City University London

Doctoral Thesis

Improving the Capacity of Radio Spectrum: Exploration of the Acyclic Orientations of a Graph

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A thesis submitted in fulfilment of the requirements for the degree of Doctor of Philosophy

in the

Faculty of Actuarial Science and Insurance
Cass Business School

April 2017

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"The secret of getting ahead is getting started. The secret of getting started is breaking your complex overwhelming tasks into manageable tasks, and then starting on the first one."

MARK TWAIN

CITY UNIVERSITY LONDON

Abstract

Faculty of Actuarial Science and Insurance
Cass Business School

Doctor of Philosophy

Improving the Capacity of Radio Spectrum: Exploration of the Acyclic Orientations of a Graph

by Robert U. SCHUMACHER

The efficient use of radio spectrum depends upon frequency assignment within a telecommunications network. The solution space of the frequency assignment problem is best described by the acyclic orientations of the network. An acyclic orientation θ of a graph (network) G is an orientation of the edges of the graph which does not create any directed cycles. We are primarily interested in how many ways this is possible for a given graph, which is the count of the number of acyclic orientations, $a(G)$. This is just the evaluation of the chromatic polynomial of the graph $\chi(G, \lambda)$ at $\lambda = -1$. Calculating (and even approximating) the chromatic polynomial is known to be #P-hard, but it is unknown whether or not the approximation at the value -1 is.

There are two key contributions in this thesis. Firstly, we obtain computational results for all graphs with up to 8 vertices. We use the data to make observations on the structure of minimal and maximal graphs, by which we mean graphs with the fewest and greatest number of acyclic orientations respectively, as well as on the distribution of acyclic orientations. Many conjectures on the structure of extremal graphs arise, of which we prove some in the theoretical part of the thesis.

Secondly, we present a compression move which is monotonic with respect to the number of acyclic orientations, and with respect to various other parameters in particular cliques. This move gives us a new approach to classifying all minimal graphs. It also enables us to tackle the harder problem of identifying maximal graphs. We show that certain Turán graphs are uniquely maximal (Turán graphs are complete multipartite graphs with all vertex classes as equal as possible), and conjecture that all Turán graphs are maximal. In addition we derive an explicit formula for the number of acyclic orientations of complete bipartite graphs.

Acknowledgements

I would like to express my special appreciation and thanks to my advisor Professor Dr. Celia Glass, you have been a wonderful mentor to me. I have learnt a lot in the last few years, and have grown in new ways as a person thanks to you. I would also like to thank professor Peter Cameron for wonderful and insightful discussions over the years.

I would also like to thank my family in particular my parents for their support and words of encouragement. Sometimes all it takes is to hear that research can be frustratingly slow and that this is normal. I would also like to thank my colleagues and friends who have supported me throughout the years.

Finally I would like to thank my examiners Professor Jan van den Heuvel and Professor Markus Linckelmann for their feedback, in particular regarding the mathematical rigour of the thesis.

I am also grateful for the Cass Business School bursary and the EPSRC Grant EP/P504872/1.

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Notation

$G(V, E)$	A graph G with vertex set V and edge set E
n	The number of vertices of a graph ($n = \|V(G)\|$)
m	The number of edges of a graph ($m = \|E(G)\|$)
x, y, u, v	These will usually denote vertices in G
e	This will usually denote an edge in G
(x, y)	The edge connecting vertices x and y
$d(x)$	The degree of a vertex x
K_k	A clique, or complete graph of size k
θ	An orientation of a graph
$a(G)$	The number of acyclic orientations of a graph G
$\chi(G, \lambda)$	The chromatic polynomial of the graph G evaluated at λ
$\mathcal{H}_{n,m}$	The set of fully compressed graphs with n vertices and m edges
$H_{n,m}$	The fully compressed and consolidated graph with n vertices and m edges
$\mathcal{N}(x)$	The neighbourhood of a vertex x
$\mathcal{O}(G)$	All orientations of G
$\mathcal{C}(G)$	All orientations of G that contain at least one cycle
$C_{xy}(G)$	The compression from x to y in graph G
$L(G)$	The line graph of a graph G
$e_{L(G)}$	The number of edges of the line graph of G ($e_{L(G)} = \|E(L(G))\|$)
V_0, V_k	The vertices in the independent set and in the clique of a split graph respectively
E^{xy}	The set of edges that get moved by the compression C_{xy}
P, P^{uv}	A path in G, sometimes with endpoints defined (u and v)
$\mathcal{S}(G)$	The set of subgraphs of G
$\mathcal{F}(G)$	The set of subgraphs of G that are forests
$\mathcal{P}_{ab}(G)$	The path switching in graph G between vertices a and b

$\mathcal{D}_{xy}(G)$	The restricted path switching in graph G between vertices x and y
$\mathcal{B}_{xy}(G)$	The crossover map in graph G between vertices x and y
$F_{n,k}$	The union of a clique, an isolated edge and isolated vertices
$T_G(x,y)$	The Tutte polynomial of a graph G
$f_G(e)$	The factor increase in acyclic orientations at edge e when added to G
$f_G(E)$	The factor increase in acyclic orientations when the edge set E is added to G
$f_G(H)$	The factor increase in acyclic orientations when the subgraph H is added to G
K_{n_1,n_2}	The complete bipartite graph on independent edge sets of size n_1 and n_2
$B_n^{(k)}$	The Poly-Bernoulli numbers of order k
$S(n,k)$	Stirling numbers of the second kind
$T(n,r)$	The r-partite Turan graph on n vertices

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p. 2, UK frequency allocations from Roke Manor Research Ltd

Dedicated to my parents Dietlind and Udo.

Chapter 1

Introduction

1.1 Frequency Assignment for Radio Spectrum

The problem of frequency assignment, or frequency allocation, arises when we have a limit on the number of frequencies available and a large demand for the frequency - a simple example being a mobile phone network. Radio signals were first predicted by Maxwell [1] in 1873, and proved to exist by Hertz [2] in 1887. It was only about a century later in the 1960's that the demand through exponential growth first exceeded the spectrum of usable wavelengths prompting 'A Report on Technical Policies and Procedures for Increased Radio Spectrum Utilization' [18] in 1968.

The radio spectrum in 2014 has such high demand that there are many regulations in place for its use. Indeed frequencies need to allocated for many different purposes. In Figure 1.1 you can see the complex assignment of frequencies in the UK to their usage category only - within each of these categories and corresponding spectra we now have the frequency allocation problem.

To attach a sense of monetary value to this we consider just one of these categories, namely the radio spectrum for mobile phones, and within this category we will only look at the auctioned 3G wavelengths. In 2000 the UK 3G mobile auction for radio spectra totalled £22.5 billion [49] (see [47] for the more recent 4G auction) and was widely described at the time as the biggest auction ever. By 2030 demand for mobile data in the UK could be 30 times higher than today [66]. According to Ofcom: "To help meet this demand and avert a possible 'capacity crunch', more mobile spectrum is needed over

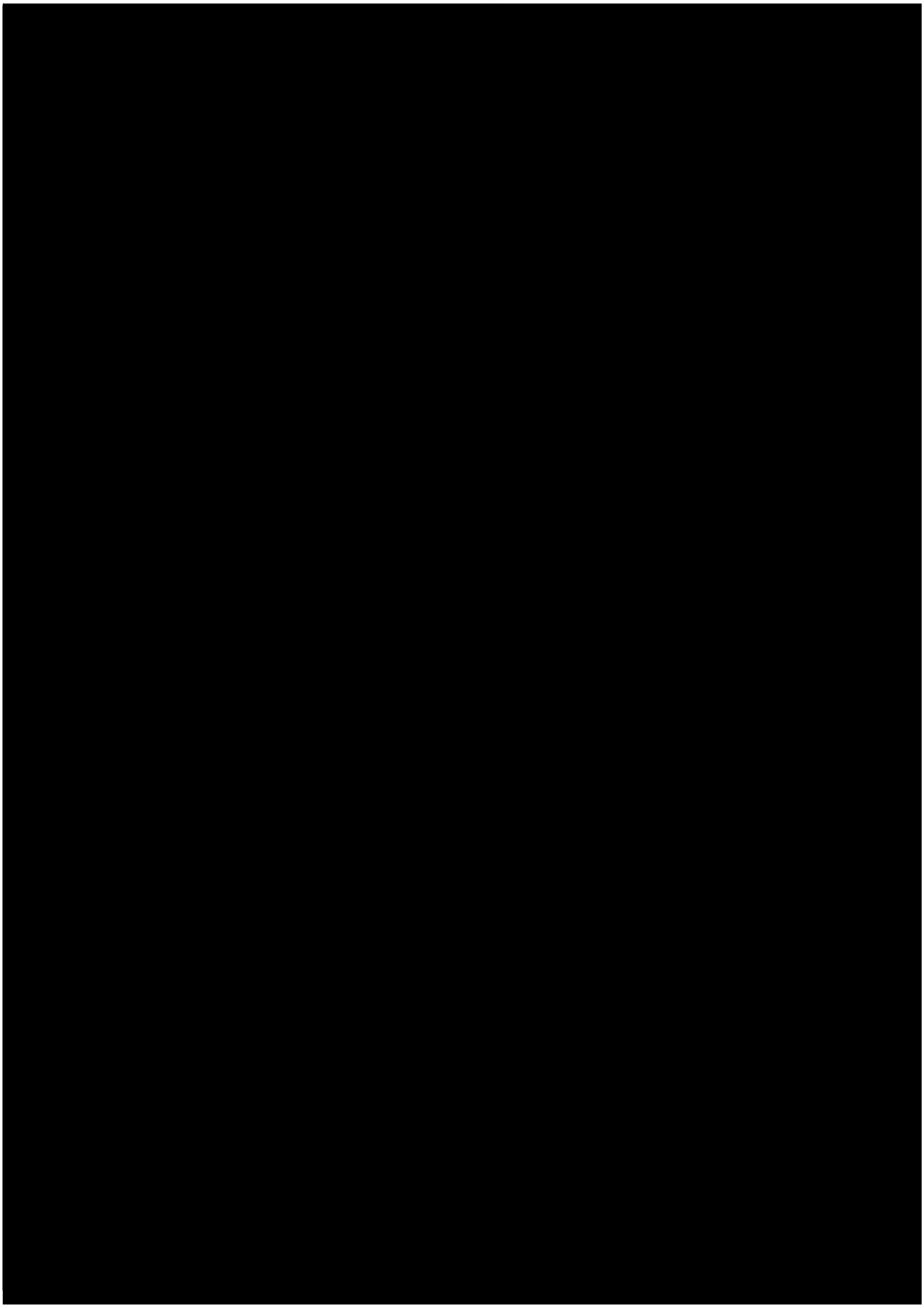

FIGURE 1.1: UK frequency allocations in 2007, Source: Roke Manor Research Ltd.

the long term, together with new technologies to make mobile broadband more efficient" [48]. The span of a frequency assignment tells us how much radio spectrum is needed for a particular assignment and is the difference between the largest and the smallest frequency used. The aim of frequency assignment is to reuse spectrum efficiently, so as to minimise the span of frequencies necessary.

While frequencies are a continuous metric, it is not possible to use frequencies that are close to each other due to interference. This means that despite the continuous nature of frequencies, we are actually dealing with a discrete problem. A lot of research is being done to minimize interference, see [44], [55] and [41] for examples of work to improve the heuristics of finding optimal frequency allocation that minimises interference.

1.2 The Link to Combinatorics

Thus far, I hope to have convinced the reader that the frequency assignment problem is one that needs solving in future. Much research is being conducted into the algorithms and efficiency of these (see as an example [32]), however less is being done on the theory of the problem. In order to make significant progress on the problem a deeper understanding is needed. In this thesis we examine the solution space, and how to move about in the space, as well as some tools that can be applied to simplify networks, after which algorithms can be applied again. We do not look at algorithms themselves, but rather wish to provide theoretical background relating to the solution space for future algorithms.

The objects of interest to us are *frequency assignments*. In the simplest case, a frequency assignment is a vertex colouring of a graph, where adjacent vertices must have different colours (without distinguishing distances between frequencies). An acyclic orientation gives us a valid frequency assignment, and the space of acyclic orientations is smaller than the space of frequency assignments (see Section 2.2). Crucially though, there is an acyclic orientation that corresponds to an optimal frequency assignment, so we are happy to work in this space.

For the purpose of this thesis we wish to study frequency assignment by studying acyclic oriented graphs. Our research looks at the solution space of all acyclic orientations of graphs and we look at how to move around in the space. We also explore maximising and minimising certain properties (including the number of forests, number of cliques and number of edges of the line graph) of graphs, and in particular the number of acyclic orientations. Any results in this area, and any theoretical insight is immediately applicable to frequency assignment.

1.3 The Mathematical Context

The history of the mathematics related to frequency assignment touches on many areas of mathematics and up-to-date research. I will give a brief history of the related problems and refer the reader to open problems that are relevant to our research in Chapter 2.

The most famous result that I will mention here is the four colour theorem. A simple statement of the theorem is: 'Every map can be coloured with four colours such that no two neighbouring countries have the same colour.' This is an easily understood, innocent statement, and if you try it out for yourself it seems true. It was mentioned as early as 1840 in lecture notes by Moebius [5], and was first made into a conjecture in 1852, which was published in 1854 [28]. Over the years there were many attempts to prove the conjecture, indeed twice a proof stood for 11 years before a flaw was found in the argument [5].

The theorem gained additional fame due to the unusual nature of its proof. Kenneth Appel and Wolfgang Haken announced in 1976 that they had proved the theorem using a computer for a large part of the proof [3]. They reduced the problem to a large number of cases and let the computer check each of these - a task impossible by hand. It is now widely accepted as the proof of the theorem and has paved the way for computer aided proofs. The four colour problem is a special case of the graph colouring problem, which in its simplest form is finding a colouring of a graph G with k colours, such that no two adjacent vertices share a colour. The aim is to find a colouring which minimizes the value of k, i.e. uses as few colours as possible. While trying to solve the four colour problem, Birkhoff introduced the chromatic polynomial in 1912 [13]. The chromatic polynomial counts the number of ways a graph can be coloured using no more than a given number of colours (amongst many other things).

The chromatic polynomial has applications in Physics: it is closely related to the zero-temperature partition function of the q-state Potts antiferromagnet, hence computing the chromatic polynomial for some classes is of interest (see for example [17]). It has been studied in its own right, for example the search for real and complex roots (see [38]), or the search for graphs with integral roots only (see [23]).

We will now give a brief informal description of some computational complexity classes in order to give an indication of the complexity of our problem based on Turing machines. A deterministic Turing machine is a state machine, which at any time is in any one of a finite number of states. Instructions for a Turing machine allow a transition between one state and another. Intuitively, a task is Turing-computable if it is possible to specify a sequence of instructions which when carried out by the Turing machine will result in the completion of the task. Using this informal definition we can describe some complexity classes that we will be using. The complexity class P consists of all problems that can be solved by a Turing machine in polynomial time. The complexity class NP, or non-deterministic polynomial time, contains all decision problems for which the correctness of an apparent solution can be verified in polynomial time by a Turing machine. A decision problem is one whose solution is either yes or no. The aforementioned famous four colour theorem [5] falls into this complexity class. It is hard to find a solution, but very easy to check if a given solution is indeed a valid one. The sub-class NP-hard contains all decision problems that are at least as hard as the hardest problems in NP. Problems in this class can be reduced from one to another by some polynomial time algorithm. Thus any solution of an NP-hard problem also solves all other NP-complete problems. An NP-hard problem is a general search problem whose decision version is NP-complete [29]. To date there is no known polynomial time algorithm for solving a problem in this class, nor has it been proved that no such is possible [26].

Next we introduce a probabilistic Turing machine. A probabilistic Turing machine is a state machine, with two transition functions (instead of one as in a Turing machine), with the one to be applied at each step chosen at random, e.g. by the toss of a fair coin. Note that if the transition functions are identical, then we obtain a deterministic Turing machine. The complexity class RP contains all problems, for which a probabilistic Turing machine exists that always runs in polynomial time, always rejects input correctly, and will accept input with a probability of at least $\frac{1}{2}$. In terms of the previously mentioned complexity classes, RP sits between P and NP (P is a subset of RP, which is a subset of NP). As we aim to count various substructures of graphs, we are also interested in the computational complexity of counting problems. The set of counting problem associated with the decision problems in the set NP are called #P. For example, asking if a proper colouring of a graph with a certain number of colours exists is a decision problem, whereas asking how many colourings there are with a certain number of colours is a counting problem.

Of particular interest to us in this thesis is that computing the number of acyclic orientations of a graph is the valuation of the chromatic polynomial $\chi(G, k)$ of a graph G at $k = -1$. Computing the chromatic polynomial is #P-hard, and evaluating it is also #P-hard, except at the points $k = 0, 1, 2$, where the evaluation is polynomial time computable [39]. The situation for approximating $\chi(G, k)$ is similar: there are no known algorithms for approximation of any k except for the three points $k = 0, 1, 2$. At the integer points $k = 3, 4, \ldots$, the corresponding decision problem of deciding if a given graph can be k-coloured is NP-hard (see [30] for an explanation of NP-Complete). The associated counting problems cannot be approximated to any multiplicative factor by a bounded-error probabilistic algorithm unless NP = RP, because any multiplicative approximation would distinguish the values 0 and 1, effectively solving the decision version in bounded-error probabilistic polynomial time. In particular, under the same assumption, this rules out the possibility of a fully polynomial time randomised approximation scheme (FPRAS). For other points, more complicated arguments are needed, which is the focus of current research. As of 2008, it is known that there is no FPRAS for computing $\chi(G, k)$ for any $k > 2$, unless NP = RP holds [33].

In turn, the chromatic polynomial can be generalized to the Tutte polynomial, a polynomial in two variables (and also the graph G) instead of one variable. It is defined for every undirected graph and contains information about how the graph is connected. For a precise definition see Biggs [10]. Of particular interest to us is the evaluation of the Tutte polynomial at $(2, 0)$ which gives us the number of acyclic orientations of a graph. In general the problem of evaluating the Tutte polynomial is #P hard, and even approximating it at some points is known to be #P hard [33].

In this thesis we will look at the points $(2, 0)$ (the number of acyclic orientations) and $(2, 1)$ (the number of forests) in great detail, for both of which finding an exact solution is #P-hard. However it is unknown how hard they are to approximate, which is a gap in the current literature.

1.4 Aims and Research Questions

The aim of this thesis is to explore the space of acyclic orientations. In particular we wish to count acyclic orientations, determine extremal values for graphs of fixed size, and examine the structure of extremal graphs. We will also try to relate the number of acyclic orientations of a graph to other graph parameters including the number of cliques, the minimal degree of the graph, and also the k-edge connectivity of a graph.

The investigation will be empirical as well as theoretical, with our theoretical results inspired by computational work. In particular we wish to provide a full set of computational results for counting the number of acyclic orientations of graphs for all graphs with at most 8 vertices. We will also explore links with other graph parameters computationally.

We wish to identify characteristics of extremal graphs with respect to the number of acyclic orientations, and will aim to classify them. We are interested in both the extremal values and the structure of extremal graphs. We introduce compression as a tool to explore the space of acyclic orientations. We will use compression to classify the minimal graphs. We want the compression move to be a unified approach for several graph parameters.

We are also interested in counting explicitly the number of acyclic orientations of Turán graphs, as they form a cornerstone in a conjecture on maximal graphs.

Finally, we also want to explore an alternative method for counting the number of acyclic orientations (as using the chromatic polynomial *might* be using a sledgehammer to crack a nut). We want to explore a factor method that has potential to run in linear time with respect to the number of edges of a graph.

1.5 Structure of Thesis

The introduction has hopefully given the reader an overview from both a mathematical side and the potential and scope for application.

Chapter 2 gives a more detailed mathematical introduction, with focus on the ground breaking work of Stanley that connected the number of acyclic orientations to the chromatic polynomial. Furthermore, an overview of computational methods for calculating the number of acyclic orientations of graphs will be given.

Chapter 3 provides results from computational experiments, and points out the most notable aspects. We use the method of complete enumeration. We list and then count the acyclic orientations for all graphs up to $n = 8$. There are two flavours of results in the chapter: first we find the minimum and maximum graphs for acyclic orientations, and second we look at the distribution of acyclic orientations for all graphs of a fixed size. Both results, but in particular the one on distribution, give new insights into the solution space of acyclic orientations. The results in this chapter are then further developed in chapters 4,5 and 7.

Chapter 4 has been submitted to the Journal of Graph Theory, and shows a new algorithm using compression to obtain the minimum graph for the number of acyclic orientations, the number of forests and the number of cliques of a graph. The advantage of the approach is in the unified method - one proof works for each of the three parameters. The tool of compression will also be used in chapter 7 to find maximal graphs.

Chapter 5 introduces a new method for counting acyclic orientations. The method looks at the contribution of each edge, vertex or subgraph in terms of a factor rather than a summand. This provides considerable computational simplification in counting the number of acyclic orientations. We also look at future directions and the potential power of this factor method.

In chapter 6 we calculate the number of acyclic orientations for complete bipartite graphs exactly, and building upon this result give formulas for complete bipartite graphs plus/minus an edge. We link these with a thus far unconnected area of mathematics: Poly-Bernoulli numbers. This chapter will be submitted to a journal with minor additions.

Chapter 7 follows on from chapter 6 with more detailed hypotheses about the maximal graphs and links these to the computational work in chapter 3. We give a picture of how we think that maximal graphs behave, in particular that they can be built in a special way out of smaller maximal graphs. We furthermore present evidence for a hypothesis based on the maximality of Turán graphs, as well as prove that a subset of Turán graphs is uniquely maximal using the compression move we developed in chapter 4.

We finish with a conclusion in chapter 8, where we clarify the mathematical progress made in this thesis as well as show how our work can be built upon in future.

1.6 Contribution to joint work

My supervisor, Professor Celia A. Glass, has been very helpful with each chapter for adding ideas, feedback and rigour. The work in Chapters 2,3,5 and 7 is my own, with input as a result of discussions with Prof. Glass and Prof. Cameron.

Chapter 4 (submitted for publication) was completed in collaboration with Prof. Glass and Prof. Cameron over the course of several meetings. The theory was developed collaboratively in meetings with all three of us, and Prof. Glass and I added the necessary details and rigour.

Chapter 6 (soon to be submitted) was also completed in collaboration with Prof. Glass and Prof. Cameron. I found the link to Poly-Bernoulli numbers, Prof. Cameron added mathematical rigour, and subsequent additional results resulted from a collaborative effort between the three of us.

Chapter 2

Background and Literature Review

Throughout the thesis we will be studying the space of acyclic orientations. Therefore we will outline the basic definitions in this chapter, as well as give an overview of what is known about acyclic orientations in the literature. We will also embed the problem of counting acyclic orientations in the more general problem of calculating the Chromatic and Tutte polynomials, and give the relevant background for each of these.

2.1 Basic definitions

In this section we present the reader with a quick introduction to the basic definitions used throughout the thesis. They can be read up in any introductory work on Graph Theory (for example in [21]). In addition to the definitions, we will also specify our usual notation.

Definition 2.1. A *graph* G is a triple consisting of a vertex set, an edge set, and a relationship that associates two vertices with each edge.

We denote the vertex set of a graph by $V(G)$ and the edge set of a graph by $E(G)$. Typically we call the two vertices associated with an edge its *endpoints*. Graphically we represent a graph by drawing a point for each vertex, and a line between between its endpoints for each edge.

Definition 2.2. A *finite* graph is a graph G for which $V(G)$ and $E(G)$ are finite sets.

We will usually denote vertices and edges by lower case letter, e.g. u, v, e, and vertex and edge sets by capital letters, e.g. V_1, V_2 and E_1, E_2 respectively. Undirected edges will sometimes be labelled by its pair of endpoints $\{x, y\}$. For labelled edges with direction from x to y, say, we use the notation (x, y).

Definition 2.3. A *simple* graph is an undirected graph both without loops and multiple edges.

There are many types of graphs, but we restrict ourselves to finite, simple graphs unless otherwise stated. Note that the endpoints of an edge uniquely identify the edge for simple graphs.

Definition 2.4. Let G be a graph. Two distinct vertices x, y of G are *adjacent* in G if $\{x, y\} \in E(G)$.

Definition 2.5. The *degree* of a vertex x, $d(x)$ is the number of edges that contain x as an endpoint.

Definition 2.6. A *path* is a sequence of edges which connects a sequence of vertices that are all distinct from one another.

For the next definition we need the idea of a separation distance between two adjacent vertices. The separation distance on an edge denotes the distance between the vertices.

Definition 2.7. *The length of a path* is the the sum of the separation distances on the sequence of edges in the path.

Thus, for graphs with separation distance 1 only the number of edges on a path is the path's length.

Definition 2.8. A *cycle* is a sequence of vertices starting and ending at the same vertex, with each two consecutive vertices in the sequence adjacent to each other in the graph. No repetitions of vertices and edges are allowed, other than the repetition of the starting and ending vertex.

Definition 2.9. A *labelled* graph is a graph whose vertices are each assigned an element from a set of symbols.

Note that in a labelled graph it is possible to distinguish between vertices without any edge information. This is unlike an unlabelled graph in which individual nodes have no distinct identifications except through their interconnectivity. We usually work with unlabelled graphs, unless otherwise stated.

Definition 2.10. A graph $G' = (V', E')$ is a *subgraph* of a graph $G = (V, E)$, if $V' \subseteq V$ and $E' \subseteq E$, and we write $G' \subseteq G$.

Definition 2.11. A subgraph $G' = (V', E') \subseteq G = (V, E)$, is an *induced subgraph* of G, if for any pair of vertices x and y in V', $\{x, y\} \in E'$ iff $\{x, y\} \in E$.

Definition 2.12. An *orientation* θ of a graph G is an assignment of direction to each edge in G.

Definition 2.13. A *directed graph* $G = (V, E)$ is a pair of a set V of vertices together with a set E of edges such that for each $e \in E$ we have $e \in V \times V$. Each edge is an ordered pair of vertices and thus has a direction associated with it.

We denote a graph G with orientation θ by (G, θ). The definition of a path and a cycle now extends naturally to that of a directed path and directed cycle as follows.

Definition 2.14. A *directed path* is a path $v_0, e_0, v_1, e_1, v_2, \ldots, v_{k-1}, e_{k-1}, v_k$ for which the edges e_i point from v_i to v_{i+1}.

Definition 2.15. A *directed cycle* is a cycle $v_0, e_0, v_1, e_1, v_2, \ldots, v_{k-1}, e_{k-1}, v_0$ for which the edges e_i for $i = 0, \ldots, k-2$ point from v_i to v_{i+1} and edge e_{k-1} points from v_{k-1} to v_0.

Definition 2.16. A *directed acyclic graph* is a directed graph which contains no directed cycles.

Definition 2.17. An *acyclic orientation* is an orientation of a graph that contains no directed cycles.

Definition 2.18. A vertex in a directed graph is called a *source* when it has no edges pointing to it in the graph.

Definition 2.19. A vertex in a directed graph is called a *sink* when it has no edges pointing away from it in the graph.

We are particularly interested in counting graphs, so will briefly explain what we mean when counting graphs. To do this we need the definition of isomorphic graphs.

Definition 2.20. Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. G is *isomorphic* to G' iff there exists a bijection $f : V \to V'$ between the vertex sets, for which $\{x, y\} \in E$ if and only if $\{f(x), f(y)\} \in E'$ for all $x, y \in V$.

We write $G \cong G'$ to denote G is isomorphic to G'. If $G = G'$ we say f is an *automorphism*. Using the notion of graph isomorphism we can now describe what we want to count.

Definition 2.21. A *graph property* is a class of graphs that is closed under isomorphism.

Examples of graph properties are: 'containing a triangle', or of particular interest to us 'containing a directed cycle'. Given a graph with a triangle with three adjacent vertices a, b, c (a triangle), any isomorphic graph G' contains the triangle $f(a), f(b), f(c)$, where f is the isomorphism from G to G', similarly the property 'containing a directed cycle' represents a class of graphs closed under isomorphism.

Definition 2.22. A map which takes graphs as arguments is a *graph invariant*, if it assigns equal values to isomorphic graphs.

A graph that will be of interest to us is the *Turán graph*, defined as follows.

Definition 2.23. The *Turán graph* $T(n, r)$ is the graph formed by partitioning a set of n vertices into r subsets, with sizes as equal as possible, and connecting two vertices by an edge whenever they belong to different subsets and not otherwise.

Throughout the thesis we are interested in the value of certain graph invariants, in particular we wish to find graphs for a fixed number of vertices and edges that minimise or maximise the number of acyclic orientations. We let $a(G)$ denote the number of acyclic orientations of the graph G.

Definition 2.24. A *vertex colouring* of a graph is a way of colouring the vertices of a graph such that no two adjacent vertices share the same colour.

Instead of assigning colours to vertices we are now assigning frequencies to vertices; let $f(x)$ be the frequency band (an integer) assigned to vertex x. As radio frequency is deployed in bands, with small separation distances between bands, they are represented as integers for modelling purposes. In this context, a radio transmitter is represented by vertex and the potential for radio signal frequency interference between transmitters by an edge between corresponding vertices. Signal interference is avoided when the transmitting frequencies are far enough apart. This minimum acceptable distance between transmission frequencies is termed the separation distance, $d_{\{x,y\}}$, along the edge $\{x, y\}$. More formally, if the separation distance on edge $\{x, y\}$ is $d_{\{x,y\}}$, then we require the constraint $|f(x) - f(y)| \geq d_{\{x,y\}}$ on the value of two adjacent frequencies.

Definition 2.25. A *frequency assignment* of a graph G is a map $f(x)$ from the vertex set $V(G)$ to $\mathbb{N}$, such that we have $|f(x) - f(y)| \geq d_{\{x,y\}}$ for each edge $\{x, y\} \in E(G)$.

Note that in the context of frequency assignments the length of a path is not the number of vertices used, but rather the sum of the distances along each edge of the path. Using this definition of the length of a path, the *span* of a frequency assignment is the length of a longest path in a graph we can obtain by using a sequence of vertices with increasing frequencies.

2.2 How acyclic orientations, vertex colourings and frequency assignments relate to each other

We will give further definitions in the context as required. We will now very briefly touch upon the Graph Colouring Problem as well as the Frequency Assignment Problem and how they are related to each other and to acyclic orientations.

The *Graph Colouring Problem* is finding a valid vertex colouring of a graph G, where we require adjacent vertices to have distinct colours. A colouring using at most k colours is called a *k-colouring.*

The *Minimum Span Frequency Assignment Problem* (MS-FAP) is finding a valid frequency assignment while the span is as small as is possible.

In chapter 1 we noted that the MS-FAP is one of great importance for a mobile phone network. We want to use as little spectrum as possible. The MS-FAP reduces to a graph colouring problem when the distance constraints take the value 1 only. Then a colouring with k colours corresponds to a frequency assignment with span $k - 1$ as we now demonstrate.

A colouring which uses k colours leads to an acyclic orientation with longest path at most length $k - 1$, for example by ordering the colours and pointing each edge from smaller colour to larger. Conversely, an acyclic orientation θ with longest path of length $k - 1$ leads to a colouring (and thus a frequency assignment) which uses k colours (or k frequencies). For example, we may colour the sources with the first colour, delete them, then colour all vertices that are now sources using the second colour, and so on.

A frequency assignment with span $k - 1$ in a graph with all distance constraints either 1 or 0, leads to an acyclic orientation with longest path of length at most $k - 1$, by orienting each edge of the graph from lower to higher frequency of its end nodes. Conversely, given an acyclic orientation with maximum path length $k - 1$, we may obtain a frequency assignment of span at most $k - 1$ by numbering the nodes from source to sink with the smallest possible number at each stage. This shows that we will certainly find an (but not necessarily all) optimal solution(s) for the frequency assignment problem in the (smaller) space of acyclic orientations.

2.3 How the chromatic polynomial relates to acyclic orientations

Acyclic orientations of a graph were first considered by Stanley [58] in 1973, who showed that $a(G)$ is closely related to the chromatic polynomial $\chi(G, \lambda)$. The chromatic polynomial of a graph counts the number of vertex colourings of a graph G as a function of the number of colours λ available. In fact if the chromatic polynomial of a graph is known, it is easy to obtain $a(G)$, it is simply the absolute value of the evaluation $\chi(G, -1)$ of the chromatic polynomial. This remarkable result is the key result that opened up the topic for most of the research that follows.

This result is incredibly useful - so useful in fact, that a good proportion of papers published in the area since cite Stanley's paper. It expresses the notion of acyclic orientations in terms of something well studied - the chromatic polynomial. Unfortunately the chromatic polynomial, while being well understood, is also hard to compute explicitly.

No general polynomial time algorithm is known for finding the chromatic polynomial of a graph. It has been shown that the evaluation of the Tutte polynomial, which is a more general form of the chromatic polynomial, is #P-hard, except for some special cases [39].

In 1986 Linial used Stanley's result to show that determining $a(G)$ is #P-complete [43]. Hence the best we can hope to do is bound $a(G)$.

As we make extensive use of the result that relates the chromatic polynomial to the number of acyclic orientations throughout, and in particular because we give a new approach to counting acyclic orientations in chapter 5, we present the chromatic polynomial and Stanleys approach in detail in Appendix A for background reading. The key feature to note in the approach by Stanley is the *additive* nature of obtaining $a(G)$, and in the Appendix we prove the following theorem, taken from Stanley [58].

Theorem 2.26. *For a graph G, $(-1)^{|G|}\chi(G,-1)$ is the number of acyclic orientations of G. This is just the sum of the modulus of the coefficients of $\chi(G,\lambda)$.*

In order to put this into computational context we will show a number of algorithms in Section 2.6 that compute the number of acyclic orientations and state their complexity.

2.4 The Theoretical Setting of evaluating $\chi(G,-1)$ in the bigger picture

In this section we will see how our particular point of interest, $\chi(G,-1)$, fits into what is known about the chromatic polynomial and more generally the Tutte polynomial.

The chromatic polynomial was first introduced in 1912 by Birkhoff [13], where he does not actually coin the term chromatic polynomial, but calls it 'A Determinant Formula for the Number of Ways of Coloring a Map'. Here he considered only planar graphs, as it was a tool to attempt to prove the four colour conjecture. The chromatic polynomial was developed over the years by Birkoff (see [11],[12]) and Whitney (see [64],[63]), and in 1946 Birkhoff and Lewis studied the chromatic polynomial in its own right [14]. In this section we explore how our work ties in with the study of the chromatic polynomial in its own right.

2.4.1 Evaluation of the coefficients of the chromatic polynomial

The deletion-contraction algorithm has usually been the tool used to compute the chromatic polynomial. Calculating the coefficients of the chromatic polynomial is #P-hard in general [50]. All progress in increasing the efficiency of algorithms relies on using some method of improving the deletion contraction method. Determining the chromatic polynomial of some special classes of graphs, e.g. chordal graphs, can be done in polynomial time (see [16]). Some algorithms make use of this and try to reduce any graph via the deletion contraction relation to chordal graphs (see [54] for example). There are also algorithms that at each stage check if we have already calculated the chromatic polynomial of an isomorphic graph (e.g. [34],[35],[36]). In 2005 an algorithm was introduced that uses edge-addition and non-edge contraction and chordal graphs as the base case (see [9]); they give a lower bound of the complexity of their algorithm in terms of the number of clique covers of the graph.

2.4.2 Evaluation and approximation of the Tutte polynomial

The Tutte polynomial is a generalization of the chromatic polynomial [61] (at $y = 0$, the Tutte polynomial specialises to the chromatic polynomial). Thus counting the number of acyclic orientations of a graph is also the valuation of the Tutte polynomial at a certain point, $(2, 0)$.

Definition 2.27. For an undirected graph $G = (V, E)$ the Tutte polynomial is

$T_G(x, y) = \sum_{E' \subseteq E}(x-1)^{k(E')-k(E)}(y-1)^{k(E')+|E'|-|V|}$,

where $k(E')$ denotes the number of connected components of the graph (V, E').

We will briefly mention what is known about the computational complexity of the Tutte polynomial and where the number of acyclic orientations fits in. Jaeger, Vertigan and Welsh have completely mapped the complexity of *exactly* computing the Tutte polynomial in [39]. Here they show that evaluating the Tutte polynomial is #P-hard, except along the hyperbola $(x-1)(y-1) = 1$ and at four special points, $\{(1, 1), (0, -1), (-1, 0), (-1, -1)\}$. In particular both the number of acyclic orientations at $(2, 0)$ and the number of forests at $(2, 1)$ are #P-hard to evaluate. Jerrum and

Goldberg have further classified for which points a fully polynomial randomised approximation scheme exists (see [33] for the detailed definition of an FPRAS as well as the result). For our particular point of interest, the number of acyclic orientations found at $T_G(2,0)$, it is unknown whether such an algorithm exists. Therefore finding an approximation of the number of acyclic orientations would be of interest in this line of research.

2.5 A generating function for the number of acyclic orientations

Let $a(n,m)$ be the number of acyclic orientations of all graphs with n vertices and m edges. Bender, Richmond, Robinson and Wormald used the following theorem in 1986 [7] to obtain an asymptotic approximation for $a(n,m)$.

Theorem 2.28. *Let* $A_n(x) = \sum_m a(n,m)x^m$. *Then*

$$A_n(x) = \sum_{i=1}^{n} (-1)^{i+1} \binom{n}{i} (1+x)^{i(n-i)} A_{n-i}(x).$$

This theorem together with the observation that there are $\binom{n(n-1)/2}{m}$ graphs on n vertices and m edges allows us to calculate the average number of acyclic orientations of a graph with n vertices and m edges.

Consider now the example of graphs with 10 vertices. It is straightforward to calculate A_{10} using this recurrence relation.

$$
\begin{aligned}
A_{10}(x) = {} & 3628800x^{45} + 146966400x^{44} \\
& + 2899411200x^{43} + 37126101600x^{42} \\
& + 346868600400x^{41} + 2520365009400x^{40} \\
& + 14823549568800x^{39} + 72525982284000x^{38} \\
& + 301056304575600x^{37} + 1076055091414800x^{36} \\
& + 3349674724515840x^{35} + 9163072757462400x^{34} \\
& + 22184317673849520x^{33} + 47807980082864190x^{32} \\
& + 92129542599754800x^{31} + 159344586974784960x^{30} \\
& + 248071275833167080x^{29} + 348409073759608260x^{28} \\
& + 442176547815875040x^{27} + 507675000725890200x^{26} \\
& + 527641018776771732x^{25} + 496515058907266500x^{24} \\
& + 422913488921810640x^{23} + 325827430873816320x^{22} \\
& + 226797475663517760x^{21} + 142397107185335940x^{20} \\
& + 80476050938371200x^{19} + 40832558916877560x^{18} \\
& + 18542265211960110x^{17} + 7508190221370540x^{16} \\
& + 2699438041234560x^{15} + 857577282883200x^{14} \\
& + 239434790091840x^{13} + 58405018216860x^{12} \\
& + 12368745491760x^{11} + 2259242749800x^{10} \\
& + 353530511420x^{9} + 47056700160x^{8} \\
& + 5284309680x^{7} + 495329520x^{6} \\
& + 38167920x^{5} + 2362500x^{4} \\
& + 113280x^{3} + 3960x^{2} \\
& + 90x + 1
\end{aligned}
$$

We may now use some of these values to obtain the average value of acyclic orientations of graphs in Table 2.1. The average value goes from 1 at $m = 0$ all the way to $10! = 3628800$ at $m = 45$. The average values in this table are rounded to the nearest integer. We will find some average values for small graphs via complete enumeration in Section 3.3.

m	5	10	15	20	25
$a(10, m)$	38167920	2.26E+12	2.70E+15	1.42E+17	5.28E+17
number of graphs	1221759	3.19E+09	3.45E+11	3.17E+12	3.45E+11
average number of a.o.s	31	708	7827	44922	166455
m	30	35	40	45	
$a(10, m)$	1.59E+17	3.35E+15	2.52E+12	3628800	
number of graphs	3.45E+11	3.19E+09	1221759	1	
average number of a.o.s	462046	1049993	2062899	3628800	

TABLE 2.1: The average number of acyclic orientations per graph for graphs with 10 vertices and m edges.

Bender found an asymptotic formula for the value of $a(n, m)$ in [7] based on Theorem 2.28 and further related the asymptotic number of acyclic orientations of labelled and unlabelled graphs to each other in [8]. Thus for large graphs the average number of acyclic orientations of a graph with n vertices and m edges is (roughly) known.

2.6 Computational Approaches to studying Acyclic Orientations

We have talked in detail about the theoretical embedding of the problem of counting acyclic orientations; we now consider the problem from a more practical side. We look at current computational approaches to counting and studying the number of acyclic orientations. Further we explore in which computational complexity class the problem lies.

2.6.1 Using the chromatic polynomial to study acyclic orientations

First we use the chromatic polynomial to study the number of acyclic orientations of graphs. We wish to find for example extremal graphs with respect to the number of acyclic orientations. In order to do so we need to compute the chromatic polynomial of all graphs with a certain number of vertices and edges. The problem of computing the number of 3-colourings of a given graph is a canonical example of a #P-complete problem. As we have already seen in chapter 1 the problem of computing the coefficients of a single chromatic polynomial is #P-hard. We thus look at algorithms that compute the chromatic polynomial and approximate on how many graphs we need to run these algorithms.

2.6.2 Counting Graphs up to Isomorphism

We wish to generate a list with all graphs with n vertices and m edges, in order to find the chromatic polynomial of each of these, and evaluate it at -1 in order to study acyclic orientations. Note that the chromatic polynomial is an invariant under isomorphism, so we need only have a list of graphs *up to isomorphism.* First let us count graphs, without worrying about potential isomorphic graphs. There are $2^{\binom{n}{2}}$ labelled graphs on n vertices. This is because there are potentially $\binom{n}{2}$ edges, one for each pair of vertices, which we may or may not choose. Each edge gives us a factor 2, resulting in $2^{\binom{n}{2}}$ possibilities. See Table 2.2 for the growth of this function. Note that already for $n = 8$ there exist a huge number of labelled graphs.

n	0	1	2	3	4	5	6	7	8
$2^{\binom{n}{2}}$	1	2	8	64	1024	32768	2097152	268435456	68719476736

TABLE 2.2: Values of $2^{\binom{n}{2}}$ for $n = 1...8$

Allowing for relabelling of the vertices makes the problem significantly more complicated. The number of unlabelled graphs is exactly the number of the labelled graphs on n vertices up to isomorphism. In order to find the number of unlabelled graphs on n vertices we need to use the Pólya Enumeration Theorem (see [37] for details), which makes use of the automorphism group of the graph. Fortunately, most automorphism groups of graphs with n vertices are trivial [52] as n gets large, so asymptotically we have $\frac{2^{\binom{n}{2}}}{n!}$ graphs on n vertices up to isomorphism. For small graphs, these lists already exist, and we will make use of them later on for our computational work. See Table 2.3 for a comparison of both approximations with S_n, the actual number of simple graphs on n vertices. Note that indeed $\frac{2^{\binom{n}{2}}}{n!}$ is a very good approximation for the number of unlabelled simple graphs. Not shown here is that for $n = 16$, the approximation differs only by a factor of 0.993 from the actual number, so in terms of order of magnitude they are equal. We conclude that for our purposes there are $\frac{2^{\binom{n}{2}}}{n!}$ graphs that we have to check for each n.

2.6.3 Computing the Chromatic Polynomial

Now that we know for how many graphs we need to calculate the chromatic polynomial, we look at the complexity of finding the chromatic polynomial for each graph. We have

n	3	4	5	6	7	8	9	10	11
$2^{\binom{n}{2}}$	8	64	1024	32768	2097152	2e8	6e10	3e13	3e16
S_n	4	11	34	156	1044	12346	274688	1.2e7	1e9
$S_n/2^{\binom{n}{2}}$	0.5	0.17	0.03	0.004	0.0004	0.00004	4e-6	3.4e-7	2.8e-8
$\frac{2^{\binom{n}{2}}}{n!}$	1.3	2.6	8.5	46	416	6658	189372	9.6e6	9e8
$S_n/\frac{2^{\binom{n}{2}}}{n!}$	0.325	0.236	0.25	0.294	0.4	0.54	0.69	0.81	0.89

TABLE 2.3: Two approximations of the number of simple graphs on n vertices

noted in general that this problem is #P-hard, therefore we will show some cases that will be of interest to us later, as they are easier.

For some basic graph classes, closed formulas for the chromatic polynomial are known, which make the computation trivial. For instance this is true for trees and cliques, where the chromatic polynomial is $\lambda(\lambda-1)^{n-1}$ and $\lambda(\lambda-1)\dots(\lambda-(n-1))$ respectively. This means that given λ colours there are $\lambda(\lambda-1)^{n-1}$ different ways of colouring any tree. This can be shown using a simple argument, which I will illustrate: Suppose we have λ colours to choose from to colour the tree. Pick a vertex in the tree and colour it in any colour. We have λ choices for this vertex. Now pick a vertex adjacent to a vertex that we have already coloured (this is possible unless we have coloured all vertices). This vertex has $\lambda-1$ choices of colour. This is true for every other vertex, and multiplying together all these possibilities gives us the answer. A similar approach works for complete graphs.

Furthermore polynomial time algorithms are known for computing the chromatic polynomial for some other classes of graphs, including chordal graphs [46] and graphs of bounded clique-width [31]. Chordal graphs are graphs for which every cycle of length longer than 3 can be cut by a chord which is in $E(G)$. A graph with bounded clique-width is a graph which can be constructed using an algorithm with only a certain number of labels available to the vertices. In [19] a precise definition is given, as well as the result that so-called cographs are exactly the graphs with clique-width at most 2. This result will be interesting to us later on, as a special Turán graph which we believe is the graph with the most acyclic orientations is one of these cographs. In particular this shows that we can calculate the number of acyclic orientations of some Turán graphs efficiently.

Apart from graphs for which calculation is simple, there are other tools we can use to simplify the calculation of $\chi(G,\lambda)$. The chromatic polynomial is multiplicative over graph components, so if G has connected components $G_1, G_2, \dots, G_k$ then the chromatic

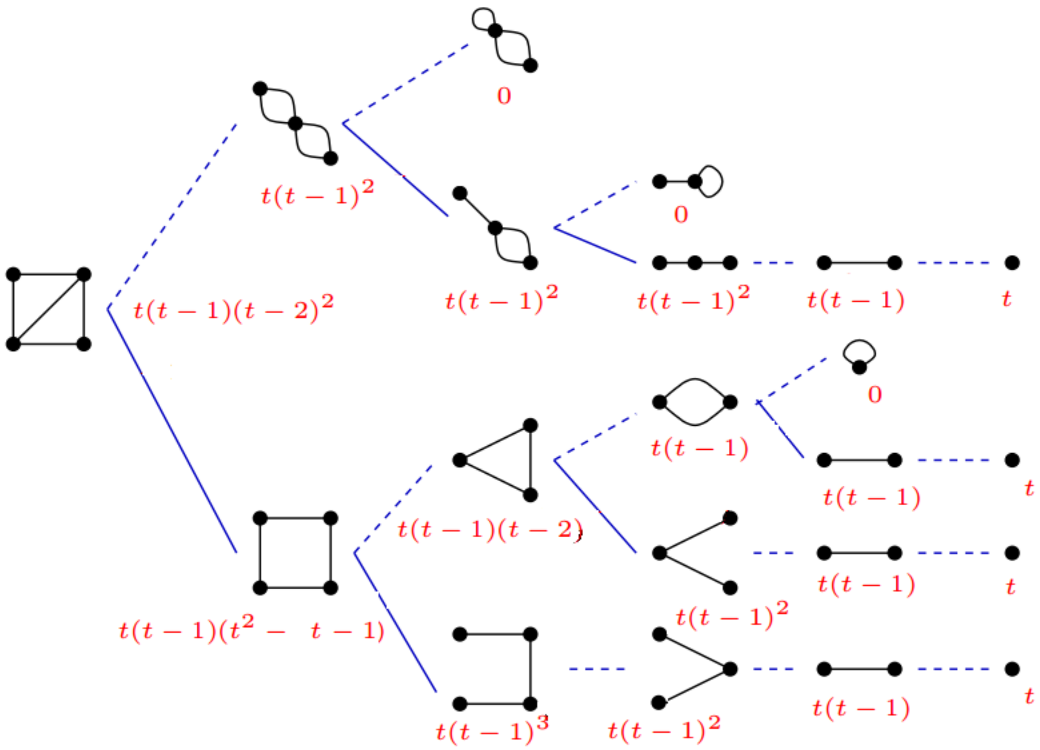

FIGURE 2.1: An example of obtaining the chromatic polynomial for $K_4 - e$ via the deletion-contraction method. Source: Peter Kaski 2008 (with edits and corrections)

polynomial of G can be worked out via $\chi(G,\lambda) = \chi(G_1,\lambda)\times\chi(G_2,\lambda)\times\cdots\times\chi(G_k,\lambda)$. We may assume that G has only one connected component, else we can apply the algorithm to each component individually. There are other tricks for certain special graphs, but in general we have to apply some sort of deletion-contraction relation, which leads to computationally complex algorithms. See Figure 2.1 for visualization of the computation of the chromatic polynomial for a graph on 4 vertices and 5 edges.

There are two potential applications of the deletion-contraction relation. First we can use the equation already presented earlier in this chapter, for a graph G and an edge $e \in G$,

$$\chi(G,\lambda) = \chi(G-e,\lambda) - \chi(G/e,\lambda).$$

Repeated application of this relation will terminate in a collection of empty graphs, from which we then build the chromatic polynomial. This algorithm makes sense for sparse graphs, but for dense graphs it is more efficient (in general) to use the following relation, in which we pick an arbitrary pair x, y of vertices with $\{x,y\} \notin E(G)$, and add $e = \{x,y\}$ to G:

$$\chi(G,\lambda) = \chi(G+e,\lambda) + \chi((G+e)/e,\lambda).$$

Note that here we add an edge $e \notin G$, and repeated application of this algorithm will give us a set of complete graphs, for which the chromatic polynomial is known, from which again we obtain the chromatic polynomial of the original graph G. Mathematica, which we use later in our computational work, uses this approach [51]. The worst case running time of either formula satisfies the same recurrence relation as the Fibonacci numbers. Thus the computational complexity of the algorithm is within a polynomial factor of

$\phi^{n+m} = \left(\frac{1+\sqrt{5}}{2}\right)^{n+m} \in O\left(1.62^{n+m}\right),$

on a graph with n vertices and m edges [65].

2.6.4 Computing the Tutte polynomial of cographs

We will briefly discuss cographs here, as an efficient algorithm exists to compute the number of acyclic orientations of cographs. Of particular interest to us are Turán graphs, which are a subset of the set of cographs. We later conjecture that all Turán graphs are maximal in Chapter 7, as well as prove that a certain subset of Turán graphs are uniquely maximal with respect to the number of acyclic orientations. Indeed we conjecture an upper bound on the number of acyclic orientations any graph can have in Section 7.6.1 which is based on the values that the closest Turán graphs have. Therefore having an efficient algorithm for the number of acyclic orientations of Turán graphs is of great interest to us.

Definition 2.29. A *cograph* is a graph that belongs to the following recursively defined family:

1. K_1 is a cograph
2. If G is a cograph, then so is its complement $\overline{G}$
3. If G and H are cographs, then so is their union $G \cup H$.

This family contains all graphs that can be generated from the single vertex graph K_1 by complementation and disjoint union.

In particular it is possible to generate the Turán graphs in such a manner. Simply create cliques by taking the complement of a disjoint union of K_1's and then complement a disjoint union of cliques of the right size to obtain any Turán graph. If we can prove that Turán graphs are graphs that realize the maximum value, then we can make use of the following theorem from [31]:

Theorem 2.30. *The Tutte polynomial of a cograph with n vertices can be computed in time* $\exp(O(n^{2/3}))$.

Proof. See [31]. □

This result shows a subexponential algorithm (running in time $\exp(O(n^{2/3}))$) for computing the Tutte polynomial on cographs. In our case we want to evaluate the chromatic polynomial at -1, which is equivalent to finding the number of acyclic orientations of a Turán graph. Remember that normally finding the number of acyclic orientations has exponential time complexity.

This result together with the hanging curtains conjecture (see Section 7.6.1) will give us a subexponential upper bound on the number of acyclic orientations a graph can have in Section 7.6.2.

2.6.5 Computing the Tutte polynomial of graphs with bounded clique-width

We now show that the algorithm in Section 2.6.4 can be extended to a subexponential algorithm computing the Tutte polynomial on all graphs of bounded clique-width, as defined here. This result will be of interest for Conjecture 7.15 which states that graphs with a Turán like property (which can be a graph with a bounded clique-width) are maximal with respect to the number of acyclic orientations. We will first need to give a definition of clique-width.

Definition 2.31. The clique-width of a graph G is defined via a graph construction process where only a certain number of vertex labels are available. Vertices that share the same label must be treated identically.

The clique-width of G is the smallest integer k such that G can be constructed by means of repeated application of the following four operations and using the labels $\{1, 2, ..., k\}$.

1. Creation of a new vertex v with label i,
2. Disjoint union of two labelled graphs,
3. Connecting all vertices labelled i to all vertices labelled j where $i \neq j$, without creating duplicate edges,
4. Changing the label of all vertices with label i to label j.

Note that we call a construction of a graph in the manner described in Definition 2.31 using k labels a *k-expression* of a graph.

Cographs are exactly the graphs of clique-width at most 2. The structure we believe is true for maximal graphs in Conjecture 7.15 has the property that we have a lot of components that are all completely connected to each other. If we limit the maximum size of these components by k, then we limit the clique-width of the graph to $k+1$ using a simple algorithm and the labels $\{1, 2, ..., k+1\}$:

1. Use the labels $\{1, 2, ..., k\}$ to create the first component of the complement, and relabel the whole component with the extra label $k+1$
2. Use the labels $\{1, 2, ..., k\}$ to create the next component of the complement, and connect each of these to everything labelled with the label $k+1$
3. Relabel the labels of the component created in step 2 with the label $k+1$
4. Repeat steps 2 & 3 until the desired graph is obtained

There are many graphs that are candidates for the maximum graph with respect to the number of acyclic orientations (at a non-Turán number of edges) that can be efficiently calculated using the following result; indeed many graphs in Conjecture 7.15 have bounded clique-width. The proof for Theorem 2.32 can be found in [31].

Theorem 2.32. *Let G be a graph with n vertices of clique-width k along with a k-expression for G as an input. The Tutte polynomial of G can be computed in time $\exp(O(n^{1-1/(k+2)}))$.*

2.6.6 Generating all Acyclic Orientations of a Graph

Various algorithms exist to generate all the acyclic orientations of an undirected graph. In 1997 Squire [57] and in 1999 Barbosa and Szwarcfiter [6] gave different algorithms to generate all the acyclic orientations of a graph G with n vertices and m edges in time $O((n+m)a(G))$ and $O(na(G))$ respectively. Unfortunately $a(G)$ is not polynomial in n or m so neither of these algorithms is efficient. The algorithms use induction on the edges, and at each stage must calculate all acyclic orientations for the current graph. This is very time consuming, however it is the most efficient way to date - no other known algorithm is faster. In practice, unless we want a list of all orientations, it is quicker to compute the chromatic polynomial than use either of these methods or similar ones.

In this chapter we hope to have given a good introduction to the problem of counting the number of acyclic orientations of a graph from a mathematical point of view as well as a computational point of view. We have also given some more detailed results that will be relevant later on in the thesis.

Chapter 3

Computational Results for Small Graphs

In this chapter we explore the space of acyclic orientations by computation. All results in this chapter are of an exploratory nature and have been derived by exhaustive search. First we describe the algorithm used and its limitations. Then we present the minimum values of $a(G)$ for a fixed number of vertices (n) and edges (m), and the graphs that attain these values. We do the same for the graphs and values that correspond to the maximum. We then look at the distribution of $a(G)$ both for a fixed n and m, and for a fixed n and all corresponding m.

We will perform the complete enumeration for values of n up to $n = 8$, and also show that this is the practical limit for our given computing power. We will briefly touch on the theoretical computational time for larger, but still small graphs (i.e. up to $n = 15$) with faster computers (i.e. the worlds fastest computer), which shows us that tackling this problem beyond small numbers of vertices is impossible using this method of complete enumeration.

Finally we analyse the resulting observations. The observations for small graphs will be built upon later throughout the thesis, in particular in Chapter 7. The graphs and values obtained are further useful to test hypotheses on small graphs, as well as gain insight into acyclic orientations in general.

3.1 The computational method

In this section we present the method used for generating all graphs with a certain number of vertices and edges, in order to find maximal and minimal graphs with respect to the number of acyclic orientations, as well as the distribution of the number of acyclic orientations. We first generate all graphs, then for each graph find the chromatic polynomial, and evaluate it at -1 to find its number of acyclic orientations. This provides the complete distribution of the number of acyclic orientations from which we can identify the minimum and maximum values.

3.1.1 Computational methodology

First - in order to generate all non-isomorphic graphs - we use the software package nauty [45] which gives us all graphs up to ten vertices up to isomorphism. The problem of finding these graphs is already non-polynomial (see the next section for details), but luckily the work here has already been done. There are $12,005,168$ graphs with 10 vertices, which is the highest number of graphs we can hope to compute. The compressed file of graphs is 31MB and the largest number of vertices we can hope to tackle is 10. The file format used is described in Appendix B.1.

The list of graphs was then imported into Mathematica [67], and the inbuilt function for computing the chromatic polynomial was used on each of the graphs in the list. Details on the code used for Mathematica is presented in Appendix B.2. Finally simple database and statistical tools were used to give us graphical output of distributions of interest and to find maximum and minimum values.

3.1.2 Complexity of the algorithm

There are $2^{\binom{n}{2}}$ graphs on n vertices, and as n gets large, most automorphism groups of graphs with n vertices are trivial [52]. Thus allowing for relabelling of the vertices, there are asymptotically $\frac{2^{\binom{n}{2}}}{n!}$ graphs on n vertices up to isomorphism. So the computational complexity is at most $\frac{2^{\binom{n}{2}}}{n!}$ times the complexity of the computation (and evaluation) of the chromatic polynomial. The theoretical running time of an algorithm that computes the chromatic polynomial is $O(2^n n^r)$ for some fixed r. The evaluation of the chromatic polynomial at a certain value is obviously polynomial. The

```
ChromaticPolynomial[ , ]
```

function in Mathematica employs deletion contraction (see [51] for details) with a running time of $O(1.62^{n+m})$ [65]. While this may seem contradictory at first glance, as the Mathematica algorithm has a smaller base and seemingly the same exponent (and thus does better than the running time), the exponent is $n+m$, where m can be as large as $\frac{n^2}{2}$, so clearly larger than n.

In Tables 3.1 and 3.2 we have shown the number of graphs (up to isomorphism) on n vertices, the order of magnitude of the worst case running time of a graph with n vertices and multiplying the two together an estimate of the number of computational steps necessary to find the chromatic polynomial of each graph on n vertices. The approximation of $O(1.62^{n+m})$ is obviously very rough, but this table is just there to give an indication of the running time. Thus the computation of the chromatic polynomial of each graph with 10 vertices has a theoretical running time of around 10^{18} steps for $n = 10$ and 10^{22} steps for $n = 11$. We used an approximation of $m = 45$ and $m = 55$ for $n = 10$ and $n = 11$, which gives us an upper bound on the value of the exponential of $O(1.62^{n+m})$. This crude upper bound could be improved by using the actual value of m rather than the maximal value of $m = \binom{n}{2}$. Even so, the general behaviour of the system will not change, we will only observe the threshold behaviour at a slightly higher number of edges. To put some of these values into perspective, the world's fastest computer, Tianhe-2, can operate at 33.86 PFLOPS (PEta FLoating point Operations Per Second), which is 10^{15} operations per second, according to the TOP500 list [59] in February 2015. Using this computer, we would be able to compute the chromatic polynomial of all graphs on $n = 10$ vertices in around 2 minutes, but the chromatic polynomial of all graphs on $n = 12$ vertices would take around 3400 years, and for $n = 14$ longer than the age of the universe.

3.1.3 Practical limitation of the computer used

The theoretical running time of just calculating (and not evaluating) the chromatic polynomial can be seen in Table 3.1 and Table 3.2 in the row labelled 'Theoretical PC

No. of vertices	2	3	4	5	6	7	8	9
No. of graphs	2	4	11	34	156	1044	12346	274668
Max $O(1.62^{n+m})$	4	18	124	1389	25108	735225	34877113	2.68×10^9
Total steps	9	72	1369	47228	3916877	7.68×10^8	4.31×10^{11}	7.36×10^{14}
Theoretical PC time	0s	0s	0s	0s	0s	0.10s	1min	1 day
Tianhe-2 time	0s	0s	0s	0s	0s	0s	0s	0.02s

Table 3.1: The theoretical time taken to perform the complete enumeration of all graphs and calculating the chromatic polynomial $2 \leq n \leq 9$

No. of vertices	10	11	12	13	14	15
No. of graphs	12005168	1×10^9	1×10^{11}	5×10^{13}	2×10^{16}	3×10^{19}
Max $O(1.62^{n+m})$	3×10^{11}	6×10^{13}	2×10^{16}	1×10^{19}	9×10^{21}	1×10^{25}
Total steps	4×10^{18}	6×10^{22}	3×10^{27}	5×10^{32}	2×10^{38}	4×10^{44}
Theoretical PC time	16 years	250k years	age of universe	ouch	ouch	ouch
Tianhe-2 time	2 mins	24 days	3400 years	500M years	ouch	ouch

Table 3.2: The theoretical time taken to perform the complete enumeration of all graphs and calculating the chromatic polynomial $10 \leq n \leq 15$

time'[1]. The computer we used was a laptop with a 16Ghz CPU and with 8GB of RAM. Further details on the computer used can be found in Appendix B.3. For the calculation relating to $n = 8$ the input data was split into several pieces, which were each calculated separately (as the data set was too big for a single reliable run). This splitting up may have affected the running time, but not greatly so, as there were only 28 separate pieces of input. The actual run times for the algorithm are reported in Table 3.3. This run time also includes evaluating each chromatic polynomial at -1, as well as reading and writing the data, none of which are included in the figure in Table 3.1 and Table 3.2. These should only be a minor factor in changing the run time. A much bigger change is in our approximation of the algorithm by $O(1.62^{n+m})$, where we have ignored all terms smaller than this exponential (including any polynomial coefficiants). We can compare the time taken with the theoretical run time for $n = 8$. The theoretical run time for $n = 8$ just to obtain each chromatic polynomial was one minute (54 seconds), the reported value was 4 hours which includes the evaluation and saving of the output value for each graph. There is a difference here of a factor of over 200, which we attribute to the terms in $O(1.62^{n+m})$ that we have not picked up in our approximation. However, overall the growth in theoretical time is similar to that of actual time, starting at 0 seconds and

[1]The actual values for 'ouch' - each of which is much greater than the age of the universe - in the 'Theoretical PC time' column are: $7 \times 10^{22}, 3 \times 10^{28}, 5 \times 10^{34}$ seconds. If we allow one calculation per shortest meaningful time unit (Planck time: 5.4×10^{-44}s), then these calculations are almost instant. In fact $n = 15$ will take around 1 second. The universe is about 4×10^{17} seconds old. A computer at this Planck speed will take around the age of the universe to calculate the number of steps required for $n = 16$.

n	1	2	3	4	5	6	7	8
t	<< 1s	<< 1s	<< 1s	<< 1s	< 1s	2.88s	59.12s	4h

TABLE 3.3: The running time of computing and evaluating the chromatic polynomial at -1 for all graphs with $n = 1, ..., 8$ vertices on my computer

then increasing rapidly around a threshold value. Judging by the theoretical run time of more than one day for $n = 9$, and applying a similar factor of growth (of 200 from theoretical to actual time) leads to an actual time of over 200 days for $n = 9$ with our current computer.

3.1.4 Full set of results for acyclic orientations

From the program described in Section 3.1.1, we have a list of all graphs for up to $n = 8$ and the corresponding number of acyclic orientations. The full set of results obtained are presented in Appendix C for up to $n = 7$, and on the CD[2] for up to $n = 8$. In addition, results for $n = 8$ for particular values of interest are presented at the end of Appendix C. All observations that follow use these results. It is simple to obtain (for us of particular interest) the minimal degree, and any other graph parameters we wish to compare alongside the number of acyclic orientations parameter and we will do so in Section 3.2.2. We will show in Chapter 4 that indeed the number of edges of the line graph, the number of cliques and the number of forests behave somewhat like the number of acyclic orientations. These parameters thus appear to be somehow related. It is also possible to use these results to test other hypothesis about graphs.

3.2 Extremal graphs for $n \leq 8$

3.2.1 The minimal graphs and values with respect to acyclic orientaitons

The minimum values for the number of acyclic orientations is extracted from results given in the tables in Appendix C. Table 3.4 lists the first 15 values and values for m up to 28 are plotted in Figure 3.1. For all minimal graphs with the same number of vertices

[2]For the pre-bound submission the data is provided in electronic form only

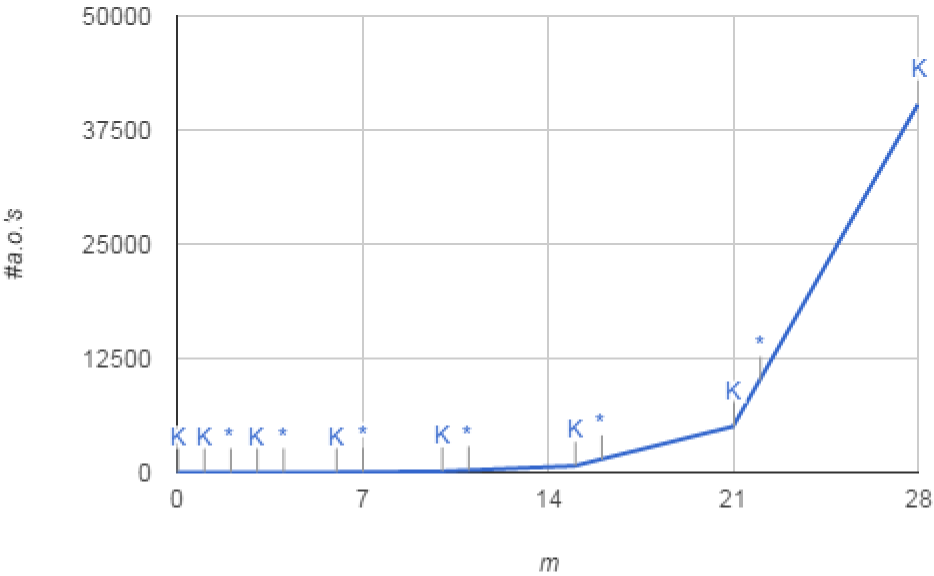

FIGURE 3.1: The minimum number of acyclic orientations possible for m edges, where K denotes a complete graph, and an asterisk $*$ denotes a non unique minimal graph.

and edges, the minimal graph is always one of at most two graphs. This applies for any number of vertices, as long as the number of edges actually fit into the graph.

The graphs to note in particular are highlighted by a K in Figure 3.1, and denote the complete graph of degree $1, 2, 3...$ respectively. The minimum grows piecewise linearly as can be seen in Figure 3.1 between the complete graphs. We will prove this piecewise linear growth in Section 5.4.1, as a consequence of the factor method introduced in Chapter 5. The only graphs that do not obtain the minimum value uniquely are those that have one more edge than a complete graph, highlighted here by an asterisk, which were obtained by inspecting values of the full set of graphs in Appendix C. We will show in Chapter 4 which of these graphs are uniquely minimal and why (Theorem 4.31). In Figure 3.2 the minimum number of acyclic orientations is attained by a triangle with an isolated edge, but the edge can be attached without changing the contribution of acyclic orientations of the edge to the overall graph.

The linear sections in Figure 3.1 arise between complete graphs, and at each complete graph there is an increase in gradient. The number of acyclic orientations of the complete graph K_k is $a(K_k) = k!$ (shown for example in Section 5.4.1) and it has $\binom{k}{2}$ edges. This

m	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$a(m)$	2	4	6	12	18	24	48	72	96	120	240	360	480	600	720
$\Delta a(m)$	2	2	6	6	6	6	24	24	24	24	120	120	120	120	720
$\frac{a(m+1)}{a(m)}$	2	$\frac{3}{2}$	2	$\frac{3}{2}$	$\frac{4}{3}$	2	$\frac{3}{2}$	$\frac{4}{3}$	$\frac{5}{4}$	2	$\frac{3}{2}$	$\frac{4}{3}$	$\frac{5}{4}$	$\frac{6}{5}$	2
K_k	K_1	K_2	K_3			K_4				K_5					K_6
Unique	✓	✓	✓		✓	✓		✓	✓	✓		✓	✓	✓	✓

TABLE 3.4: The minimum number of acyclic orientations for $m = 1, ..., 15$, where $\Delta a(m) = a(m+1) - a(m)$

FIGURE 3.2: Two graphs with 4 edges and the minimal number of acyclic orientations

allows us to compute the gradient between two neighbouring complete graphs K_{n-1} and K_n as follows

$$\frac{a(K_n) - a(K_{n-1})}{\binom{n}{2} - \binom{n-1}{2}} = \frac{n! - (n-1)!}{n-1} = (n-1)!.$$

This gradient $\Delta a(m) = (n-1)!$ can be observed in Table 3.4. The graphs that give the minimal value are in Figure 3.3, and the values are in Table 3.4. Where the minimal graph is not unique, we have opted to show the minimal graph with an extra edge connected to the clique, rather than a floating edge. Note that if we choose the minimal graph with a connected edge as a representative of the two minimal graphs, then each minimal graph is obtained from the previous one by the addition of an edge. We will use a compression method in Chapter 4 to prove this.

3.2.2 The maximal graphs and values with respect to acyclic orientations

We have just seen that the minimal graphs grow nicely, which is unlike the maximal graphs. This also means that the maximum values for the number of acyclic orientations

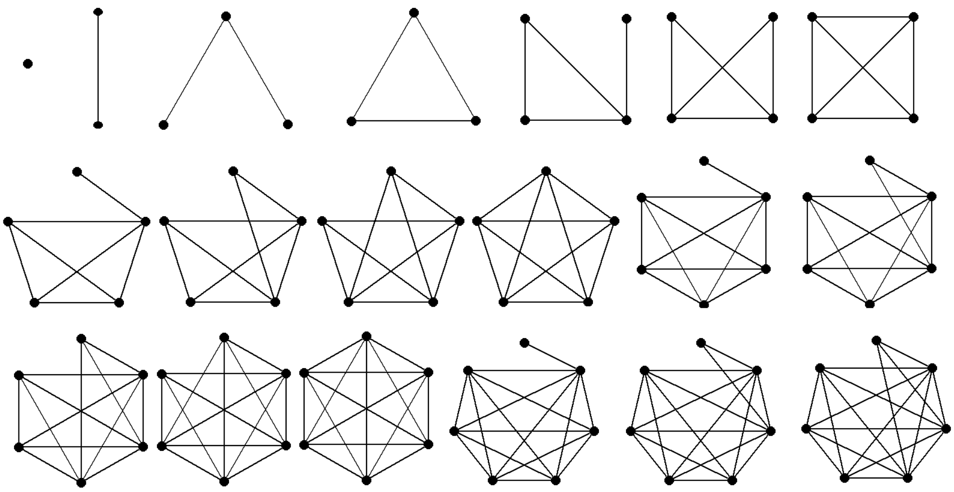

FIGURE 3.3: Some graphs with the minimum number of acyclic orientations

is far more interesting. We have given the graphs that attain the maximal value for $n = 7$ in Figure 3.4. The maximal graph depends both on n and m, and the structure of maximal graphs does not seem to build up from previous maximal graphs. Note that unlike for the minimal graphs there is no immediate overarching structure that can be seen by looking at these graphs. We note that the first couple of maximal graphs are forests, and also that unlike the minimal graphs the edges in maximal graphs seem to be spread out. By this we mean that the vertex degrees are as even as possible, i.e. $|d(x) - d(y)| \leq 1$ for all $x, y \in V(G)$, which we can observe in Table 3.5 (obtained from the results in Table C.6 to Table C.11). We will give an example in Figure 7.2, to show that having all vertex degrees as even as possible is not necessary for a graph to be maximal. There is always one (as far as we checked, i.e. up to $n = 8$) with this condition for each pair n, m, by checking each graph corresponding to the maximal value in the Tables in Appendix C.

Furthermore we have shown another parameter in Table 3.5 which will be of interest in Chapter 7. It will turn out that the condition on degrees is not as helpful as initially expected in order to describe the maximal graphs, as there are some graphs with this property that are not at all close to the maximum, as we will see for example in Figure 7.4. Instead we will make a conjecture using the edge-connectedness of a graph (defined in detail in Definition 7.7) in Conjecture 7.10. For now we note only that the edge connectedness parameter is also extremal in the extremal graph for acyclic orientations for all possible values of n and m up to $n = 8$.

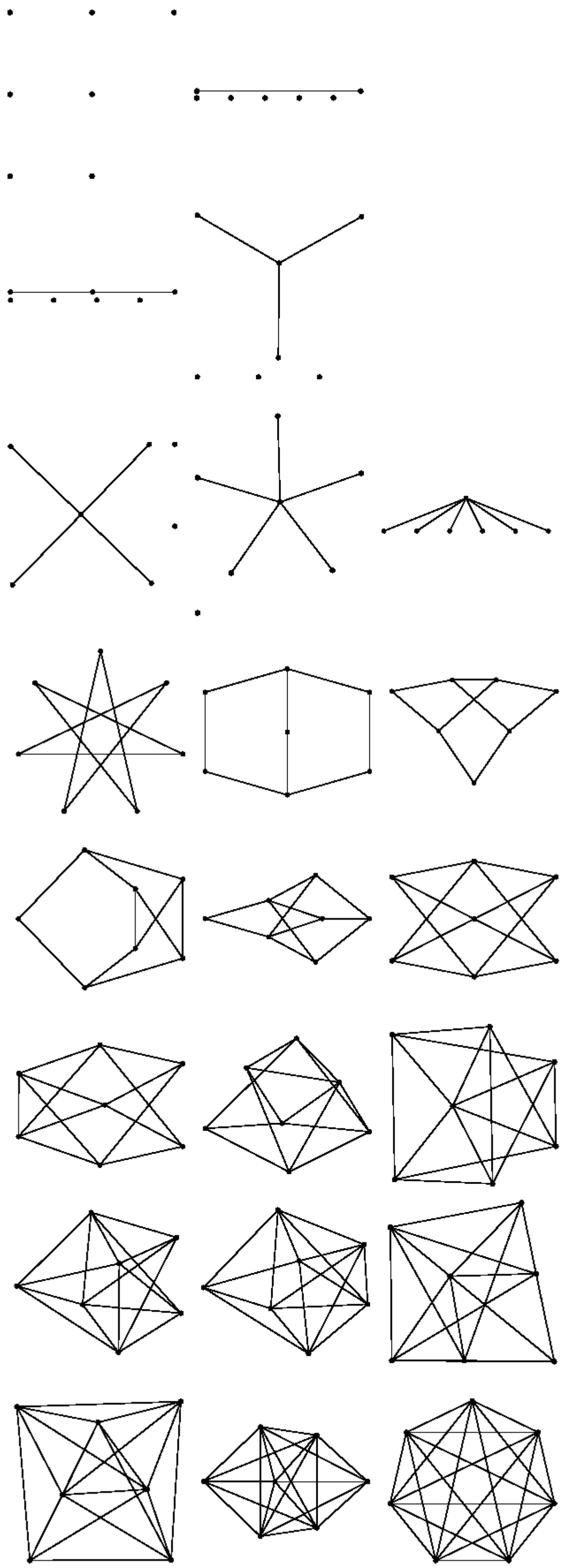

FIGURE 3.4: Maximal graphs for $n = 7$

m	a_{max}	$\frac{a(m+1)}{a(m)}$	Turán	max $\delta(G)$	max $\delta(G_{max})$	max edge-conn of G	max edge-conn of G_{max}
8	254	1.85		2	2	2	2
9	470	1.69		2	2	2	2
10	792	1.55		2	2	2	2
11	1230	1.56		2	2	2	2
12	1920	1.39		3	3	3	3
13	2670	1.35		3	3	3	3
14	3602	1.39		3	3	3	3
15	5000	1.38		3	3	3	3
16	6902	1.15	$T(8,2)$	4	4	4	4
17	7968	1.16		4	4	4	4
18	9264	1.16		4	4	4	4
19	10752	1.19		4	4	4	4
20	12840	1.20		5	5	5	5
21	15402	1.15	$T(8,3)$	5	5	5	5
22	17688	1.15		5	5	5	5
23	20400	1.18		5	5	5	5
24	24024	1.13	$T(8,4)$	6	6	6	6
25	27240	1.14	$T(8,5)$	6	6	6	6
26	30960	1.14	$T(8,6)$	6	6	6	6
27	35280	1.14	$T(8,7)$	6	6	6	6
28	40320	-	$T(8,8)$	7	7	7	7

Table 3.5: Greatest minimal degrees and maximal edge-connectivity of select graphs up to $n = 8$.

Instead of looking at the graphs, it is most instructional to look at the adjacency matrices, and further instead of using 1's and 0's to use a black square for a 1 and a white square for a 0. It is possible to permute the rows and columns to obtain an isomorphic graph, so here the adjacency matrices are ordered in such a way that you can see some structure. Indeed this structure is not obvious from other orderings. You can see the progression for $n = 7$, $m = 0...21$ in Figure 3.5. This structure is much (much) harder to see in Figure 3.4, so in Chapter 7 when we examine the maximal graphs in more detail, we will mainly focus on the representation of graphs used in Figure 3.5. In order to talk about this structure we need the following definition.

Definition 3.1. An *independent subset* of vertices in a graph is a set of vertices $\{x_1, ..., x_k\}$ such that $(x_i, x_j) \notin E(G)$ for all $i, j \in 1, ..., k$, i.e. a set of vertices that have no edges between any pair in its set.

There are several interesting observations we can make using this representation of the

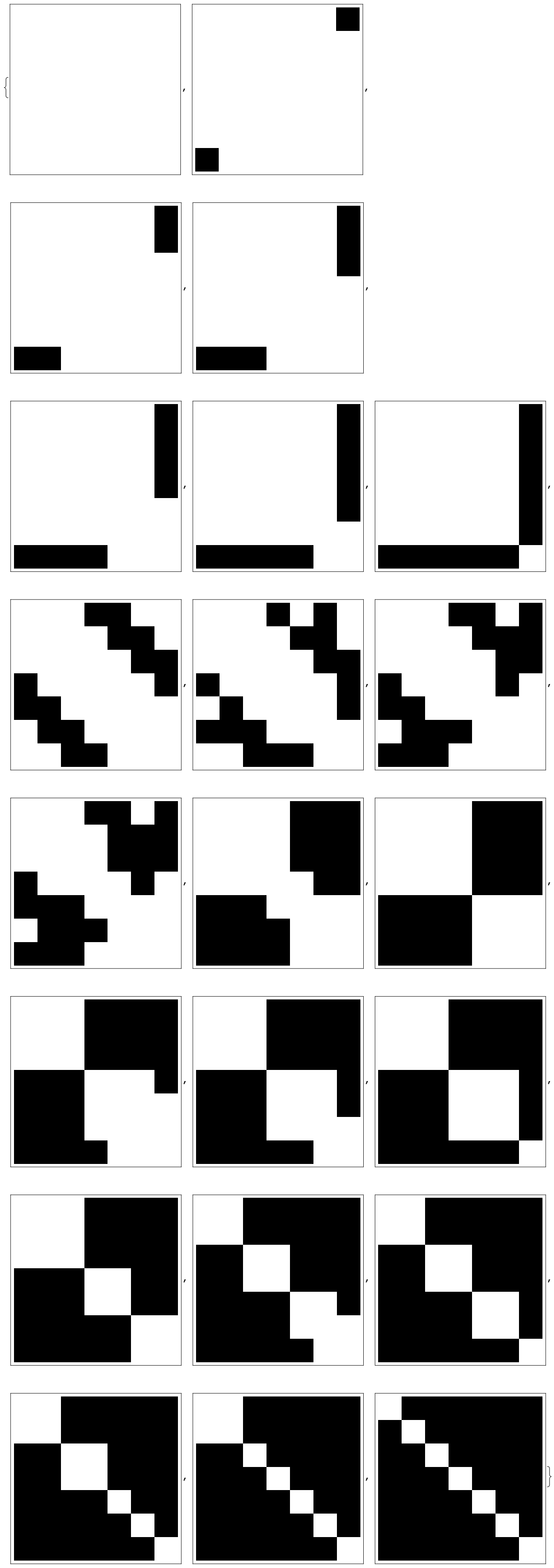

FIGURE 3.5: Maximal graphs for $n = 7$

maximal graphs. We first note that there does indeed seem to be some structure underlying the maximum graphs. For graphs with a higher density, corresponding to a more filled square here, we see that along the diagonal there are a series of non overlapping white blocks. These white blocks are independent subsets of the graph, meaning that we can partition the maximal graphs into independent blocks corresponding to the independent subsets in a Turán like structure (more detail on this structure is given in Chapter 7).

Furthermore, if we allow a white block to contain"a few" black squares also, then we can observe that the number of these white blocks monotonically increases as the number of edges in the graph increase. When the non overlapping white blocks are as equal as possible, and none of them contain any black squares, we actually have a Turán graph. For a more detailed explanation and discussion see Chapter 7. We conjecture Turán graphs to be maximal, and prove the conjecture in some cases, in Chapter 7.

The last observation that we make here is that the maximal graphs between $m = 16$ and $m = 20$, which are the graphs with two white squares and 3 white squares in the representation of Figure 3.5 respectively, are all built upon the maximal graph at $m = 16$ by the addition of an edge at a time. It is possible to obtain all the graphs up to $m = 19$ in this manner, by adding one edge to the previous maximal graph. In Figure 3.6 we have shown how it would be possible to also build up the graph for $m = 20$ from $m = 18$ onwards in such a manner, but then the graph at $m = 19$ is not maximal. The lower two graphs in the middle of the figure are the graphs that build nicely, the top graph is the (only) maximal graph which does not build nicely up to the next maximal graph. We later conjecture that Turán graphs are maximal, and it is not possible to simply add edges between each pair of Turán graphs. This example shows that even when it would be possible to build nicely, the maximal graphs do not always do so, so building nicely up to the next maximum is not something that is important structure-wise for maximal graphs.

In Figure 3.7 we can see the minimum and maximum values for the number of acyclic orientations plotted alongside for the full range of m for $n = 8$. We have here highlighted the Turán graphs, and it is just about possible to see that in a sense the maximum curve 'hangs' from the Turán graphs. What we mean by this we firm up in Chapter 7. Some

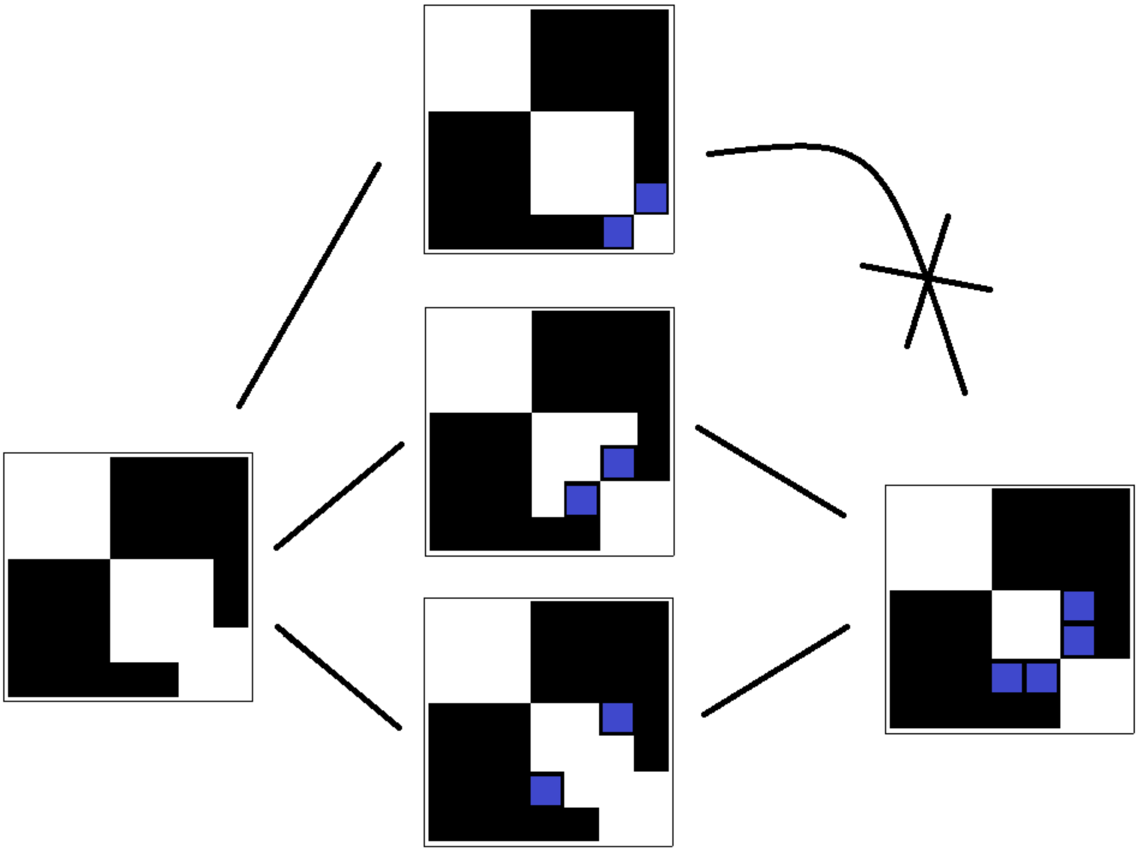

FIGURE 3.6: Maximal graphs not built by simple edge addition

further examples of these hanging curves for other values of n can be found in Appendix D.1.1.

Most of the minimal values in Figure 3.7 are obtained uniquely by one graph, those that are not are the ones directly following a complete graph, marked by an asterisk. Turán graphs are always (verified up to $n = 8$) uniquely maximal. We can see that there is often a large difference between the maximum and the minimum number of acyclic orientations for fixed n and m, which we will further examine in the next section. More maximal values for $n = 6, 7$ (the minimal value is the same for all n) can be found in Appendix D.1 and D.2.

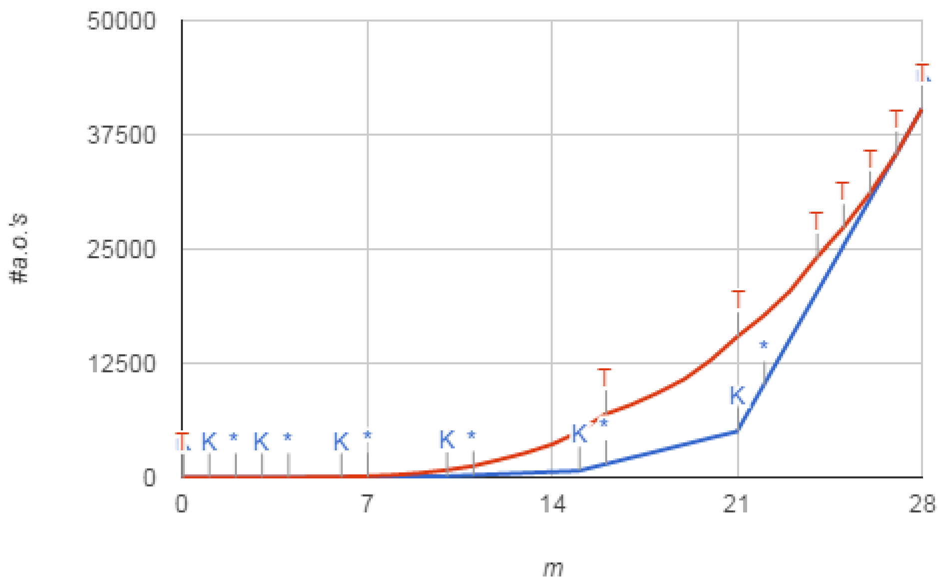

FIGURE 3.7: Maximal and minimal graphs for $n = 8$, where K denotes a complete graph, an asterisk a non-unique minimal graph and T a Turán graph.

3.3 Distribution of the number of acyclic orientations (for $n \leq 8$)

We have now looked in detail at both the maximal and minimal graphs with respect to the number of acyclic orientations. We observe in Section 3.2.1 that the minimal graphs are packed tightly, i.e. vertex degrees are either 0 or as large as possible. For the maximal graphs we observe in Section 3.2.2 that the maximal graphs are spread out, i.e. all vertex degrees are as equal as possible. In this section we look at the distribution of the number of acyclic orientations of all graphs with a fixed number of vertices and edges. Bender et al. calculated the average number of acyclic orientations asymptotically in 1986 [7], an example of which we gave in Section 2.5. In Table 3.6 we have give the minimum, average and maximum values a graph on $n = 8$ vertices can have for each fixed number of edges m, $0 \leq m \leq 28$, rounded to the nearest integer. Unlike the values obtained in Table 2.1, these are the actual values and not an approximation.

It is first interesting to note that for very sparse graphs, a random graph will actually be a maximal graph or very close to one with high probability, as we can observe from

m	min		ave	max	
0	1	K	1	1	T
1	2	K	2	2	
2	4		4	4	
3	6	K	8	8	
4	12		15	16	
5	18		29	32	
6	24	K	54	64	
7	48		100	128	
8	72		176	254	
9	96		296	470	
10	120	K	479	792	
11	240		744	1230	
12	360		1112	1920	
13	480		1606	2670	
14	600		2251	3602	
15	720	K	3070	5000	
16	1440		4089	6902	T
17	2160		5332	7968	
18	2880		6825	9264	
19	3600		8589	10752	
20	4320		10654	12840	
21	5040	K	12976	15402	T
22	10080		15717	17688	
23	15120		18797	20400	
24	20160		22366	24024	T
25	25200		26184	27240	T
26	30240		30600	30960	T
27	35280		35280	35280	T
28	40320	K	40320	40320	T

TABLE 3.6: The minimum, average and maximum values for $n = 8$.

Table 3.6. In Table C.14 we look in detail at all graphs with $m = 4, 5, 6$ edges. For $m = 4, 5$ more than half the graphs attain the maximal value, and for $m = 6$ nearly half of them do. All of these maximal graphs are forests (this follows from Lemma 5.2). Most graphs in the density range $m < n$ are either forests or contain very few cycles, which means that they are close to the maximum. We have plotted the average value against the minimum and maximum value in Figure 3.8, where we can observe that while for sparse graphs ($m < n$) the average is very close to the maximum, this is not true for denser graphs.

We can summarize that for our small examples the minimum grows piecewise linearly between complete graphs (blue line), the maximum hangs from the Turán graphs (yellow

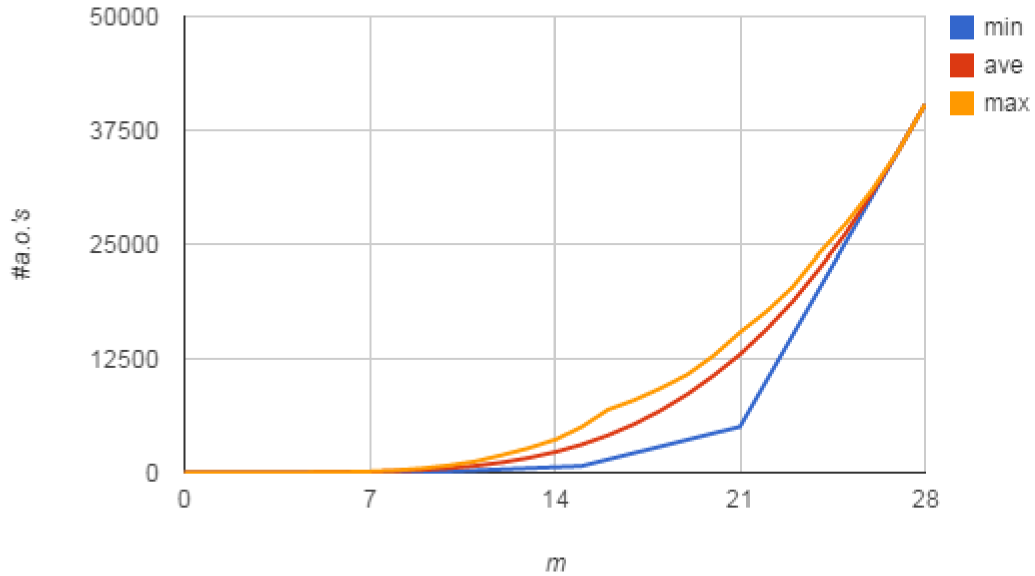

FIGURE 3.8: Minimum, average and maximum number of acyclic orientations of graphs with 8 vertices.

line), and the average value grows smoothly (red line) and is closer to the maximum than the minimum.

3.3.1 Shape of the distribution of the number of acyclic orientations for fixed n, m

We have thus far looked at the number of acyclic orientations for a fixed number of vertices and all possible numbers of edges. We will now fix both the number of vertices and edges. Figure 3.9 depicts the number of acyclic graphs for the particular fixed number of vertices and edges of $n = 8$ and $m = 15$. We have used the *actual* distribution and not just an approximation here, using the data in Appendix C. We have depicted it as a histogram with a bin size of 100 here, in order to make the data easier to read.

The minimal graph for $n = 8$ and $m = 15$ is a K_6 union two isolated vertices with 720 acyclic orientations (found in Table C.12). The graph with the second fewest number of acyclic orientations obtained from the results in Table C.12 has has 1200 acyclic orientations. The isolation on the lower end, which can be measured by a factor of $1200/720 = 1.2$, can be explained by the factor method in Chapter 5. The exact increase

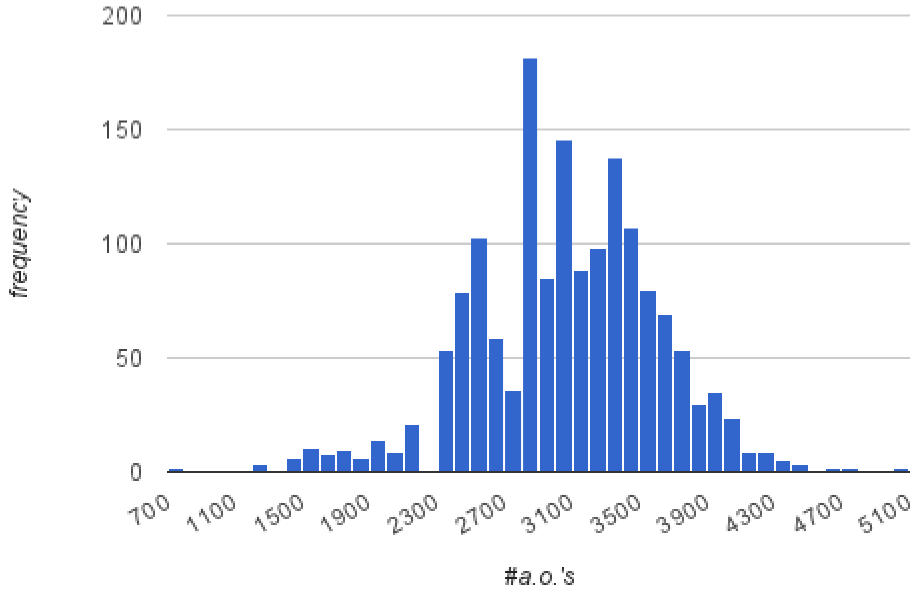

FIGURE 3.9: The distributions for $n = 8, m = 15$

is given by $\frac{2k}{k-1}$, where k is 6 in this case, see Lemma 5.8. Thus we obtain the factor $\frac{2\times 6}{6-1} = 1.2$ as observed. In the case where the minimal graph is not a complete graph, this factor is slightly smaller than the formula suggests, but we will not go into detail why here. We will study such factors in greater detail in Chapter 5, in particular starting with Definition 5.1.

The isolation on the upper end of the spectrum of the number of acyclic orientations is less obvious. For the particular case of $n = 8$ and $m = 15$, the maximum value is 5000 and the second largest value is 4718 read from Table C.12. This makes the gap between the maximum and the second largest value 282 (compared to the minimum gap of 480), and in terms of a ratio or factor is < 1.06 (compared to the factor at the minimum of 1.20). The gap is significantly smaller than the gap at the minimum value, and we have no formula or approximation as we did for the minimum value for the size of the gap.

We now consider $n = 8$ and $m = 16$ instead, as here a Turán graph exists. We have given the distribution in Figure 3.10 with a bin size of 150. The 100 graphs with the fewest, and the 100 graphs with the most acyclic orientations are given in Table C.13. The minimum value is 1440 and the second smallest value is 1800, which gives us a gap

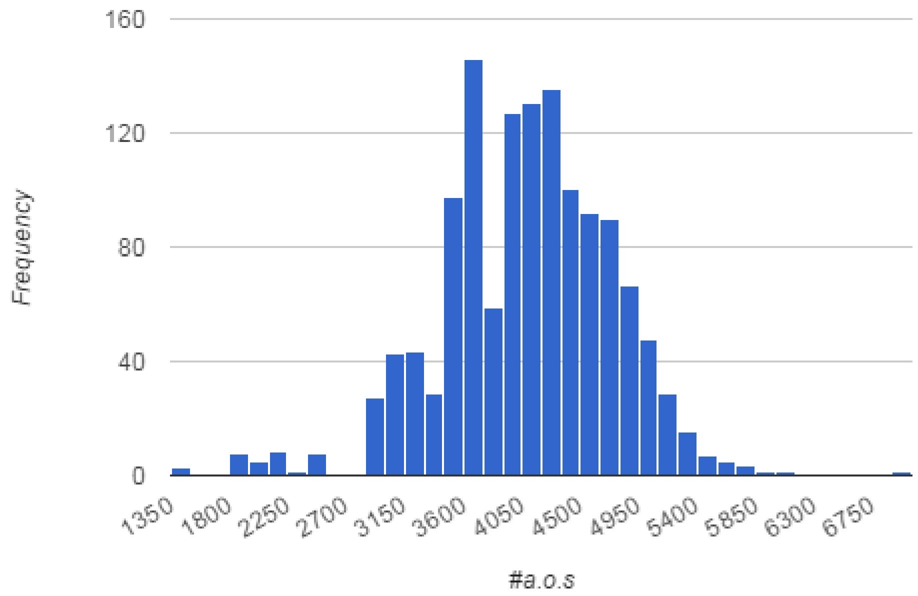

FIGURE 3.10: The distributions for $n = 8, m = 16$

of 360 between the minimum and second smallest value, or in terms of ratio 1.25. The maximum value is 6902 and the second largest value is 6066 which is a gap of 836, or in terms of ratio 1.14. For these values of n and m, the maximum is more isolated than the minimum in terms of absolute value, but not in terms of ratio to the second most extremal value.

There are actually two graphs that attain the minimum value, so the difference could also be viewed as 0. The graphs are a K_6 with either a connected or a disconnected edge respectively. We will show in Chapter 4 that actually it is quite unusual for a minimal graph not to be unique and why for this particular value this happens in Theorem 4.31.

The graph that attains the maximal value here is the complete bipartite graph $K_{4,4}$, which is also the Turán graph $T(8, 2)$. For $n = 8, m = 15$ the maximum was not as isolated, which can hint at a stability result of some form close to Turán graphs. By a stability result we mean a result that says roughly: 'if we are given a graph with the same number of edges and vertices as a Turán graph, it cannot have close to the number of acyclic orientations of the Turán graph without being the Turán graph itself'. We conjecture something similar in Chapter 7, namely that Turán graphs are the best kind of maximum in a sense as explained in Section 7.6.1.

We do know that there are stability results for other parameters around Turán graphs, for example in the number of triangles. In 1941 Rademacher proved that for even values of n every graph G with one more edge than the number of edges of the bipartite Turán graph, contains at least $\frac{n}{2}$ triangles and that this is the best possible. Rademacher's proof was not published, so see [24] and [25] for details. While this is not exactly the result that we want here (we would prefer any graph with as many edges and at least one triangle to have many triangles), it is still relevant to us later on. In Chapter 5 we attempt to make a connection between triangles and the number of acyclic orientations, and we observe that Turán graphs are the conjectured maximal graphs on which the hanging curtains conjecture is based in Section 7.6.1. The result we do want is used as a Lemma in a proof of a result due to Erdős in [24] and can be reduced to the Lemma we require as follows.

Lemma 3.2. *Let $\delta > 0$ be a fixed number. Consider any graph G on n vertices and $m = \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$ edges, for a sufficiently large $n > n_0(\delta)$, which contains a triangle. Then G contains at least $\lfloor cn \rfloor + 1$ triangles for some constants $c = c(\delta)$ and $n_0(\delta)$.*

We have reduced the Lemma to our needs here, the full Lemma states that we can find such a set of $\lfloor cn \rfloor + 1$ triangles that share one edge. This result states that if there exists a triangle in the graph, then there exist many. Thus the graph with no triangles are isolated in terms of the number of triangles of a graph.

Therefore it is not surprising to see that the maximal graph is somewhat isolated. In particular triangles are the worst local structure (using only 3 edges) for maximising the number of acyclic orientations, further supporting a corresponding stability result for the number of acyclic orientations. In some cases, where no Turán graph exists there is also no gap (as can be observed in Figure 3.11), or indeed several non isomorphic graphs will give the maximal value (an example is given in 7.2). It is also useful to compare the factors obtained in Table 3.5, where each factor at every Turán graph is significantly smaller than the factor at the preceeding non-Turán graph. We look into more detail at the upper end of the spectrum in Chapter 7.

3.3.2 Shape of the distribution of the number of acyclic orientations for a fixed n and all corresponding m

We now look at the distribution of the number of acyclic orientations plotted in Figure 3.11 with respect to the number of edges. Each vertical cross section corresponding to a specific value of m can be viewed as a distribution as illustrated in Figure 3.9 for $n = 7$. Here we have only plotted the values, multiple graphs with the same value only appear as one point in this figure. It is interesting to note that the minimal values are isolated between 13 and 17, and particularly so for $m = 15$ which corresponds to the graph K_6, indicated by a K in Figure 3.9. The isolation increases from $m = 13$ to $m = 15$, as we grow the graph towards the complete graph. At $m = 16$ the difference is smaller again. This behaviour is fully explained by the factor method in Chapter 5, in particular relying on Lemma 5.2.

We are less interested in the isolated maximum values at $m = 18$ and $m = 19$, as here all values are isolated from each other. The maximum value on the other hand appears to be isolated in particular at $m = 12$. The value $m = 12$ corresponds exactly to the complete bipartite graph on 3 and 4 vertices. This is the first (non-trivial) Turán graph $T(n, 2)$, highlighted in Figure 3.11. We have already mentioned that we believe that the Turán graphs are maximal in general, furthermore this isolation will be interesting to examine later on in Chapter 7. In particular we will use the Turán graphs in Section 7.6.1 to conjecture an upper bound on the number of acyclic orientations, as well as prove maximality of a subset of Turán graphs. The graphs we have highlighted here, the complete graph and the Turán graph here tell a nice story, each piece of which we have touched upon here we will explain in more depth the later chapters.

3.3.3 Overlap between the possible number of acyclic orientations for m and $m + 1$

We note that in Figure 3.11 there is a very significant overlap between the range of acyclic orientations for neighbouring numbers of edges, and thus that the structure of the graph is far more important than the addition of an extra edge. We thus can not hope to find a formula that gives us a good approximation on the number of acyclic orientations just by knowing the number of vertices and edges. This is not the case

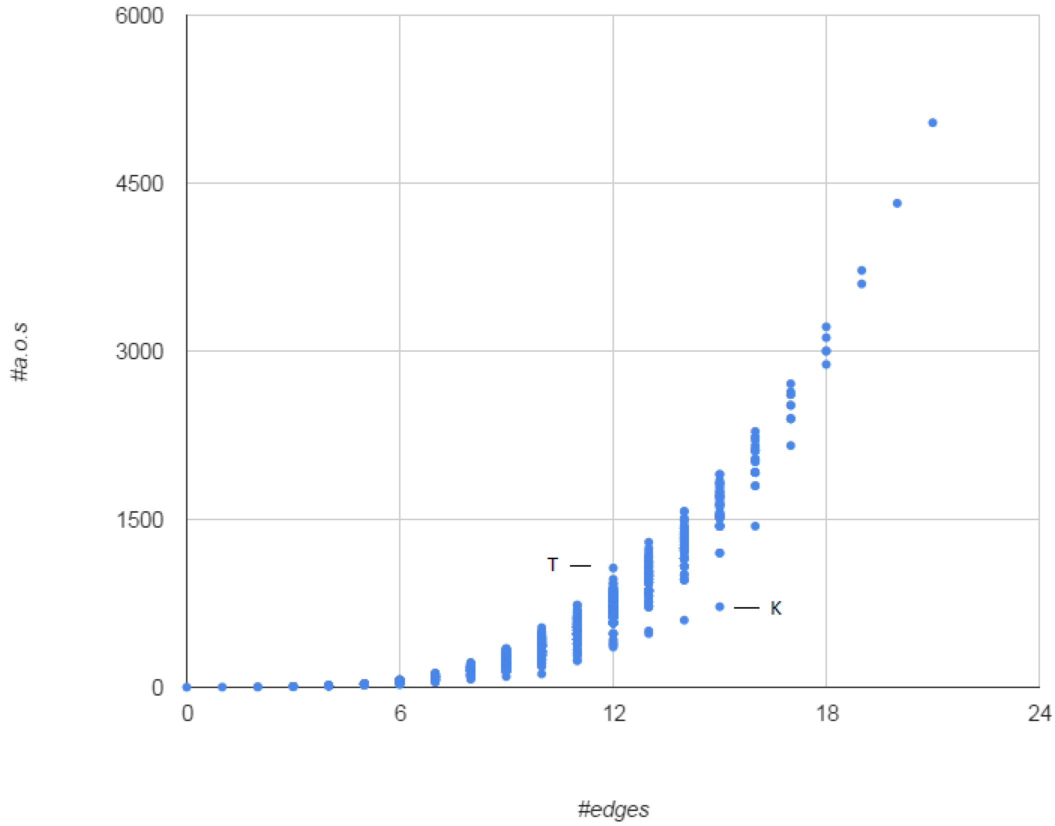

FIGURE 3.11: Distribution of the number of acyclic orientations for each value of m for $n = 7$

for graphs very close to the complete graph, or graphs very close to the empty graph, because here so much of the structure is forced by the density that only the number of edges matters. For example, there is only one graph with exactly $\binom{n}{2} - 1$ edges, and only two graphs with $\binom{n}{2} - 2$ edges. Most of the structure of the graph thus must be forced. We therefore conclude that apart from these extreme examples the structure is indeed the main feature of the graph that we should be investigating. Thus this is where the focus throughout the thesis lies, as it tells us much more then just the number of edges.

3.4 Conclusion

In this chapter we have given a complete set of results for graphs with up to $n = 8$ vertices. We have identified complete graphs as good candidates for the minimal graphs, and Turán graphs as good candidates for maximal graphs. We have further shown that minimal graphs grow from one complete graph to the next. Maximal graphs on the other hand do not, but we have identified an underlying structure which we will explore further in chapter 7.

Further we have identified that the structure of a graph is more important than the number of edges it has in determining the number of acyclic orientations. We have identified some relevant characteristics of maximal graphs, such as a minimal degree condition as well as an edge-connectedness condition.

In addition, the computational results have identified instances for which multiple graphs attain a maximum value as well as multiple graphs attaining a minimum value. Thus we have shown that, for given n, m neither a maximal graph nor a minimal graph are necessarily unique.

On the distribution of acyclic orientations we establish a large overlap of distributions of graphs with a similar number of edges. We note that some extremal graphs are isolated. In the case of the minimal graphs the complete graphs and in the case of the maximal graphs the Turán graphs are particularly isolated, as well as unique.

Chapter 4

Moving to Extremal Graph Parameters

This chapter addresses the minimum number of acyclic orientations and introduces a powerful method for exploring the space of acyclic orientations. We also give insight into the extremal graphs for other parameters. The compression tool will be used again in chapter 7 to find the maximal graphs with respect to the number of acyclic orientations. This chapter is self contained and has been submitted as a paper to the Journal of Graph Theory in a slightly abridged form.

4.1 Introduction

In this chapter we consider graphs with given numbers of vertices and edges, and investigate which of them have the smallest numbers of acyclic orientations or edges in the line graph, or the greatest numbers of cliques or forests. We develop a technique which handles all these parameters, and possibly others.

It is known that calculating the exact values for the number of acyclic orientations and the number of forests is #P-hard [39]. However, we are able to find the extremal graph for these parameters. The result for the number of acyclic orientations was found by Linial [42], based on work by Stanley [58]. Similarly Wood [68] found the extremal graph for maximising the number of cliques in 2007.

In section 4.3, we define an operation called *compression* for reducing graphs to a standard structure. In Section 4.3.2, we classify the *fully compressed graphs*, those which cannot be further compressed using this move. These are *split graphs* (the vertex set is the disjoint union of a clique and an independent set) in which the neighbourhoods of vertices in the independent set are nested. The structure of these graphs allows us to apply a further *consolidation* move in Section 4.4 to obtain a single graph $H_{n,m}$, in which a clique and a set of isolated vertices together contain all but at most one vertex of the graph. We will first show that compression and consolidation terminate, and then in Section 4.5 show that both moves are monotone in several graph parameters. The terminal graph $H_{n,m}$ we obtain for these moves is therefore also extremal for each of the graph parameters.

Compression techniques are used widely in Graph Theory, the most famous and relevant use is in the Kruskal-Katona Theorem, which is closely related to our results on cliques. The power of our method is the unified approach to several parameters.

In their paper "Extremal graphs for homomorphisms" [20], closely related to this chapter, Cutler and Radcliffe use the same compression move as ours to find the graph (with given number of vertices and edges) which has the maximum number of homomorphisms to a path of length 2 with a loop at each vertex. They find that the extremal graph is one of five specific graphs, depending on the values of m and n. By contrast we show that the compression move is monotonic for the number of acyclic orientations, the number of forests and the number of cliques of a graph, and classify the set of fully compressed graphs. We then use consolidation to constructively obtain the graph which minimizes the number of acyclic orientations and maximises the number of cliques simultaneously.

4.2 Definitions

We need some additional definitions for this chapter not given in Chapter 2.

Definition 4.1. The *neighbourhood* of a vertex x, $\mathcal{N}(x)$ is the set of vertices of G that are connected to x by an edge in G, i.e. $\mathcal{N}(x) = \{v \in V(G) | \{x, v\} \in E(G)\}$.

Definition 4.2. Let $\mathcal{O}(G)$ be all orientations of G and $\mathcal{C}(G)$ be the orientations of G that contain at least one cycle respectively.

Definition 4.3. A *Euler subgraph* is a subgraph in which every vertex has even degree.

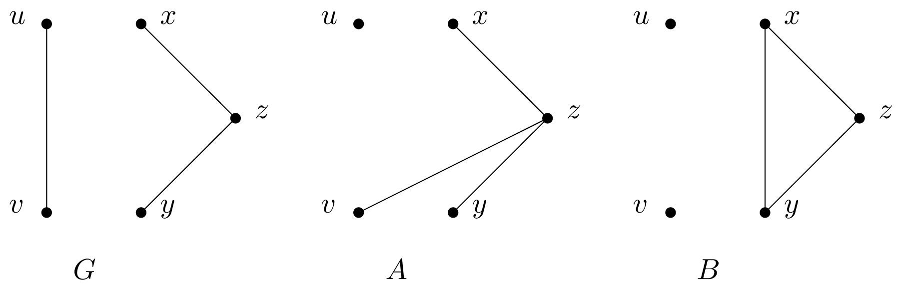

FIGURE 4.1: An example of two different compressions of a graph.

4.3 Compression

Definition 4.4. A *compression* is a function on the set of all graphs. The compression associated with a graph G from x to y for $x, y \in V(G)$, denoted by $C_{xy}(G)$, moves all edges from $\{v, x\}$ to $\{v, y\}$ for $v \in V$ that it is possible to move without creating any double edges, preserving orientation if one exists.

Formally compression constitutes the following mapping on edges in G:

$$\{a,b\} \mapsto \begin{cases} \{y,b\}, & \text{if } a = x, b \neq y, b \notin \mathcal{N}(y). \\ \{a,y\}, & \text{if } b = x, a \neq y, a \notin \mathcal{N}(y). \\ \{a,b\}, & \text{otherwise.} \end{cases}$$

Note that this map may be applied both to a directed and an undirected graph G. Sometimes we will refer to the orientation of the compressed graph as the *compression of an orientation*. Moreover, G when compressed by C_{xy} is isomorphic to G compressed by C_{yx} under the map interchanging x and y and fixing all other vertices. Similarly, we refer to a compression $C_{xy}(G)$ as *isomorphic* if $C_{xy}(G)$ is isomorphic to G. However, the graph we obtain from G after compressing repeatedly is not unique. Consider the following small example.

Example 4.1. *We can compress from G to A in one move, $C_{uz}(G) = A$. From G to B we compress twice, $C_{ux}(C_{vy}(G)) = B$. Neither of these two graphs can be compressed further.*

Note the following interesting result.

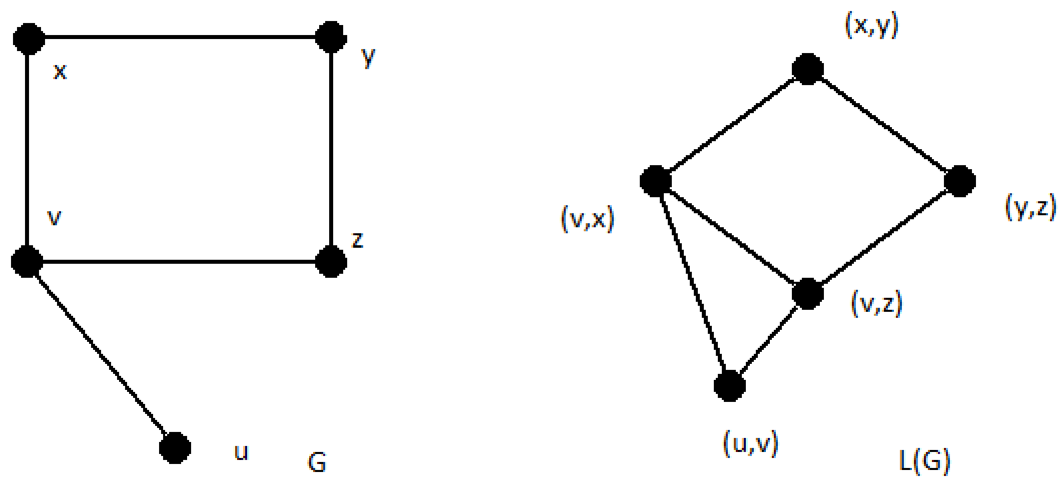

Figure 4.2: An illustration of the line graph transformation of a graph

Lemma 4.5. *The complement of a compressed graph is identical to the compression of the graph's complement, i.e.*

$$C_{xy}(\overline{G}) = \overline{C_{yx}(G)}$$

Proof. Moving all edges that can be moved in one direction is the same as moving all non-edges that can be moved in the opposite direction. □

4.3.1 Partial Order on Graphs imposed by Compression

Definition 4.6. Given a graph G, the *line graph* $L(G)$ is the graph whose vertices are the edges of G, and two vertices x and y in $L(G)$ are connected iff the corresponding edges share a common endpoint in G.

In Figure 4.2 there is an example of obtaining a line graph from a graph. On the left we have G and on the right $L(G)$. Each edge is replaced by a vertex, which here we have labelled in the standard fashion.

We will be interested in the number of edges of the line graph of G, $|E(L(G))|$, which we will simplify to $e_{L(G)}$. We will show that the number $e_{L(G)}$ strictly increases with each compression that gives us a graph that is not isomorphic to the original graph. This in turn implies that compression must terminate, as the number of edges of the line graph of a graph with n vertices and m edges is bounded.

Lemma 4.7. *For a graph G and a non-isomorphic compression $C_{xy}(G)$ we have $e_{L(G)} < e_{L(C_{xy}(G))}$.*

Proof. Let x and y be vertices in G and $H := C_{xy}(G)$ and x' and y' be the vertices x and y in H. By counting each edge in the line graph as a connection between edges sharing a vertex in G we obtain the following formula for $e_{L(G)}$, where $d(v)$ denotes the degree of the vertex v.:

$$e_{L(G)} = \sum_{v \in V(G)} \binom{d(v)}{2}.$$

The function $f(x) = \binom{x}{2}$ is a strictly convex function, and for the compression move we note that $d(x) + d(y) = d(x') + d(y')$. The difference $|d(x) - d(y)|$ is increased by a non-isomorphic compression move, thus we have that

$$\binom{d(x)}{2} + \binom{d(y)}{2} < \binom{d(x')}{2} + \binom{d(y')}{2}.$$

Since $d(v)$ is unchanged for $v \neq x, y$, the result follows from the above expression for the values of $e_{L(G)}$ and $e_{L(H)}$.

□

Corollary 4.8. *The set of all possible compression moves on all possible graphs gives us a partial order on the set of all graphs with n vertices and m edges.*

4.3.2 An extremal set of graphs with respect to consolidation

We see that compression gives us a partial order on the set of all graphs with n vertices and m edges. It is natural to ask which graphs are at the bottom and which at the top of this ordering. We first wish to classify so called *fully compressed graphs* to which we cannot apply any further non-isomorphic compression.

An example of such a partial order is given in Figure 4.3 with a compression mapping indicated by a downward line for graphs up to isomorphism with $n = 5$ and $m = 3$. This is a deliberately simple example which already has three interesting properties. There is no unique minimal or maximal element with respect to compression. The set of minimal

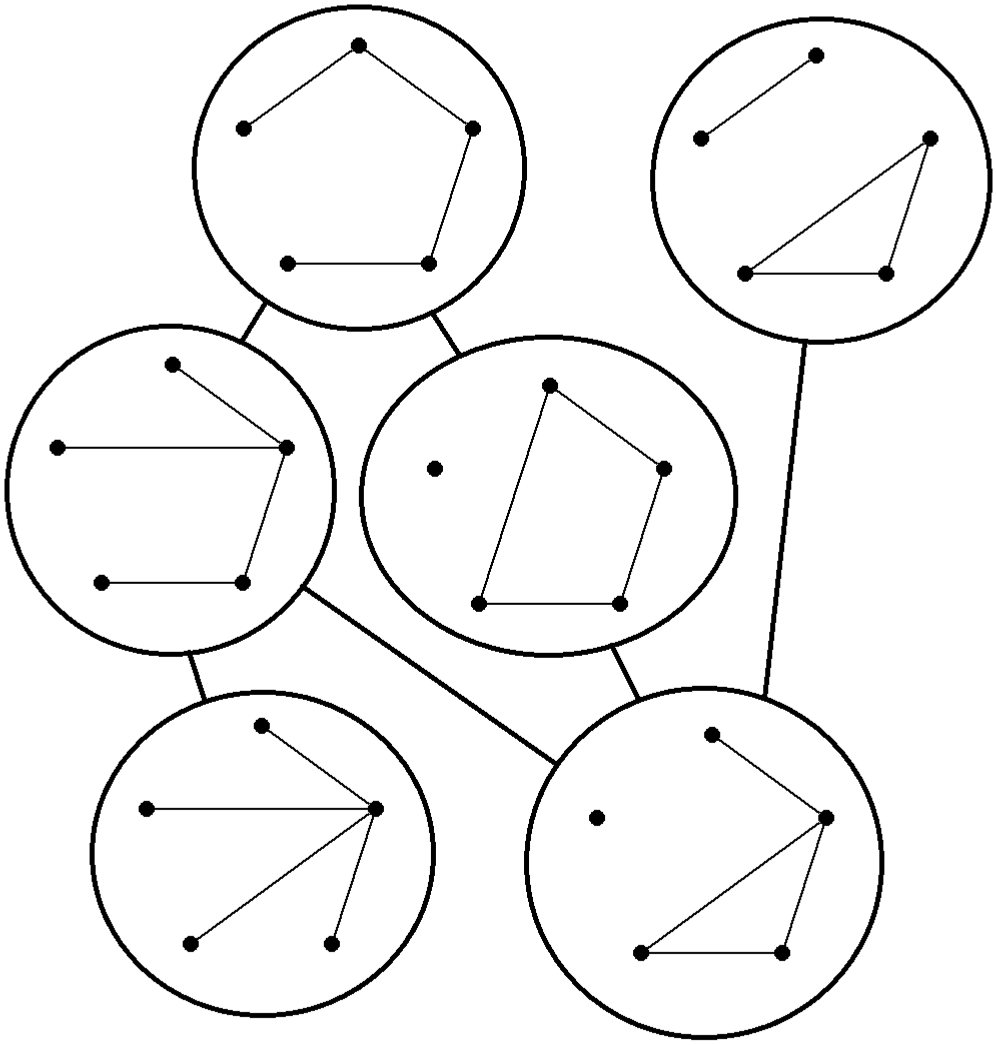

FIGURE 4.3: The Hasse diagram of the partial order given by compression on all graphs with 3 edges and 5 vertices

graphs is the set of fully compressed graphs. Moreover it is possible to arrive at the same fully compressed graph by a number of different routes. It is helpful here to introduce the following definitions.

Definition 4.9. A *split graph* is a graph in which the vertices can be partitioned into a clique, V_k, and an independent set, V_0.

Definition 4.10. Let $H(k, n_0, n_1, \ldots, n_{k-1})$ be the graph with a complete subgraph K_k, and with n_i extra vertices of degree i whose neighbourhoods are in K_k and are nested.

This definition identifies a graph uniquely up to isomorphism. Observe that $n = k + \sum n_i$ and $m = \sum i n_i + \binom{k}{2}$ and that there are n_0 isolated vertices. In Figure 4.4 we have an example of such a graph with $G = H(5, 1, 1, 0, 2)$. We call these graphs *nested split*

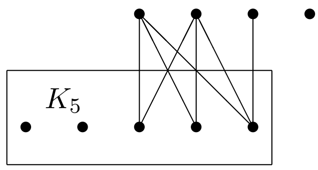

FIGURE 4.4: An example of a nested split graph, omitting edges in K_5.

graphs as they are split graphs, in which the neighbourhoods of the vertices in the independent set are nested.

Note that the nested aspect of the definition may equivalently be stated as the neighbourhoods of the vertices in the clique are nested in the independent set. We denote the set of all nested split graphs with n vertices and m edges by $\mathcal{H}_{n,m}$.

Theorem 4.11. *The set $\mathcal{H}_{n,m}$ is precisely the set of fully compressed graphs with n vertices and m edges.*

Proof. We first show that any graph $H = (V_0 \cup V_k, E) \in \mathcal{H}_{n,m}$ is fully compressed by considering all forms of compression on H. Any compression on H to a vertex in its complete subgraph V_k does not change the graph. Any compression from a vertex in V_k to another vertex in V_0 gives us an isomorphic graph. Any compression from a vertex in V_0 to one in V_0 does not change the graph, or gives us an isomorphic graph. So indeed the graph H is fully compressed.

Now suppose we have a fully compressed graph, G. By definition G gives an isomorphic graph under any compression. It follows that there can only be one component of G that has any edges, else any compression from one component to another reduces the number of components by one and we obtain a non-isomorphic graph. From the cliques of largest size in G pick one with the highest sum of degrees of its vertices. Observe that each vertex not in the clique cannot be completely connected to the clique, else we would have a larger clique. All edges in G are connected directly to this clique, since otherwise a non-isomorphic graph results from compressing one end-point of an edge unconnected to the clique into the clique, as it gives us a clique with a higher sum of degree. Thus G is a split graph. Suppose now that the neighbourhoods of two points not in this clique are not nested. Then a compression from one to the other will give us a non-isomorphic graph, and hence a contradiction. So G is a nested split graph, i.e. $G \in \mathcal{H}_{n,m}$.

□

It is interesting to note that the complement of a (nested) split graph is also a (nested) split graph and that hence the complement of a fully compressed graph is also a fully compressed graph.

Corollary 4.12. *There exists a graph G, with n vertices and m edges, that maximises the number of edges of the line graph of G, in $\mathcal{H}_{n,m}$.*

Proof. By Lemma 4.7 we know that compression increases the number of edges of the line graph, and every graph with n vertices and m edges can be compressed to a graph in $\mathcal{H}_{n,m}$ by Theorem 4.11. □

4.4 Consolidation

We have thus far shown that repeatedly using the compression move results in a graph in the set $\mathcal{H}_{n,m}$. We now introduce a second move, a *consolidation move*, which applies only to graphs in $\mathcal{H}_{n,m}$. This move will allow us to consolidate further to obtain a single graph with n vertices and m edges.

Definition 4.13. The *consolidation* move acts on a nested split graph by removing an edge that is connected to a smallest non-zero degree vertex, and then adding an edge between the complete graph and a largest degree vertex in the independent vertex set.

Note that this operation is well defined. If there are several choices for the removal of an edge or addition of an edge, then it is easy to check that they all give us isomorphic graphs. In the case where we only have one non-zero degree vertex in the independent set of vertices the consolidation moves does nothing (gives us an isomorphic graph).

Lemma 4.14. *Consolidation cannot be repeatedly applied without obtaining isomorphic graphs. The only graphs in $\mathcal{H}_{n,m}$ which are not affected by consolidation are those with at most one vertex connected to the clique.*

Proof. Suppose we are given a graph $H \in \mathcal{H}_{n,m}$. We observe that consolidation can increase the size of the clique K_k in the graph H but not decrease it. Any move that does not increase the size of the clique increases the degree of the largest vertex not in the clique. Thus there can only be a finite number of moves before the clique size is increased, and there is a largest clique for the number of edges in H, so consolidation must terminate. By definition consolidation acts as an isomorphism only if all $m - \binom{k}{2}$ edges not in the clique are incident on an single vertex not in the clique, which leaves us with the unique graph unchanged by consolidation. $\square$

It is possible to obtain a partial order on the set $\mathcal{H}$ using consolidation. This order is similar to (can be extended to) the order that is lexicographic first on the number of isolated vertices and then on the number of vertices of high degrees in increasing degree order.

4.4.1 The extremal graph with respect to consolidation

Now that we have shown that consolidation as well as compression terminates, we are interested in the graph that we obtain at the end of the compression and consolidation process. For $m = \binom{k}{2}$ we define $H_{n,m}$ to be the graph consisting of a clique of size k and $n - k$ isolated vertices. Otherwise $H_{n,m}$ consists of a clique of size k determined by $\binom{k}{2} < m < \binom{k+1}{2}$, one vertex of degree $m - \binom{k}{2}$ connected only to the clique, and $n - k - 1$ isolated vertices.

We give an example here in Figure 4.5 with $n = 7$ and $m = 6$. We show all graphs in the set $\mathcal{H}_{n,m}$, and it is easy to see the consolidation move acting on the graphs from top left to bottom right.

4.5 Application to cliques, forests and acyclic orientations

We have shown that the two transformations, compression followed by consolidation, move us to a unique graph with n vertices and m edges $H_{n,m}$. If for any invariant, we can show demonstrate monotonic behaviour with respect to both these moves, i.e. is increasing or decreasing in value, then the extremal value of that graph parameter is obtained by the graph $H_{n,m}$. If we can further show that the parameter is strictly

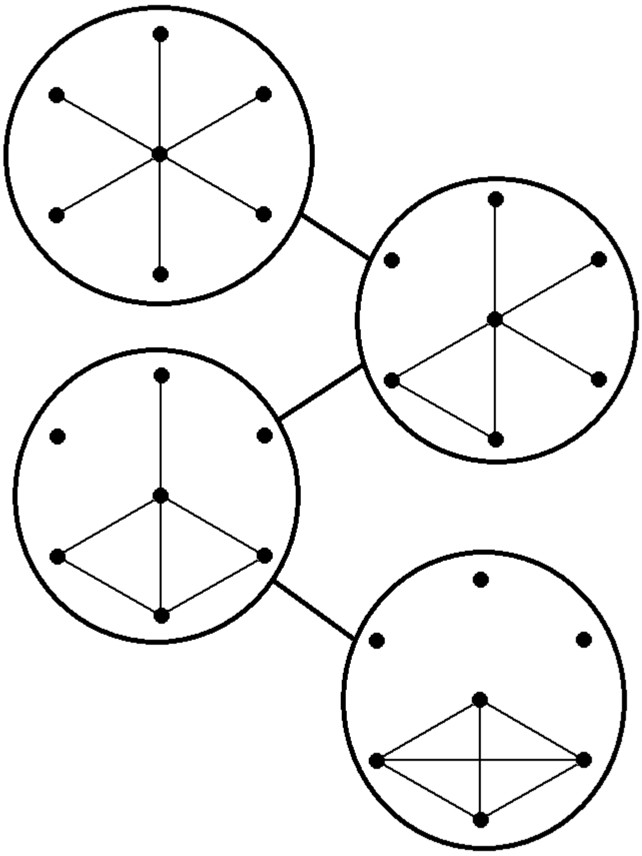

FIGURE 4.5: The Hasse diagram of the ordering given by consolidation on the set $\mathcal{H}_{7,6}$.

increasing or strictly decreasing, then $H_{n,m}$ is the unique extremal graph. We consider the following parameters:

- the number of cliques of G
- the number of forests of G
- the number of acyclic orientations of G.

4.5.1 Monotonicity of the compression move for the number of cliques, Euler subgraphs and acyclic orientations respectively

We have already shown that for the number of edges of the line graph compression is strictly monotone (in Lemma 4.7) and that the extremal graph must lie in $\mathcal{H}$ (see Corollary 4.12). In Lemmas 4.15, 4.19 and 4.26 we shall show that compression is monotone in the number of cliques, Euler subgraphs and acyclic orientations respectively.

The proof for each of the first two is quite short, but proving monotonicity for acyclic orientations is harder and requires several other lemmas.

Lemma 4.15. *Compression cannot decrease the number of cliques in a graph. In fact the number of cliques of every size does not decrease.*

Proof. Take a graph G and vertices $x, y \in G$. It is sufficient to find an injection from the set of cliques of G to the set of cliques of $C_{xy}(G)$. The only cliques in G which can be affected by compression are those that contain x and not y. Take a clique consisting of the vertices $\{a_1, ..., a_k, x\}$ say. If $\{a_i, y\}$ is an edge for all a_i, then this clique is unaffected by compression. Otherwise some edge $\{a_i, x\}$ in G is mapped to $\{a_i, y\}$ by compression and all vertices a_i are connected to y after compression. Thus we can map the original clique in G to the clique containing the vertices $\{a_1, ..., a_k, y\}$ in $C_{xy}(G)$. We map each other clique to itself. No two cliques get mapped to the same clique. Hence we have an injection and we are done. $\square$

Definition 4.16. Given a graph G and two vertices $x, y \in G$ we let $E^{xy}(G) = \{\{u, x\} | u \in \mathcal{N}(x) \backslash \mathcal{N}(y) - \{y\}\}$.

The edges $E^{xy}(G)$ are precisely the edges that get moved by the compression C_{xy}. Thus any cycle in G that gets destroyed by compression must go through such an edge $\{x, u\}$. Note that if G is acyclic then $G - E^{xy}(G)$ is acyclic.

Lemma 4.17. *Given an oriented graph G and two vertices $x, y \in G$. For every vertex $u \in E^{xy}(G)$ for which (u, x) is contained in a directed cycle in G but (u, y) is not contained in a directed cycle in $C_{xy}(G)$, there exists a vertex v such that $uxvy$ forms a directed path in G. Moreover all such paths are oriented in the same direction, from x to y or vice versa.*

Proof. Take such a vertex u. There is a cycle $uxvP$ in G for some vertex v and path P, which is destroyed by compression. Suppose that the vertex v is not connected to y in G. Then the compression moves the edge (x, v) as well as (u, x) and the cycle is preserved. So v must be connected to both x and y. Furthermore, if (y, v) has the same orientation as (x, v), then $uyvP$ is a cycle in $C_{xy}(G)$ contradicting the property of u. So there must exist at least one vertex v as described in the lemma. Suppose that u is not

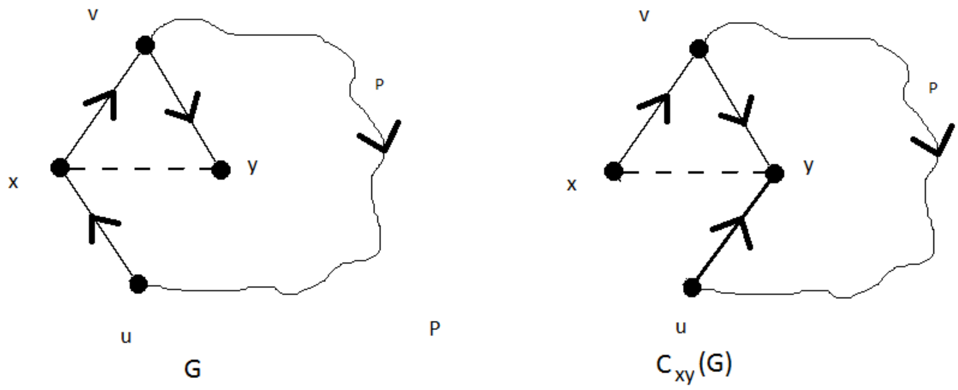

FIGURE 4.6: Illustration of a directed cycle in G that is not mapped to a cycle in $C_{xy}(G)$.

unique, and we have u_1 and u_2 with corresponding paths u_1xv_1y and yv_2xu_2 pointing in different directions. Then we have the cycle xv_1yv_2x in $C_{xy}(G)$, a contradiction. □

Figure 4.6 illustrates the path construction in the first part of Lemma 4.17. The dashed edge may or may not exist in G and all other edges were omitted.

Definition 4.18. Given a graph G, let $\mathcal{S}(G)$ and $\mathcal{F}(G)$ be the number of subgraphs of G and the number of subgraphs of G that are forests respectively.

Lemma 4.19. *Given any graph G and any pair of vertices $x, y \in G$, compression C_{xy} does not increase the number of forests of G.*

Proof. Take a graph G and vertices $x, y \in G$. It is sufficient to find an injection from $\mathcal{F}(C_{xy}(G))$ to $\mathcal{F}(G)$ in order to prove the lemma. Subgraphs of the form $H \in \mathcal{F}(G)$ that are forests in $C_{xy}(G)$ we inject via the inverse of the compression map C_{xy}.

Now take a graph $H \in \mathcal{S}(G) - \mathcal{F}(G)$ for which the compression map gives us a forest H' in $C_{xy}(G)$. From this H we construct a unique $I \in \mathcal{F}(G)$ that maps to a graph I' with a cycle in $C_{xy}(G)$ under compression. This allows us to complete the injection, by injecting from H' to I.

H contains a cycle in G as $H \in \mathcal{S}(G) - \mathcal{F}(G)$. Without loss of generality let this cycle consist of $uxvP$ where P is a path from u to v. This cycle is the only such cycle, hence both u and v are unique. Furthermore, in order for the cycle to be destroyed by compression the edge $\{v, y\} \in G$ exists and the $\{u, y\} \notin G$ doesn't exist, relabelling v

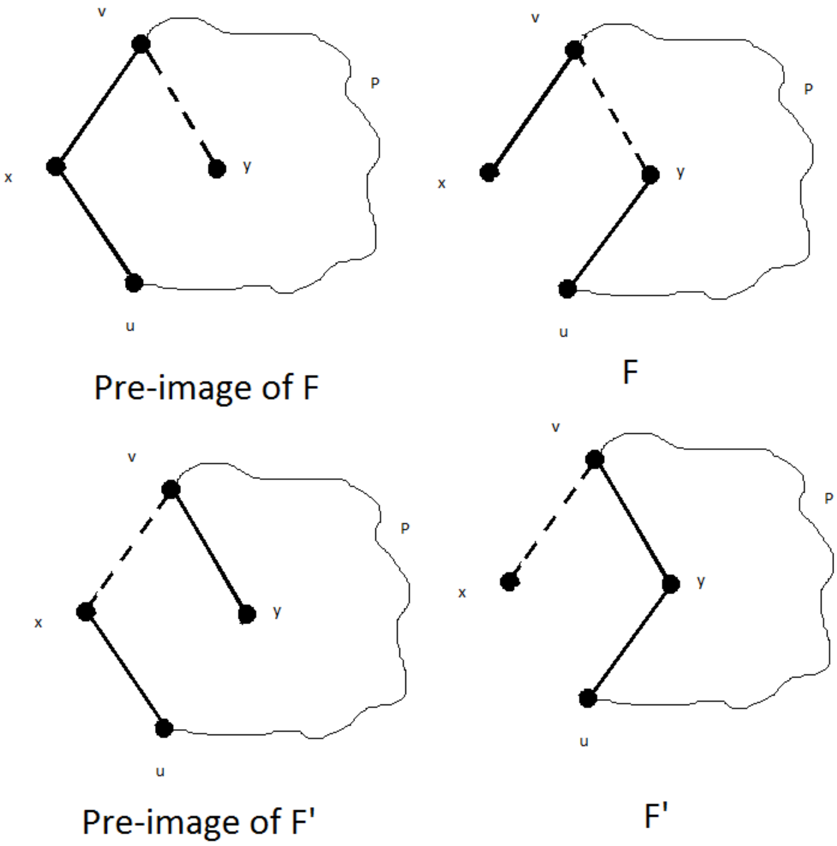

Figure 4.7: The graph F and F' in the proof of Lemma 4.19

and u if necessary. If there is another cycle in H that is destroyed by compression, then by piecing together the two destroyed cycles we obtain a cycle in H', a contradiction to the assumption on H'. Let $I = H - \{v, x\} + \{v, y\}$. I maps to a subgraph I' with cycle $uyvP$ in $C_{xy}(G)$, but is itself a forest. Each of the moves to obtain I from H are reversible, so for $H' \in \mathcal{F}(C_{xy}(G))$ we have now a unique $I \in \mathcal{F}(G)$ completing the injection.

□

In Figure 4.7 we show graphs F and F' and their uncompressed forms. Dashed lines are in G or $C_{xy}(G)$ but not in F or F'. Any other parts of the forest are unchanged or do not matter so are omitted in the drawing. The pre-image of F' in the bottom left becomes the forest that we use to find an injection for the 'bad' forest F.

We now show that a compression does not increase the number of acyclic orientations. The proof requires some lemmas, which aid the construction of an orientation used in the proof of the main result, the Compression Theorem for acyclic orientations Theorem 4.26.

Definition 4.20. A *path switching* $P_{ab}(G)$ in an acyclic graph G is the reversal of direction on every edge that is on a directed path between a and b. This operation is well defined, as each edge is either on a directed path between a and b or it is not. Furthermore this includes switching the direction of the edge between a and b if it exists.

Lemma 4.21 (Path Switching Lemma)**.** *For a given acyclic graph G and any two vertices $a, b \in G$, $P_{ab}(G)$ is acyclic.*

Proof. Suppose otherwise. Then there is a graph G with orientation θ which is acyclic, while $P_{ab}(G)$ contains a cycle. Take a cycle in $P_{ab}(G)$. It must have a proper subpath which has been path-switched, else G would not be acyclic. Hence the cycle can be viewed as a sequence of subpaths, $Q_1, P_1, Q_2, P_2, ...$ pointing in alternating directions in G, where the Q_j are path switched in $P_{ab}(G)$. For every edge e in a path segment Q_j there must exist a path from one end to a, and from the other end to b. All of these paths must point in the same direction from a to b, else there would be a cycle in G. Thus there exists a path in G from a to b containing Q_j, $A_jQ_jB_j$ say for each Q_j, $q \geq 1$. If there exists a Q_j for $j > 1$, then we note that $A_1P_1B_2$ is a path in G from a to b containing P_1, a contradiction since P_1 is not switched by P_{ab}. Thus, the cycle in $P_{ab}(G)$ is of the form Q_1P_1. But then $A_1P_1B_1$ is a path from a to b containing P_1 providing the required contradiction. □

We now use path switching and the set of edges E^{xy} to define a map D.

Definition 4.22. Given a graph G with orientation θ and vertices $x, y \in G$ a *restricted path switching* $D_{xy}(G)$ acts as follows: For θ such that $G - E^{xy}$ is acyclic, D removes the edges moved by compression, applies path-switching and then puts the edges back in, and is the identity map otherwise, i.e.

$$D_{xy}(G, \theta) = \begin{cases} P_{xy}(G - E^{xy}, \theta) \cup E^{xy}, & \text{if } (G - E^{xy}, \theta) \text{ is acyclic} \\ (G, \theta), & \text{otherwise.} \end{cases}$$

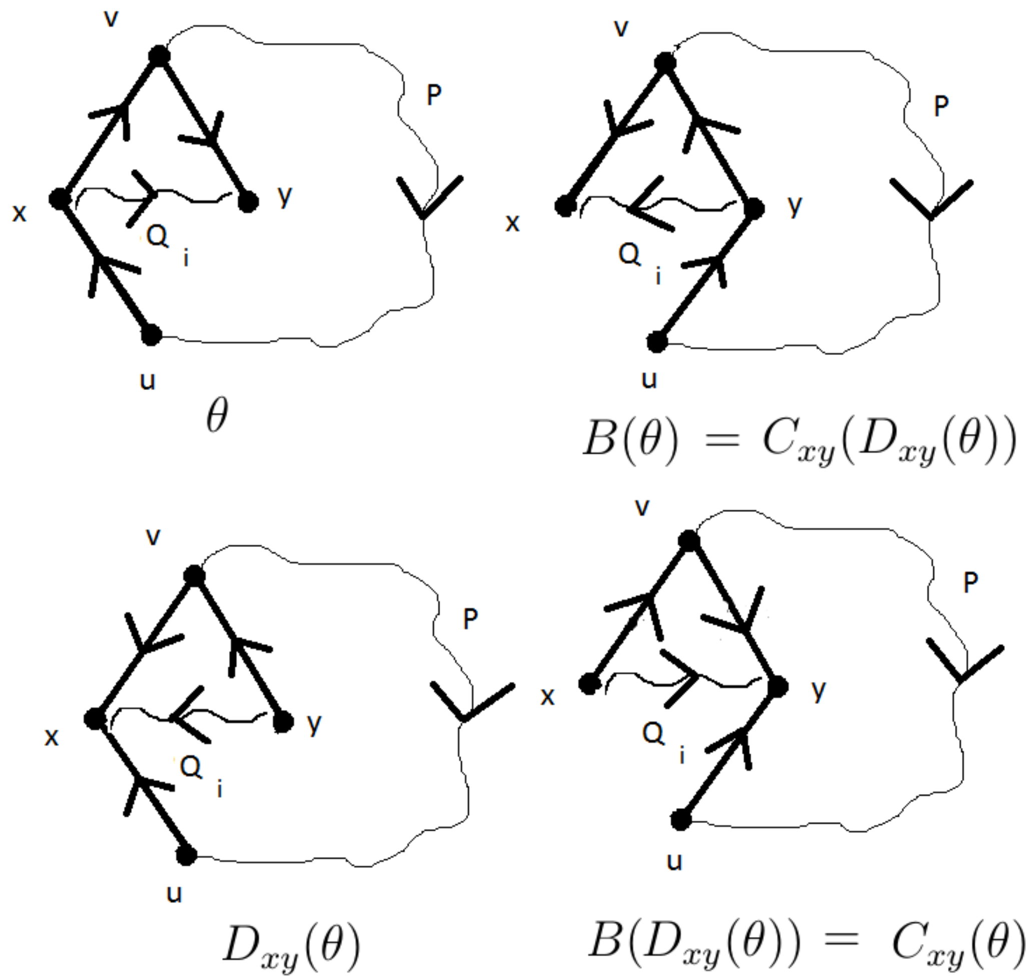

FIGURE 4.8: The action of map B acting on the pair of acyclic orientations θ and $D_{xy}(\theta)$, when $\theta \in \mathcal{C}(G)$ and $C_{xy}(\theta) \notin \mathcal{C}(C_{xy}(G))$

Note that D_{xy} is invertible and is its own inverse.

Definition 4.23. Given a graph G with orientation θ and vertices $x, y \in G$ we define the map $B_{xy}(G, \theta)$ as follows:

$$B_{xy}(G,\theta) = \begin{cases} C_{xy}(D_{xy}(G,\theta)), & \text{if } \theta \in \mathcal{C}(G) \text{ and } C_{xy}(G,\theta) \notin \mathcal{C}(C_{xy}(G)) \\ C_{xy}(D_{xy}(G,\theta)), & \text{if } D_{xy}(G,\theta) \in \mathcal{C}(G) \text{ and } C_{xy}(D_{xy}(G,\theta)) \notin \mathcal{C}(C_{xy}(G)) \\ C_{xy}(G,\theta), & \text{otherwise.} \end{cases}$$

Note that for $(G, \theta') = D_{xy}(G, \theta)$ we have $B(G, \theta) = C_{xy}(G, \theta')$ and $B(G, \theta') = C_{xy}(G, \theta)$. The map B can be seen in Figure 4.8 acting on such a pair (G, θ) and $D_{xy}(G, \theta)$. This shows us intuitively that the map B is a bijection, we now give a formal proof.

Lemma 4.24. *For given G, $x, y \in G$ the map B_{xy} is a bijection between the orientations of G and the orientations of $\mathcal{C}_{xy}(G)$.*

Proof. $|\mathcal{O}(G)| = |\mathcal{O}(C_{xy}(G)|$, so it is sufficient to prove that the map B_{xy} is an surjection. Consider the orientations of G in pairs, (G, θ) and $D_{xy}(G, \theta)$, allowing for cases where $(G, \theta) = D_{xy}(G, \theta)$. Then B_{xy} acts either as C_{xy} or as $C_{xy} \circ D_{xy}$ on both orientations in the pair, by construction. Thus, since D_{xy} is its own inverse, $\{B_{xy}(G, \theta), B_{xy}(D_{xy}(G, \theta)\} = \{C_{xy}(G, \theta), C_{xy}(D_{xy}(G, \theta))\}$. The map C_{xy} is a surjection and the image is the same as the image of B_{xy} so we have a surjection. □

Lemma 4.25. *The map B is an injection of orientations with cycles to orientations with cycles, and a surjection of acyclic orientations to acyclic orientations.*

Proof. If $(G, \theta) \in \mathcal{C}(G)$ and $C_{xy}(G, \theta) \notin \mathcal{C}(C_{xy}(G))$ then $B_{xy}(G, \theta) = C_{xy}(D_{xy}(G, \theta))$, which contains the cycle $uyvP$, with u, v, P as constructed in Lemma 4.17.

If $(G, \theta) \in \mathcal{C}(G)$ and $B_{xy}(G, \theta) \in \mathcal{C}(C_{xy}(G))$, then $C_{xy}(G, \theta)$ (and by symmetry $C_{xy}(B_{xy}(G, \theta)))$ $\in \mathcal{C}(C_{xy}(G))$, as the construction in Lemma 4.17 required for $C_{xy}(G, \theta)$ to be acyclic would also imply $B_{xy}(G, \theta) \notin \mathcal{C}(C_{xy}(G))$.

Now we have an injection of orientations with cycles to orientations with cycles, thus it follows from Lemma 4.24 that we must have a surjection of acyclic orientations to acyclic orientations. □

Theorem 4.26 (Compression Theorem for acyclic orientations)**.** *For any graph G, and any $x, y \in V(G)$, applying C_{xy} to G cannot increase the number of acyclic orientations, i.e.*

$$a(G) \geq a(C_{xy}(G)).$$

Proof. In Lemma 4.25 we have shown that we have a surjection of acyclic orientations from G to $C_{xy}(G)$ in map B. Thus the number of acyclic orientations cannot have increased. □

4.5.2 Monotonicity of the consolidation move for the number of cliques and acyclic orientations

Lemma 4.27. *The number of cliques is strictly increased by consolidation.*

Proof. Suppose we are given a fully compressed graph G, and after one consolidation move on G we obtain a non isomorphic graph H. We simply find an injection from the cliques of G to the cliques of H and show that one new clique exists in H. The only cliques affected are the cliques through the edge that is moved. Suppose the vertices of the clique are $\{a, b, x_1, ..., x_l\}$ and that the edge $\{a, b\}$ got moved to $\{c, d\}$ where a and c are not in the clique respectively. Then we simply inject to the clique $\{c, d, x_1, ..., x_l\}$. Now we have injected all cliques, but more is true. There exists a new clique containing the edge $\}c, d\}$ which has size of the neighbourhood of c, which did not exist previously, so in fact we have a strict increase.

□

Proposition 4.28. *The number of acyclic orientations of a graph $G \in \mathcal{H}$ with a maximal clique of size k and n_i vertices of degree i not in the clique is*

$$a(G) = k! \cdot \prod_{i=0}^{k-1} (i+1)^{n_i}.$$

Proof. Observe that an acyclic orientation of a graph defines an ordering on the vertices of the graph and vice versa. Thus for a clique we have a bijection between the acyclic orientations of the clique and the permutations of its vertices. First we order all the vertices in the k-clique and obtain an acyclic orientations of the clique in $k!$ ways. Then any vertex x not in the clique can be placed in $d(x)+1$ positions in the ordering amongst its neighbours. Each one of these leads to a unique extension of the acyclic orientation since x and $\mathcal{N}(x)$ form a clique. Each such extension is independent of the others, which gives us:

$$a(G) = k! \cdot \prod_{x \notin K_k} (d(x)+1) \tag{4.1}$$

$$= k! \cdot \prod_{i=0}^{k-1} (i+1)^{n_i}.$$

□

Corollary 4.29. *The number of acyclic orientations is strictly decreased by consolidation.*

Proof. Each non-trivial consolidation move replaces two factors in expression (4.1) $s+1$ and $l+1$ by two new factors s and $l+2$. Since $l \geq s$ we have a strict decrease in the number of acyclic orientations following consolidation. □

Note that the number of edges of the line graph of G is not monotone decreasing with respect to consolidation. This is easiest seen by considering a star, which maximises the number of edges in the line graph since any two edges in G share a common vertex. Consolidation takes us away from the star, reducing the number of edges in the line graph.

4.5.3 Extremal Graphs with respect to the number of acyclic orientations and the number of cliques

We would now like to show that the extremal graph is unique for both the number of acyclic orientations and the number of cliques, except for a special case.

Definition 4.30. For any n and $k \leq n-2$, the graph $F_{n,k}$ is the disjoint union of a complete graph K_k, one edge and $n-k-2$ isolated vertices.

Observe that $F_{n,k}$ has n vertices and $m = \binom{k}{2} + 1$ edges. The corresponding $H_{n,m}$ is a complete graph of size k with a single edge attached to it and $n-k-1$ isolated vertices. It is possible to compress from $F_{n,k}$ to $H_{n,m}$ in a single compression move (which is the only non-isomorphic move).

Theorem 4.31. *For graphs with n vertices and m edges, the number of cliques is maximised and the number of acyclic orientations is minimised by graph of the following form: $H_{n,m}$ for all m, $F_{n,k}$ when $m = \binom{k}{2} + 1$ for some integer k and no other graphs.*

Proof. Given n, m, the extremal values of cliques and acyclic orientations is achieved within $\mathcal{H}$ uniquely by $H_{n,m}$ by Theorem 4.11 Lemma 4.14, Lemma 4.27 and Corollary 4.29. Note that $F_{n,k}$ has the same number of acyclic orientations and cliques as $H_{n,\binom{k}{2}+1}$.

Thus we need only show that the last compression move that takes us to $H_{n,m}$ or to $F_{n,m}$ strictly increases or strictly decreases the parameters respectively.

Take a graph G and vertices x and y for which G is not in $\mathcal{H}$ but $C_{xy}(G) = H_{n,m}$. The graph $H_{n,m}$ is $H(k, n-k-1, 0, \dots, 0, n_{m-\binom{k}{2}} = 1, 0, \dots, 0)$ for k determined by $\binom{k}{2} < m < \binom{k+1}{2}$ and K_k together with a set of $n-k$ isolated vertices for values of k such that $\binom{k}{2} = m$. Let V_0 be the set of isolated vertices and V_k be the set of vertices in the clique. Finally let w be the vertex of degree $m - \binom{k}{2}$ connected only to the clique for the values of n, m such that it exists.

Observe that $y \notin V_0$, as there are no edges in $C_{xy}(G)$ going to the set V_0. In addition, $x \notin V_k$, as a compression to another vertex in V_k would have been an isomorphism, and a compression to a vertex in V_0 or to vertex w would have moved some other edges too, a contradiction. We are left with the following cases, which we need to check for both graph parameters:

1. $x \in V_0$, $y = w$ for k such that $\binom{k}{2} < m < \binom{k+1}{2}$
2. $x \in V_0$, $y \in V_k$
3. $x = w$, $y \in V_k$ for k such that $\binom{k}{2} < m < \binom{k+1}{2}$.

If no cycles or cliques are destroyed by compression, it is sufficient to show the existence of a new clique for a strict increase in the number of cliques. Furthermore this shows a strict decrease in the number of acyclic orientations, as the new clique adds an extra restriction to the orientations.

Move (1) does not destroy any cycles or cliques and creates a new clique of size $d(w)+1$ in $C_{xy}(G)$. Note that the move that creates an isomorphic graph we exclude by having insisted that G is not in $\mathcal{H}$.

For move (2) four possible things can happen. First, the move creates an isomorphic graph, which we have excluded. Second, the edge $\{w, x\}$ is moved to $\{w, y\}$. Here we note that we have a new clique of size $d(w)+1$ containing the vertices $w \cup \mathcal{N}(w)$ unless $d(w) = 1$ in $C_{xy}(G)$. Third, at least one edge was moved from $\{z, x\}$ to $\{z, y\}$ for $z \in V_k$, which creates the clique V_k that is not in G. Finally, if $d(w) = 1$ in $C_{xy}(G)$, and we have not created the clique V_k as in the third case, then $G = F_{n,k}$, which we also know is extremal.

For move (3) to be a non-isomorphic move there must exist a vertex $z_1 \in V_k$ such that $\{z_1, w\} \in E(G)$ but $\{z_1, y\} \notin E(G)$. Since we have k such that $\binom{k}{2} < m < \binom{k+1}{2}$, we deduce that $d(w) \geq 1$ in $C_{wy}(G)$. Thus, there must also exist a vertex $z_2 \in V_k$ such that $\{z_2, w\}, \{z_2, y\} \in E(G)$. If all vertices in the clique connected to y in G are also connected to w, then the compression $C_{wy}(G)$ is an isomorphism which swaps the labels of w and y. Hence there exists a vertex $z_3 \in V_k$, $\{z_3, y\} \in E(G), \{z_3, w\} \notin E(G)$.

Thus for a compression move of type (3) to be non-isomorphic, the vertices in V_k form a clique of size k in $C_{xy}(G)$ but not in G, since $\{z_1, y\} \notin E(G)$. So by Lemma 4.15 we have a strict increase in the number of cliques.

To show a strict decrease in the number of acyclic orientations we need only find an acyclic orientation in $C_{wy}(G)$ that is mapped to an orientation with a cycle by the map B_{wy}, by Corollary 4.25. Consider the acyclic orientation θ defined by the vertex ordering in which the first five vertices are z_2, y, z_3, z_1, w in that order and all other vertices are after this in any order. If the edge $(w, y) \notin G$ then $D_{wy}(G, \theta)$ is the same as θ and thus $B_{wy}(G, \theta) = C_{wy}(G, \theta)$ which contains the cycle yz_3z_1. On the other hand, if $(w, y) \in G$, then both $C_{wy}(G, \theta)$ and $C_{wy}(D_{wy}(G, \theta))$ contain the cycle yz_3z_1. Thus $B_{wy}(G, \theta)$ contains a cycle.

Consider a graph G and a compression for which $C_{xy}(G) = F_{n,k}$. If compression C_{xy} involves the isolated edge in $F_{n,k}$ then this edge is of the form $\{v, y\}$ and thus $\{v, x\}$ is an isolated edge in G, hence $G \cong F_{n,k}$. Now we can ignore this edge and are left with the graph $H_{n,\binom{k}{2}}$.

□

4.6 Other parameters

Some other graph parameters, such as the number of connected components and the number of Euler subgraphs, are also maximised by $H_{n,m}$, but it is not a unique maximising graph. We give brief arguments for this, independent of the main thrust of the chapter.

Proposition 4.32. *Given n and m with $m \le \binom{n}{2}$, let k be the largest integer such that $m > \binom{k}{2}$. If $m \neq \binom{k}{2} + 1$, then the number of connected components of a graph with n vertices and m edges is maximised by precisely those graphs with a single component of size $k+1$ and $n-k-1$ isolated vertices.*

If $m = \binom{k}{2} + 1$, then the maximising graph are all of the form of a complete graph K_k, one further edge (which may or may not be connected to the complete graph) isolated vertices (if any). Thus the maximum number of connected components c is $c = n - k$.

Proof. Suppose that G has the maximum number of components. If G has more than two connected components components of size $l, l' \ge 3$ then the edges in connected components of sizes l and l' can be fitted inside a component of size $l + l' - 2$, increasing the number of components by 1, as $\binom{l}{2} + \binom{l'}{2} \le \binom{l+l'-2}{2}$ for $l, l' \ge 3$. Thus the maximum is achieved by a graph G with (at most) one component H on more than two vertices and some isolated edges. But two isolated edges can be replaced by two edges joining a single vertex to a component of size at least 2, again reducing the number of components; so there is at most one isolated edge. If $m \neq \binom{k+1}{2}$, then H has a non-edge which can be replaced by an edge and the isolated edge deleted. $\square$

Note that the graphs in the set $\mathcal{H}$ are of the form described in Proposition 4.32, which maximise the number of components. Thus $H_{n,m}$ in particular maximises the number of components of a graph with n vertices and m edges. The same conclusion holds for Euler subgraphs since the number of these is a monotonic function of the number of connected components, as we now show.

Proposition 4.33. *A graph G with n vertices, m edges and c connected components has 2^{m-n+c} Euler subgraphs.*

Proof. A subgraph of G can be represented by its characteristic function, a vector in $\mathbb{Z}_2^m$. The subgraph is Eulerian if and only if the vector is orthogonal to all vectors of vertex stars in G. There are n vertex stars, satisfying c independent linear relations (the binary sum of the stars of all vertices in a connected component is zero). So the Euler subgraphs form a subspace of dimension $m - n + c$. $\square$

4.7 Further directions

There are a number of other graph parameters which are related to those we have considered here. For example, an orientation of a graph is *totally cyclic* if it has the property that each edge lies in a directed cycle. A graph has a totally cyclic orientation if and only if it is bridgeless. Similarly, there is a lot of interest in the number of *spanning trees* of a graph; this is non-zero if and only if the graph is connected.

The number of totally cyclic orientations is not monotonic under compression. If we start with a 5-cycle, compress to a 4-cycle with a pendant edge, and compress again to a 4-cycle with a chord, the number of totally cyclic orientations goes from 2 to 0 and then to 6.

The number of acyclic orientations, the number of totally cyclic orientations and the number of spanning trees of G are all evaluations of the *Tutte polynomial* T_G, a two-variable polynomial introduced by Tutte in 1947 [61]. The Tutte polynomial has a number of specializations of interest, for example

- $T(2, 0)$ is the number of acyclic orientations [58];
- $T(1, 1)$ is the number of spanning trees;
- $T(0, 2)$ is the number of totally cyclic orientations [62];
- $T(1, 2)$ is the number of spanning subgraphs.

See Sokal [56] for a survey.

The number of spanning trees is also one of a class of parameters depending on the Laplacian eigenvalues of a graph, which are used in statistical design theory as measures of the optimality of block designs; a design is D-optimal if it maximizes the number of spanning trees in its concurrence graph. Related concepts are A-optimality (minimizing the average resistance between vertices, when the edges of the graph are one-ohm resistors), and E-optimality (which is connected with isoperimetric number). See [4] for a survey.

There are some problems in applying our methods to these parameters, apart from the non-monotonicity mentioned above. First, several of them (such as numbers of totally cyclic orientations and spanning trees) are zero for disconnected graphs, so the minimization problem is trivial. For these and other parameters, including the ones we treated in this chapter, we could pose the question:

Question 4.34. *Which graphs realize the extremal values of the parameters in question if we restrict to* connected *graphs?*

The big problem is that, for many of these parameters, the extremum in the other direction is considerably more interesting. Which graphs maximize the number of acyclic orientations, or of spanning trees, or of totally cyclic orientations, or minimize the average resistance between pairs of vertices? These may not be so easy to deal with, but we could ask:

Question 4.35. *Is there an "anti-compression" move which finds the graphs at the opposite extreme?*

Chapter 5

The Factor Method

In this chapter we present a new approach using factors to count the number of acyclic orientations of a graph. We define the factor method used in Section 5.2. We then show some properties of the factor method in Section 5.3, as well as apply the factor method and properties to some graphs in Section 5.4. We then try to apply the factor method in order to obtain some meaningful bounds to the number of acyclic orientations in Section 5.5.

There are two reasons to use the factor method described in this chapter to count the number of acyclic orientations of a graph instead of the usual approach we have described in Chapter 2 using Stanley's formula. First, it may be possible to use the factor method to obtain bounds/estimates for the number of acyclic orientations of a graph relatively easily. Second the method leads to a number of simple but powerful observations, which we will use in Chapter 7 to establish an edge-connectedness property of maximal graphs.

5.1 Comparison of Algorithm

In Stanley's method of counting the number of acyclic orientations outlined in Section 2.3, one inductively calculates the chromatic polynomial of a graph using Proposition A.3. Unfortunately calculating the chromatic polynomial of a graph requires calculating the chromatic polynomial of two smaller graphs and the total iterative process is exponential, as illustrated in Figure 2.1 for the graph K_4 minus an edge. In this example we start with a $K_4 - \{e\}$, and using the deletion contraction relation, after 14 non-trivial

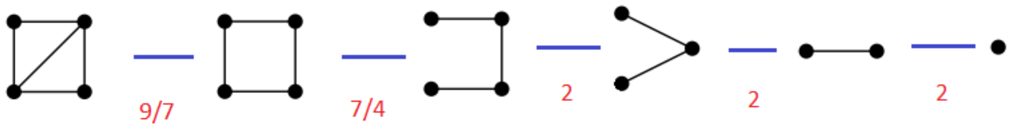

FIGURE 5.1: Illustration of the factor method algorithm for graph $K_4 - e$.

contraction moves (dotted blue lines) and 6 non-trivial deletion moves (blue lines) we obtain the chromatic polynomial as $\chi(G) = \lambda(\lambda-1)(\lambda-2)^2$. The last step is to insert $\lambda = -1$ into $\lambda(\lambda-1)(\lambda-2)^2$ to obtain 18. You can gather from Figure 2.1 and the branching process that calculating the chromatic polynomial is a computationally complex problem as we have already noted in Section 3.1.2.

I wish to go about things differently. We do not want to obtain the chromatic polynomial, only the information that is relevant for acyclic orientations. We assign an order to the edges (cleverly), and we use this order to remove edges from the graph one at a time. At each stage, before removing the designated edge we calculate the factor increase of the reduced graph attributable to this selected edge. Then we remove this edge leaving us with one graph, rather than two as in Stanley's method. A visualization of this algorithm on the same graph $K_4 - \{e\}$ is given in Figure 5.1, with factors $\frac{9}{7}, \frac{7}{4}, 2, 2, 2$. Multiplying out these factors gives us the total number of acyclic orientations as 18 as before.

Now if we can make finding the factor at each step as simple as possible, or even bounding the factor, we will have either a more efficient algorithm, or at least a way of finding bounds for the number of acyclic orientations of the graph. There is a limit on how efficient an algorithm can be (See Section 2.4.1), since the evaluation of $\chi(G, -1)$ is #P-hard. This means that under some widely believed hypotheses about complexity classes, no polynomial-time algorithm for the exact result can exist. Instead we see where we can use the factor method in order to compute the number of acyclic orientations of special cases. It is not known for $k = -1$, whether or not it is hard to approximate $\chi(G, -1)$, so we will be also interested in using the factor method to find bounds on the number of acyclic orientations a graph can have.

We will also use some of the methods here to provide alternate proofs to some results on the minimum number of acyclic orientations in Chapter 4, and apply the factor method to the set of cycles, complete graphs, and the set of split graphs. Even for simple graphs such as the complete graph, the factor method gives insight into the contribution of an individual edge to the total number of acyclic orientations, which other methods do not give.

5.2 Definition of the Factor

First note that adding an extra isolated vertex to a graph does not change the number of acyclic orientations, so we are just interested in edges. Consider adding an edge to a graph. In every *specific* acyclic orientation of the graph the additional edge either has a choice of two orientations or else has its direction fixed. Calculating the factor of increase in the number of acyclic orientations is just finding the average of all the 2's and 1's for every acyclic orientation. We will now provide an equivalent formal definition of the factor. We may use either, depending on which is easier to apply. It is convenient here to denote by $\overline{E}$ the missing edges in the graph $G = (V, E)$.

Definition 5.1. Given a graph $G = (V, E)$ and an $e \in \overline{E}$, we define the *factor* $f_G(e)$ as $f_G(e) = \frac{a(G+e)}{a(G)}$, i.e. the ratio between the number of acyclic orientations in the graph $G + e$ and the original graph G.

We can thus view the factor on the edge to be added $e \in \overline{E}(G)$ as the average number of acyclic orientations spawned from each of the existing acyclic orientations of G. Note that every acyclic orientation θ of G may be extended to an acyclic orientation of $G+e$ in at least one way. Suppose $e = \{a, b\}$, and that θ has no directed paths between a and b, then adding edge e with either orientation cannot create a cycle. If on the other hand θ has directed paths between a and b, they must all be in the same direction between a and b (else we have a cycle), so we give the edge e the same orientation as these paths, and do not create a cycle. For any given acyclic orientation θ of G, if the edge e has a forced direction in order for the orientation to remain acyclic, then the contribution is a 1, if both directions lead to an acyclic orientation then the contribution is a 2.

If we let $b(G, e)$ be the number of acyclic orientations of G which fix the direction of e and $c(G, e)$ be those for which both directions of e lead to an acyclic orientation, then noting that $a(G) = b(G,e) + c(G,e)$ we obtain

$$f_G(e) = \frac{b(G,e) + 2c(G,e)}{a(G)} = 1 + \frac{c(G,e)}{a(G)}$$

and similarly

$$f_G(e) = \frac{b(G,e) + 2c(G,e)}{a(G)} = 2 - \frac{b(G,e)}{a(G)}.$$

Note that we look at the factor on edge e when we add e last. For simplicity when it is clear to which graph we are adding the edge we simplify the notation to $f(e)$. The factor concept may be generalized from adding one edge to adding many edges, we define $f_G(E') = \frac{a(G+E')}{a(G)}$ where $E(G) \cap E' = \emptyset$. Thus, for example

$$f_G(\{e_1, e_2\}) = \frac{a(G + e_1 + e_2)}{a(G)} = \frac{a(G + e_1 + e_2)}{a(G + e_1)} \cdot \frac{a(G + e_1)}{a(G)} = f_G(e_1) f_G(e_2)$$

Indeed if we add all the edges of a graph in an index order, $e_1, ..., e_m$ say, and multiply the factors together we obtain

$$\frac{a(G - e_2...e_m)}{a(G - e_1...e_m)} \cdot \frac{a(G - e_3...e_m)}{a(G - e_2...e_m)} \cdot ... \cdot \frac{a(G - e_m)}{a(G - e_{m-1} - e_m)} \cdot \frac{a(G)}{a(G - e_m)} = a(G)$$

as expected, since $a(G - e_1...e_m) = 1$ as this is the graph with no edges.

5.3 Properties of the Factor

Lemma 5.2. *Suppose we are given a graph G and we wish to add an edge $e \in \overline{E}$. Then $f_G(e) \leq 2$, furthermore $f_G(e) = 2$ if and only if e is a bridge.*

Proof. The factor at an edge is an average of 1's and 2's. Therefore the maximum factor is 2.

Suppose e is not a bridge. Then there exists a path from one end of e to the other. Number the vertices in the path in order, and orient the edges accordingly. Orient all other edges, s.t. G is acyclic (e.g. pick an ordering for the remaining vertices, and orient the edges accordingly). Now in this specific orientation the orientation of e is fixed in order for G to remain acyclic, and thus this specific factor is 1. Therefore the average factor over all orientations cannot be 2.

Now suppose e is a bridge. Then there can be no possible directed cycle going through e, as there is no cycle containing e. Thus both directions can be assigned to e for every acyclic orientation of G, and we get a factor of 2. □

As the factor is an average of 1's and 2's, finding an orientation that has a contribution of 2 for a given edge shows that the average cannot be 1. We will do so for any edge in any graph, proving that no edge can have a factor of 1.

Lemma 5.3. *An edge e with $f_G(e) = 1$ is not possible.*

Proof. Given a graph G, take an edge $e = (u, v) \in \overline{E}$ that we will add to G. Label the vertices from 1 to n, beginning with u and v. Orient the graph from smaller labelled vertex to larger labelled vertex to obtain an acyclic orientation of G. This acyclic orientation allows both directions for e, thus the contribution to the factor is 2 for this specific orientation, and thus the average cannot be 1. □

We may also prove this using Stanley's method - though with less insight.

Alternate Proof. Take any graph G and any edge e in G. We consider the factor at e when added to $G - e$. We simply apply Stanley's formula to obtain the following.

$$a(G) = \overline{\chi}(G, -1) = \overline{\chi}(G - e, -1) + \overline{\chi}(G/e, -1) = a(G - e) + a(G/e)$$

The last term $a(G/e)$ is always at least 1, as it counts the number of a.o.'s of G/e (note that the one vertex graph has one acyclic orientation!). Thus the increase in acyclic orientations when we add an edge is always at least 1, i.e. $a(G) > a(G - e)$, therefore the factor is always greater than 1. □

We can now bound the factors.

Lemma 5.4. *The factors are rational numbers in the interval* $(1, 2]$.

Proof. The upper bound for the interval follows from Lemma 5.2 and the lower bound from Lemma 5.3. As the factor is always just $\frac{a(G+e)}{a(G)}$ and both of these are non-zero natural numbers, the factor must be a rational number. □

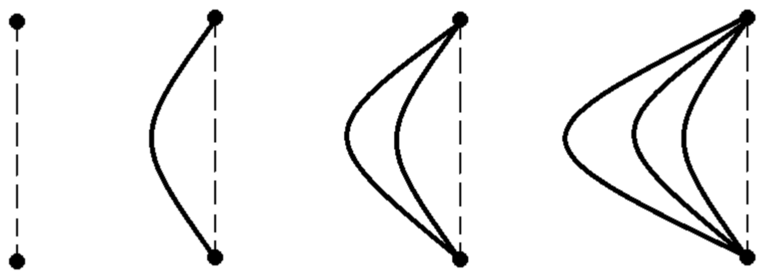

FIGURE 5.2: The graphs $G_0(0,l), G_0(1,l), G_0(2,l), G_0(3,l)$ where the thick curved lines are paths of length l and the dotted line is a non-edge.

We finish off the section on basic properties with a little result on the interval $(1,2]$ in which the factors can lie.

Lemma 5.5. *Let G be any graph, and $e \in \overline{E}$ any non-edge in G. The factors $f_G(e)$ that an edge e can have are dense in the interval* $(1,2]$.

I will demonstrate the construction that shows this. I will build a sequence of graphs for which a special edge has factor 2, and as we progress along the sequence, we move arbitrarily close to the special edge having factor 1, with the gap between the factors in neighbouring graphs becoming arbitrarily small. We can thus build a 'grid' of decreasing size covering the interval, which shows that factors are dense in $(1,2]$. The graphs that give us the grid are graphs with two special vertices connected by p paths of length l, and no other edges. Each additional path $(p \to p+1)$ decreases the factor. Increasing the length of the paths $(l \to l+1)$ decreases the grid size (but increases the factor).

Proof. Let $G_0(p,l)$ be the graph which consists of two vertices, a and b say, and p independent paths of length l connecting a and b, and no other edges or vertices. $G_0(p,l)$ has $m = p * l$ edges and $n = 2 + (l-1) * p$ vertices. Let $G_1(p,l)$ be $G_0(p,l)$ with an additional edge e connecting a and b.

We claim that $G_1(p,l)$ has $2 \times (2^l - 1)^p$ acyclic orientations. Apply the factor method to $G_1(p,l)$, taking the edges in the following order. First take edge e, which gives us a factor of 2. For each other path of length l, we have a factor of $2^l - 1$, as we can extend the acyclic orientation in every way except the one where all edges are pointed in the opposite direction to e.

By applying the deletion-contraction relation to edge e in G_1, we obtain the equation $a(G_1(p,l)) = a(G_0(p,l)) + a(H(p,l))$, where $H(p,l)$ is the graph consisting of p cycles of length l connected only by a single shared vertex, obtained by contracting edge e. The graph $H(p,l)$ has $(2^l-2)^p$ acyclic orientations, as each cycle of length l has 2^l-2 acyclic orientations, and there are (as far as acyclic orientations are concerned) p independent cycles. Thus,

$$\begin{aligned} a(G_0(p,l)) &= a(G_1(p,l)) - a(H(p,l)) \\ &= 2 \times (2^l-1)^p - (2^l-2)^p. \end{aligned}$$

Thus the factor at the special edge $\{a,b\}$ of the graph $G_1(p,l)$ is a function of l and p, namely

$$f(p,l) := f_{G_1(p,l)}(\{a,b\}) = \frac{a(G_1(p,l))}{a(G_0(p,l))} = \frac{2 \times (2^l-1)^p}{2 \times (2^l-1)^p - (2^l-2)^p}.$$

Now we let $L = (2^l - 1)$ to simplify the expression to

$$f(p,l) = \frac{2 \times L^p}{2 \times L^p - (L-1)^p} = \frac{2}{2 - (\frac{L-1}{L})^p}.$$

We now build our 'grid' of factor values of $f(p,l)$ for p between 1 and 2. The factor at $p=0$ is $f(0,l)=2$, the factor at $p=1$ is $f(1,l) = \frac{2\times L}{2\times L-(L-1)} = 2 \times \frac{L}{L+1}$, which we can make arbitrarily close to 2 by increasing l. Furthermore $f(p,l) = \frac{2}{2-(\frac{L-1}{L})^p} \xrightarrow{p\to\infty} 1$ for $p \geq 1$, since $(\frac{L-1}{L})^p \xrightarrow{p\to\infty} 0$. Thus, the value of $f(p,1)$ starts at 2 and moves arbitrarily close to 1 as $p \to \infty$. It thus remains to show only that the steps are sufficiently small for all values of l.

In order to do so, we now consider the underlying continuous function related to $f(k,l)$. For $g(x) = \frac{1}{2-a^x}$, and $0 < a < 1$, $x > 1$, we have the first derivative

$$g'(x) = \frac{a^x log(a)}{(2-a^x)^2},$$

which is always negative for the given parameters. Furthermore the second derivative

$$g''(x) = \frac{2a^{2x}log(a)^2}{(2-a^x)^3} + \frac{a^x log(a)^2}{(2-a^x)^2}$$

is always positive. Thus the function $g(x)$ is strictly decreasing (for the given parameters), but the slope is strictly increasing, which means that it is decreasing less and less as x grows. Now we let $a = \frac{L-1}{L}$ and $x = p$ which shows that each step increasing p following the first one has a smaller difference than the first step. We can make the first step as small as we wish as this is just the factor at an edge in an arbitrarily large cycle of length $l+1$ (the exact factor is shown in Lemma 5.6 to be $\frac{2^{l+1}-2}{2^l}$). Thus we can make a grid of arbitrarily small size covering $(1,2]$ showing that factors are indeed dense in $(1,2]$. □

5.3.1 A more general factor

We will now abuse the notation of a factor as follows. The factor of a subgraph H of a graph G is defined as the product of the factors of the edges of H in G as follows:

$$f_{G-H}(H) = f_{G-e_1}(e_1) f_{G-e_1-e_2}(e_2) \dots f_{G-E(H)}(e_h),$$

for $E(H) = \{e_1, ..., e_h\}$.

Similarly we can define a factor on a vertex $v \in G$ as the product of the factors of the edges connected to v.

5.3.2 Some simple examples of factors

Lemma 5.6. *The factor at the last edge of a cycle of length n is $\frac{2^n-2}{2^{n-1}}$.*

Proof. The number of acyclic orientations of a cycle of length n is $2^n - 2$ (all possible orientations minus the only two orientations that are cyclic), the number of acyclic orientations of a path on length $n-1$ is 2^{n-1}, together these give the result. □

When adding a vertex to a graph, which is only connected to a complete subgraph of that graph, we can compute the factor of each edge easily, and thus obtain the exact factor for that vertex.

Lemma 5.7. *The factor of a vertex x and all adjacent edges, when added to a graph G such that all edges are connected to a complete subgraph of G is $d(x) + 1$.*

Proof. There are $a(G)$ acyclic orientations of G. We consider all possible acyclic orientations of G, each of which gives us a complete ordering of the vertices in the complete subgraph of G. Now the vertex x can be inserted in $d(x) + 1$ places into this ordering of the vertices of the subgraph, each of which gives a unique acyclic orientation in the enlarged graph, so we have $(d(x) + 1) \times a(G)$ acyclic orientations of G with the added vertex. This gives us a factor of $f(x) = \frac{a(G)\times(d(x)+1)}{a(G)} = d(x) + 1$. □

5.4 Applications of the Factor Method

We first give a simple lemma that has applications for complete graphs and split graphs.

Lemma 5.8 (Factors on edges connected only to a clique)**.** *For any graph G with a clique K_k, and any vertex $x \in G\backslash K_k$, s.t. $\mathcal{N}(x) \subset K_k$ the following holds for every edge e connected to x:*

for any H s.t. $K_k \cup \{x\} \subset H \subset G$, $f_H(e) = \frac{d_H(x)+2}{d_H(x)+1}$, where $d_H(x)$ is the degree of x in H.

Proof. Stanley's formula gives us the number of acyclic orientations of the graph before and after we add the edges, so the factor at the ith edge e_i added to the vertex x, $f(e_i)$ is

$$\begin{aligned} f(e_i) &= \frac{|\chi(H+e_1+...+e_i,-1)|}{|\chi(H+e_1+...+e_{i-1},-1)|} && (5.1)\\ &= \frac{|\chi(H+e_1+...+e_{i-1},-1)-\chi((H+e_1+...+e_i)/e_i,-1)|}{|\chi(H+e_1+...+e_{i-1},-1)|} && (5.2)\\ &= \frac{|\chi(H+e_1+...+e_{i-1},-1)+\chi(H,-1)|}{|\chi(H+e_1+...+e_{i-1},-1)|} && (5.3)\\ &= \frac{(i+1)|\chi(H,-1)|}{i|\chi(H,-1)|} && (5.4)\\ &= \frac{(i+1)}{i} && (5.5) \end{aligned}$$

The first step uses the deletion contraction relation for the chromatic polynomial, the second step observes that $(H+e_1+...+e_i)/e_i$ is just H with one isolated vertex deleted, the third step is repeated application of the first two steps for the numerator, and an application of Lemma 5.7 to the vertex x of graph $H+e_1+...+e_{i-1},-1)$ for the denominator. This completes the proof. □

5.4.1 Application to K_n

We can apply Lemma 5.8 to a clique and thus obtain the factors at every edge in a clique, which also gives some insight into the relative 'isolation' of several minimal graphs compared to others.

If we build the graph K_n by first building a K_3, then a K_4 and so on, then we are always adding edges which connect only to a complete graph as is the requirement for Lemma 5.8. Thus the factors for a K_n when added in this order are:

$$2, 2, \frac{3}{2}, 2, \frac{3}{2}, \frac{4}{3}, 2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4} \ldots, \frac{n-1}{n-2}, \frac{n}{n-1}$$

As a check, when we multiply out each vertex we get $2, 3, 4, ..., n$, so we do obtain $n!$ as the number of acyclic orientations of K_n. We can also note that the first edge to be added to each vertex has by far the greatest contribution, and the last one only has a tiny contribution.

We will build on this in Chapter 7 where we will examine graphs that attain the maximum number of acyclic orientations. From the factor method it is clear that a maximal graph with more than n edges must be connected (as a bridge has maximal edge factor), and we will hope to generalise this and find some more properties which maximise the number of acyclic orientations building on this insight.

5.4.2 Application to split graphs

For a split graph Lemma 5.8 tells us that the factors at the edges connected to a vertex not in the clique are $2, \frac{3}{2}, \frac{4}{3}, ..., \frac{d_i+1}{d_i}$, as long as the edges are added after the edges of the clique. This holds for all vertices in the independent set simultaneously, so we obtain the following corollary.

Corollary 5.9. *For a split graph with d_i vertices of degree i in the independent set, and a clique of size k, the number of acyclic orientations is*

$$a(G) = k! \cdot \prod_{i=0}^{k-1} (i+1)^{d_i}.$$

Proof. Add the edges of the complete graph K_k first. The factors in the K_k multiplied together are $k!$. The remaining factors follow from Lemma 5.7:

$$\begin{aligned} a(G) &= k! \cdot \prod_{x \notin K_k} (d(x) + 1) \\ &= k! \cdot \prod_{i=0}^{k-1} (i+1)^{d_i} \end{aligned}$$

□

We have given an alternate proof for this in the proof of Lemma 4.28. Our method is the generalization of the method used in Lemma 4.28, and thus the proof of Lemma 4.28 simply follows from Lemma 5.8.

5.5 Work in progress on using the factor as a bounding tool

In the remainder of this chapter we examine the contribution that an edge makes, when we consider only paths of length 2, i.e. triangles which that edge would complete. Using a hypothesis which is false (we will discuss how to weaken it appropriately), this factor gives us an upper bound on the actual factor, which can be used in proving local optimality of $K_{n,n}$, or other graphs with few triangles. We use some results that we have already obtained in previous chapters. The complete bipartite graph $K_{2,n}$ has $2 \times 3^n - 2^n$ acyclic orientations. This is just the evaluation of B_n^{-2} as shown in Chapter 6. Also shown in Chapter 6 is that the bipartite graph $K_{2,n}$ with an extra edge connecting the vertices in the class of size 2 has 2×3^n acyclic orientations. It can also be proved using Lemma 5.7 on connecting vertices to complete subgraphs. The first edge that is the special edge has factor 2, each following vertex with a pair of edges attached has factor $2 \times 3/2 = 3$, thus we get the result.

5.5.1 Using a plausible hypothesis to bound factors

In order to make factors useful we really want the following to be true, and for the remainder of this subsection we will assume it is, to give an idea of the power that this method could have. We will follow through assuming the hypothesis to be true (even though we have a counterexample), to show why a similar, but weaker hypothesis can be powerful, or why a deeper understanding of how factors behave can be useful in bounding.

Statement 5.10 (The false Factor Hypothesis: H1)**.** *If we add two edges to a graph G, say e and e', the factor on e when added to G is at least as big as the factor on e when added to $G + e'$, i.e.*

$$f_G(e) \geq f_{G+e'}(e).$$

An equivalent result is the following.

Statement 5.11 (Equivalent Hypothesis: H2). *For $G = (V, E)$ and a set of edges $E' \subset \overline{E}$ we have*

$$f_G(e) \geq f_{G+E'}(e) \;\; \forall e \in \overline{E} \backslash E'$$

We have a counterexample for both statements. We have given an attempt at a proof in Appendix E, which we hope can be fixed when the hypotheses are weakened. Alternatively it might be possible to show that the proof holds in most cases, which would also make the factor hypothesis useful.

We will now obtain an upper bound on any edge in any graph by considering the number of triangles that edge completes in the graph. Furthermore we obtain a lower bound for any edge in a complete bipartite graph K_{n_1,n_2}. Putting the two together we show that complete bipartite graphs are locally maximal with respect to edge moves.

Hypothesis 5.12 (based on Hypothesis H1). *The factor of an edge e in any graph G can be bounded above by $\frac{1}{1-\frac{1}{2}(\frac{2}{3})^t}$ where t is the number of triangles containing edge e, i.e. $f_G(e) \leq \frac{1}{1-\frac{1}{2}(\frac{2}{3})^t}$.*

This hypothesis is not true, but we will demonstrate our proof approach below.

Proof approach. If we can show that the factor at edge e in a graph which only contains the number of triangles to be completed is $f_{K_{2,t}}(e) \leq \frac{1}{1-\frac{1}{2}(\frac{2}{3})^t}$, then using the Factor Hypothesis 5.10 we have shown that this upper bound holds in general. The complete bipartite graph $K_{2,n}$ has $2 \times 3^n - 2^n$ acyclic orientations by Section 6.1.4. The complete bipartite graph $K_{2,n}$ with an extra edge connecting the vertices in the class of size 2 has 2×3^n acyclic orientations also shown in Section 6.1.4. Putting the two together as the factor of edge e gives us the result. □

This neat result would give us a bound on any edge in any graph. Note that the existence of any other path means that the upper bound is not tight, the upper bound is only attained when there are no other paths that connect the two end vertices of e other than paths of length 2. We will now show the power of this method by considering the complete bipartite graph K_{n_1,n_2}, and the factor of the last edge added to it. Some values for the last factor are shown in Table 5.1, which we have calculated using the formulas obtained in Chapter 6.

$n_1 \setminus n_2$	2	3	4	5	6	7	$>> 7$
2	1.75	1.643	1.587	1.555	1.535	1.523	$\to 3/2$
3		1.513	1.448	1.410	1.387	1.371	$\to 4/3$
4			1.380	1.341	1.316	1.299	$\to 5/4$
5				1.300	1.274	1.257	$\to 6/5$
6					1.248	*Error*	$\to 7/6$
7						1.210	$\to 8/7$

TABLE 5.1: Factor of last edge added to K_{n_1,n_2}

Hypothesis 5.13 (based on Hypothesis H1)**.** *The factor of any edge of K_{n_1,n_2} is bounded below by $1 + \frac{1}{min(n_1,n_2)}$.*

We believe that this is true, and hope that the proof below can be fixed using an adapted factor hypothesis.

Proof approach. Note first that all edges have the same factor. Suppose $n_1 \leq n_2$. We simply add in edges to fill up the smaller vertex set of K_{n_1,n_2}. We may do this by Hypothesis 5.10. Now the edge is the last edge of a vertex that is connected to a complete subgraph only, and thus by Lemma 5.7 we know that the factor of that edge is $\frac{n_1+1}{n_1}$. The Factor Lemma shows that this must be a lower bound, as we now remove edges to obtain the factor in K_{n_1,n_2}. $\square$

Note that there is strong computational evidence given to us in Table 5.1. It seems that as n gets large and we fix the size of one of the components, we tend to the value $1+\frac{1}{min(n_1,n_2)}$ from above, as indicated by the last column, which is currently unproven.

Note that in this case we *do not* have a counterexample to the factor hypothesis, and indeed this is strong evidence that in this special case the factor hypothesis holds! We now put these who bounds together to obtain the following lemma.

Hypothesis 5.14 (based on Hypothesis H1)**.** *Given $n_1, n_2 > 1$, K_{n_1,n_2} and graph $G = K_{n_1,n_2} - e + f$ for any $e \in E(K_{n_1,n_2})$ and any $f \in \overline{E}(K_{n_1,n_2})$, we have $a(K_{n_1,n_2}) > a(G)$, i.e. we say that K_{n_1,n_2} is locally maximal for acyclic orientations with respect to edge moves.*

We believe that this is true, and hope that the proof below can be fixed, or we can find an alternate proof.

Proof approach. Suppose $n_1 \le n_2$. The first edge e we move in a K_{n_1,n_2} creates at least $n_1 - 1$ triangles. Now we have an upper bound for the factor the moved edge e' with k triangles, and a lower bound for e that got moved from the K_{n_1,n_2}. As long as the upper bound of the triangle factor is smaller than the lower bound of the factor in K_{n_1,n_2}, we are locally maximal for edge moves. The upper bound for the edge in its new place is

$$f_G(e') \le \frac{1}{1 - \frac{1}{2}(\frac{2}{3})^{n_1 - 1}},$$

as shown in Section 6.1.4. The lower bound for e is

$$f_G(e) \ge \frac{n}{n-1},$$

by using the factor hypothesis and referring to a K_n. Both of these bounds require some form of the factor hypothesis. Putting the two together would show

$$\frac{n}{n-1} \le f_G(e') \le \frac{1}{1 - \frac{1}{2}(\frac{2}{3})^{n_1 - 1}},$$

which does not always hold, e.g. for $n = 10, n_1 = 5$ we would get

$$1.1 < \frac{10}{9} \le f_G(e') \le \frac{1}{1 - \frac{1}{2}(\frac{2}{3})^9} < 1.02,$$

which does not hold, so the proof cannot work.

□

Our brief excursion into building on this hypothesis ends here - but I hope to have shown the direction in which I want to take the factor method. As mentioned before, we believe that one lemma (Lemma 5.13) obtained using the factor hypothesis is true, and this is supported by computational evidence.

We will now present a counterexample to Lemma 5.12. We wish to find a graph for which "The factor of an edge e in any graph G can be bounded above by $\frac{1}{1-\frac{1}{2}(\frac{2}{3})^t}$ where t is the number of triangles containing edge e" does not hold. Consider the graph K_5. The last edge in K_5 has factor $\frac{5}{5-1}$. The edge is in 3 triangles, so it is bounded above by $\frac{1}{1-\frac{1}{2}(\frac{2}{3})^3}$, which gives us an upper bound of 1.174, a contradiction, as the factor is actually 1.2.

5.5.2 A possible approach to fix The Factor Hypothesis

Unfortunately the factor hypothesis is not true. It seems that adding in certain edges will increase the factor at an edge. One possible way to fix the hypothesis, is to assume that we have given an ordering (or we find one) such that we *do* have this property, which will allow us to use the factor method with some restrictions. We start by giving some simple ordering, from which we hope to build an order that will give us the desired property.

Lemma 5.15. *It is possible to index the edges of a graph $e_1, ..., e_m$, such that $f_{G-e_i}(e_i) \geq f_{G-e_{i+1}}(e_{i+1})$ for all i.*

Proof. First find the factor $f_{G-e}(e)$ at each edge $e \in G$ in turn. Order the edges in factor size order, beginning with an edge of largest factor. For all edges with equal factors simply pick any sub-ordering. Now label label the ith vertex in this ordering e_i and we are done. □

Conjecture 5.16. *It is possible to index the edges of a graph $e_1, ..., e_m$, such that $f_{\cup_{k<i}e_k}(e_i) \geq f_{\cup_{k<i+1}e_k}(e_{i+1})$ for all i.*

This is not as obvious as the first lemma. The ordering of Lemma 5.15 will not necessarily (in fact almost never) do the trick. Consider this counterexample: a union of a triangle and a square. Then the ordering obtained from Lemma 5.15 will put all the edges in the triangle first in any order followed by the edges in the square in any order. The ordering obtained from Conjecture 5.16 on the other hand will first pick any six edges as long as neither the square or the triangle are completed, then the missing edge from the triangle, then the final edge from the square. This shows that neither ordering can be obtained from the other in general. For many proof ideas Conjecture 5.16 might be sufficient to fill the gap, we do not need the full strength of the Factor Hypothesis. As noted before we also have an attempt at a proof in Appendix E. In particular we have noted where the proof breaks down, which will hopefully also give insight into the problem of fixing the factor hypothesis.

5.6 Conclusion

We have provided a new approach to counting acyclic orientations - the factor method. We develop the tool and demonstrate some simple use cases. We show that the factors are dense in $(1,2]$.

We explore the range of potential factors, and attempt to use the factor as a bounding tool to prove local optimality of the complete bipartite graph. Our work is based on a false hypothesis, which needs weakening in order to complete the proof.

We will later use the concept of the factor in Chapter 7 to show that a subset of Turán graphs is uniquely maximal.

Chapter 6

The Number of Acyclic Orientations of Complete Bipartite Graphs

This chapter is a self contained piece of work, and has been submitted as a pre-print on the arXiv. It has been pointed out by Stanley that Theorem 6.1 follows from Exercise 5.6 of his book "Enumerative Combinatorics", vol. 2, by putting $q = -1$ and changing the sign of x and y to obtain the generating function $1/(e^{-x} + e^{-y} - 1)$. Nonetheless our Theorem 6.2 and Theorem 6.3 are new, as is the connection to lonesum matrices.

6.1 The number of acyclic orientations of certain graphs

Our main results are given in the next three theorems. We define $S(n,k)$ to be the *Stirling numbers of the second kind* which count the number of ways to partition a set of n objects into k non-empty subsets.

Theorem 6.1. *The number of acyclic orientations of the complete bipartite graph K_{n_1,n_2} is*

$$\sum_{k=1}^{\min\{n_1+1,n_2+1\}} (k-1)!^2 S(n_1+1,k)S(n_2+1,k),$$

where S denotes Stirling numbers of the second kind.

Theorem 6.2. *Let G be the graph obtained from K_{n_1,n_2} by adding an edge e_1 joining two vertices in the bipartite block of size n_1, where $n_1 > 1$. Then*

$$a(G) = a(K_{n_1 n_2} + e_1) = a(K_{n_1,n_2}) + a(K_{n_1-1,n_2}).$$

Theorem 6.3. *Let G be the graph obtained by deleting an edge from K_{n_1,n_2}. Then*

$$a(G) = a(K_{n_1 n_2} - e) = a(K_{n_1,n_2}) - \frac{1}{2}X,$$

where

$$\begin{aligned} X = 1 + \sum_{k=2}^{\min\{n_1,n_2\}+1} \quad & ((k-2)!)^2 \, [S(n_1+1,k)S(n_2+1,k)(2k-3) \\ & -(S(n_1+1,k)S(n_2,k) + S(n_1,k)S(n_2+1,k))(k-2) \\ & -S(n_1,k)S(n_2,k)\,]. \end{aligned}$$

We will prove these three theorems in the next three subsections.

6.1.1 Proof of Theorem 6.1

Let A and B be the two bipartite blocks; we will imagine their vertices as coloured amber and blue respectively. Now any acyclic orientation of the graph can be obtained by ordering the vertices and making the edges point from smaller to greater. If we do this, we will have alternating amber and blue intervals; the ordering within each interval is irrelevant in identifying the orientation, but the ordering of the intervals themselves matters.

In terms of structure for a given orientation, call two points $a_1, a_2 \in A$ *equivalent* if the orientations of $\{a_1, b\}$ and $\{a_2, b\}$ are the same for all $b \in B$. Points are equivalent if and only if they are not separated by a point of B in any ordering giving rise to the acyclic orientation. Similarly for B. This gives us the intervals, which are interleaved.

It is left to count alternating intervals. To get around the problem that the first interval in the ordering might be in either A or B, and similarly for the last interval, we use the following trick. Add a dummy amber vertex a_0 to A and a dummy blue vertex b_0 to B.

Now partition $A \cup \{a_0\}$ and $B \cup \{b_0\}$ into the same number, say k, of intervals. This can be done in $S(n_1+1,k)S(n_2+1,k)$ ways. Now we order the intervals so that

- the interval containing a_0 is first;
- the colours of the intervals alternate;
- the interval containing b_0 is last.

This can be done in $(k-1)!^2$ ways. Finally, delete the dummy points.

Summing over k gives the total number claimed.

6.1.2 Proof of Theorem 6.2

Let G be the graph consisting of K_{n_1,n_2} (with bipartite blocks A and B) together with an edge joining two vertices in A. Now any acyclic orientation of K_{n_1,n_2} can be extended to either one or two acyclic orientations of G; so $a(G) = a(K_{n_1,n_2}) + Z$, where Z is the number of acyclic orientations of K_{n_1,n_2} for which the added edge $\{a_1, a_2\}$ can be oriented in either direction.

The edge $\{a_1, a_2\}$ can be oriented in either direction without creating a cycle, iff $\{a_1, b\}$ has the same orientations as $\{a_2, b\}$, for each vertex b of B. Thus, we are effectively finding an acyclic orientation of K_{n_1-1,n_2}. So $Z = a(K_{n_1-1,n_2})$.

6.1.3 Proof of Theorem 6.3

Deleting an edge is a little more difficult. Suppose that we calculate the number X of acyclic orientations of K_{n_1,n_2} which remain acyclic when the orientations at a given edge e is reversed. (This number clearly does not depend on the chosen edge.) Then we let the number of acyclic orientations of $G = K_{n_1,n_2} - e$ be $F(n_1, n_2) - \frac{1}{2}X$ for some unknown function $F(n_1, n_2)$. Now let $Y = a(G)$ $(= F(n_1, n_2) - \frac{1}{2}X)$, then $\frac{1}{2}X$ of the acyclic orientations of G extend to two acyclic orientations of K_{n_1,n_2}, while the remaining $Y - \frac{1}{2}X$ extend to a unique acyclic orientation; so $F(n_1, n_2) = 2 \times \frac{1}{2}X + (Y - \frac{1}{2}X)$, giving the result.

So we have to verify the formula for X given in the statement of the theorem.

Let $e = \{a_1, b_1\}$. Then e can be flipped if and only if the part of the partition of B containing b_1 immediately precedes or follows the part of the partition of A containing a_1. (For example, if a part of B containing b_2 and a part of A containing a_2 intervene, then we have arcs (a_1, b_2), (b_2, a_2) and (a_2, b_1), so the arc (a_1, b_1) is forced.)

We follow the proof of Theorem 6.1. If $k = 1$, then all edges are directed from A to B, and (a_1, b_1) can be flipped. So this contributes 1 to the sum.

Suppose that $k > 2$. We distinguish four cases, according as a_0 and a_1 are or are not in the same part, and similarly for b_0 and b_1. Of the $S(n_1 + 2, k)$ partitions of $A \cup \{a_0\}$, $S(n_1, k)$ have a_0 and a_1 in the same part: this is found by regarding a_0 and a_1 as the same element, partitioning the resulting set of size n_1, and then separating them again.

Case 1 a_0 and a_1 in the same part, b_0 and b_1 in the same part. Since $k > 1$, the parts containing a_1 and b_1 are not consecutive, so the contribution from this case is 0.

Case 2 a_0 and a_1 in the same part, b_0 and b_1 not. There are $S(n_1, k)(S(n_2 + 1, k) - S(n_2, k))$ pairs of partitions with this property. Now the part containing b_1 must come immediately after the part containing a_1, so there are only $(k-2)!$ orderings of the parts of B, while still $(k - 1)!$ for the parts of A.

Case 3 b_0 and b_1 in the same part, a_0 and a_1 not. This case is the same as Case 2, with n_1 and n_2 interchanged.

Case 4 a_0 and a_1 in different parts, b_0 and b_1 in different parts. There are $(S(n_1 + 1, k) - S(n_1, k))(S(n_2 + 1, k) - S(n_2, k))$ such pairs of partitions. Now the parts containing a_1 and b_1 must be adjacent, so must occur as $(3, 2), (3, 4), (5, 4), \ldots, (2k - 1, 2k - 2)$ in the ordering of parts: there are $(2k - 3)$ possibilities. Once one possibility has been chosen, the position of two parts for both A and B are fixed, so there are $((k - 2)!)^2$ possible orderings.

Combining all this and rearranging, we find the result of the theorem.

6.1.4 Some numerical values

It is instructive to view the numerical values of the number of acyclic orientations of bipartite graphs K_{n_1,n_2}. When $n_1 = 1$, the graph is a tree, and we have $a(K_{1,n}) = 2^n$. For n_1 between 2 and 7 Table 6.1 gives the number of acyclic orientations of the complete bipartite graphs and Tables 6.2 and 6.3 those graphs with an edge added or removed, calculated from the formulae in Theorems 6.1, 6.2 and 6.3. In Table 6.2 for $K_{n_1,n_2} + e_1$, the added edge e_1 is in the bipartite block of size n_1. All of these values have been checked by calculating the chromatic polynomial of the graph. (A theorem of Stanley [58] asserts that the number of acyclic orientations of an n-vertex graph G is $(-1)^n P_G(-1)$, where P_G is the chromatic polynomial of G.)

$n_1 \setminus n_2$	2	3	4	5	6	7
2	14	46	146	454	1394	4246
3		230	1066	4718	20266	85310
4			6902	41506	237686	1315666
5				329462	2441314	17234438
6					22934774	22934774
7						2193664790

TABLE 6.1: The number of acyclic orientations of K_{n_1,n_2}

$n_1 \setminus n_2$	2	3	4	5	6	7
2	18	54	162	486	1458	4374
3	60	276	1212	5172	21660	89556
4	192	1296	7968	46224	257952	1400976
5	600	5784	48408	370968	2679000	18550104
6	1848	24984	279192	2770776	25376088	219463704
7	5640	105576	1553352	19675752	225164040	2395894056

TABLE 6.2: The number of acyclic orientations of $K_{n_1,n_2} + e_1$

$n_1 \setminus n_2$	2	3	4	5	6	7
2	8	28	92	292	908	2788
3		152	736	3344	14608	62192
4			5000	30952	180632	1012936
5				253352	1915672	13715144
6					18381608	164501368
7						1812141032

TABLE 6.3: The number of acyclic orientations of $K_{n_1,n_2} - e$

Note that as well as the formula $a(K_{1,n}) = 2^n$ we have $a(K_{2,n} + e_1) = 2 \cdot 3^n$. This is because the graph $K_{2,n} + e_1$ consists of n triangles sharing a common edge e_1, there are two ways to orient the edge e_1, and then three ways to choose the orientations of the remaining edges of each triangle to avoid a cycle. Putting these two results together in Theorem 6.2 gives us $a(K_{2,n}) = 2 \cdot 3^n - 2^n$. Is there a closed formula for $a(K_{n_1,n_2})$ in general?

6.2 Complete multipartite graphs

A similar method computes the number of acyclic orientations of complete multipartite graphs. Here is what happens for a complete tripartite graph.

Theorem 6.4. *The number of acyclic orientations of the complete tripartite graph K_{n_1,n_2,n_3} can be computed as follows. First list all strings of the symbols a, b, c with the properties*

- *the first symbol is a;*
- *adjacent symbols are different;*
- *the last symbol is c;*
- *the numbers k_1, k_2, k_3 of occurrences of the symbols a, b, c are all non-zero and do not exceed $n_1 + 1$, n_2, $n_3 + 1$ respectively.*

For each such string, calculate the term

$$S(n_1 + 1, k_1)S(n_2, k_2)S(n_3 + 1, k_3)(k_1 - 1)!k_2!(k_3 - 1)!\,.$$

Sum all these terms to obtain the result.

Proof. We follow the proof of Theorem 6.1. Any acyclic orientation of a complete multipartite graph is obtained by partitioning the three multipartite blocks and then ordering the pieces so that successive pieces belong to different blocks. (The set of sources is an independent set, so is contained in a block; delete it and repeat.) In the case of a complete tripartite block, label the blocks A, B, C; as in Theorem 1, add a point a_0 to A and a block; delete it and repeat.) In the case of a complete tripartite block, label the blocks A, B, C; as in Theorem 1, add a point a_0 to A and a point c_0 to C and specify that these should be in the first and last piece respectively. Then the sequence of symbols a, b, c

records the order in which the blocks come, and it is easy to see that the properties of the theorem hold. Now the partitions and reordering of pieces are as in Theorem 1, except that arbitrary permutations of the pieces in B are permitted. $\square$

This formula is not useful, however, since listing all the required character strings is a non-trivial task.

6.3 Poly-Bernoulli numbers and lonesum matrices

The formulae for the number of acyclic orientations of a bipartite graph K_{n_1,n_2} in Theorem 6.1 appear to be obscure. However, we now show that it is actually the poly-Bernoulli number in the variables n_1 and n_2, and furthermore represents the number of lonesum matrices of dimension n_1, n_2.

6.3.1 Poly-Bernoulli numbers

This is only a very brief introduction to the poly-Bernoulli numbers, which were introduced by Masanobu Kaneko [40] in 1997. Kaneko gave the following definitions. Let

$$\mathrm{Li}_k(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^k},$$

and let

$$\frac{\mathrm{Li}_k(1-\mathrm{e}^{-x})}{1-\mathrm{e}^{-x}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{x^n}{n!}.$$

The numbers $B_n^{(k)}$ are the *poly-Bernoulli numbers of order k*. Kaneko gave a couple of nice formulae for the poly-Bernoulli numbers of negative order, of which one is relevant here.

Theorem 6.5 (Kaneko)**.**

$$B_n^{(-k)} = \sum_{j=0}^{\min(n,k)} (j!)^2 S(n+1, j+1) S(k+1, j+1).$$

This formula has the (entirely non-obvious) corollary that these numbers have a symmetry property: $B_n^{(-k)} = B_k^{(-n)}$ for all non-negative integers n and k. Use Kaneko's Theorem together with Theorem 6.1 to obtain the following result.

Theorem 6.6. *The number of acyclic orientations of K_{n_1,n_2} is the poly-Bernoulli number $B_{n_1}^{(-n_2)} = B_{n_2}^{(-n_1)}$.*

6.3.2 Lonesum matrices

Another combinatorial interpretation was given by Chad Brewbaker [15] in 2008. A zero-one matrix is a *lonesum matrix* if it is uniquely determined by its row and column sums. Clearly a lonesum matrix cannot contain either $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ as a submatrix (in not necessarily consecutive rows or columns). (Since if one such submatrix occurred it could be flipped into the other without changing the row and column sums.) Ryser [53] showed that, conversely, a matrix containing neither of these is a lonesum matrix. Brewbaker showed that the number of $n_1 \times n_2$ lonesum matrices is given by the poly-Bernoulli number $B_{n_1}^{(-n_2)}$. We give the simple argument why this number is equal to the number of acyclic orientations of K_{n_1,n_2}.

In one direction, number the vertices in the bipartite blocks from 1 to n_1 (in A) and from 1 to n_2 (in B). Now given an orientation of the graph, we can describe it by a matrix whose (i, j) entry is 1 if the edge from vertex i of A to vertex j of B goes in the direction from A to B, and 0 otherwise. The two forbidden submatrices for lonesum matrices correspond to directed 4-cycles; so any acyclic orientation gives us a lonesum matrix.

Conversely, if an orientation of a complete bipartite graph contains no directed 4-cycles, then it contains no directed cycles at all. For suppose that there are no directed 4-cycles, but there is a directed cycle $(a_1, b_1, a_2, b_2, \ldots, a_k, b_k, a_1)$. Then the edge between a_1 and b_2 must be directed from a_1 to b_2, since otherwise there would be a 4-cycle $(a_1, b_1, a_2, b_2, a_1)$. But then we have a shorter directed cycle $(a_1, b_2, a_3, \ldots, b_k, a_1)$. Continuing this shortening process, we would eventually arrive at a directed 4-cycle, a contradiction. (This simply says that the cycle space of the complete bipartite graph is generated by 4-cycles.)

6.4 Maximizing the number of acyclic orientations

We believe that the graphs K_{n_1,n_2}, for $|n_1 - n_2| \leq 1$, maximise the number of acyclic orientations for graphs with as many vertices and edges. There are several conjectures of varying strengths which we will now discuss.

6.4.1 Conjectures

The Turán graph $T(n, r)$ as defined in detail in 2.23 maximises the number of edges of a graph with n vertices without the graph containing an $r + 1$ clique [60]. Turán graphs have many nice properties including $|d(x) - d(y)| \leq 1 \forall x, y \in V(T(n, r))$, and are also extremal for many graph properties for example colouring (the Turán graph is the unique maximal n-vertex graph with r-colour classes as equal as possible).

Conjecture 6.7. *Turán graphs maximise the number of acyclic orientations for all graphs with as many vertices and edges.*

This seems believable as cliques are the worst local structure for maximising the number of acyclic orientations (and forests are the best), and Turán graphs minimise the number of cliques.

Conjecture 6.8. *For a given n, m, such that $m \geq n$, there exists a maximal graph with respect to acyclic orientations, with the property $|d(x) - d(y)| \leq 1$ for all $x, y \in V(G)$.*

This reduces the search space, and in the next chapter we narrow it even further.

6.4.2 Computational support for conjectures

The above two hypotheses have been verified up to $n = 8$ as part of a broader computational study in [Chapter 3]. All graphs up to isomorphism were generated by McKay's nauty software [45]. For each graph the number of acyclic orientations was computed by evaluating the chromatic polynomial at -1 using Mathematica [67]. Finally, for each pair of n, m the maximal value and each corresponding maximal graph was identified.

Chapter 7

A Result on the Maximum Number of Acyclic Orientations and some Hypotheses and Counterexamples

In this chapter we will summarize our knowledge on the graphs with the maximum number of acyclic orientations as well as the maximum value attained by these. Depending on the density of the graph, we know more or less about the structure and the values, as is summarized in Figure 7.1. We know what the maximal graphs are for $0 \leq m \leq n$. This result is neither new nor interesting. We deal with the range $n \leq m \leq \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$ in Section 7.2, and here we know the least about the structure of the maximum, we only give a general conjecture that also applies in this range in Conjecture 7.10. For graphs with $m \geq \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$, we have a nice 'hanging curtains conjecture' in Section 7.6.1, which gives us an upper bound using Turán graphs. Finally for $\binom{n}{2} - \frac{n}{2} \leq m \leq \binom{n}{2}$ we find all maximal graphs in Theorem 7.31 which also strengthens the hanging curtains conjecture.

From the computational results reported in Chapter 3 we make conjectures about the graphs with the maximum number of acyclic orientations. For each conjecture we explain why we believe it to be true and also how certain of this we are. We may also, where appropriate, mention why a certain approach to prove optimality that seems natural

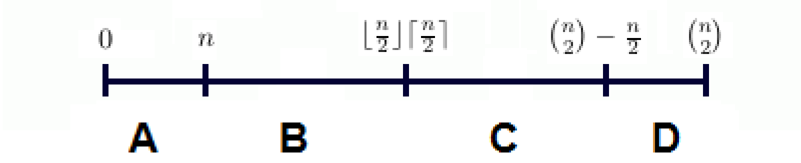

FIGURE 7.1: A summary of our knowledge on the maximum with respect to the number of edges. For sections A and D we know the maximum and have proved what it is. For C we have a good idea about the structure, and for B we have the least idea of what the graphs look like.

at that stage does not work. We begin with a property that we believe holds for most maximal graphs, no matter how dense the graph is.

7.1 A general property to help identify a maximal graph with respect to acyclic orientations

We wish to find a property for which we can find a graph exhibiting the maximal value for each pair n, m with respect to the number of acyclic orientations. This is not useful in completely classifying the maximum, but it can greatly aid the search for the maximal value. In fact, finding only one graph for each pair n, m that attains the maximal value would already be a great result. In some cases (for very dense graphs) we can prove that only one maximal graph exists and that it has this property.

Conjecture 7.1. *For any given n, m, for which there exists a graph on n vertices and m edges, there is a graph G on n vertices and m edges which maximises the number acyclic orientations and has all vertex degrees as even as possible, i.e. $|d(u) - d(v)| \leq 1$ for all $u, v \in G$.*

Note that such a construction always exists for every valid pair n, m. You can see this simply by taking any graph on n vertices and m edges. Now take a vertex v with largest degree and a vertex w with smallest degree, and remove an edge $\{v, x\}, x \in N(v) - N(w)$ and add the edge $\{w, x\}$. Repeat this until you obtain a graph with all vertex degrees as even as possible. Until such a graph is obtained this move is always possible, as the neighbourhood of a vertex with smaller degree cannot include all of the vertices in the

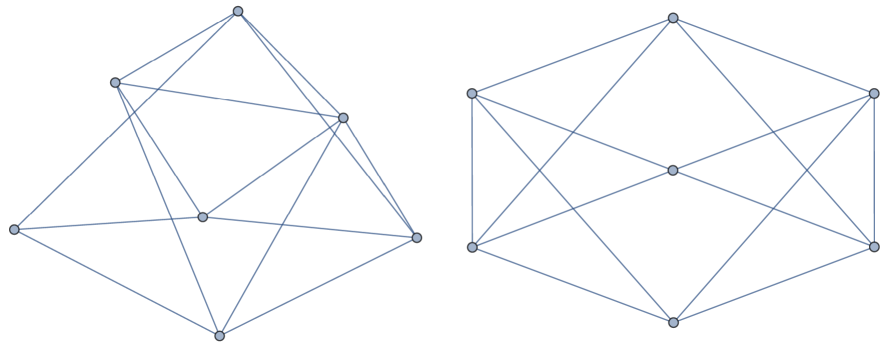

FIGURE 7.2: An example of two maximal graphs with $n = 7$, $m = 14$, with distinct degree sequences

neighbourhood of a vertex with larger degree. Conjecture 7.1 claims that some such graph maximises the number of acyclic orientations.

Conjecture 7.1 has been verified for values of n up to 8 by the computational work reported in Chapter 3. A maximal graph is not necessarily unique, hence we limit ourselves to saying that one maximal graph has even vertex degree. Any forest is maximal by Lemma 7.18, thus Conjecture 7.1 clearly cannot be extended to apply to all maximal graphs.

A more interesting example can be found in Figure 7.2, which shows two graphs that have 7 vertices 14 edges and the maximum number of acyclic orientations, but distinct degree sequences, namely $(5, 4, 4, 4, 4, 4, 3)$ and $(4, 4, 4, 4, 4, 4, 4)$. This is a very interesting counterexample, as it is somewhat unexpected, and will later on be a counterexample to some other properties we would expect of all maximal graphs.

We believe Conjecture 7.1 to be true for a number of reasons, beyond that it is true for up to $n = 8$. The graphs that minimise the number of acyclic orientations are obtained via the compression move defined in Section 4.3. There we looked at graphs that can be compressed no further. Now we look at the other end - we look at graphs that can not have been compressed from any other graph.

Definition 7.2. Given a graph G, let $\mathcal{U}_G = \{H | C_{xy}(H) = G, \text{ for some } x, y \in V(H)\}$ be the *upper shadow* of G with respect to compression.

We can generalize this to finding the upper shadow of a set of graphs:

Definition 7.3. Given a set of graphs $\mathcal{G}$, let $\mathcal{U}_{\mathcal{G}} = \{H | C_{xy}(H) = G, \text{ for some } x, y \in V(H), \text{ and some } G \in \mathcal{G}\}$ be the *upper shadow* of $\mathcal{G}$ with respect to compression.

Note that every graph in the upper shadow of G has at least as many acyclic orientations as G by Theorem 4.26. Further note that repeatedly finding the upper shadow of a set of graphs will eventually terminate, as the upper shadow of a set includes the set itself.

Now we are interested in graphs for which the upper shadow of the graph is only the graph itself (or an isomorphic graph). One of these graphs must be a graph that attains the maximum value.

Definition 7.4. For a given n, m, let $\mathcal{U}_{n,m} = \{G| \; |V(G)| = n, |E(G)| = m$ and $H \cong G$ for all $H \in \mathcal{U}_G\}$, i.e. those graphs with n vertices and m edges whose upper shadow is itself only.

We have given in Definition 7.4 a set of graphs that certainly contain a maximal graph. We will first prove this, and then give some properties of graphs in this set.

Lemma 7.5. *For a given n, m, the set $\mathcal{U}_{n,m}$ contains a graph that attains the maximum number of acyclic orientations for all graphs with n vertices and m edges.*

Proof. Suppose we are given any graph H_0 with n vertices and m edges, such that H_0 has the maximum number of acyclic orientations. If $H_0 \in \mathcal{U}_{n,m}$ we are done. If not, then pick any graph H_1 in the upper shadow of H_0 that is not isomorphic to H_0. This graph also has the maximum number of acyclic orientations by Theorem 4.26. Repeat this process as long as the upper shadow of H_i contains at least one other graph, which becomes H_{i+1}. This process must terminate as the number of edges of the line graph of H_i is strictly greater than the number of edges of the line graph of H_{i+1} by Lemma 4.7. Finally we are left with a graph H_k for some k (not necessarily uniquely determined by H_0), which has the maximum number of acyclic orientations as well as the property $H_k \in \mathcal{U}_{n,m}$. □

In this proof we also have a more general method of moving from any graph to a maximal one, simply by picking an element of the upper shadow at random. Next we give an equivalent definition of the set $\mathcal{U}_{n,m}$.

Lemma 7.6. *A graph G is in $\mathcal{U}_{n,m}$ if and only if for each pair of vertices $x, y \in G$ we have at least one of the following properties:*

1. $\mathcal{N}(x) \not\subset \mathcal{N}(y)$ *and* $\mathcal{N}(y) \not\subset \mathcal{N}(x)$

2. $\mathcal{N}(x) = \mathcal{N}(y) \cup \{w\}$ *for some* $w \in G$

3. $\mathcal{N}(y) = \mathcal{N}(x) \cup \{w\}$ *for some* $w \in G$

4. $\mathcal{N}(y) = \mathcal{N}(x)$

Proof. Take a graph G with the four properties. We must show that the upper shadow of G is only G. Take a graph H and vertices x, y, such that $C_{xy}(H) = G$. Now x and y can not be vertices in G with property 1, as after the compression $C_xy(H)$ we have $\mathcal{N}(x) \subset \mathcal{N}(y)$ in G. If x and y have properties 2 in G, then $C_xy(H)$ gives us a graph isomorphism of G which relabels x and y. If x and y have property 3 or 4 in G, then $C_xy(H) = G$, so the upper shadow of G is only G itself.

Now suppose we have a graph G in $\mathcal{U}_{n,m}$. Then every compression move such that $C_{xy}(H) = G$ for all H and $x, y \in H$ must be either an isomorphism or the identity map. For all pairs of nested vertices this means that they can differ by at most one vertex, giving us properties 2,3 and 4. For all not nested pairs, no compression is possible which ends with this pair, which gives us property 1. □

Our intuition that tells us to bunch all edges as close together as possible to be restrictive to obtain the minimum also tells us to spread the edges out as evenly as possible to allow the greatest freedom to obtain the maximum, i.e. it seems likely that the vertex degrees should also be similar. Finally there are many examples where moving an edge from a lower degree vertex to a higher one increases the number of acyclic orientations, for example in Figure 7.3 every such move works except for one, and we will go into more detail with this example.

Unfortunately, this cannot be used as a method of proof (in a simple way), as making such a move does not always increase the number of acyclic orientations. Finding a counterexample is not easy, and it seems that this is actually a rare occurrence. The graph we show in Figure 7.3 shows a graph where it is impossible to move any edge from vertex a with degree 4 to vertex b with degree 2 without decreasing the number of acyclic orientations. This is a highly symmetrical example, and even in this example it is possible to move an edge from, say c to b in order to increase the number of acyclic orientations, while evening out the vertex degrees. Indeed in this example every other 'evening out of degrees edge move' strictly increases the number of acyclic orientations.

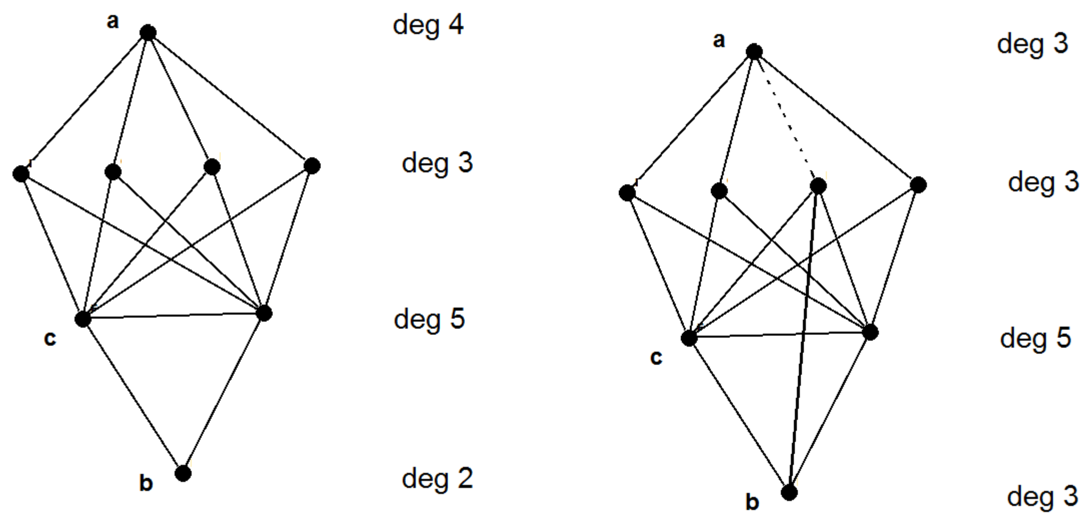

FIGURE 7.3: Example of a move that evens out the vertex degrees but decreases the number of acyclic orientations

It may be possible that if we specify and say 'from highest degree vertex to lowest degree vertex' such a proof will work, but I think that even then we could run into trouble - a local optimum - even though I have not found such an example of getting stuck yet.

Thus we conclude that there is considerable evidence that all vertex degrees must be as equal as possible, and we furthermore conjecture an algorithm that will take us to a graph with all vertex degrees as equal as possible while increasing (not strictly) the number of acyclic orientations.

7.2 The structure of maximal graphs with respect to acyclic orientations for $n < m < \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$

We know relatively little about the structure of maximal graphs with respect to acyclic orientations in the range $n < m < \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$. In this range we do not have enough edges in the graph to build a complete bipartite graph with independent sets as equal as possible. We may apply some results from the factor method in Chapter 5 which allow some very simple observations, but really this is the section we have the least idea of what is going on. First we need a definition.

Definition 7.7. Let $G = (V, E)$ be an arbitrary graph. If subgraph $G' = (V, E \setminus X)$ is connected for all $X \subseteq E$ with $|X| < k$, then G is said to be *k-edge-connected.*

This means that a graph G is k-edge-connected iff we can remove any set of $k - 1$ edges without disconnecting the graph.

Lemma 7.8. *Given a fixed number of vertices n and edges m with $m \geq n$, all graphs which maximise the number of acyclic orientations are 1-edge-connected, which is the same as being connected.*

Proof. Suppose we have a graph G with $m \geq n$ which is *not* connected. The G has at least 2 components, and contains a cycle. Now there exists an edge within the cycle, which has a factor strictly less than 2 by Lemma 5.2. Remove this and replace it by a bridge between two separate components of G, which has edge factor 2 (again by Lemma 5.2), strictly increasing the number of acyclic orientations. □

We further believe that actually all maximal graphs are at least 2-edge-connected for $m \geq n$.

Conjecture 7.9. *A graph G with n vertices and m edges, where $m \geq n$, which is maximal in the number of acyclic orientations, is 2-edge-connected.*

It is not possible for a graph with fewer edges, i.e. $m < n$, to be 2-edge-connected. We can thus note that for $m = n$ this is the threshold number of edges for a graph that is 2-edge-connected to exist. Similarly when we hit the threshold number of edges for a graph to exist that is 1-edge-connected (which is at $m = n - 1$, with any tree that has only one connected component), then this graph is a maximal graph, so it is tempting to generalize this conjecture to the following:

Conjecture 7.10 (Edge Connectivity Conjecture)**.** *For a given n, m, let k be the largest number, such that there exists a k-edge-connected graph with n vertices and m edges. Then* there exists a *graph that maximises the number of acyclic orientations and is k-edge-connected.*

In order to back up this conjecture we have tested it up to $n = 8$, and you can see the measures for $n = 8$ in Table 7.1. For each m a maximal graph with the nicest possible

Turán?	m	a_{max}	max $k(G)$	$k(G_{max})$
	8	254	2	2
	9	470	2	2
	10	792	2	2
	11	1230	2	2
	12	1920	3	3
	13	2670	3	3
	14	3602	3	3
	15	5000	3	3
T	16	6902	4	4
	17	7968	4	4
	18	9264	4	4
	19	10752	4	4
	20	12840	5	5
T	21	15402	5	5
	22	17688	5	5
	23	20400	5	5
T	24	24024	6	6
T	25	27240	6	6
T	26	30960	6	6
T	27	35280	6	6
T	28	40320	7	7

TABLE 7.1: Some measures for maximal graphs for $n = 8$.

measures was selected. The column max $k(G)$ is the largest value k for which we can find a graph with that many vertices and edges that is k-edge-connected. This number is clearly monotonically increasing, and we can see the threshold values that we mention in Conjecture 7.10. You can see that indeed these threshold values correspond exactly with the k-edge-connectivity of the maximum graphs, in the column $k(G_{max})$.

Conjecture 7.10 is somewhat daring, as the only reason we believe it to be true is that it holds for small n ($n \leq 8$) as can be seen in Table 3.5, and that a high edge-connectivity leads to 'spread out graphs'. We will now present a corollary of Conjecture 7.10 that ties in nicely with an Conjecture 7.1 - the conjecture that all vertex degrees are as equal as possible for a set of maximal graphs.

Corollary 7.11. *For a given n, m, let k be the largest number, such that there exists a graph with n vertices and m edges with minimum vertex degree k. Then* there exists a *graph that maximises the number of acyclic orientations that has minimum vertex degree k.*

This can be seen both as a corollary of the vertex degree conjecture (Conjecture 7.1) and also of the edge connectivity conjecture (Conjecture 7.10). It is straight forward to

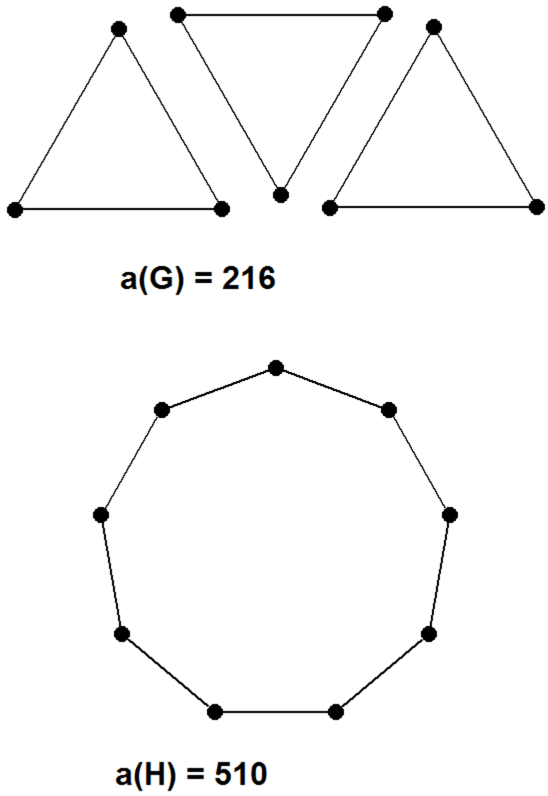

Figure 7.4: Example of two graphs with minimum vertex degree 2 and edge connectivity 2 respectively with $n = 8$ and $m = 9$.

construct a graph with all degrees as even as possible, which is very far from maximal with respect to acyclic orientations. We give an example in Figure 7.4 where the graph G has the same vertex degrees as the maximal graph, but only around half as many acyclic orientations (216 vs 510). On the other hand, the only graph with the maximal edge-connectivity and $n = 9, m = 9$ is be graph H in Figure 7.4, which is maximal with respect to acyclic orientations. Similar larger examples exist, and the pattern is the same. While for larger examples the graph obtained is not unique, the values for the number of acyclic orientations obtained are still is much closer to the maximum value than the values obtained via the condition on the minimal degrees. Vertex degrees only restrict the local structure, which on its own does not maximise the number of acyclic orientations in any meaningful way, whereas edge-connectivity ensures that the edges are 'spaced out' nicely.

It is not true that all maximal graphs have the property that they are as edge-connected as possible. We may reuse the old counterexample given to us in Figure 7.2. The graph on the left is 3-edge-connected but not 4-edge-connected, the graph on the right is 4-edge-connected, so for the conjecture the maximal graph must be 4-edge-connected. But both graphs are maximal. This is a very surprising result, as acyclic orientations and edge-connectivity are very closely related otherwise. We therefore may not generalize the conjecture in this way, but note that a maximal graph which is 4-edge-connected does exists. Finishing off this section we show that k-edge-connectivity does not increase with compression.

Lemma 7.12. *Given a graph G, vertices $x, y \in G$ and k such that G is k-edge-connected, but not $k+1$-edge-connected, the compressed graph $C_{xy}(G)$ is not $k+1$-edge-connected.*

Proof. It is the same to say that a graph G is $k+1$-edge-connected, as to say that for each pair of vertices in G we can find $k+1$ edge-disjoint paths between them.

□

7.3 The structure of maximal graphs with respect to acyclic orientations for $m \geq \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$

We now consider the other end of the density range, namely $m > \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$, i.e. when we have enough edges to make at least a balanced complete bipartite graph. We give the definition of a Turán graph, along with some useful parameters that we will use in following conjectures.

Definition 7.13. The Turán graph $T(n, r)$ is the complete multipartite graph formed by partitioning a set of n vertices into r subsets, with sizes as equal as possible, and connecting two vertices by an edge whenever they belong to different subsets. When the value of n is clear, we sometimes reduce this notation to $T(r)$.

The Turán graph $T(n, r)$ has $m = \lfloor \frac{(r-1)n^2}{2r} \rfloor$ edges and is split into r independent sets of vertices. We have given an example of a Turán graph in Figure 7.5. Now it is possible to say, for a given n and m between which Turán graph(s) we are, and also the corresponding r_l and r_u respectively, where these denote the number of independent sets of vertices of the Turán graph with fewer vertices and more vertices respectively. Note

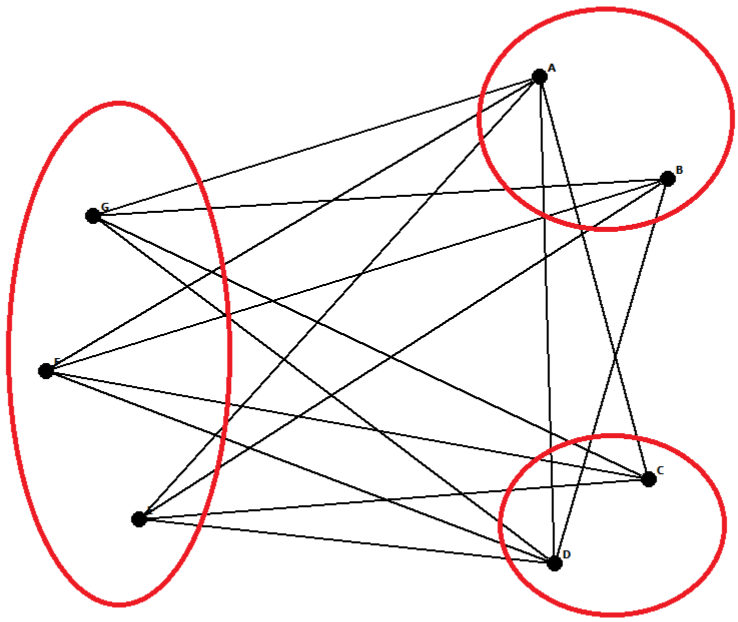

Figure 7.5: The Turán graph $T(7,3)$.

that $T(n,2)$ is the complete bipartite graph that has $m = \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$ edges, which is the fewest number of edges we consider in this section.

We start with a two part conjecture:

Conjecture 7.14. *For those n, m for which Turán graphs exist, the Turán graph is a) maximal and b) uniquely maximal.*

It is easier to explain why we believe this conjecture is true, after we show some other conjectures, and give some computational 'evidence' for small n.

Conjecture 7.15. *The maximum graph for n, m can be split into r vertex sets that are all completely connected to each other, where r is the number of independent sets of the closest Turán graph that has as many or fewer edges, i.e. the largest r such that $m \geq \lfloor \frac{(r-1)n^2}{2r} \rfloor$*

Note that the first time a graph can be split into two vertex sets that are completely connected to each other is the star that utilises all vertices. The vertex sets are the single vertex that is connected to all others, and all other vertices comprise the other set. This graph has the structure we are talking about, if we put $m = n-1$ into our calculation of r, we see that this is 'too early', i.e. $r = 1$, and not $r = 2$. We think that this is the only

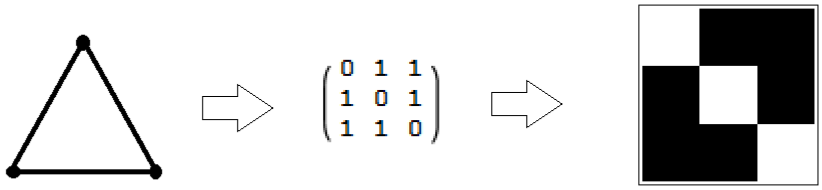

Figure 7.6: Alternative representations of a triangle.

time that a maximal graph accidentally already has the structure before the respective Turán graph, as it doesn't happen in any other cases from $n = 1...8$. Furthermore the structure of this particular tree is very special and as all trees are maximal so must the one with this property.

The best way to visualize Hypothesis 7.15 is to use a different way of looking at graphs, as it is hard to make sense of drawings of dense graphs. Instead we look at the adjacency matrix, and to make structure even clearer, we use a black square in place of a '1' and a white square in place of a '0'. This is called an *array plot* of a matrix. An example of this representation is given in Figure 7.6. Note that in this example (which is the complete graph on 3 vertices - a triangle) the diagonal is empty/white. It will be along this white diagonal from which the maximum builds.

A subset of consecutively indexed vertices $\{x_1, ..., x_l\}$ is completely connected to all other vertices, if the adjacency matrix has the value 1 at all (i, j) for $i \leq l, j > l$ (and so also for $j \leq l, i > l$. In an array plot this looks like a black rectangle of size $i \times j$ which is mirrored along the diagonal. This is easily generalized to arbitrary subsets.

Now we can illustrate Hypothesis 7.15 in Figure 7.7 where we show array plots of the maximal graphs for $n = 8$ and increasing m from 16 to 28 with judiciously indexed vertices. Each square in this figure is the array plot of the adjacency matrix of the maximal graph. In this example r ranges from 2 to 8, and increases by 1 at each Turán graph. Note that we start with the complete, balanced bipartite graph (which is also a Turán graph), and here we have a 4×4 black square in the top right and bottom left, which shows us that the first 4 vertices are completely connected to the second 4 vertices. Similarly the maximal graph with one more edge ($m = 17$) has the same structure, except with an additional edge in one of the components, which is an extra small black square in the array plot and otherwise identical structure.

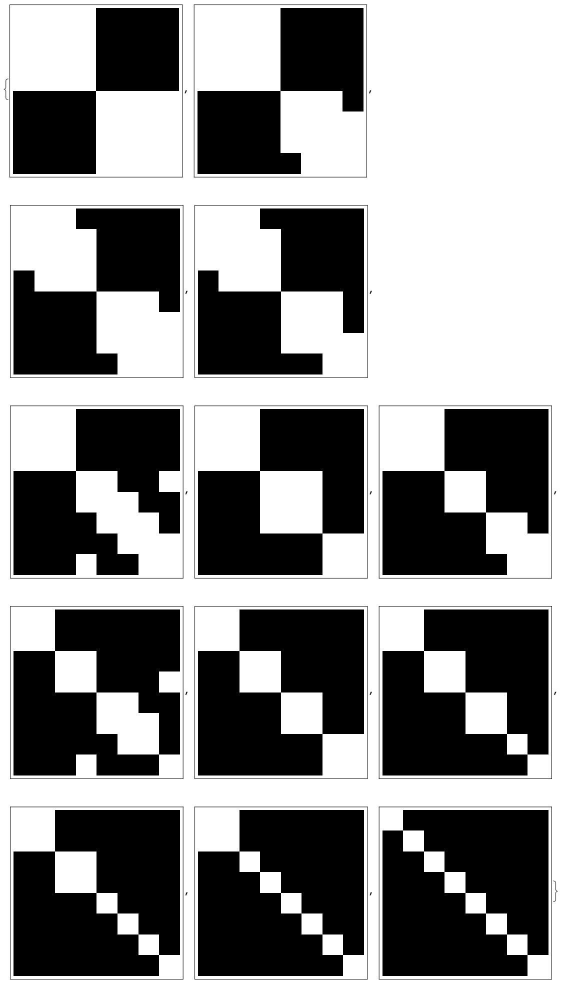

FIGURE 7.7: Analysis for maximum graphs for $n = 8, m \geq 16$.

Now the Conjecture 7.15 states that the number of these sets of vertices connected to all other vertices are monotonically increasing. You can observe this nicely in Figure 7.7, and further it is easy to pick out the Turán graphs as the only 'clean' graphs with even sized subsets.

Note that we cannot insist that the completely connected subsets of maximal graphs of all intermediary m have as even size as possible (excluding the star graph). Consider the maximum graph for $n = 8, m = 20$ depicted in Figure 7.8. It has two vertex sets of size 3 and 5 respectively, that are completely connected to each other. What is remarkable here is that there exists a graph with even vertex set sizes, and which also is built up of maximum squares (more detail to follow), so evening out the size of the components is not the top priority for the maximum.

In fact to show that making the components as equal as possible is not even close to

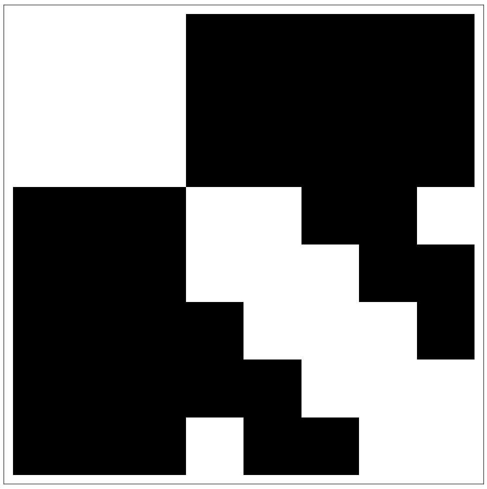

FIGURE 7.8: The maximum graph for $n = 8, m = 20$ with vertex sets of size 3 and size 5

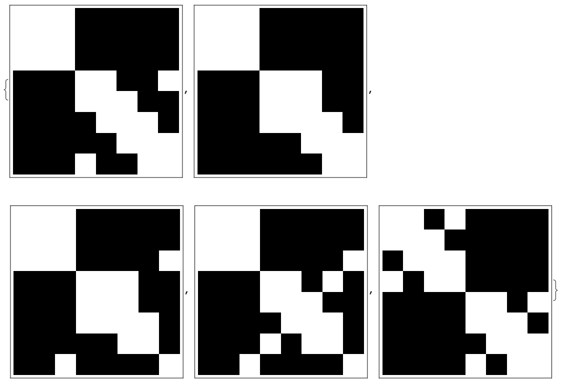

FIGURE 7.9: The top 5 graphs for $n = 8, m = 20$

being important, we look at the top 5 graphs with respect to the number of acyclic orientations, for $n = 8, m = 20$ up to isomorphism in Figure 7.9. The first graph is the one we already know, a $K_{3,5}$ where we have added a C_5 to the 5 set, with 12840 acyclic orientations. Remarkably, the second best graph on 5 vertices and 5 edges (a C_4 union an edge), when added to the $K_{3,5}$ on top of the 5 set gives us the second best graph with 12786 acyclic orientations. The remaining three graphs all have 12576 acyclic orientations. Only here, tied third place, does the $K_{4,4}$ with four added edges show up.

There are structures where it is beneficial to minimize the number of consecutive 1 blocks, for example in [27] (see [22] for a recent survey). In these papers a binary matrix is defined as having the *consecutive-ones property* if there is a permutation of its columns that places the 1s consecutively in every row. Observe that most of the graphs for $n = 8$

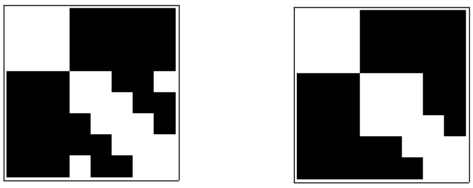

FIGURE 7.10: A maximal graph, and a graph that minimizes consecutive 0 and 1 blocks.

depicted in Figure 7.7 have this property, but we provide a counterexample for this depicted in Figure 7.10, where the maximal graph for $n = 8$ and $m = 20$ on the left has no representation that minimises the number of consecutive 0 and 1 blocks, one example of which is given on the right. This maximal graph is unique (which can be observed in the computational work resulting from Chapter 3), which shows that we can't even make the hypothesis that a maximal graph with this property exists for each pair n, m.

We have thus far only looked at the broad structure of maximal graphs. We now look at each of the sub-squares of the maximal graphs, which we obtain by looking only at those vertices which form components in the complement, i.e. each set of vertices that is connected to all other sets individually.

One way of taking an induced subgraph (defined in 2.11) of a graph from an array plot is by cutting out a square from the array plot which is along the diagonal. This is of course dependent on the vertex indexing, but our indexing allows us to pick the induced subgraphs we are interested in in such a manner.

Conjecture 7.16. *A subgraph H of a maximal graph G is also a maximum graph, when H is an induced subgraph, and every vertex in H is connected to every vertex in $G - H$.*

Obtaining an induced subgraph is exactly cutting out a square from an array plot, but now we also require all vertices not in the square to be connected to the vertices in the square. As mentioned before, this does not give us all induced subgraphs, but does give us all the ones we are interested in. We have visualized these requirements in Figure 7.11. The induced subgraph to be cut out is the 3×3 square in the middle, and the black rectangles surrounding the square in all directions show that every vertex in the square is connected to all other vertices. Here the induced subgraph would be a V shape on 3 vertices and 2 edges.

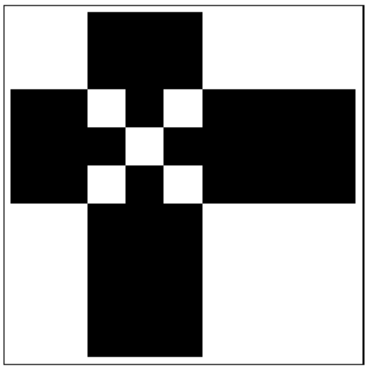

FIGURE 7.11: An induced subgraph connected to all other vertices

The conjecture states that every subgraph that we can obtain from a maximal graph in such a manner is itself a maximal graph. This is true for all observed examples, and would mean that only graphs with locally optimal structure are contenders for the maximum, which is much easier to check and decreases the search space if the method of finding the maximum is by exhaustive search. Now we have a conjecture of which we are a little less sure, which is a very powerful generalization of this conjecture. This conjecture has also been observed to hold in every case that was tested up to $n = 8$, but exhaustive testing of this conjecture is much harder, so it has not been done yet.

Conjecture 7.17. *Suppose we are given a graph G and a subgraph H, that is an induced subgraph, such that every vertex in H is connected to every vertex in $G - H$. Now in G replace H by some other graph H' with as many vertices and edges, to obtain G'. Then $a(H) > a(H') \implies a(G) > a(G')$, $a(H) = a(H') \implies a(G) = a(G')$ and $a(H) < a(H') \implies a(G) < a(G')$*

We use a special case of this conjecture to prove that certain graphs are uniquely maximal later on in this chapter. We will show that if H is a 4 cycle and if H' is a triangle with an edge attached, and if H, H' are a subgraphs of G, G' respectively as described in the conjecture, that then $a(G) > a(G')$.

7.4 The graphs that attain the maximum number of acyclic orientations for $m \leq n$

When $m \leq n$ the graph is very sparse, and we can describe the extremal graph with respect to several parameters including the maximum number of acyclic orientations.

The tools provided in Chapter 5, in particular the factor of edges that are bridges and the maximum edge factor value, lead us to the complete classification for $m \leq n$.

Lemma 7.18. *For $m < n$ any forest maximises the number of acyclic orientations.*

Proof. The factor at every edge in a forest is 2 and hence maximal by Lemma 5.2, so the graph must be maximal overall. □

Theorem 7.19. *The n-cycle C_n is the unique graph that maximises the number of acyclic orientations for graphs on n vertices and n edges.*

Proof. Any graph with n vertices and m edges contains a cycle, since it has $m \geq n$ edges. Every edge in a cycle can have factor 2, except for the last edge in the cycle. The factor at this edge is $\frac{2^k}{2^k-2}$, where k is the length of the cycle by Lemma 5.6. Now suppose we have a graph G which is not C_n. G has a cycle of length $k < n$. Now the factors at the edges of this cycle when added first are exactly $2, 2, \ldots, \frac{2^k}{2^k-2}$. We can bound the remainder of the factors of the edges of G above by 2, giving us an upper bound of $a(G) \leq 2^{n-1} \times \frac{2^k}{2^k-2}$ acyclic orientations. The n-cycle has $a(C_n) = 2^{n-1} \times \frac{2^n}{2^n-2}$ acyclic orientations and as $n > k$, $2^{n-1} \times \frac{2^n}{2^n-2} > 2^{n-1} \times \frac{2^k}{2^k-2} \implies a(C_n) > a(G)$, completing the proof. □

Thus we have completely classified the maximal graphs for the bottom end of edge density up to $n = m$: Any forest is maximal for $m < n$ and the n-cycle is uniquely maximal for $n = m$. Since we can always construct a forest with vertex degrees as even as possible, and the cycle has all degrees equal to 2, the results are consistent with Conjecture 7.1.

7.5 The graphs that attain the maximum number of acyclic orientations for $\binom{n}{2} - \frac{n}{2} \leq m \leq \binom{n}{2}$

In this section we consider the set of high density graphs with at most $\frac{n}{2}$ independent edges removed. Note that we include the complete graph in this set. Furthermore note that these graphs are exactly $T(n, r)$ for $r \geq \frac{n}{2}$ (this corresponds to $m \geq \binom{n}{2} - \frac{n}{2}$). We do not show that they uniquely attain the maximum value in this section, but will do so in Section 7.7.

Lemma 7.20. *For a given n, m, with $m \geq \binom{n}{2} - \frac{n}{2}$, there is a sequence of compression moves from $T(n, r)$ for $r \geq \frac{n}{2}$ to any other graph on n vertices and m edges.*

Proof. Observe that a compression $C_{xy}(G)$ can either be seen as a compression from x to y on the edges of G, or as a compression from y to x on the non-edges of G. As in $T(n, r)$ for $r \geq \frac{n}{2}$ non-edges are independent, it is possible to compress to any configuration of non-edges of a graph with n vertices and m edges. But this is exactly the set of all graphs with n vertices and m edges, so we are done. □

This Lemma leads to the following Theorem.

Theorem 7.21 (General Maximum Theorem)**.** *For a given number of vertices n and edges m with $m \geq \binom{n}{2} - \frac{n}{2}$, the corresponding graph $T(n, r)$ has the extremal value with respect to compression for every graph parameter that is monotonic with respect to compression.*

Proof. Given any graph G with n vertices and m edges such that $m \geq \binom{n}{2} - \frac{n}{2}$, use the compression sequence in Lemma 7.20 to show that for any parameter that is monotonic in compression the graph $T(n, r)$ attains the extremal value. □

Note that in the theorem we can do no better than remove $\frac{n}{2}$ independent edges from a set of n vertices by this method. We do not show that the graphs $T(n, r)$ for $r \geq \frac{n}{2}$ are the only maximal graphs (though the proof will later be extended to show that they are for some parameters), but simply show that other graphs can do no better.

Corollary 7.22. *The maximum number of acyclic orientations for a graph with a given number of vertices and edges, say n, m, for which $m \geq \binom{n}{2} - \frac{n}{2}$ is obtained by a complete graph with $\binom{n}{2} - m$ independent edges removed.*

Proof. This follows from Theorem 7.21 which states that M is extremal for parameters that are monotonic with respect to compression, and Theorem 4.26 which states that the number of acyclic orientations is monotonic for compression. □

It is possible to apply this method for some other parameters, indeed as we have shown in Lemma 4.7 that compression strictly increases the number of edges in the line graph we also have the following corollary.

Corollary 7.23. *The graphs $T(n,r)$ for $r \geq \frac{n}{2}$ minimise the number of edges of the line graph for graphs with as many vertices and edges. There are no other graphs with as many vertices and edges that do this.*

Similarly we have the result that compression is monotonic (but not strictly so) in both the number of forests in Lemma 4.19 and the number of cliques of a graph in Lemma 4.15, giving us the following simple corollary.

Corollary 7.24. *The graphs $T(n,r)$ for $r \geq \frac{n}{2}$ minimise the number of cliques and maximise the number forests simultaneously for graphs with as many vertices and edges.*

We will further prove that the set $T(n,r)$ for $r \geq \frac{n}{2}$ is actually uniquely minimal for the number of cliques a graph can have, by looking at the first compression move possible on any graph $T(n,r)$ for $r \geq \frac{n}{2}$.

Lemma 7.25. *The only non-isomorphic compression move on a graph $T(n,r)$ for $r \geq \frac{n}{2}$ is from $T(n,r)$ to a graph that has one pair of missing edges that are connected by a common vertex and all other missing edges are independent.*

Proof. We remind ourselves that a compression $C_{xy}(T(n,r))$ can either be seen as a compression from x to y on the edges of $T(n,r)$, or as a compression from y to x on the non-edges of $T(n,r)$. The only non-isomorphic compression on the non-edges of $T(n,r)$ moves one (isolated) missing edge to share a vertex with another missing edge. □

The only possible non-isomorphic compression move on $T(n,r)$ for $r \geq \frac{n}{2}$, say $C_{xy}(T(n,r))$, moves one a non-edge, say $\{b,y\}$ to $\{b,x\}$ creating two adjacent non-edges at x, say $\{a,x\}$ and $\{b,x\}$. For clarity, in the graph $T(n,r)$ we have the following neighbourhoods: $\mathcal{N}_{T(n,r)}(x) = V - \{x\} - \{a\}$ and $\mathcal{N}_{T(n,r)}(y) = V - \{y\} - \{b\}$, and in the graph $C_{xy}(T(n,r))$ we have the following neighbourhoods: $\mathcal{N}_{T(n,r)}(x) = V - \{x\} - \{a\} - \{b\}$ and $\mathcal{N}_{T(n,r)}(y) = V - \{y\}$. We will use this labelling in the proof of the lemma below.

Lemma 7.26. *The graphs $T(n,r)$ for $r \geq \frac{n}{2}$ minimize the number of cliques for all graphs with as many vertices and edges. All other graphs have strictly more cliques.*

Proof. Take n,m such that $m \geq \binom{n}{n} - \frac{n}{2}$. Let T be the Turán graph with n vertices and m edges. It minimizes the number of cliques by Corollary 7.24. We can obtain any other graph with n vertices and m edges via a sequence of compression moves from T by Corollary 7.20.

Take a pair of vertices x and y, and consider the compression $C_{xy}(T)$. If this compression is non-trivial, then by Lemma 7.25 x must have a non-edge to a, say, and y must have a non-edge to b, say, and x, y, a, b are connected to all other vertices in T. Thus, all other edges in the set of vertices $\{a, b, x, y\}$ exist and form a cycle in T. Furthermore the rest of the graph looks identical from any vertex in the set $\{a, b, x, y\}$, as each of the vertices is connected to everything else in T.

Now the compression $C_{xy}(T)$ moves the edge $\{x, b\}$ to the edge $\{y, b\}$ and nothing else by Lemma 7.25. The only cliques that have been destroyed are those that contain $\{x, b\}$ (as nothing else was moved), i.e. cliques of the form $\{x, b, x_1, ..., x_r\}$ for all possible $\{x_1, ..., x_r\}$ that make up a clique together with the vertices x, b. Neither y nor a can be in this set of vertices, as because of the missing edges $\{x, a\}$ and $\{y, b\}$ they cannot be in a clique together with both x and b. And as the remainder of the graph looks identical to all vertices in the set $\{a, b, x, y\}$, we can simply find the new cliques $\{y, b, x_1, ..., x_r\}$, one for each destroyed clique. But we also have the new clique $\{a, b, y\}$, so we have a strict increase after compression, and thus we have shown that no other graphs can be minimal. □

7.6 The Maximal Values for all m and fixed n

7.6.1 The hanging curtains conjecture

There is a section on this conjecture, as we have had this conjecture for a long time, and every result that we have obtained and every bit of computational work has led to making this conjecture more precise, which essentially says 'Turán graphs are the best kind of maximum graph'.

Suppose we plot a graph G on the following graph: The x-axis counts m, the number of edges of G, and the y-axis $a(G)$, the number of acyclic orientations of G. Now suppose we only plot the maximal values of $a(G)$. We have done so in Figure 7.12. This has been done by using the data in Appendix C. The shape of this curve (and others not shown here), have lead to the following conjectures.

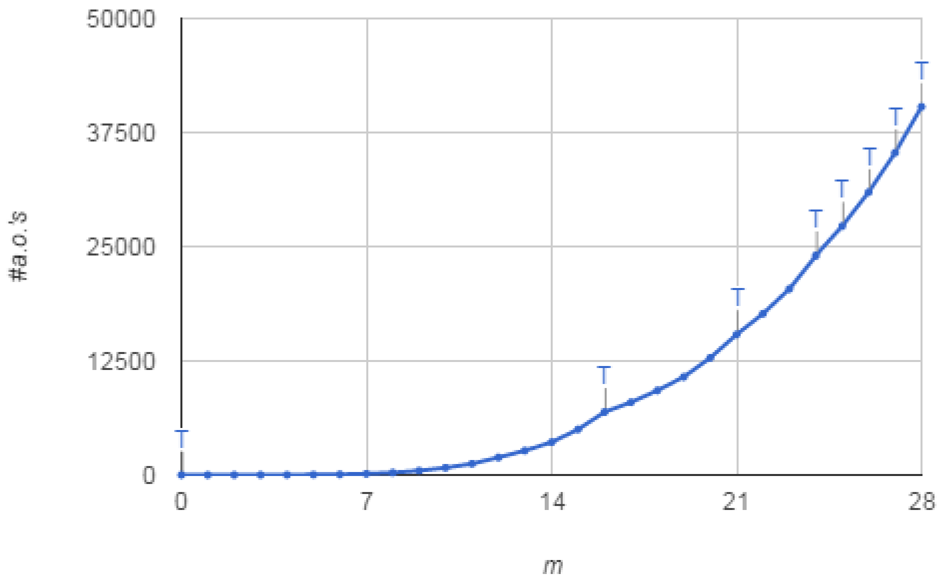

FIGURE 7.12: Hanging curtains effect for $n = 8$

Conjecture 7.27. *For each number of vertices n, we can constuct an upper bound for the maximum number of acyclic orientations of any graph by the piecewise linear concave hull, made up of lines connecting adjacent Turán graphs.*

Conjecture 7.28. *This upper bound is only achieved by the Turán graphs themselves.*

The two last conjectures together give us the hanging curtains effect. By the hanging curtains effect we mean that between neighbouring Turán graphs the slope of the curve of $a_{max}(n, m)$ is increasing, and that after every Turán graph there is a decrease in slope. You can see this effect in Figure 7.12. It is most pronounced at the complete bipartite graph, which is also the Turán graph $T(8, 2)$, abbreviated here to T at the value of $m = 16$, where you can see where the conjecture got its name. We have given the maximal values as well as some differences in Table 7.2. The first difference $a_{max}(m+1) - a_{max}(m)$ is the slope immediately after that value of m in the figure. The second difference $a_{max}(m + 1) - 2a_{max}(m) + a_{max}(m - 1)$ is the change in slope at the value m. Only right after a Turán graph does the slope decrease, otherwise the slope is increasing.

We have given the last parameter $(a_{max}(m+1) - 2a_{max}(m) + a_{max}(m - 1))$, the change in slope again in Figure 7.13. This is essentially the second derivative of the curve in

Turan	m	$a_{max}(m)$	$a_{max}(m+1) - a_{max}(m)$	$a_{max}(m+1) - 2a_{max}(m) + a_{max}(m-1)$
T	0	1	1	
	1	2	2	1
	2	4	4	2
	3	8	8	4
	4	16	16	8
	5	32	32	16
	6	64	64	32
	7	128	126	62
	8	254	216	90
	9	470	322	106
	10	792	438	116
	11	1230	690	252
	12	1920	750	60
	13	2670	932	182
	14	3602	1398	466
	15	5000	1902	504
T	16	6902	1066	-836
	17	7968	1296	230
	18	9264	1488	192
	19	10752	2088	600
	20	12840	2562	474
T	21	15402	2286	-276
	22	17688	2712	426
	23	20400	3624	912
T	24	24024	3216	-408
T	25	27240	3720	504
T	26	30960	4320	600
T	27	35280	5040	720
T	28	40320		

TABLE 7.2: The hanging curtains effect in numbers for $n = 7$.

Figure 7.12 in graphical form. If we were to choose to ignore the Turán graphs here, it would look like a gradual, nearly smooth increase. No other values are negative, and our data shows that this is the case for every value up to $n = 8$. This shows that Turán graphs are certainly special in some way. We give the same table and two figures for $n = 7$ in Table D.3, Figure D.3 and Figure D.2. Here we can observe exactly the same behaviour. It is clear that in these small examples Turán graphs are the best kind of maximum, as shown by the change in slope at isolated Turán graphs.

For every $n > 4$ we have a very pronounced 'hang' before and after the second Turán graph $T(n, 2)$, but as m increases, the Turán graphs are spaced closer and closer to each other. In Figure 7.12 and Figure 7.13 for example the last four Turán graphs are all next to each other, so it is impossible to see any 'hanging cutains' here. As n gets

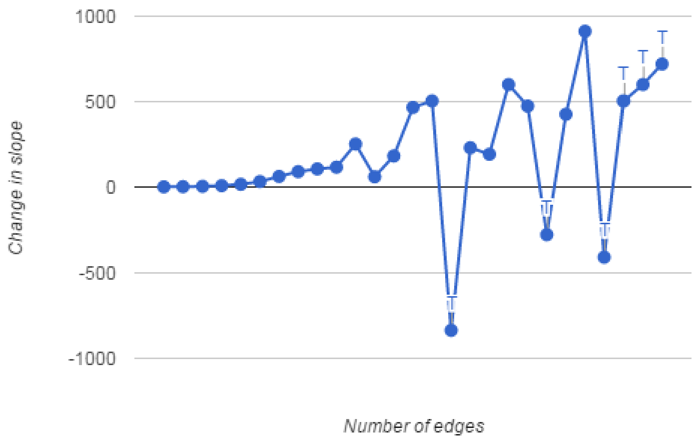

FIGURE 7.13: The change in the increase of acyclic orientations for growing m and $n = 8$, with Turán graphs highlighted.

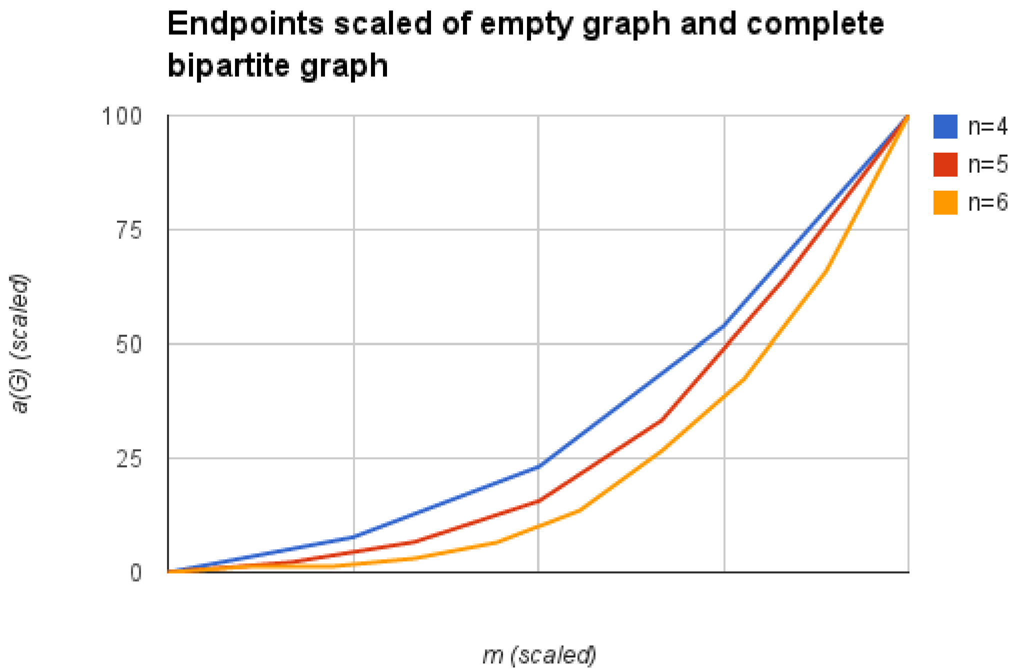

FIGURE 7.14: Rescaled maximum values $a_{max}(n, m)$ between the first and the second Turán graph (i.e. $0 \le m \le \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$ for $n = 4, 5, 6$

large though, the spacing between the Turán graphs will also increase, and so the effect should be visible here too. Unfortunately our computational power only allows us to get to $n = 8$ using the exhaustive search method, see Table 3.3 for the limits on our computational method.

We further investigate the shape of the a_{max} curve in Figure 7.14. Here we show the maximum values that $a(G)$ can take for $n = 4, 5, 6$ and for m ranging from the empty graph to the complete balanced bipartite graph, i.e. $0 \leq m \leq \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$, and we have rescaled the endpoints to coincide. For each maximum curve you can see the 'hanging' effect quite clearly, and also that the hang becomes steeper as we increase n. In particular the shape of the curve is similar for all n, so this makes it less likely that the growth of the curve here is just random. This strengthens the hanging curtains conjecture, and in particular the conjecture that Turán graphs are maximal.

7.6.2 The computational complexity of the upper bound achieved by the hanging curtains conjecture

The hanging curtains conjecture aims to describe the behaviour of the maximum value, and in doing so can also provide an upper bound for the number of acyclic orientations of a graph with n vertices and m edges that is easily calculable for every n, m. Together with the known lower bound this bounds the number of acyclic orientations a graph can have.

In Section 2.6.4 we show that we can (slightly more) efficiently calculate the number of acyclic orientations of Turán graphs than of graphs in general. In Theorem 2.30 we state that the Tutte polynomial of all graphs with bounded clique width can be computed in subexponential time. This means that we can (somewhat more efficiently) calculate the number of acyclic orientations of the Turán graphs. Putting this result together with the hanging curtains conjecture from Section 7.6.1 gives us a piecewise linear upper bound for the number of acyclic orientations that we can compute in subexponential time. We have shown what this upper bound would look like compared to the actual maximum values in Figure 7.15.

Note that the upper bound obtained in Figure 7.15 is not particularly good for the graphs with fewer edges than the Turán graph $T(7, 2)$. The upper bound becomes increasingly accurate as the number of edges increases, as the Turán graphs are closer and closer together and there is less of a deep hang between them.

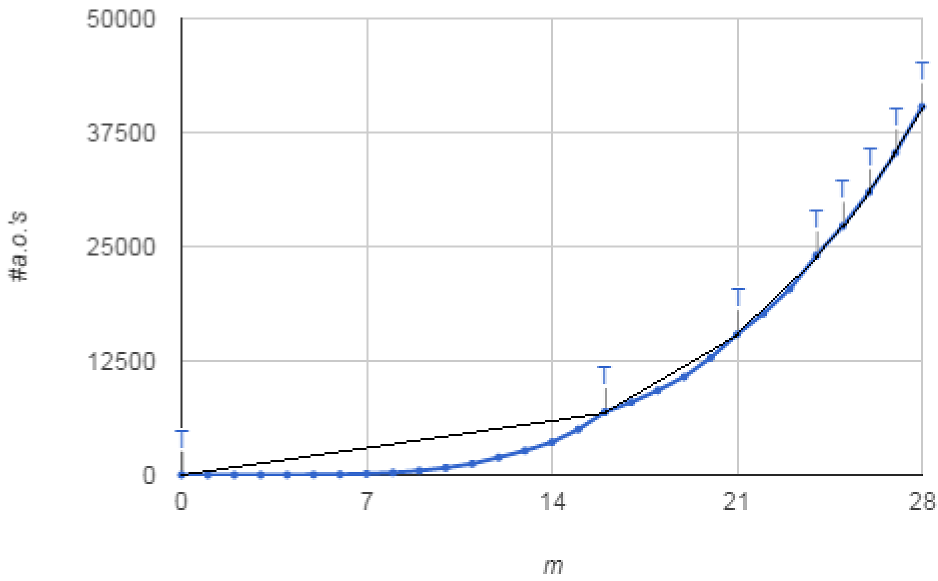

FIGURE 7.15: The upper bound achieved using a piece-wise linear approximation between Turán graphs and the actual maximum values.

7.6.3 The computational complexity of Turán like graphs

In Conjecture 7.15 we say that each maximal graph can be spilt into r subgraphs, which are all completely connected to each other. We wish to bound the size of these subgraphs above, and in order to do so, we insist that $r > \frac{n}{k}$ for some constant k. This will not necessarily mean that each subgraph of the maximum has size at most k, but might mean that each subgraph has at most size say $2k$. We may then use the algorithm given in Theorem 2.32, which gives us a subexponential algorithm for computing the number of acyclic orientations of graphs bounded clique width. In our case the candidates for maximal graphs have bounded clique width $2k$.

In some special cases we might have a good idea as to what the maximal graph candidates are, and if they all have bounded clique width then it may be possible to find the best of the candidates for slightly larger graphs than previously possible.

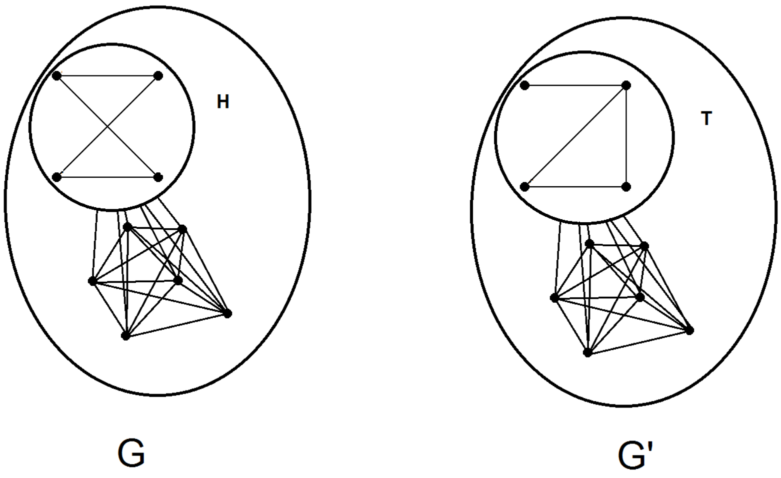

FIGURE 7.16: The compression move from G to G

7.7 A proof showing the unique maximality of a set of dense Turán graphs

To finish this chapter we give a proof that every graph in the set $T(n,r)$ for $r \geq \frac{n}{2}$ is uniquely maximal with respect to acyclic orientations.

Theorem 7.29. *Suppose we are given a graph G and a graph G' that differ only by their induced subgraphs H and H' on 4 vertices, and all other vertices are connected to each of these 4 vertices, where H is the 4-cycle, and H' a triangle with an edge attached. All other vertices are completely connected to each other.*

Then $a(G) > a(G')$, and further $a(G) = a(G') \times (1 + \frac{n-2}{n(62+(n-11)n)-124})$, where n is the number of vertices of G.

Proof. For each acyclic orientation θ of $G - H = G' - H'$ we pick a representative permutation of vertices $\pi = \pi^\theta$, which gives us the acyclic orientation θ. We will now embed the vertices of H and H' in each permutation π^θ in every possible way and count how many unique acyclic orientations we obtain through each embedding.

Note that redundancy of permutations with respect to acyclic orientations arise only when there is a missing edge between two adjacent vertices. There are no missing edges between a vertex of H and a vertex of $G - H$. Thus, it does not matter into which acyclic orientation and permutation π^θ of vertices of $G - H$ we embed the subgraph H,

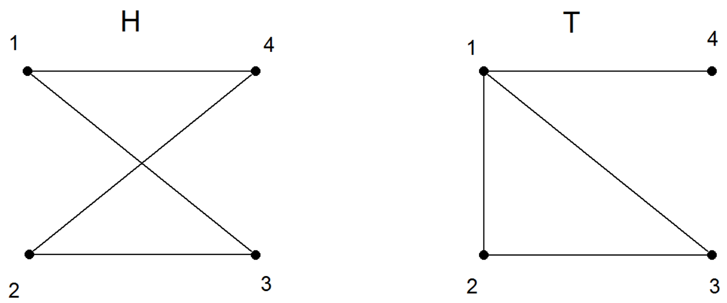

FIGURE 7.17: The labels of the graph H and T

the number of ways of extending the orientation is the same in each case. Similarly for H'.

Then redundancy of permutations with respect to acyclic orientations arise only when there is a missing edge between two adjacent vertices in H or H' which are adjacent in the permutation when it is embedded into π^θ. All other embeddings of permutations lead to unique acyclic orientations.

If we embed each vertex of H in π^θ such that it is separated from the others by vertices in π^θ, then the vertex sequence looks like this, where each square represents a vertex in H and the dots represent the remainder of the vertices:

$$1)\ldots\square\ldots\square\ldots\square\ldots\square\ldots.$$

The first and the last dots may contain no vertices, but the dots in the middle *must* contain vertices of $G - H$. Each embedding gives a different (unique) orientation for G and G', so the number of acyclic orientations we obtain in this way are the same for G and G'.

Next we consider what happens when we have two vertices of the subgraph adjacent in the vertex ordering, but no others, as shown here:

$$2a)\ldots\boxed{}\ldots\square\ldots\square\ldots$$

$$2b)\ldots\square\ldots\boxed{}\ldots\square\ldots,$$

$$2c)\ldots\square\ldots\square\ldots\boxed{}\ldots.$$

H	H'	Unique a.o.s of H	Unique a.o.s of H'
$\boxed{\quad\quad} \quad \boxed{\quad\quad}$	$\boxed{\quad\quad} \quad \boxed{\quad\quad}$		
12 34	12 34		
21 34	21 34		
12 43	12 43		
21 43	21 43	1	2
13 24	13 24		
31 24	31 24		
13 42	13 42		
31 42	31 42	4	2
14 23	14 23		
41 23	41 23		
14 32	14 32		
41 32	41 32	4	4
		9	8

TABLE 7.3: The counting of case 3.

Redundancy here arises when the two vertices that are adjacent in the final embedding have a missing edge, and in no other way. As H and H' have the same number of missing edges (i.e. neighbouring vertices), again there is an identical number of acyclic orientations with H and H' as induced subgraphs (with the remainder of G/G' the same). Note that where the pair is does not matter.

We have three cases left to consider.

$$3) \ldots \boxed{\quad\quad} \ldots \boxed{\quad\quad} \ldots,$$

$$4a) \ldots \boxed{\quad\quad\quad} \ldots \boxed{\;} \ldots,$$

$$4b) \ldots \boxed{\;} \ldots \boxed{\quad\quad\quad} \ldots,$$

$$5) \ldots \boxed{\quad\quad\quad\quad} \ldots.$$

For case 3 we will count in how many ways each embedding of two pairs of vertices gives us a new graph. In Table 7.3 we have counted in how many ways each embedding of the subgraphs H and H' gives us acyclic orientations when embedded into a $G - H$. In total H gives 9 and H' gives 8. Thus there are more acyclic orientations that arise from H than from H' here.

For case 4 where a subset of 3 adjancent vertices of H (or H') is embedded into π^θ we use a similar argument in Table 7.4. In this case actually there are the same number of ways in total to extend each graph, i.e. both subgraphs H and H' extend $G - H = G' - H'$ by the same number of acyclic orientations.

H	H'	Unique a.o.s of H	Unique a.o.s of H'
1 234	1 234		
1 243	1 243		
1 324	1 324		
1 342	1 342		
1 432	1 432		
1 423	1 423	4	2
2 134	2 134		
2 143	2 143		
2 314	2 314		
2 341	2 341		
2 413	2 413		
2 431	2 431	4	4
3 124	3 124		
3 142	3 142		
3 214	3 214		
3 241	3 241		
3 412	3 412		
3 421	3 421	4	4
4 123	4 123		
4 132	4 132		
4 213	4 213		
4 231	4 231		
4 312	4 312		
4 321	4 321	4	6
		16	16

TABLE 7.4: The counting of case 4.

For case 5 where all 4 vertices of H (or H') are adjancent in the embedding we note that this is simply a count of the number of acyclic orientations of the subgraph, which is $a(H) = 14$ and $a(H') = 12$ respectively (calculated for example by using the formula in Lemma 5.6). So H does strictly better than H' here.

Finally we can say that when embedded in every possible way in every acyclic graph G, H gives us strictly more acyclic orientations than H'. Now we will count exactly by how many acyclic orientations G does better than G' and thus obtain the factor difference between $a(G)$ and $a(G')$. We will use Table 7.5 to work out the increase in the number of acyclic orientations that H has compared to $G - H$, where H is completely connected

Case	Number of unique a.o.s of H	Number of unique a.o.s of H'	Number of permutations of parts of embedding	Number of embeddings of parts into π
1	1	1	4!	$\binom{n-3}{4}$
2	4	4	3!	$\binom{n-3}{4}$
3	9	8	2!	$\binom{n-3}{2}$
4	16	16	2!	$\binom{n-3}{2}$
5	14	12	1!	$\binom{n-3}{1}$

TABLE 7.5: Counting the increase in acyclic orientations due to H and H' respectively.

to $G - H$. Putting together all the pieces we have calculated so far gives us:

$$\begin{aligned} a(G) = a(G-H) \times \Bigg[& 4! \times \binom{n-4}{4} + 4 \times 3! \times \binom{n-3}{3} \\ & + 9 \times 2 \times \binom{n-3}{2} + 16 \times 2 \times \binom{n-3}{2} \\ & + 14 \times \binom{n-3}{1} \Bigg] \end{aligned}$$

Similarly we have

$$\begin{aligned} a(G') = a(G'-H') \times \Bigg[& 4! \times \binom{n-4}{4} + 4 \times 3! \times \binom{n-3}{3} \\ & + 8 \times 2 \times \binom{n-3}{2} + 16 \times 2 \times \binom{n-3}{2} \\ & + 12 \times \binom{n-3}{1} \Bigg]. \end{aligned}$$

But $a(G-H) = a(G'-H')$, and thus the factor increase can be calculated as $1 + \frac{-2+n}{-124+n(62+(-11+n)n)}$.

□

We now compare two graphs H and H' in the next theorem, which we will use in our proof of showing that $T(n,r)$ for $r \geq \frac{n}{2}$ is uniquely maximal with respect to the number of acyclic orientations.

Theorem 7.30. *Let H be the complete graph on n vertices with $r+1$ independent edges missing, for $r \geq 1$, i.e. at least two missing edges. Let H' be the complete graph on n vertices with $r-1$ independent edges missing as well as a pair of connected edges missing. Then $a(H) > a(H')$, i.e. H has more acyclic orientations than H'.*

Proof. We choose to look at the increase in acyclic orientations from the graph G to H and G to H', where G is the complete graph on $n-1$ vertices with r independent edges missing. We obtain H from G by adding a vertex v_n to G, which is connected to all vertices in G except for one vertex v_{n-1}, which also connected to all other vertices in G. On the other hand we obtain H' by adding the vertex v_n, which is connected to all vertices in G, except to one vertex which already has a missing edge in G, say v_{n-2}. The vertex in G which v_{n-2} is not connected to we label v_{n-3}.

We pick any acyclic orientation θ of G. All vertices are either ordered with respect to all other vertices, or all other vertices bar one. Suppose there are k such pairs of vertices that are not ordered by this acyclic orientation ($0 \leq k \leq r$). We consider these pairs of vertices together with the single vertices of G as *parts*. There are $n-k-1$ parts.

We now consider in how many ways we can expand this acyclic orientation θ by the addition of the extra vertex v_n both in H and H'. There are $n-k$ places to insert the vertex n_n in-between parts, and for each part of size 2 there are 2 ways of inserting the extra vertex v_n such that it splits up the part. We now count how many *unique* acyclic orientations are grown from G by these insertions.

For the graph H, inserting the vertex in-between parts, when v_n is inserted directly before vertex v_{n-1} we obtain the same acyclic orientation as directly after, leading to a '-1' term, giving us $n-k-1$ extensions of this particular acyclic orientation.

For the graph H, if the vertex v_n splits a part of size 2, this leads to 2 possible acyclic orientations. All splits are possible, hence we obtain $2k$ acyclic orientations.

In H', when v_{n-2} is adjacent to v_{n-3} in θ, and we are inserting in-between parts, each insertion leads to a unique acyclic orientations, hence there are $n-k$ acyclic orientations gained.

In H', when v_{n-2} is adjacent to v_{n-3} in θ, and v_n splits a part of size 2, we may split each part in 2 ways, except for the part (v_{n-2}, v_{n-3}). Thus $k-1$ parts may be split, each in 2 possible ways, giving us $2k-2$ acyclic orientations.

In H', when v_{n-2} is not adjacent to v_{n-3} in θ, and we are inserting in-between parts, we obtain the same acyclic orientation by inserting v_n directly before or after v_{n-2}. Thus we are left with $n-k-1$ new acyclic orientations.

In H', when v_{n-2} is not adjacent to v_{n-3} in θ, and v_n splits a part of size 2, each possible split gives us 2 new acyclic orientations, so we have $2k$ new acyclic orientations.

We have summarized these counts in Table 7.6.

	H	H'	
	$\forall\theta$	v_{n-2} not adj to v_{n-3} in θ	v_{n-2} adj to v_{n-3} in θ
In-between parts	$n-k-1$	$n-k-1$	$n-k$
Splitting parts	$2k$	$2k$	$2k-2$
Total	$n+k-1$	$n+k-1$	$n+k-2$

Table 7.6: The increase in acyclic orientations for a specific acyclic orientation.

We can obtain the total increase of acyclic orientations by summing over all acyclic orientations of G, but we simply note that the increase is always greater of equal for each individual orientation. Thus as one increase is strict (pick any orientation where v_{n-2} is adjacent to v_{n-3}), we have shown that $a(H) > a(H')$.

□

Theorem 7.31. *For n, m with $m > \binom{n}{2} - \frac{n}{2}$, the graph $T(n,r)$ with n vertices and m edges uniquely maximises the number of acyclic orientations for graphs with n vertices and m edges.*

Proof. For $m = \binom{n}{2} - 1$ or $m = \binom{n}{2}$ there is only one graph with m edges, so the maximum is necessarily unique. Thus we need only consider $m \leq \binom{n}{2} - 2$. We know by Theorem 7.22 that the graph $T(n,r)$ maximises the number of acyclic orientations for $m \geq \binom{n}{2} - \frac{n}{2}$. In Theorem 4.26 we showed that compression is monotonic with respect to the number of acyclic orientations. Further we can find a sequence of compression moves to obtain all other graphs on n vertices and m edges by Lemma 7.20. Thus it is sufficient to prove that every compression move from $T(n,r)$ strictly decreases the number of acyclic orientations. The only compression move possible on $T(n,r)$ is given to us in Lemma 7.25 and acts on a subgraph of $T(n,r)$ as the compression of a 4-cycle to a 3-cycle (a triangle) with an extra edge attached. No other edges can be moved, as x and y are completely connected to all other vertices. Thus this compression move fulfils the criteria necessary in Theorem 7.30, so we know that there is a strict decrease in the number of acyclic orientations by this compression, completing the proof. □

7.8 Summary of results and further research

We have given insight into both the structure of maximal graphs and the maximal values in this chapter. An upper bound for maximal values is conjectured to be the piecewise linear hull obtained by connecting neighbouring Turán graphs. We have computational support for this conjecture for up to $n = 8$. Furthermore we have a proof that the Turán graphs $T(n, r)$ for $r \geq \frac{n}{2}$ are uniquely maximal.

We have also given an insight into the structure of the maximal graphs, which leads to conjecture for graphs with lower densities, in terms of edge-connectedness in Conjecture 7.10, and in terms of the Turán-like structure in Conjecture 7.15. We have given a general result that show that in any graph G under certain conditions it is optimal to use the Turán-like property in Theorem 7.29.

In future work we would like to expand on the methodology used in Theorem 7.29 in a number of ways. We would like to generalize to the following conjecture.

Conjecture 7.32. *Suppose we are given a graph G and a graph G' that differ only by their induced subgraphs H and H', and all other vertices are connected to each of these 4 vertices. If H and H' have the same number of vertices and edges, and $a(H) > a(H')$, then $a(G) > a(G')$.*

This result together with the Turán-like property could then be used to greatly reduce the search space for finding maximal graphs for graphs with at least $m \geq \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$ edges, as we can then reduce the problem to only checking graphs with all optimal subgraphs, effectively giving us a way to inductively find maximal graphs.

Chapter 8

Contribution and Future Work

At the end of individual chapters more detailed conclusions and/or suggestions for future work are given as appropriate. To conclude the thesis we will summarize the key points here.

8.1 Contribution

We have obtained a full set of computational results for the number of acyclic orientations for up to $n = 8$ in Chapter 3. We have identified candidates for the minimal and maximal graphs as complete graphs and Turán graphs respectively. We give conditions on the minimal degree and the edge-connectedness of a graph that might aid the search for maximal graphs.

We introduced the compression method as a tool for working in the space of acyclic orientations in Chapter 4. We completely classified the minimum graphs in Chapter 4, which had previously been done, but our compression method added new insight into the structure of minimal graphs. In particular the compression move is monotonic with respect to the number of acyclic orientations, number of forests and number of cliques. Thus we have successfully linked these parameters together and unified an approach to studying them.

We also make some progress in the much harder problem of classifying the maximal graphs with respect to acyclic orientations in Chapter 7. In particular we prove that

Turán graphs for $m \geq \binom{n}{2} - \frac{n}{2}$ are maximal using the tool of compression developed in Chapter 4. We go on to show that each of these Turán graph uniquely maximises the number of acyclic orientations for graphs with as many vertices and edges in Theorem 7.29. The result was obtained by using our new factor approach developed in Chapter 5 on two special subgraphs in a Turán-like graph.

We explicitly count the number of acyclic orientations of complete bipartite graphs in Chapter 6, which includes the Turán graph $T(n, 2)$, which we conjecture to be maximal in Chapter 7. We further link up counting acyclic orientations of complete bipartite graphs with lonesum matrices and poly-Bernoulli numbers. We also count the number of acyclic orientations of complete bipartite graphs minus and plus an edge.

We provide an alternate framework, the factor method, for counting the number of acyclic orientations in Chapter 5. We develop basic properties and give some insight into how factors can be useful. In particular we develop a proof which shows that the complete bipartite graph is locally optimal with respect to edge moves based on a hypothesis. We use the concept of the factor in Chapter 7 to prove unique maximality of certain Turán graphs.

We have linked acyclic orientations to other graph parameters, which gives some (easier to realize) conditions to optimize network design for example evening out the vertex degree.

8.2 Future work

Many new questions arise from this work. We highlight only the ones of particular interest, and where we feel results will be easiest to obtain.

- It may be possible to apply the compression technique to further graph parameters, which will then prove that the same set of graphs is minimal/maximal, in particular to k-edge-connectedness.
- Our proof showing that very dense Turán graphs ($m \geq \binom{n}{2} - \frac{n}{2}$) are uniquely maximal might be extended to proving which graphs are maximal up to $\binom{n}{2} - \frac{n}{2} \geq m \geq \binom{n}{2} - n$. In this range not all graphs are Turán graphs, so any result here could support the hanging curtains conjecture.

- We want to be able to bound factors better in order to obtain bounds on the number of acyclic orientations of graphs.
- We would like to find a version of the factor hypothesis that is true, in order to prove local optimality of complete bipartite graphs
- We want to prove that the complete balanced bipartite graph is maximal with respect to the number of acyclic orientations
- We want to prove that all Turán graphs are (uniquely) maximal with respect to the number of acyclic orientations

Appendix A

Using the Chromatic Polynomial to Count Acyclic Orientations

A.0.1 The Chromatic Polynomial

First I shall talk about the chromatic polynomial of a graph $\chi(G, \lambda)$, the number of proper colourings of a given graph G with λ colours.

$\chi(G, \lambda)$ can be (inductively) defined as follows:

$$\chi(G_0, \lambda) = \lambda,$$

where G_0 is the graph with one vertex and zero edges.

$$\chi(G \sqcup H, \lambda) = \chi(G, \lambda)\chi(H, \lambda),$$

where $G \sqcup H$ denotes the disjoint union of G and H.

$$\chi(G, \lambda) = \chi(G - e, \lambda) - \chi(G/e, \lambda),$$

where this is just the deletion contraction relation, and e is an edge of G.

A.0.2 Another Interpretation of the Chromatic Polynomial

As usual, let $V(G)$ be the vertex set of the graph G.

Proposition A.1. *$\chi(G, \lambda)$ is the number of pairs (σ, θ) for any map $\sigma : V(G) \to \{1, 2, ..., \lambda\}$ and θ is an acyclic orientation of G such that: If $u \to v$, i.e. if there is a directed edge from vertex u to vertex v, then $\sigma(u) > \sigma(v)$.*

Proof. The last condition ensures that every allowed σ will be a proper colouring (adjacent vertices get different colours). On the other hand, every proper colouring gives us a unique acyclic orientation of G. Hence the number of allowed σ's is the number of proper colourings with the colours 1,2,...,λ, which is $\chi(G, \lambda)$. □

A.0.3 A Related Polynomial

Now we define $\overline{\chi}(G, \lambda)$, as defined by Stanley:

Definition A.2. $\overline{\chi}(G, \lambda)$ is the number of pairs (σ, θ) for any map $\sigma : V(G) \to \{1, 2, ..., \lambda\}$ and θ is an acyclic orientation of G such that: If $u \to v$, i.e. if there is a directed edge from vertex u to vertex v, then $\sigma(u) \geq \sigma(v)$.

The only difference to $\chi(G, \lambda)$ is the condition that $\sigma(u) \geq \sigma(v)$, rather than $\sigma(u) > \sigma(v)$. When this (weaker) condition is fulfilled, we say σ is *compatible* with θ. The relationship between $\chi(G, \lambda)$ and $\overline{\chi}(G, \lambda)$ is as follows:

Proposition A.3.

$$\begin{aligned} \overline{\chi}(G_0, \lambda) &= \lambda & \text{(A.1)} \\ \overline{\chi}(G \sqcup H, \lambda) &= \overline{\chi}(G, \lambda)\overline{\chi}(H, \lambda) & \text{(A.2)} \\ \overline{\chi}(G, \lambda) &= \overline{\chi}(G - e, \lambda) + \overline{\chi}(G/e, \lambda) & \text{(A.3)} \end{aligned}$$

These equations hold for $\overline{\chi}(G, \lambda)$.

Proof. (A.1) and (A.2) both follow immediately from the definition of $\overline{\chi}(G, \lambda)$. (A.3) will be shown in the following:

We start by considering the Graph $G - e$, where e is an edge from vertex u to vertex v.

Let $\sigma^* : V(G-e) \to \{1,2,...,\lambda\}$ be any map and let θ^* be an acyclic orientation compatible with σ^*, both as before. Now to get from $G-e$ to G, we must add the edge e. Let θ_1 be the orientation when we add the edge $u \to v$ and θ_2 the orientation when we add $v \to u$. Let σ^* be the same, as we are not adding any vertices. Note that the increase here is exactly the increase in the number of possible acyclic orientations we were interested in in earlier talks! We can now explicitly show what this growth is.

When we add the edge e between the vertices u and v, we have already defined the values $\sigma^*(u)$ and $\sigma^*(v)$. There are three cases that can happen, and in each of them we will see how many acyclic orientations are added.

$$\sigma^*(u) > \sigma^*(v) \tag{A.4}$$

$$\sigma^*(u) < \sigma^*(v) \tag{A.5}$$

$$\sigma^*(u) = \sigma^*(v) \tag{A.6}$$

In (A.4) and (A.5) only the edge pointing in the direction from $u \to v$ or $v \to u$ respectively gives us an acyclic orientatain, so here we are not adding anything.

The interesting case is (A.6). When we have equality, at least one orientation of the edge e will give us a valid acyclic orientation; suppose not. Then there exist cycles along the following vertices: u, v, w_1, ... , w_n and also along v, u, v_1, ... , v_m. But patching these together (possibly by removing double edges) we can obtain a cycle in $G-e$ which contradicts our assumption the $G-e$ is acyclic.

Now we have shown that every acyclic orientation of $G-e$ can be extended to an acyclic orientation of G, this gives us the term $\overline{\chi}(G-e,\lambda)$ in (A.3). It remains to be shown that precisely $\overline{\chi}(G/e,\lambda)$ edges can have both orientations. We can easily establish this fact, by bijecting these cases with all acyclic orientations of G/e;

Every acyclic orientation of G, where both directions are possible for e must have condition (A.6). Thus u and v have the same label. Now in G/e let z be the vertex at the fused edge, and let $\sigma^*(z) = \sigma^*(v)$. Keep all other orientations of edges the same as in G to obtain a unique acyclic orientation of G/e.

On the other hand, every acyclic orientation of G/e can be extended to a unique orientation of G, up to the orientation of the edge e in the obvious way; keep all orientations the same, and let $\sigma^*(u) = \sigma^*(v) = \sigma^*(z)$. We have now injected both sets into each other, hence they must be the same size. We cannot have created a cycle in this step, otherwise the cycle without e would be a cycle in G/e.

□

Thus we have shown the remarkable fact, that adding an edge to a graph increases the number of possible acyclic orientations by:

$$\overline{\chi}(G/e, \lambda)$$

We now further show that the following holds: (where $|G|$ denotes $|V(G)|$)

Proposition A.4.

$$\overline{\chi}(G, \lambda) = (-1)^{|G|}\chi(G, -\lambda)$$

for $\lambda \in \mathbb{N}$

Proof. We shall show that this is true by induction. We assume that the result is true for all G with either fewer edges, fewer vertices, or both fewer edges or vertices. To start the induction we have:

$$\overline{\chi}(G_0, \lambda) = \lambda = (-1)^1(-\lambda) = (-1)^{|G_0|}\chi(G_0, -\lambda)$$

If we wish to add a disconnected graph, we may use (A.2):

$$\begin{aligned}
\overline{\chi}(G \sqcup H, \lambda) &= \overline{\chi}(G, \lambda)\overline{\chi}(H, \lambda) \\
&= (-1)^{|G|}\chi(G, -\lambda)(-1)^{|H|}\chi(H, -\lambda) \\
&= (-1)^{|G|+|H|}\chi(G \sqcup H, -\lambda)
\end{aligned}$$

If we wish to add an edge, we may use (A.3):

$$\begin{aligned}
\overline{\chi}(G \cup e, \lambda) &= \overline{\chi}(G, \lambda) + \overline{\chi}((G \cup e)/e, \lambda) \\
&= (-1)^{|G|}\chi(G, -\lambda) + (-1)^{|V|-1}\chi(G \cup e)/e, -\lambda) \\
&= (-1)^{|G|}(\chi(G, -\lambda) - \chi(G \cup e)/e, -\lambda)) \\
&= (-1)^{|G|}\chi(G \cup e, -\lambda)
\end{aligned}$$

So now we can build all graphs and the eqation holds true for each step, hence it must be true in general. □

Now let us consider what we are counting in $\overline{\chi}(G, \lambda)$ when we only use one colour, i.e. $\lambda = 1$. Then *every* acyclic orientation is compatible with the one unique colouring, hence we are just counting the number of acyclic orientations!

Appendix B

Computational Details

The purpose of this appendix is to describe how we carried out our computational work of enumerating graphs and the the number of acyclic orientations for small values of n ($n \leq 8$) in a sufficient level of detail to facilitate replication.

B.1 User guide part 1: Using nauty and the graph6 (.g6) file format

The full set of graphs up to isomorphism for any value of n have been provided by the nauty software [45]. They are given in the graph6 file format, which we briefly explain here. A typical graph in graph6 format is displayed as follows.

```
G?~vf_
```

This is the complete bipartite graph on 8 vertices and 16 edges, with each partition of size 4. Using the package "showg" which is part of nauty, we can turn this into an adjacency matrix, which is a human readable form as seen in Table B.1. You can see that the file format g6 is both completely unreadable for humans (in any meaningful way) and also vastly more efficient then an adjacency matrix. Where appropriate throughout the thesis we give a human readable form of any graph data points that are of interest. We will either use the adjacency matrix, the graph diagram, or other forms of representing the graph to give as much insight as possible.

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

TABLE B.1: The output of showg -A graph1.g6

In order to use the .g6 files in Mathematica they had to first be converted into another format (.dat), with the separators between entries a space, the separator for the next row a new line, and an empty line as the separator for a new graph. The output of running nauty with -A give us the adjacency matrix of a graph with spaces. Also adding -q, suppresses all additional output, which gives nearly the required output, except instead of an empty line it gives the order of the graph, so a simple find and replace text editor gives us the required format.

B.2 User guide part 2: The program using Mathematica Combinatorica

To obtain the number of acyclic orientations of a specific graph we have used the Combinatorica package of Mathematica version 9 [67]. It calculates the chromatic polynomial of a graph, and evaluates it at any value, for us at -1. Explicit steps of execution follow. To load Combinatorica in Mathematica simply write the following code and press Shift+Enter:

```
<<Combinatorica`
```

The Combinatorica package contains many useful graph theory tools, amongst them the computation of the chromatic polynomial. The chromatic polynomial of any graph G can be computed via the line of code:

```
ChromaticPolynomial[G,x]
```

Unfortunately this particular function does not allow files in the form g6, and does also not work on imported files of multiple graphs (I am unsure why). Using the following code in Mathematica for the file graphs4.dat, which contains all graphs in the adjacency matrix form described above, we obtain a file with the number of acyclic orientations for each respective graph in a new line:

```
f[x_] = ChromaticPolynomial[x, -1]
aostable4 =
 f /@ FromAdjacencyMatrix /@
   Partition[
    Import["/home/schuie/nauty25r9/graph4.dat", "Table",
     "IgnoreEmptyLines" -> True], 4]
```

The input is a .dat file with 0-1 matrices, with separators space, new line and empty line which we have demonstated how to obtain in the user guide part 1. Mathematica cannot read these as matrices, hence we import the file as a table and ignore empty lines:

```
Import[...,"IgnoreEmptyLines" -> True]
```

The imported file is now a table of vectors of length $n = 4$. The partition function allows us to create sets of equal size ($n = 4$) of vectors:

```
Partition[..., 4]
```

We now have sets of size $n = 4$ with elements vectors of size $n = 4$. These are turned into graph format, which Mathematica requires for the ChromaticPolynomial function using FromAdjacencyMatrix on each of the sets of sets, which gives us these sets as adjacency matrices:

```
FromAdjacencyMatrix /@ ...
```

Finally we apply the function f to each of these matrices, and save the output as aostable4:

G	a(G)
C?	1
CC	2
CE	4
CF	8
CQ	4
CT	6
CU	8
CV	12
C]	14
C^	18
C~	24

TABLE B.2: The number of acyclic orientations for all graphs with 4 vertices

```
aostable4 =  f /@ ...
```

The output of the process is a complete list of all graphs up to isomorphism and their number of acyclic orientations, such as the Table B.2 for graphs of 4 vertices, the full set of which is is Appendix C.

B.3 Computing details

The computer used had the following technical specifications.

```
RAM: 8122508 kB
CPU: i7-2630QM CPU @ 2.00GHz x 8
OS: Ubuntu 12.04.4 LTS
Kernel version and system architecture: 3.13.0-32-generic
#57~precise1-Ubuntu SMP Tue Jul 15 03:51:20
UTC 2014 x86_64 x86_64 x86_64 GNU/Linux
```

Run times using this computer can be found in Table 3.3. For the complete enumeration of $n = 8$ the data-input was split into 28 parts (1,2,3...,28), each part corresponding to all graphs with as many edges.

Appendix C

Computational Data

In this part of the appendix we present the number of edges and number of acyclic orientations of each graph with up to 7 vertices. The graphs are provided with their short-hand graph6 identifier, see Appendix A for details. The complete data for $n = 8$ are attached in digital form, as there are too many graphs to print here (For $n = 7$ there are roughly 1000 graphs, for $n = 8$ there are more than $12,000$ i.e. more than 60 pages in this format).

G	m	a(G)
A?	0	1
A_	1	2

TABLE C.1: List of graphs with 2 vertices

G	m	a(G)
B?	0	1
BO	1	2
BW	2	4
Bw	3	6

TABLE C.2: List of graphs with 3 vertices

G	m	a(G)
C?	1	1
CC	2	2
CE	3	4
CF	4	8
CQ	2	4
CT	3	6
CU	4	8
CV	4	12
C]	4	14
Cˆ	4	18
C	5	24

TABLE C.3: List of graphs with 4 vertices

G	m	a(G)	G	m	
D??	0	1	DEk	5	24
D?_	1	2	DQw	5	24
D?o	2	4	DEw	5	28
DCO	2	4	DUW	5	30
DCc	3	6	DTk	6	24
D?w	3	8	DE{	6	36
DCW	3	8	DQ{	6	36
DCo	3	8	DTw	6	36
DCs	4	12	DUw	6	42
DQg	4	12	DFw	6	46
DEo	4	14	DT{	7	48
D?{	4	16	DF{	7	54
DCw	4	16	DU{	7	54
DQo	4	16	D]w	7	60
DEs	5	18	DV{	8	72
DC{	5	24	D]{	8	78
DEk	5	24	Dˆ{	9	96
DQw	5	24	D {	10	120

TABLE C.4: List of graphs with 5 vertices

G	m	a(G)	G	m	a(G)	G	m	
E???	0	1	ECpo	6	60	EEuw	9	144
E?A?	1	2	EEh_	6	62	EQjw	9	144
E?B?	2	4	ECfW	7	48	EQzg	9	144
E¿?	2	4	EQig	7	48	EErw	9	156
E?aG	3	6	E?zg	7	54	EEvo	9	156
E?B_	3	8	ECrW	7	54	E? w	9	162
E¿_	3	8	EErO	7	60	ECzw	9	162
E?b?	3	8	E?rw	7	72	EEjw	9	162
ECO_	3	8	E?zW	7	72	EElw	9	162
E?bG	4	12	ECRw	7	72	EQzW	9	162
ECQO	4	12	ECZW	7	72	EQyw	9	168
E?r?	4	14	ECZg	7	72	EEno	9	180
E?Bo	4	16	ECfo	7	72	EEzg	9	180
E¿o	4	16	ECrg	7	72	EQzo	9	180
E?b_	4	16	EEiW	7	72	EUZo	9	186
E?oo	4	16	EQjO	7	72	EC o	9	192
ECQ_	4	16	EQj_	7	72	EEzo	9	198
ECR?	4	16	ECZo	7	84	EUxo	9	204
E?rG	5	18	ECro	7	84	EFz_	9	230
E?bg	5	24	EEhW	7	84	ETmw	10	120
E?qg	5	24	EEio	7	84	EEvw	10	192
ECQo	5	24	EEho	7	90	ETno	10	192
ECRO	5	24	E?zo	7	92	EC w	10	216
ECYO	5	24	EEr_	7	92	EEnw	10	216
E?r_	5	28	EEj_	7	98	EQzw	10	216
ECX_	5	28	ECxo	7	102	ETzg	10	216
ECpO	5	30	ECvW	8	72	EEzw	10	234
E?Bw	5	32	EErW	8	78	EUzW	10	234
E?bo	5	32	ECfw	8	96	ETzo	10	240
E?ow	5	32	ECuw	8	96	EUZw	10	240
E?qo	5	32	EQjg	8	96	EFzW	10	252
ECR_	5	32	E?zw	8	108	EQ o	10	252
ECZ?	5	32	ECZw	8	108	EUzo	10	258
ECeW	6	24	ECrw	8	108	EFzo	10	276
E?rg	6	36	ECzW	8	108	ETnw	11	240
ECRW	6	36	ECzg	8	108	EE w	11	288
ECXg	6	36	EEjW	8	108	EQ w	11	288
ECrG	6	36	EQjo	8	108	ETzw	11	288
EQhO	6	36	ECvo	8	120	EUzw	11	312
ECrO	6	42	EEro	8	120	E]zg	11	312
E?z_	6	46	EEzO	8	120	EFzw	11	330
E?bw	6	48	ECxw	8	126	E]zo	11	330
E?qw	6	48	EEhw	8	126	E]yw	11	336
ECRo	6	48	EEjo	8	126	ET w	12	360
ECYW	6	48	EQzO	8	126	EF w	12	384
ECZG	6	48	EUZO	8	132	EU w	12	384
ECZO	6	48	ECzo	8	138	E]zw	12	408
ECqg	6	48	EUZ_	8	144	E] o	12	426
E?ro	6	56	E? o	8	146	EV w	13	480
E?zO	6	56	EEz_	8	152	E] w	13	504
ECZ_	6	56	EEvW	9	96	Eˆ w	14	600
ECr_	6	56	ECvw	9	144	E w	15	720

Table C.5: List of graphs with 6 vertices

G	m	a(G)	G	m	a(G)	G	m	a(G)	G	m	a(G)
F????	0	1	F¿F?	6	32	F¿fG	7	72	F?qaW	8	84
F??C?	1	2	F¿c_	6	32	F?aNO	7	72	F?qdO	8	84
F??E?	2	4	F¿EW	6	36	F?bFG	7	72	F?qbO	8	90
F??F?	2	8	F¿bG	6	36	F?bMW	7	72	F?osW	8	96
F?AA?	3	4	F?Bf?	6	46	F¿fO	7	84	F?otO	8	96
F?AB?	3	8	F¿Do	6	48	F?bFO	7	84	F?qf?	8	98
F??F_	3	16	F¿FO	6	48	F¿r_	7	92	F?bbO	8	102
F??Fo	3	32	F¿ag	6	48	F¿cw	7	96	F?beW	8	108
F??Fw	3	64	F¿cW	6	48	F¿eg	7	96	F?bfG	8	108
F?ACG	4	6	F?BfG	6	54	F¿eo	7	96	F?qeW	8	108
F?AE?	4	8	F¿b_	6	56	F¿uO	7	96	F?opo	8	112
F?AEG	4	12	F¿F_	6	64	F?aNW	7	96	F?ov?	8	112
F?AB_	4	16	F?Beg	6	72	F?aN_	7	96	F?q_w	8	112
F?AF?	4	16	F¿FW	6	72	F?bDg	7	96	F?bao	8	120
F?AFG	4	24	F¿bg	6	72	F?bDo	7	96	F?q‘o	8	120
F?ABo	4	32	F?Bf_	6	92	F?bLW	7	96	F?qb_	8	124
F?AF_	4	32	F¿Fo	6	96	F¿fW	7	108	F?bbW	8	126
F?AFg	4	48	F?Bfg	6	108	F¿rg	7	108	F?qbW	8	126
F?AFo	4	64	F?Beo	6	112	F?bFW	7	108	F?qfO	8	126
F?BE?	5	14	F?Bew	6	144	F¿f_	7	112	F?bfO	8	138
F?B@_	5	16	F¿Fw	6	144	F¿v?	7	112	F?bcw	8	144
F?BEG	5	18	F?Bv_	6	146	F?bF_	7	112	F?beg	8	144
F?BDG	5	24	F?Bvg	6	162	F?bBo	7	120	F?otW	8	144
F?BF?	5	28	F?Bfo	6	184	F?bNO	7	120	F?ouW	8	144
F?B@g	5	32	F?BvO	6	184	F¿ew	7	144	F?ovO	8	144
F?B@o	5	32	F?Bfw	6	216	F¿fg	7	144	F?qcw	8	144
F?BD_	5	32	F?BvW	6	216	F¿uW	7	144	F?qeo	8	144
F?BFG	5	36	F?Bvo	6	292	F¿vG	7	144	F?qmW	8	144
F?BDg	5	48	F?Bvw	6	324	F?aNo	7	144	F?bfW	8	162
F?BF_	5	56	F?B o	6	454	F?bFg	7	144	F?qfW	8	162
F?Be_	5	56	F?B w	6	486	F?bLo	7	144	F?qjW	8	162
F?B@w	5	64	F?aKW	7	24	F?bNW	7	144	F?baw	8	168
F?BDo	5	64	F?bAO	7	30	F¿fo	7	168	F?beo	8	168
F?Bco	5	64	F¿e?	7	32	F¿vO	7	168	F?oto	8	168
F?BFg	5	72	F?bEG	7	36	F?bFo	7	168	F?qaw	8	168
F?AFw	5	96	F?bEO	7	42	F?bN_	7	168	F?qdo	8	168
F?BDw	5	96	F¿co	7	48	F¿v_	7	184	F?qbo	8	180
F?Bcw	5	96	F¿eG	7	48	F?aNw	7	192	F?qnO	8	180
F?BFo	5	112	F¿eO	7	48	F?bLw	7	192	F?qrO	8	180
F?BFw	5	144	F?aJ_	7	48	F¿fw	7	216	F?bf_	8	184
F¿@?	6	8	F?aMW	7	48	F¿vW	7	216	F?bnO	8	192
F¿CO	6	12	F?bDG	7	48	F¿vg	7	216	F?qkw	8	192
F¿@_	6	16	F?bEW	7	54	F?bFw	7	216	F?ov_	8	196
F¿D?	6	16	F¿f?	7	56	F?bNg	7	216	F?qf_	8	196
F¿E?	6	16	F?bF?	7	56	F¿vo	7	276	F?bbo	8	204
F¿DO	6	24	F?bBO	7	60	F¿vw	7	324	F?qpo	8	204
F¿EO	6	24	F¿cg	7	64	F?ou?	8	56	F?qr_	8	214
F¿cO	6	24	F¿e_	7	64	F?qb?	8	62	F?bew	8	216
F¿b?	6	28	F?bB_	7	64	F?ouO	8	72	F?bfg	8	216
F¿D_	6	32	F¿eW	7	72	F?qcW	8	72	F?bnW	8	216

TABLE C.6: Part 1 of list of graphs with 7 vertices

G	m	a(G)	G	m	a(G)	G	m	a(G)	G	m	a(G)
F?ovW	8	216	FCQ‘_	9	60	F?rdo	9	288	F?zVo	9	594
F?o—W	8	216	FCOf?	9	64	F?rnW	9	288	F?rvo	9	630
F?o}W	8	216	FCQb?	9	64	FCQVw	9	288	F?q w	9	648
F?qew	8	216	FCOeo	9	72	F?rf_	9	304	F?z\w	9	648
F?qmo	8	216	FCQRO	9	72	F?qro	9	306	F?zuo	9	648
F?qnW	8	216	FCQT_	9	72	F?rFw	9	312	F?zfw	9	660
F?bNo	8	240	FCQUO	9	72	F?rNo	9	312	F?zno	9	660
F?bmo	8	240	FCQU_	9	72	F?rew	9	312	F?rvw	9	702
F?qlo	8	240	F?rEW	9	78	F?zUW	9	312	F?zVw	9	702
F?bbw	8	252	F?rF?	9	92	F?qv_	9	322	F?zv_	9	736
F?ovo	8	252	F?rMW	9	96	F?qtw	9	324	F?zˆo	9	756
F?o O	8	252	FCOf_	9	96	F?qvW	9	324	F?znw	9	768
F?qbw	8	252	FCQTg	9	96	F?zfW	9	330	F?r o	9	792
F?qfo	8	252	FCQUg	9	96	F?rdw	9	360	F?zvg	9	828
F?qjo	8	252	FCQbO	9	96	F?rfg	9	360	F?r w	9	864
F?bfo	8	276	FCQUo	9	108	F?zTW	9	360	F?zˆw	9	864
F?bvO	8	276	F?rFO	9	120	F?zVO	9	360	F?zvo	9	882
F?o—o	8	276	F?reO	9	120	F?qrw	9	378	F?zvw	9	990
F?bNw	8	288	FCQb_	9	120	F?qvg	9	378	F? vW	9	1044
F?bmw	8	288	FCOfo	9	144	F?rNw	9	384	F? v_	9	1066
F?qmw	8	288	FCQUw	9	144	F?rmw	9	384	F?z w	9	1152
F?qn_	8	288	FCQVO	9	144	F?zcw	9	384	F? vo	9	1212
F?bv_	8	292	FCQV_	9	144	F?znW	9	384	F? vw	9	1374
F?bfw	8	324	F?rf?	9	152	F?rfo	9	396	F? w	9	1536
F?bvW	8	324	F?rFW	9	156	F?rn_	9	396	FCQd_	10	84
F?bvg	8	324	F?rNO	9	156	F?zUo	9	396	FCQeG	10	96
F?ovw	8	324	F?rDo	9	168	F?qvo	9	414	FCQeO	10	96
F?o W	8	324	F?rfG	9	180	F?zPw	9	414	FCQe_	10	96
F?qfw	8	324	F?rF_	9	184	F?q—w	9	432	FCQdg	10	108
F?qjw	8	324	F?re_	9	184	F?q W	9	432	FCQf?	10	112
F?bro	8	330	F?rNW	9	192	F?zTo	9	432	FCXbO	10	120
F?o _	8	342	FCQVg	9	192	F?zf_	9	460	FCRUO	10	126
F?qno	8	360	F?rfO	9	198	F?q _	9	468	FCQeW	10	144
F?brw	8	378	F?qtg	9	216	F?rfw	9	468	FCQeo	10	144
F?bno	8	384	F?quW	9	216	F?rng	9	468	FCQfG	10	144
F?o o	8	414	FCOfw	9	216	F?zVW	9	468	FCQrO	10	144
F?bnw	8	432	FCQVo	9	216	F?rv_	9	484	FCQtg	10	144
F?qnw	8	432	F?zf?	9	230	F?qvw	9	486	FCRUg	10	144
F?bvo	8	438	F?rfW	9	234	F?qzw	9	486	FCXbW	10	156
F?bvw	8	486	F?rFo	9	240	F?zV_	9	502	FCRUW	10	162
F?o w	8	486	F?rN_	9	240	F?rno	9	504	FCQfO	10	168
F?b o	8	600	F?reg	9	240	F?zew	9	504	FCQf_	10	168
F?b w	8	648	F?reo	9	240	F?q g	9	540	FCRSw	10	168
FCOc_	9	24	F?qvG	9	252	F?rvg	9	540	FCRV?	10	168
FCOe?	9	32	F?qvO	9	252	F?zTw	9	540	FCXe_	10	168
FCQQO	9	36	F?zeW	9	252	F?zfo	9	552	FCQbo	10	180
FCOe_	9	48	F?qrg	9	270	F?zvO	9	552	FCRUo	10	180
FCQSg	9	48	F?qto	9	276	F?q o	9	576	FCXf?	10	184
FCQaO	9	48	F?zfO	9	276	F?rnw	9	576	FCQug	10	192
F?rEO	9	60	F?rLw	9	288	F?zmw	9	576	FCXjW	10	192

TABLE C.7: Part 2 of list of graphs with 7 vertices

G	m	a(G)	G	m	a(G)	G	m	a(G)	G	m	a(G)
FCQfW	10	216	FCRTw	10	336	FCZTw	10	504	FCpeo	11	216
FCQfg	10	216	FCY]g	10	336	FCZVg	10	504	FCrKw	11	216
FCQrW	10	216	FCZIw	10	336	FCY^o	10	540	FCrUW	11	234
FCQuW	10	216	FCQvo	10	360	FCZNg	10	540	FCZbO	11	240
FCQuo	10	216	FCRVo	10	360	FCZNo	10	540	FCe]w	11	240
FCRUw	10	216	FCRuo	10	360	FCZVo	10	540	FCpUw	11	240
FCRdg	10	216	FCXfo	10	360	FCY^g	10	552	FCrIw	11	240
FCReg	10	216	FCXn_	10	360	FCR^w	10	576	FCdf_	11	252
FCRfG	10	216	FCY]o	10	360	FCRvo	10	576	FCpeg	11	252
FCR‘o	10	222	FCZMo	10	360	FCXnw	10	576	FCpfG	11	252
FCQv_	10	240	FCZNG	10	360	FCZ]w	10	576	FCpfO	11	252
FCRV_	10	240	FCZTg	10	360	FCZ^_	10	612	FCrUo	11	258
FCXfO	10	240	FCZUo	10	360	FCRvw	10	648	FCpVO	11	264
FCQfo	10	252	FCRdw	10	378	FCY^w	10	648	FCpdg	11	270
FCQvO	10	252	FCY^O	10	378	FCZNw	10	648	FCpdo	11	270
FCRVO	10	252	FCZHw	10	378	FCZVw	10	648	FCZeO	11	288
FCR]o	10	252	FCZLg	10	378	FCZ\w	10	648	FCdew	11	288
FCRbg	10	252	FCZLo	10	378	FCZ^g	10	720	FCpV_	11	288
FCRcw	10	252	FCZVO	10	378	FCZ^o	10	756	FCrLW	11	288
FCReo	10	252	FCRv_	10	384	FCR o	10	792	FCrMw	11	288
FCXeo	10	252	FCXnW	10	384	FCR w	10	864	FCpbo	11	294
FCZUO	10	252	FCZJo	10	396	FCZ^w	10	864	FCpf_	11	294
FCRTo	10	264	FCZTo	10	396	FCdb?	11	84	FCZf?	11	304
FCZSo	10	264	FCRfo	10	414	FCdco	11	96	FCZco	11	306
FCZTO	10	270	FCRvO	10	414	FCdbG	11	108	FCrUw	11	312
FCRf_	10	276	FCY^G	10	414	FCe[w	11	120	FCreW	11	312
FCXf_	10	276	FCZJg	10	420	FCp‘_	11	126	FCdfg	11	324
FCQuw	10	288	FCZV_	10	420	FCpUO	11	132	FCpfW	11	324
FCQvg	10	288	FCQvw	10	432	FCdcg	11	144	FCptO	11	324
FCRVg	10	288	FCRVw	10	432	FCdeG	11	144	FCZb_	11	336
FCR]w	10	288	FCRto	10	432	FCde_	11	144	FCdfo	11	336
FCXmW	10	288	FCRuw	10	432	FCpU_	11	144	FCpv?	11	348
FCY[w	10	288	FCRvg	10	432	FCpeO	11	144	FCZfG	11	360
FCZMW	10	288	FCXmw	10	432	FCpbO	11	168	FCf]w	11	360
FCZUg	10	288	FCY]w	10	432	FCpeG	11	168	FCrHw	11	360
FCR‘w	10	294	FCZMw	10	432	FCpe_	11	168	FCrVG	11	360
FCZV?	10	294	FCZNW	10	432	FCpdG	11	180	FCreo	11	360
FCRdo	10	306	FCZN_	10	432	FCpdO	11	180	FCpVo	11	372
FCXfW	10	312	FCZUw	10	432	FCpUo	11	186	FCpug	11	372
FCXnO	10	312	FCY^_	10	444	FCpd_	11	186	FCpuo	11	372
FCQfw	10	324	FCXfw	10	468	FCdcw	11	192	FCrJo	11	372
FCQvW	10	324	FCXno	10	468	FCdeo	11	192	FCZeg	11	378
FCRVW	10	324	FCRfw	10	486	FCrMW	11	192	FCpfg	11	378
FCRew	10	324	FCRvW	10	486	FCpf?	11	196	FCpfo	11	378
FCRfg	10	324	FCY^W	10	486	FCpV?	11	204	FCe^o	11	384
FCXkw	10	324	FCZLw	10	486	FCrQo	11	204	FCrNW	11	384
FCXmo	10	324	FCZVW	10	486	FCpb_	11	210	FCr]w	11	384
FCZKw	10	324	FCR^o	10	504	FCdeg	11	216	FCZfO	11	396
FCZLW	10	324	FCRtw	10	504	FCdfG	11	216	FCprg	11	396
FCZMg	10	324	FCZJw	10	504	FCpeW	11	216	FCrfO	11	396

TABLE C.8: Part 3 of list of graphs with 7 vertices

G	m	a(G)	G	m	a(G)	G	m	a(G)	G	m	a(G)
FCrRo	11	408	FCZjw	11	672	FCvdg	12	558	FCxuw	12	792
FCZbg	11	420	FCZfw	11	702	FEhtg	12	558	FCxvW	12	798
FCZeo	11	432	FCZng	11	702	FEhto	12	558	FEhvo	12	798
FCdfw	11	432	FCfvW	11	702	FCxrW	12	564	FCvVw	12	816
FCqnW	11	432	FCrfw	11	702	FEhfo	12	564	FCvˆo	12	816
FCqn_	11	432	FCZvg	11	720	FCuuw	12	576	FCrvo	12	828
FCrLw	11	432	FCfˆw	11	720	FCvdo	12	576	FCzfo	12	828
FCrdo	11	432	FCpvw	11	720	FCzcw	12	576	FCzbw	12	834
FCZbo	11	438	FCrjw	11	720	FEhuo	12	582	FEhzo	12	834
FCpvO	11	450	FCZno	11	756	FCv‘w	12	594	FCrnw	12	864
FCZf_	11	456	FCfvo	11	768	FCvfO	12	594	FCuvw	12	864
FCrf_	11	456	FCrˆw	11	768	FCxsw	12	606	FCz\w	12	864
FCZfW	11	468	FCZvo	11	828	FCvbg	12	612	FCzmw	12	864
FCrNo	11	468	FCZnw	11	864	FCxuW	12	612	FCxvo	12	870
FCrVW	11	468	FCfvw	11	864	FEiro	12	612	FCzro	12	924
FCrfW	11	468	FCZvw	11	936	FEhro	12	618	FCrvw	12	936
FCpv_	11	474	FCf o	11	984	FCruw	12	624	FCvfw	12	936
FCeˆw	11	480	FCZ o	11	1044	FCvTw	12	624	FCvtw	12	936
FCf\w	11	480	FCf w	11	1080	FCvew	12	624	FCx}w	12	936
FCpVw	11	480	FCZ w	11	1152	FCvbo	12	630	FEh}w	12	936
FCpuw	11	480	FEhe_	12	276	FCxvO	12	636	FCvˆw	12	960
FCrJw	11	480	FCvSw	12	312	FCzaw	12	636	FCzjw	12	960
FCrVg	11	480	FEhd_	12	312	FEhvO	12	636	FEhvw	12	960
FCpfw	11	486	FCvUW	12	330	FCvVW	12	660	FEhzw	12	960
FCrbo	11	486	FCvUo	12	336	FCzfW	12	660	FCzˆo	12	990
FCrVo	11	516	FEhf?	12	340	FCrro	12	666	FCzfw	12	990
FCZew	11	540	FCusw	12	384	FCvVo	12	672	FCzno	12	990
FCZfg	11	540	FEheo	12	402	FCzbo	12	672	FCvvg	12	1008
FCqno	11	540	FCvUw	12	408	FCzf_	12	690	FCzˆg	12	1008
FCrfg	11	540	FEhbo	12	438	FCrng	12	702	FCxvw	12	1032
FCZbw	11	546	FEhf_	12	438	FCuvW	12	702	FCr o	12	1044
FCpvW	11	558	FEhuO	12	456	FCvfW	12	702	FCu w	12	1080
FCpvg	11	558	FCuuW	12	468	FCrvg	12	720	FCvvo	12	1080
FCZnW	11	576	FEitW	12	468	FCuvo	12	720	FCzrw	12	1086
FCfuw	11	576	FCuto	12	480	FCv\w	12	720	FEh o	12	1086
FCrNw	11	576	FCv]w	12	480	FCvdw	12	720	FCzvg	12	1104
FCrmw	11	576	FCvcw	12	480	FEhtw	12	720	FCr w	12	1152
FCrnW	11	576	FCveo	12	480	FEhfw	12	726	FCzˆw	12	1152
FCZfo	11	594	FEhv?	12	492	FCxv_	12	732	FCznw	12	1152
FCZn_	11	594	FCvRW	12	504	FEhuw	12	744	FCzvo	12	1158
FCrfo	11	594	FCzbW	12	504	FEhvg	12	744	FCvvw	12	1224
FCpvo	11	612	FCzeW	12	504	FCrno	12	756	FCx w	12	1248
FCfˆo	11	624	FCvaw	12	516	FCzew	12	756	FEh w	12	1248
FCrVw	11	624	FCveg	12	516	FCz]w	12	768	FC uw	12	1296
FCZv_	11	636	FCxv?	12	534	FCznW	12	768	FCzvw	12	1320
FCZmw	11	648	FCurW	12	540	FCrrw	12	774	FC vW	12	1374
FCfvg	11	648	FCvbW	12	540	FCvbw	12	774	FCv w	12	1440
FCqnw	11	648	FCvdW	12	540	FCvfg	12	774	FC vo	12	1488
FCrlw	11	648	FCvfG	12	540	FEh}o	12	774	FCz w	12	1536
FCrˆo	11	660	FCzfO	12	552	FCvfo	12	792	FC vw	12	1704

TABLE C.9: Part 4 of list of graphs with 7 vertices

FC w	12	1920	FEjfw	13	1014	FQilW	14	240	FE uw	14	1608
FEr]o	13	426	FEjvW	13	1014	FQhVO	14	288	FE tw	14	1704
FEr]w	13	504	FEjrw	13	1032	FQhV_	14	324	FE vW	14	1782
FEjeg	13	546	FEnew	13	1032	FQinO	14	384	FE vo	14	1824
FEjdg	13	600	FEzVg	13	1032	FQjR_	14	396	FFzvg	14	1848
FEjfG	13	600	FErvo	13	1068	FQhVo	14	432	FFzfw	14	1902
FEv]w	13	600	FEzfW	13	1068	FQjUg	14	432	FFzvo	14	1902
FEjeo	13	618	FEu—w	13	1080	FQjfG	14	432	FE vw	14	2112
FEitw	13	624	FEjvo	13	1086	FQinW	14	480	FFzvw	14	2232
FEzUg	13	624	FEnbw	13	1104	FQjVG	14	486	FFz o	14	2286
FEjf_	13	654	FEnfg	13	1110	FQjVO	14	504	FE w	14	2400
FEzUo	13	660	FEnfo	13	1128	FQjdg	14	504	FFz w	14	2616
FEzSw	13	672	FEjˆw	13	1152	FQjdo	14	528	FF w	14	3000
FEjbo	13	690	FEl}w	13	1152	FQjV_	14	540	FQzUW	12	702
FEivg	13	720	FEzfo	13	1164	FQhVw	14	576	FQzVO	12	720
FEzVO	13	756	FEzvO	13	1164	FQino	14	576	FTm—w	15	720
FEj]w	13	768	FEvˆw	13	1200	FQjRo	14	576	FQzUo	15	774
FEivo	13	774	FErvw	13	1224	FQjfW	14	576	FQzTo	15	816
FEivW	13	780	FEu g	13	1224	FQjVg	14	648	FQzV_	15	852
FEjfg	13	780	FEvvg	13	1224	FQjfg	14	648	FQzVW	15	936
FEncw	13	792	FEzVw	13	1224	FQjdw	14	672	FQznW	15	960
FEruw	13	816	FEjvw	13	1248	FQytW	14	696	FQzvO	15	1008
FEzTg	13	816	FEu o	13	1248	FQjew	14	702	FQzVo	15	1032
FEzUw	13	816	FEznW	13	1248	FQinw	14	720	FQyvo	15	1056
FEzeW	13	816	FEvvW	13	1278	FQjVo	14	720	FQzuo	15	1104
FEzPw	13	828	FEl o	13	1320	FQjfo	14	720	FQzmw	15	1152
FEzV_	13	834	FEvvo	13	1320	FQjnW	14	720	FQzlw	15	1200
FEnaw	13	846	FEnfw	13	1344	FQyvO	14	768	FTm o	15	1200
FEjfo	13	852	FEnvg	13	1344	FQyqw	14	792	FQzno	15	1224
FEndg	13	852	FEr o	13	1356	FQyuW	14	798	FQy}w	15	1248
FErˆo	13	852	FEj o	13	1374	FQyuo	14	816	FQzVw	15	1248
FEj\w	13	864	FEnvW	13	1398	FQjlw	14	840	FQyvw	15	1272
FEnbg	13	870	FEzfw	13	1398	FQjVw	14	864	FQzˆo	15	1320
FEnbo	13	888	FEzno	13	1398	FQjfw	14	864	FQy o	15	1344
FEzfO	13	888	FEnvo	13	1416	FQjvW	14	864	FQzvg	15	1344
FEnf_	13	930	FEu w	13	1440	FQjvg	14	864	FQzvo	15	1416
FEivw	13	936	FEzvg	13	1440	FQyv_	14	876	FQznw	15	1440
FEjtw	13	936	FEzvo	13	1494	FQjno	14	936	FTm w	15	1440
FErvg	13	936	FEr w	13	1512	FQjuw	14	936	FTnvg	15	1440
FEjuw	13	960	FEvvw	13	1512	FQyvW	14	984	FQzˆw	15	1536
FEjvg	13	960	FEj w	13	1536	FQjvo	14	1008	FQy w	15	1560
FEv\w	13	960	FEl w	13	1536	FQyuw	14	1032	FTnvo	15	1560
FEzf_	13	966	FEnvw	13	1632	FQjnw	14	1080	FQzvw	15	1632
FEjˆo	13	990	FEznw	13	1632	FQjvw	14	1152	FTzvW	15	1632
FErvW	13	990	FEzvw	13	1728	FFzeo	14	1242	FTzvg	15	1680
FEzVo	13	990	FEv w	13	1800	FFzf_	14	1296	FQ vW	15	1704
FErˆw	13	1008	FEn w	13	1920	FQj o	14	1296	FTzvo	15	1752
FErtw	13	1008	FEz w	13	2016	FQj w	14	1440	FTnvw	15	1800
FEvˆo	13	1008	FQhTO	14	144	FFzfo	14	1572	FQ vo	15	1824
FEzTw	13	1008	FQhV?	14	216	FFzvO	14	1572	FQz w	15	1920

TABLE C.10: Part 5 of list of graphs with 7 vertices

FTznw	15	1920
FQ vw	15	2112
FTn w	15	2160
FQ w	15	2400
FUZv_	16	1086
FUxuo	16	1128
FUxvO	16	1128
FUxv_	16	1182
FUZvg	16	1272
FUZvW	16	1344
FUZuw	16	1368
FUZvo	16	1416
FUxvo	16	1440
FUz]w	16	1512
FUzro	16	1512
FUZvw	16	1656
FUz^o	16	1704
FUzvW	16	1728
FUxvw	16	1752
FUZ o	16	1800
FUz^w	16	2016
FTzvw	16	2040
FUZ w	16	2040
FTz w	16	2400
FT w	16	2880
FUzvo	17	1824
FUzvw	17	2136
F]zlw	17	2160
FU vW	17	2208
FU vo	17	2232
FUz w	17	2520
FU vw	17	2616
F]y w	17	2640
FU w	17	3000
FV w	17	3600
F]znW	18	1920
F]zno	18	2112
F]znw	18	2520
F] vo	18	2712
F]z w	18	3120
F] vw	19	3216
F] w	19	3720
F^ w	20	4320
F w	21	5040

TABLE C.11: Part 6 of list of graphs with 7 vertices

a(G)	G	a(G)	G	a(G)	G	a(G)	G
720	GCe[{{	1920	GCZM][	3828	GCZVfo	3996	GEhvBs
1200	G?rM ˆ [	1920	GCZUmk	3834	GEhvEs	4002	G?qvvo
1200	GCXjZ[	1920	GCrM [	3834	GEhvUo	4002	G?rv‘{
1200	GCrM][	2016	G?rEˆ{	3840	GQjRro	4002	GCRfvo
1440	G?bNˆ[	2016	G?rF]{	3858	G?brvw	4008	GCqnbw
1440	G?qmˆ[	2016	G?rMˆw	3858	G?qr o	4020	G?bvrw
1440	GCQU} {	2016	G?rNU{	3858	GCRd o	4026	GCrbvg
1440	GCRUm{	2016	G?rmu[	3858	GCrfRw	4032	GCYˆfg
1440	GCrK}[	2016	GCXbˆ[	3864	GCYˆNg	4032	GCpvdw
1512	G?rFˆ[	2016	GCXjˆW	3864	GEhvSw	4032	GCxvEw
1512	G?rNV[	2016	GCre][	3876	GEjeqw	4044	GEjbvG
1512	GCrU][	2160	G?aNˆ{	3888	G?q—vo	4050	GEhvRo
1536	G?Bvnk	2160	G?bLˆ{	3888	G?zffc	4050	GEjbtg
1536	G?bfˆ[	2160	G?bL—{	3888	GCRvfo	4056	G?qzvo
1536	G?qfˆ[	2160	G?bN{	3888	GCrbrs	4056	G?zfew
1536	G?qjˆ[	2160	G?qk [	3888	GEhvEw	4056	GCRvVo
1536	GCRU]{	2160	G?qm{	3888	GEhvQw	4056	GEhvA{
1560	GCRS} {	2160	GCQTn{	3900	GCZNfg	4062	GCZffo
1560	GCrMY{	2160	GCQUn{	3900	GCZNfo	4062	GCZvfO
1608	G?rnU[	2160	GCQVm{	3900	GCZbno	4062	GCrffo
1632	G?qnV[	2160	GCQVnk	3900	GCZfbw	4068	GEjbuo
1632	G?rfN[	2160	GCQtm{	3900	GCqnfo	4074	GCxvBw
1632	GCRUu{	2160	GCQum{	3900	GCxvFW	4080	G?bnvo
1632	GCrI{ {	2160	GCQunk	3900	GCxvVG	4092	G?o vg
1656	GCpUu{	2160	GCXm][	3912	G?rvfg	4104	G?zVfc
1680	GCrIy{	2160	GCdcv{	3918	G?zTvo	4134	G?q fg
1704	G?bnV[	2160	GCdc} {	3918	GCZevo	4140	GCpvfo
1704	G?rnT[	2160	GCdeu{	3918	GCrdvo	4152	GCRtvo
1704	GCR]uk	2160	GCrL [	3924	G?rfvo	4164	G?zffo
1704	GCrU]w	2160	GQhTVs	3924	GCpvbs	4164	GCZvf_
1728	G?rfV[	2160	GQhVVS	3930	G?o fw	4188	GCrfbw
1728	GCrI}s	2304	G?Bfn{	3930	G?q bs	4194	GCxvFg
1752	GCrI}w	2304	G?Bvn[	3930	GCYˆNo	4206	G?zVdw
1752	GCrQu{	2304	G¿fˆ{	3942	GCZbvo	4212	GEjf‘w
1782	G?rnVK	2304	G¿rn{	3954	GCZJno	4230	GCrbvo
1800	GCpU}w	2304	G¿vˆ[	3954	GCZfew	4236	GCxvFo
1824	G?rnVW	2304	G¿vnk	3954	GCpvfW	4236	GCxvfO
1824	GCR]uw	2304	G?bFˆ{	3954	GCrdrw	4272	GCxvBs
1824	GCrUuw	2304	G?bNn[	3954	GCrfdw	4284	GCxveW
1848	G?zfVS	2304	G?beˆ{	3972	G?q fc	4290	GCxveo
1902	G?zfF[	2304	G?be} {	3972	G?zTvg	4320	GCxvcw
1902	G?zfVW	2304	G?be [	3972	GCYˆfo	4326	GEjbro
1920	G?bMˆ{	2304	G?bfN{	3972	GEhvFo	4338	G?zVfo
1920	G?bN]{	2304	G?bfn[	3972	GEhvV_	4386	G?q fo
1920	G?qm]{	2304	G?bfnk	3978	G?rnfo	4410	G?bvvo
1920	G?rM{	2304	G?ovˆ[	3978	GCpvbw	4470	GCxvbo
1920	GCQtnk	2304	G?o—ˆ[	3978	GEhvC{	4494	G?rvfo
1920	GCRUnk	2304	G?o}ˆ[	3984	G?q bw	4668	GCxvf_
1920	GCXj][	2304	G?qeˆ{	3996	GCpvVo	4718	G?B vo
1920	GCY[{ {	2304	G?qe} {	3996	GCxvbW	5000	G?zvf_

Table C.12: The best and worst 100 graphs with $n = 8$ and $m = 15$

a(G)	G	a(G)	G	a(G)	G	a(G)	
1440	GCe[} {	2880	GCrL^[	4968	GCR^vo	5136	GEhrvW
1440	GQil [	2880	GCrM {	4968	GCZvc {	5136	GEirvW
1800	G?rN^[	2880	GQhTV{	4968	GCpvfw	5136	GEjbvc
1800	GCrM]{	2880	GQinVS	4968	GCpvno	5154	GEjbvg
1920	G?bn^[	2880	GQjUmk	4968	GCzfes	5160	GCxves
1920	G?qn^[	2880	GQjfNK	4968	GQyurg	5160	GCzbrk
1920	GCRU} {	3024	G?rF^{	4986	GCZb o	5166	G?zvek
1920	GCf]s{	3024	G?rNV{	4986	GCvbng	5166	GCZfvo
1920	GCrK} {	3024	G?rNv[	4986	GCvfRk	5166	GCrv‘{
2016	G?rf^[	3024	G?re} {	4986	GCvfVg	5172	G?B vw
2016	GCrU]{	3024	G?rf]{	4986	GEjdno	5178	G?q vg
2040	GCpU} {	3024	G?rmv[	4986	GEjep{	5184	GCzffc
2040	GCrI} {	3024	GCXb^{	4992	G?zvfK	5208	GEnbvG
2112	G?rnV[	3024	GCXf^[	4992	GEjfI{	5220	G?zVfw
2112	GCR]u{	3024	GCXj^w	5010	G?bvvw	5220	G?zVvg
2112	GCf]uk	3024	GCXnV[	5010	GEhru[	5220	GCZnfo
2136	GCrUu{	3024	GCf uk	5016	G?q fw	5220	GCrvVo
2160	GCf]uw	3024	GCrM^w	5016	GCY^vg	5226	GEjf‘{
2208	GCr]uk	3024	GCrU^[	5016	GEjbrs	5244	GCpvvo
2232	G?zfV[	3024	GCre^[	5016	GQzTrg	5274	GCrfrw
2232	GCr]uw	3024	GEit[	5028	GCxvb[	5298	GCrvbw
2286	G?zf^W	3072	G?Bvn{	5028	GEhrvK	5298	GCzfew
2400	G?rM^{	3072	G?bf^{	5040	GCpv^o	5304	GEjfa{
2400	G?rN]{	3072	G?bv^[	5040	GCrtrw	5316	G?z^fc
2400	GCXj^[	3072	G?bvnk	5040	GCvfRw	5316	GCZ^fo
2400	GCe[w	3072	G?o ^[	5046	G?zffw	5316	GCxve[
2400	GCf s{	3072	G?qf^{	5046	G?zfvg	5340	GCxvbs
2400	GCrM^[	3072	G?qj^{	5046	G?znfo	5352	GCxvc{
2400	GQil^W	3072	G?qnZ{	5058	GCrbvw	5358	GCzbrw
2880	G?bN^{	3072	G?qt—{	5058	GEjbvK	5370	GEjfbw
2880	G?bn]{	3072	G?qv^[	5058	GEjerw	5376	G?zvc{
2880	G?qm^{	3072	GCRU^{	5064	GCZ^fg	5376	GCRvvo
2880	G?qm} {	3072	GCRV]{	5064	GCrvdw	5394	GCxvfW
2880	G?qm [	3072	GCRe} {	5064	GCxvFw	5394	GCzbvg
2880	G?rL [	3072	GCRfnk	5064	GCxvVg	5400	GCxvfc
2880	GCQU {	3072	GCZK} {	5064	GEjbtk	5412	GEjbvo
2880	GCQtn{	3072	GCZL^[	5082	GCvbrw	5448	GCxvew
2880	GCQu} {	3072	GCZMm{	5082	GCvbvc	5460	GCZvfo
2880	GCQvnk	3072	GCpf^[	5082	GEhvVo	5460	GCrvfo
2880	GCRUn{	3120	GCRS {	5088	G?z^dw	5466	GEnfbW
2880	GCRU k	3120	GCRU—{	5088	GCfvRw	5550	G?z^fo
2880	GCRVnk	3120	GCY]m{	5106	GCxvFs	5628	GCrrvo
2880	GCXj]{	3120	GCZI} {	5106	GCxvfS	5628	GCzfbw
2880	GCXm^[	3120	GCf tw	5112	GCrtrs	5634	GCxvfo
2880	GCY[} {	3120	GCf^c{	5112	GCrvRs	5706	G?rvvo
2880	GCZM]{	3120	GCrI [	5124	G?rvfw	5736	G?zvfg
2880	GCZM^[	3120	GCrMZ{	5124	G?rvvg	5784	G?b vo
2880	GCZUm{	3120	GCveus	5136	GCrrvg	5964	GCzvbo
2880	GCde} {	3216	G?rN^w	5136	GCvbno	6066	G?zvfo
2880	GCrK [	3216	G?rnU{	5136	GCvbvg	6902	G? vf_

TABLE C.13: The best and worst 100 graphs with $n = 8$ and $m = 16$

m=4		m=5		m=6			
a(G)	g	a(G)	G	a(G)	G	a(G)	
8	G???F?	16	G???F_	24	G?ACKK	56	G?B@e?
8	G??CB?	16	G??CB_	36	G??EFC	56	G¿@‘_
6	G??CCC	12	G??CEC	36	G?AAEK	60	G?AEBG
8	G??CE?	16	G??CF?	36	G?ABBC	60	G¿D@O
8	G?AA@?	16	G??E@_	36	G?AEEC	62	G?BDB?
		14	G??EE?	36	G¿CQG	64	G???Fw
		16	G?AA@_	42	G?AEEG	64	G??CBw
		12	G?AACG	46	G??FF?	64	G??CFo
		16	G?AAD?	48	G??CFc	64	G??E@s
		16	G?AAE?	48	G??EDc	64	G??E@w
		16	G¿@?_	48	G?AADg	64	G??EDo
				48	G?AAFG	64	G??F?w
				48	G?ABAc	64	G??FCo
				48	G?ABCK	64	G?AADo
				48	G?ABCg	64	G?AAF_
				48	G?ABEC	64	G?AB?s
				48	G?ABEG	64	G?ABAo
				48	G?ACJ_	64	G?ABCc
				48	G?AEDC	64	G?ABCo
				48	G¿@Co	64	G?ABE_
				48	G¿@EO	64	G?AE@o
				48	G¿@cO	64	G?AEB_
				48	G¿DAG	64	G¿@E_
				56	G??EF_	64	G¿@F?
				56	G??FE_	64	G¿@d?
				56	G?ABB_	64	G¿@e?
				56	G?ABF?	64	G¿DA_
				56	G?AEF?	64	G¿DB?

TABLE C.14: All graphs and the number of acyclic orientations for $n = 8$, $m = 4, 5, 6$

Appendix D

Additional Detailed Results (Tables and Figures)

D.1 A selection of maximum graphs and tables

Tables D.1 and D.2 present the maximum number of acyclic orientations that a graph with 6 and 7 vertices can have respectively. The values in the table were obtained from the data in Appendix C, and we have highlighted where a Turán graph provides the maximum value with a T. These results align with those for $n = 8$ presented in chapter 3 and strengthen the conjectures in Section 7.6.1.

D.1.1 Some Hanging Curtains

In Figures D.1 and D.2 we have plotted the maximum number of acyclic orientations that a graph with 6 and 7 vertices can have respectively with respect to the number of edges. In both cases we have highlighted the Turán graphs with a T. It is possible to see the first 'hang' of a curtain in each case, the remainder of the Turán graphs are too close together to produce the effect.

m	a.o.s	
0	1	T
1	2	
2	4	
3	8	
4	16	
5	32	
6	62	
7	102	
8	152	
9	230	T
10	276	
11	336	
12	426	T
13	480	T
14	600	T
15	720	T

Table D.1: Maximum number of acyclic orientations for $n = 6$

m	a.o.s	
0	1	T
1	2	
2	4	
3	8	
4	16	
5	32	
6	64	
7	126	
8	222	
9	348	
10	534	
11	736	
12	1066	T
13	1296	
14	1572	
15	1902	
16	2286	
17	2712	
18	3216	T
19	3720	T
20	4320	T
21	5040	T

Table D.2: Maximum number of acyclic orientations for $n = 7$

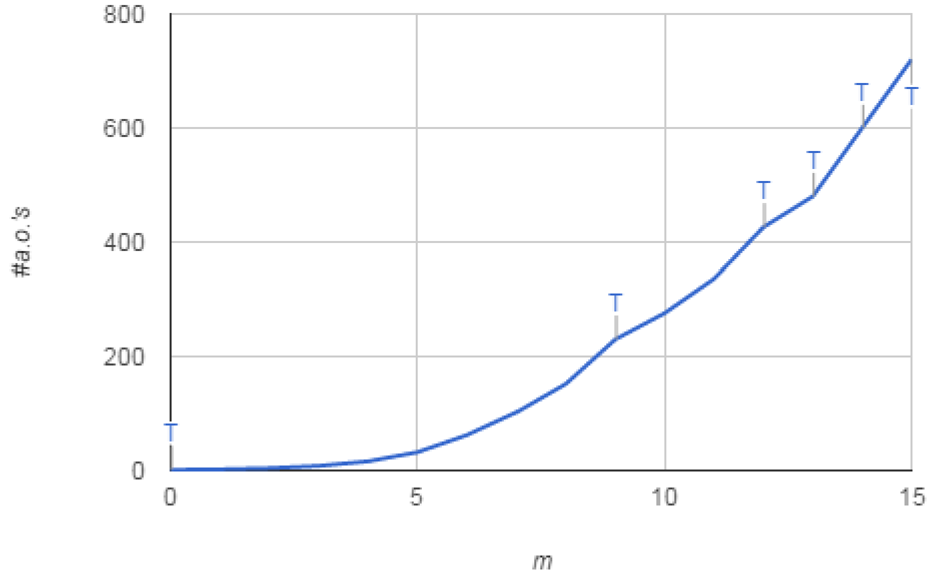

FIGURE D.1: Maximum number of acyclic orientations for $n = 6$

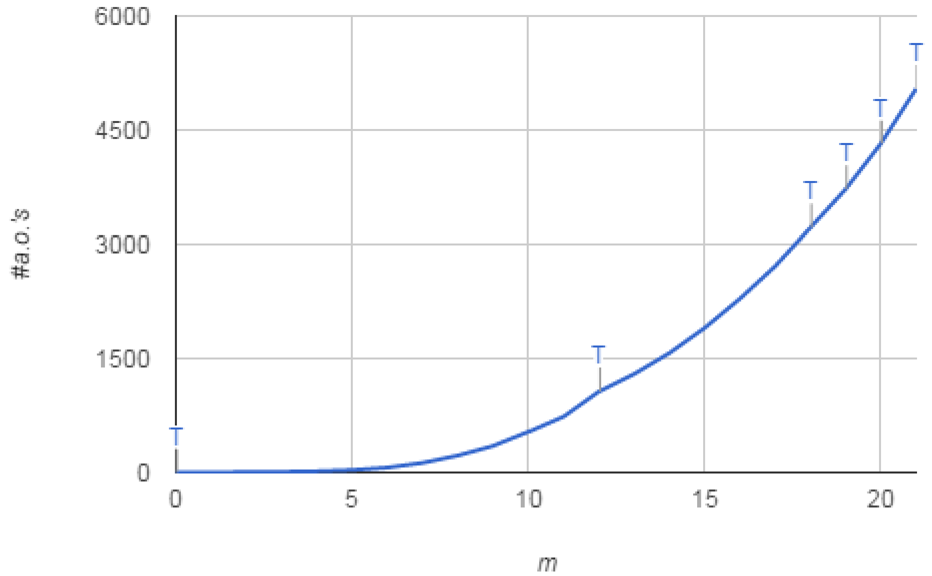

FIGURE D.2: Maximum number of acyclic orientations for $n = 7$

m	Slope	Change in Slope	Turan
0	1		T
1	2	1	
2	4	2	
3	8	4	
4	16	8	
5	32	16	
6	62	30	
7	96	34	
8	126	30	
9	186	60	
10	202	16	
11	330	128	
12	230	-100	T
13	276	46	
14	330	54	
15	384	54	
16	426	42	
17	504	78	
18	504	0	T
19	600	96	T
20	720	120	T
21			T

TABLE D.3: The slope and change in slope of the number of acyclic orientations as the number of edges increases for maximal graphs with $n = 7$.

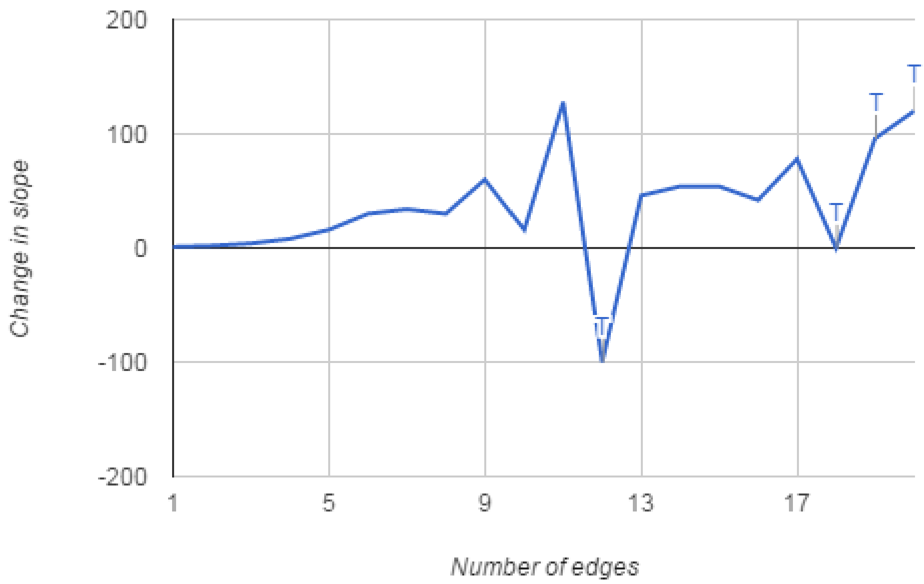

FIGURE D.3: The change in slope for $n = 7$, with Turán graphs highlighted.

Appendix E

Proof Approach for Factor Method

In this section we give an incomplete proof of the factor method. As a reminder, we wish to prove the following Conjecture.

Conjecture E.1 (The Factor Lemma). *If we add two edges to a graph G, say e and g, the factor on e when added to G is at least as big as the factor on e when added to $G+g$, i.e.*

$$f_G(e) \geq f_{G+g}(e).$$

Proof. We look at only one a.o. θ of G. We consider G, $G+e$, $G+g$ and $G+e+g$, i.e. we add e and g one at a time in both orders and look at the factor increase in a.o.'s for (G,θ). The factors $f_G(e), f_{G+g}(e)$ will be a weighted average of all these factor increases, so if for each individual factor we can show that $f_G(e) \geq f_{G+g}(e)$ restricted to θ holds, then it must hold for $f_G(e) \geq f_{G+g}(e)$. As we look at only one a.o. of G, $f_G(e), f_G(g) \in \{1,2\}$. Hence we only have the following four cases.

$$\begin{tikzcd} & G \arrow[dl, dash, "1"'] \arrow[dr, dash, "1"] & \\ G+g \arrow[dr, dash, "f_{G+e}(g)"'] & & G+e \arrow[dl, dash, "f_{G+g}(e)"] \\ & G+e+g & \end{tikzcd}$$

(a)

$$\begin{tikzcd} & G \arrow[dl, dash, "1"'] \arrow[dr, dash, "2"] & \\ G+g \arrow[dr, dash, "f_{G+e}(g)"'] & & G+e \arrow[dl, dash, "f_{G+g}(e)"] \\ & G+e+g & \end{tikzcd}$$

(b)

$$\begin{tikzcd} & G \arrow[dl, dash, "2"'] \arrow[dr, dash, "1"] & \\ G+g \arrow[dr, dash, "f_{G+e}(g)"'] & & G+e \arrow[dl, dash, "f_{G+g}(e)"] \\ & G+e+g & \end{tikzcd}$$

(c)

$$\begin{tikzcd} & G \arrow[dl, dash, "2"'] \arrow[dr, dash, "2"] & \\ G+g \arrow[dr, dash, "f_{G+e}(g)"'] & & G+e \arrow[dl, dash, "f_{G+g}(e)"] \\ & G+e+g & \end{tikzcd}$$

(d)

These are the only options. From (a) it follows that $f_{G+g}(e) = f_{G+e}(g) = 1$ and the lemma holds. From (b) it follows that $f_{G+g}(e) = 2$ and $f_{G+e}(g) = 1$, as $2 * f_{G+g}(e) = 1 * f_{G+e}(g)$, so the lemma holds. From (c) similar to (b), lemma holds. For (d) we observe that $f_{G+g}(e), f_{G+e}(g) \leq 2$, so the lemma must hold. For completeness the only possible options for $f_{G+g}(e)$ and $f_{G+e}(g)$ are $f_{G+g}(e) = f_{G+e}(g) = 1, 1.5, 2$. The lemma holds for each of the possible cases (a) - (d), so it holds in general. $\square$

As mentioned previously this proof does not work. The reason it does not work is quite well hidden. In case (d) of the proof it is possible that for a particular acyclic orientation the edges e and g have factor 2 in each case. In this case we have $f_{G+g}(e) = f_G(e) = 2$, which seems ok. Unfortunately, as this also means that $f_G(g) = 2$, the factor $f_{G+g}(e) = 2$ is weighted twice as often as the factor $f_G(e) = 2$ invalidating the proof, which relied on the same weighting of $f_{G+g}(e)$ and $f_G(e)$ for each particular orientation.

We now want to further investigate for which edges the property $f_G(e) \geq f_{G+g}(e)$ holds. The change of a factor at an edge e when moved after edge g is influenced in (a subset of) the following three ways.

1. g and e are not both in a cycle at the same time

2. g is in a cycle with e

3. g connects two cycles that contain e

The first property means that the addition of g does not change the factor at e, equality $f_G(e) = f_{G+g}(e)$ holds. The second property adds a further restriction on e, which means that the factor decreases, if property 2 holds and not property 3, then we have a strict decrease in factor, i.e. $f_G(e) > f_{G+g}(e)$ holds. If property 3 holds, then this is responsible for an increase in the factor, i.e. the contribution of property 3 gives $f_G(e)|_{Property3} < f_{G+g}(e)|_{Property3}$. Unfortunately whenever property 3 holds, property 2 holds. It is now the task to find out when the contribution due to property 2 is enough to cancel out the contribution due to property 3, and in all these cases the factor hypothesis may be applied, which will allow for the results in Chapter 5 to be used.

An example of where the conjecture fails and the properties above do not hold is for the following graphs. Let G be the 4-cycle, $G + e = G + f$ are the 4-cycle with a chord, and $G + e + f = K_4$. Now $a(G) = 14$, $a(G + e) = a(G + f) = 18$ and $a(G + e + f) = 24$. Then $f_G(e) = \frac{18}{14}$ and $f_{G+f}(e) = \frac{24}{18}$, i.e. $f_G(e) < f_{G+f}(e)$.

Bibliography

[1] Year 13 – 1873: A Treatise on Electricity and Magnetism by James Clerk Maxwell. *MIT Libraries*, 1873.

[2] The most important Experiments and their Publication between 1886 and 1889. *Fraunhofer Heinrich Hertz Institute*, 1887.

[3] K. Appel and W. Haken. The Solution of the Four-Colour-Map Problem. *Scientific American*, 237:108–121, 1977.

[4] R. A. Bailey and P. J. Cameron. Combinatorics of optimal designs. *Surveys in combinatorics*, pages 19–73, 2009.

[5] W. W. Rouse Ball. *The Four Color Theorem.* Macmillan, 1960.

[6] V. C. Barbosa and J. L. Szwarcfiter. Generating all the acyclic orientations of an undirected graph. *Information Processing letters*, 72:71–74, 1999.

[7] E. A. Bender, L. B. Richmond, R. W. Robinson, and N. C. Wormald. The Asymptotic Number of Acyclic Digraphs. *Combinatorica*, 6(1):15–22, 1986.

[8] E. A. Bender and R. W. Robinson. The asymptotic number of acyclic digraphs, ii. *Journal of Combinatorial Theory, Series B*, 44(3):363 – 369, 1988.

[9] P. Berthome, S. Lebresne, and K. Nguyen. Computation of chromatic polynomials using triangulations and clique trees. In Dieter Kratsch, editor, *Graph-Theoretic Concepts in Computer Science*, volume 3787 of *Lecture Notes in Computer Science*, pages 362–373. Springer Berlin Heidelberg, 2005.

[10] N. L. Biggs. *The Tutte Polynomial.* Cambridge University Press, 1993.

[11] G. Birkhoff. On the number of ways of coloring a map. *Proceedings of the Edinburgh Mathematical Society (2)*, 2:83–91, 1930.

[12] G. Birkhoff. On the polynomial expressions for the number of ways of coloring a map. *Annali delia R. Scuola Normale Superiore di Pisa (Scienze Fisiche e Matematiche) (2)*, 3:1–19, 1934.

[13] G. D. Birkhoff. A Determinant Formula for the Number of Ways of Coloring a Map. *Ann. Math.*, 14:42–46, 1912.

[14] G.D. Birkhoff and D.C. Lewis. Chromatic polynomials. *Transactions of the American Mathematical Society*, 60:355–451, 1946.

[15] C. Brewbaker. A combinatorial interpretation of the poly-Bernoulli numbers and two Fermat analogues. *Integers*, 8:9pp, 2008.

[16] N. Chandrasekharan, C.E.V. Madhavan, and R. Laskar. Chromatic polynomials of chordal graphs. *Congressus Numerantium*, 61:133–142, 1988.

[17] S-C. Chang. Exact chromatic polynomials for toroidal chain of complete graphs. *Physica A.*, 313:397–426, 2002.

[18] Joint Technical Advisory Committee. *Spectrum Engineering: the Key to Progress: A Report on Technical Policies and Procedures for Increased Radio Spectrum Utilization*. IEEE, 1968.

[19] B. Courcelle and S. Olariu. Upper bounds to the clique width of graphs. *Discrete Applied Mathematics*, 101(1–3):77 – 114, 2000.

[20] J. Cutler and J. Radciffe. Extremal Graphs for Homomorphisms. *J. Graph Theory*, 67:261–284, 2011.

[21] R. Diestel. *Graph Theory*. Springer, 2005.

[22] M. Dom. Algorithmic aspects of the consecutive-ones property. *EATCS Bulletin*, 98:27–59, 2008.

[23] F.M. Dong, K.L. Tep, K.M. Koh, and M.D. Hendy. Non-chordal graphs having integral root chromatic polynomial ii. *Discrete Mathematics*, 2002.

[24] P. Erdős. On a theorem of rademacher-turán. *Illinois J. Math.*, 6:122–127, 1962.

[25] P. Erdős and M. Simonovits. A limit theorem in graph theory. *Studia Sci. Math. Hungar.*, 1:51–57, 1966.

[26] L. Fortnow. The status of the P versus NP problem. *Communications of the ACM*, 52 (9):78–86, 2009.

[27] D.R. Fulkerson and O.A. Gross. Incidence matrices and interval graphs. *Pacific Journal of Mathematics*, 15:835–855, 1965.

[28] F. G. Tinting Maps. *The Athenaeum: 726*, 1854.

[29] M. Garey and D. S. Johnson. Computers and Intractability: A Guide to the Theory of NPCompleteness. *W. H. Freemand and Company*, 1979.

[30] M.R. Garey and D.S. Johnson. *Computers and Intractability: A guide to the Theory of BP-Completeness.* W.H. Freeman and Company, 1979.

[31] O. Giménez, P. Hliněný, and M. Noy. Computing the Tutte Polynomial on Graphs of Bounded Clique-Width. In Dieter Kratsch, editor, *Graph-Theoretic Concepts in Computer Science*, volume 3787 of *Lecture Notes in Computer Science*, pages 59–68. Springer Berlin Heidelberg, 2005.

[32] C. Glass, S. A. Callaghan, J. L. Agnew, D. Bacon, I. Davies, and S. W. Wales. Improving spectral efficiency of fixed link services - final report. Radio Communications Research Unit, Rutherford Appleton Laboratory, CCLRC Rutherford Appleton Laboratory, Cass Business School and Roke Manor Research Ltd., Ofcom Contract Number: 830000131, 2006.

[33] L. A. Goldberg and M. Jerrum. Inapproximability of the Tutte polynomial. *Information and Computation*, 206(7):908–929, 2008.

[34] G. Haggard. Computing chromatic polynomials of large graphs i. *Journal of Combinatorial M*, 13:175–186, 1993.

[35] G. Haggard. Using thresholds to compute chromatic polynomials. *Ars Combinatoria*, 58:85–95, 2001.

[36] G. Haggard and T.R. Mathies. Note on the computation of chromatic polynomials. *Discrete Mathematics*, 199:227–231, 1999.

[37] F. Harary and E. M Palmer. Graphical enumeration. Technical report, DTIC Document, 1973.

[38] B. Jackson. Zeros of chromatic and flow polynomials of graphs. *Journal of Geometry*, 76:95–109, 2003.

[39] F. Jaeger, D. L. Vertigan, and D. J. A. Welsh. On the computational complexity of the Jones and Tutte polynomials. *Mathematical Proceedings of the Cambridge Philosophical Society*, 108:35–53, 1990.

[40] M. Kaneko. Poly-Bernoulli numbers. *Journal de Théorie des Nombres de Bordeaux*, 9:221–228, 1997.

[41] A.E. Koller and S.D. Noble. Domination analysis of greedy heuristics for the frequency assignment problem. *Discrete Mathematics*, 275(1–3):331–338, 2004.

[42] N. Linial. Legal Colourings of Graphs. In *Proc. 24th Symp. on Foundations of Computer Science*, pages 470–472, 1983.

[43] N. Linial. Hard enumeration problems in geometry and combinatorics. *SIAM J. Algorithmic Discrete Methods*, 7:185–204, 1986.

[44] V. Maniezzo and A. Carbonaro. An ANTS heuristic for the frequency assignment problem. *Future Generation Computer Systems*, 16(8):927–935, 2000.

[45] B. D. McKay and A. Piperno. Practical graph isomorphism, II. *Journal of Symbolic Computation*, 60(0):94–112, 2014.

[46] J. Naor, M. Naor, and A. Schaffer. Fast Parallel Algorithms for Chordal Graphs. In *Proceedings of the Nineteenth Annual ACM Symposium on Theory of Computing*, STOC '87, pages 355–364, New York, NY, USA, 1987. ACM.

[47] Ofcom. Ofcom announces winners of the 3G mobile auction, 2000.

[48] Ofcom. Ofcom unveils plans to avoid mobile capacity crunch, November 2012.

[49] Ofcom. Ofcom announces winners of the 4G mobile auction, February 2013.

[50] J. Oxley and D. Welsh. Chromatic, flow and reliability polynomials: the complexity of their coefficients. *Combinatorics, Probability and Computing*, 11:403–426, 2002.

[51] S. Pemmaraju and S. Skiena. Computational Discrete Mathematics: Combinatorics and Graph Theory with Mathematica. 2003.

[52] G. Pólya. Kombinatorische Anzahlbestimmungen für Gruppen, Graphen und chemische Verbindungen. *Acta Mathematica*, 68(1):145–254, 1937.

[53] H. J. Ryser. Combinatorial properties of matrices of zeros and ones. *Canadian J. Math.*, 9:371–377, 1957.

[54] D.R. Shier and N. Chandrasekharan. Algorithms for computing the chromatic polynomial. *Journal of Combinatorial Mathematics and Combinatorial Computing*, 4:213–222, 1988.

[55] D.H. Smith, S. Hurley, and S.U. Thiel. Improving heuristics for the frequency assignment problem. *European Journal of Operational Research*, 107(1):76–86, 1998.

[56] A. D. Sokal. The multivariate Tutte polynomial (alias Potts model) for graphs and matroids. *Surveys in combinatorics*, pages 173–226, 2005.

[57] M. B. Squire. Generating the Acyclic Orientations of a Graph. *Journal of Algorithms*, 26:275–290, 1998.

[58] R. P. Stanley. Acyclic orientations of graphs. *Discrete Mathematics*, 5:171–178, 1973.

[59] Top500.org. Top500 supercomputer sites, 2015.

[60] P. Turán. On an extremal problem in graph theory. *Matematikai és Fizikai Lapok*, 48:436–452, 1941.

[61] W. T. Tutte. A ring in graph theory. *Proceedings of the Cambridge Philosophical Society*, 43:26–40, 1947.

[62] M. L. Vergnas. Convexity in Oriented Matroids. *Journal of Combinatorial Theory, Series B*, 29:231–243, 1980.

[63] H. Whitney. The coloring of graphs. *Ann. of Math.*, 33:688–718, 1932.

[64] H. Whitney. A logical expansion in mathematics. *Bull. Amer. Math. Soc.*, 38:572–579, 1932.

[65] H. S. Wilf. *Algorithms and Complexity.* Prentice–Hall, 1986.

[66] Real Wireless. Techniques for increasing the capacity of wireless broadband networks: UK, 2012-2030, April 2012.

[67] Inc. Wolfram Research. Mathematica. 2014.

[68] D. R. Wood. On the maximum number of cliques in a graph. *Springer-Verlag*, 23(3):337–352, 2007.