Optimal performance of endoreversible quantum refrigerators

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The derivation of general performance benchmarks is important in the design of highly optimized heat engines and refrigerators. To obtain them, one may model phenomenologically the leading sources of irreversibility ending up with results that are model independent, but limited in scope. Alternatively, one can take a simple physical system realizing a thermodynamic cycle and assess its optimal operation from a complete microscopic description. We follow this approach in order to derive the coefficient of performance at maximum cooling rate for any endoreversible quantum refrigerator. At striking variance with the universality of the optimal efficiency of heat engines, we find that the cooling performance at maximum power is crucially determined by the details of the specific system-bath interaction mechanism. A closed analytical benchmark is found for endoreversible refrigerators weakly coupled to unstructured bosonic heat baths: an ubiquitous case study in quantum thermodynamics.

I. INTRODUCTION

Energy conversion systems, including heat engines and refrigerators, encompass a broad variety of devices that find widespread uses in the domestic, industrial, and academic domains. Design optimization of such systems is crucial for their implementation to be cost efficient and the determination of general performance benchmarks to assess their optimality is a very active research area [1,2]. A familiar example of heat engine is a nuclear power station. The relevant figure of merit to benchmark its optimality is the output power rather than the efficiency of energy conversion [3]: In fact, capital costs are by far the dominant contribution to the price of the kWh, while the nuclear fuel itself is comparatively inexpensive. Hence, ideally, a nuclear energy station will be designed to operate at the maximum power output $\mathscr{P}_*$ corresponding to some heat input $\dot{\mathscr{Q}}_{h,*}$, which defines an optimal efficiency $\eta_* \equiv -\mathscr{P}_*/\dot{\mathscr{Q}}_{h*}$.

As a working assumption, one may treat a nuclear power station as a perfect Carnot engine running between heat reservoirs at temperatures $T_c < T'_h (< T_h)$, where T'_h is the effective temperature of the working fluid at the hot end of the cycle. This amounts to saying that the leading source of irreversibility in atomic power generation is the imperfect thermal contact of the working fluid with the reactor, to the point that internal friction and heat leaks may be completely disregarded. This is known as the endoreversible approximation [2]. If one further assumes a simple Newtonian heat transfer law for the heat current $\dot{\mathscr{Q}}_h = C_h(T_h - T'_h)$, where C_h is a constant, then the effective temperature maximizing the power may be found to be the geometric mean of T_h and T_c. Consequently, the optimal efficiency reads

$$\eta_* = 1 - T_c/T'_{h*} = 1 - \sqrt{T_c/T_h} = 1 - \sqrt{1 - \eta_C}, \quad (1)$$

where $\eta_C = 1 - T_c/T_h$ is the ultimate Carnot efficiency [4]. This formula, introduced by Yvon [5] and Novikov [3] in the mid 1950s in the context of atomic energy generation,

was rederived 20 years later by Curzon and Ahlborn [6] in their 1975 seminal paper.[1] In principle, it should be nothing but a crude approximation to optimality, but it turns out to be in good agreement with the observed efficiency of actual thermal power plants and proves to be remarkably independent of the specific design [5]. Indeed, it agrees with the optimal efficiency of any engine operating close to equilibrium [8,9] and applies quite generally to symmetric low-dissipation engines [10], even if these are realized on a quantum mechanical support [11,12]. Equation (1) is therefore a useful design guideline, as it reliably benchmarks the optimal operation of a large class of heat engines. Besides, it is clearly model independent.

In the past few decades, many attempts have been made to answer the fundamental question of whether a similar model-independent benchmark can be obtained for optimal cooling. That would certainly be very useful in the design optimization of refrigerators, but unfortunately the straightforward endoreversible approach together with the assumption of a linear heat transfer law does not help in this case: The cooling rate $\dot{\mathscr{Q}}_c$, which replaces $\mathscr{P}$ as figure of merit, is maximal only at a vanishing coefficient of performance (COP) $\varepsilon \equiv \dot{\mathscr{Q}}_c/\mathscr{P}$. This problem might be circumvented by resorting to alternative heat transfer laws, though these usually lead to involved (nonuniversal) formulas for the optimal COP, explicitly depending on phenomenological heat conductivities [13].

Benchmarks analogous to Eq. (1) may still be obtained by retaining the simple Newtonian ansatz and changing instead the definition of optimality. Practical considerations may advise, e.g., that one pay the same attention to the COP and the cooling rate so that the meaningful figure of merit would be $\chi \equiv \varepsilon \dot{\mathscr{Q}}_c$ rather than $\dot{\mathscr{Q}}_c$ alone. In this case, one would find an optimal performance of $\varepsilon_* = \sqrt{1 + \varepsilon_C} - 1$ [13], which holds in fact for any symmetric low-dissipation Carnot

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[1]Interestingly, the origins of Eq. (1) can be traced back to a book by Reitlinger, first published in 1929 [7].

refrigerator [14]. Here $\varepsilon_C = T_c/(T_h - T_c)$ stands for the Carnot COP. Other criteria for optimality [15,16] would lead of course to different performance benchmarks.[2]

In this paper we analyze the COP at maximum cooling rate for endoreversible quantum refrigerators, generally modeled as tricycles [19]. We find that the details of the system-bath interaction mechanism place a tight upper bound on the cooling performance, which automatically precludes the derivation of any model-independent benchmarks. We then look into the paradigmatic case of a three-level compression refrigerator [20,21] operating between unstructured bosonic heat baths to obtain a simple closed expression for $\varepsilon_*(\varepsilon_C)$, which is further shown to bound and closely reproduce the optimal COP of *any* multistage endoreversible refrigerator within the same dissipative scheme. Our analysis unveils fundamental differences between heat engines and refrigerators from the point of view of their optimal performance and highlights the key importance of reservoir engineering in the optimization of technologically relevant quantum models.

This paper is structured as follows. The generic template of a quantum tricycle is briefly described Sec. II. Then, our main result, concerning the nonuniversality of the optimal cooling performance, is derived in Sec. III and illustrated with a simple example in Sec. IV. Finally, in Sec. V we summarize and draw our conclusions. For the sake of clarity, the technical details of the derivation of quantum master equations for periodically driven systems are postponed until the Appendix.

II. ENDOREVERSIBLE QUANTUM TRICYCLES

A generic energy conversion device may be thought of as a stationary black box in simultaneous thermal contact with three heat reservoirs at different temperatures $T_w > T_h > T_c$ or, alternatively, with two heat reservoirs $T_h > T_c$ and a work repository ($T_w \to \infty$), which, in principle, accounts for the case of a heat engine or a power-driven refrigerator (we shall elaborate more on this equivalence in an example below). This template, termed tricycle [22], is suitable to describe averaged finite-time cycles or continuous processes and is represented by the triple $\{\dot{\mathcal{Q}}_w, \dot{\mathcal{Q}}_h, \dot{\mathcal{Q}}_c\}$ of steady-state rates of incoming (positive) and outgoing (negative) energy flow in the system through each of the thermal contact ports. In order to comply with the first and second laws of thermodynamics, these must satisfy

$$\dot{\mathcal{Q}}_w + \dot{\mathcal{Q}}_h + \dot{\mathcal{Q}}_c = 0, \tag{2a}$$

$$\frac{\dot{\mathcal{Q}}_w}{T_w} + \frac{\dot{\mathcal{Q}}_h}{T_h} + \frac{\dot{\mathcal{Q}}_c}{T_c} \equiv -\dot{S} \leqslant 0. \tag{2b}$$

If the black box encloses a quantum system, thermal contact with the heat reservoir may be selectively established through filters at frequencies ω_α with $\alpha \in \{w,h,c\}$. This is the distinctive feature of a quantum tricycle [19] (see Fig. 1). In the

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[2]Another option would be to relax the endoreversible approximation, allowing for heat leaks and internal friction, while keeping $\dot{\mathcal{Q}}_c$ as the figure of merit and a simple linear model for the heat currents [17,18]. Generally, this also leads to model-dependent benchmarks.
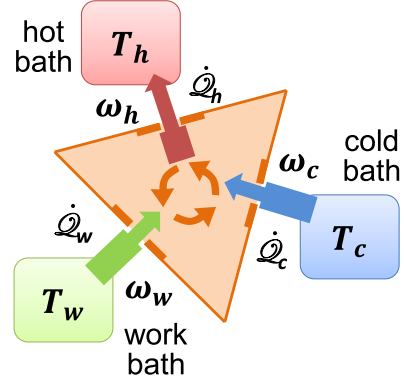

FIG. 1. (Color online) Quantum tricycle. A quantum system selectively coupled through frequency filters to three heat baths (with temperatures $T_w > T_h > T_c$) embodies the prototype of any thermal device. Here the direction of the heat currents (depicted by arrows) correspond to a refrigerator. Reversing them realizes a heat transformer or heat engine.

absence of heat leaks or internal friction, a quantum tricycle exchanges quanta with all three baths at a single stationary rate $\mathscr{I}$, i.e., $\dot{\mathcal{Q}}_\alpha = -\omega_\alpha \mathscr{I}_\alpha$, with $\mathscr{I}_h = -\mathscr{I}_c = -\mathscr{I}_w \equiv \mathscr{I}$ (in what follows $\hbar = k_B = 1$). Thence, the fulfillment of the first law in Eq. (2a) demands that the filters are tuned in resonance so that $\omega_w = \omega_h - \omega_c$.

Such ideal devices have two complementary modes of operation compatible with Eq. (2b): the absorption or compression refrigerator $\{\dot{\mathcal{Q}}_w > 0, \dot{\mathcal{Q}}_h < 0, \dot{\mathcal{Q}}_c > 0\}$ and the heat transformer or heat engine $\{\dot{\mathcal{Q}}_w < 0, \dot{\mathcal{Q}}_h > 0, \dot{\mathcal{Q}}_c < 0\}$ [22,23]. Let us consider, for instance, a compression refrigerator ($T_w \to \infty$) at fixed ω_h, for which the inequality (2b) may be rewritten as $\omega_c \leqslant \omega_{c,\text{rev}} \equiv \omega_h T_c/T_h$. As $\omega_c \to \omega_{c,\text{rev}}$, the contact ports simultaneously reach local thermal equilibrium with their respective heat reservoirs and the COP is maximized ($\varepsilon \to \varepsilon_C$) [24]. In general, however, the effective temperatures T'_α defined from the stationary state of the contacts do not coincide with the corresponding equilibrium values T_α and the COP is strictly smaller than ε_C.

The irreversibility hindering the cooling performance of ideal quantum tricycles might be thus understood as if only arising from imperfect thermal contact with the heat baths. It is in this sense that we refer to them as endoreversible. Alternatively, ideal energy conversion systems may be tagged strongly coupled [9,25], referring to the fact that their energy fluxes remain at all times proportional to each other. This is a necessary prerequisite for any device to achieve maximum efficiency, although at vanishing energy-conversion rates [25].

III. OPTIMAL COP FOR HIGH TEMPERATURES

Next we tune the frequency filters of a generic endoreversible power-driven tricycle in the refrigerator configuration so as to maximize its cooling power in search for the optimal COP. From Eq. (2b) it follows that the entropy production can be written as $\dot{S} = x_h \mathscr{I}_h + x_c \mathscr{I}_c$, where the fluxes are $\mathscr{I}_h = -\mathscr{I}_c \equiv \mathscr{I}$ and their conjugate thermodynamic forces are given by $x_\alpha \equiv \omega_\alpha/T_\alpha$. Note that refrigeration is achieved whenever $x_c < x_h$, according to Eq. (2b). Even though we

shall concentrate on the dependence of the flux on the thermodynamic forces, it will generally be a function of other independent dimensionless combinations of parameters, describing the system-bath interactions and the spectrum of thermal fluctuations of the heat reservoirs.

The cold heat current is written as $|\dot{\mathcal{Q}}_c| = T_c x_c \mathscr{I}(x_h, x_c)$ and its local maximization with respect to x_c at fixed x_h follows from

$$x_{c,*} \frac{\partial \mathscr{I}}{\partial x_c}(x_h, x_{c,*}) + \mathscr{I}(x_h, x_{c,*}) = 0. \tag{3}$$

Little more can be said without disclosing the full Hamiltonian of the tricycle, except if one is restricted to a certain regime of parameters. Here we take the high-temperature limit ($x_\alpha \to 0$) where, e.g., symmetric quantum heat engines are known to operate at the Yvon-Novikov-Curzon-Ahlborn efficiency [26,27] and different models of absorption refrigerators achieve their maximal performance [28,29].

We thus approximate $\mathscr{I}(x_h, x_c)$ around $x_\alpha = 0$, retaining only the first nonzero term in its Taylor expansion

$$\mathscr{I}(x_h, x_c) = \sum_i \left(\frac{\partial \mathscr{I}}{\partial x_i}\right)_{\vec{0}} x_i + \sum_{ij} \left(\frac{\partial^2 \mathscr{I}}{\partial x_i \partial x_j}\right)_{\vec{0}} x_i x_j + \cdots , \tag{4}$$

and express the optimal cold force as $x_{c,*} \simeq C x_h$ to first order in x_h. The coefficient C may be obtained by substituting the approximated current of Eq. (4) into Eq. (3) and will thus depend explicitly on the partial derivatives of the stationary heat current evaluated in $x_\alpha = 0$. Noting that the COP of an endoreversible refrigerator is written as

$$\varepsilon = \frac{\omega_c}{\omega_h - \omega_c} = \left(\frac{\varepsilon_C + 1}{\varepsilon_C}\frac{x_h}{x_c} - 1\right)^{-1}, \tag{5}$$

the optimal performance, normalized by ε_C, is finally

$$\frac{\varepsilon_*}{\varepsilon_C} = \frac{C}{(1 - C)\varepsilon_C + 1}. \tag{6}$$

Here C must be positive and upper bounded by 1, so $0 \leqslant \varepsilon_* \leqslant \varepsilon_C$. In general, it will be a function of parameters such as the dissipation rate γ, Ohmicity s, high-frequency cutoff Ω_c, dimensionality of the baths d, or their equilibrium temperatures (through ε_C). Thus, and unlike Eq. (1), $\varepsilon_*/\varepsilon_C$ converges to $C(\varepsilon_C = 0, \gamma, s, \Omega_c, d, \ldots)$ as $\varepsilon_C \to 0$ rather than to a universal constant value.

The above discussion can be compared to the one in Ref. [9] for a generic heat engine in the linear regime: There the first-order term in the expansion of the optimal force $x_{c,*}$ (in that case, around $x_c - x_h \to 0$) contributed to η_*/η_C with a universal constant value of $1/2$, while the second-order term added a correction, explicitly involving the first- and second-order partial derivatives of the heat current. In contrast, as we have just seen, the optimal cooling performance is already nonuniversal to the lowest order in x_h.

In order to intuitively understand this fundamental difference between engines and refrigerators, we remark that the useful effect in a heat engine is sought at the interface of the working substance with an infinite-temperature heat reservoir, implying that the corresponding contact transitions will be saturated regardless of the details of the system-bath interaction. In contrast, in a refrigerator, the useful effect takes

place in the interface with a bath at some finite temperature. Therefore, it is not so surprising that the spectral properties of the environmental fluctuations play a relevant role in establishing the optimal cooling performance. Indeed, the situation resembles that of the maximization of the cooling power of endoreversible (classical) refrigerators, for which the optimal performance is generally set by the heat conductivities, and depends critically on the specific heat transfer law assumed [13].

Finally, let us comment on the optimal COP in the complementary limit of $\varepsilon_C \to \infty$, that is, in the linear regime. Close to equilibrium, we may assume a linear relation between fluxes $\mathscr{I}_\alpha$ and forces x_α such that $\mathscr{I}_h = L_{11} x_h + L_{12} x_c$ and $\mathscr{I}_c = L_{21} x_h + L_{22} x_c$. The Onsager coefficients L_{ij} satisfy $L_{11} \geqslant 0$, $L_{22} \geqslant 0$, $L_{12} = L_{21}$, and $q^2 \equiv L_{12}^2 / L_{11} L_{22}$. Here the parameter $-1 \leqslant q \leqslant 1$ stands for the tightness of the coupling between input and output fluxes [25], where $q^2 \to 1$ implies endoreversiblity. We can maximize again $|\dot{\mathcal{Q}}_c| = T_c x_c \mathscr{I}_c$ in x_c for fixed x_h, obtaining $x_{c,*} = -L_{21} x_h / 2 L_{22}$. This yields an optimal COP of

$$\varepsilon_* = \frac{q^2 \varepsilon_C}{(4 - 3q^2)\varepsilon_C + (4 - 2q^2)}, \tag{7}$$

which converges to $\varepsilon_* = q^2/(4 - 3q^2)$ as $\varepsilon_C \to \infty$. Hence, the ratio $\varepsilon_*/\varepsilon_C$ simply vanishes close to equilibrium, regardless of the magnitude of x_α and the details system-bath coupling.

IV. EXAMPLE: UNSTRUCTURED BOSONIC BATHS

In order to get a closed expression for $C(\varepsilon_C, \gamma, s, \Omega_c, d, \ldots)$, specific instances have to be considered. Here we focus on a simple and paradigmatic endoreversible device, such as a three-level maser [20] subject to a weak periodic driving, in contact with unstructured bosonic baths (i.e., characterized by a flat spectral density) in d_α dimensions. Its Hamiltonian is written as

$$H = \omega_c |2\rangle\langle 2| + \omega_h |3\rangle\langle 3| + \lambda(e^{i\omega_w t}|2\rangle\langle 3| + e^{-i\omega_w t}|3\rangle\langle 2|), \tag{8}$$

where λ is the intensity of the driving at the power input transition $|2\rangle \leftrightarrow |3\rangle$. The remaining transitions ($|1\rangle \leftrightarrow |3\rangle$ and $|1\rangle \leftrightarrow |2\rangle$) are linearly connected with the hot and cold heat reservoirs, through terms of the form $\sigma_\alpha \otimes \mathscr{B}_\alpha$, where

$$\mathscr{B}_\alpha \equiv \sum_\mu g_{\alpha\mu}(b_{\alpha\mu} + b_{\alpha\mu}^\dagger), \tag{9a}$$

$$\sigma_h \equiv |1\rangle\langle 3| + |3\rangle\langle 1|, \tag{9b}$$

$$\sigma_c \equiv |1\rangle\langle 2| + |2\rangle\langle 1|. \tag{9c}$$

The constants $g_{\alpha\mu} \propto (\gamma_\alpha \omega_\mu)^{1/2}$ indicate the intensity of the coupling between the mode ω_μ of bath α and the corresponding contact transition of the working substance and γ_α stands for the dissipation strength [28].

We shall assume very weak dissipation (i.e., $\gamma_\alpha \ll T_\alpha$) and parameters well into the quantum optical regime so as to consistently derive a quantum master equation such as $\dot{\varrho} = \sum_\alpha \sum_\omega \sum_{q \in \mathbb{Z}} \mathscr{L}_{\omega,q}^\alpha \varrho$, with dissipators $\mathscr{L}_{\omega,q}^\alpha$ of the Lindblad-Gorini-Kossakowski-Sudarshan type [30,31]. Their explicit form is given in the Appendix.

The nonequilibrium limit cycle state ϱ_∞ may be found from $\sum_\alpha \sum_\omega \sum_{q \in \mathbb{Z}} \mathscr{L}_{\omega,q}^\alpha \varrho_\infty = 0$, while the corresponding

(time-averaged) heat currents are [32]

$$\dot{\mathcal{Q}}_\alpha = -T_\alpha \sum_\omega \sum_{q \in \mathbb{Z}} \mathrm{tr}\big\{\mathcal{L}^\alpha_{\omega,q}\varrho_\infty \ln \tilde{\varrho}^\alpha_{\omega,q}\big\}. \tag{10}$$

The states $\tilde{\varrho}^\alpha_{\omega,q}$ are the unique local stationary states of each dissipator, i.e., $\mathcal{L}^\alpha_{\omega,q}\tilde{\varrho}^\alpha_{\omega,q} = 0$.

In general, a power-driven three-level maser does not realize a tricycle as it features closed performance characteristics, which is a clear indicator of irreversibility [19]. We shall take, however, the limit of weak driving, i.e., $\lambda \to 0$, in which the time-averaged limit flux $\mathscr{I}$ reads

$$\mathscr{I} \simeq \frac{\Gamma_{\omega_h}\Gamma_{-\omega_c} - \Gamma_{-\omega_h}\Gamma_{\omega_c}}{\Gamma_{\omega_h} + \Gamma_{\omega_c} + 2(\Gamma_{-\omega_h} + \Gamma_{-\omega_c})}. \tag{11}$$

The excitation and relaxation rates $\Gamma_{\pm\omega_\alpha}$ are given by $\Gamma_{\omega_\alpha} \equiv \gamma_\alpha \omega^{d_\alpha}[N(\omega_\alpha)+1]$ and $\Gamma_{-\omega_\alpha} = e^{-\omega_\alpha/T_\alpha}\Gamma_{\omega_\alpha}$, with $N(\omega_\alpha) \equiv (e^{\omega_\alpha/T_\alpha}-1)^{-1}$. Here d_α stands for the physical dimensionality of bath α [33].

Taking now the high-temperature limit would result in $\Gamma_{\omega_\alpha} \simeq \gamma_\alpha T_\alpha \omega_\alpha^{d_\alpha-1}$ and $\Gamma_{-\omega_\alpha} \simeq \gamma_\alpha T_\alpha \omega_\alpha^{d_\alpha-1}(1-x_\alpha)$, so

$$\mathscr{I} \simeq \frac{\gamma_h\gamma_c}{3}T_c\omega_c^{d_c-1}\frac{\omega_h/T_h - \omega_c/T_c}{\gamma_h + \omega_c^{d_c-1}\omega_h^{1-d_h}\gamma_c T_c/T_h}. \tag{12}$$

We shall discard the second term in the denominator of Eq. (12) by assuming that the coupling to the entropy sink is much stronger than the interaction with the cold bath (i.e., $\gamma_c \ll \gamma_h$). Setting up a comparatively efficient heat rejection mechanism is indeed very important for the maximization of the stationary flux in a refrigerator, which justifies this assumption as a first step towards optimality. Nonetheless, noting that $T_c/T_h = \varepsilon_C/(\varepsilon_C + 1)$, we see that this would be justified anyway, as long as $\varepsilon_C \to 0$. The stationary flux may be thus written as

$$\mathscr{I} \simeq \mathscr{I}_0\big(x_c^{d_c-1}x_h - x_c^{d_c}\big), \tag{13}$$

with $\mathscr{I}_0 = \gamma_c T_c^{d_c}/3$. From here it follows that $C \equiv d_c/(d_c+1)$, i.e., $x_{c,*} = d_c/(d_c+1)\omega_h$, which once substituted in Eq. (6) yields the simple performance benchmark

$$\frac{\varepsilon_*}{\varepsilon_C} = \frac{d_c}{d_c + 1 + \varepsilon_C}. \tag{14}$$

In Fig. 2 the optimal normalized COP of a large number of single-stage and multistage endoreversible absorption refrigerators [34] is compared with Eq. (14), considering unstructured bosonic baths with dimensionality $d_\alpha = 3$. We observe remarkable agreement, especially at low ε_C. Notice, however, that Eq. (14) was obtained for a specific model of compression refrigerator[3] and under the assumption of asymmetric dissipation: There is in principle no reason why it should remain tight nor an upper bound to the performance of other endoreversible models. Therefore, it should be thought of just as a reasonable approximation to their generic behavior. Upon further thought, however, the excellent agreement observed may not be so surprising, provided the optimal performance

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[3]Alternatively, we can consider an absorption three-level maser, driven by heat from a third reservoir at T_w [21], and then take the limit $T_w \to \infty$. Equation (12) would be thus exactly reproduced.
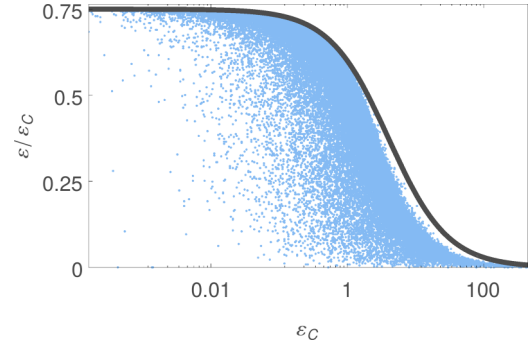

FIG. 2. (Color online) Optimal normalized COP (blue dots) versus ε_C for about 2×10^5 n-stage endoreversible absorption refrigerators [34] with $n \in \{1,\ldots,10\}$ and coupled to unstructured three-dimensional bosonic baths. All three temperatures T_α, dissipation rates γ_α, and hot frequencies ω_h were picked at random and the COP was optimized in ω_c so as to maximize the cooling power $\dot{\mathcal{Q}}_c$ in each case. Equation (14) is plotted in solid gray.

is set by the dissipative scheme alone. Solving for the limit cycle of a (weakly driven) compression three-level maser in different types of environment would be thus enough to come up with generally valid benchmarks for any endoreversible refrigerator in each case. This is one of the main messages of the present paper.

The optimal performance of single-stage and multistage quantum absorption refrigerators is indeed known to be limited by $\varepsilon_*/\varepsilon_C \leq d/(d+1)$ when attached to unstructured baths in d dimensions, with a saturation occurring precisely in the limit of high temperatures [29,34]. Equation (14) can be thus regarded for these models as a stronger bound that sharpens the one given in Ref. [34] for any finite ε_C. Remarkably, also another model of nonideal refrigerator, with the same dissipative scheme, has been shown to have an optimal performance below $\varepsilon_*/\varepsilon_C = d/(d+1)$ [28]. Note, however, that Eq. (14) should not be expected to hold quantitatively (and not even as a qualitative indicator of optimality) when moving away from endoreversibility.

V. CONCLUSION

We have shown from first principles how the COP at maximum cooling power of endoreversible quantum tricycles is not universal in the high-temperature limit, but fundamentally constrained by the details of their interaction with the external heat reservoirs. For quantum refrigerators coupled to unstructured bosonic baths, we obtained a compact expression for their optimal performance, only dependent on the Carnot COP and the dimensionality of the baths.

Our results highlight the importance of reservoir engineering [35] in the design of quantum thermal devices: While squeezed-thermal and other types of engineered nonequilibrium environments are known to be capable of enhancing both the performance and power of heat engines and quantum refrigerators [29,36–38], the exploration of more exotic and highly tunable reservoirs, such as cold atomic gases [39], might bring about new possibilities for the physical realization of superefficient thermodynamic cycles, especially interesting for practical applications to quantum technologies.

APPENDIX: MASTER EQUATION FOR A PERIODICALLY DRIVEN THREE-LEVEL MASER

In the following we derive a quantum master equation for a three-level maser weakly coupled to two unstructured bosonic reservoirs in d dimensions and driven by a periodic perturbation. As already stated in the main text, the full Hamiltonian of system and baths (excluding their mutual interactions) is given by

$$
\begin{aligned}
H_0(t) = H(t) + H_B = {} & \omega_c|2\rangle\langle2| + \omega_h|3\rangle\langle3| \\
& + \lambda(e^{i\omega_w t}|2\rangle\langle3| + e^{-i\omega_w t}|3\rangle\langle2|) \\
& + \sum_{\alpha=\{h,c\}}\sum_\lambda \omega_\lambda b_{\alpha\lambda}^\dagger b_{\alpha\lambda},
\end{aligned}
\tag{A1}
$$

while the system-bath coupling is written as

$$
H_I = \sum_{\alpha=\{h,c\}} \sigma_\alpha \otimes \left(\sum_\lambda g_{\alpha\lambda}(b_{\alpha\lambda} + b_{\alpha\lambda}^\dagger)\right).
\tag{A2}
$$

Recall that the thermal contact operators σ_α were just $\sigma_h \equiv |1\rangle\langle3| + |3\rangle\langle1|$ and $\sigma_c \equiv |1\rangle\langle2| + |2\rangle\langle1|$. The standard recipe to derive a Lindblad-Gorini-Kossakovsky-Sudarshan quantum master equation [33] demands that the two σ_α in the interaction picture are expressed with respect to $H_0(t)$ and then are suitably decomposed.

In the present case, the unitary evolution operator associated with $H_0(t)$ is formally given by the time-ordered exponential

$$
U_0(t) = \mathbb{T}\exp\left(-i\int_0^t ds\, H_0(s)\right)
$$

and may be written as $U_0(t) = U_1(t)U_2(t) \otimes e^{-iH_B t}$, where

$$
U_1(t) \equiv \exp[-it(\omega_c|2\rangle\langle2| + \omega_h|3\rangle\langle3|)],
\tag{A3a}
$$

$$
U_2(t) \equiv \exp[-it\lambda(|2\rangle\langle3| + \omega_h|3\rangle\langle2|)].
\tag{A3b}
$$

This may be easily checked by noticing that $\frac{d}{dt}[U_1(t)U_2(t)] = -iH(t)U_1(t)U_2(t)$.

A time-independent (or time-averaged) Hamiltonian $\bar{H}$ can be defined that generates the same unitary dynamics as $U(t)$ [i.e., $e^{-i\bar{H}t} \equiv U_1(t)U_2(t)$]. For our three-level maser, this would be

$$
\bar{H} = \omega_c(|2\rangle\langle2| + |3\rangle\langle3|) + \lambda(|2\rangle\langle3| + |3\rangle\langle2|),
\tag{A4}
$$

with eigenvalues $\epsilon = \{0, \omega_c - \lambda, \omega_c + \lambda\}$. Its corresponding set of positive Bohr quasifrequencies ($\epsilon_j - \epsilon_i > 0$) is thus $\bar\omega = \{0, 2\lambda, \omega_c \pm \lambda\}$, where we have assumed without loss of generality that $\omega_c > \lambda$.

In general, we would have to resort now to Floquet theory [32,40] in order to decompose the interaction picture

thermal contact operators as

$$
U(t)^\dagger \sigma_\alpha U(t) = \sum_{\bar\omega}\sum_{q\in\mathbb{Z}} A_{\bar\omega,q}^\alpha e^{-i(\bar\omega+q\omega_w)t}.
\tag{A5}
$$

Fortunately for us, this may be done by mere inspection of the left-hand side of Eq. (A5), resulting in

$$
A_{\omega_c+\lambda,1}^h = \tfrac{1}{2}(|1\rangle\langle3| + |1\rangle\langle2|),
\tag{A6a}
$$

$$
A_{\omega_c-\lambda,1}^h = \tfrac{1}{2}(|1\rangle\langle3| - |1\rangle\langle2|),
\tag{A6b}
$$

$$
A_{\omega_c+\lambda,0}^c = \tfrac{1}{2}(|1\rangle\langle2| + |1\rangle\langle3|),
\tag{A6c}
$$

$$
A_{\omega_c-\lambda,0}^c = \tfrac{1}{2}(|1\rangle\langle2| - |1\rangle\langle3|),
\tag{A6d}
$$

$$
A_{-\bar\omega,-q}^\alpha = A_{\bar\omega,q}^{\alpha\ \dagger}.
\tag{A6e}
$$

There are therefore two open decay channels for each thermal contact, corresponding to frequencies $\omega_\alpha \pm \lambda$ ($\omega_w = \omega_h - \omega_c$).

Provided with the decomposition of Eq. (A6e), we can now successively apply the Born, Markov, and rotating-wave (or secular) approximations on the effective equation of motion of the reduced density operator of the system in the interaction picture $\varrho(t)$ [32,33]. We thus arrive at a quantum master equation in the standard form

$$
\begin{aligned}
\frac{d\varrho}{dt} = {} & \sum_\alpha\sum_{\bar\omega}\sum_{q\in\mathbb{Z}} \mathscr{L}_{\bar\omega,q}^\alpha[\varrho] \\
\equiv {} & \sum_\alpha\sum_{\bar\omega}\sum_{q\in\mathbb{Z}}\left[\Gamma_{\bar\omega,q}^\alpha\left(A_{\bar\omega,q}^\alpha \varrho A_{\bar\omega,q}^{\alpha\ \dagger} - \frac{1}{2}\{A_{\bar\omega,q}^{\alpha\ \dagger}A_{\bar\omega,q}^\alpha, \varrho\}_+\right)\right. \\
& \left. + \Gamma_{-\bar\omega,-q}^\alpha\left(A_{\bar\omega,q}^{\alpha\ \dagger}\varrho A_{\bar\omega,q}^\alpha - \frac{1}{2}\{A_{\bar\omega,q}^\alpha A_{\bar\omega,q}^{\alpha\ \dagger}, \varrho\}_+\right)\right].
\end{aligned}
\tag{A7}
$$

The assumption of factorized initial conditions between system and environmental degrees of freedom is implicit in the above, as is thermal equilibrium for the hot and cold heat reservoirs. Also note that the Lamb-shift term has been neglected in Eq. (A7).

The relaxation rates

$$
\Gamma_\omega^\alpha = 2\,\mathrm{Re}\left(\int_0^\infty ds\, e^{i\omega s}\langle B_\alpha(t)B_\alpha(t-s)\rangle\right)
$$

are determined by the power spectrum of the environmental fluctuations and satisfy the Kubo-Martin-Schwinger condition [41,42] $\Gamma_{-\omega}^\alpha = e^{-\omega/T_\alpha}\Gamma_\omega^\alpha$. Here angular brackets stand for equilibrium averaging. As already advanced in the main text, for our choice of the system-bath coupling scheme, i.e., bosonic baths with constant spectral density $J_\alpha(\omega) \sim \gamma_\alpha$, the relaxation rates are explicitly given by $\Gamma_\omega^\alpha = \gamma_\alpha\omega^{d_\alpha}[N_\alpha(\omega) + 1]$, with $N_\alpha(\omega) \equiv (e^{\omega/T_\alpha} - 1)^{-1}$. Physically, this is compatible with weak coupling to the quantized electromagnetic field in thermal equilibrium, inside a d_α-dimensional box [33].

Equipped with Eqs. (A6e) and (A7), we are now in the position of finding the limit cycle state ϱ_∞, which is defined as

$$
\sum_\alpha\sum_{\bar\omega}\sum_{q\in\mathbb{Z}} \mathscr{L}_{\bar\omega,q}^\alpha[\varrho_\infty] = 0.
\tag{A8}
$$

The dissipators $\mathscr{L}^\alpha_{\bar\omega,q}$ have local steady states (i.e., $\mathscr{L}^\alpha_{\bar\omega,q}[\tilde\varrho^\alpha_{\bar\omega,q}] = 0$) of the form [32]

$$\tilde\varrho^\alpha_{\bar\omega,q} = Z^{-1} \exp\left(-\frac{\bar\omega + q\omega_w}{\bar\omega}\bar H\right).$$

Given their standard Lindblad form, each $\mathscr{L}^\alpha_{\bar\omega,q}$ individually generates a fully contractive reduced dynamics towards $\tilde\varrho^\alpha_{\bar\omega,q}$, which is reflected in the monotonic decrease of the distance, as measured by the relative entropy, from any locally evolved state $\frac{d}{dt}\varrho(t) = \mathscr{L}^\alpha_{\bar\omega,q}[\varrho(t)]$ to $\tilde\varrho^\alpha_{\bar\omega,q}$ (i.e., $\frac{d}{dt}S[\varrho(t)||\tilde\varrho^\alpha_{\omega,q}] \leqslant 0$) [33,43]. Such a contractivity property applied to the actual steady state of the full Eq. (A7) eventually leads to the inequality [32]

$$\sum_\alpha \frac{1}{T_\alpha}\left(-T_\alpha \sum_{\bar\omega}\sum_{q\in\mathbb{Z}}\mathrm{Tr}\,\mathscr{L}^\alpha_{\bar\omega,q}[\varrho_\infty]\ln\tilde\varrho^\alpha_{\bar\omega,q}\right) \leqslant 0 \quad (A9)$$

or equivalently

$$\sum_\alpha \frac{1}{T_\alpha}\left(\sum_{\bar\omega}\sum_{q\in\mathbb{Z}}\frac{\bar\omega + q\omega_w}{\bar\omega}\mathrm{Tr}\,\bar H\mathscr{L}^\alpha_{\bar\omega,q}[\varrho_\infty]\right) \leqslant 0. \quad (A10)$$

This can be understood as a statement of the second law of thermodynamics upon defining the limit cycle heat currents as [32]

$$\dot{\mathscr{Q}}_\alpha \equiv \sum_{\bar\omega}\sum_{q\in\mathbb{Z}}\frac{\bar\omega + q\omega_w}{\bar\omega}\mathrm{Tr}\{\bar H\mathscr{L}^\alpha_{\bar\omega,q}[\varrho_\infty]\}.$$

As it is probably useful for the interested reader, we now detail the specific form of the hot and cold dissipators. These are given by

$$\mathscr{L}^\alpha_{\omega_\alpha+\lambda}[\varrho] = \frac{\Gamma_{\omega_\alpha+\lambda}}{4}\mathscr{D}_{++}[\varrho] + \frac{\Gamma_{-\omega_\alpha-\lambda}}{4}\mathscr{D}_{--}[\varrho],$$

$$\mathscr{L}^\alpha_{\omega_\alpha-\lambda}[\varrho] = \frac{\Gamma_{\omega_\alpha-\lambda}}{4}\mathscr{D}_{+-}[\varrho] + \frac{\Gamma_{-\omega_\alpha+\lambda}}{4}\mathscr{D}_{-+}[\varrho]. \quad (A11)$$

To simplify the notation, we have introduced the superoperators $\mathscr{D}$, which act on ϱ as

$$\mathscr{D}_{++}[\varrho] = (|1\rangle\langle 2| + |1\rangle\langle 3|)\varrho(|2\rangle\langle 1| + |3\rangle\langle 1|) - \frac{1}{2}(|2\rangle\langle 2| + |2\rangle\langle 3| + |3\rangle\langle 2| + |3\rangle\langle 3|)\varrho - \frac{1}{2}\varrho(|2\rangle\langle 2| + |2\rangle\langle 3| + |3\rangle\langle 2| + |3\rangle\langle 3|),$$

$$\mathscr{D}_{--}[\varrho] = (|2\rangle\langle 1| + |3\rangle\langle 1|)\varrho(|1\rangle\langle 2| + |1\rangle\langle 3|) - |1\rangle\langle 1|\varrho - \varrho|1\rangle\langle 1|,$$

$$\mathscr{D}_{+-}[\varrho] = (|1\rangle\langle 2| - |1\rangle\langle 3|)\varrho(|2\rangle\langle 1| - |3\rangle\langle 1|) - \frac{1}{2}(|2\rangle\langle 2| - |2\rangle\langle 3| - |3\rangle\langle 2| + |3\rangle\langle 3|)\varrho - \frac{1}{2}\varrho(|2\rangle\langle 2| - |2\rangle\langle 3| - |3\rangle\langle 2| + |3\rangle\langle 3|),$$

$$\mathscr{D}_{-+}[\varrho] = (|2\rangle\langle 1| - |3\rangle\langle 1|)\varrho(|1\rangle\langle 2| - |1\rangle\langle 3|) - |1\rangle\langle 1|\varrho - \varrho|1\rangle\langle 1|. \quad (A12)$$

The populations of the limit cycle state ϱ_∞ (expressed in vector form as $\mathbf{n} = \{n_1, n_2, n_3\}$) may be found by combining the relation $\mathbf{M}^{D3L}\cdot\mathbf{n} = \mathbf{0}$ with the normalisation condition $\sum_i n_i = 1$. The coefficient matrix $\mathbf{M}^{D3L}$ is given by

$$\mathbf{M}^{D3L} = \begin{pmatrix} -2G^+_{-\omega} + 2\frac{G^-_\omega G^-_{-\omega}}{G^+_\omega} & G^+_\omega - \frac{G^-_\omega G^-_\omega}{G^+_\omega} & G^+_\omega - \frac{G^-_\omega G^-_\omega}{G^+_\omega} \\ G^+_{-\omega} - \frac{G^-_\omega G^-_{-\omega}}{G^+_\omega} & -G^+_\omega + \frac{G^-_\omega G^-_\omega}{2G^+_\omega} & \frac{G^-_\omega G^-_\omega}{2G^+_\omega} \\ G^+_{-\omega} - \frac{G^-_\omega G^-_{-\omega}}{G^+_\omega} & \frac{G^-_\omega G^-_\omega}{2G^+_\omega} & -G^+_\omega + \frac{G^-_\omega G^-_\omega}{2G^+_\omega} \end{pmatrix}, \quad (A13)$$

where the constants $G^\pm_\omega$ are defined as

$$G^+_\omega \equiv \frac{\Gamma^S_{\omega+\lambda} + \Gamma^S_{\omega-\lambda}}{4}, \quad G^+_{-\omega} \equiv \frac{\Gamma^S_{-\omega-\lambda} + \Gamma^S_{-\omega+\lambda}}{4},$$

$$G^-_\omega \equiv \frac{\Gamma^S_{\omega+\lambda} - \Gamma^S_{\omega-\lambda}}{4}, \quad G^-_{-\omega} \equiv \frac{\Gamma^S_{-\omega-\lambda} - \Gamma^S_{-\omega+\lambda}}{4}, \quad (A14)$$

and $\Gamma^S_{\pm\omega\pm\lambda} \equiv \Gamma^\alpha_{\pm\omega_h\pm\lambda} + \Gamma^\alpha_{\pm\omega_c\pm\lambda}$.

Finally, we also give the explicit form of the cycle-averaged stationary heat flows. In particular, $\dot{\mathscr{Q}}_h$ is given by

$$\dot{\mathscr{Q}}_h = \dot{\mathscr{Q}}_{\omega_h+\lambda} + \dot{\mathscr{Q}}_{\omega_h-\lambda},$$

$$\dot{\mathscr{Q}}_{\omega_h+\lambda} = \frac{\omega_h + \lambda}{\omega_c + \lambda}\mathrm{Tr}\{\bar H\mathscr{L}_{\omega_h+\lambda}[\tilde\rho]\}$$

$$= (\omega_h + \lambda)\left[\frac{\Gamma_{-\omega_h-\lambda}}{2}n_1 - \frac{\Gamma_{\omega_h+\lambda}}{4}(n_2 + n_3 + n_c)\right],$$

$$\dot{\mathscr{Q}}_{\omega_h-\lambda} = \frac{\omega_h - \lambda}{\omega_c - \lambda}\mathrm{Tr}\{\bar H\mathscr{L}_{\omega_h-\lambda}[\tilde\rho]\}$$

$$= (\omega_h - \lambda)\left[\frac{\Gamma_{-\omega_h+\lambda}}{2}n_1 - \frac{\Gamma_{\omega_h-\lambda}}{4}(n_2 + n_3 - n_c)\right]$$

and $\dot{\mathscr{Q}}_c$ is written as

$$\dot{\mathscr{Q}}_c = \dot{\mathscr{Q}}_{\omega_c+\lambda} + \dot{\mathscr{Q}}_{\omega_c-\lambda},$$

$$\dot{\mathscr{Q}}_{\omega_c+\lambda} = \mathrm{Tr}\{\bar H\mathscr{L}_{\omega_c+\lambda}[\tilde\rho]\}$$

$$= (\omega_c + \lambda)\left[\frac{\Gamma_{-\omega_c-\lambda}}{2}n_1 - \frac{\Gamma_{\omega_c+\lambda}}{4}(n_2 + n_3 + n_c)\right],$$

$$\dot{\mathscr{Q}}_{\omega_c-\lambda} = \mathrm{Tr}\{\bar H\mathscr{L}_{\omega_c-\lambda}[\tilde\rho]\}$$

$$= (\omega_c - \lambda)\left[\frac{\Gamma_{-\omega_c+\lambda}}{2}n_1 - \frac{\Gamma_{\omega_c-\lambda}}{4}(n_2 + n_3 - n_c)\right].$$

In these expressions, the constant n_c is defined as

$$n_c \equiv (n_{23} + n_{32}) = 2\frac{G^-_{-\omega}}{G^+_\omega}n_1 - \frac{G^-_\omega}{G^+_\omega}(n_2 + n_3), \quad (A15)$$

where $n_{ij} = \langle i|\varrho_\infty|j\rangle$ stands for steady-state coherences. Getting the steady-state populations from Eq. (A13) and using the expressions for the heat currents above allows us to check the validity of Eq. (11) in the limit of $\lambda \to 0$.

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