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No-activation theorem for Gaussian nonclassical correlations by Gaussian operations

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We study general quantum correlations of continuous variable Gaussian states and their interplay with entanglement. Specifically, we investigate the existence of a quantum protocol activating all nonclassical correlations between the subsystems of an input bipartite continuous variable system, into output entanglement between the system and a set of ancillae. For input Gaussian states, we prove that such an activation protocol cannot be accomplished with Gaussian operations, as the latter are unable to create any output entanglement from an initial separable yet nonclassical state in a worst-case scenario. We then construct a faithful non-Gaussian activation protocol, encompassing infinite-dimensional generalizations of controlled-NOT gates to generate entanglement between system and ancillae, in direct analogy with the finite-dimensional case. We finally calculate the negativity of quantumness, an operational measure of nonclassical correlations defined in terms of the performance of the activation protocol, for relevant classes of two-mode Gaussian states.

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I. INTRODUCTION

Quantum correlations in composite systems transcend entanglement [1]. A bipartite quantum state ρ_{AB} can be defined as nonclassical or nonclassically correlated if it cannot be expressed as a convex mixture of local basis states of subsystems A and B [2]. Consequently, all inseparable (entangled) states as well as the majority of separable states are nonclassical.

General nonclassical correlations, however, can be mapped to entanglement in a very precise sense, which provides an insightful framework for their characterization and operational interpretation. Specifically, it was proven in [3–5] and very recently experimentally observed in [6] that all nonclassical states of a finite-dimensional system can be turned into states with distillable entanglement between the system and a set of ancillae by an *activation protocol*. Focusing on a bipartite setting, the protocol runs as follows. The subsystems A and B are first subject to arbitrary local unitary transformations $U_{A,B}$; then, each system $j = A,B$ interacts via a controlled-NOT (CNOT) operation $U_{jj'}^{\mathrm{CNOT}}$ (i.e., a so-called premeasurement interaction) with an auxiliary system j', $j = A,B$, initialized in a pure state $|0\rangle_{j'}$. The activation protocol then possesses two key properties: (1) for all classical states ρ_{AB} at the input of the protocol, there exist local unitaries $U_{A,B}$ for which the output state $\rho_{ABA'B'}$ is separable across the $AB|A'B'$ splitting, and (2) for all nonclassical states ρ_{AB} and for all local unitaries the output state is entangled across the $AB|A'B'$ splitting.

Let us stress that both criteria 1 and 2 must be met by any scheme in order to be a valid activation protocol. In particular, they allow us to define faithful measures of nonclassical correlations for the input state ρ_{AB} in terms of the output $AB|A'B'$ entanglement, minimized over $U_{A,B}$. One such measure, when the output entanglement is quantified by the negativity [7], has been termed negativity of quantumness [3,8] and has been experimentally investigated in [6,9].

In this paper we study activation of nonclassical correlations in multimode bipartite Gaussian states ρ_{AB} of continuous variable systems [10]. Nonclassical correlations of Gaussian states have been studied extensively both theoretically and experimentally [11–14] but their interplay with entanglement has not been pinned down so far in terms of the activation framework. Attempts to devise activationlike protocols for Gaussian states have been explored in [15]. However, these differed significantly from the original prescription in that nonunitary operations were employed between system and ancillae, so that the entanglement generation was obtained as a dynamical feature, and conditions 1 and 2 were not generally verified.

Here we consider a general Gaussian activation protocol in which $U_{A,B}$ are Gaussian unitaries and the CNOT gates are replaced with a global Gaussian unitary on subsystems A,B,A',B'. In Sec. II we then prove that any such protocol satisfying condition 1 will unavoidably violate condition 2, which implies that activation of Gaussian nonclassical correlations by Gaussian operations is impossible. This fact establishes a no-go theorem for Gaussian quantum information processing, which can be enlisted alongside other well known no-go results such as the no-distillation theorem, according to which distilling entanglement from Gaussian states by using only Gaussian operations is impossible [16,17]. We then show in Sec. III how, by using non-Gaussian operations which properly extend the CNOT to infinite dimensions, one can construct the continuous variable counterpart of the activation protocol of [3], verifying criteria 1 and 2. This allows us to define the negativity of quantumness for Gaussian states and to calculate it for relevant examples in Sec. IV. This work provides an operational setting to understand and manipulate nonclassical correlations in paradigmatic infinite-dimensional systems. We draw our conclusions in Sec. V, while some technical derivations (which can be of independent interest) are deferred to the Appendices.

II. GAUSSIAN NO-ACTIVATION THEOREM

Gaussian states are quantum states of systems with an infinite-dimensional Hilbert space (continuous variable

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systems), e.g., a collection of harmonic oscillators, which possess a Gaussian-shaped Wigner function in phase space [10]. L modes are described by a vector $\mathbf{r} = (x_1, p_1, \ldots, x_L, p_L)^T$ of quadrature operators x_j and p_j satisfying the canonical commutation rules expressible in terms of elements of the vector $\mathbf{r}$ as $[r_j, r_k] = i\Omega_{jk}$ and $j,k = 1, \ldots, L$ with $\Omega = \oplus_{j=1}^{L} i\sigma_y$, where σ_y is the Pauli y matrix. An L-mode Gaussian state ρ is fully characterized by a $2L \times 1$ vector $\langle \mathbf{r} \rangle$ of the first moments with elements $\langle r_i \rangle = \mathrm{Tr}(\rho r_i)$ and by its $2L \times 2L$ covariance matrix (CM) γ with elements $\gamma_{ij} = \langle \Delta r_i \Delta r_j + \Delta r_j \Delta r_i \rangle / 2$ and $i, j = 1, \ldots, L$, where $\Delta r_i = r_i - \langle r_i \rangle$. Gaussian unitaries are generated by Hamiltonians that are quadratic in the quadrature operators and they preserve the Gaussian characteristic of quantum states. An L-mode Gaussian unitary $U(S)$ is represented in phase space by a $2L \times 2L$ real symplectic transformation S satisfying the condition $S\Omega S^T = \Omega$, which transforms a CM γ to $S\gamma S^T$.

Here we are interested in the question of whether an activation protocol exists satisfying conditions 1 and 2 which would rely solely on Gaussian states and Gaussian unitaries. We therefore assume the state ρ_{AB} to be a Gaussian state of $(N + M)$ modes with CM γ_{AB} and the state $\rho_{A'B'}$ of the ancilla to be also a Gaussian state with CM $\gamma_{A'B'}$. The local unitaries $U_{A,B}$ of the original discrete protocol are replaced with local Gaussian unitaries $U_A(S_A)$ and $U_B(S_B)$ represented by the symplectic matrices S_A and S_B, respectively. Likewise, the global operation $U_{AA'}^{\mathrm{CNOT}} \otimes U_{BB'}^{\mathrm{CNOT}}$ on the whole system $ABA'B'$ is replaced with one global Gaussian unitary $U(S)$ represented by a symplectic matrix S.

Let us recall the definition of a fully classical state [2,3,5]. Suppose ρ_{AB} is a bipartite state containing two subsystems A and B with N and M modes, respectively, and let $\mathcal{B}_j = \{|\mathcal{B}_j(\mathbf{n}_j)\rangle\}$ be a basis of subsystem j, with $\mathbf{n}_A = (n_{A_1}, \ldots, n_{A_N})$, $\mathbf{n}_B = (n_{B_1}, \ldots, n_{B_M})$, and $n_{j_i} \in \mathbb{N}_0$. If there exists a basis $\mathcal{B}$ consisting of the tensor products of all elements of $\mathcal{B}_A$ with all elements of $\mathcal{B}_B$, then ρ_{AB} is a classical state if it is diagonal with respect to $\mathcal{B}$. It has been shown in [11,18] that a two-mode Gaussian state is classical if and only if it is a product state, i.e., its CM is represented by a direct sum $\gamma_A \oplus \gamma_B$ of local CMs $\gamma_{A,B}$. One can prove that this statement remains valid for the generic case of bipartite $(N + M)$-mode Gaussian states (see Appendix A for the proof). Therefore, all nonproduct bipartite Gaussian states (including separable ones) are nonclassical. According to condition 1, in any Gaussian activation protocol with an input Gaussian product state there must exist local Gaussian unitaries $U_{A,B}$ for which one gets a separable state $\rho_{ABA'B'}$ across the $AB|A'B'$ splitting at the output of the protocol. We will show, however, that this implies that for *all* separable Gaussian states including nonclassical ones there exist local Gaussian unitaries $U_{A,B}$ for which the output state is separable. That is, condition 2 is not satisfied. Thus, any Gaussian activation protocol described above cannot meet simultaneously criteria 1 and 2 and hence does not exist.

The proof of this no-go theorem is depicted in Fig. 1 and follows from the decomposability of any Gaussian separable state into a product state and noise [19] and from the linearity of symplectic transformations. Namely, for any separable Gaussian state with CM γ_{AB} there exist local CMs $\gamma_{A,B}$ such

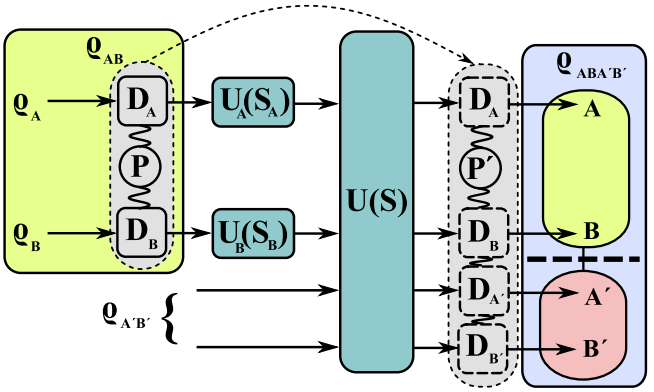

FIG. 1. (Color online) Pictorial representation of the no-activation theorem. ρ_{AB} is a separable Gaussian state prepared from a Gaussian product state $\rho_A \otimes \rho_B$ by correlated displacements D_A and D_B distributed according to a Gaussian distribution with correlation matrix P, Eq. (1). $U_A(S_A)$ and $U_B(S_B)$ are local Gaussian unitaries which are adjusted such that without the displacements D_A and D_B the activation protocol produces from the product state $\rho_A \otimes \rho_B$ an output state which is separable across the $AB|A'B'$ splitting. As the unitaries $U_A(S_A)$, $U_B(S_B)$, and $U(S)$ induce a linear transformation of quadrature operators of the input modes, the displacements D_A and D_B can be relocated behind the global transformation $U(S)$ (dotted arrow). The new displacements $D_A, D_B, D_{A'}$, and $D_{B'}$, Eq. (2), cannot turn a separable state into an entangled state and therefore the protocol transforms the separable state ρ_{AB} into a state which is separable across the $AB|A'B'$ cut (thick dashed line). See text for details.

that

$$P = \gamma_{AB} - \gamma_A \oplus \gamma_B \geqslant 0. \tag{1}$$

In other words, any separable Gaussian state with CM γ_{AB} can be prepared from a suitable product state with CM $\gamma_A \oplus \gamma_B$ by the addition of noise, represented by a positive-semidefinite matrix P, i.e., $\gamma_{AB} = \gamma_A \oplus \gamma_B + P$. The noise can be created by displacing the vector of quadratures $\mathbf{r} = (x_{A_1}, p_{A_1}, \ldots, x_{A_N}, p_{A_N}, x_{B_1}, p_{B_1}, \ldots, x_{B_M}, p_{B_M})^T$ of the product state as $\mathbf{r} \to \mathbf{r} + \mathcal{V}\mathbf{R}$. Here $\mathcal{V}$ is a $2(N + M) \times K$ matrix given by the first K columns of the matrix V bringing the matrix P to the diagonal form $V^T P V = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_K, 0, 0, \ldots, 0)$, where $\lambda_1, \ldots, \lambda_K$ denote $K \leqslant 2(N + M)$ strictly positive eigenvalues of the matrix (1), and $\mathbf{R} = (R_1, \ldots, R_K)^T$ is the vector of classical displacements uncorrelated with the vector of quadratures $\mathbf{r}$ and distributed according to the Gaussian distribution with zero means and the diagonal correlation matrix $\mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_K)$.

Let us now consider a separable state with CM γ_{AB} at the input of a Gaussian activation protocol and let $\gamma_A \oplus \gamma_B$ be a CM of the product state from which the state can be prepared using the aforementioned algorithm. Assume that the local symplectic matrices $S_{A,B}$ are chosen such that the CM $\gamma^{(0)}_{ABA'B'} \equiv S(S_A \oplus S_B \oplus \mathbb{1}_{A'B'})\gamma_A \oplus \gamma_B \oplus \gamma_{A'B'}(S_A^T \oplus S_B^T \oplus \mathbb{1}_{A'B'})S^T$ of the output state, where $\gamma_{A'B'}$ is the CM of the state of the ancilla, is separable across the $AB|A'B'$ splitting. Hence, for the original separable state with CM γ_{AB}, the output of the activation protocol is obtained by displacing the vector

of quadratures $\mathbf{r}^{(0)}$ for the state with CM $\gamma^{(0)}_{ABA'B'}$ by

$$\mathbf{r}^{(0)} \to \mathbf{r}^{(0)} + S\begin{pmatrix}(S_A \oplus S_B)\mathcal{V}\mathbf{R}\\ \mathbb{O}\end{pmatrix}, \tag{2}$$

where $\mathbb{O}$ is a $2T \times 1$ zero vector with T being the number of modes of the ancilla $A'B'$. However, for a separable state with CM $\gamma^{(0)}_{ABA'B'}$, where AB is separable from $A'B'$, the local displacements (2) cannot create a state in which the system AB is entangled with the system $A'B'$. Consequently, for any separable state (even nonclassical) it is always possible to find local Gaussian unitaries for which the output is separable, thus accomplishing the proof of the no-go theorem.

Therefore, Gaussian operations are unable to activate nonclassical correlations of Gaussian separable states into entanglement in the worst-case scenario: assuming condition 1 holds, then for any Gaussian separable state there exist local Gaussian unitaries for which the output of the activation protocol remains a separable Gaussian state. This indicates that a non-Gaussian element, like a non-Gaussian global unitary U or a non-Gaussian state of the ancilla, is necessary for faithful activation of nonclassical correlations in Gaussian states. In the following we design such an activation protocol involving a non-Gaussian CNOT gate in the Fock basis and an ancillary system in a Gaussian state.

III. NON-GAUSSIAN ACTIVATION PROTOCOL

The main benefit of the activation protocol is that it allows one to quantify the amount of nonclassical correlations in a given quantum state as the potential to create entanglement in the activation protocol [3–5]. More precisely, if $E_{AB|A'B'}(\rho_{ABA'B'})$ denotes an entanglement measure quantifying the amount of entanglement between systems AB and $A'B'$ in a quantum state $\rho_{ABA'B'}$, then we can define a measure of nonclassical correlations on the input state ρ_{AB} as

$$Q_E(\rho_{AB}) = \min_{U_A,U_B} E_{AB|A'B'}(\rho_{ABA'B'}), \tag{3}$$

where the minimization is carried out over all local unitaries U_A and U_B on subsystems A and B. It has been proven in [5] that $Q_E(\rho_{AB}) \geqslant E(\rho_{AB})$, with equality if ρ_{AB} is pure.

From now on we assume that systems A and B each contain one mode. The non-Gaussian activation protocol is obtained as a direct generalization of the finite-dimensional protocol [3]. At the input we allow for generally non-Gaussian states ρ_{AB} of continuous variable systems, local unitaries U_A and U_B, and the global Gaussian unitary $U(S)$ of the preceding protocol is replaced with the tensor product $V \equiv U^{\text{CNOT}}_{AA'} \otimes U^{\text{CNOT}}_{BB'}$ of the infinite-dimensional generalizations of CNOT gates in the Fock basis:

$$U^{\text{CNOT}}_{jj'}|m,n\rangle_{jj'} = |m,m+n\rangle_{jj'}, \quad j = A,B, \tag{4}$$

where $|m,n\rangle_{jj'} \equiv |m\rangle_j \otimes |n\rangle_{j'}$, $m,n = 0,1,\ldots$, and $|k\rangle_l$ is the kth Fock state of mode l. We also assume the initial state $\rho_{A'B'}$ of the ancilla $A'B'$ to be the vacuum state $|0\rangle_{A'}\langle 0| \otimes |0\rangle_{B'}\langle 0|$. Hence, the final output state can be expressed as

$$\rho_{ABA'B'} = V(\tilde{\rho} \otimes |0\rangle_{A'}\langle 0| \otimes |0\rangle_{B'}\langle 0|)V^\dagger, \tag{5}$$

where

$$\tilde{\rho} \equiv (U_A \otimes U_B)\rho_{AB}(U_A^\dagger \otimes U_B^\dagger). \tag{6}$$

By following arguments similar to the finite-dimensional case [3], one can show that the non-Gaussian activation protocol defined above satisfies both criteria 1 and 2. For condition 1 we assume that ρ_{AB} is classically correlated and hence there exist local unitaries U_A and U_B such that the density matrix $\tilde{\rho}$, Eq. (6), takes the form $\tilde{\rho} = \sum_{n,m=0}^{\infty} p_{n,m}|n,m\rangle_{AB}\langle n,m|$. Making use of Eqs. (4) and (5) it then follows that the output state of the protocol is the following convex mixture of product states, $\rho_{ABA'B'} = \sum_{n,m=0}^{\infty} p_{n,m}|n,m\rangle_{AB}\langle n,m| \otimes |n,m\rangle_{A'B'}\langle n,m|$ and is thus a separable state across the $AB|A'B'$ splitting as required.

For the proof of condition 2 we now suppose that the density matrix ρ_{AB} is nonclassical and show that the density matrix $\rho_{ABA'B'}$ given in Eq. (5) is entangled across the $AB|A'B'$ cut for all local unitaries U_A and U_B. To prove the presence of entanglement in $\rho_{ABA'B'}$ we will use the negativity $\mathcal{N}$ defined in [7] as

$$\mathcal{N}(\rho_{ABA'B'}) = \tfrac{1}{2}\left(\left\|\rho^{T_{AB}}_{ABA'B'}\right\|_1 - 1\right). \tag{7}$$

Here $\|.\|_1$ denotes the trace norm, $\rho^{T_{AB}}_{ABA'B'}$ is the partial transpose [20] of the state $\rho_{ABA'B'}$ with respect to subsystem AB, and a strictly positive value of negativity implies that the state $\rho_{ABA'B'}$ is (distillable) entangled with respect to the $AB|A'B'$ splitting. The specific feature of the present activation protocol is that the output state $\rho_{ABA'B'}$ is a so-called maximally correlated state and therefore, following results in [3,8], the output negativity can be expressed as

$$\mathcal{N}(\rho_{ABA'B'}) = \frac{1}{2}\sum_{\mathbf{m}\neq\mathbf{n}=\mathbf{0}}^{\infty}|\tilde{\rho}_{\mathbf{m},\mathbf{n}}| = \frac{1}{2}\left(\sum_{\mathbf{m},\mathbf{n}=\mathbf{0}}^{\infty}|\tilde{\rho}_{\mathbf{m},\mathbf{n}}| - 1\right), \tag{8}$$

where $\mathbf{m} = (m_1,m_2)$, $\mathbf{n} = (n_1,n_2)$, and $\tilde{\rho}_{\mathbf{m},\mathbf{n}} = {}_{AB}\langle m_1m_2|\tilde{\rho}|n_1n_2\rangle_{AB}$ are elements of the density matrix $\tilde{\rho}$, Eq. (6), in the Fock basis.

Since our input state ρ_{AB} is nonclassical, the state $\tilde{\rho}$ is also nonclassical for any choice of unitaries U_A and U_B. Thus, there must be at least one nonzero off-diagonal element $\tilde{\rho}_{\mathbf{m},\mathbf{n}}$ for every choice of U_A and U_B. Hence, Eq. (8) implies $\mathcal{N}(\rho_{ABA'B'}) > 0$ and the output state $\rho_{ABA'B'}$ is entangled for any nonclassical input state. This completes the proof of our non-Gaussian activation protocol.

IV. EXAMPLES

The optimization in Eq. (3) is generally carried out over all local unitary operations U_A and U_B, including non-Gaussian ones, which is not a tractable task. Here we consider input Gaussian states with CM in standard form [21] and consider the nonoptimized output entanglement $E_{AB|A'B'}(\rho_{ABA'B'})$ obtained when the local unitaries $U_{A,B}$ are selected to be identity matrices. Therefore, the state (6) remains a Gaussian state in standard form with the following CM:

$$\tilde{\gamma} = \begin{pmatrix}A & C\\ C^T & B\end{pmatrix}, \tag{9}$$

where $A = \text{diag}(a,a)$, $B = \text{diag}(b,b)$, and $C = \text{diag}(c_1,c_2)$ are diagonal matrices. In what follows we determine the nonoptimized quantity for several classes of two-mode Gaussian states by considering the negativity (7) as an entanglement measure E and using Eq. (8). The corresponding measure

of nonclassical correlations $Q_{\mathcal{N}}(\rho_{AB})$ is called the negativity of quantumness [3] accordingly. Although our choice of local unitaries $U_{A,B}$ gives in general an upper bound on $Q_{\mathcal{N}}$, we find that it coincides with the true measure on pure states, leading us to conjecture that our choice is optimal for calculating the negativity of quantumness of all two-mode Gaussian states in standard form. Verifying this conjecture numerically is beyond the scope of this work.

A. Pure states

A closed form of the output negativity can be found for pure two-mode Gaussian states. The density matrix $\tilde{\rho}$ amounts to that of the two-mode squeezed vacuum state, with $\tilde{\rho}_{\mathbf{m},\mathbf{n}} = [1-\tanh^2(r)](\tanh r)^{m_1+n_1}\delta_{m_1,m_2}\delta_{n_1,n_2}$, where $r \geqslant 0$ is the squeezing parameter. Hence, by a direct substitution into Eq. (8) we get

$$\mathcal{N}_{\mathrm{p}} = \tfrac{1}{2}(e^{2r}-1). \tag{10}$$

Consequently, as the output negativity $\mathcal{N}(\rho_{ABA'B'})$ is equal to the negativity of the input state ρ_{AB}, it coincides with the true optimized negativity of quantumness $Q_{\mathcal{N}}(\rho_{AB})$ [5], and our choice of local unitaries is thus optimal for pure states. The negativity (10) is depicted by a solid red line in Fig. 2.

B. Unbiased mixtures of coherent states

These Gaussian states are of the form

$$\rho_{AB} = \int_{\mathbb{C}} P(\alpha)|\alpha\rangle_A\langle\alpha| \otimes |\alpha\rangle_B\langle\alpha| d^2\alpha \tag{11}$$

and can be prepared by splitting a thermal state with a mean number of thermal photons $2\sigma^2$ on a balanced beam splitter. Here $\alpha \in \mathbb{C}$, $P(\alpha) = \exp(-|\alpha|^2/\sigma^2)/(\pi\sigma^2)$ and $d^2\alpha = d(\mathrm{Re}\alpha)d(\mathrm{Im}\alpha)$. The states are already in standard form with a CM (9) specified by $a = b = \sigma^2 + 1/2$ and $c_1 = c_2 = \sigma^2$.

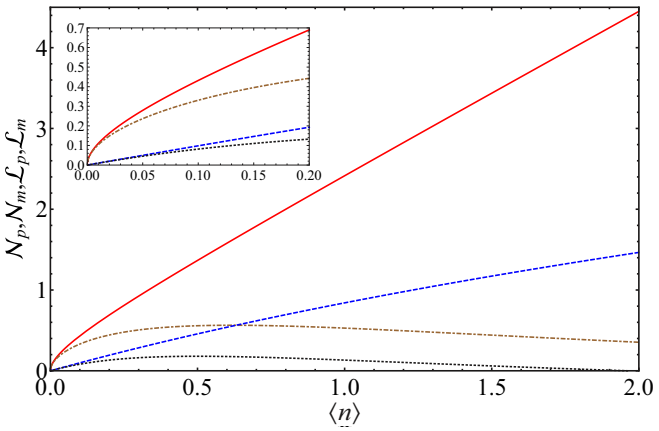

FIG. 2. (Color online) Negativity of quantumness $\mathcal{N}_{\mathrm{p}}$ [Eq. (10)] (solid red line) and its lower bound $\mathcal{L}_{\mathrm{p}}$ [Eq. (28)] (dash-dotted brown line) for pure squeezed vacuum states, plotted as a function of the local mean number of thermal photons $\langle n\rangle = \sinh^2(r)$. The upper bound on the negativity of quantumness $\mathcal{N}_{\mathrm{m}}$ [Eq. (13)] (dashed blue line) and its lower bound $\mathcal{L}_{\mathrm{m}}$ [Eq. (29)] (dotted black line) for separable mixed states which are obtained as unbiased mixtures of coherent states, plotted as a function of the local mean number of thermal photons $\langle n\rangle = \sigma^2$. Inset: Close-up for $\langle n\rangle \ll 1$, where the lower bounds become tight.

Making use of the components of a coherent state in the Fock basis $\langle m|\alpha\rangle = \exp(-|\alpha|^2/2)\alpha^m/\sqrt{m!}$ we get the following matrix elements of the state (11):

$$\tilde{\rho}_{\mathbf{m},\mathbf{n}} = \frac{(m_1+m_2)!}{\sqrt{m_1!m_2!n_1!n_2!}}\frac{\delta_{m_1+m_2,n_1+n_2}}{s(m_1+m_2)}, \tag{12}$$

where $s(j) = \sigma^2(1/\sigma^2+2)^{j+1}$. By substitution of the latter expression into Eq. (8) we get after some algebra

$$\mathcal{N}_{\mathrm{m}} = \frac{1}{2}\left\{\sum_{M=0}^{\infty}\frac{1}{s(M)}\left[\sum_{J=0}^{M}\sqrt{\binom{M}{J}}\right]^2 - 1\right\}. \tag{13}$$

The negativity (13) is depicted by a dashed blue line in Fig. 2 and is generally smaller than the one of pure states calculated in Eq. (10). Both classes of Gaussian states have a nonzero negativity of quantumness which increases with $\langle n\rangle > 0$; this is in agreement with earlier studies of nonclassical correlations based on entropic measures of quantum discord [11,12].

C. Standard-form two-mode Gaussian states

In general we need the Fock basis elements $\tilde{\rho}_{\mathbf{m},\mathbf{n}}$ for an arbitrary two-mode Gaussian state with zero first moments. Combining the results of [22–24] we can express them as

$$\tilde{\rho}_{\mathbf{m},\mathbf{n}} = \frac{H^{(R)}_{m_1,m_2,n_1,n_2}(0)}{\sqrt{\det\left(\tilde{\gamma}+\frac{1}{2}\mathbb{1}\right)}\sqrt{m_1!m_2!n_1!n_2!}}, \tag{14}$$

where $\tilde{\gamma}$ is the CM of the state, $\mathbb{1}$ is the 4×4 identity matrix, and $H^{(R)}_{m_1,m_2,n_1,n_2}(0)$ is the four-dimensional Hermite polynomial [25] at the origin; see Appendix B for a complete derivation of Eq. (14). Here

$$R = WO\left[\left(\tilde{\gamma}+\tfrac{1}{2}\mathbb{1}\right)^{-1} - \mathbb{1}\right]O^{\dagger}V \tag{15}$$

is the symmetric matrix defining the polynomial, where

$$W = \begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}, \quad V = \begin{pmatrix}0&0&1&0\\1&0&0&0\\0&0&0&1\\0&1&0&0\end{pmatrix}, \tag{16}$$

and

$$O = \bigoplus_{j=1}^{2}\frac{1}{\sqrt{2}}\begin{pmatrix}1&i\\1&-i\end{pmatrix}. \tag{17}$$

For the standard-form CM $\tilde{\gamma}$, Eq. (9), we get in particular

$$R = \begin{pmatrix}R_1 - R_2 & R_1 + R_2 - \mathbb{1}_2\\ R_1 + R_2 - \mathbb{1}_2 & R_1 - R_2\end{pmatrix}, \tag{18}$$

with $\mathbb{1}_2$ being the 2×2 identity matrix,

$$R_j = \frac{1}{2d_j}\begin{pmatrix}b+\frac{1}{2} & -c_j\\ -c_j & a+\frac{1}{2}\end{pmatrix}, \tag{19}$$

and $d_j = (a+1/2)(b+1/2) - c_j^2$ $(j = 1,2)$. One can then evaluate the negativity (8) by performing a numerical summation of the absolute values of the elements in Eq. (14). The higher-order Hermite polynomials can be calculated from the lower-order ones by using, e.g., the recurrence formula derived in Appendix B.

We remark that the compact expression in Eq. (14) is of independent interest and can be useful for the characterization of hybrid information processing involving conversion between continuous and discrete variable entanglement [26], or particularly for studies of Bell nonlocality of arbitrary two-mode Gaussian states by means of dichotomic pseudospin measurements [27], whose expectation value can be conveniently evaluated at the Fock-space level.

In the context of the present paper, apart from the utility for numerical evaluation of the output negativity (8), Eq. (14) also enables us to derive a simple analytical lower bound on the output negativity. The bound results from the following chain of inequalities:

$$\begin{aligned}\sum_{\mathbf{m},\mathbf{n}=\mathbf{0}}^{\infty}|\tilde{\rho}_{\mathbf{m},\mathbf{n}}| &= \sum_{\mathbf{m},\mathbf{n}=\mathbf{0}}^{\infty}\frac{\left|H^{(R)}_{m_1,m_2,n_1,n_2}(0)\right|}{\sqrt{\det\left(\tilde{\gamma}+\frac{1}{2}\mathbb{1}\right)}\sqrt{m_1!m_2!n_1!n_2!}}\\ &\geqslant \sum_{\mathbf{m},\mathbf{n}=\mathbf{0}}^{\infty}\frac{\left|H^{(R)}_{m_1,m_2,n_1,n_2}(0)\right|}{\sqrt{\det\left(\tilde{\gamma}+\frac{1}{2}\mathbb{1}\right)m_1!m_2!n_1!n_2!}}\\ &\geqslant \left|\sum_{\mathbf{m},\mathbf{n}=\mathbf{0}}^{\infty}\frac{H^{(R)}_{m_1,m_2,n_1,n_2}(0)}{\sqrt{\det\left(\tilde{\gamma}+\frac{1}{2}\mathbb{1}\right)m_1!m_2!n_1!n_2!}}\right|\\ &= \frac{e^{-\frac{1}{2}\sum_{i,j=1}^{4}R_{ij}}}{\sqrt{\det\left(\tilde{\gamma}+\frac{1}{2}\mathbb{1}\right)}},\end{aligned} \tag{20}$$

where the first inequality follows from the inequality $1/\sqrt{n!}\geqslant 1/n!$ which holds for any $n\geqslant 0$, the second inequality is a consequence of the triangular inequality for absolute values, and the last equation follows from the expression for the generating function of the four-dimensional Hermite polynomials at the origin [25]:

$$e^{-\frac{1}{2}h^{T}Rh}=\sum_{\mathbf{m},\mathbf{n}=\mathbf{0}}^{\infty}\frac{\alpha_1^{*m_1}\alpha_2^{*m_2}\alpha_1^{n_1}\alpha_2^{n_2}}{m_1!m_2!n_1!n_2!}H^{(R)}_{m_1,m_2,n_1,n_2}(0), \tag{21}$$

where $h=(\alpha_1^{*},\alpha_2^{*},\alpha_1,\alpha_2)^{T}$ and R is the matrix (15). A comparison between the right-hand side (RHS) of the previous equation and the expression of the Husimi Q-quasiprobability distribution $\Phi_{\mathcal{A}}(\alpha_1,\alpha_2)=\langle\alpha_1\alpha_2|\tilde{\rho}|\alpha_1\alpha_2\rangle/\pi^2$ in the Fock basis further yields

$$\frac{e^{-\frac{1}{2}h^{T}Rh}}{\sqrt{\det\left(\tilde{\gamma}+\frac{1}{2}\mathbb{1}\right)}}=\pi^2 e^{|\alpha_1|^2+|\alpha_2|^2}\Phi_{\mathcal{A}}(\alpha_1,\alpha_2), \tag{22}$$

as can be easily seen from the results of Appendix B. Therefore, the last expression in the chain of inequalities (20) can be written in the following compact form:

$$\frac{e^{-\frac{1}{2}\sum_{i,j=1}^{4}R_{ij}}}{\sqrt{\det\left(\tilde{\gamma}+\frac{1}{2}\mathbb{1}\right)}}=(\pi e)^2\,\Phi_{\mathcal{A}}(1,1). \tag{23}$$

Now, making use of the inequalities (20) and equality (23) one finds that the sum in Eq. (8) has a lower bound

$$\sum_{\mathbf{m},\mathbf{n}=\mathbf{0}}^{\infty}|\tilde{\rho}_{\mathbf{m},\mathbf{n}}|\geqslant(\pi e)^2\,\Phi_{\mathcal{A}}(1,1), \tag{24}$$

which finally gives the following bound on the output negativity (8):

$$\mathcal{N}(\rho_{ABA'B'})\geqslant\tfrac{1}{2}[(\pi e)^2\Phi_{\mathcal{A}}(1,1)-1]. \tag{25}$$

The bound (25) can be evaluated for any zero-mean two-mode Gaussian state with CM $\tilde{\gamma}$ by calculating the matrix (15) and substituting it into the formula (23). To test the tightness of this bound we evaluate it for the previous examples of pure states and mixtures of coherent states and compare the obtained lower bounds with the exact values of the negativities (10) and (13), respectively. The CM $\tilde{\gamma}$ is in the standard form of Eq. (9) in both cases and therefore one can evaluate easily the matrix (15) using Eqs. (18) and (19), which gives, after substitution into Eq. (23),

$$(\pi e)^2\,\Phi^{\mathrm{p}}_{\mathcal{A}}(1,1)=\frac{e^{2\tanh r}}{\cosh^2(r)} \tag{26}$$

for pure states and

$$(\pi e)^2\,\Phi^{\mathrm{m}}_{\mathcal{A}}(1,1)=\frac{e^{\frac{4\sigma^2}{2\sigma^2+1}}}{2\sigma^2+1} \tag{27}$$

for unbiased mixtures of coherent states. The corresponding negativities then satisfy

$$\mathcal{N}_{\mathrm{p}}\geqslant\frac{1}{2}\left[\frac{e^{2\tanh r}}{\cosh^2(r)}-1\right]\equiv\mathcal{L}_{\mathrm{p}} \tag{28}$$

and

$$\mathcal{N}_{\mathrm{m}}\geqslant\frac{1}{2}\left(\frac{e^{\frac{4\sigma^2}{2\sigma^2+1}}}{2\sigma^2+1}-1\right)\equiv\mathcal{L}_{\mathrm{m}}. \tag{29}$$

The bounds $\mathcal{L}_{\mathrm{p}}$ and $\mathcal{L}_{\mathrm{m}}$ as well as the negativities $\mathcal{N}_{\mathrm{p}}$, Eq. (10), and $\mathcal{N}_{\mathrm{m}}$, Eq. (13), are depicted in Fig. 2. The figure shows that both bounds are tight in the region of small $\langle n\rangle$ (see the inset), which also proves that Eq. (13) gives to the exact value of the negativity of quantumness for mixtures of coherent states with a small mean number of thermal photons in each mode. Both lower bounds are then shown to increase with increasing $\langle n\rangle$ and the gap between the bounds $\mathcal{L}_{\mathrm{p,m}}$ and the numerically evaluated values of the output negativities $\mathcal{N}_{\mathrm{p,m}}$ becomes larger. Further analysis reveals however that the lower bounds $\mathcal{L}_{\mathrm{p}}$ and $\mathcal{L}_{\mathrm{m}}$ are nonmonotonic for larger $\langle n\rangle$; they both attain a maximum at $\langle n\rangle\approx 0.62$ and 0.52, respectively, and then both monotonically decrease for larger values of $\langle n\rangle$. Eventually, both lower bounds become trivial as they enter the region of negative values, namely, $\mathcal{L}_{\mathrm{p}}<0$ for $\langle n\rangle\gtrsim 5.26$ and $\mathcal{L}_{\mathrm{m}}<0$ for $\langle n\rangle\gtrsim 1.97$.

As a final remark, note that the sum in the negativity (8) just amounts to the so-called ℓ_1 norm of the density matrix $\tilde{\rho}$ [8], i.e., $\sum_{\mathbf{m},\mathbf{n}=\mathbf{0}}^{\infty}|\tilde{\rho}_{\mathbf{m},\mathbf{n}}|=\|\tilde{\rho}\|_{\ell_1}$. The results of the present section therefore describe how to calculate numerically the ℓ_1 norm for an arbitrary two-mode Gaussian state with zero means, and the inequality (24) gives a simple analytical lower bound $\|\tilde{\rho}\|_{\ell_1}\geqslant(\pi e)^2\,\Phi_{\mathcal{A}}(1,1)$ on such a norm.

V. CONCLUSIONS

We have shown that a protocol capable of activating nonclassical correlations in bipartite Gaussian states based

solely on Gaussian operations cannot exist. We have also constructed a non-Gaussian activation protocol and subsequently investigated quantitatively its performance using the negativity of quantumness as a figure of merit. Our analysis suggests that optimal performance of the protocol is achieved if the input Gaussian state is in the standard form. By restricting to local Gaussian unitaries, the conjecture can be proved or disproved with help from Eq. (14), using numerical minimization of the negativity (8) with respect to the unitaries, which is left for further research.

We hope that our results will stimulate further exploration of the negativity of quantumness and its interplay with other nonclassicality indicators [11,13] in the context of Gaussian states.

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APPENDIX A: CLASSICALLY CORRELATED BIPARTITE GAUSSIAN STATES ARE PRODUCT STATES

This section is dedicated to the proof that a bipartite Gaussian state ρ_{AB} of an N-mode subsystem A and an M-mode subsystem B is classically correlated across the $A|B$ splitting if and only if it is a product state $\rho_A \otimes \rho_B$.

The proof of the "only if" part is trivial because any product state is diagonal in the product of eigenbases of local states.

The "if" part can be proved using the necessary and sufficient condition for zero quantum discord [18]. Quantum discord $D_B(\rho_{AB})$ of a quantum state ρ_{AB} with a measurement on subsystem B is zero if and only if the state can be expressed as [28]

$$\rho_{AB} = \sum_i p_i \rho_A^{(i)} \otimes |i\rangle_B\langle i|, \quad 0 \leqslant p_i \leqslant 1, \tag{A1}$$

where $\{|i\rangle_B\}$ is an orthonormal basis of subsystem B. The zero-discord criterion [18] then says that a quantum state ρ_{AB} can be expressed in the form of Eq. (A1) if and only if for an informationally complete positive operator valued measurement (ICPOVM) on subsystem A, the conditional states $\rho_{B|k}$ of subsystem B corresponding to the measurement outcomes k mutually commute, i.e.,

$$[\rho_{B|k}, \rho_{B|k'}] = 0, \quad \text{for all } k \text{ and } k'. \tag{A2}$$

We consider a Gaussian state ρ_{AB} with zero means and covariance matrix (CM) γ_{AB}. Modes $A_1, A_2, \ldots, A_N$ comprising subsystem A are subject to a Gaussian measurement characterized by a CM γ_{m} and a vector of measurement outcomes $k = (x_{A_1}, p_{A_1}, \ldots, x_{A_N}, p_{A_N})^T \in \mathbb{R}_{2N}$. If a measurement outcome k occurs then the state ρ_{AB} collapses into the M-mode state $\rho_{B|k}$ of subsystem B with CM σ and vector of first moments d_k of the form [17]

$$\sigma = B - C^T \frac{1}{A + \gamma_{\mathrm{m}}} C, \tag{A3}$$

$$d_k = C^T \frac{1}{A + \gamma_{\mathrm{m}}} k, \tag{A4}$$

where A, B, and C are blocks of the CM γ_{AB} expressed with respect to the $A|B$ splitting:

$$\gamma_{AB} = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}. \tag{A5}$$

As in [18] we will now express criterion (A2) in terms of the characteristic function. For this purpose we will first use the fact that an M-mode quantum state ρ_j can be expressed as [17]

$$\rho_j = \frac{1}{(2\pi)^M} \int_{\mathbb{R}_{2M}} C_j(\xi) W^\dagger(\xi) d\xi, \tag{A6}$$

where $C_j(\xi)$ is the characteristic function of the state ρ_j and $W(\xi) = \exp(-i\xi^T \mathbf{r})$ is the displacement operator with $\xi = (\xi_{x_1}, \xi_{p_1}, \ldots, \xi_{x_M}, \xi_{p_M})^T \in \mathbb{R}_{2M}$ and $\mathbf{r} = (x_1, p_1, \ldots, x_M, p_M)^T$ is the vector of quadratures. Due to the validity of the relation $\mathrm{Tr}[W^\dagger(\xi')W(\xi)] = (2\pi)^M \delta(\xi - \xi')$ we get from Eq. (A6) immediately the following expression for the characteristic function of the state ρ_j:

$$C_j(\xi) = \mathrm{Tr}[\rho_j W(\xi)]. \tag{A7}$$

Making use of Eq. (A6) we can express the commutator on the left-hand side (LHS) of Eq. (A2) as

$$[\rho_{B|k}, \rho_{B|k'}] = \frac{1}{(2\pi)^{2M}} \iint_{\mathbb{R}_{2M}} C_k(\xi) C_{k'}(\xi') \times \left(e^{-\frac{i}{2}\xi^T \Omega \xi'} - e^{\frac{i}{2}\xi^T \Omega \xi'}\right) W^\dagger(\xi + \xi') d\xi d\xi', \tag{A8}$$

where $C_k(\xi)$ and $C_{k'}(\xi')$ are the characteristic functions of the states $\rho_{B|k}$ and $\rho_{B|k'}$, respectively, and where we have used the relation

$$W^\dagger(\xi') W(\xi) = e^{\frac{i}{2}\xi'^T \Omega \xi} W(\xi - \xi'), \tag{A9}$$

with

$$\Omega = \bigoplus_{i=1}^{M} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{A10}$$

From Eqs. (A3) and (A4) it follows that the Gaussian states $\rho_{B|k}$ and $\rho_{B|k'}$ possess the same CM σ and first moments d_k and $d_{k'}$, respectively, and therefore their characteristic functions read

$$C_k(\xi) = e^{-\frac{1}{2}\xi^T \sigma \xi - i\xi^T d_k}, \quad C_{k'}(\xi') = e^{-\frac{1}{2}\xi'^T \sigma \xi' - i\xi'^T d_{k'}}. \tag{A11}$$

Equation (A8) allows us to calculate the characteristic function of the commutator $[\rho_{B|k}, \rho_{B|k'}]$ given by

$$C_{kk'}(\xi) = \mathrm{Tr}\{[\rho_{B|k}, \rho_{B|k'}] W(\xi)\}. \tag{A12}$$

By inserting the RHS of the commutator from Eq. (A8) into Eq. (A12), using Eq. (A9) and carrying out the integration, we

arrive at the characteristic function (A12) in the form

$$C_{kk'}(\xi) = 2\frac{e^{-\frac{1}{4}\xi^T(\sigma+\frac{1}{4}\Omega^T\sigma^{-1}\Omega)\xi-\frac{1}{4}(d_k-d_{k'})^T\sigma^{-1}(d_k-d_{k'})-\frac{i}{2}\xi^T(d_k+d_{k'})}}{2^M\sqrt{\det\sigma}} \times \sinh\left[\frac{1}{4}(d_{k'}-d_k)^T\sigma^{-1}\Omega\xi\right]. \quad \text{(A13)}$$

From Eq. (A12) and the formula

$$[\rho_{B|k},\rho_{B|k'}] = \frac{1}{(2\pi)^M}\int_{\mathbb{R}_{2M}} C_{kk'}(\xi)W^\dagger(\xi)d\xi \quad \text{(A14)}$$

it follows that $[\rho_{B|k},\rho_{B|k'}] = 0$ if and only if $C_{kk'}(\xi) = 0$ for all ξ. Assuming that the CM σ in Eq. (A3) has finite second moments and that the measurement outcomes k and k' and hence also the displacements d_k and $d_{k'}$ defined by Eq. (A4) are finite, the condition $C_{kk'}(\xi) = 0$ for all ξ is equivalent to the condition

$$(d_{k'}-d_k)^T\sigma^{-1}\Omega\xi = 0, \quad \text{for all } \xi, \quad \text{(A15)}$$

which can be rewritten using Eq. (A4) as

$$(k'-k)^T\frac{1}{A+\gamma_{\rm m}}C\sigma^{-1}\Omega\xi = 0. \quad \text{(A16)}$$

Previous results allow us to rephrase the zero-discord criterion of [18] for bipartite Gaussian states and Gaussian ICPOVMs as follows. An $N+M$-mode Gaussian state ρ_{AB} can be expressed in the form of Eq. (A1) if and only if the condition (A16) is satisfied for all k and k', where k and k' are measurement outcomes of a Gaussian ICPOVM on subsystem A characterized by the CM $\gamma_{\rm m}$. Condition (A16) is satisfied for all k and k' ($k \neq k'$) if and only if the matrix

$$\frac{1}{A+\gamma_{\rm m}}C\sigma^{-1}\Omega = 0. \quad \text{(A17)}$$

Consider now the heterodyne measurement which is an example of a Gaussian ICPOVM [29]. Then $\gamma_{\rm m} = (1/2)\mathbb{1}$, the matrix $\frac{1}{A+\gamma_{\rm m}}$ is invertible, and therefore condition (A17) is equivalent with the equation $C\sigma^{-1}\Omega = 0$. As both the matrices Ω and σ^{-1} are also invertible the latter condition is equivalent with the condition $C = 0$. For the heterodyne detection the condition (A17) is thus equivalent with the vanishing of the off-diagonal block C given in Eq. (A5), which carries intermodal correlations. This means in other words that a bipartite $(N+M)$-mode Gaussian state can be expressed in the form of Eq. (A1) if and only if it is a product state, i.e., $\rho_{AB} = \rho_A \otimes \rho_B$.

Let us now move to the necessary and sufficient condition for a bipartite $(N+M)$-mode Gaussian state to be a classically correlated state. A quantum state ρ_{AB} is classically correlated if and only if $D_A(\rho_{AB}) = D_B(\rho_{AB}) = 0$ [30], where $D_A(\rho_{AB})$ is the discord of ρ_{AB} for measurement on subsystem A. A quantum state ρ_{AB} is therefore classically correlated if and only if it can be expressed simultaneously in the form of Eq. (A1) and in the form

$$\rho_{AB} = \sum_i p_i|i\rangle_A\langle i| \otimes \rho_B^{(i)}. \quad \text{(A18)}$$

According to the criterion given in [18] a quantum state ρ_{AB} can be expressed in the form of Eq. (A18) if and only if for an ICPOVM on subsystem B the conditional states $\rho_{A|k}$ of subsystem A corresponding to the measurement outcomes k mutually commute, i.e.,

$$[\rho_{A|k},\rho_{A|k'}] = 0, \quad \text{for all } k \text{ and } k'. \quad \text{(A19)}$$

Like in the previous case we can express the latter condition in terms of a characteristic function. We can proceed exactly along the same lines as in the case of the commutator (A8) with the only difference being that now we consider a measurement on the M-mode subsystem B. Consequently, the formulas which we get for the present case of the commutator (A19) are obtained from the formulas derived in the context of commutator (A8) by the replacements $A \leftrightarrow B$ and $C \leftrightarrow C^T$ of the blocks of the matrix γ_{AB} and by the replacement $M \to N$. Thus we find that the commutator (A19) vanishes if and only if $C^T = 0$. Therefore, the condition $C = 0$ is necessary and sufficient for an $(N+M)$-mode Gaussian state to be classical, which concludes our proof.

APPENDIX B: MATRIX ELEMENTS OF A GAUSSIAN STATE IN THE FOCK BASIS IN TERMS OF HERMITE POLYNOMIALS

Our aim is to express the elements of a density matrix of a Gaussian state ρ of two modes A and B in the Fock basis. Here and in what follows we assume that the state has all first moments equal to zero. The present derivation combines the results obtained in [22–24]. First we express the elements of the density matrix in the basis of coherent states as

$$e^{|\alpha_1|^2+|\alpha_2|^2}\langle\alpha_1\alpha_2|\rho|\alpha_1\alpha_2\rangle = \sum_{m_1,m_2,n_1,n_2=0}^{\infty}\frac{\alpha_1^{*m_1}\alpha_2^{*m_2}\alpha_1^{n_1}\alpha_2^{n_2}}{\sqrt{m_1!m_2!n_1!n_2!}}\langle m_1m_2|\rho|n_1n_2\rangle, \quad \text{(B1)}$$

where we have used the expression of the components of a coherent state $|\alpha\rangle$ in the Fock basis:

$$\langle m|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}}\frac{\alpha^m}{\sqrt{m!}}. \quad \text{(B2)}$$

The matrix element on the LHS of Eq. (B1) can be further expressed as

$$\langle\alpha_1\alpha_2|\rho|\alpha_1\alpha_2\rangle = \pi^2\Phi_{\mathcal{A}}(\alpha_1,\alpha_2), \quad \text{(B3)}$$

where

$$\Phi_{\mathcal{A}}(\alpha_1,\alpha_2) = \frac{1}{\pi^2\sqrt{\det\gamma_{\mathcal{A}}^{(c)}}}e^{-\frac{1}{2}\alpha^\dagger[\gamma_{\mathcal{A}}^{(c)}]^{-1}\alpha} \quad \text{(B4)}$$

is the Husimi Q-quasiprobability distribution of the Gaussian state ρ [31]. Here, $\alpha = (\alpha_1,\alpha_1^*,\alpha_2,\alpha_2^*)^T$ and $\gamma_{\mathcal{A}}^{(c)}$ is the complex CM corresponding to antinormal ordering of the canonical operators. Substituting now from Eq. (B3) into the LHS of Eq. (B1) and making use of Eq. (B4) we arrive at the following equality:

$$\frac{1}{\sqrt{\det\gamma_{\mathcal{A}}^{(c)}}}e^{-\frac{1}{2}\alpha^\dagger\{[\gamma_{\mathcal{A}}^{(c)}]^{-1}-\mathbb{1}\}\alpha} = \sum_{m_1,m_2,n_1,n_2=0}^{\infty}\frac{\alpha_1^{*m_1}\alpha_2^{*m_2}\alpha_1^{n_1}\alpha_2^{n_2}}{\sqrt{m_1!m_2!n_1!n_2!}}\langle m_1m_2|\rho|n_1n_2\rangle. \quad \text{(B5)}$$

The LHS of the latter equation can be expressed in terms of the multidimensional Hermite polynomials [25]. Specifically, the generating function of the four-dimensional Hermite polynomials is

$$e^{-\frac{1}{2}h^T Rh+h^T Rx} = \sum_{m_1,m_2,n_1,n_2=0}^{\infty} \frac{\alpha_1^{*m_1}\alpha_2^{*m_2}\alpha_1^{n_1}\alpha_2^{n_2}}{m_1!m_2!n_1!n_2!} H^{(R)}_{m_1,m_2,n_1,n_2}(x), \tag{B6}$$

where $h=(\alpha_1^*,\alpha_2^*,\alpha_1,\alpha_2)^T$, $x=(x_1,x_2,x_3,x_4)^T$, and R is a symmetric matrix of order 4. The LHS of Eq. (B5) then can be rewritten in terms of the LHS of Eq. (B6) as follows. The complex CM $\gamma^{(c)}_{\mathcal{A}}$ can be expressed as

$$\gamma^{(c)}_{\mathcal{A}} = O\left(\gamma+\tfrac{1}{2}\mathbb{1}\right)O^\dagger, \tag{B7}$$

where $\mathbb{1}$ is the 4×4 identity matrix,

$$O = \bigoplus_{j=1}^{2} \frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ 1 & -i\end{pmatrix} \tag{B8}$$

is a 4×4 unitary matrix, and γ is the standard real symmetrically ordered CM of the state ρ, with elements $\gamma_{ij}=\langle r_i r_j + r_j r_i\rangle/2$ and $i,j=1,\ldots,4$, where r_i is the ith component of the vector of quadratures $\mathbf{r}=(x_A,p_A,x_B,p_B)^T$. Hence we get

$$\left[\gamma^{(c)}_{\mathcal{A}}\right]^{-1} - \mathbb{1} = O\left[\left(\gamma+\tfrac{1}{2}\mathbb{1}\right)^{-1}-\mathbb{1}\right]O^\dagger. \tag{B9}$$

Furthermore, we can write

$$\alpha = Vh, \quad \alpha^\dagger = h^T W, \tag{B10}$$

where

$$V=\begin{pmatrix}0&0&1&0\\1&0&0&0\\0&0&0&1\\0&1&0&0\end{pmatrix}, \quad W=\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}.$$

Consequently,

$$\alpha^\dagger\left\{\left[\gamma^{(c)}_{\mathcal{A}}\right]^{-1}-\mathbb{1}\right\}\alpha = h^T Rh,$$

where

$$R = WO\left[\left(\gamma+\tfrac{1}{2}\mathbb{1}\right)^{-1}-\mathbb{1}\right]O^\dagger V. \tag{B11}$$

As $(WO)^T = O^\dagger V$ and the CM γ is symmetric, one finds immediately that $R^T=R$ and therefore R is symmetric as required. Making use of Eqs. (B5) and (B6) we get

$$\frac{1}{\sqrt{\det\gamma^{(c)}_{\mathcal{A}}}} \sum_{m_1,m_2,n_1,n_2=0}^{\infty} \frac{\alpha_1^{*m_1}\alpha_2^{*m_2}\alpha_1^{n_1}\alpha_2^{n_2}}{m_1!m_2!n_1!n_2!} H^{(R)}_{m_1,m_2,n_1,n_2}(0)$$
$$= \sum_{m_1,m_2,n_1,n_2=0}^{\infty} \frac{\alpha_1^{*m_1}\alpha_2^{*m_2}\alpha_1^{n_1}\alpha_2^{n_2}}{\sqrt{m_1!m_2!n_1!n_2!}} \langle m_1m_2|\rho|n_1n_2\rangle,$$

where the matrix R defining the Hermite polynomial $H^{(R)}_{m_1,m_2,n_1,n_2}$ is given in Eq. (B11). By equating each term in the summation we are left with the elements of the density matrix ρ in the Fock basis:

$$\langle m_1m_2|\rho|n_1n_2\rangle = \frac{H^{(R)}_{m_1,m_2,n_1,n_2}(0)}{\sqrt{\det\gamma^{(c)}_{\mathcal{A}}}\sqrt{m_1!m_2!n_1!n_2!}}, \tag{B12}$$

where

$$\det\gamma^{(c)}_{\mathcal{A}} = \det\left(\gamma+\tfrac{1}{2}\mathbb{1}\right). \tag{B13}$$

Equation (B12) allows us to calculate any element of a density matrix in the Fock basis for an arbitrary two-mode Gaussian state with zero first moments.

To calculate matrix (B11) it is convenient to express the CM γ in the block form:

$$\gamma = \begin{pmatrix}A & C\\ C^T & B\end{pmatrix}. \tag{B14}$$

This allows us to express the inverse matrix $(\gamma+\mathbb{1}/2)^{-1}$, appearing in Eq. (B11), in block form using the following blockwise inversion formula [32]:

$$\begin{pmatrix}A & C\\ C^T & B\end{pmatrix}^{-1} = \begin{pmatrix}(A-CB^{-1}C^T)^{-1} & A^{-1}C(C^TA^{-1}C-B)^{-1}\\ (C^TA^{-1}C-B)^{-1}C^TA^{-1} & (B-C^TA^{-1}C)^{-1}\end{pmatrix}. \tag{B15}$$

1. Recurrence relations

Higher-order Hermite polynomials can be calculated from lower-order polynomials using a recurrence relation. It is derived from the generating function (B6), where we set $x=0$. By deriving both sides of Eq. (B6) with respect to the ith element of the vector $h=(\alpha_1^*,\alpha_2^*,\alpha_1,\alpha_2)^T$, substituting the RHS of Eq. (B6) for the exponential function $\exp(-h^TRh/2)$ appearing on the LHS of the obtained expression, and equating each term in the summation, we arrive at the following recurrence relation:

$$H^{(R)}_{\mu+e_i}(0) = -\sum_{j=1}^{4} R_{ij}\mu_j H^{(R)}_{\mu-e_j}(0), \tag{B16}$$

where $H^{(R)}_{\mu}(0)$ is the four-dimensional Hermite polynomial at the origin with multi-index $\mu=(m_1,m_2,n_1,n_2)$. The coefficients R_{ij} correspond to the (i,j)th element of the matrix R, Eq. (B11), and e_i is the ith canonical basis vector with 1 in the ith component and zeros everywhere else. Here, any Hermite polynomial with a negative index is zero, i.e., $H_\mu(0)=0$ for all μ with $\mu_i<0$ for some i. Every Hermite polynomial at the origin can be found from the latter recurrence formula and by using the first few cases:

$$H^{(R)}_{0,0,0,0}(0) = 1, \tag{B17}$$

$$H^{(R)}_{e_i}(0) = 0, \tag{B18}$$

$$H^{(R)}_{e_i+e_j}(0) = -R_{ij}, \tag{B19}$$

$$H^{(R)}_{e_i+e_j+e_k}(0) = 0, \tag{B20}$$

$$H^{(R)}_{1,1,1,1}(0) = R_{12}R_{34}+R_{23}R_{41}+R_{13}R_{24}, \tag{B21}$$

with $i \neq j \neq k$. These can be derived by a direct calculation from the expression

$$H_{\mu}^{(R)}(x) = (-1)^{\sum_{i=1}^{2} n_i + m_i} \exp\left(\frac{1}{2} x^T R x\right) \frac{\partial^{\sum_{i=1}^{2} n_i + m_i}}{\partial x_1^{m_1} \partial x_2^{m_2} \partial x_3^{n_1} \partial x_4^{n_2}} \exp\left(-\frac{1}{2} x^T R x\right),$$

found in [22]. Note that it is sufficient to calculate only the polynomials where the parity of the multi-index μ is even. When the parity of the multi-index μ is odd, i.e., $P(\mu) = m_1 + m_2 + n_1 + n_2 = 2\ell + 1$, where $\ell \in \mathbb{N}_0$, then $H_{\mu}^{(R)}(0) = 0$.

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