕ is completely contractive if and only if φ is.*
- (b) *If \mathbb{C} is an operator system coalgebra and \mathbb{V} is an operator system then ϕ is real (respectively, completely positive, or unital) if and only if φ is.*
- (c) *If \mathbb{C} is a C^* -bialgebra and \mathbb{V} is a C^* -algebra then ϕ is multiplicative if and only if φ is.*

A convolution semigroup of functionals on an operator space coalgebra \mathbb{C} is a one-parameter family $\lambda = (\lambda_t)_{t \geq 0}$ in C^* satisfying

$$\lambda_0 = \epsilon \text{ and } \lambda_{s+t} = \lambda_s \star \lambda_t.$$

In other words a convolution semigroup of functionals on \mathbb{C} is a one-parameter semigroup in the unital algebra (C^*, \star) .

Proposition 2.5. *Let \mathbb{C} be an operator space coalgebra. The map $\lambda \mapsto P := (R_{\star}\lambda_t)_{t \geq 0}$ is a bijection from the set of convolution semigroups of functionals on \mathbb{C} to the set of one-parameter semigroups in $CB^{\Delta}(\mathbb{C})$. Moreover, the conditions in (a) below are equivalent, and so are the conditions in (b):*

- (a) (i) $\lambda_t \rightarrow \epsilon$ pointwise as $t \rightarrow 0$;
- (ii) P is a C_0 -semigroup on \mathbb{C} .
- (b) (i) λ is norm continuous in t ;
- (ii) P is norm continuous in t ;
- (iii) P is cb-norm continuous in t ;
- (iv) P has a completely bounded generator.

Proof. The first part follows from Corollary 2.2.

Since $\epsilon \circ P_t = \lambda_t$, (iii) implies (ai). Suppose therefore that (ai) holds. Then, for any $\varphi \in C^*$,

$$\varphi \circ P_t = \lambda_t \circ (\varphi \otimes \text{id}_{\mathbb{C}}) \circ \Delta \text{ and } \epsilon \circ (\varphi \otimes \text{id}_{\mathbb{C}}) \circ \Delta = \varphi,$$

so $P_t x \rightarrow x$ weakly as $t \searrow 0$, for all $x \in \mathbb{C}$. But this implies that P is strongly continuous ([Dav], Proposition 1.23) and thus a C_0 -semigroup, so (iii) holds.

By Corollary 2.3

$$\|P_t - \text{id}_{\mathbb{C}}\|_{\text{cb}} = \|\lambda_t - \epsilon\| = \|P_t - \text{id}_{\mathbb{C}}\|,$$

and so (b) follows. □

Thus each norm-continuous convolution semigroup of functionals λ on \mathbb{C} has a generator:

$$\gamma := \lim_{t \searrow 0} t^{-1}(\lambda_t - \epsilon)$$

from which the convolution semigroup of functionals may be recovered

$$\lambda_t = \exp_{\star} t\gamma := \sum_{n \geq 0} \frac{t^n}{n!} \gamma^{\star n}.$$

The corresponding one-parameter semigroup on \mathbb{C} has completely bounded generator:

$$R_{\star}\lambda_t = e^{t\tau}, \text{ where } \tau = R_{\star}\gamma \in CB^{\Delta}(\mathbb{C}).$$

3. OPERATOR SPACE COALGEBRAIC QS DIFFERENTIAL EQUATIONS

In this section we consider operator space coalgebraic quantum stochastic differential equations with completely bounded coefficients, and relate their solutions to those of standard QS differential equations by means of R -maps. In particular we show that the complete boundedness property is preserved when moving between these two kinds of solutions. *For this section \mathbb{C} is a fixed operator space coalgebra.*

Let $\varphi \in CB(\mathbb{C}; B(\widehat{\mathbf{k}}))$. A weakly initial space bounded process $k \in \mathbb{P}_*(\mathbb{C})$ is a *weak solution* of the operator space coalgebraic quantum stochastic differential equation

$$dk_t = k_t \star d\Lambda_\varphi(t), \quad k_0 = \iota_{\mathcal{F}} \circ \epsilon, \quad (3.1)$$

if

$$s \mapsto (k_s^{\varepsilon', \varepsilon} \star \varphi^{\zeta', \zeta})(x) \text{ is continuous, and} \quad (3.2)$$

$$k_t^{\varepsilon', \varepsilon}(x) = \langle \varepsilon', \varepsilon \rangle \kappa(x) + \int_0^t (k_s^{\varepsilon', \varepsilon} \star \varphi^{\widehat{f}'(s), \widehat{f}(s)})(x) dx \quad (3.3)$$

for $\zeta, \zeta' \in \widehat{\mathbf{k}}, \varepsilon = \varepsilon(f), \varepsilon' = \varepsilon(f') \in \mathcal{E}, x \in \mathbb{C}$ and $t \in \mathbb{R}_+$.

Remark. By the Banach-Steinhaus Theorem

$$\sup \left\{ |(\omega_{\varepsilon(f'), \varepsilon(f)} \circ k_s) \star (\omega_{\widehat{f}'(s), \widehat{f}(s)} \circ \varphi)(x)| \mid x \in \mathbb{C}, \|x\| \leq 1, s \in [0, t] \right\} < \infty$$

for each $f, f' \in \mathbb{S}$ and $t \in \mathbb{R}_+$. It follows therefore that weak solutions of the operator space coalgebra QS differential equation in the above sense are automatically weakly regular.

Theorem 3.1. *Let $\varphi \in CB(\mathbb{C}; B(\widehat{\mathbf{k}}))$. Then the operator space coalgebraic quantum stochastic differential equation (3.1) has a unique weak solution.*

Proof. Let $k \in \mathbb{P}_*(\mathbb{C})$ be weakly regular. Then

$$k_t^{\varepsilon', \varepsilon} \star \varphi^{\zeta', \zeta} = k_t^{\varepsilon', \varepsilon} \circ \phi^{\zeta', \zeta}$$

($\varepsilon, \varepsilon' \in \mathcal{E}, \zeta, \zeta' \in \widehat{\mathbf{k}}, t \in \mathbb{R}_+$) where $\phi \in R_{B(\widehat{\mathbf{k}})}\varphi$. It follows that k weakly satisfies the operator space coalgebraic QS differential equation (3.1) if and only if k weakly satisfies the operator space QS differential equation $dk_t = k_t \bullet d\Lambda_\phi(t)$, $k_0 = \iota_{\mathcal{F}} \circ \epsilon$. Since $\phi \in CB(\mathbb{C}; \mathbb{C} \otimes B(\widehat{\mathbf{k}}))$ and $\epsilon \in \mathbb{C}^* = CB(\mathbb{C}; \mathbb{C})$ the result therefore follows from Theorem 1.1, and the automatic weak regularity of weak solutions of (3.1). \square

Notation. We denote the unique weak solution of (3.1), for completely bounded φ , by l^φ . From the above proof we see that $l^\varphi = k^{\epsilon, \phi}$ where

$$\phi = R_{B(\widehat{\mathbf{k}})}\varphi \in CB(\mathbb{C}; \mathbb{C} \otimes B(\widehat{\mathbf{k}})).$$

Note that Proposition 2.1 implies that

$$\epsilon \bullet \phi^{*n} = \epsilon \bullet R_{B(\widehat{\mathbf{k}})^{\otimes n}}\varphi^{*n} = \varphi^{*n}, \quad n \geq 0.$$

The properties of solutions of operator space QS differential equations listed in Section 1 entail the following for $l = l^\varphi$ where $\varphi \in CB(\mathbb{C}; B(\widehat{\mathbf{k}}))$:

- 1'. $l \in \mathbb{P}_*(\mathbb{C})$ and $l^\dagger = l^\psi$ where $\psi = \varphi^\dagger \in CB(\mathbb{C}^\dagger; B(\widehat{\mathbf{k}}))$.
- 2'. $l_{t, |\varepsilon\rangle} \in CB(\mathbb{C}; |\mathcal{F}\rangle)$ and the map $s \mapsto l_{s, |\varepsilon\rangle}$ is locally Hölder continuous with exponent $\frac{1}{2}$, moreover $k_{t, |\varepsilon\rangle}^\phi(\mathbb{C}) \subset \mathbb{C} \otimes |\mathcal{F}\rangle$ for all $\varepsilon \in \mathcal{E}$ and $t \in \mathbb{R}_+$.
- 3'. Since $l = k^{\epsilon, \phi}$, where $\phi = R_{B(\widehat{\mathbf{k}})}\varphi$,

$$l_{t, |\varepsilon\rangle} = \epsilon \bullet k_{t, |\varepsilon\rangle}^\phi \quad (3.4)$$

($\varepsilon \in \mathcal{E}, t \in \mathbb{R}_+$); also if k^ϕ is completely bounded then l is too and

$$l_t = \varepsilon \bullet k_t^\phi, \quad t \in \mathbb{R}_+.$$

4'. In the notation of Property 4,

$$l_t^{\varepsilon', \varepsilon} = \langle \varepsilon', \varepsilon \rangle \int_{\Gamma_{[0,t]}} d\sigma \Omega_\sigma \circ \tau_{\#\sigma} \circ \varphi^{*\#\sigma} \quad (3.5)$$

for $\varepsilon = \varepsilon(f), \varepsilon' \in \varepsilon(f') \in \mathcal{E}$ and $t \in \mathbb{R}_+$.

5'. In the notation of Property 5,

$$l_t^{\varepsilon', \varepsilon} = \lambda_{\varepsilon'} \circ l_{t, |\varepsilon \rangle}$$

and, if l is completely bounded so that $l_t \in CB(\mathbb{C}; B(\mathcal{F}))$, then

$$l_{t, |\varepsilon \rangle} = \rho_\varepsilon \circ l_t$$

($\varepsilon, \varepsilon' \in \mathcal{E}, t \in \mathbb{R}_+$).

6'. l is a strong solution of the operator space coalgebraic QS differential equation:

$$l_t = \iota_{\mathcal{F}} \circ \varepsilon + \int_0^t l_s \star d\Lambda_\varphi(s)$$

is valid in a strong sense.

7'. l is given explicitly by

$$l_t = \Lambda_t \circ v \text{ where } v = (v)_{n \geq 0} \text{ and } v_n = \tau_n \circ \varphi^{*n}, \quad n \in \mathbb{Z}_+. \quad (3.6)$$

Remark. In view of the injectivity of the quantum stochastic operation ([LW₄], Proposition 2.3), Property 7' implies that

$$\text{the map } \varphi \mapsto l^\varphi \text{ is injective.} \quad (3.7)$$

The next two results strengthen Property 3'.

Proposition 3.2. *Let $l = l^\varphi$ and $k = k^\phi$ where $\varphi \in CB(\mathbb{C}; B(\widehat{\mathbf{k}}))$ and $\phi = R_{B(\widehat{\mathbf{k}})}\varphi$. Then $l_{t, |\varepsilon \rangle} \in CB(\mathbb{C}; |\mathcal{F}\rangle)$ and*

$$k_{t, |\varepsilon \rangle} = R_{|\mathcal{F}\rangle} l_{t, |\varepsilon \rangle}, \quad t \in \mathbb{R}_+, \varepsilon \in \mathcal{E}.$$

In particular, k satisfies

$$k_{t, |\varepsilon \rangle} \in CB(\mathbb{C}; \mathbb{C} \otimes |\mathcal{F}\rangle), \quad \varepsilon \in \mathcal{E}, t \in \mathbb{R}_+.$$

Proof. The first (and last) part has already been noted in Property 2'. Write $\widetilde{k} \in \mathbb{P}_\star(\mathbb{C})$ for the process defined by

$$\widetilde{k}_{t, |\varepsilon \rangle} = R_{|\mathcal{F}\rangle} l_{t, |\varepsilon \rangle} \in CB(\mathbb{C}; \mathbb{C} \otimes |\mathcal{F}\rangle) \quad (\varepsilon \in \mathcal{E}, t \in \mathbb{R}_+).$$

Let $\varepsilon = \varepsilon(f), \varepsilon' \in \varepsilon(f') \in \mathcal{E}$ and $t \in \mathbb{R}_+$, and consider the ‘form representation’ of l given in Property 7' and the corresponding representation of k . Writing R_σ for $R_{\mathbb{V}}$ where $\mathbb{V} = B(\widehat{\mathbf{k}})^{\otimes \#\sigma}$, Proposition 2.1 yields the identity

$$R(\Omega_\sigma \circ \tau_{\#\sigma} \circ \varphi^{*\#\sigma}) = \Omega_\sigma \bullet R_\sigma(\tau_{\#\sigma} \circ \varphi^{*\#\sigma}) = \Omega_\sigma \bullet (\tau_{\#\sigma} \circ \phi^{*\#\sigma})$$

($\sigma \in \Gamma$). Thus, integrating over $\Gamma_{[0,t]}$,

$$R(l_t^{\varepsilon', \varepsilon}) = k_t^{\varepsilon', \varepsilon}.$$

Therefore, using Property 5',

$$k_t^{\varepsilon', \varepsilon} = R_\star(\lambda_{\varepsilon'} \circ l_{t, |\varepsilon \rangle}) = (\text{id}_{\mathbb{C}} \otimes \lambda_{\varepsilon'}) \circ R_{|\mathcal{F}\rangle} l_{t, |\varepsilon \rangle} = \widetilde{k}_t^{\varepsilon', \varepsilon}.$$

The result follows. \square

Proposition 3.3. *Let $l = l^\varphi$ and $k = k^\phi$ where $\varphi \in CB(\mathbb{C}; B(\widehat{\mathbf{k}}))$ and $\phi = R_{B(\widehat{\mathbf{k}})}\varphi$. Then the process l is completely bounded if and only if k is, and in this case*

$$k_t = R_{B(\mathcal{F})}l_t, \quad t \in \mathbb{R}_+, \quad (3.8)$$

in particular k is $\mathbb{C} \otimes B(\mathcal{F})$ -valued.

Proof. Suppose that l is completely bounded and define the process \tilde{k} by $\tilde{k}_t = R_{B(\mathcal{F})}l_t$. By Proposition 3.2 and Properties 5' and 5,

$$k_{t,|\varepsilon} = R_{|\mathcal{F}}(\rho_\varepsilon \circ l_t) = (\text{id}_{\mathbb{C}} \otimes \rho_\varepsilon) \circ \tilde{k}_t = \tilde{k}_{t,|\varepsilon}$$

($\varepsilon \in \mathcal{E}, t \in \mathbb{R}_+$), and so k is the completely bounded $\mathbb{C} \otimes B(\mathcal{F})$ -valued process $(R_{B(\mathcal{F})}l_t)_{t \geq 0}$. Conversely if k is completely bounded then l is too, by Property 3'. \square

4. QUANTUM STOCHASTIC CONVOLUTION COCYCLES

In this section we study quantum stochastic convolution cocycles on an operator space coalgebra by applying the R -map to the theory of quantum stochastic cocycles on an operator space ([LW₂]). For this section an operator space coalgebra \mathbb{C} is fixed.

Definition. A completely bounded process $l \in \mathbb{P}_*(\mathbb{C})$ is called a *quantum stochastic convolution cocycle* if it satisfies

$$l_0 = \iota_{\mathcal{F}} \circ \epsilon \text{ and } l_{s+t} = l_s \star (\sigma_s \circ l_t) \text{ for } s, t \in \mathbb{R}_+. \quad (4.1)$$

QS convolution cocycles therefore satisfy

$$l_{s+t}^{\varepsilon', \varepsilon} = \langle \varepsilon'_3, \varepsilon_3 \rangle l_s^{\varepsilon'_1, \varepsilon_1} \star l_t^{\varepsilon'_2, \varepsilon_2} \quad (4.2)$$

for $\varepsilon = \varepsilon(f), \varepsilon' = \varepsilon(f')$ and $s, t \in \mathbb{R}_+$, where $\varepsilon_1, \dots, \varepsilon'_3$ are defined by (1.15). More generally, if l is a weakly initial space bounded process $\mathbb{C} \rightarrow \mathbb{C}$ satisfying (4.2) then it is called a *weak quantum stochastic convolution cocycle*. Compare this with the cocycle property for a weakly initial space bounded process on an operator space (see Property 8 in the list of properties of solutions of QS differential equations).

For a weak QS convolution cocycle l on \mathbb{C} define

$$\lambda_t^{c', c} := e^{-t(c', c)} l_t^{\varepsilon', \varepsilon} \text{ where } \varepsilon = \varepsilon(c_{[0, t]}) \text{ and } \varepsilon' = \varepsilon(c'_{[0, t]})$$

($c, c' \in \mathbf{k}, t \in \mathbb{R}_+$). Then $\lambda^{c', c} := (\lambda_t^{c', c})_{t \geq 0}$ is a convolution semigroup and we refer to $\{\lambda^{c', c} : c, c' \in \mathbf{k}\}$ as the cocycle's *associated convolution semigroups of functionals* and call l *Markov-regular* if $\lambda^{0, 0}$ is norm continuous, in analogy to Markov-regular quantum stochastic cocycles ([LW₂]).

As for standard QS cocycles, if the cocycle is contractive then Markov-regularity implies that all of its associated convolution semigroups of functionals are norm continuous. Repeated application of the defining property (4.2) shows that, for each $\varepsilon = \varepsilon(f), \varepsilon' = \varepsilon(f') \in \mathcal{E}$ and $t \in \mathbb{R}_+$, $\langle \varepsilon', \varepsilon \rangle^{-1} l_t^{\varepsilon', \varepsilon}$ is the convolute of a finite number of associated convolution semigroups of functionals of l . In particular two weak QS convolution cocycles are the same if each of their corresponding associated convolution semigroups of functionals coincide.

Lemma 4.1. *Let $l \in \mathbb{P}_*(\mathbb{C})$ and $k \in \mathbb{P}_*(\mathbb{C})$ be weakly initial space bounded processes related by*

$$k_t^{\varepsilon', \varepsilon} = R_\star l_t^{\varepsilon', \varepsilon}, \quad (4.3)$$

for $\varepsilon, \varepsilon' \in \mathcal{E}, t \in \mathbb{R}_+$. Then l is a weak QS convolution cocycle if and only if k is a weak QS cocycle, and in this case l is Markov-regular if and only if k is.

Proof. In view of the identity

$$R_\star(l_s^{\varepsilon'_1, \varepsilon_1} \star l_t^{\varepsilon'_2, \varepsilon_2}) = k_s^{\varepsilon'_1, \varepsilon_1} \circ k_t^{\varepsilon'_2, \varepsilon_2}$$

(in the notation (1.15)) the result follows from the complete isometry of R_\star . \square

Proposition 4.2. *Let $\varphi \in CB(\mathbb{C}; B(\widehat{\mathbf{k}}))$. Then l^φ is a Markov-regular weak QS convolution cocycle, each of whose convolution semigroups of functionals is norm continuous.*

Proof. Let $k = k^\phi$ where $\phi = R_{B(\widehat{\mathbf{k}})}\varphi$. Then k is a Markov-regular quantum stochastic cocycle all of whose associated semigroups are norm continuous ([LW₂]). Since, by Proposition 3.2, l and k are related by (4.3) the result therefore follows from Lemma 4.1. \square

In the next section we obtain a converse by restricting to completely positive, contractive QS convolution cocycles on a C^* -hyperbialgebra. In view of the identity

$$\int_{\Gamma_{[0,t]}} d\sigma \langle \pi_{\widehat{c}}(\sigma), \varphi^{\star\#\sigma}(\cdot) \pi_{\widehat{c}}(\sigma) \rangle = \sum_{n \geq 0} \frac{t^n}{n!} (\omega_{\widehat{c}, \widehat{c}} \circ \varphi)^{\star n},$$

the convolution semigroup of functionals $\lambda^{\cdot, c}$ associated with the weak QS convolution cocycle l^φ has generator

$$\omega_{\widehat{c}, \widehat{c}} \circ \varphi. \quad (4.4)$$

This corresponds to the fact that the semigroups associated with a Markov-regular QS cocycle k^ϕ on an operator space have generators

$$\omega_{\widehat{c}, \widehat{c}} \bullet \phi. \quad (4.5)$$

Below we initiate a traffic between properties of a QS convolution cocycle and those of its stochastic generator. Recall Property 1' for processes l^φ . The following is easily proved either using the R -map, or directly.

Proposition 4.3. *Let $l = l^\varphi$ where $\varphi \in CB(\mathbb{C}; B(\widehat{\mathbf{k}}))$ and \mathbb{C} is an operator system coalgebra. Then*

- (a) l is unital if and only if $\varphi(1) = 0$,
- (b) l is real if and only if φ is real.

Opposite QS convolution cocycles. The *opposite* QS convolution cocycle relation, for processes in $\mathbb{P}_{\star \text{cb}}(\mathbb{C})$, is

$$l_0 = \iota_{\mathcal{F}} \circ \epsilon \text{ and } l_{s+t} = (\sigma_s \circ l_t) \star l_s,$$

which involves the natural identifications $B(\mathcal{F}_{[s, s+t]}) \overline{\otimes} B(\mathcal{F}_{[0, s]}) = B(\mathcal{F}_{[0, s+t]})$, for $s, t \in \mathbb{R}_+$. Completely bounded processes which satisfy a QS differential equation of the form

$$dl_t = d\Lambda_\varphi(t) \star l_t, \quad l_0 = \iota_{\mathcal{F}} \circ \epsilon,$$

for $\varphi \in CB(\mathbb{C}; B(\widehat{\mathbf{k}}))$, are opposite QS convolution cocycles; they are given explicitly by

$$l_t = \Lambda_t \circ v \text{ where } v_n = \varphi^{\star n}, n \in \mathbb{Z}_+$$

(cf. Properties 6' and 7' in Section 3), with ${}^\varphi l$ being an appropriate notation. There is a bijective correspondence between the set of QS convolution cocycles treated in this paper and the set of opposite QS convolution cocycles. This is effected by time-reversal, as in [LW₂]. Opposite QS convolution cocycles have convolution semigroup representation as QS convolution cocycles do, but with the semigroups appearing in the reverse order. In particular time-reversal exchanges l^φ and ${}^\varphi l$. One may also view the correspondence in terms of the opposite coproduct Δ^{op} .

In [LS₁] we actually worked with opposite cocycles (thus the convolvands in (5.1) and the $a_{(i)}$'s in (5.2), on p. 595 of that paper, should both have appeared in the reverse order, with the notation φl being more appropriate for the opposite QS convolution cocycles generated there). The results of that paper are equally valid for QS convolution cocycles on coalgebras defined as here through the relations (4.1) and (4.2).

5. COMPLETELY POSITIVE QS CONVOLUTION COCYCLES

In this section we characterise the Markov-regular QS convolution cocycles amongst the completely positive and contractive processes on a C^* -hyperbialgebra \mathbf{E} , as those which satisfy a coalgebraic quantum stochastic differential equation with completely bounded coefficient of a particular form. We also give the general form of the coefficient of the QS differential equation. Recall Theorem 1.2 and the notations (1.8).

Theorem 5.1. *Let \mathbf{E} be a C^* -hyperbialgebra and let $l \in \mathbb{P}_*(\mathbf{E})$. Then the following are equivalent:*

- (i) l is a Markov-regular, completely positive, contractive QS convolution cocycle;
- (ii) $l = l^\varphi$ where $\varphi \in CB(\mathbf{E}; B(\widehat{\mathbf{k}}))$ satisfies $\varphi(1) \leq 0$ and may be decomposed as follows:

$$\varphi = \psi - \epsilon(\cdot) (\Delta^{QS} + |e_0\rangle\langle\chi| + |\chi\rangle\langle e_0|) \quad (5.1)$$

for some completely positive map $\psi : \mathbf{E} \rightarrow B(\widehat{\mathbf{k}})$ and vector $\chi \in \widehat{\mathbf{k}}$;

- (iii) there is a $*$ -representation (ρ, \mathbf{K}) of \mathbf{E} , a contraction $D \in B(\mathbf{k}; \mathbf{K})$ and a vector $\xi \in \mathbf{K}$, such that $l = l^\varphi$ where

$$\varphi(x) = \begin{bmatrix} \langle \xi | \\ D^* \end{bmatrix} (\rho(x) - \epsilon(x)I_{\mathbf{K}}) \begin{bmatrix} | \xi \rangle & D \end{bmatrix} + \epsilon(x)\varphi(1) \quad (5.2)$$

($x \in \mathbf{E}$), and $\varphi(1)$ is nonpositive with block matrix of the form

$$\begin{bmatrix} * & * \\ * & D^*D - I_{\mathbf{k}} \end{bmatrix}.$$

Proof. For the proof of the equivalence of (i) and (ii) we may suppose that \mathbf{E} is faithfully and nondegenerately represented in $B(\mathfrak{h})$, say, in such a way that the counit extends to a normal state ϵ'' on \mathbf{E}'' . (This may be achieved by taking the direct sum of an arbitrary faithful nondegenerate representation and the GNS representation $(\mathfrak{h}_\epsilon, \pi_\epsilon, \xi_\epsilon)$, so that ϵ is extended by the vector state $\omega_{(0, \xi_\epsilon)}$. Alternatively, take the universal representation and bidual map ϵ^{**} .) Note that ϵ'' is necessarily $*$ -homomorphic.

Suppose first that (i) holds and let $\{\gamma_{c',c} : c', c \in \mathbf{k}\}$ be the generators of the associated convolution semigroups of functionals of l . Let $k \in \mathbb{P}^\ddagger(\mathbf{E} \rightarrow \mathbf{E})$ be the process $(R_{B(\mathcal{F})}l_t)_{t \geq 0}$. By Proposition 2.4, k is completely positive and contractive. Moreover (4.3) holds so that k is a Markov-regular QS cocycle on \mathbf{E} . In view of Theorem 1.2, it follows that $k = k^\phi$ for some map ϕ of the form (1.16) with constituents $\Psi \in CP(\mathbf{E}; \mathbf{E}'' \overline{\otimes} B(\widehat{\mathbf{k}}))$ and $J \in \mathbf{E}'' \overline{\otimes} \widehat{\mathbf{k}}$, say. Thus, from the definition of k ,

$$\omega_{\widehat{c}, \widehat{c}} \bullet \phi = (\text{id}_{\mathbf{E}} \otimes \gamma_{c',c}) \circ \Delta \quad (5.3)$$

($c, c' \in \mathbf{k}$). Now define $\varphi, \psi \in CB(\mathbf{E}; B(\widehat{\mathbf{k}}))$ and $\chi \in \widehat{\mathbf{k}}$ by

$$\varphi = \epsilon \bullet \phi, \quad \psi = (\epsilon'' \overline{\otimes} \text{id}_{B(\widehat{\mathbf{k}})}) \circ \Psi \text{ and } \langle \chi | = (\epsilon'' \overline{\otimes} \text{id}_{\widehat{\mathbf{k}}})(J),$$

noting that, by the complete positivity of ϵ , $\varphi(1) \leq 0$ and ψ is completely positive. We claim that $l = l^\varphi$ and that φ has the decomposition (5.1). By (5.3), $\omega_{\widehat{c}, \widehat{c}} \circ \varphi =$

$\gamma_{c',c}$ and so, by (4.4), the QS convolution cocycles l^φ and l have the same associated convolution semigroups and are therefore equal. In view of the multiplicativity of ϵ'' ,

$$(\epsilon'' \overline{\otimes} \text{id}_{B(\widehat{\mathbf{k}})})(x \otimes |e_0\rangle)J = \epsilon(x)|e_0\rangle\langle\chi|,$$

for $x \in \mathbf{E}$. Now, using the fact that ϵ'' is real to obtain the adjoint identity, collecting terms yields the decomposition (5.1), so (ii) holds.

Suppose conversely that (ii) holds. As in the proof of Proposition 4.2, let $k = k^\phi$ where $\phi = R_{B(\widehat{\mathbf{k}})}\varphi \in CB(\mathbf{E}; \mathbf{E} \otimes B(\widehat{\mathbf{k}}))$. Then ϕ has the form (1.16), with $\Psi = R_{B(\widehat{\mathbf{k}})}\psi$ and $J = I_{\mathfrak{h}} \otimes \langle\chi|$, moreover it follows from Proposition 2.4 that Ψ is completely positive and $\phi(1) \leq 0$. Thus, by Theorem 1.2, the Markov-regular weak QS cocycle k is completely positive and contractive. Therefore, by Proposition 3.3, Proposition 2.4 and Lemma 4.1, (i) holds.

Again suppose that (ii) holds. Let

$$\begin{bmatrix} \langle\xi| \\ D^* \end{bmatrix} \rho(\cdot) \begin{bmatrix} | \xi \rangle & D \end{bmatrix} \quad (5.4)$$

be a minimal Stinespring decomposition of ψ . Thus (ρ, \mathbf{K}) is a unital C^* -representation of \mathbf{E} , ξ is a vector in \mathbf{K} , D is an operator in $B(\mathbf{k}; \mathbf{K})$ (and (5.5) below holds). Identity (5.2) follows, with

$$\varphi(1) = \begin{bmatrix} \|\xi\|^2 - 2 \operatorname{Re} \alpha & \langle D^* \xi - c | \\ |D^* \xi - c \rangle & D^* D - I_{\mathbf{k}} \end{bmatrix},$$

where $\begin{pmatrix} \alpha \\ c \end{pmatrix} = \chi$, so (iii) holds.

Conversely, suppose that (iii) holds. Then, writing

$$\begin{bmatrix} t & \langle d | \\ |d \rangle & D^* D - I_{\mathbf{k}} \end{bmatrix}$$

for the block matrix form of $\varphi(1)$, φ has the form (5.1) where ψ is given by (5.4) and

$$\chi = \begin{pmatrix} \frac{1}{2}(\|\xi\|^2 - t) \\ D^* \xi - d \end{pmatrix}$$

so (ii) holds. This completes the proof. \square

Remarks. An alternative proof of the above theorem, which directly establishes the equivalence of (i) and (iii) without appeal to Theorem 1.2 on standard QS cocycles (whose proof depends on the Christensen-Evans Theorem), is given in [S].

In (iii) the following *minimality* condition on the quadruple $(\rho, \mathbf{K}, D, \xi)$ may be assumed:

$$\rho(\mathbf{E})(C\xi + \operatorname{Ran} D) \text{ is dense in } \mathbf{K}. \quad (5.5)$$

Under minimality there is uniqueness too: if $(\rho', \mathbf{K}', D', \xi')$ is another quadruple as in (iii) then there is a unique isometry $V \in B(\mathbf{K}; \mathbf{K}')$ (unitary if this quadruple is also minimal) satisfying

$$VD = D', V\xi = \xi' \text{ and } V\rho(x) = \rho'(x)V \text{ for } x \in \mathbf{E}.$$

By a characterisation of nonnegative block matrix operators (see, for example, Lemma 2.2 in [GLSW]), if φ is the stochastic generator of a Markov-regular, completely positive, contractive QS convolution cocycle then $\varphi(1)$ has the form

$$\begin{bmatrix} t & \langle C^{1/2}e | \\ |C^{1/2}e \rangle & -C \end{bmatrix}$$

for a nonnegative contraction C , a unique vector $e \in \overline{\operatorname{Ran} C}$ and a real number t satisfying $t \leq -\|e\|^2$. Moreover, with respect to any decomposition (5.2), $C = I_{\mathbf{k}} - D^* D$.

Unitality for the cocycle is equivalent to its stochastic generator being expressible in the form

$$\begin{bmatrix} \langle \xi | \\ D^* \end{bmatrix} (\rho - \iota_{\mathbf{K}} \circ \epsilon)(\cdot) \begin{bmatrix} | \xi \rangle \\ D \end{bmatrix}$$

where D is isometric and $\rho(1)$ coincides with the identity operator on $\mathbb{C}\xi + \text{Ran } D$.

It also follows from the above proof that if k is the QS cocycle on a C^* -hyperbialgebra \mathbf{E} , given by $k_t = R_{B(\mathcal{F})} l_t$ where l is a Markov-regular, completely positive, contractive QS convolution cocycle on \mathbf{E} , then $k = k^\phi$ where the stochastic generator ϕ is expressible in the form

$$x \mapsto \psi(x) - x \otimes (\Delta^{QS} + |\chi\rangle\langle e_0| + |e_0\rangle\langle \chi|)$$

for some completely positive map $\psi : \mathbf{E} \rightarrow \mathbf{E} \otimes B(\widehat{\mathbf{k}})$ and vector $\chi \in \widehat{\mathbf{k}}$. Note that no appeal to a concrete realisation of \mathbf{E} is needed in this decomposition.

6. HOMOMORPHIC QS CONVOLUTION COCYCLES

In this section we characterise the stochastic generators of Markov-regular $*$ -homomorphic convolution cocycles on a C^* -bialgebra, by applying the R -map to the characterisation of the generators of Markov-regular multiplicative cocycles obtained in [LW₄]. Thus let \mathbf{B} be a C^* -bialgebra.

Weak multiplicativity for a process $l \in \mathbb{P}_*^\dagger(\mathbf{B})$ is the following property:

$$l_t^{\varepsilon', \varepsilon}(x^*y) = l_{t, |\varepsilon'\rangle}^\dagger(x)^* l_{t, |\varepsilon\rangle}(y)$$

($\varepsilon, \varepsilon' \in \mathcal{E}, x, y \in \mathbf{B}, t \in \mathbb{R}_+$). If the C^* -bialgebra is concretely realised on a Hilbert space then weak multiplicativity for a process $k \in \mathbb{P}^\dagger(\mathbf{B} \rightarrow \mathbf{B})$ reads

$$k_t(x^*y) = k_t^\dagger(x)^* k_t(y)$$

($x, y \in \mathbf{B}, t \in \mathbb{R}_+$), an identity in Hilbert space operators. In view of the remark at the end of the introduction, if $k \in \mathbb{P}^\dagger(\mathbf{B} \rightarrow \mathbf{B})$ is both weakly multiplicative and real then it is bounded, and so $*$ -homomorphic — in particular it is completely bounded.

Theorem 6.1. *Let $l = l^\varphi$ where $\varphi \in CB(\mathbf{B}; B(\widehat{\mathbf{k}}))$. Then the following are equivalent*

- (a) l is weakly multiplicative;
- (b) φ satisfies

$$\varphi(xy) = \varphi(x)\epsilon(y) + \epsilon(x)\varphi(y) + \varphi(x)\Delta^{QS}\varphi(y) \quad (6.1)$$

$$(x, y \in \mathbf{B}).$$

Proof. For the proof we may suppose without loss of generality that the C^* -bialgebra \mathbf{B} is concretely realised, in $B(\mathfrak{h})$ say. Let $\phi = R_{B(\widehat{\mathbf{k}})}\varphi$ and set $k = k^\phi$. Since $\text{Ran } \phi \subset \mathbf{B} \otimes B(\widehat{\mathbf{k}})$, Theorem 3.4 and Corollary 4.2 of [LW₄] imply that k is weakly multiplicative if and only if ϕ satisfies

$$\phi(xy) = \phi(x)\iota_{\widehat{\mathbf{k}}}(y) + \iota_{\widehat{\mathbf{k}}}(x)\phi(y) + \phi(x)(I_{\mathfrak{h}} \otimes \Delta^{QS})\phi(y). \quad (6.2)$$

If (6.2) holds then, applying the homomorphism $\epsilon \otimes \text{id}_{B(\widehat{\mathbf{k}})}$ to both sides yields (6.1). Conversely, suppose that (6.1) holds and set $\tilde{\varphi} = \text{id}_{\mathbf{B}} \otimes \varphi$ and $\tilde{\epsilon} = \text{id}_{\mathbf{B}} \otimes (\iota_{\widehat{\mathbf{k}}} \circ \epsilon)$, so that

$$\phi = \tilde{\varphi} \circ \Delta \text{ and } \tilde{\epsilon} \circ \Delta = \iota_{\widehat{\mathbf{k}}}. \quad (6.3)$$

Then, for simple tensors $X = x_1 \otimes x_2$ and $Y = y_1 \otimes y_2$ in $\mathbf{B} \otimes \mathbf{B}$,

$$x_1 y_1 \otimes \varphi(x_2 y_2) = x_1 y_1 \otimes \{ \varphi(x_2)\epsilon(y_2) + \epsilon(x_2)\varphi(y_2) + \varphi(x_2)\Delta^{QS}\varphi(y_2) \},$$

or

$$\tilde{\varphi}(XY) = \tilde{\varphi}(X)\tilde{\epsilon}(Y) + \tilde{\epsilon}(X)\tilde{\varphi}(Y) + \tilde{\varphi}(X)(I_{\mathfrak{h}} \otimes \Delta^{QS})\tilde{\varphi}(Y).$$

By linearity and continuity this holds for all $X, Y \in \mathbf{B} \otimes \mathbf{B}$, in particular, for $X = \Delta x$ and $Y = \Delta y$. Therefore, by (6.3) and the multiplicativity of Δ , (6.2) holds.

It therefore remains only to show that l is weakly multiplicative if and only if k is. Recall that $l \in \mathbb{P}_{\star}^{\dagger}(\mathbf{B})$ and $l^{\dagger} = l^{\psi}$ where $\psi = \varphi^{\dagger}$. Let $u, u' \in \mathfrak{h}$, $t \in \mathbb{R}_{+}$ and $\varepsilon, \varepsilon' \in \mathcal{E}$. If l is weakly multiplicative then $\langle \varepsilon', l_t(xy)\varepsilon \rangle = \langle l_t^{\dagger}(x^*)\varepsilon', l_t(y)\varepsilon \rangle$, so

$$\langle u'\varepsilon', (\text{id}_{\mathbf{B}} \odot l_t)(XY)u\varepsilon \rangle = \langle (\text{id}_{\mathbf{B}} \odot l_t^{\dagger})(X^*)u'\varepsilon', (\text{id}_{\mathbf{B}} \odot l_t)(Y)u\varepsilon \rangle \quad (6.4)$$

holds for all $X, Y \in \mathbf{B} \odot \mathbf{B}$. Now it follows, from the identity

$$(\text{id}_{\mathbf{B}} \odot l_t^{\#})(X)u\varepsilon = (\text{id}_{\mathbf{B}} \otimes l_{t,|\varepsilon}^{\#})(X)u$$

(where $l^{\#}$ stands for l or l^{\dagger}) and Property 2', that both sides of (6.4) are continuous in both X and Y , giving an identity for all $X, Y \in \mathbf{B} \otimes \mathbf{B}$. Setting $X = \Delta x$ and $Y = \Delta y$ and using the multiplicativity of Δ , this identity becomes a statement of the weak multiplicativity of k .

Suppose conversely that k is weakly multiplicative. Set $k^{\dagger} = k^{\psi}$ for $\psi = \phi^{\dagger}$, and let $x, y \in \mathbf{B}$. First note that the identity

$$(\varepsilon \otimes \text{id}_{|\mathcal{F}})(X^*)(\varepsilon \otimes \text{id}_{|\mathcal{F}})(Y) = \varepsilon(X^*Y)$$

is obvious for $X, Y \in \mathbf{B} \odot |\mathcal{F}$ and so holds for $X, Y \in \mathbf{B} \otimes |\mathcal{F}$ by continuity. Set $X = k_{t,|\varepsilon}^{\dagger}(x^*)$ and $Y = k_{t,|\varepsilon'}(y)$. Then $X, Y \in \mathbf{B} \otimes |\mathcal{F}$ and so

$$\begin{aligned} \langle l_t^{\dagger}(x^*)\varepsilon, l_t(y)\varepsilon' \rangle &= (\varepsilon \otimes \text{id}_{|\mathcal{F}})(X)^*(\varepsilon \otimes \text{id}_{|\mathcal{F}})(Y) \\ &= \varepsilon(X^*Y) \\ &= (\varepsilon \circ \omega_{\varepsilon, \varepsilon'} \bullet k_t)(xy) \\ &= (\omega_{\varepsilon, \varepsilon'} \circ \varepsilon \bullet k_t)(xy) = \langle \varepsilon, l_t(xy)\varepsilon' \rangle. \end{aligned}$$

Thus l is weakly multiplicative. This completes the proof. \square

Combining this result with Proposition 4.3, Theorem 5.1 and Theorem A.6 we obtain the advertised characterisation of the stochastic generators of Markov-regular *-homomorphic convolution cocycles on a C^* -bialgebra.

Theorem 6.2. *Let \mathbf{B} be a C^* -bialgebra and let $l \in \mathbb{P}_{\star}(\mathbf{B})$. Then the following are equivalent:*

- (i) l is a Markov-regular, *-homomorphic (and unital) QS convolution cocycle on \mathbf{B} ;
- (ii) $l = l^{\varphi}$ where $\varphi \in CB(\mathbf{B}; B(\hat{\mathbf{k}}))$ satisfies

$$\varphi(x^*y) = \varphi(x)^*\varepsilon(y) + \varepsilon(x)^*\varphi(y) + \varphi(x)^*\Delta^{QS}\varphi(y) \quad (\text{and } \varphi(1) = 0); \quad (6.5)$$

- (iii) there is a vector $c \in \mathbf{k}$ and (unital) *-homomorphism $\pi : A \rightarrow B(\mathbf{k})$ such that $l = l^{\varphi}$ where

$$\varphi(x) = \begin{bmatrix} \langle c | \\ I_{\mathbf{k}} \end{bmatrix} (\pi(x) - \varepsilon(x)I_{\mathbf{k}}) \begin{bmatrix} |c \rangle & I_{\mathbf{k}} \end{bmatrix}, \quad x \in \mathbf{B}. \quad (6.6)$$

Remark. In fact, as is shown in the appendix, the relation (6.5) for a linear map φ (an ε -structure map in the terminology used there) entails the implemented form (6.6), in particular the complete boundedness of φ .

The characterisations of stochastic generators of completely positive, contractive QS convolution cocycles and *-homomorphic QS convolution cocycles in Theorems 5.1 and 6.2 may be used to derive dilation theorems for QS convolution cocycles (see [S]), of the type obtained for standard QS cocycles in [GLW] and [GLSW].

These characterisations are also used to establish the main result in [FrS], that every Markov-regular Fock space quantum Lévy process can be realised as a limit of (suitably scaled) random walks.

7. AXIOMATISATION OF TOPOLOGICAL QUANTUM LÉVY PROCESSES

Defining quantum Lévy process on a C^* -bialgebra requires certain modifications of the original, purely algebraic, definition of Accardi, Schürmann and von Waldenfels ([ASW], [Sch]). The problem is how to build convolution increments of the process given that, in general, multiplication $\mathbb{B} \odot \mathbb{B} \rightarrow \mathbb{B}$ need not extend continuously to $\mathbb{B} \otimes \mathbb{B}$. (This is a commonly met difficulty in the theory of topological quantum groups, see [Kus₁]). Below we outline two ways of overcoming this obstacle.

The simplest idea is to define a quantum Lévy process using only the concept of distributions.

Definition. A *weak quantum Lévy process* on a C^* -bialgebra \mathbb{B} over a unital $*$ -algebra-with-state (\mathcal{A}, ω) is a family $(j_{s,t}: \mathbb{B} \rightarrow \mathcal{A})_{0 \leq s \leq t}$ of unital $*$ -homomorphisms such that the functional $\lambda_{s,t} := \omega \circ j_{s,t}$ is continuous and satisfies the following conditions, for $0 \leq r \leq s \leq t$:

- (wQLPi) $\lambda_{r,t} = \lambda_{r,s} \star \lambda_{s,t}$;
- (wQLPii) $\lambda_{t,t} = \epsilon$;
- (wQLPiii) $\lambda_{s,t} = \lambda_{0,t-s}$;
- (wQLPiv)

$$\omega \left(\prod_{i=1}^n j_{s_i, t_i}(x_i) \right) = \prod_{i=1}^n \lambda_{s_i, t_i}(x_i)$$

whenever $n \in \mathbb{N}$, $x_1, \dots, x_n \in \mathbb{B}$ and the intervals $[s_1, t_1[, \dots, [s_n, t_n[$ are disjoint;

- (wQLPv) $\lambda_{0,t} \rightarrow \epsilon$ pointwise as $t \rightarrow 0$.

A weak quantum Lévy process on a C^* -bialgebra \mathbb{B} is called *Markov-regular* if $\lambda_{0,t} \rightarrow \epsilon$ in norm, as $t \rightarrow 0$.

The family $\lambda := (\lambda_{0,t})_{t \geq 0}$ is a pointwise continuous convolution semigroup of functionals on \mathbb{B} , called the *one-dimensional distribution* of the process; if the process is Markov-regular then λ has a convolution generator which is also referred to as the *generator* of the weak quantum Lévy process. Two weak quantum Lévy processes on \mathbb{B} , j^1 over $(\mathcal{A}^1, \omega^1)$ and j^2 over $(\mathcal{A}^2, \omega^2)$, are said to be *equivalent* if they satisfy

$$\omega^1 \circ j_{s,t}^1 = \omega^2 \circ j_{s,t}^2$$

for all $0 \leq s \leq t$, in other words if their one-dimensional distributions coincide; if they are Markov-regular then this is equivalent to equality of their generators.

Remarks. Note that the above definition of a weak quantum Lévy process, in contrast to the definition of a quantum Lévy process on an algebraic $*$ -bialgebra, does not yield a recipe for expressing the joint moments of the process increments corresponding to overlapping time intervals, such as

$$\omega(j_{r,t}(x)j_{s,t}(y)) \text{ where } r, s < t.$$

To achieve the latter, one would have to formulate the weak convolution increment property (wQLPi) in greater generality and assume certain commutation relations between the increments corresponding to disjoint time intervals. For other investigations of the notion of independence in noncommutative probability, in the absence of commutation relations being imposed, we refer to the recent paper [HKK].

As in the algebraic case, the generator of a Markov-regular weak quantum Lévy process vanishes on $1_{\mathbb{B}}$, is real and is *conditionally positive*, that is positive on the kernel of the counit. Observe that if $l \in \mathbb{P}_*(\mathbb{B})$ is a unital $*$ -homomorphic QS convolution cocycle then, defining $\mathcal{A} := B(\mathcal{F})$, $\omega := \omega_{\varepsilon(0)}$, and $j_{s,t} := \sigma_s \circ l_{t-s}$ for all $0 \leq s \leq t$, we obtain a weak quantum Lévy process on \mathbb{B} , called a *Fock space quantum Lévy process*, Markov-regular if l is. The proof of the following theorem closely mirrors the proof of Schürmann's reconstruction theorem for the purely algebraic case ([Sch], see also [LS₁]); all the necessary continuity properties follow from the results in the appendix.

Theorem 7.1. *Let γ be a real, conditionally positive linear functional on \mathbb{B} vanishing at $1_{\mathbb{B}}$. Then there is a (Markov-regular) Fock space quantum Lévy process with generator γ .*

Proof. The proof uses a GNS-style construction. Let $D = \text{Ker } \epsilon / N$ where N is the following subspace of $\text{Ker } \epsilon$:

$$\{x \in \text{Ker } \epsilon \mid \gamma(x^*x) = 0\}.$$

Then $([x], [y]) \mapsto \gamma(x^*y)$ defines an inner product on D . Let \mathfrak{k} be the Hilbert space completion of D . The prescription $\pi(x) : [z] \mapsto [xz]$ defines bounded operators on D , whose extensions make up a unital representation of \mathbb{B} on \mathfrak{k} satisfying

$$\langle \pi(x)[y], [z] \rangle = \langle [y], \pi(x^*)[z] \rangle.$$

Furthermore the linear map $\delta : x \mapsto |d(x)\rangle$, where $d(x) = [x - \epsilon(x)I_{\mathfrak{k}}]$, is easily seen to be a $(\pi - \epsilon)$ -derivation $\mathbb{B} \rightarrow |\mathfrak{k}\rangle$ satisfying

$$\delta(x)^* \delta(y) = \gamma(x^*y) - \gamma(x)^* \epsilon(y) - \epsilon(x)^* \gamma(y).$$

Theorem A.6 therefore implies that the map $\varphi : \mathbb{B} \rightarrow B(\widehat{\mathfrak{k}})$, with block matrix form given by the prescription (A.8) (with $\lambda = \gamma$ and $\chi = \epsilon$), is completely bounded. Setting $l = l^\varphi$, Theorem 6.2 implies that the Markov-regular weak QS convolution cocycle l is unital and $*$ -homomorphic. Since $\varphi_0^0 = \gamma$ the result follows. \square

Corollary 7.2. *Every Markov-regular weak quantum Lévy process is equivalent to a Fock space quantum Lévy process.*

Another notion, in a sense intermediate between weak quantum Lévy processes and Fock space quantum Lévy processes, can be formulated in terms of product systems — a similar idea is mentioned in a recent paper of Skeide ([Ske]). Recall that a *product system of Hilbert spaces* is a ‘measurable’ family of Hilbert spaces $E = \{E_t : t \geq 0\}$, together with unitaries $U_{s,t} : E_s \otimes E_t \rightarrow E_{s+t}$ ($s, t \geq 0$) satisfying associativity relations:

$$U_{r+s,t}(U_{r,s} \otimes I_t) = U_{r,s+t}(I_r \otimes U_{s,t}) \quad (7.1)$$

($r, s, t \in \mathbb{R}_+$), where I_s denotes the identity operator on E_s . A *unit* for the product system E is a ‘measurable’ family $\{u(t) : t \geq 0\}$ of vectors with $u(t) \in E_t$ and $u(s+t) = U_{s,t}(u(s) \otimes u(t))$ for all $s, t \geq 0$ (the unit is *normalised* if, for all $t \geq 0$, $\|u(t)\| = 1$). For the precise definition we refer to [Arv]. The unitaries $U_{s,t}$ implement isomorphisms $\sigma_{s,t} : B(E_s \otimes E_t) \rightarrow B(E_{s+t})$.

Definition. A *product system quantum Lévy process* on \mathbb{B} over a product-system-with-normalised-unit (E, u) is a family $(j_t : \mathbb{B} \rightarrow B(E_t))_{t \geq 0}$ of unital $*$ -homomorphisms satisfying the following conditions:

- (psQLPi) $j_{r+s} = \sigma_{r,s} \circ (j_r \star j_s)$,
- (psQLPii) $j_0 = \iota_0 \circ \epsilon$,
- (psQLPiii) $\omega_{u(t)} \circ j_t \rightarrow \epsilon$ pointwise as $t \rightarrow 0$,

for $r, s \geq 0$, where ι_0 denotes the ampliation $\mathbb{C} \rightarrow B(E_0)$.

The ‘exponential’ product system is given by $E_t = \mathcal{F}_{[0,t]}$ and $U_{s,t} = I_s \otimes S_{s,t}$ where $S_{s,t}$ denotes the natural shift $\mathcal{F}_{[0,t]} \rightarrow \mathcal{F}_{[s,s+t]}$ and the exponential property of symmetric Fock space is invoked. Clearly every Fock space quantum Lévy process may be viewed as a product system quantum Lévy process over (E, Ω) where Ω is the normalised unit given by $\Omega(t) = \varepsilon(0) \in \mathcal{F}_{[0,t]}$, $t \geq 0$.

Proposition 7.3. *Each product system quantum Lévy process on \mathbb{B} naturally determines a weak quantum Lévy process on \mathbb{B} with the same one-dimensional distribution.*

Proof. Let j be a quantum Lévy process on \mathbb{B} over a product-system-with-normalised-unit (E, u) . We use an inductive limit construction. Define $\tilde{\mathcal{A}} := \bigcup_{t \geq 0} (B(E_t), t)$ and introduce on $\tilde{\mathcal{A}}$ the relation: $(T, r) \equiv (S, s)$ if there is $t \geq \max\{r, s\}$ such that $\sigma_{r,t-r}(T \otimes I_{t-r}) = \sigma_{s,t-s}(S \otimes I_{t-s})$, in other words we identify operators with common ampliations. The associativity relations (7.1) imply that \equiv is an equivalence relation. Define $\mathcal{A} = \tilde{\mathcal{A}}/\equiv$ and introduce the structure of a unital $*$ -algebra on \mathcal{A} , consistent with the pointwise operations:

$$(T, t) + (S, t) = (T + S, t), \quad (S, t) \cdot (T, t) = (ST, t), \quad (T, t)^* = (T^*, t)$$

($t \geq 0, S, T \in B(E_t)$). The map $\tilde{\omega} : \tilde{\mathcal{A}} \rightarrow \mathbb{C}$ defined by $\tilde{\omega}(T, t) = \omega_{u(t)}(T)$ induces a state ω on \mathcal{A} . For $s, t \in \mathbb{R}_+$ define

$$j_{s,t} : \mathbb{B} \rightarrow \mathcal{A} \text{ by } x \mapsto [\sigma_{s,t-s}(I_s \otimes j_{t-s}(x))]_{\equiv}.$$

It is easy to see that the family $(j_{s,t})_{0 \leq s \leq t}$ is a weak quantum Lévy process on \mathbb{B} over (\mathcal{A}, ω) . \square

The construction in the above proof, informed by the case of QS convolution cocycles, is a special case of the familiar construction of C^* -algebraic inductive limits. The completion of \mathcal{A} with respect to the norm induced from $\tilde{\mathcal{A}}$ is a unital C^* -algebra that may be called the *C^* -algebra of finite range operators on the product system E* .

Remark. A form of reconstruction theorem also holds for completely positive QS convolution cocycles. It is easily seen that if $l \in \mathbb{P}_*(\mathbb{E})$ is a Markov-regular, unital, completely positive QS convolution cocycle on a C^* -hyperbialgebra \mathbb{E} , then the generator of its Markov convolution semigroup is real, vanishes at $1_{\mathbb{E}}$ and is conditionally positive. The GNS-type construction from the proof of Theorem 7.1 yields a completely bounded map $\varphi : \mathbb{A} \rightarrow B(\hat{\mathbf{k}})$ for which the cocycle l^φ is unital and completely positive according to Proposition 4.3 and Theorem 5.1 (of course there is no reason why it should be $*$ -homomorphic, if \mathbb{E} is not a C^* -bialgebra). Clearly the Markov convolution semigroup of l^φ coincides with that of l .

8. EXAMPLES

In this section we consider $*$ -homomorphic convolution cocycles on three types of C^* -bialgebra, namely algebras of continuous functions on compact semigroups, universal C^* -algebras of discrete groups, and full compact quantum groups. We focus on connections between the results obtained in this paper and the case of purely algebraic convolution cocycles analysed in its predecessor, [LS₁]. Recall that in [LS₁] the basic object is an algebraic $*$ -bialgebra (or even coalgebra) \mathcal{B} , and coalgebraic QS differential equations are driven by coefficients in $L(\mathcal{B}; \mathcal{O}^\dagger(\hat{D}))$, where D is some dense subspace of the noise dimension space \mathbf{k} . Processes $V \rightarrow \mathbb{C}$, now for a vector space V , are families $k = (k_t)_{t \geq 0}$ of maps $V \rightarrow \mathcal{O}(\mathcal{E}_D)$; we denote the space of these by $\mathbb{P}_*(V : \mathcal{E}_D)$, and write $\mathbb{P}_*^\dagger(\mathcal{B} : \mathcal{E}_D)$ for the subspace of $\mathcal{O}^\dagger(\mathcal{E}_D)$ -valued processes. *Pointwise Hölder-continuity* for such a process k means that each

of the vector-valued functions $t \mapsto k_t(x)\varepsilon$ should be locally Hölder-continuous with exponent $1/2$. Note that it is a weaker form of continuity than the one that arises when V is an operator space (cf. Properties 2 and 2' after Theorems 1.1 and 3.1).

The notation introduced after Theorem 3.1 extends as follows: for $\varphi \in L(\mathcal{B}; \mathcal{O}(\widehat{D}))$, (3.6) still defines a process $\mathbb{P}_*(\mathcal{B} : \mathcal{E}_D)$ (again written l^φ) which (uniquely) satisfies the QS differential equation (3.1), now understood in the sense of [LS₁], and is a QS convolution cocycle with respect to the purely algebraic coalgebra structure. If the coefficient φ lies in $L(\mathcal{B}; \mathcal{O}^\dagger(\widehat{D}))$ then $l^\varphi \in \mathbb{P}_*^\dagger(\mathcal{B} : \mathcal{E}_D)$.

Commutative case: continuous functions on a semigroup. Let H be a compact semigroup with identity e and let \mathbf{B} denote $C(H)$, the algebra of continuous complex-valued functions on H . Then \mathbf{B} has the structure of a C^* -bialgebra with comultiplication and counit given by

$$\Delta(F)(h, h') = F(hh') \quad \text{and} \quad \epsilon(F) = F(e)$$

($h, h' \in H, F \in \mathbf{B}$), courtesy of the natural identification $\mathbf{B} \otimes \mathbf{B} \cong C(H \times H)$.

Following standard practice in quantum probability (going back to [AFL] and beyond), any H -valued stochastic process $X = (X_t)_{t \geq 0}$ on the probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, may be described by a family of unital $*$ -homomorphisms $(l_t)_{t \geq 0}$ given by

$$l_t : \mathbf{B} \rightarrow L^\infty(\Omega, \mathfrak{F}, \mathbb{P}), \quad F \mapsto F \circ X_t,$$

in turn these homomorphisms uniquely determine the original process.

Recall that a process X on a semigroup with identity is called a *Lévy process* if it has identically distributed, independent increments, $\mathbb{P}(\{X_0 = e\}) = 1$ and the distribution of X_t converges weakly to the Dirac measure $\delta_{\{e\}}$ (the distribution of X_0) as t tends to 0. In general every Lévy process on a semigroup may be equivalently realised, in the sense of equal finite-dimensional distributions (see [Sch], [LS₁]), as a quantum Lévy process on a $*$ -bialgebra ([Sch], [FrScho]).

As is well known, not all Lévy processes have stochastic generators defined on the whole of \mathbf{B} . In our language, this corresponds to the fact that not all $*$ -homomorphic processes on \mathbf{B} are Markov-regular. Now Markov-regularity of the process corresponds to norm continuity of the convolution semigroup given by

$$\lambda_t(F) = \int_{\Omega} F \circ X_t d\mathbb{P}$$

($F \in \mathbf{B}, t \geq 0$). Note that the usual notion of weak continuity for this semigroup corresponds, in the algebraic formulation, to pointwise continuity of the Markov semigroup. We therefore obtain the following result.

Proposition 8.1. *Let X be a Lévy processes on a compact semigroup with identity H . Suppose that as a topological space H is normal. Then X is equivalent to a Markov-regular $*$ -homomorphic QS convolution cocycle on \mathbf{B} if and only if it satisfies the following condition:*

$$\mathbb{P}(\{X_t = e\}) \rightarrow 1 \text{ as } t \rightarrow 0. \quad (8.1)$$

Proof. It is easily seen that condition (8.1) implies the existence of a bounded generator $\gamma : \mathbf{B} \rightarrow \mathbb{C}$ from which the process can be reconstructed. The other direction can be seen by considering the Markov semigroup of a given QS convolution cocycle and judiciously choosing continuous functions on H with values in $[0, 1]$, which are equal 1 at e and vanish outside of some neighbourhood of the identity element e . \square

Processes satisfying (8.1) were investigated for example in [Gre]. They are called homogenous processes of discontinuous type and their laws are compound Poisson distributions ([Gre], Theorem 2.3.5).

Cocommutative case: group algebras. Let Γ be a discrete group. Denote by $\mathbb{B} = C^*(\Gamma)$ the enveloping C^* -algebra of the Banach algebra $l^1(\Gamma)$ ([Ped]), called the universal (or full) C^* -algebra of Γ . By construction (the algebra of functions on Γ with finite support being dense in \mathbb{B}), there is a universal unitary representation $L : \Gamma \rightarrow \mathbb{B}$ such that $\mathcal{B} := \text{Lin}\{L_g : g \in \Gamma\}$ is dense in \mathbb{B} . Due to universality the mappings Δ and ϵ defined on the image of L by

$$\Delta(L_g) = L_g \otimes L_g \text{ and } \epsilon(L_g) = 1,$$

extend to $*$ -homomorphisms on \mathbb{B} . It is easily checked that \mathbb{B} , equipped with the resulting comultiplication and counit, becomes a cocommutative C^* -bialgebra.

Theorem 8.2. *Let $\mathbb{B} = C^*(\Gamma)$ for a discrete group Γ . Then*

$$W(t, g) = l_t(L_g) \quad (g \in \Gamma, t \geq 0) \quad (8.2)$$

defines a bijective correspondence between unital $$ -homomorphic QS convolution cocycles on the C^* -bialgebra \mathbb{B} and maps $W : \mathbb{R}_+ \times \Gamma \rightarrow B(\mathcal{F})$ satisfying the following conditions:*

- (i) *for each $g \in \Gamma$ the family $\{W(t, g) : t \geq 0\}$ is a left QS operator cocycle;*
- (ii) *for each $t \geq 0$ the family $\{W(t, g) : g \in \Gamma\}$ is a unitary representation of Γ on \mathcal{F} .*

Proof. Let $l \in \mathbb{P}_*(\mathbb{B})$ is a $*$ -homomorphic QS convolution cocycle and define a map $W : \mathbb{R}_+ \times \Gamma \rightarrow B(\mathcal{F})$ by (8.2). Then, for all $g, h \in \Gamma$ and $s, t \geq 0$,

$$\begin{aligned} l_{s+t}(L_g) &= (l_s \otimes (\sigma_t \circ l_s))(\Delta L_g) = l_s(L_g) \otimes \sigma_s(l_t(L_g)) = W(s, g) \otimes \sigma_s(W(t, g)), \\ l_t(L_g)l_t(L_h) &= l_t(L_g L_h) = l_t(L_{gh}) = W(t, gh), \\ l_t(L_g)^* &= l_t(L_g^*) = l_t(L_{g^{-1}}) = W(t, g^{-1}), \\ l_t(L_e) &= l_t(1_{\mathbb{B}}) = I_{\mathcal{F}} \text{ and} \\ l_0(L_g) &= I_{\mathcal{F}}, \end{aligned}$$

so W satisfies (i) and (ii). Conversely, suppose that $W : \mathbb{R}_+ \times \Gamma \rightarrow B(\mathcal{F})$ is a map satisfying conditions (i) and (ii). Due to universality there are maps $l_t : \mathbb{B} \rightarrow B(\mathcal{F})$, $t \geq 0$, satisfying (8.2). The properties of W imply that they are unital $*$ -homomorphisms and that they satisfy

$$l_0(x) = \epsilon(x)I_{\mathcal{F}} \text{ and } l_{s+t}(x) = (l_s \otimes (\sigma_s \circ l_t))(\Delta x)$$

for $s, t \geq 0$ and $x \in \mathbb{B}$. Continuity ensures that these remain valid for $x \in \mathbb{B}$ and so the result follows. \square

On the level of stochastic generators the above correspondence takes the following form.

Theorem 8.3. *Let $\mathcal{B} := \text{Lin}\{L_g : g \in \Gamma\}$ for a discrete group Γ . Then*

$$\psi_g = \varphi(L_g), \quad g \in \Gamma,$$

determines a bijective correspondence between maps $\varphi \in L(\mathcal{B}; B(\widehat{\mathbf{k}}))$ satisfying

$$\varphi(ab) = \varphi(a)\epsilon(b) + \epsilon(a)\varphi(b) + \varphi(a)\Delta^{QS}\varphi(b), \quad \varphi(a)^* = \varphi(a^*), \quad \varphi(1) = 0, \quad (8.3)$$

and maps $\psi : \Gamma \rightarrow B(\widehat{\mathbf{k}})$ satisfying

$$\psi_{gh} = \psi_g + \psi_h + \psi_g \Delta^{QS} \psi_h, \quad (\psi_g)^* = \psi_{g^{-1}}, \quad \psi_e = 0; \quad (8.4)$$

Proof. Elementary calculation. \square

Remarks. Identities (8.4) may be considered as a special (time-independent) case of formulae (4.2-4) in [HLP]. They are equivalent to ψ having the block matrix form

$$\psi_g = \begin{bmatrix} i\lambda_g - \frac{1}{2}\|\xi_g\|^2 & -\langle \xi_g | U_g \rangle \\ |\xi_g\rangle & U_g - I_{\mathfrak{k}} \end{bmatrix}, \quad (8.5)$$

for a unitary representation U of Γ on $\widehat{\mathfrak{k}}$ and maps $\lambda : \Gamma \rightarrow \mathbb{R}$ and $\xi : \Gamma \rightarrow \mathfrak{k}$ satisfying

$$\xi_{gh} = \xi_g + U_g \xi_h \text{ and } \lambda_{gh} = \lambda_g + \lambda_h - \text{Im}\langle \xi_g, U_g \xi_h \rangle.$$

Note that, according to Theorem 6.3 of [LS₁], each map $\varphi \in L(\mathcal{B}; B(\widehat{\mathfrak{k}}))$ satisfying (8.3) generates a unital, real and weakly multiplicative QS convolution cocycle l^φ on \mathcal{B} . The process l^φ continuously extends to a *-homomorphic QS convolution cocycle on \mathcal{B} (see Lemma 8.7 below). On the other hand, given a map $\psi : \Gamma \rightarrow B(\widehat{\mathfrak{k}})$ satisfying (8.4), for each fixed $g \in \Gamma$ the unique (weakly regular, weak) solution of the operator QS differential equation

$$X_0 = I_{\mathcal{F}}, \quad dX_t = X_t d\Lambda_L(t),$$

where $L = \psi_g$, is a unitary left QS cocycle W^g ([LW₂]). The map $W : \mathbb{R}_+ \times \Gamma$ given by $W(t, g) = W_t^g$ satisfies the conditions of Theorem 8.2. One can easily see that the correspondences described in Theorems 8.2 and 8.3 are consistent with this construction.

Proposition 8.4. *A unital *-homomorphic QS convolution cocycle l on \mathcal{B} is equal to l^φ for some $\varphi \in L(\mathcal{B}; B(\widehat{\mathfrak{k}}))$ if and only if it is pointwise weakly measurable.*

Proof. One direction is trivial. For the other consider the unitary cocycles $\{W(\cdot, g) : g \in \Gamma\}$ associated with l by Theorem 8.2. Theorem 6.7 of [LW₂] implies that each of these cocycles is stochastically generated (as it is weakly measurable). Denoting the respective generators by ψ_g one can see that the map $\psi : \Gamma \rightarrow B(\widehat{\mathfrak{k}})$ so obtained satisfies the conditions (8.4). The desired conclusion therefore follows from Theorem 8.3 and the subsequent discussion. \square

If a *-homomorphic QS convolution cocycle l on \mathcal{B} is Markov-regular, the automatic implementedness of its stochastic generator φ (Theorem 6.2) implies in particular that the triple (λ, ξ, U) corresponding to φ by (8.5) and Theorem 8.3 must also be *implemented*, in the following sense: there is a vector $\eta \in \mathfrak{k}$ such that

$$\xi_g = U_g \eta - \eta \text{ and } \lambda_g = \text{Im}\langle \eta, U_g \eta \rangle, \quad g \in G.$$

In the language of group cohomology, the first order cocycle ξ is a coboundary. In this connection, see [PaS].

Elements of a C^* -bialgebra \mathcal{B} are called *group-like* when they satisfy $\Delta b = b \otimes b$, as the L_g 's do. On such elements the solution $(k_t(b))_{t \geq 0}$, of the mapping QS differential equation (3.1), is given by the solution of the operator QS differential equation

$$dX_t = X_t d\Lambda_L(t), \quad X_0 = I_{\mathcal{F}},$$

where $L = \varphi(b) \in B(\widehat{\mathfrak{k}})$. For more on this we refer to Section 4.1 of [Sch].

Full compact quantum groups. A concept of compact quantum groups was introduced by Woronowicz, in [Wor₁]. For our purposes it is most convenient to adopt the following definition:

Definition ([Wor₂]). A *compact quantum group* is a pair (\mathcal{B}, Δ) , where \mathcal{B} is a unital C^* -algebra, and $\Delta : \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{B}$ is a unital, *-homomorphic map which is coassociative and satisfies the quantum cancellation properties:

$$\overline{\text{Lin}}((1 \otimes \mathcal{B})\Delta(\mathcal{B})) = \overline{\text{Lin}}((\mathcal{B} \otimes 1)\Delta(\mathcal{B})) = \mathcal{B} \otimes \mathcal{B}.$$

For the concept of Hopf $*$ -algebras and their unitary corepresentations, as well as unitary corepresentations of compact quantum groups, we refer the reader to [KLS]. For our purposes it is sufficient to note the facts contained in the following theorem.

Theorem 8.5 ([Wor₂]). *Let \mathbb{B} be a compact quantum group and let \mathcal{B} denote the linear span of the matrix coefficients of irreducible unitary corepresentations of \mathbb{B} . Then \mathcal{B} is a dense $*$ -subalgebra of \mathbb{B} , the coproduct of \mathbb{B} restricts to an algebraic coproduct Δ_0 on \mathcal{B} and there is a natural counit ϵ and coinverse \mathcal{S} on \mathcal{B} which makes it a Hopf $*$ -algebra.*

Remark ([BMT]). In the above theorem $(\mathcal{B}, \Delta_0, \epsilon, \mathcal{S})$ is the unique dense Hopf $*$ -subalgebra of \mathbb{B} , in the following sense: if $(\mathcal{B}', \Delta'_0, \epsilon', \mathcal{S}')$ is a Hopf $*$ -algebra, in which \mathcal{B}' is a dense $*$ -subalgebra of \mathbb{B} and the coproduct of \mathbb{B} restricts to the algebraic coproduct Δ'_0 on \mathcal{B}' , then $(\mathcal{B}', \Delta'_0, \epsilon', \mathcal{S}')$ equals $(\mathcal{B}, \Delta_0, \epsilon, \mathcal{S})$.

The Hopf $*$ -algebra arising here is called the *associated Hopf $*$ -algebra* of (\mathbb{B}, Δ) . When $\mathbb{B} = C(G)$ for a compact group G , \mathcal{B} is the algebra of all matrix coefficients of unitary representations of G ; when \mathbb{B} is the universal C^* -algebra of a discrete group Γ , $\mathcal{B} = \text{Lin}\{L_g : g \in \Gamma\}$ (see the beginning of the previous subsection). Dijkhuizen and Koornwinder observed that the Hopf $*$ -algebras arising in this way have intrinsic algebraic structure.

Definition. A Hopf $*$ -algebra \mathcal{B} is called a *CQG algebra* if it is the linear span of all matrix elements of its finite dimensional unitary corepresentations.

Theorem 8.6 ([DiK]). *Each Hopf $*$ -algebra associated with a compact quantum group is a CQG algebra. Conversely, if \mathcal{B} is a CQG algebra then*

$$\|x\| := \sup \{ \|\pi(x)\| : \pi \text{ is a } * \text{-representation of } \mathcal{B} \text{ on a Hilbert space} \} \quad (8.6)$$

defines a C^ -norm on \mathcal{B} and the completion of \mathcal{B} with respect to this norm is a compact quantum group whose comultiplication extends that of \mathcal{B} .*

The compact quantum group obtained from a Hopf $*$ -algebra \mathcal{B} in this theorem is called its *universal compact quantum group* and is denoted \mathcal{B}_u .

For later use note the following extension of Lemma 11.31 in [KLS]:

Lemma 8.7. *Let E be a dense subspace of a Hilbert space H and let \mathcal{B} be a CQG algebra. Suppose that $\pi : \mathcal{B} \rightarrow \mathcal{O}^\natural(E)$ is real, unital and weakly multiplicative. Then π is bounded-operator-valued and admits a continuous extension to a unital $*$ -homomorphism from \mathcal{B}_u to $B(H)$.*

Proof. Let $[x_{i,j}]_{i,j=1}^n$ be any finite dimensional unitary corepresentation of \mathcal{B} . Then, since $\sum_{k=1}^n x_{k,j}^* x_{k,j} = 1_{\mathcal{B}}$ for $j \in \{1, \dots, n\}$,

$$\begin{aligned} \|\pi(x_{i,j})\xi\|^2 &\leq \sum_{k=1}^n \|\pi(x_{k,j})\xi\|^2 = \sum_{k=1}^n \langle \pi(x_{k,j})\xi, \pi(x_{k,j})\xi \rangle \\ &= \left\langle \xi, \pi \left(\sum_{k=1}^n x_{k,j}^* x_{k,j} \right) \xi \right\rangle = \|\xi\|^2 \end{aligned}$$

for $i, j \in \{1, \dots, n\}$ and $\xi \in E$. This implies that, for each $x \in \mathcal{B}$, $\pi(x)$ is bounded — let $\pi_1(a)$ denote its continuous extension to a bounded operator on H . The resulting map $\pi_1 : \mathcal{B} \rightarrow B(H)$ is then a unital $*$ -homomorphism, moreover it is clearly contractive with respect to the canonical norm on \mathcal{B} , given by (8.6); the result follows. \square

Definition. A compact quantum group (\mathbb{B}, Δ) is called *full* if the C^* -norm it induces on its associated CQG algebra \mathcal{B} coincides with its canonical norm defined in (8.6) — equivalently, if \mathbb{B} is $*$ -isomorphic to \mathcal{B}_u .

The notion of full compact quantum groups was introduced in [BMT] and in [BaS] (in the first paper they were called universal compact quantum groups). It is very relevant for our context, as the above facts imply the following

Proposition 8.8. *Let \mathbb{B} be a full compact quantum group with associated Hopf $*$ -algebra \mathcal{B} . Then \mathbb{B} is a C^* -bialgebra whose counit is the continuous extension of the counit on \mathcal{B} . Moreover restriction induces a bijective correspondence between unital, $*$ -homomorphic QS convolution cocycles on \mathbb{B} and unital, real, weakly multiplicative QS convolution cocycles (in the sense of [LS₁]) on \mathcal{B} .*

Both families of examples described in the previous two subsections, namely algebras of continuous functions on compact groups and full C^* -algebras of discrete groups, are full compact quantum groups. Moreover most of the genuinely quantum (i.e. neither commutative nor cocommutative) compact quantum groups considered in the literature also fall into this category, including the queen of examples, $SU_q(2)$.

Reconnecting further with our previous work, we obtain the following result.

Theorem 8.9. *Let $k \in \mathbb{P}_{\star \text{cb}}(\mathbb{B})$ where \mathbb{B} is a full compact quantum group with associated Hopf $*$ -algebra \mathcal{B} . Then the following are equivalent:*

- (i) k and k^\dagger are pointwise Hölder-continuous QS convolution cocycles;
- (ii) $k|_{\mathcal{B}} = l^\varphi$ for some map $\varphi \in L(\mathcal{B}; B(\widehat{\mathfrak{k}}))$.

Proof. One direction follows from the fact that \mathcal{B} is an (algebraic) coalgebra and Theorem 5.8 of [LS₁]. The other is trivial. \square

Specialising to $*$ -homomorphic cocycles yields the following much stronger result.

Theorem 8.10. *Let $k \in \mathbb{P}_\star(\mathbb{B} : \mathcal{E}_D)$ where \mathbb{B} is a full compact quantum group with associated Hopf $*$ -algebra \mathcal{B} and D is a dense subspace of \mathfrak{k} . Then the following are equivalent:*

- (i) k is pointwise Hölder-continuous, unital and $*$ -homomorphic (thus bounded) and $a \mapsto k_t(a)$ defines a QS convolution cocycle;
- (ii) k is bounded and $k|_{\mathcal{B}} = l^\varphi$ for some $\varphi \in L(\mathcal{B}; \mathcal{O}^\dagger(\widehat{D}))$ satisfying the structure relations (6.5).

Proof. The implication (i) \Rightarrow (ii) follows from the previous theorem and implication (i) \Rightarrow (ii) of Theorem 6.3 of [LS₁] (note that it even yields $\varphi \in L(\mathcal{B}; \mathcal{O}^\dagger(\widehat{\mathfrak{k}})) = L(\mathcal{B}; B(\widehat{\mathfrak{k}}))$).

Suppose conversely that (ii) holds. Theorem 6.3 of [LS₁] guarantees that $l = k|_{\mathcal{B}}$ is real, unital, and weakly multiplicative. Lemma 8.7 shows that l admits a continuous extension to a $*$ -homomorphic unital process $\mathbb{B} \rightarrow \mathbb{C}$ defined on \mathcal{E}_D , which must coincide with k . Application of the previous theorem therefore completes the proof. \square

The above theorem may be equivalently formulated in the following way.

Theorem 8.11. *Let $k \in \mathbb{P}_\star(\mathcal{B} : \mathcal{E}_D)$ where \mathcal{B} is the Hopf $*$ -algebra associated with a full compact quantum group \mathbb{B} and D is a dense subspace of \mathfrak{k} . Then the following are equivalent:*

- (i) k extends to a pointwise Hölder-continuous, unital, $*$ -homomorphic QS convolution cocycle on \mathbb{B} ;
- (ii) $k = l^\varphi$ for some $\varphi \in L(\mathcal{B}; \mathcal{O}^\dagger(\widehat{D}))$ satisfying the structure relations (6.5).

Remark. In the course of the proof of the previous theorem it was established that each map φ defined on a CQG algebra \mathcal{B} with values in $\mathcal{O}^\dagger(\widehat{D})$ satisfying the conditions (6.5) must be bounded-operator-valued. We stress however, that φ need not extend continuously to \mathbb{B} (for examples see [SchS]). On the other hand if φ is continuous, then it is necessarily completely bounded.

APPENDIX: (π', π) -DERIVATIONS AND χ -STRUCTURE MAPS

In this appendix we give an extension of the innerness theorem of Christensen, for completely bounded derivations on a C^* -algebra, to (π', π) -derivations, and prove automatic complete boundedness for (π, χ) -derivations, when χ is a character. These are then applied to prove the innerness of what we call χ -structure maps. We first recall the relevant theorems on derivations.

Theorem A.1 ([Sak₁], [Rin]). *Let $\delta : A \rightarrow X$ be a derivation from a C^* -algebra A into a Banach A -bimodule. Then δ is bounded.*

Theorem A.2 ([Chr]). *Let A be a C^* -algebra in $B(\mathfrak{h})$ and let $\delta : A \rightarrow B(\mathfrak{h})$ be a derivation. If δ is completely bounded then it is inner: there is $R \in B(\mathfrak{h})$ such that $\delta(a) = aR - Ra, a \in A$.*

A simple proof of the first theorem in the case $X = A$ (Sakai's Theorem), due to Kishimoto, may be found in [Sak₂], and a good reference for the second, along with connections to not-necessarily-involutive homomorphisms between C^* -algebras, is [Pis₁]. We are interested in the particular class of Banach A bimodule-valued derivations captured by the following definition.

Definition. Let A be a C^* -algebra with representations (π, \mathfrak{h}) and (π', \mathfrak{h}') . A map $\delta : A \rightarrow B(\mathfrak{h}; \mathfrak{h}')$ is called a (π', π) -derivation if it satisfies

$$\delta(ab) = \delta(a)\pi(b) + \pi'(a)\delta(b);$$

it is *inner* if it is implemented by an operator $T \in B(\mathfrak{h}; \mathfrak{h}')$ in the sense that

$$\delta : a \rightarrow \pi'(a)T - T\pi(a).$$

Theorem A.3. *Let A be a C^* -algebra with representations (π, \mathfrak{h}) and (π', \mathfrak{h}') , and let $\delta : A \rightarrow B(\mathfrak{h}; \mathfrak{h}')$ be a completely bounded (π', π) -derivation. Then δ is inner.*

Proof. Let (ρ, \mathfrak{K}) be a faithful representation of A and set $\mathfrak{H} = \mathfrak{h} \oplus \mathfrak{h}' \oplus \mathfrak{K}$ and $\tilde{A} = \tilde{\pi}(A)$ where $\tilde{\pi}$ is the faithful representation $\pi \oplus \pi' \oplus \rho$. Then \tilde{A} is a C^* -subalgebra of $B(\mathfrak{H})$ and it is easily verified that

$$\tilde{\pi}(a) \mapsto \begin{bmatrix} 0 & & \\ \delta(a) & 0 & \\ & & 0 \end{bmatrix}$$

defines a derivation $\tilde{\delta} : \tilde{A} \rightarrow B(\mathfrak{H})$. It is also clear that $\tilde{\delta}$ is completely bounded if and only if δ is. Moreover, if $\tilde{\delta}$ is inner then the (π', π) -derivation δ is implemented by $S_{21} \in B(\mathfrak{h}; \mathfrak{h}')$ for any operator $S = [S_{ij}] \in B(\mathfrak{H})$ implementing the derivation $\tilde{\delta}$. The result therefore follows from Theorem A.2. \square

Theorem A.4. *Let A be a C^* -algebra with representation (π, \mathfrak{h}) and character χ , and let $\delta : A \rightarrow |\mathfrak{h}|$ be a (π, χ) -derivation. Then δ is inner.*

Proof. Without loss of generality we may suppose that the C^* -algebra A and representation π are both unital; if necessary by extending π , χ and δ to the unitisation of A in the following natural way:

$$(a, z) \mapsto \pi(a) + zI_{\mathfrak{h}}, \quad (a, z) \mapsto \chi(a) + z \quad \text{and} \quad (a, z) \mapsto \delta(a).$$

By Theorem A.1, δ is bounded. Let $A_0 = \text{Ker } \chi$ and let $\psi : A \rightarrow A_0$ be the projection $a \mapsto a - \chi(a)1$. Then A_0 is a C^* -subalgebra of A , ψ is completely bounded and $\delta = \tilde{\delta} \circ \psi$, where $\tilde{\delta} = \delta|_{A_0}$. Therefore, by the previous theorem, it suffices to show that $\tilde{\delta}$ is completely bounded. Now $\tilde{\delta}(ab) = \pi(a)\delta(b)$ for all $a, b \in A_0$. Since δ is bounded this implies that

$$\tilde{\delta}^{(n)}(A) = \lim_{\lambda} \pi^{(n)}(A)(\delta(e_{\lambda}) \otimes I_n)$$

($n \in \mathbb{N}$, $A \in M_n(\mathbf{A}_0)$), for any C^* -approximate identity (e_λ) for \mathbf{A}_0 , and so $\|\tilde{\delta}^{(n)}\| \leq \|\delta\|$. The result follows. \square

We note two consequences; the first is used in [S].

Corollary A.5. *Let \mathbf{A} be a C^* -algebra with characters (i.e. nonzero multiplicative linear functionals) χ and χ' . Then every (χ', χ) derivation on \mathbf{A} vanishes.*

For the second the following definitions are convenient. If \mathbf{A} is a C^* -algebra with character χ , then a χ -structure map on \mathbf{A} is a linear map $\varphi : \mathbf{A} \rightarrow B(\mathbb{C} \oplus \mathfrak{h})$, for some Hilbert space \mathfrak{h} , satisfying

$$\varphi(a^*b) = \varphi(a)^*\chi(b) + \chi(a)^*\varphi(b) + \varphi(a)^*\Delta\varphi(b) \quad (\text{A.7})$$

where $\Delta := \begin{bmatrix} 0 & \\ & I_{\mathfrak{h}} \end{bmatrix}$. For any C^* -representation (π, \mathfrak{h}) and vector $\xi \in \mathfrak{h}$,

$$a \mapsto \begin{bmatrix} \langle \xi | \\ I_{\mathfrak{h}} \end{bmatrix} (\pi(a) - \chi(a)I_{\mathfrak{h}}) \begin{bmatrix} | \xi \rangle & I_{\mathfrak{h}} \end{bmatrix}$$

defines a χ -structure map. Such χ -structure maps are said to be *implemented*. Thus implementation involves a pair (π, ξ) . Note that implemented χ -structure maps are completely bounded.

Theorem A.6. *Let \mathbf{A} be a C^* -algebra with character χ and let φ be a χ -structure map on \mathbf{A} . Then φ is implemented.*

Proof. Without loss of generality we may suppose that \mathbf{A} is unital, since otherwise (invoking the reality of φ) the prescriptions

$$(a, z) \mapsto \chi(a) + z, \quad \text{respectively} \quad (a, z) \mapsto \varphi(a),$$

extend χ and φ to the unitisation of \mathbf{A} , maintaining the χ -structure relation (A.7). Now the χ -structure relation is equivalent to φ having block matrix form

$$\begin{bmatrix} \lambda & \delta^\dagger \\ \delta & \nu \end{bmatrix} \quad (\text{A.8})$$

where $\nu = \pi - \iota_{\mathfrak{k}} \circ \chi$ for a $*$ -homomorphism $\pi : \mathbf{A} \rightarrow B(\mathfrak{h})$, δ is a (π, χ) -derivation and the linear functional λ satisfies

$$\lambda(a^*b) = \lambda(a)^*\chi(b) + \chi(a)^*\lambda(b) + \delta(a)^*\delta(b)$$

($a, b \in \mathbf{A}$) — in particular, λ is real and satisfies

$$\lambda(1) = -\delta(1)^*\delta(1) \text{ and } \lambda(a^*b) = \delta(a)^*\delta(b) \text{ for } a, b \in \mathbf{A}_0, \quad (\text{A.9})$$

where $\mathbf{A}_0 = \text{Ker } \chi$. By Theorem A.4, there is a vector $\xi \in \mathfrak{k}$ such that $\delta(a) = \nu(a)|\xi\rangle$. Now define a bounded linear functional $\tilde{\lambda}$ on \mathbf{A} by $\tilde{\lambda}(a) = \langle \xi, \nu(a)\xi \rangle$. It is easily checked that $\tilde{\lambda}$ also satisfies (A.9), thus $\tilde{\lambda}$ agrees with λ on $\mathbf{A}_{00} + \mathbb{C}1_{\mathbf{A}}$ where $\mathbf{A}_{00} = \text{Lin}\{a^*b : a, b \in \mathbf{A}_0\}$. But \mathbf{A}_{00} is dense in \mathbf{A}_0 and $\mathbf{A} = \mathbf{A}_0 \oplus \mathbb{C}1_{\mathbf{A}}$, so $\tilde{\lambda}$ equals λ . The result follows. \square

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