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ON THE STRUCTURE OF GENERAL MEAN-VARIANCE HEDGING STRATEGIES

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We provide a new characterization of mean-variance hedging strategies in a general semimartingale market. The key point is the introduction of a new probability measure $P^\star$ which turns the dynamic asset allocation problem into a myopic one. The minimal martingale measure relative to $P^\star$ coincides with the variance-optimal martingale measure relative to the original probability measure P.

Contents

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1. Introduction.

1.1. *Overview.* In incomplete market models perfect replication of contingent claims is typically impossible. A classical way out is to minimize the *mean squared hedging error*

$$E\big((v + \vartheta \bullet S_T - H)^2\big)$$

over all reasonable hedging strategies ϑ and possibly all initial endowments v. Here, the random variable H denotes the discounted payoff of the claim, the semimartingale S stands for the discounted price process of the underlying, the dot refers to stochastic integration, and T is the time horizon. Mathematically speaking, one seeks to compute the orthogonal projection of H on some space of stochastic integrals.

This problem has been extensively studied both as far as general theory as well as concrete results in specific setups are concerned. In order to render equal justice (or rather injustice) to most contributions, we refer the reader to [38] and [45] for excellent overviews of the literature. More recent publications in this context include [2–4, 7–11, 17, 23–26, 32–36, 46].

The purpose of this piece of research is to provide a deeper understanding of the structure of the mean-variance hedging problem in a general semimartingale context. More specifically, we aim at concrete formulas for the objects of interest—to the extent that this is possible without restricting to more specific situations.

If S is a square-integrable martingale, the answer to the above hedging problem is provided by the *Galtchouk–Kunita–Watanabe decomposition* of the claim (cf. [19]). In particular, the optimal hedge ϑ is of the form

$$\vartheta_t = \frac{d\langle V, S\rangle_t}{d\langle S, S\rangle_t}, \tag{1.1}$$

where $V_t = E(H|\mathscr{F}_t)$ denotes the martingale generated by the contingent claim H.

If S fails to be a martingale, the hedging problem becomes much more involved. Relatively explicit results have been obtained by Schweizer [42] under the condition of *deterministic mean-variance tradeoff*, which can be intepreted as a certain homogeneity property of the asset price process S. In this case the optimal hedge is the sum of two terms. The first satisfies an equation resembling (1.1). The second can be interpreted in terms of a pure investment problem under quadratic utility.

In the current paper we reduce the general case to the expressions of [42]. This is done by a specific nonmartingale change of measure. If the formulas of [42] are evaluated relative to the new *opportunity-neutral measure* $P^\star$ rather than P, they yield the optimal hedge relative to the original probability measure P. We discuss the links to the literature more thoroughly in Section 4.3.

The paper is structured as follows. Section 2 explains the setup of the mean-variance problem at hand. In particular, we define a notion of admissibility which ensures the existence of an optimal hedge. The measure change alluded to above and related objects are introduced in Section 3. Subsequently, we turn to the hedging problem itself. Finally, the appendix contains and summarizes auxiliary statements on semimartingales. In particular, we prove a sufficient condition for square integrability of exponential semimartingales which is needed in Section 4.

1.2. *Semimartingale characteristics and notation.* Unexplained notation is typically used as in [28]. Superscripts refer generally to coordinates of a vector or vector-valued process rather than powers. The few exceptions should be obvious from the context. If X is a semimartingale, $L(X)$ denotes the set of X-integrable predictable processes in the sense of [28], III.6.17.

In the subsequent sections, optimal hedging strategies are expressed in terms of semimartingale characteristics.

DEFINITION 1.1. Let X be an $\mathbb{R}^d$-valued semimartingale with characteristics (B, C, ν) relative to some truncation function $h : \mathbb{R}^d \to \mathbb{R}^d$. By [28], II.2.9 there exists some predictable process $A \in \mathscr{A}^+_{\mathrm{loc}}$, some predictable $\mathbb{R}^{d\times d}$-valued process c whose values are nonnegative, symmetric matrices, and some transition kernel F from $(\Omega \times \mathbb{R}_+, \mathscr{P})$ into $(\mathbb{R}^d, \mathscr{B}^d)$ such that

$$B_t = b \bullet A_t, \qquad C_t = c \bullet A_t, \qquad \nu([0,t] \times G) = F(G) \bullet A_t \qquad \text{for } t \in [0,T], G \in \mathscr{B}^d.$$

We call (b, c, F, A) *differential characteristics* of X.

One should observe that the differential characteristics are not unique: for example, $(2b, 2c, 2F, \frac{1}{2}A)$ yields another version. Especially for $A_t = t$, one can interpret b_t or rather $b_t + \int (x - h(x))F_t(dx)$ as a drift rate, c_t as a diffusion coefficient, and F_t as a local jump measure. The differential characteristics are typically derived from other "local" representations of the process, for example, in terms of a stochastic differential equation.

From now on, we choose the same fixed process A for all the (finitely many) semimartingales in this paper. The results do not depend on its particular choice. In concrete models, A is often taken to be $A_t = t$ (e.g., for Lévy processes, diffusions, Itô processes, etc.) and $A_t = [t] := \max\{n \in \mathbb{N} : n \leq t\}$ (discrete-time processes). Since almost all semimartingales of interest in this paper are actually special semimartingales, we use from now on the (otherwise forbidden) "truncation" function

$$h(x) := x,$$

which simplifies a number of expressions considerably.

By $\langle X, Y\rangle$ we denote the P-compensator of $[X, Y]$ provided that X, Y are semimartingales such that $[X, Y]$ is P-special (cf. [27], page 37). If X and Y are vector-valued, then $[X, Y]$ and $\langle X, Y\rangle$ are to be understood as matrix-valued processes with components $[X^i, Y^j]$ and $\langle X^i, Y^j\rangle$, respectively. Moreover, if both Y and a predictable process ϑ are $\mathbb{R}^d$-valued, then the notation $\vartheta \bullet [X, Y]$ (and accordingly $\vartheta \bullet \langle X, Y\rangle$) refers to the vector-valued process whose components $\vartheta \bullet [X^i, Y]$ are the vector-stochastic integral of $(\vartheta^j)_{j=1,\dots,d}$ relative to $([X^i, Y^j])_{j=1,\dots,d}$. If $P^\star$ denotes another probability measure, we write $\langle X, Y\rangle^{P^\star}$ for the $P^\star$-compensator of $[X, Y]$.

In the whole paper, we write M^X for the local martingale part and A^X for the predictable part of finite variation in the canonical decomposition

$$X = X_0 + M^X + A^X$$

of a special semimartingale X. If $P^\star$ denotes another probability measure, we write accordingly

$$X = X_0 + M^{X\star} + A^{X\star}$$

for the $P^\star$-canonical decomposition of X.

If (b, c, F, A) denote differential characteristics of an $\mathbb{R}^d$-valued special semimartingale X, we use the notation $\tilde{c}, \hat{c}$ for *modified second characteristics* in the following sense (provided that the integrals exist):

$$\tilde{c} := c + \int xx^\top F(dx), \tag{1.2}$$

$$\hat{c} := c + \int xx^\top F(dx) - bb^\top \Delta A. \tag{1.3}$$

Observe that $x^\top \hat{c} x \le x^\top \tilde{c} x$ for any $x \in \mathbb{R}^d$. The notion of modified second characteristics is motivated by the following:

PROPOSITION 1.2. *Let X be an $\mathbb{R}^d$-valued special semimartingale with differential characteristics (b, c, F, A) and modified second characteristics as in* (1.2) *and* (1.3). *If the corresponding integrals exist, then*

$$\langle X, X\rangle = \tilde{c} \bullet A,$$
$$\langle M^X, M^X\rangle = \hat{c} \bullet A.$$

PROOF. The first equation follows from [28], I.4.52, the second from [28], II.2.17 (adjusted for the truncation function). □

From now on we use the notation (b^X, c^X, F^X, A) to denote differential characteristics of a special semimartingale X. Accordingly, $\tilde{c}^X, \hat{c}^X$ stands for the modified second characteristics of X. If they refer to some probability measure $P^\star$ rather than P, we write instead $(b^{X\star}, c^{X\star}, F^{X\star}, A)$ and $\tilde{c}^{X\star}, \hat{c}^{X\star}$, respectively. We denote the joint characteristics of two special semimartingales X, Y [i.e., the characteristics of (X, Y)] as

$$(b^{X,Y}, c^{X,Y}, F^{X,Y}, A) = \left(\begin{pmatrix} b^X \\ b^Y \end{pmatrix}, \begin{pmatrix} c^X & c^{XY} \\ c^{YX} & c^Y \end{pmatrix}, F^{X,Y}, A\right)$$

and

$$\tilde{c}^{X,Y} = \begin{pmatrix} \tilde{c}^X & \tilde{c}^{XY} \\ \tilde{c}^{YX} & \tilde{c}^Y \end{pmatrix}, \qquad \hat{c}^{X,Y} = \begin{pmatrix} \hat{c}^X & \hat{c}^{XY} \\ \hat{c}^{YX} & \hat{c}^Y \end{pmatrix}.$$

In the whole paper, we write c^{-1} for the Moore–Penrose pseudoinverse of a matrix or matrix-valued process c, which is a particular matrix satisfying $cc^{-1}c = c$ (cf. [1]). From the construction it follows that the mapping $c \mapsto c^{-1}$ is measurable. Moreover, c^{-1} is nonnegative and symmetric if this holds for c.

Finally, we write $X \sim Y$ (resp. $X \sim^\star Y$) if two semimartingales differ only by some P-σ-martingale (or some $P^\star$-σ-martingale, resp.). Some facts on σ-martingales are summarized in Appendix A.2.

2. Admissible strategies and quadratic hedging. We work on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, P)$, where $T \in \mathbb{R}_+$ denotes a fixed terminal time. The $\mathbb{R}^d$-valued process $S = (S_t^1, \dots, S_t^d)_{t\in[0,T]}$ represents the discounted prices of d securities. We assume that

$$\sup\{E((S_\tau^i)^2) : \tau \text{ stopping time }, i = 1, \dots, d\} < \infty, \tag{2.1}$$

that is, S is a $L^2(P)$*-semimartingale* in the sense of [15].

Moreover, we make the following standing:

ASSUMPTION 2.1. There exists some equivalent σ-martingale measure with square-integrable density, that is, some probability measure $Q \sim P$ with $E((\frac{dQ}{dP})^2) < \infty$ and such that S is a Q-σ-martingale.

This can be interpreted as a natural no-free-lunch condition in the present quadratic context. More specifically, Théorème 2 in [47] and standard arguments show that Assumption 2.1 is equivalent to the *absence of L^2-free lunches* in the sense that

$$\overline{K_2^s(0) - L_+^2} \cap L_+^2 = \{0\},$$

where $K_2^s(0)$ denotes the set of payoffs of simple trading defined below, L_+^2 contains the nonnegative square-integrable random variables, and the closure is to be taken in $L^2(P)$.

2.1. *Admissible strategies.* The choice of the set of admissible trading strategies in continuous time is a delicate point. If it is too large, arbitrage opportunities occur even in the Black–Scholes model, if it is too small, optimal strategies as, for example, the replicating portfolio of a European call in the Black–Scholes model fail to exist. Inspired by Delbaen and Schachermayer [15], we consider the closure (in a proper L^2-sense) of the set of simple strategies.

More specifically, an $\mathbb{R}^d$-valued process ϑ is called *simple* if it is a linear combination of processes of the form $Y1_{]\!]\tau_1,\tau_2]\!]}$, where $\tau_1 \le \tau_2$ denote stopping times and Y a bounded $\mathscr{F}_{\tau_1}$-measurable random variable. We call a payoff *attainable by simple trading with initial endowment* $v \in L^2(\Omega, \mathscr{F}_0, P)$ if it belongs to the set

$$K_2^s(v) := \{v + \vartheta \bullet S_T : \vartheta \text{ simple}\}.$$

If the initial endowment v is not fixed beforehand, we consider instead the set

$$K_2^s(\mathscr{F}_0) := \{v + \vartheta \bullet S_T : v \in L^2(\Omega, \mathscr{F}_0, P), \vartheta \text{ simple}\}.$$

Since the hedging problems in this paper concern the approximation of arbitrary payoffs H in $L^2(P)$ by attainable outcomes, it makes perfect sense from an economical point of view to call the elements of the $L^2(P)$-closures $K_2(v) := \overline{K_2^s(v)}$, respectively, $K_2(\mathscr{F}_0) := \overline{K_2^s(\mathscr{F}_0)}$ *attainable* as well. These outcomes can be written as a stochastic integral $v + \vartheta \bullet S_T$ with some strategy $\vartheta \in L(S)$ that can be approximated in the following sense by simple strategies (cf. Lemmas 2.4 and 2.6 below).

DEFINITION 2.2. We call $\vartheta \in L(S)$ *admissible strategy* if there exists some sequence $(\vartheta^{(n)})_{n \in \mathbb{N}}$ of simple strategies such that

$$\vartheta^{(n)} \bullet S_t \to \vartheta \bullet S_t \qquad \text{in probability for any } t \in [0, T] \quad \text{and}$$
$$\vartheta^{(n)} \bullet S_T \to \vartheta \bullet S_T \qquad \text{in } L^2(P).$$

Similarly, we call $(v, \vartheta) \in L^0(\Omega, \mathscr{F}_0, P) \times L(S)$ *admissible endowment/strategy pair* if there exist some sequences $(v^{(n)})_{n \in \mathbb{N}}$ in $L^2(\Omega, \mathscr{F}_0, P)$ and $(\vartheta^{(n)})_{n \in \mathbb{N}}$ of simple strategies such that

$$v^{(n)} + \vartheta^{(n)} \bullet S_t \to v + \vartheta \bullet S_t \qquad \text{in probability for any } t \in [0, T] \quad \text{and}$$
$$v^{(n)} + \vartheta^{(n)} \bullet S_T \to v + \vartheta \bullet S_T \qquad \text{in } L^2(P).$$

We set

$$\overline{\Theta} := \{\vartheta \in L(S) : \vartheta \text{ admissible}\},$$
$$\overline{L^2(\mathscr{F}_0) \times \Theta} := \{(v, \vartheta) \in L^0(\Omega, \mathscr{F}_0, P) \times L(S) : (v, \vartheta) \text{ admissible}\}.$$

One easily verifies that $\overline{L^2(\mathscr{F}_0) \times \Theta} = \mathbb{R} \times \overline{\Theta}$ if the initial σ-field $\mathscr{F}_0$ is trivial. Admissible strategies are linked via duality to martingale measures of the following kind:

DEFINITION 2.3. We call a signed measure $Q \ll P$ with $Q(\Omega) = 1$ *absolutely continuous signed σ-martingale measure (SσMM)* if SZ^Q is a P-σ-martingale for the density process

$$Z_t^Q := E\left(\frac{dQ}{dP}\middle|\mathscr{F}_t\right)$$

of Q.

A probability measure $Q \sim P$ is a SσMM if and only if S is a Q-σ-martingale (cf. Lemma A.8).

LEMMA 2.4. *For $H \in L^2(P)$ and $v \in L^2(\Omega, \mathscr{F}_0, P)$ the following statements are equivalent*:

1. $H \in K_2(v)$.
2. $E_Q(H - v) = 0$ *for any SσMM Q with* $\frac{dQ}{dP} \in L^2(P)$.
3. $H = v + \vartheta \cdot S_T$ *with some* $\vartheta \in \overline{\Theta}$.
4. $H = v + \vartheta \cdot S_T$ *with some* $\vartheta \in L(S)$ *such that* $(\vartheta \cdot S)Z^Q$ *is a martingale for any SσMM Q with density process Z^Q and* $\frac{dQ}{dP} \in L^2(P)$.

In particular, we have $K_2(v) = \{v + \vartheta \cdot S_T : \vartheta \in \overline{\Theta}\}$.

PROOF. It suffices to consider the case $v = 0$.

$1 \Rightarrow 3, 4$: *Step* 1: We start by showing that statement 4 holds for $H \in K_2^s(0)$, that is, for $H = \vartheta \cdot S_T$ with some simple ϑ. Integration by parts yields

$$(2.2)\qquad \begin{aligned}(\vartheta \cdot S)Z^Q &= (\vartheta \cdot S)_- \cdot Z^Q + \vartheta \cdot (Z^Q_- \cdot S + [Z^Q, S])\\ &= (\vartheta \cdot S_- - \vartheta^\top S_-) \cdot Z^Q + \vartheta \cdot (SZ^Q),\end{aligned}$$

which implies that $(\vartheta \cdot S)Z^Q$ is a σ-martingale. Since $\sup_{t\in[0,T]} |Z_t^Q| \in L^2(P)$ by Doob's inequality and $\vartheta \cdot S$ is a L^2-semimartingale in the sense of (2.1), we have that $(\vartheta \cdot S)Z^Q$ is of class (D) and hence a martingale (cf. Lemma A.7).

Step 2: Let $H^n = \vartheta^{(n)} \cdot S_T$ be an approximating sequence for $H \in K_2(0)$. From [15], Theorem 1.2, it follows that H has a representation $H = \vartheta \cdot S_T$ for some $\vartheta \in L(S)$. In the proof of this theorem it is actually shown that ϑ can be chosen such that $\vartheta^{(n)} \cdot S_t$ converges in probability to $\vartheta \cdot S_t$ for any $t \in [0, T]$.

Since $H^n Z_T^Q \to H Z_T^Q$ in $L^1(P)$, we have that

$$E\big((\vartheta^{(n)} \cdot S_T)Z_T^Q|\mathscr{F}_t\big) \to E\big((\vartheta \cdot S_T)Z_T^Q|\mathscr{F}_t\big)$$

in $L^1(P)$ and hence in probability. Step 1 yields $E((\vartheta^{(n)} \cdot S_T)Z_T^Q|\mathscr{F}_t) = (\vartheta^{(n)} \cdot S_t)Z_t^Q$. Together, it follows that $E((\vartheta \cdot S_T)Z_T^Q|\mathscr{F}_t) = (\vartheta \cdot S_t)Z_t^Q$.

$3 \Rightarrow 1$: This is obvious.

$4 \Rightarrow 2$: This is obvious as well.

$2 \Rightarrow 1$: It suffices to show that $K_2(0)^\perp \subset (V^\perp)^\perp$ for

$$V := \left\{\frac{dQ}{dP} : Q \text{ S}\sigma\text{MM with } \frac{dQ}{dP} \in L^2(P)\right\},$$

where the orthogonal complements refer to $L^2(P)$. Let $Y \in K_2(0)^\perp$ and set $Z_t := E(Y|\mathscr{F}_t)$. For $s \le t$ and $F \in \mathscr{F}_s$ we have

$$\begin{aligned}&E\big(1_F(S_t Z_t - S_s Z_s)\big)\\ &\quad = E\big(1_F(S_t - S_s)Y\big) - E\big(1_F S_t(Z_T - Z_t)\big) + E\big(1_F S_s(Z_T - Z_s)\big)\\ &\quad = 0\end{aligned}$$

because Z is a martingale and $1_{F\times(s,t]} \bullet S_T \in K_2(0)$. If $E(Y) \neq 0$, then Y is a multiple of a SσMM and hence in $(V^\perp)^\perp$. If $E(Y) = 0$, then $Y + \frac{dQ}{dP} \in V \subset (V^\perp)^\perp$ for the SσMM Q from Assumption 2.1, which implies that $Y \in (V^\perp)^\perp$ as well. □

This leads to the following characterization of admissible strategies:

COROLLARY 2.5. *We have equivalence between*:

1. *ϑ is an admissible strategy.*
2. *$\vartheta \in L(S)$, $\vartheta \bullet S_T \in L^2(P)$, and $(\vartheta \bullet S)Z^Q$ is a martingale for any SσMM Q with density process Z^Q and $\frac{dQ}{dP} \in L^2(P)$.*

PROOF. $1 \Rightarrow 2$: This follows from the argument in step 2 of the proof of Lemma 2.4.

$2 \Rightarrow 1$: We have $\vartheta \bullet S_T \in K_2(0)$ by Lemma 2.4. Let Q be a σ-martingale measure as in Assumption 2.1. By the proof of Lemma 2.4 ($1 \Rightarrow 3$,4) there exists some $\tilde{\vartheta} \in \overline{\Theta}$ such that $\tilde{\vartheta} \bullet S$ is a Q-martingale with $\tilde{\vartheta} \bullet S_T = \vartheta \bullet S_T$. Since $\vartheta \bullet S$ is a Q-martingale as well, we have $\tilde{\vartheta} \bullet S = \vartheta \bullet S$ and hence $\vartheta \in \overline{\Theta}$. □

In the case without fixed initial endowment we have:

LEMMA 2.6. *There exists*

1. $K_2(\mathscr{F}_0) = \{v + \vartheta \bullet S_T : (v, \vartheta) \in \overline{L^2(\mathscr{F}_0) \times \Theta}\}$.
2. *If $(v, \vartheta) \in \overline{L^2(\mathscr{F}_0) \times \Theta}$, then $(v + \vartheta \bullet S)Z^Q$ is a martingale for any SσMM Q with density process Z^Q and $\frac{dQ}{dP} \in L^2(P)$.*

PROOF. This follows by rather obvious extension of the proof of Lemma 2.4 ($1 \Rightarrow 3, 4$) and the underlying arguments in [15]. □

REMARK 2.7. An inspection of the proof reveals that statement 2 in Corollary 2.5 and Lemma 2.6 holds for any square-integrable martingale Z^Q such that SZ^Q is a σ-martingale, that is, the property $E(Z_T^Q) = 1$ is not needed.

If necessary the whole setup can be relaxed to slightly more general price processes:

REMARK 2.8. Instead of (2.1), Delbaen and Schachermayer [15] assume only that S is a *local $L^2(P)$-semimartingale*, that is, that there is a localizing sequence of stopping times $(U_n)_{n\in\mathbb{N}}$ such that:

$$\sup\{E((S^i_\tau)^2) : \tau \le U_n \text{ stopping time}, i = 1, \dots, d\} < \infty$$

for any $n \in \mathbb{N}$. Equivalently, $S^1, \dots, S^d$ are locally square-integrable semimartingales (cf. Definition A.1 and Lemma A.2 in the Appendix). In this case Delbaen and Schachermayer [15] call a linear combination of strategies $Y 1_{]\!]\tau_1,\tau_2]\!]}$ *simple* if the corresponding stopping times $\tau_1 \leq \tau_2$ are dominated by some U_n. One easily verifies that *all results in this paper extend to this slightly more general setup*.

The corresponding admissible sets $\overline{\Theta}$ and $\overline{L^2(\mathscr{F}_0) \times \Theta}$ from Definition 2.2 do not depend on the chosen sequence $(U_n)_{n\in\mathbb{N}}$: For $\overline{\Theta}$ this follows from the characterization in Corollary 2.5. Moreover, $K_2(\mathscr{F}_0) = \overline{L^2(\Omega, \mathscr{F}_0, P) + K_2(0)}$ does not depend on $(U_n)_{n\in\mathbb{N}}$ by Lemma 2.4. Using Lemma 2.6 and arguing similarly as in the proof of Corollary 2.5 ($2 \Rightarrow 1$), we have that the same is true for $\overline{L^2(\mathscr{F}_0) \times \Theta}$.

Many results in the subsequent sections could also be expressed in terms of the generally different set of strategies considered in [42] and other papers on mean-variance hedging, namely

$$\Theta := \{\vartheta \in L(S) : \vartheta \bullet S \in \mathscr{S}^2\},$$

where $\mathscr{S}^2$ denotes the set of square-integrable semimartingales (cf. Definition A.1). In contrast to $\{v + \vartheta \bullet S_T : \vartheta \in \overline{\Theta}\}$, the set $\{v + \vartheta \bullet S_T : \vartheta \in \Theta\}$ is not necessarily closed. This issue is discussed in detail by Monat and Stricker [37], Delbaen et al. [13] and Choulli, Krawczyk and Stricker [12]. By considering L^2-closures in the above sense, one avoids the problem that optimal hedging strategies may fail to exist. In the context of continuous processes, our notion of admissible strategies coincides with the one of Gourieroux, Laurent and Pham [20] and Laurent and Pham [30]. Recently, the question of how to choose a reasonable set of strategies in a quadratic context has been discussed by Xia and Yan [48]. Their notion of admissibility differs from ours but their set of terminal payoffs coincides with $K_2(0)$.

The relationship between $\overline{\Theta}$ and Θ is clarified by the following result. The first assertion is inspired by a similar statement in Grandits and Rheinläender [21], Lemma 2.1 for continuous processes. Loosely speaking, it says that $\overline{\Theta}$ is a kind of L^2-closure of Θ.

COROLLARY 2.9. *We have*

1. $\Theta \subset \overline{\Theta}$ *and* $\overline{\{\vartheta \bullet S_T : \vartheta \in \Theta\}} = K_2(0) = \{\vartheta \bullet S_T : \vartheta \in \overline{\Theta}\}$.
2. $L^2(\Omega, \mathscr{F}_0, P) \times \Theta \subset \overline{L^2(\mathscr{F}_0) \times \Theta}$ *and*

$$\begin{aligned}&\overline{\{v + \vartheta \bullet S_T : v \in L^2(\Omega, \mathscr{F}_0, P), \vartheta \in \Theta\}}\\ &\qquad = K_2(\mathscr{F}_0) = \{v + \vartheta \bullet S_T : (v, \vartheta) \in \overline{L^2(\mathscr{F}_0) \times \Theta}\}.\end{aligned}$$

In both cases the closure $\overline{\{\cdots\}}$ *refers to the* $L^2(P)$*-norm.*

PROOF. 1. For $\vartheta \in \Theta$ we have $E(\sup_{t\in[0,T]} |\vartheta \bullet S_t|^2) < \infty$ by Protter [39], Theorem IV.5. $\vartheta \in \overline{\Theta}$ now follows easily from Corollary 2.5 ($2 \Rightarrow 1$) together with (2.2) and Lemma A.7. The second equality is shown in Lemma 2.4. In order to verify the first equality, it suffices to prove that any simple strategy is in Θ. This may not be true in the first place. But if the sequence $(U_n)_{n\in\mathbb{N}}$ in Remark 2.8 is chosen such that $(S^i)^{U_n} \in \mathscr{S}^2$ for $n \in \mathbb{N}$, $i = 1, \dots, d$, then $\vartheta \in \Theta$ for any simple ϑ. Since $\overline{\Theta}$ does not depend on the chosen sequence $(U_n)_{n\in\mathbb{N}}$, the claim follows.

2. By statement 1 we have

$$L^2(\Omega, \mathscr{F}_0, P) \times \Theta \subset L^2(\Omega, \mathscr{F}_0, P) \times \overline{\Theta} \subset \overline{L^2(\mathscr{F}_0) \times \Theta}.$$

The equalities follow similarly as in statement 1, this time using Lemma 2.6. $\square$

2.2. *Mean-variance hedging.* The goal of this paper is to hedge a fixed contingent claim with discounted payoff $H \in L^2(\Omega, \mathscr{F}, P)$. We consider two closely related optimization problems.

DEFINITION 2.10. 1. We call an admissible endowment/strategy pair (v_0, φ) *optimal* if $(v, \vartheta) = (v_0, \varphi)$ minimizes the expected squared hedging error

$$E\big((v + \vartheta \bullet S_T - H)^2\big) \tag{2.3}$$

over all admissible endowment/strategy pairs (v, ϑ).

2. If the initial endowment $v = v_0 \in L^2(\Omega, \mathscr{F}_0, P)$ is given beforehand, a minimizer $\vartheta = \varphi$ of (2.3) over all $\vartheta \in \overline{\Theta}$ is called *optimal hedging strategy for given initial endowment* v_0.

Due to the chosen notion of admissibility, optimal hedges always exist:

LEMMA 2.11. *There exist optimal hedges in the sense of Definition* 2.10(1) *and* (2). *In both cases, the value process $v_0 + \varphi \bullet S$ of the optimal hedge is unique up to a P-null set.*

PROOF. The existence follows from Lemmas 2.4, 2.6 and the closedness of $K_2(\mathscr{F}_0)$ and $K_2(v_0)$, respectively.

Denote by $v_0 + \varphi \bullet S$ and $\tilde{v}_0 + \tilde{\varphi} \bullet S$ value processes of two optimal hedges [which implies that $v_0 = \tilde{v}_0$ in the situation of Definition 2.10(2)]. A simple convexity argument yields $v_0 + \varphi \bullet S_T = \tilde{v}_0 + \tilde{\varphi} \bullet S_T$. It remains to be shown that this implies $v_0 + \varphi \bullet S = \tilde{v}_0 + \tilde{\varphi} \bullet S$ up to a P-null set. Otherwise, there exists some $n \in \mathbb{N}$ such that $P(\tau < T) > 0$ for the stopping time

$$\tau := \inf\left\{t \in [0, T] : v_0 + \varphi \bullet S_t \geq \tilde{v}_0 + \tilde{\varphi} \bullet S_t + \frac{1}{n}\right\} \wedge T$$

(or possibly with exchanged roles of $\varphi, \tilde{\varphi}$). From Corollary 2.5 and Lemma 2.6 if follows that $M := v_0 - \tilde{v}_0 + (\varphi - \tilde{\varphi}) \bullet S$ is a martingale with respect to the σ-martingale measure Q from Assumption 2.1. Consequently, $E_Q(M_\tau) = E_Q(M_T) = 0$, which is impossible if $P(\tau < T) > 0$. $\square$

3. On the pure investment problem. In many papers the mean-variance hedging problem is partially reduced to pure portfolio optimization with quadratic utility. This is done here as well.

3.1. *Opportunity process.* In the spirit of Markowitz, we call an admissible strategy $\lambda^{(\tau)}$ *efficient* on a stochastic interval $]\!]\tau, T]\!]$ if it minimizes

$$E\big((1-\vartheta \bullet S_T)^2\big) \tag{3.1}$$

over all $\vartheta \in \overline{\Theta}$ vanishing on $[\![0,\tau]\!]$. Indeed, by standard arguments there exists no strategy with at most the same variance yielding a higher expected return. Alternatively, one may view $\lambda^{(\tau)}$ as optimal hedging strategy on $]\!]\tau, T]\!]$ for the constant option $H = 1$. A crucial role will be played by the related *opportunity process*

$$L_t = E\big((1-\lambda^{(t)} \bullet S_T)^2 | \mathscr{F}_t\big),$$

whose existence and properties are yet to be derived.

LEMMA 3.1. 1. *For any stopping time τ there exists an efficient strategy $\lambda^{(\tau)}$ on $]\!]\tau, T]\!]$. Its value process $1-\lambda^{(\tau)} \bullet S$ is uniquely determined.*

2. $1-\lambda^{(\varrho)} \bullet S_\tau = (1-\lambda^{(\varrho)} \bullet S_\sigma)(1-\lambda^{(\sigma)} \bullet S_\tau)$ *for all stopping times $\varrho \le \sigma \le \tau$.*

3. *If $1-\lambda^{(\sigma)} \bullet S_\tau = 0$, then $1-\lambda^{(\sigma)} \bullet S_T = 0$ for all stopping times $\sigma \le \tau$.*

4. $E((1-\lambda^{(\tau)} \bullet S_T)^2 | \mathscr{F}_\sigma) \le E((1-\vartheta \bullet S_T)^2 | \mathscr{F}_\sigma)$ *for all stopping times $\sigma \le \tau$ and any strategy $\vartheta \in \overline{\Theta}$ with $\vartheta 1_{[\![0,\tau]\!]} = 0$.*

5. $E(1-\lambda^{(\tau)} \bullet S_T | \mathscr{F}_\sigma) = E((1-\lambda^{(\tau)} \bullet S_T)^2 | \mathscr{F}_\sigma) \in (0,1]$ *almost surely for all stopping times $\sigma \le \tau$.*

PROOF. 1. If G denotes the orthogonal projection of 1 on

$$\overline{\{\vartheta \bullet S_T : \vartheta \in \overline{\Theta} \text{ and } \vartheta 1_{[\![0,\tau]\!]} = 0\}} \subset K_2(0),$$

then there is a sequence $(\vartheta^{(n)})_{n\in\mathbb{N}}$ of strategies in $\overline{\Theta}$ that vanish on $[\![0,\tau]\!]$ and satisfy $\vartheta^{(n)} \bullet S_T \to G$ in $L^2(P)$. By Lemma 2.4 we have $G = \vartheta \bullet S_T$ for some $\vartheta \in \overline{\Theta}$. Moreover, $\vartheta^{(n)} \bullet S_T \to \vartheta \bullet S_T$ in $L^1(Q)$ for the σ-martingale measure Q from Assumption 2.1. This implies $0 = \vartheta^{(n)} \bullet S_t \to \vartheta \bullet S_t$ in $L^1(Q)$ because both $\vartheta^{(n)} \bullet S$ and $\vartheta \bullet S_t$ are Q-martingales by Corollary 2.5. Hence we have $\vartheta 1_{[\![0,\tau]\!]} = 0$ without loss of generality. Uniqueness follows as in the proof of Lemma 2.11.

2. We start by showing that

$$\begin{aligned} &E\big((1-\lambda^{(\varrho)} \bullet S_T)^2 | \mathscr{F}_\sigma\big) \\ &\quad \le E\Big(\big(1-\big(\lambda^{(\varrho)} 1_{[\![0,\sigma]\!]} + (1-\lambda^{(\varrho)} \bullet S_\sigma)\vartheta\big) \bullet S_T\big)^2 | \mathscr{F}_\sigma\Big) \end{aligned} \tag{3.2}$$

holds almost surely for any $\vartheta \in \overline{\Theta}$ with $\vartheta 1_{[\![0,\sigma]\!]} = 0$. Otherwise, there exists some $\vartheta \in \overline{\Theta}$ with $\vartheta 1_{[\![0,\sigma]\!]} = 0$ such that the reverse inequality holds on some set $F \in \mathscr{F}_\sigma$ with $P(F) > 0$. Define the strategy

$$\psi := \begin{cases} \lambda^{(\varrho)} 1_{[\![0,\sigma]\!]} + (1-\lambda^{(\varrho)} \bullet S_\sigma)\vartheta, & \text{on } F, \\ \lambda^{(\varrho)}, & \text{on } F^C. \end{cases}$$

We have

$$
\begin{aligned}
&E\big((1-\psi \bullet S_T)^2\big)\\
&\quad = E\big(E\big((1-\lambda^{(\varrho)} \bullet S_T)^2 | \mathscr{F}_\sigma\big) 1_{F^C}\big)\\
&\qquad + E\big(E\big((1-(\lambda^{(\varrho)} 1_{[\![0,\sigma]\!]} + (1-\lambda^{(\varrho)} \bullet S_\sigma)\vartheta) \bullet S_T)^2 | \mathscr{F}_\sigma\big) 1_F\big)\\
&\quad < E\big((1-\lambda^{(\varrho)} \bullet S_T)^2\big).
\end{aligned}
\tag{3.3}
$$

This contradicts the optimality of $\lambda^{(\varrho)}$ if $\psi \in \overline{\Theta}$.

In order to show $\psi \in \overline{\Theta}$, let Z be the density process of some SσMM with square-integrable density. Integration by parts yields that $(\psi \bullet S)Z$ is a σ-martingale [cf. (2.2)]. Since $P(|\lambda^{(\varrho)} \bullet S_\sigma| \le n) \uparrow 1$ and $P(|\vartheta \bullet S_\sigma| \le n) \uparrow 1$ for $n \uparrow \infty$, we may assume w.l.o.g. that $|\lambda^{(\varrho)} \bullet S_\sigma|$ and $|\vartheta \bullet S_\sigma|$ are bounded on F, say by $n \in \mathbb{N}$. On $]\!]\sigma, T]\!]$ we have

$$
\begin{aligned}
&\big|(\psi \bullet S - \lambda^{(\varrho)} \bullet S)Z\big|\\
&\quad \le \big(\big|\lambda^{(\varrho)} \bullet S - \lambda^{(\varrho)} \bullet S_\sigma\big| + \big|1-\lambda^{(\varrho)} \bullet S_\sigma\big||\vartheta \bullet S - \vartheta \bullet S_\sigma|\big)|Z| 1_F\\
&\quad \le \big(\big|(\lambda^{(\varrho)} \bullet S)Z\big| + n|Z| + (n+1)\big(|(\vartheta \bullet S)Z| + n\big)\big) 1_F.
\end{aligned}
$$

The processes in the last line are of class (D) by Corollary 2.5. This in turn implies that $(\psi \bullet S)Z$ is of class (D) as well and hence a martingale. Another application of Corollary 2.5 yields $\psi \in \overline{\Theta}$. Thus (3.3) yields a true contradiction, which means that (3.2) holds.

Note that (3.2) implies

$$
E\left(\left(1 - \frac{\lambda^{(\varrho)} 1_{]\!]\sigma,T]\!]}}{1-\lambda^{(\varrho)} \bullet S_\sigma} \bullet S_T\right)^2 \middle| \mathscr{F}_\sigma\right) \le E\big((1-\vartheta \bullet S_T)^2 | \mathscr{F}_\sigma\big)
\tag{3.4}
$$

almost surely on $\{1-\lambda^{(\varrho)} \bullet S_\sigma \neq 0\}$ for any $\vartheta \in \overline{\Theta}$ with $\vartheta 1_{[\![0,\sigma]\!]} = 0$. Moreover, we have on the set $\{1-\lambda^{(\varrho)} \bullet S_\sigma = 0\}$ that

$$
E\big((1-\lambda^{(\varrho)} \bullet S_T)^2 | \mathscr{F}_\sigma\big) \le E\big((1-(\lambda^{(\varrho)} 1_{[\![0,\sigma]\!]}) \bullet S_T)^2 | \mathscr{F}_\sigma\big) = 0
$$

and hence $1-\lambda^{(\varrho)} \bullet S_T = 0$.

Similarly as (3.2), one shows that

$$
E\big((1-\lambda^{(\sigma)} \bullet S_T)^2 | \mathscr{F}_\sigma\big) \le E\left(\left(1 - \left(\alpha \frac{\lambda^{(\varrho)} 1_{]\!]\sigma,T]\!]}}{1-\lambda^{(\varrho)} \bullet S_\sigma} + \vartheta\right) \bullet S_T\right)^2 \middle| \mathscr{F}_\sigma\right)
\tag{3.5}
$$

holds almost surely on $\{1-\lambda^{(\varrho)} \bullet S_\sigma \neq 0\}$ for any $\alpha \in \mathbb{R}_+$ and any $\vartheta \in \overline{\Theta}$ with $\vartheta 1_{[\![0,\sigma]\!]} = 0$. Using a convexity argument, (3.4) and (3.5) yield that

$$
1 - \lambda^{(\sigma)} \bullet S_T = 1 - \frac{\lambda^{(\varrho)} 1_{]\!]\sigma,T]\!]}}{1-\lambda^{(\varrho)} \bullet S_\sigma} \bullet S_T
$$

on $\{1-\lambda^{(\varrho)}\bullet S_\sigma \neq 0\}$ and hence

$$\lambda^{(\sigma)}\bullet S_T\big(1-\lambda^{(\varrho)}\bullet S_\sigma\big)=\big(\lambda^{(\varrho)}1_{]\!]\sigma,T]\!]}\big)\bullet S_T.$$

By taking conditional expectation relative to the σ-martingale measure Q from Assumption 2.1, it follows that

$$\lambda^{(\sigma)}\bullet S_\tau\big(1-\lambda^{(\varrho)}\bullet S_\sigma\big)=\big(\lambda^{(\varrho)}1_{]\!]\sigma,T]\!]}\big)\bullet S_\tau$$

for any $\tau\geq\sigma$ (cf. Corollary 2.5), which yields the claim.

3. This is shown in the proof of statement 2.
4. This follows from (3.2) for $\varrho=\sigma$.
5. If $E((1-\lambda^{(\tau)}\bullet S_T)^2|\mathscr{F}_\sigma)=0$ on some set $F\in\mathscr{F}_\sigma$ with $P(F)>0$, then

$$\lambda^{(\tau)}\bullet S_T-\lambda^{(\tau)}\bullet S_\sigma=1,$$

which contradicts the fact that $\lambda^{(\tau)}\bullet S$ is a Q-martingale for the σ-martingale measure Q from Assumption 2.1 (cf. Corollary 2.5). Hence, $E((1-\lambda^{(\tau)}\bullet S_T)^2|\mathscr{F}_\sigma)>0$ almost surely. Moreover,

$$\begin{aligned}E\big(\big(1-(1+\varepsilon)\lambda^{(\tau)}\bullet S_T\big)^2|\mathscr{F}_\sigma\big)&=E\big(\big(1-\lambda^{(\tau)}\bullet S_T\big)^2|\mathscr{F}_\sigma\big)\\&\quad-2\varepsilon E\big(\lambda^{(\tau)}\bullet S_T\big(1-\lambda^{(\tau)}\bullet S_T\big)|\mathscr{F}_\sigma\big)\\&\quad+\varepsilon^2E\big(\big(\lambda^{(\tau)}\bullet S_T\big)^2|\mathscr{F}_\sigma\big)\end{aligned}$$

for any $\varepsilon\in\mathbb{R}$. By statement 4 this implies $E(\lambda^{(\tau)}\bullet S_T(1-\lambda^{(\tau)}\bullet S_T)|\mathscr{F}_\sigma)=0$. Together, the assertion follows. $\square$

LEMMA 3.2. 1. *There exists a unique semimartingale L with $L_T=1$ such that the process $M^{(\tau)}-(M^{(\tau)})^\tau$ is a martingale for any stopping time τ, where*

$$M^{(\tau)}:=\big(1-\lambda^{(\tau)}\bullet S\big)L. \tag{3.6}$$

2. *The process $1_{]\!]\tau,T]\!]}\bullet(SM^{(\tau)})$ is a martingale for any stopping time τ. (In the slightly more general setup of Remark 2.8, the upper bound T is to be replaced by U_n for arbitrary n.)*

3. *The process $((v+\vartheta\bullet S_s)M_s^{(t)})_{s\in[t,T]}$ is a martingale for any $(v,\vartheta)\in\overline{L^2(\mathscr{F}_0)\times\Theta}$ and any $t\in[0,T]$.*

PROOF. 1. Our reasoning relies heavily on the proofs of Lemma 3.4 and Theorem 1.3 in [16]. For any stopping time σ we introduce the process

$${}^\sigma M_t:=\frac{E(1-\lambda^{(\sigma)}\bullet S_T|\mathscr{F}_t)}{E(1-\lambda^{(\sigma)}\bullet S_T|\mathscr{F}_{\sigma\wedge t})}.$$

Define stopping times $(\tau_n)_{n\in\mathbb{N}}$ recursively by $\tau_0:=0$ and

$$\tau_{n+1}:=\inf\left\{t\geq\tau_n:\left|\frac{1-\lambda^{(\tau_n)}\bullet S_t}{E(1-\lambda^{(\tau_n)}\bullet S_T|\mathscr{F}_{\tau_n})}\right|\leq\frac{1}{2}\right\}\wedge T.$$

Then

$$|{}^{\tau_n}M_{\tau_{n+1}}| = \frac{|1-\lambda^{(\tau_n)} \bullet S_{\tau_{n+1}}|}{E(1-\lambda^{(\tau_n)} \bullet S_T|\mathscr{F}_{\tau_n})}\big|E\big(1-\lambda^{(\tau_{n+1})} \bullet S_T|\mathscr{F}_{\tau_{n+1}}\big)\big| \leq \frac{1}{2}$$

on $\{\tau_{n+1} < T\}$ by Lemma 3.1. Using Lemma 3.1(2) one easily verifies that

$${}^{\tau_n}M_t = {}^{\tau_n}M_{\tau_{n+1}}\,{}^{\tau_{n+1}}M_t$$

for $t \geq \tau_{n+1}$. Consequently, $\lim_{m\to\infty} {}^{\tau_n}M_{\tau_m} = 0$ on $D := \{\tau_n < T \text{ for all } n \in \mathbb{N}\}$. Letting

$$\widetilde{M}_t^{(\tau_n)} := E\big(1-\lambda^{(\tau_n)} \bullet S_T|\mathscr{F}_t\big)$$

we have

$$\begin{aligned}
1 &= \lim_{m\to\infty} \frac{E(\widetilde{M}_{\tau_m}^{(\tau_n)}|\mathscr{F}_{\tau_n})}{\widetilde{M}_{\tau_n}^{(\tau_n)}} \\
&= E\left(\frac{\lim_{m\to\infty}\widetilde{M}_{\tau_m}^{(\tau_n)}}{\widetilde{M}_{\tau_n}^{(\tau_n)}}\middle|\mathscr{F}_{\tau_n}\right) \\
&= E\left(\lim_{m\to\infty} {}^{\tau_n}M_{\tau_m}\middle|\mathscr{F}_{\tau_n}\right) \\
&= E({}^{\tau_n}M_T 1_{D^C}|\mathscr{F}_{\tau_n}) \\
&\leq \sqrt{E(({}^{\tau_n}M_T)^2|\mathscr{F}_{\tau_n})}\sqrt{E(1_{D^C}|\mathscr{F}_{\tau_n})}.
\end{aligned}$$

Since the last term converges to 0 on D, it follows that

$$\lim_{n\to\infty} E(({}^{\tau_n}M_T)^2|\mathscr{F}_{\tau_n}) = \infty \qquad \text{on } D. \tag{3.7}$$

Denote by Z the density process of the measure Q from Assumption 2.1. By Corollary 2.5 we have

$$E\left((1-\lambda^{(\tau_n)} \bullet S_T)\frac{Z_T}{Z_{\tau_n}}\middle|\mathscr{F}_{\tau_n}\right) = 1. \tag{3.8}$$

Observe that

$$\begin{aligned}
E\left(\left(\frac{Z_T}{Z_{\tau_n}}\right)^2\middle|\mathscr{F}_{\tau_n}\right) &= E(({}^{\tau_n}M_T)^2|\mathscr{F}_{\tau_n}) + 2E\left({}^{\tau_n}M_T\left(\frac{Z_T}{Z_{\tau_n}} - {}^{\tau_n}M_T\right)\middle|\mathscr{F}_{\tau_n}\right) \\
&\quad + E\left(\left(\frac{Z_T}{Z_{\tau_n}} - {}^{\tau_n}M_T\right)^2\middle|\mathscr{F}_{\tau_n}\right)
\end{aligned}$$

for any $n \geq 1$. Due to (3.8) and Lemma 3.1(5) the second term on the right-hand side vanishes. It follows that

$$E(({}^{\tau_n}M_T)^2|\mathscr{F}_{\tau_n}) \leq E\left(\left(\frac{Z_T}{Z_{\tau_n}}\right)^2\middle|\mathscr{F}_{\tau_n}\right). \tag{3.9}$$

Together we have $P(D)=0$: Indeed, otherwise (3.7) yields

$$P\left(\sup_{t\in[0,T]}\frac{E(Z_T^2|\mathscr{F}_t)}{Z_t^2}<E(({}^{\tau_n}M_T)^2|\mathscr{F}_{\tau_n})\right)>0$$

for large n. Consequently,

$$\left\{\frac{E(Z_T^2|\mathscr{F}_{\tau_n})}{Z_{\tau_n}^2}<E(({}^{\tau_n}M_T)^2|\mathscr{F}_{\tau_n})\right\}\in\mathscr{F}_{\tau_n}$$

has positive probability as well in contradiction to (3.9).

Now define the semimartingale L by

$$L_t:=\frac{E(1-\lambda^{(\tau_n)}\bullet S_T|\mathscr{F}_t)}{1-\lambda^{(\tau_n)}\bullet S_t}\qquad\text{for } \tau_n\leq t<\tau_{n+1}.$$

The claimed martingale property follows from Lemma 3.1(2).

Uniqueness of L follows from

$$E\big(1-\lambda^{(t)}\bullet S_T|\mathscr{F}_t\big)-L_t=E\big(M_T^{(t)}|\mathscr{F}_t\big)-M_t^{(t)}=0.$$

2. It suffices to verify that $E(1_{\rrbracket\tau,T\rrbracket}\bullet(SM^{(\tau)})_\sigma)=0$ for any stopping time σ. By substituting $\sigma\vee\tau$ for σ, we may assume $\sigma\geq\tau$ w.l.o.g. Since $1_{\rrbracket\tau,T\rrbracket}\bullet M^{(\tau)}$ is a square-integrable martingale, we have $E(S_\sigma(M_T^{(\tau)}-M_\sigma^{(\tau)}))=0$ and similarly $E(S_\tau(M_T^{(\tau)}-M_\tau^{(\tau)}))=0$. Consequently,

$$E\big(S_\sigma M_\sigma^{(\tau)}-S_\tau M_\tau^{(\tau)}\big)=E\big((S_\sigma-S_\tau)M_T^{(\tau)}\big)=E\big((\psi\bullet S_T)M_T^{(\tau)}\big)$$

for $\psi:=1_{\rrbracket\tau,\sigma\rrbracket}$. The optimality of $\lambda^{(\tau)}$ implies that

$$\begin{aligned}0&\leq E\big(\big(1-\big(\lambda^{(\tau)}+\varepsilon\psi\big)\bullet S_T\big)^2\big)-E\big(\big(1-\lambda^{(\tau)}\bullet S_T\big)^2\big)\\&=2\varepsilon E\big((\psi\bullet S_T)M_T^{(\tau)}\big)+\varepsilon^2E\big((\psi\bullet S_T)^2\big)\end{aligned}$$

for any $\varepsilon\in\mathbb{R}$ and hence $E((\psi\bullet S_T)M_T^{(\tau)})=0$.

3. By statement 2 we have that $1_{\rrbracket t,T\rrbracket}\bullet(SM^{(t)})$ and hence $(S_sM_s^{(t)})_{s\in[t,T]}$ is a martingale. Consequently, the signed measure with density process $(M^{(t)}/E(L_t))_{s\in[t,T]}$ is a SσMM in the sense of Definition 2.3 if the time set $[0,T]$ is replaced with $[t,T]$. By Lemma 2.6 (also adapted to $[t,T]$ instead of $[0,T]$ as time set), the assertion follows. □

DEFINITION 3.3. We call the process L from Lemma 3.2 *opportunity process*.

The terminology is inspired by the fact that L is linked to optimal investment opportunities. Indeed, the following corollary states that L represents both first and second moments of efficient strategies in the sense of (3.1).

COROLLARY 3.4. *For any $t \in [0, T]$ we have*

$$\begin{aligned} L_t &= E\big(1 - \lambda^{(t)} \bullet S_T | \mathscr{F}_t\big) \\ &= E\big((1 - \lambda^{(t)} \bullet S_T)^2 | \mathscr{F}_t\big) \qquad (3.10) \\ &= \inf\big\{E\big((1 - \vartheta \bullet S_T)^2 | \mathscr{F}_t\big) : \vartheta \in \overline{\Theta} \textit{ with } \vartheta 1_{[\![0,t]\!]} = 0\big\}. \end{aligned}$$

In particular, L is a submartingale.

PROOF. This follows from Lemmas 3.2 and 3.1. □

These equations can be interpreted in terms of dynamic Sharpe ratios (cf. also [31], (5.16)):

DEFINITION 3.5. For $t \in [0, T]$ we call

$$\varrho_t := \sup\left\{\frac{E(\vartheta \bullet S_T | \mathscr{F}_t)}{\sqrt{\operatorname{Var}(\vartheta \bullet S_T | \mathscr{F}_t)}} : \vartheta \in \overline{\Theta} \text{ with } \vartheta 1_{[\![0,t]\!]} = 0\right\} \qquad (3.11)$$

maximal Sharpe ratio on $(t, T]$, where we set $\operatorname{Var}(X|\mathscr{F}_t) := E(X^2|\mathscr{F}_t) - (E(X|\mathscr{F}_t))^2$ and $\frac{0}{0} := 0$.

PROPOSITION 3.6. *The relation between opportunity process L and maximal Sharpe ratio ϱ is given by*

$$\varrho = \sqrt{\frac{1}{L} - 1}$$

and

$$L = \frac{1}{1 + \varrho^2},$$

respectively.

PROOF. On the set

$$D := \big\{\omega \in \Omega : E(\vartheta \bullet S_T | \mathscr{F}_t)(\omega) = 0 \text{ for all } \vartheta \in \overline{\Theta} \text{ with } \vartheta 1_{[\![0,t]\!]} = 0\big\}$$

we have $\varrho_t = 0$. Moreover, the infimum in (3.10) is attained in $\lambda^{(t)} = 0$, which implies that $L_t = 1$ on D.

For $\omega \in D^C$ there exists some $\vartheta \in \overline{\Theta}$ with $\vartheta 1_{[\![0,t]\!]=0}$ and $E(\vartheta \bullet S_T | \mathscr{F}_t)(\omega) > 0$. For sufficiently small $\varepsilon > 0$ we have that $E((1 - \varepsilon\vartheta \bullet S_T)^2 | \mathscr{F}_t)(\omega) < 1$, which implies that $L_t < 1$ on D^C (cf. Corollary 3.4). By scaling invariance it suffices to consider ϑ with $E(\vartheta \bullet S_T | \mathscr{F}_t) = 1 - L_t$ in the supremum of (3.11). For these ϑ we have

$$\frac{E(\vartheta \bullet S_T | \mathscr{F}_t)}{\sqrt{\operatorname{Var}(\vartheta \bullet S_T | \mathscr{F}_t)}} = \frac{1 - L_t}{\sqrt{E((1 - \vartheta \bullet S_T)^2 | \mathscr{F}_t) - L_t^2}},$$

which implies that the supremum is attained in $\vartheta = \lambda^{(t)}$. The assertion follows now from Corollary 3.4. $\square$

3.2. *Adjustment process.* The optimal number of shares $\lambda^{(\tau)}$ in (3.1) depends on τ. However, the optimal number of shares *per unit of wealth* does not. It is denoted by $\tilde{a}$ in the following lemma.

LEMMA 3.7. *We use the notation from Lemma 3.1. There exists some $\tilde{a} \in L(S)$ such that*

$$1 - \lambda^{(\tau)} \bullet S = \mathscr{E}\big((-\tilde{a}1_{]\!]\tau,T]\!]}) \bullet S\big) = 1 - \big(\tilde{a}1_{]\!]\tau,T]\!]}\mathscr{E}\big((-\tilde{a}1_{]\!]\tau,T]\!]}) \bullet S\big)_{-}\big) \bullet S$$

for any stopping time τ. Consequently, we may assume

$$\lambda^{(\tau)} = \tilde{a}1_{]\!]\tau,T]\!]}\mathscr{E}\big((-\tilde{a}1_{]\!]\tau,T]\!]}) \bullet S\big)_{-}. \tag{3.12}$$

PROOF. Let

$$\tilde{a} := \sum_{n=0}^{\infty} \frac{\lambda^{(\tau_n)}}{1 - \lambda^{(\tau_n)} \bullet S_{-}} 1_{]\!]\tau_n,\tau_{n+1}]\!]},$$

where $(\tau_n)_{n\in\mathbb{N}}$ denotes the sequence of stopping times from the proof of Lemma 3.2. On $[\![0, \tau_{n+1}]\!]$ we have

$$1 - \lambda^{(\tau_n)} \bullet S = 1 - \big((1 - \lambda^{(\tau_n)} \bullet S_{-})\tilde{a}1_{]\!]\tau_n,T]\!]}\big) \bullet S$$

and hence

$$1 - \lambda^{(\tau_n)} \bullet S = \mathscr{E}\big((-\tilde{a}1_{]\!]\tau_n,T]\!]}) \bullet S\big). \tag{3.13}$$

From

$$1 - \lambda^{(\tau_n)} \bullet S_t = \big(1 - \lambda^{(\tau_n)} \bullet S_{\tau_{n+1}}\big)\big(1 - \lambda^{(\tau_{n+1})} \bullet S_t\big)$$

and

$$\mathscr{E}\big((-\tilde{a}1_{]\!]\tau_n,T]\!]}) \bullet S\big)_t = \mathscr{E}\big((-\tilde{a}1_{]\!]\tau_n,T]\!]}) \bullet S\big)_{\tau_{n+1}}\mathscr{E}\big((-\tilde{a}1_{]\!]\tau_{n+1},T]\!]}) \bullet S\big)_t$$

for $t \in]\!]\tau_{n+1}, \tau_{n+2}]\!]$ it follows recursively that (3.13) holds on $[0, T]$. Now let τ be arbitrary. On $\{\tau_n \leq \tau < \tau_{n+1}\}$ we have

$$1 - \lambda^{(\tau)} \bullet S = \frac{1 - \lambda^{(\tau_n)} \bullet S}{1 - \lambda^{(\tau_n)} \bullet S_\tau} = \frac{\mathscr{E}((-\tilde{a}1_{]\!]\tau_n,T]\!]}) \bullet S)}{\mathscr{E}((-\tilde{a}1_{]\!]\tau_n,T]\!]}) \bullet S)_\tau} = \mathscr{E}\big((-\tilde{a}1_{]\!]\tau,T]\!]}) \bullet S\big)$$

as claimed. $\square$

DEFINITION 3.8. The (not necessarily unique) process $\tilde{a}$ from Lemma 3.7 is called *adjustment process*. Moreover, we call

$$\hat{a} := (1 + \Delta A^K)\tilde{a}$$

extended adjustment process.

The name *adjustment process* is taken from [44]:

COROLLARY 3.9. *$E(\vartheta \bullet S_T \mathscr{E}(-\tilde{a} \bullet S)_T) = 0$ for any $\vartheta \in \overline{\Theta}$, i.e., $\tilde{a}$ is an adjustment process in the sense of* [44], *Section* 3 *with $\overline{\Theta}$ substituted for Θ.*

PROOF. This follows from Lemma 3.2(3). □

LEMMA 3.10. *L, L_- are* $(0, 1]$*-valued.*

PROOF. Lemma 3.1(5) implies that $L_t = E(1 - \lambda^{(t)} \bullet S_T | \mathscr{F}_t) \in (0, 1]$ almost surely for fixed t, which yields by right-continuity that L is $[0, 1]$-valued outside some evanescent set.

Let $\tau := \inf\{t \in [0, T] : L_t = 0\} \wedge T$. Again by Lemma 3.1(5), we have

$$0 = L_{\tau\wedge T} = E\big((1 - \lambda^{(\tau\wedge T)} \bullet S_T)^2 | \mathscr{F}_{\tau\wedge T}\big) \in (0, 1]$$

on $\{L_t = 0 \text{ for some } t \in [0, T]\}$, which implies that

$$P(L_t = 0 \text{ for some } t \in [0, T]) = 0. \tag{3.14}$$

Finally let $\tau := \inf\{t \in [0, T] : L_{t-} = 0\} \wedge T$. Define an increasing sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ via $\tau_n := \inf\{t \in [0, T] : L_t \leq \frac{1}{n}\} \wedge T$. By (3.14) we have $\tau_n \uparrow\uparrow \tau$ on $\{L_{\tau-} = 0\}$. Lemma 3.1(5) implies

$$E\big((1 - \lambda^{(\tau_n)} \bullet S_T)^2 1_{\{\tau_n<T\}}\big) = E\big(L_{\tau_n} 1_{\{\tau_n<T\}}\big).$$

By [39], Theorem V.13 we have that

$$1 - \lambda^{(\tau_n)} \bullet S_T = \mathscr{E}\big((-\tilde{a} 1_{\rrbracket\tau_n,T\rrbracket}) \bullet S\big)_T \to \mathscr{E}\big((-\tilde{a} 1_{\llbracket\tau,T\rrbracket\cap\{L_{\tau-}=0\}}) \bullet S\big)_T$$

in probability for $n \to \infty$. In view of Fatou's lemma and dominated convergence, we obtain

$$0 \leq E\big((\mathscr{E}((-\tilde{a} 1_{\llbracket\tau,T\rrbracket}) \bullet S)_T)^2 1_{\{L_{\tau-}=0\}}\big) \leq E\big(L_{\tau-} 1_{\{L_{\tau-}=0\}}\big) = 0.$$

Suppose that $\{L_- = 0\}$ is not evanescent. Then there is some n such that $P(D) > 0$ for

$$\begin{aligned} D &:= \{L_{\tau-} = 0 \text{ and } 1 - \lambda^{(\tau_n)} \bullet S_{\tau-} > 0\} \\ &\supset \{L_{\tau-} = 0 \text{ and } \Delta(-\tilde{a} \bullet S) > -1 \text{ on } \rrbracket\tau_n, \tau\llbracket\}. \end{aligned}$$

On $D \in \mathscr{F}_{\tau-}$ we have

$$\frac{1 - \lambda^{(\tau_n)} \bullet S_T}{1 - \lambda^{(\tau_n)} \bullet S_{\tau-}} = \frac{\mathscr{E}((-\tilde{a} 1_{\rrbracket\tau_n,T\rrbracket}) \bullet S)_T}{\mathscr{E}((-\tilde{a} 1_{\rrbracket\tau_n,T\rrbracket}) \bullet S)_{\tau-}} = \mathscr{E}\big((-\tilde{a} 1_{\llbracket\tau,T\rrbracket}) \bullet S\big)_T = 0.$$

Consequently, the process $\lambda^{(\tau_n)} \bullet S$ cannot be a martingale under the σ-martingale measure from Assumption 2.1, which yields a contradiction to Corollary 2.5. □

Since L_- does not vanish, the stochastic logarithm of L is well defined:

DEFINITION 3.11. We call

$$K := \mathscr{L}(L) := \frac{1}{L_-} \bullet L$$

modified mean-variance tradeoff (MMVT) process.

The modified mean-variance tradeoff process is related to the *mean-variance tradeoff (MVT) process* of [42] (cf. Section 3.6).

3.3. *Variance-optimal signed martingale measure.* With the help of the modified mean-variance tradeoff process K and the adjustment process $\tilde{a}$ we can define a signed measure $Q^\star$ which plays an important role in the context of quadratic hedging. This *variance-optimal signed martingale measure* appears more or less explicitly in many papers on the subject.

DEFINITION 3.12. We call

$$N := K - \tilde{a} \bullet S - [\tilde{a} \bullet S, K] \tag{3.15}$$

variance-optimal logarithm process and the signed measure $Q^\star$ defined via

$$\frac{dQ^\star}{dP} := \frac{L_0}{E(L_0)} \mathscr{E}(N)_T = \frac{\mathscr{E}(-\tilde{a} \bullet S)_T}{E(L_0)} = \frac{1 - \lambda^{(0)} \bullet S_T}{E(1 - \lambda^{(0)} \bullet S_T)} \tag{3.16}$$

variance-optimal signed martingale measure (variance-optimal SσMM).

The following result explains the terminology.

PROPOSITION 3.13. 1. *$Q^\star$ is a SσMM (cf. Definition 2.3) with density process*

$$Z^{Q^\star} := \frac{L_0}{E(L_0)} \mathscr{E}(N) = \frac{L \mathscr{E}(-\tilde{a} \bullet S)}{E(L_0)}.$$

2. *$Q^\star$ minimizes $Q \mapsto E((\frac{dQ}{dP})^2)$ over all SσMM's Q. Hence it is the variance-optimal signed $\overline{\Theta}$-martingale measure in the sense of* [44], *Section* 1, *with Θ replaced by $\overline{\Theta}$ in the definition.*

PROOF. 1. Note that $L_0 \mathscr{E}(N) = M^{(0)}$ is a martingale by Lemma 3.2. Lemma 3.2(3) implies that $Q^\star$ is a SσMM.

2. For any other SσMM Q with $\frac{dQ}{dP} \in L^2(P)$ we have

$$\begin{aligned}
&E\left(\left(\frac{dQ}{dP}\right)^2\right) - E\left(\left(\frac{dQ^\star}{dP}\right)^2\right) \\
&\quad \geq 2E\left(\left(\frac{dQ}{dP} - \frac{dQ^\star}{dP}\right)\frac{dQ^\star}{dP}\right) \\
&\quad = 2E\left(\frac{dQ}{dP} \frac{1 - \lambda^{(0)} \bullet S_T}{E(L_0)}\right) - 2E\left(\frac{dQ^\star}{dP} \frac{1 - \lambda^{(0)} \bullet S_T}{E(L_0)}\right) = 0
\end{aligned}$$

by Corollary 2.5. $\square$

If $Q \sim P$ is a probability measure with density process $Z = \mathscr{E}(M)$, then the density $\frac{dQ}{dP}$, the density process Z, and its stochastic logarithm M uniquely determine one another. This is not true for the variance-optimal SσMM $Q^\star$ because $\mathscr{E}(N)$ may vanish and hence N cannot be fully recovered from $\mathscr{E}(N)$ or $\frac{dQ^\star}{dP}$. Therefore the following result does not follow immediately from the fact that $Q^\star$ is a SσMM whose density process is a multiple of $\mathscr{E}(N)$.

LEMMA 3.14. *The variance-optimal logarithm process N and also $S+[S,N]$ are σ-martingales. Consequently, $S\mathscr{E}(N)$ is a σ-martingale as well.*

PROOF. Denote by $(\tau_n)_{n\in\mathbb{N}}$ the sequence of stopping times from the proof of Lemma 3.2. Since $\bigcup_{n\in\mathbb{N}}]\!]\tau_n, \tau_{n+1}]\!] = \Omega \times (0,T]$, it suffices to show that $1_{]\!]\tau_n,\tau_{n+1}]\!]} \bullet N$ and $1_{]\!]\tau_n,\tau_{n+1}]\!]} \bullet (S+[N,S])$ are σ-martingales for any $n \in \mathbb{N}$. Since

$$\begin{aligned}
\mathscr{E}(N - N^{\tau_n}) &= \mathscr{E}\big(1_{]\!]\tau_n,T]\!]} \bullet (K - \tilde{a} \bullet S - [\tilde{a} \bullet S, K])\big) \\
&= \mathscr{E}\big(1_{]\!]\tau_n,T]\!]} \bullet K\big)\mathscr{E}\big((-\tilde{a}1_{]\!]\tau_n,T]\!]}) \bullet S\big) \\
&= \frac{L(1 - \lambda^{(\tau_n)} \bullet S)}{L^{\tau_n}} \\
&= 1 + \frac{1_{]\!]\tau_n,T]\!]}}{L_{\tau_n}} \bullet M^{(\tau_n)}
\end{aligned}$$

is a σ-martingale, we have that

$$1_{]\!]\tau_n,\tau_{n+1}]\!]} \bullet N = \frac{1_{]\!]\tau_n,\tau_{n+1}]\!]}}{\mathscr{E}(N - N^{\tau_n})_-} \bullet \mathscr{E}(N - N^{\tau_n})$$

is a σ-martingale as well. Similarly,

$$\begin{aligned}
1_{]\!]\tau_n,T]\!]} \bullet \big(\mathscr{E}(N - N^{\tau_n})S\big) &= 1_{]\!]\tau_n,T]\!]} \bullet \frac{L(1 - \lambda^{(\tau_n)} \bullet S)S}{L^{\tau_n}} \\
&= \frac{1_{]\!]\tau_n,T]\!]}}{L_{\tau_n}} \bullet \big(1_{]\!]\tau_n,T]\!]} \bullet \big(M^{(\tau_n)}S\big)\big)
\end{aligned}$$

is a σ-martingale by Lemma 3.2(2). Integration by parts yields

$$\begin{aligned}
&1_{]\!]\tau_n,T]\!]} \bullet \big(\mathscr{E}(N - N^{\tau_n})S\big) - S_- \bullet \mathscr{E}(N - N^{\tau_n}) \\
&\qquad = \big(\mathscr{E}(N - N^{\tau_n})_- 1_{]\!]\tau_n,T]\!]}\big) \bullet (S + [N,S]),
\end{aligned}$$

which implies that

$$\begin{aligned}
&1_{]\!]\tau_n,\tau_{n+1}]\!]} \bullet (S + [N,S]) \\
&\quad = \frac{1_{]\!]\tau_n,\tau_{n+1}]\!]}}{\mathscr{E}(N - N^{\tau_n})_-} \bullet \big(\big(\mathscr{E}(N - N^{\tau_n})_- 1_{]\!]\tau_n,T]\!]}\big) \bullet (S + [N,S])\big)
\end{aligned}$$

is a σ-martingale as well. Finally,

$$S\mathscr{E}(N) = S_0 + \mathscr{E}(N)_- \bullet (S_- \bullet N + S + [S, N])$$

yields the last assertion. □

3.4. *Opportunity-neutral measure.* In this section we define a measure $P^\star$ in terms of its density process

$$Z^{P^\star} := \frac{L}{E(L_0)\mathscr{E}(A^K)}.$$

For $Z^{P^\star}$ to be truly a density process, we need the following

LEMMA 3.15. *The process $Z^{P^\star}$ is a bounded positive martingale and satisfies*

$$Z^{P^\star} = \frac{L_0}{E(L_0)}\mathscr{E}\left(\frac{1}{1+\Delta A^K} \bullet M^K\right).$$

PROOF. Since L is a submartingale by Corollary 3.4, we have $b^L \geq 0$ and hence $b^K = \frac{1}{L_-}b^L \geq 0$ outside some $P \otimes A$-null set. This implies that $A^K = b^K \bullet A$ and hence also $\mathscr{E}(A^K)$ are increasing processes. Thus we have $0 < Z^{P^\star} \leq \frac{1}{E(L_0)}$. The equality of the two expressions for $Z^{P^\star}$ follows from Yor's formula. From the second representation we conclude that $Z^{P^\star}$ is a local martingale and hence a martingale because it is bounded. □

DEFINITION 3.16. We call the probability measure $P^\star \sim P$ with density process $Z^{P^\star}$ *opportunity-neutral probability measure*.

The opportunity-neutral probability measure is typically not a martingale measure. In some instances it actually equals P (cf. Section 3.6). For later use we determine the $P^\star$-characteristics of S.

LEMMA 3.17. *The components of S are locally $P^\star$-square integrable semimartingales. Moreover,*

$$b^{S\star} = \frac{\bar{b}}{1+\Delta A^K}, \tag{3.17}$$

$$\tilde{c}^{S\star} = \frac{\bar{c}}{1+\Delta A^K}, \tag{3.18}$$

$$\big(1+(b^{S\star})^\top(\hat{c}^{S\star})^{-1}b^{S\star}\Delta A\big)\big(1-(b^{S\star})^\top(\tilde{c}^{S\star})^{-1}b^{S\star}\Delta A\big) = 1 \tag{3.19}$$

and

$$\hat{c}^{S\star}(\hat{c}^{S\star})^{-1}b^{S\star} = b^{S\star}, \tag{3.20}$$

$$\tilde{c}^{S\star}(\tilde{c}^{S\star})^{-1}b^{S\star} = b^{S\star}, \tag{3.21}$$

$$\bar{c}\bar{c}^{-1}\bar{b} = \bar{b}. \tag{3.22}$$

$P \otimes A$-almost everywhere, where

$$\bar{b} := b^S + c^{SL}\frac{1}{L_-} + \int x\frac{y}{L_-}F^{S,L}(d(x,y)) \tag{3.23}$$

$$= b^S + c^{SK} + \int xyF^{S,K}(d(x,y)) \tag{3.24}$$

and

$$\bar{c} := c^S + \int xx^\top\left(1+\frac{y}{L_-}\right)F^{S,L}(d(x,y)) \tag{3.25}$$

$$= c^S + \int xx^\top(1+y)F^{S,K}(d(x,y)). \tag{3.26}$$

PROOF. The components of S are locally $P^\star$-square-integrable semimartingales because $\frac{dP^\star}{dP} = Z_T^{P^\star}$ is bounded (cf. Lemma A.2). Let

$$M := \frac{1}{Z_-^{P^\star}} \bullet Z^{P^\star} = \frac{1}{1+\Delta A^K} \bullet M^K.$$

Observe that

$$K^c - M^c = K^c - \frac{1}{1+\Delta A^K} \bullet K^c = \frac{\Delta A^K}{1+\Delta A^K} \bullet K^c.$$

Since

$$\begin{aligned}\left\langle \frac{\Delta A^K}{1+\Delta A^K} \bullet K^c, \frac{\Delta A^K}{1+\Delta A^K} \bullet K^c\right\rangle_T &= \left(\frac{\Delta A^K}{1+\Delta A^K}\right)^2 \bullet \langle K^c, K^c\rangle_T \\ &= \sum_{t\le T}\left(\frac{\Delta A_t^K}{1+\Delta A_t^K}\right)^2 \Delta\langle K^c, K^c\rangle_t \\ &= 0\end{aligned}$$

by continuity of K^c, we have $\frac{\Delta A^K}{1+\Delta A^K} \bullet K^c = 0$ and hence $M^c = K^c$. Moreover, M is a local martingale with $\Delta M = \frac{1}{1+\Delta A^K}\Delta K - \frac{\Delta A^K}{1+\Delta A^K}$. Together, it follows that $b^{S,M} = (b^S, 0)^\top$, $c^{S,M} = c^{S,K}$,

$$F^{S,M}(G) = \int 1_G\left(x, \frac{y-\Delta A^K}{1+\Delta A^K}\right)F^{S,K}(d(x,y))$$

for $G \in \mathscr{B}^{d+1}$ with $G \cap (\{0\}^d \times \mathbb{R}) = \varnothing$. By the Girsanov theorem as in Lemma A.9, $P^\star$-characteristics $(b^{S\star}, c^{S\star}, F^{S\star}, A)$ of S are given by

$$\begin{aligned} b^{S\star} &= b^S + c^{SM} + \int xyF^{S,M}(d(x,y)) \\ &= b^S + c^{SK} + \int x\frac{y-\Delta A^K}{1+\Delta A^K}F^{S,K}(d(x,y)) \\ &= \frac{1}{1+\Delta A^K}\left(b^S + c^{SK} + \int xyF^{S,K}(d(x,y))\right),\end{aligned}$$

$c^{S\star} = c^S$ and

$$F^{S\star}(G) = \int 1_G(x)(1+y)F^{S,M}\big(d(x,y)\big) = \frac{1}{1+\Delta A^K}\int 1_G(x)(1+y)F^{S,K}\big(d(x,y)\big)$$

for $G \in \mathscr{B}^d$ with $0 \notin G$. This yields (3.17), (3.18).

Using the same argument as in the proof of [14], Theorem 3.5, it follows that $b_t^{S\star} \in \hat{c}_t^{S\star}\mathbb{R}^d$ and also $b_t^{S\star} \in \tilde{c}_t^{S\star}\mathbb{R}^d$ $(P \otimes A)$-almost everywhere on $\Omega \times [0,T]$. (Due to Assumption 2.1 local boundedness is not needed in our setup.) This implies (3.20), (3.21), and hence also (3.22) outside some $P \otimes A$-null set. Consequently,

$$\begin{aligned}&\big(1 + (b^{S\star})^\top(\hat{c}^{S\star})^{-1}b^{S\star}\Delta A\big)\big(1 - (b^{S\star})^\top(\tilde{c}^{S\star})^{-1}b^{S\star}\Delta A\big)\\ &\quad = 1 + (b^{S\star})^\top\big((\hat{c}^{S\star})^{-1} - (\tilde{c}^{S\star})^{-1} - (\hat{c}^{S\star})^{-1}b^{S\star}(b^{S\star})^\top(\tilde{c}^{S\star})^{-1}\Delta A\big)b^{S\star}\Delta A\\ &\quad = 1 + (b^{S\star})^\top(\hat{c}^{S\star})^{-1}\big(\tilde{c}^{S\star} - \hat{c}^{S\star} - b^{S\star}(b^{S\star})^\top\Delta A\big)(\tilde{c}^{S\star})^{-1}b^{S\star}\Delta A\\ &\quad = 1.\end{aligned}$$ $\square$

REMARK 3.18. An inspection of the proofs of Lemmas 3.15 and 3.17 yields that L need not be the opportunity process for (3.22) to hold. We only used the fact that $L = L_0\mathscr{E}(K)$ is a bounded semimartingale with $b^L \geq 0$ and $L, L_- > 0$.

3.5. *Characterization of L and $\tilde{a}$.* The opportunity process L and the adjustment process $\tilde{a}$ play a crucial role in quadratic hedging. For example, they yield the density processes of the variance-optimal SσMM $Q^\star$ and the opportunity-neutral measure $P^\star$, which in turn lead to formulas for the optimal hedge in Section 4. The characterizations of L and $\tilde{a}$ in this section help to determine these processes in concrete models.

LEMMA 3.19. *We have*

$$b^L = L_-\tilde{a}^\top\bar{b}, \tag{3.27}$$

$$\bar{b} = \bar{c}\tilde{a}, \tag{3.28}$$

$$b^K = \bar{b}^\top\bar{c}^{-1}\bar{b} = (b^{S\star})^\top(\hat{c}^{S\star})^{-1}b^{S\star} \tag{3.29}$$

outside some $P \otimes A$-null set, where $\bar{b}$, $\bar{c}$ are defined in (3.23) *and* (3.25).

PROOF. We denote by τ_n the stopping times in the proof of Lemma 3.2. Fix $n \in \mathbb{N}$. Integration by parts and Lemma 3.2 yield that

$$\big(\mathscr{E}\big((-\tilde{a}1_{\rrbracket\tau_n,T\rrbracket})\bullet S\big)_-1_{\rrbracket\tau_n,T\rrbracket}\big)\bullet\big(L - (L_-\tilde{a})\bullet S - \tilde{a}\bullet[L,S]\big) = 1_{\rrbracket\tau_n,T\rrbracket}\bullet M^{(\tau_n)}$$

is a martingale. Consequently, its compensator

$$\big(\mathscr{E}\big((-\tilde{a}1_{\rrbracket\tau_n,T\rrbracket})\bullet S\big)_-(b^L - L_-\tilde{a}^\top b^S - \tilde{a}^\top\tilde{c}^{SL})1_{\rrbracket\tau_n,T\rrbracket}\big)\bullet A$$

vanishes. Since $\mathscr{E}((-\tilde{a}1_{\rrbracket\tau_n,T\rrbracket})\bullet S)_- \neq 0$ on $\rrbracket\tau_n,\tau_{n+1}\rrbracket$, this implies that

$$b^L - \tilde{a}^\top L_- b^S - \tilde{a}^\top \tilde{c}^{SL} = 0$$

$P\otimes A$-almost everywhere on $\rrbracket\tau_n,\tau_{n+1}\rrbracket$. This yields (3.27).

Fix $n\in\mathbb{N}$. From Lemma 3.2(2) and integration by parts it follows that

$$\begin{aligned}
0 &\sim 1_{\rrbracket\tau_n,T\rrbracket}\bullet\big(SM^{(\tau_n)}\big)\\
&= 1_{\rrbracket\tau_n,T\rrbracket}\bullet\big(S_-\bullet M^{(\tau_n)} + M_-^{(\tau_n)}\bullet S + \big[S, M^{(\tau_n)}\big]\big)\\
&\sim 1_{\rrbracket\tau_n,T\rrbracket}\bullet\big(\big(\mathscr{E}\big((-a1_{\rrbracket\tau_n,T\rrbracket})\bullet S\big)_- L_-\big)\bullet S + \big[S, \mathscr{E}\big((-a1_{\rrbracket\tau_n,T\rrbracket})\bullet S\big)L\big]\big)\\
&= \big(\mathscr{E}\big((-a1_{\rrbracket\tau_n,T\rrbracket})\bullet S\big)_- 1_{\rrbracket\tau_n,T\rrbracket}\big)\\
&\qquad\bullet\big(L_-\bullet S + [S,L] - \tilde{a}\bullet\big(L_-\bullet[S,S] - \big[[L,S],S\big]\big)\big)\\
&\sim \bigg(\mathscr{E}\big((-a1_{\rrbracket\tau_n,T\rrbracket})\bullet S\big)_- 1_{\rrbracket\tau_n,T\rrbracket}\\
&\qquad\times\bigg(L_- b^S + \tilde{c}^{SL} - \bigg(L_-\tilde{c}^S + \int xx^\top y F^{S,L}\big(d(x,y)\big)\bigg)\tilde{a}\bigg)\bigg)\bullet A.
\end{aligned}$$

Since $\mathscr{E}((-\tilde{a}1_{\rrbracket\tau_n,T\rrbracket})\bullet S)_-$ does not vanish on $\rrbracket\tau_n,\tau_{n+1}\rrbracket$, we have

$$L_- b^S + \tilde{c}^{SL} - \bigg(L_-\tilde{c}^S + \int xx^\top y F^{S,L}\big(d(x,y)\big)\bigg)\tilde{a} = 0$$

and hence (3.28) outside some $P\otimes A$-null set.

Finally, (3.27), (3.28), (3.22) yield

$$L_-\bar{b}^\top\bar{c}^{-1}\bar{b} = L_-\bar{b}^\top\bar{c}^{-1}\bar{c}\tilde{a} = L_-\bar{b}^\top\tilde{a} = b^L,$$

which in turn implies the first equality in (3.29).

On the set $\{\Delta A = 0\}\supset\{\Delta A^K = 0\}$, the second equality follows from (3.17), (3.18). On $\{\Delta A^K \neq 0\}$ the same equations yield

$$1 = (1+\Delta A^K) - b^K\Delta A = (1+\Delta A^K)\big(1-(b^{S\star})^\top(\tilde{c}^{S\star})^{-1}b^{S\star}\Delta A\big).$$

In view of (3.19) we have

$$1 + b^K\Delta A = 1 + \Delta A^K = 1 + (b^{S\star})^\top(\hat{c}^{S\star})^{-1}b^{S\star}\Delta A,$$

which in turn implies $b^K = (b^{S\star})^\top(\hat{c}^{S\star})^{-1}b^{S\star}$ on the set $\{\Delta A^K \neq 0\}$. □

COROLLARY 3.20. *The adjustment process and the extended adjustment process satisfy the equations*

$$b^{S\star} = \tilde{c}^{S\star}\tilde{a} = \hat{c}^{S\star}\hat{a} \tag{3.30}$$

or, put differently,

$$A^{S\star} = \tilde{a}\bullet\langle S,S\rangle^{P^\star} = \hat{a}\bullet\langle M^{S\star}, M^{S\star}\rangle^{P^\star}.$$

In the univariate case, this can be written more intuitively in terms of pathwise Radon–Nikodym derivatives:

$$\tilde{a}_t = \frac{dA_t^{S\star}}{d\langle S, S\rangle_t^{P\star}}, \qquad \hat{a}_t = \frac{dA_t^{S\star}}{d\langle M^{S\star}, M^{S\star}\rangle_t^{P\star}}.$$

PROOF. $b^{S\star} = \tilde{c}^{S\star}\tilde{a}$ follows from (3.28), (3.17), (3.18). Together with (3.21), (3.19), (3.29) we have

$$\begin{aligned}
\hat{c}^{S\star}\tilde{a} &= \big(\tilde{c}^{S\star} - b^{S\star}(b^{S\star})^\top \Delta A\big)\tilde{a} \\
&= b^{S\star}\big(1 - (b^{S\star})^\top (\tilde{c}^{S\star})^{-1}\tilde{c}^{S\star}\tilde{a}\Delta A\big) \\
&= b^{S\star}\big(1 - (b^{S\star})^\top (\tilde{c}^{S\star})^{-1} b^{S\star}\Delta A\big) \\
&= \frac{b^{S\star}}{1 + (b^{S\star})^\top (\hat{c}^{S\star})^{-1} b^{S\star}\Delta A} \\
&= \frac{b^{S\star}}{1 + \Delta A^K,}
\end{aligned}$$

which yields $b^{S\star} = \hat{c}^{S\star}\hat{a}$. □

LEMMA 3.21. *We have $\hat{a} \in L(M^{S\star})$.*

PROOF. Equations (3.30), (3.17), (3.27) imply that

$$(\hat{a}^\top \hat{c}^{S\star}\hat{a}) \bullet A_T = \big((1 + \Delta A^K)\tilde{a}^\top b^{S\star}\big) \bullet A_T = (\tilde{a}^\top \bar{b}) \bullet A_T = \frac{1}{L_-} \bullet A_T^L < \infty$$

and hence $\hat{a} \in L^2_{\mathrm{loc}}(M^{S\star}) \subset L(M^{S\star})$ relative to $P^\star$. □

DEFINITION 3.22. We call

$$N^\star := -\hat{a} \bullet M^{S\star}$$

$P^\star$-minimal logarithm process.

The terminology is motivated by the fact that $\mathscr{E}(N^\star)$ is essentially the density process of the so-called *minimal signed martingale measure* relative to $P^\star$ instead of P (in the sense of [44], (3.14)).

LEMMA 3.23. *We have*

$$\frac{L_0}{E(L_0)}\mathscr{E}(N) = Z^{P^\star}\mathscr{E}(N^\star).$$

Consequently, $\mathscr{E}(N^\star)$ is the density process of $Q^\star$ relative to $P^\star$.

PROOF. Integration by parts yields

$$\frac{L_0\mathscr{E}(N)}{E(L_0)Z^{P^\star}} = \frac{\mathscr{E}(-\tilde{a}\bullet S)L\mathscr{E}(A^K)}{L} = \mathscr{E}(-\tilde{a}\bullet S + A^K - [\tilde{a}\bullet S, A^K]).$$

The term in parentheses on the right-hand side equals

$$\begin{aligned} &x - \tilde{a}\bullet M^{S\star} - (\tilde{a}^\top b^{S\star})\bullet A + b^K \bullet A \\ &\qquad - (\tilde{a}\Delta A^K)\bullet M^{S\star} - (\tilde{a}^\top b^{S\star}\Delta A^K)\bullet A \end{aligned} \tag{3.31}$$

(cf. [28], I.4.49b). Since

$$b^K = \frac{1}{L_-}b^L = \tilde{a}^\top \bar{b} = \tilde{a}^\top b^{S\star}(1+\Delta A^K)$$

by (3.27), (3.17), the expression in (3.31) equals $-\hat{a}\bullet M^{S\star} = N^\star$. □

Roughly speaking, the next statement is another way of saying that S is a $Q^\star$-σ-martingale.

LEMMA 3.24. *$N^\star$ and $S+[S,N^\star]$ are $P^\star$-σ-martingales, which implies that $S\mathscr{E}(N^\star)$ is a $P^\star$-σ-martingale as well.*

PROOF. $N^\star$ is a $P^\star$-σ-martingale by definition. Moreover,

$$\begin{aligned} S+[S,N^\star] &= S - \hat{a}\bullet[S, M^{S\star}] \\ &= S - \hat{a}\bullet[M^{S\star}, M^{S\star}] - \hat{a}\bullet\big((\Delta A^{S\star})\bullet M^{S\star}\big) \\ &\sim^\star (b^{S\star} - \hat{c}^{S\star}\hat{a})\bullet A = 0 \end{aligned}$$

by (3.30). The last statement follows as in Lemma 3.14. □

Corollary 3.20 expresses the adjustment process in terms of the $P^\star$-characteristics of S. Of course this only helps if the opportunity-neutral measure is known in the first place. The following important result characterizes L and $\tilde{a}$ directly in terms of P-characteristics.

THEOREM 3.25. *The opportunity process is the unique semimartingale L such that*:

1. *L, L_- are $(0,1]$-valued*,
2. *$L_T = 1$*,
3. *The joint characteristics of (S, L) solve the equation*

$$b^L = L_- \bar{b}^\top \bar{c}^{-1}\bar{b} \tag{3.32}$$

outside some $P\otimes A$-null set, where $\bar{b}, \bar{c}$ are defined as in (3.23), (3.25), 4.

$$a\mathscr{E}\big((-a1_{\rrbracket\tau,T\rrbracket})\bullet S\big)_-1_{\rrbracket\tau,T\rrbracket}\in\overline{\Theta},\tag{3.33}$$

$$\mathscr{E}\big((-a1_{\rrbracket\tau,T\rrbracket})\bullet S\big)L\text{ is of class }(D)\tag{3.34}$$

hold for $a:=\bar{c}^{-1}\bar{b}$ *and any stopping time* τ.

In this case $a=\bar{c}^{-1}\bar{b}$ *meets the requirement of an adjustment process* $\tilde{a}$ *in Lemma* 3.7.

PROOF. Suppose that L is the opportunity process. Properties 1 and 2 are shown in Lemmas 3.2 and 3.10. Equation (3.29) and $b^L=L_-b^K$ yield (3.32). By (3.17), (3.18), (3.21), (3.29) we have

$$\begin{aligned}(a^\top\hat{c}^{S\star}a)\bullet A_T&\le(a^\top\tilde{c}^{S\star}a)\bullet A_T=((b^{S\star})^\top(\tilde{c}^{S\star})^{-1}b^{S\star})\bullet A_T\\&=\frac{1}{1+\Delta A^K}\bullet A_T^K<\infty,\end{aligned}$$

which implies $a\in L^2_{\mathrm{loc}}(M^{S\star})$ relative to $P^\star$ by [28], III.4.3. Similarly, we have $a\in L(A^{S\star})$ because $|a^\top b^{S\star}|\bullet A_T\le\frac{1}{1+\Delta A^K}\bullet A_T^K<\infty$. Together, it follows that $a\in L(S)$.

More specifically, we have

$$a\bullet A^{S\star}=(a^\top b^{S\star})\bullet A=\frac{b^L}{(1+\Delta A^K)L_-}\bullet A$$

and likewise for $\tilde{a}$ by (3.17), (3.27). Similarly, (3.27–3.29) yield

$$\langle(a-\tilde{a})\bullet M^{S\star},(a-\tilde{a})\bullet M^{S\star}\rangle^{P^\star}\le\big((a-\tilde{a})^\top\tilde{c}^{S\star}(a-\tilde{a})\big)\bullet A=0,$$

which implies $(a-\tilde{a})\bullet M^{S\star}=0$. Together, we have $a\bullet S=\tilde{a}\bullet S$. Hence one may choose $\tilde{a}=a$ in Lemma 3.7.

Finally, (3.33) follows from (3.12) and (3.34) from Lemma 3.2.

Conversely, let L' be a semimartingale satisfying properties 1–4 with $\bar{b}',\bar{c}'$ as in (3.23) and (3.25). Define $K':=\frac{1}{L'_-}\bullet L'$ and $N':=K'-a\bullet S-[a\bullet S,K']$. We use the notation $L',\bar{b}',\bar{c}',K',N'$ in this part of the proof because L' is yet to be shown to coincide with the true opportunity process. From

$$[S,K']=\frac{1}{L'_-}\bullet[M^S,M^{L'}]+(\Delta A^S)\bullet M^{K'}+(\Delta A^{K'})\bullet S$$

and standard results (cf. [28], I.4.24, III.3.14) it follows that

$$[S,K']=[S^c,K'^c]+\int_{[0,\cdot]\times\mathbb{R}^d\times\mathbb{R}}xy\mu^{(S,K')}(d(t,x,y))$$

is an $\mathbb{R}^d$-valued special semimartingale with compensator $(c^{SK'} + \int xyF^{S,K'} \times (d(x,y))) \bullet A$. For $n \in \mathbb{N}$ define the predictable set $D_n := \{|a| \leq n\}$. Since 1_{D_n} and $a1_{D_n}$ are bounded, we have that

$$1_{D_n} \bullet N' = 1_{D_n} \bullet K' - (1_{D_n} a) \bullet S - (1_{D_n} a) \bullet [S, K']$$

is a special semimartingale as well with compensator

$$\begin{aligned}&\left(1_{D_n} b^{K'} - 1_{D_n} a^\top \left(b^S + c^{SK'} + \int xyF^{S,K'}\big(d(x,y)\big)\right)\right) \bullet A\\ &\quad = \left(\left(\frac{b^{L'}}{L'_-} - \bar{b}'^\top \bar{c}'^{-1} \bar{b}'\right) 1_{D_n}\right) \bullet A = 0.\end{aligned}$$

Consequently, $1_{D_n} \bullet N'$ is actually a local martingale. Since $D_n \uparrow \Omega \times [0, T]$ up to an evanescent set, N' is a σ-martingale (cf. Remark A.5).

Similarly, we have that

$$\begin{aligned}&1_{D_n} \bullet (S^i + [S^i, N'])\\ &\quad = 1_{D_n} \bullet S^i + 1_{D_n} \bullet [S^i, K']\\ &\qquad - \sum_{j=1}^n (1_{D_n} a^j) \bullet [S^i, S^j] - \sum_{j=1}^n (1_{D_n} a^j) \bullet \big[S^i, [S^j, K']\big]\end{aligned}$$

is a special semimartingale with compensator

$$\begin{aligned}&\left(1_{D_n}\left(b^S + c^{SK'} + \int xyF^{S,K'}\big(d(x,y)\big)\right.\right.\\ &\qquad \left.\left. - c^S a - \int x(x^\top a)(1+y)F^{S,K'}\big(d(x,y)\big)\right)^i\right) \bullet A\\ &\quad = \big(1_{D_n}(\bar{b}' - \bar{c}'a)^i\big) \bullet A\end{aligned}$$

for $i = 1, \ldots, d$. Since $\bar{b}' - \bar{c}'a = \bar{b}' - \bar{c}'\bar{c}'^{-1}\bar{b}' = 0$ by Remark 3.18, it follows that the process $1_{D_n} \bullet (S^i + [S^i, N'])$ is a local martingale. This implies that $S + [S, N']$ is a σ-martingale as well.

Fix a stopping time τ. Let $\vartheta := a\mathscr{E}(-a1_{]\!]\tau,T]\!]} \bullet S)_- 1_{]\!]\tau,T]\!]}$ and

$$Z := (1 - \vartheta \bullet S)L' = \mathscr{E}\big((-a1_{]\!]\tau,T]\!]}) \bullet S\big)L'.$$

[In (3.33) and (3.34) it is implicitly assumed that $a \in L(S)$ for the integral to make sense. By similar arguments as in the first part of the proof one can show that this integrability condition is in fact implied by properties 1–3 of Theorem 3.25.]

Since N' and $S + [S, N']$ are σ-martingales,

$$\frac{Z}{Z^\tau} = \mathscr{E}\big(1_{]\!]\tau,T]\!]} \bullet K'\big)\mathscr{E}\big((-a1_{]\!]\tau,T]\!]}) \bullet S\big) = \mathscr{E}(N' - N'^\tau)$$

and

$$\frac{Z}{Z^\tau}(S-S^\tau)$$
$$= \mathscr{E}(N'-N'^\tau)_- \bullet \big((S-S^\tau)_- \bullet (N'-N'^\tau) + 1_{]\!]\tau,T]\!]} \bullet (S+[S,N'])\big)$$

are σ-martingales as well.

We show that ϑ is efficient on $]\!]\tau,T]\!]$. Indeed, from (3.34) and Lemma A.7 it follows that $Z-Z^\tau = (Z_\tau 1_{]\!]\tau,T]\!]}) \bullet \frac{Z}{Z^\tau}$ is a martingale. It is even a square-integrable martingale because $Z_T - Z_\tau \in L^2(P)$. Let ψ be a simple strategy with $\psi 1_{[\![0,\tau]\!]} = 0$. The same arguments as in step 1 of the proof of Lemma 2.4 yield that $(\psi \bullet S)Z = ((Z_\tau\psi) \bullet (S-S^\tau))\frac{Z}{Z^\tau}$ is a martingale. Consequently,

$$\begin{aligned} &E\big((1-(\vartheta+\psi)\bullet S_T)^2\big) \\ &\quad\geq E\big((1-\vartheta\bullet S_T)^2\big) - 2E\big((1-\vartheta\bullet S_T)L'_T(\psi\bullet S_T)\big) \\ &\quad= E\big((1-\vartheta\bullet S_T)^2\big), \end{aligned}$$

which implies the optimality of ϑ. Since $Z-Z^\tau$ is a martingale, Lemma 3.2 yields that L' is the opportunity process. □

Condition (3.33) looks somewhat unpleasant because of the involved definition of $\overline{\Theta}$. The following example shows that uniqueness in Theorem 3.25 does not generally hold without this condition. For related considerations see also [44] and [10].

EXAMPLE 3.26. Let $T=1$ and S be a standard Wiener process. By Theorem 3.25 the opportunity and adjustment processes are $L=1$ and $\tilde{a}=0$. Choose some doubling-type strategy $\psi \in L(S)$ with $1-\psi\bullet S \geq \frac{1}{2}$ and $1-\psi\bullet S_T = \frac{1}{2}$. Of course, ψ cannot be admissible. We write $1-\psi\bullet S = \mathscr{E}(-\bar{a}\bullet S)$ with $\bar{a} := \frac{\psi}{1-\psi\bullet S_-}$. Define

$$\overline{L} := \frac{1}{2\mathscr{E}(-\bar{a}\bullet S)} = \frac{1}{2}\mathscr{E}(\bar{a}\bullet S + \bar{a}^2\bullet[S,S]).$$

Straightforward calculations yield that $\overline{L}$ satisfies conditions 1–3 in Theorem 3.25. Moreover, $\bar{a}$ is the corresponding process in condition 4. Since $\mathscr{E}((-\bar{a}1_{]\!]\tau,T]\!]})\bullet S)\overline{L} = \overline{L}^\tau$ is bounded, (3.34) is satisfied as well.

It is interesting to note that the "variance-optimal logarithm process" $\overline{N}$ corresponding to this wrong choice of $\overline{L}, \bar{a}$ satisfies $\mathscr{E}(\overline{N}) = \frac{\overline{L}}{\overline{L}_0}\mathscr{E}(-\bar{a}\bullet S) = 1$, that is, it coincides with the true variance-optimal logarithm process. In particular,

$$\frac{dQ^\star}{dP} = \frac{1-\psi\bullet S_T}{E(1-\psi\bullet S_T)},$$

which parallels the last expression in (3.16). Nevertheless, ψ is not an efficient strategy on $]\!]0,T]\!]$ because it is not admissible.

In concrete models, it may be easier to verify the following sufficient condition instead of (3.33), (3.34).

LEMMA 3.27. *Let L be a special semimartingale satisfying conditions* 1–3 *in Theorem* 3.25 *with* $\bar{b}, \bar{c}$ *defined as in* (3.23), (3.25). *If* $a := \bar{c}^{-1}\bar{b}$ *satisfies*

$$\sup\{E(\mathscr{E}((-a1_{]\!]\tau,T]\!]})\bullet S)^2_\sigma) : \sigma \text{ stopping time}\} < \infty$$

for any stopping time τ, *then condition* 4 *holds as well*, *that is*, L *is the opportunity process*.

PROOF. Condition (3.34) is obvious because L is bounded. Let Q be an SσMM with density process Z^Q and $\frac{dQ}{dP} \in L^2(P)$. Integration by parts yields that $(\vartheta \bullet S)Z^Q$ is a σ-martingale for

$$\vartheta := a\mathscr{E}((-a1_{]\!]\tau,T]\!]})\bullet S)_- 1_{]\!]\tau,T]\!]}$$

[cf. (2.2)]. Since $\sup_{t\in[0,T]}|Z^Q_t| \in L^2(P)$ by Doob's inequality and $1 - \vartheta \bullet S$ is an L^2-semimartingale, we have that $(\vartheta \bullet S)Z^Q$ is of class (D) and hence a martingale (cf. Lemma A.7). Using Corollary 2.5 we obtain (3.33). □

3.6. *When does* $P^\star = P$ *hold*? The opportunity-neutral measure plays a key role in quadratic hedging. Therefore we want to have a closer look at the question when $P^\star$ equals P. In line with [42], we call

$$\widehat{K} := ((b^S)^\top(\hat{c}^S)^{-1}b^S) \bullet A$$

mean-variance tradeoff (*MVT*) *process*. Similarly, the MVT process relative to $P^\star$ is denoted by $\widehat{K}^\star$, that is,

$$\widehat{K}^\star := ((b^{S\star})^\top(\hat{c}^{S\star})^{-1}b^{S\star}) \bullet A.$$

Observe that $\widehat{K}^\star = A^K$ by (3.29).

PROPOSITION 3.28. *The following statements are equivalent*:

1. $P^\star = P$.
2. K (*or equivalently* L) *is a predictable process of finite variation and* L_0 *is deterministic*.
3. $K = \widehat{K}$ *and* L_0 *is deterministic*.
4. $K = \widehat{K}^\star$ *and* L_0 *is deterministic*.
5. $\mathscr{E}(\widehat{K})_T$ *is finite and deterministic*.
6. $\mathscr{E}(\widehat{K}^\star)_T$ *is deterministic*.

In this case the opportunity process equals $L = \mathscr{E}(\widehat{K})/\mathscr{E}(\widehat{K})_T$.

PROOF. $1 \Rightarrow 2$: Since $1 = Z^{P^\star} = L/(E(L_0)\mathscr{E}(A^K))$, we have that L and hence also $K = \mathscr{L}(L)$ are predictable processes of finite variation. L_0 is deterministic because $Z_0^{P^\star} = 1$.

$2 \Rightarrow 4$: This is obvious because $K = A^K = \widehat{K}^\star$.

$4 \Rightarrow 1$: This follows from

$$Z^{P^\star} = \frac{L}{E(L_0)\mathscr{E}(A^K)} = \frac{L_0\mathscr{E}(K)}{E(L_0)\mathscr{E}(\widehat{K}^\star)} = 1.$$

$1 \Rightarrow 6$: This follows from $Z_T^{P^\star} = 1$ and $\widehat{K}^\star = A^K$.

$6 \Rightarrow 1$: This holds because $Z_T^{P^\star} = 1/(E(L_0)\mathscr{E}(\widehat{K}^\star)_T)$ is deterministic.

$1 \Rightarrow 3$: In view of $(1 \Rightarrow 2)$, this follows from $K = A^K = \widehat{K}^\star = \widehat{K}$.

$3 \Rightarrow 5$: This follows from $1 = L_T = L_0\mathscr{E}(K)_T$.

$5 \Rightarrow 2$: Let $L := \mathscr{E}(\widehat{K})/\mathscr{E}(\widehat{K})_T$. Since $\widehat{K}$ is an increasing predictable process, L is a $(0, 1]$-valued increasing predictable process. The predictability of L implies $c^{SL} = 0$ and $y = \Delta L_t$ $(F_t^{S,L}(d(x, y))A(dt))$-almost everywhere. If $\bar{b}$, $\bar{c}$ are defined as in (3.23), (3.25), we have $\bar{b} = (1 + \Delta\widehat{K})b^S$, $\bar{c} = (1 + \Delta\widehat{K})\tilde{c}^S$ and hence

$$L_-\bar{b}^\top\bar{c}^{-1}\bar{b} = L_-\bigl(1 + (b^S)^\top(\hat{c}^S)^{-1}b^S\Delta A\bigr)(b^S)^\top(\tilde{c}^S)^{-1}b^S.$$

Observe that (3.19–3.21) can be derived literally for P instead of $P^\star$. We obtain

$$L_-\bar{b}^\top\bar{c}^{-1}\bar{b} = L_-(b^S)^\top(\hat{c}^S)^{-1}b^S = b^L,$$

which implies that L satisfies conditions 1–3 in Theorem 3.25. If we can show that L is the true opportunity process, then $P^\star = P$ follows from Lemma 3.15.

Fix any stopping time τ. For $a := \bar{c}^{-1}\bar{b} = (\tilde{c}^S)^{-1}b^{S\star}$ and $X := (-a1_{]\!]\tau,T]\!]}) \bullet S$ we have

$$\begin{aligned}
\langle M^X, M^X\rangle_T &= \bigl(a^\top\hat{c}^S a1_{]\!]\tau,T]\!]}\bigr) \bullet A_T \\
&\le \bigl(a^\top\tilde{c}^S a1_{]\!]\tau,T]\!]}\bigr) \bullet A_T \\
&= \bigl((b^S)^\top(\tilde{c}^S)^{-1}\tilde{c}^S(\tilde{c}^S)^{-1}b^S1_{]\!]\tau,T]\!]}\bigr) \bullet A_T \\
&= \left(\frac{1_{]\!]\tau,T]\!]}}{1 + \Delta\widehat{K}}\bar{b}^\top\bar{c}^{-1}\bar{b}\right) \bullet A_T \\
&\le ((b^S)^\top(\hat{c}^S)^{-1}b^S) \bullet A_T \\
&= \widehat{K}_T \le \mathscr{E}(\widehat{K})_T.
\end{aligned}$$

Similarly, we have

$$\operatorname{var}(A^X)_T = \bigl|a^\top b^S1_{]\!]\tau,T]\!]}\bigr| \bullet A_T = \left(\frac{1_{]\!]\tau,T]\!]}}{1 + \Delta\widehat{K}}\bar{b}^\top\bar{c}^{-1}\bar{b}\right) \bullet A_T \le \mathscr{E}(\widehat{K})_T$$

for the variation process of A^X. In view of Lemmas A.3 and 3.27, L is the opportunity process. $\square$

To relate the condition $P^\star = P$ to earlier literature, we define (myopic) portfolio weights

$$\begin{aligned}\tilde\lambda &:= (\tilde c^S)^{-1} b^S,\\ \hat\lambda &:= (1+\Delta\widehat K)\tilde\lambda\end{aligned} \tag{3.35}$$

in accordance with [42]. Repeating the arguments leading to (3.30) under P rather than $P^\star$ yields $\hat c^S\hat\lambda = b^S$ [which implies that $\hat\lambda = (\hat c^S)^{-1}b^S$ if $\hat c^S$ is invertible]. By Theorem 1 of [43] we have $\hat\lambda \in L(M^S)$.

DEFINITION 3.29. If $\mathscr{E}(-\hat\lambda \bullet M^S)$ is of class (D) and hence a martingale, then it is the density process of some SσMM Q. Only slightly extending [44], (3.14) we call Q the *minimal signed martingale measure* (*minimal SσMM*).

In view of Proposition 3.28, the following corollary can be interpreted as an extension of Proposition 5.1 in [30]. It also extends sufficient conditions for $Q^\star = Q$ given in [44], Examples 1 and 2.

COROLLARY 3.30. *Suppose $\mathscr{E}(-\tilde a \bullet S)_T \neq 0$ almost surely. Then there is equivalence between*:

1. $P^\star = P$,
2. *$\widehat K_T$ is finite, the minimal SσMM Q exists, $Q^\star = Q$, and $\tilde a$ can be chosen as $\tilde\lambda$.*

The implication $1 \Rightarrow 2$ still holds without the assumption on $\mathscr{E}(-\tilde a \bullet S)$.

PROOF. $1 \Rightarrow 2$: This follows from Lemma 3.23, Theorem 3.25, and (3.17), (3.18).

$2 \Rightarrow 1$: As in the proof of Lemma 3.17 it follows that $b^S = \hat c^S(\hat c^S)^{-1}b^S$ and hence $\hat\lambda^\top b^S = (b^S)^\top(\hat c^S)^{-1}b^S$. Hence, the density process of Q equals

$$\begin{aligned}\mathscr{E}(-\hat\lambda \bullet M^S) &= \mathscr{E}\big((\hat\lambda^\top b^S)\bullet A - \hat\lambda \bullet S\big)\\ &= \mathscr{E}\big(((b^S)^\top(\hat c^S)^{-1}b^S)\bullet A - \big((1+\Delta\widehat K)\tilde\lambda\big)\bullet S\big)\\ &= \mathscr{E}\big(\widehat K - \tilde\lambda\bullet S - (\Delta\widehat K)\bullet(\tilde\lambda\bullet S)\big)\\ &= \mathscr{E}(\widehat K - \tilde\lambda\bullet S - [\widehat K, \tilde\lambda\bullet S])\\ &= \mathscr{E}(\widehat K)\mathscr{E}(-\tilde\lambda\bullet S),\end{aligned}$$

where the fourth equality follows from [28], I.4.49b and the last from Yor's formula. This density process equals $\frac{L}{E(L_0)}\mathscr{E}(-\tilde a\bullet S)$ by $Q^\star = Q$ and Proposition 3.13. Since $\tilde a = \tilde\lambda$ and $\mathscr{E}(-\tilde a\bullet S)$ never vanishes (cf. [28], I.4.61), we have that $L = E(L_0)\mathscr{E}(\widehat K)$ is predictable with $L_0 = E(L_0)$. The assertion follows now from Proposition 3.28 ($2 \Rightarrow 1$). $\square$

Finally, we consider the situation of deterministic mean-variance tradeoff, which is the focus of [42].

COROLLARY 3.31. *If the MVT process* $\widehat{K}$ *is finite and deterministic, then* $L := \mathscr{E}(\widehat{K})/\mathscr{E}(\widehat{K})_T$ *is the opportunity process,* $K := \widehat{K}$ *is the modified mean-variance tradeoff process, and* $P^\star = P$.

PROOF. This follows from Proposition 3.28 ($5 \Rightarrow 1$, 3) and from $1 = L_T = L_0\mathscr{E}(K)_T$. $\square$

3.7. *Determination of the opportunity process.* Unless we are in the fortunate situation of Corollary 3.31 or at least Proposition 3.28, the crucial step in concrete applications is to determine the opportunity process L. This is relatively easy in discrete time.

EXAMPLE 3.32. Suppose that we are actually considering a discrete-time model, that is, $A_t = [t] := \max\{n \in \mathbb{N} : n \leq t\}$ and $\mathscr{F}_t = \mathscr{F}_{[t]}$ for $t \in [0, T]$ with $T \in \mathbb{N}$. In this case all processes in this paper are (or can be chosen) piecewise constant between integer times. For ease of notation suppose that $d = 1$ (only one tradable asset). By [28], II.3.11 we have $b^L_t = E(\Delta L_t|\mathscr{F}_{t-1})$, $\bar{b}_t = E(\Delta S_t L_t/L_{t-1}|\mathscr{F}_{t-1})$, and $\bar{c}_t = E((\Delta S_t)^2 L_t/L_{t-1}|\mathscr{F}_{t-1})$ for $t \in \{1, 2, \ldots, T\}$. Consequently, (3.32) can be rewritten as

$$L_{t-1} = E(L_t|\mathscr{F}_{t-1}) - \frac{(E(\Delta S_t L_t|\mathscr{F}_{t-1}))^2}{E((\Delta S_t)^2 L_t|\mathscr{F}_{t-1})}, \tag{3.36}$$

that is, the opportunity process is determined by a simple backward recursion starting in $L_T = 1$. For the adjustment process we have

$$\tilde{a}_t = \frac{\bar{b}_t}{\bar{c}_t} = \frac{E(\Delta S_t L_t|\mathscr{F}_{t-1})}{E((\Delta S_t)^2 L_t|\mathscr{F}_{t-1})}.$$

The previous example indicates that the characteristic equation (3.32) may be interpreted as the continuous-time analogue of a backward recursion. True continuous-time models are typically Markovian in S_t or at least (S_t, Y_t) with some additional process Y as, for example, stochastic volatility. If one makes the natural assumption $L_t = f(t, S_t, Y_t)$ with some C^2-function f, then (3.32) can be rewritten as an integro-differential equation for f by means of Itô's formula. But as it is not obvious whether the smoothness assumption is justified, it may require substantial effort to make this statement precise. In [11] and ongoing research, L is determined explicitly by an ansatz of the above type in specific stochastic volatility models.

Alternatively, the process L can be interpreted as the solution to some backward stochastic differential equation (BSDE). To this end, we use the martingale representation theorem (cf. [28], III.4.24) to write the martingale part of L as

$$M^L = J \bullet S^c + W * (\mu^S - \nu^S) + U$$

with some $J \in L^2_{\mathrm{loc}}(S^c)$, $W \in G_{\mathrm{loc}}(\mu^S)$ and some local martingale $U \in \mathscr{H}^2_{\mathrm{loc}}$ such that $\langle U^c, S^c \rangle = 0$ and $M^P_{\mu^S}(\Delta U | \widetilde{\mathscr{P}}) = 0$ in the sense of [28], III.3c. Using the notation

$$\widehat{W}_t := E\big(W(t, \Delta S_t) | \mathscr{F}_{t-}\big),$$

the quadruple (J, W, L, U) solves the BSDE

$$\begin{aligned} L &= J \bullet S^c + W * (\mu^S - \nu^S) + U \\ &\quad + \left(\left(b^S + c^S \frac{J}{L_-} + \int \frac{W(x) - \widehat{W}}{L_-} x F^S(dx)\right)^\top \right. \\ &\qquad \times \left(c^S + \int x x^\top \left(1 + \frac{W(x) - \widehat{W}}{L_-}\right) F^S(dx)\right)^{-1} \\ &\qquad \left. \times \left(b^S + c^S \frac{J}{L_-} + \int \frac{W(x) - \widehat{W}}{L_-} x F^S(dx)\right) L_- \right) \bullet A, \\ L_T &= 1. \end{aligned} \tag{3.37}$$

However, it is not obvious whether this representation is of any use.

One should note that (3.37) is not related to the BSDEs (3.6) and (4.10) in [44], which characterize the adjustment process and the optimal hedge. The latter are hard to use in practice because their terminal values involve the L^2-projection of 1, respectively, H on $K_2(0)$, which is generally unknown. If at all, one may rather observe a certain similarity between (3.36) and the recursive expression (2.1) in [44] for the adjustment process in discrete time. Mania and Tevzadze [34, 36] derive BSDE's for $1/L$ in the case of a continuous asset price process S. These equations are quite different from both (3.37) and (3.32).

4. On the quadratic hedging problem. We now come back to the hedging problem from Definition 2.10. The processes and measures $\lambda^{(\tau)}$, $M^{(\tau)}$, L, $\tilde{a}$, $\hat{a}$, K, N, $Q^\star$, $Z^{P^\star}$, $P^\star$, $\bar{b}$, $\bar{c}$ are defined as in the previous section. Recall that P is the default probability measure for expectations, martingales and so forth.

4.1. *Mean value process and pure hedge coefficient.* If S is a martingale, the mean value process $V_t = E(H | \mathscr{F}_t)$ leads to the optimal hedge via (1.1). If S fails to be a martingale, a similar role is played by the conditional expectation of H relative to the variance-optimal SσMM $Q^\star$. By the generalized Bayes formula we have

$$E_{Q^\star}(H | \mathscr{F}_t) = E\big(H \mathscr{E}(N - N^t)_T | \mathscr{F}_t\big) \tag{4.1}$$

if $Q^\star$ is a true probability measure. In the general case we use the right-hand side of (4.1) as a substitute for the possibly undefined conditional expectation.

LEMMA 4.1. *There is a unique semimartingale V satisfying*

$$V_t = E\big(H\mathscr{E}(N - N^t)_T | \mathscr{F}_t\big) \tag{4.2}$$

$$= E_{P^\star}\big(H\mathscr{E}\big(N^\star - (N^\star)^t\big)_T | \mathscr{F}_t\big) \tag{4.3}$$

for $t \in [0, T]$. Moreover, $(V_s M_s^{(t)})_{s\in[t,T]}$ is a martingale for any $t \in [0, T]$.

PROOF. In this proof φ denotes an optimal hedging strategy for arbitrary initial endowment $v_0 \in L^2(\Omega, \mathscr{F}_0, P)$ or, alternatively, (v_0, φ) denotes an optimal endowment/strategy pair. Moreover, let $G := v_0 + \varphi \bullet S$ and define a square-integrable martingale Z by its terminal value $Z_T := G_T - H$. Finally, we set $V := G - \frac{Z}{L}$. The optimality of φ implies that

$$\begin{aligned} 0 &\le E\big((G_T + \varepsilon\vartheta \bullet S_T - H)^2\big) - E\big((G_T - H)^2\big) \\ &= 2\varepsilon E\big((\vartheta \bullet S_T) Z_T\big) + \varepsilon^2 E\big((\vartheta \bullet S_T)^2\big) \end{aligned}$$

for any simple strategy ϑ and any $\varepsilon \in \mathbb{R}$. Therefore

$$E\big((\vartheta \bullet S_T) Z_T\big) = 0 \tag{4.4}$$

for any simple ϑ, which implies that SZ is a σ-martingale. By Remark 2.7 we have that $(\vartheta \bullet S)Z$ is a martingale for any $\vartheta \in \overline{\Theta}$. In particular, $(G - V)M^{(t)} = (1 - \lambda^{(t)} \bullet S)Z$ is a martingale for any fixed $t \in [0, T]$. By Lemma 3.2(3), $(G_s M_s^{(t)}))_{s\in[t,T]}$ and hence also $(V_s M_s^{(t)}))_{s\in[t,T]}$ is a martingale. Using Lemma 3.7, we have

$$\begin{aligned} E\big(H\mathscr{E}(N - N^t)_T | \mathscr{F}_t\big) L_t &= E\big(V_T L_T\big(1 - \lambda^{(t)} \bullet S_T\big) | \mathscr{F}_t\big) \\ &= V_t L_t\big(1 - \lambda^{(t)} \bullet S_t\big) \\ &= V_t L_t, \end{aligned}$$

which shows (4.2).

Along the same lines as Lemma 3.23 it follows that

$$\mathscr{E}(N - N^t) = \mathscr{E}\big(N^\star - (N^\star)^t\big) \frac{Z^{P^\star}}{(Z^{P^\star})^t}. \tag{4.5}$$

Consequently,

$$\begin{aligned} E_{P^\star}\big(H\mathscr{E}\big(N^\star - (N^\star)^t\big)_T | \mathscr{F}_t\big) &= E\left(H\mathscr{E}\big(N^\star - (N^\star)^t\big)_T \frac{Z_T^{P^\star}}{Z_t^{P^\star}} \middle| \mathscr{F}_t\right) \\ &= E\big(H\mathscr{E}(N - N^t)_T | \mathscr{F}_t\big), \end{aligned}$$

which yields (4.3). The uniqueness (up to indistinguishability) of V is obvious. $\square$

DEFINITION 4.2. We call V from Lemma 4.1 *mean value process* of the option.

The following technical statements mean essentially that V is a $Q^\star$-σ-martingale.

LEMMA 4.3. *We have* 1. $V+[V,N]$ *and hence* $V\mathscr{E}(N)$ *are* σ*-martingales*.
2. $V+[V,N^\star]$ *and hence* $V\mathscr{E}(N^\star)$ *are* $P^\star$-σ*-martingales*.

PROOF. 1. Fix $n\in\mathbb{N}$. If $(\tau_n)_{n\in\mathbb{N}}$ denotes the sequence of stopping times from the proof of Lemma 3.2, then

$$\mathscr{E}(N-N^{\tau_n})_- = \frac{L_-}{L_-^{\tau_n}}(1-\lambda^{(\tau_n)}\bullet S_-)\neq 0$$

on $\rrbracket\tau_n,\tau_{n+1}\rrbracket$. For $t\in\rrbracket\tau_n,T\rrbracket$ we have

$$\mathscr{E}(N-N^t)\mathscr{E}(N-N^{\tau_n})_t=\mathscr{E}(N-N^{\tau_n}),$$

which implies that $(L_{\tau_n}1_{\rrbracket\tau_n,T\rrbracket})\bullet(V\mathscr{E}(N-N^{\tau_n}))$ is a martingale by (4.2). Integration by parts and Lemma 3.14 yield that

$$\begin{aligned}1_{\rrbracket\tau_n,\tau_{n+1}\rrbracket}&\bullet(V+[V,N])\\ &=\frac{1_{\rrbracket\tau_n,\tau_{n+1}\rrbracket}}{\mathscr{E}(N-N^{\tau_n})_-}\bullet\big(V\mathscr{E}(N-N^{\tau_n})\big)-\big(1_{\rrbracket\tau_n,\tau_{n+1}\rrbracket}V_-\big)\bullet N\end{aligned}$$

is a σ-martingale, which implies the first claim. The second follows as in Lemma 3.14.

2. By Lemma A.8 we must show that $(V+[V,N^\star])Z^{P^\star}$ is a P-σ-martingale. Integration by parts yields

$$(V+[V,N^\star])Z^{P^\star}\sim Z_-^{P^\star}\bullet\left(V+[V,N^\star]+\left[V+[V,N^\star],\frac{1}{1+\Delta A^K}\bullet M^K\right]\right).$$

Hence we must show that the integrator is a σ-martingale. It equals

$$\begin{aligned}V&+\left[V,N^\star+\frac{1}{1+\Delta A^K}\bullet M^K+\left[N^\star,\frac{1}{1+\Delta A^K}\bullet M^K\right]\right]\\ &=V+[V,N]+\Big[V,\tilde a\bullet S-K+[\tilde a\bullet S,K]-\big(\tilde a(1+\Delta A^K)\big)\bullet M^{S\star}\\ &\qquad\qquad+\frac{1}{1+\Delta A^K}\bullet M^K-[\tilde a\bullet M^{S\star},M^K]\Big].\end{aligned}$$

Since $V+[V,N]$ is a σ-martingale by statement 1, it suffices to show that the right-hand side of the long covariation term vanishes. To this end, observe that using (3.27), (3.17) we get

$$\begin{aligned}
\tilde{a} \bullet S &= \tilde{a} \bullet M^{S\star} + (\tilde{a}^\top b^{S\star}) \bullet A \\
&= \tilde{a} \bullet M^{S\star} + \frac{b^K}{1+\Delta A^K} \bullet A \\
&= \tilde{a} \bullet M^{S\star} + \frac{1}{1+\Delta A^K} \bullet A^K
\end{aligned}$$

and hence

$$\begin{aligned}
[\tilde{a} \bullet S, K] &= [\tilde{a} \bullet M^{S\star}, M^K] + [\tilde{a} \bullet M^{S\star}, A^K] + \left[\frac{1}{1+\Delta A^K} \bullet A^K, K\right] \\
&= [\tilde{a} \bullet M^{S\star}, M^K] + (\tilde{a}\Delta A^K) \bullet M^{S\star} + \frac{\Delta A^K}{1+\Delta A^K} \bullet K.
\end{aligned}$$

This yields the first claim. The second follows again as in Lemma 3.14. □

In general we do not know whether V is locally square integrable, or even special, under P. Crucially, this integrability holds under $P^\star$, which is important for evaluation of the expected squared hedging error in Section 4.

LEMMA 4.4. *We have* 1. V^2L, $(v + \vartheta \bullet S)^2 L$, *and* $(v + \vartheta \bullet S - V)^2 L$ *are submartingales for any admissible endowment/strategy pair* (v, ϑ).

2. V *is a locally square-integrable semimartingale relative to* $P^\star$.

PROOF. 1. Let $G := v + \vartheta \bullet S$ and fix $s \le t$. From Lemmas 3.2(3), 4.1 and Hölder's inequality it follows that

$$\begin{aligned}
(G_s - V_s)^2 L_s^2 &= \bigl((G_s - V_s) M_s^{(s)}\bigr)^2 \\
&= \bigl(E\bigl((G_t - V_t) M_t^{(s)} | \mathscr{F}_s\bigr)\bigr)^2 \\
&\le E\bigl((G_t - V_t)^2 L_t | \mathscr{F}_s\bigr) E\bigl((1 - \lambda^{(s)} \bullet S_t) M_t^{(s)} | \mathscr{F}_s\bigr) \\
&= E\bigl((G_t - V_t)^2 L_t | \mathscr{F}_s\bigr) L_s.
\end{aligned}$$

Integrability follows by setting $t = T$. The claim for V^2L and G^2L follows analogously.

2. For any stopping time τ we have

$$E_{P^\star}(V_\tau^2) = E(Z_\tau^{P^\star} V_\tau^2) \le \frac{E(L_\tau V_\tau^2)}{E(L_0)} \le \frac{E(H^2)}{E(L_0)}$$

by statement 1, which implies the claim (cf. Lemma A.2). □

LEMMA 4.5. *Outside some $P \otimes A$-null set we have*

$$b^{V\star} = \tilde{c}^{VS\star}\tilde{a}, \tag{4.6}$$

$$\tilde{c}^{S\star}(\tilde{c}^{S\star})^{-1}\tilde{c}^{SV\star} = \tilde{c}^{SV\star}. \tag{4.7}$$

PROOF. By Lemma 4.3 and (3.27), (3.17) we have

$$\begin{aligned}
0 &\sim^\star V + [V, N^\star] \\
&= V - \hat{a} \bullet [V, M^{S^\star}] \\
&= V - \hat{a} \bullet [V, S] + \hat{a} \bullet [V, A^{S^\star}] \\
&\sim^\star A^{V\star} - \hat{a} \bullet \langle V, S\rangle^{P^\star} + (\hat{a}^\top \Delta A^{S\star}) \bullet V \\
&\sim^\star (b^{V\star} - \tilde{c}^{VS\star}\hat{a}) \bullet A + \big((1 + \Delta A^K)\tilde{a}^\top b^{S\star}\Delta A\big) \bullet A^{V\star} \\
&= \big(b^{V\star} - (1 + \Delta A^K)\tilde{c}^{VS\star}\tilde{a} + \Delta A^K b^{V\star}\big) \bullet A \\
&= \big((1 + \Delta A^K)(b^{V\star} - \tilde{c}^{VS\star}\tilde{a})\big) \bullet A,
\end{aligned}$$

which yields the first assertion.

Fix $(\omega, t) \in \Omega \times [0, T]$. Since

$$\tilde{c}_t^{S,V\star}(\omega) = \begin{pmatrix} \tilde{c}_t^{S\star} & \tilde{c}_t^{SV\star} \\ (\tilde{c}_t^{SV\star})^\top & \tilde{c}_t^{V\star} \end{pmatrix}(\omega)$$

is a symmetric, nonnegative matrix, we have (4.7) by Albert [1], Theorem 9.1.6. □

The next definition constitutes a first step toward optimal hedging.

DEFINITION 4.6. We call the process

$$\xi := (\tilde{c}^{S\star})^{-1}\tilde{c}^{SV\star}$$

pure hedge coefficient.

The following representations of ξ establish the link to (1.1).

PROPOSITION 4.7. *The pure hedge coefficient ξ satisfies*

$$\xi \bullet \langle S, S\rangle^{P^\star} = \langle S, V\rangle^{P^\star} \tag{4.8}$$

and

$$\xi \bullet \langle M^{S\star}, M^{S\star}\rangle^{P^\star} = \langle M^{S\star}, M^{V\star}\rangle^{P^\star}. \tag{4.9}$$

In the univariate case, (4.8) *and* (4.9) *can be written more plainly as*

$$\xi_t = \frac{d\langle S, V\rangle_t^{P^\star}}{d\langle S, S\rangle_t^{P^\star}} = \frac{d\langle M^{S\star}, M^{V\star}\rangle_t^{P^\star}}{d\langle M^{S\star}, M^{S\star}\rangle_t^{P^\star}}. \tag{4.10}$$

PROOF. Lemma 4.5 yields

$$\langle S, V\rangle^{P^\star} = \tilde{c}^{SV\star} \bullet A = (\tilde{c}^{S\star}(\tilde{c}^{S\star})^{-1}\tilde{c}^{SV\star}) \bullet A = \xi \bullet \langle S, S\rangle^{P^\star}.$$

By (3.30) and (4.6) we have

$$\begin{aligned}\langle M^{S\star}, M^{S\star}\rangle^{P^\star} &= \big(\tilde{c}^{S\star} - b^{S\star}(b^{S\star})^\top \Delta A\big) \bullet A \\ &= \big((1_d - \tilde{c}^{S\star}\tilde{a}\tilde{a}^\top \Delta A)\tilde{c}^{S\star}\big) \bullet A\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}\langle M^{S\star}, M^{V\star}\rangle^{P^\star} &= \big(\tilde{c}_t^{SV\star} - b_t^{S\star} b_t^{V\star} \Delta A\big) \bullet A \\ &= \big((1_d - \tilde{c}^{S\star}\tilde{a}\tilde{a}^\top \Delta A)\tilde{c}^{SV\star}\big) \bullet A,\end{aligned} \tag{4.12}$$

where 1_d denotes the identity matrix. Equations (4.11), (4.12), (4.7) yield (4.9). □

The pure hedge coefficient appears in the following decomposition:

LEMMA 4.8. *There exists a $P^\star$-local martingale M with $M_0 = 0$ that is $P^\star$-orthogonal to $M^{S\star}$ (in the sense that $M^{S\star}M$ is a $P^\star$-local martingale) and such that*

$$V = V_0 + \xi \bullet S + M \tag{4.13}$$

holds.

PROOF. By (4.7), (4.6), (3.30) we have

$$0 = \big(\tilde{a}^\top \tilde{c}^{SV\star} - \tilde{a}^\top \tilde{c}^{S\star}(\tilde{c}^{S\star})^{-1}\tilde{c}^{SV\star}\big) \bullet A = \big(b^{V\star} - (b^{S\star})^\top \xi\big) \bullet A,$$

which implies that $M := V - V_0 - \xi \bullet S$ is a $P^\star$-σ-martingale. By bilinearity and (4.7) the modified second $P^\star$-characteristics of M in the sense of (1.2) equals

$$\begin{aligned}\tilde{c}^{M\star} &= \tilde{c}^{V\star} - 2\xi^\top \tilde{c}^{SV\star} + \xi^\top \tilde{c}^{S\star}\xi \\ &= \tilde{c}^{V\star} - 2(\tilde{c}^{SV\star})^\top (\tilde{c}^{S\star})^{-1}\tilde{c}^{SV\star} + (\tilde{c}^{SV\star})^\top (\tilde{c}^{S\star})^{-1}\tilde{c}^{S\star}(\tilde{c}^{S\star})^{-1}\tilde{c}^{SV\star} \\ &= \tilde{c}^{V\star} - (\tilde{c}^{SV\star})^\top (\tilde{c}^{S\star})^{-1}\tilde{c}^{SV\star} \le \tilde{c}^{V\star}.\end{aligned}$$

Since V is a locally square-integrable semimartingale relative to $P^\star$, it follows that M is a locally square-integrable martingale relative to $P^\star$ (cf. [28], II.2.29). Since

$$\langle M^{S\star}, M\rangle^{P^\star} = \langle S, V - \xi \bullet S\rangle^{P^\star} = \big(\tilde{c}^{SV\star} - \tilde{c}^{S\star}(\tilde{c}^{S\star})^{-1}\tilde{c}^{SV\star}\big) \bullet A = 0$$

by (4.7), we have that $M^{S\star}M$ is a $P^\star$-local martingale. □

Equation (4.13) can be interpreted as a process version of the $P^\star$-Föllmer–Schweizer decomposition of H. The integrand in the latter yields the *locally risk-minimizing hedging strategy* in the sense of [41] or [18] relative to $P^\star$.

4.2. *Main results.*

LEMMA 4.9. *For any $\mathscr{F}_0$-measurable random variable v, the feedback equation*

$$\varphi_t = \xi_t - (v + \varphi \bullet S_{t-} - V_{t-})\tilde{a}_t \tag{4.14}$$

has a unique solution $\varphi(v) := \varphi \in L(S)$.

PROOF. In the proof of Theorem 4.10 below it is shown that $\xi \in L(S)$. The stochastic differential equation

$$G = \big(\xi - (v - V_-)\tilde{a}\big) \bullet S - G_- \bullet (\tilde{a} \bullet S) \tag{4.15}$$

has a unique solution G by Jacod [27], (6.8). If we set $\varphi_t := \xi_t - (v + G_{t-} - V_{t-})\tilde{a}_t$, then $\varphi \in L(S)$ solves (4.14).

If, on the other hand, some $\tilde{\varphi} \in L(S)$ solves (4.14) as well, then $\widetilde{G} := \tilde{\varphi} \bullet S$ is a solution to (4.15). This implies $\widetilde{G} = G$ and hence $\tilde{\varphi} = \varphi$. □

We are now ready to state our first main result.

THEOREM 4.10. 1. *The process $\varphi := \varphi(v_0)$ given by the feedback expression* (4.14) *is an optimal hedging strategy for initial endowment $v_0 \in L^2(\Omega, \mathscr{F}_0, P)$.*

2. $(v_0, \varphi) := (V_0, \varphi(V_0))$ *is an optimal endowment/strategy pair.*

PROOF. 1. Denote by

$$\left(\begin{pmatrix} b^S \\ b^V \\ b^K \end{pmatrix}, \begin{pmatrix} c^S & c^{SV} & c^{SK} \\ c^{VS} & c^V & c^{VK} \\ c^{KS} & c^{KV} & c^K \end{pmatrix}, F^{S,V,K}, A\right)$$

P-differential characteristics of (S, V, K) relative to the "truncation" function $h(x, z, y) := (x, z1_{\{|z|\leq 1\}}, y)$ on $\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}$. [Should V be a P-special semimartingale, we could also choose the identity $h(x, z, y) = (x, z, y)$ as usual in this paper.] Along the same lines as in the proof of Lemma 3.17 it follows that

$$\tilde{c}^{SV\star} = \frac{1}{1 + \Delta A^K}\left(c^{SV} + \int xz(1 + y)F^{S,V,K}\big(d(x, z, y)\big)\right), \tag{4.16}$$

$$\tilde{c}^{V\star} = \frac{1}{1 + \Delta A^K}\left(c^V + \int z^2(1 + y)F^{S,V,K}\big(d(x, z, y)\big)\right). \tag{4.17}$$

Let $\bar{\varphi}$ be an optimal hedging strategy for initial endowment v_0, denote by $G := v_0 + \bar{\varphi} \bullet S$ its value process, and set $\bar{\xi} := \bar{\varphi} + (G_- - V_-)\tilde{a}$. Moreover, let $\vartheta \in \overline{\Theta}$ and $\widetilde{G} := \vartheta \bullet S$. In the proof of Lemma 4.1 it is shown that $Z\widetilde{G}$ is a martingale for

$Z := (G-V)L$. Integration by parts yields $Z\widetilde{G} = L_0\mathscr{E}(K)(G-V)\widetilde{G} = L_- \bullet U$ with

$$\begin{aligned}
U &= (G-V)\widetilde{G} + \big((G-V)\widetilde{G}\big)_- \bullet K + [(G-V)\widetilde{G}, K] \\
&= (G-V)_- \bullet \widetilde{G} + \widetilde{G}_- \bullet (G-V) + [G-V, \widetilde{G}] + \big((G-V)\widetilde{G}\big)_- \bullet K \\
&\quad + (G-V)_- \bullet [\widetilde{G}, K] + \widetilde{G}_- \bullet [G-V, K] + \big[G-V, [\widetilde{G}, K]\big] \\
&= (G-V)_- \bullet \big(\widetilde{G}_- \bullet N + \vartheta \bullet (S + [S,N])\big) \\
&\quad + \bar{\xi} \bullet \big(\widetilde{G}_- \bullet (S + [S,K]) + \big[\widetilde{G}, S + [S,K]\big]\big) \\
&\quad - \widetilde{G}_- \bullet (V + [V,K]) - \big[\widetilde{G}, V + [V,K]\big].
\end{aligned}$$

The first term on the right-hand side is a σ-martingale by Lemma 3.14. By (3.15), the remaining two terms equal

$$\begin{aligned}
&G_- \bullet \big(\bar{\xi} \bullet (S + [S,N]) - (V + [V,N])\big) \\
&\qquad + (\tilde{a}G_- + \vartheta) \bullet \big(\bar{\xi} \bullet \big[S, S + [S,K]\big] - \big[V, S + [S,K]\big]\big).
\end{aligned}$$

The first line is a σ-martingale by Lemmas 3.14 and 4.3. By (3.26), (3.18) we have

$$\begin{aligned}
\big[S, S + [S,K]\big] &= [S^c, S^c] + \int xx^\top (1+y)\mu^{S,K}\big(d(x,y)\big) \\
&\sim \left(c^S + \int xx^\top (1+y) F^{S,K}\big(d(x,y)\big)\right) \bullet A \\
&= \bar{c} \bullet A \\
&= \big((1 + \Delta A^K)\tilde{c}^{S\star}\big) \bullet A.
\end{aligned} \tag{4.18}$$

Similarly, (4.16) yields

$$\begin{aligned}
\big[S + [S,K], V\big] &\sim \left(c^{SV} + \int xz(1+y) F^{S,V,K}\big(d(x,z,y)\big)\right) \bullet A \\
&= \big((1 + \Delta A^K)\tilde{c}^{SV\star}\big) \bullet A.
\end{aligned} \tag{4.19}$$

For later use, we observe that

$$\big[V + [V,K], V\big] \sim \big((1 + \Delta A^K)\tilde{c}^{V\star}\big) \bullet A1 \tag{4.20}$$

by (4.17). Altogether, we have that $((\tilde{a}G_- + \vartheta)(1 + \Delta A^K)(\tilde{c}^{S\star}\bar{\xi} - \tilde{c}^{SV\star})) \bullet A$ is a σ-martingale. This being true for any ϑ, we have

$$\tilde{c}^{S\star}\bar{\xi} - \tilde{c}^{SV\star} = 0 \tag{4.21}$$

$P \otimes A$-almost everywhere.

For $n \in \mathbb{N}$ define the predictable set $D_n := \{|\xi| \vee |\bar{\xi}| \leq n\}$. Corollary 3.20 and (4.21), (4.7) yield

$$\big((\bar{\xi} - \xi)1_{D_n}\big) \bullet A^{S\star} = \big((\bar{\xi} - (\tilde{c}^{S\star})^{-1}\tilde{c}^{SV\star}\big)^\top \tilde{c}^{S\star}\tilde{a}1_{D_n}\big) \bullet A = 0$$

as well as

$$\begin{aligned}\big\langle\big((\bar{\xi} - \xi)1_{D_n}\big) \bullet M^{S\star}, \big((\bar{\xi} - \xi)1_{D_n}\big) \bullet M^{S\star}\big\rangle^{P\star} \\ = \big((\bar{\xi} - \xi)^\top \hat{c}^{S\star}(\bar{\xi} - \xi)1_{D_n}\big) \bullet A \\ \leq \big((\bar{\xi} - \xi)^\top \tilde{c}^{S\star}(\bar{\xi} - \xi)1_{D_n}\big) \bullet A = 0.\end{aligned}$$

Consequently, $((\bar{\xi} - \xi)1_{D_n}) \bullet S = 0$ for any n, which in turn implies $\bar{\xi} - \xi \in L(S)$ and $(\bar{\xi} - \xi) \bullet S = 0$ by Lemma A.11. In particular, we have $\xi = \bar{\xi} - (\bar{\xi} - \xi) \in L(S)$. The proof of Lemma 4.9 yields that $\varphi \bullet S = \bar{\varphi} \bullet S$ as well. In particular, φ is admissible and an optimal hedging strategy for initial endowment v_0.

2. This follows essentially as statement 1. We only have to determine the optimal initial endowment. Denote by $(v_0, \bar{\varphi})$ an optimal endowment/strategy pair and let Z be as in the first part of the proof. Parallel to (4.4), we conclude that $E(vZ_T) = 0$ for any $v \in L^2(\Omega, \mathscr{F}_0, P)$, which implies $0 = E(Z_T|\mathscr{F}_0) = Z_0 = L_0(v_0 - V_0)$. Consequently, $v_0 = V_0$ as claimed. □

As is well known, the gains process $\varphi \bullet S$ can be expressed more explicitly.

COROLLARY 4.11. *The gains process of the optimal hedge in Theorem 4.10 equals*

$$\varphi \bullet S = \mathscr{E}(-\tilde{a} \bullet S)\left(\frac{\xi + (V_- - v_0)\tilde{a}}{\mathscr{E}(-\tilde{a} \bullet S)_-} \bullet \left(S + \frac{\tilde{a}}{1 - \tilde{a}^\top \Delta S} \bullet [S, S]\right)\right)$$

unless $\mathscr{E}(-\tilde{a} \bullet S)$ *jumps to* 0.

PROOF. By [27], (6.8) the stochastic differential equation $X = Y + X_- \bullet Z$ with two semimartingales Y, Z such that $Y_0 = 0$ is uniquely solved by

$$X = \mathscr{E}(Z)\left(\frac{1}{\mathscr{E}(Z)_-} \bullet Y - \frac{1}{\mathscr{E}(Z)} \bullet [Y, Z]\right)$$

unless $\mathscr{E}(Z)$ jumps to 0. Since

$$\varphi \bullet S = \big(\xi - (v_0 - V_-)\tilde{a}\big) \bullet S + (\varphi \bullet S)_- \bullet (-\tilde{a} \bullet S),$$

the assertion follows. □

Finally, we state formulas for the hedging error.

THEOREM 4.12. *The expected squared hedging error of the optimal hedge in Theorem 4.10 equals*

$$
\begin{aligned}
&E\big((v_0+\varphi\bullet S_T-H)^2\big)\\
&\quad= E\big((v_0-V_0)^2L_0+\big((\tilde{c}^{V\star}-(\tilde{c}^{SV\star})^\top(\tilde{c}^{S\star})^{-1}\tilde{c}^{SV\star})L\big)\bullet A_T\big)\\
&\quad= E\big((v_0-V_0)^2L_0+L\bullet(\langle V,V\rangle^{P^\star}-\xi\bullet\langle V,S\rangle^{P^\star})_T\big)\\
&\quad= E\big((v_0-V_0)^2L_0+L\bullet\langle V-\xi\bullet S,V-\xi\bullet S\rangle_T^{P^\star}\big) &&(4.22)\\
&\quad= E_{P^\star}\big((v_0-V_0)^2\\
&\qquad\qquad+\big((\tilde{c}^{V\star}-(\tilde{c}^{SV\star})^\top(\tilde{c}^{S\star})^{-1}\tilde{c}^{SV\star})\mathscr{E}(A^K)\big)\bullet A_T\big)E(L_0)\\
&\quad= E_{P^\star}\big((v_0-V_0)^2+\mathscr{E}(A^K)\bullet\langle V-\xi\bullet S,V-\xi\bullet S\rangle_T^{P^\star}\big)E(L_0). &&(4.23)
\end{aligned}
$$

PROOF. In view of Proposition 1.2, the second equality is obvious. The third and the last follow from

$$\langle S,V-\xi\bullet S\rangle^{P^\star}=\big(\tilde{c}^{SV\star}-\tilde{c}^{S\star}(\tilde{c}^{S\star})^{-1}\tilde{c}^{SV\star}\big)\bullet A=0.$$

Define $G:=v_0+\varphi\bullet S$ and $Z:=(G-V)L$ as in the proof of Theorem 4.10. Since $(G-V)^2L$ is a submartingale by Lemma 4.4, there exists a unique increasing predictable process B with $B_0=0$ and such that $(G-V)^2L-B$ is a martingale. Since

$$E\big((v_0+\varphi\bullet S_T-H)^2\big)=E\big((G_T-V_T)^2L_T\big)=E\big((G_0-V_0)^2L_0\big)+E(B_T),$$

the first equality in the theorem holds if

$$B=\big((\tilde{c}^{V\star}-\xi^\top\tilde{c}^{SV\star})L\big)\bullet A. \tag{4.24}$$

Since GZ and Z are martingales, we have

$$
\begin{aligned}
-(G-V)^2L&\sim VZ\\
&\sim Z_-\bullet V+[V,Z]\\
&=\big((G-V)_-L_-\big)\bullet V\\
&\qquad+\big[V,(G-V)_-\bullet L+L_-\bullet(G-V)+[G-V,L]\big].
\end{aligned}\tag{4.25}
$$

In view of

$$G-V=v_0+\xi\bullet S-\big((G-V)_-\tilde{a}\big)\bullet S-V$$

(4.25) equals

$$
\begin{aligned}
&\big((G-V)_-L_-\big)\bullet\big(V+\big[V,K-\tilde{a}\bullet S-[\tilde{a}\bullet S,K]\big]\big)\\
&\qquad+L_-\bullet\big[V+[V,K],\xi\bullet S-V\big].
\end{aligned}
$$

By Lemma 4.3 the first term is a σ-martingale and hence

$$\begin{aligned}(G-V)^2L &\sim -L_-\bullet\big(\xi\bullet[V+[V,K],S]-[V+[V,K],V]\big)\\ &= -L_-\bullet\big(\xi\bullet[V,S+[S,K]]-[V+[V,K],V]\big)\\ &\sim \big(L_-(1+\Delta A^K)(\tilde{c}^{V\star}-\xi^\top\tilde{c}^{SV\star})\big)\bullet A\\ &= \big((L-L_-\Delta M^K)(\tilde{c}^{V\star}-\xi^\top\tilde{c}^{SV\star})\big)\bullet A\end{aligned}$$

by (4.19) and (4.20). Since $\Delta M^K\bullet U=[M^K,U]=\Delta U\bullet M^K$ is a σ-martingale for any predictable process U of finite variation (cf. [28], I.4.49), we obtain

$$(G-V)^2L\sim\big(L(\tilde{c}^{V\star}-\xi^\top\tilde{c}^{SV\star})\big)\bullet A.$$

Therefore the difference of both sides of (4.24) is a predictable σ-martingale of finite variation and hence 0.

It remains to be shown that (4.23) equals (4.22). Integration by parts yields

$$\begin{aligned}&Z^{P^\star}\big(E(L_0)\mathscr{E}(A^K)\bullet\langle V-\xi\bullet S,V-\xi\bullet S\rangle^{P^\star}\big)\\ &\quad=\big(Z^{P^\star}E(L_0)\mathscr{E}(A^K)\big)\bullet\langle V-\xi\bullet S,V-\xi\bullet S\rangle^{P^\star}\\ &\qquad+\big(E(L_0)\mathscr{E}(A^K)\bullet\langle V-\xi\bullet S,V-\xi\bullet S\rangle^{P^\star}\big)_-\bullet Z^{P^\star}\\ &\quad=L\bullet\langle V-\xi\bullet S,V-\xi\bullet S\rangle^{P^\star}+M\end{aligned}$$

with some P-local martingale M. Hence

$$\begin{aligned}&E_{P^\star}\big(\mathscr{E}(A^K)\bullet\langle V-\xi\bullet S,V-\xi\bullet S\rangle^{P^\star}_{T_n}\big)E(L_0)\\ &\quad=E\big(Z^{P^\star}_{T_n}\big(E(L_0)\mathscr{E}(A^K)\bullet\langle V-\xi\bullet S,V-\xi\bullet S\rangle^{P^\star}_{T_n}\big)\big)\\ &\quad=E(L\bullet\langle V-\xi\bullet S,V-\xi\bullet S\rangle^{P^\star}_{T_n}),\end{aligned}$$

where $(T_n)_{n\in\mathbb{N}}$ denotes a localizing sequence for M. Monotone convergence yields that (4.23) equals (4.22). □

If the results in this paper are to be applied to concrete models, it is not necessary to determine all the processes that have been introduced. Instead, one may proceed as follows: first one determines the opportunity process L and the adjustment process $\tilde{a}$ using the characterization in Theorem 3.25. These processes yield the modified mean-variance tradeoff process K, the opportunity-neutral measure $P^\star$ and the variance-optimal logarithm process N. Finally, the mean-value process V leads to the pure hedge coefficient ξ and hence to the optimal hedge φ.

4.3. *Connections to the literature.* In this section we clarify the link of our results to the literature. If S is a martingale, we are in the setup of Föllmer and Sondermann [19]. In our notation, they show that the optimal hedge φ satisfies

$$\varphi_t=\frac{d\langle S,V\rangle_t}{d\langle S,S\rangle_t},\tag{4.26}$$

where

$$V_t = E(H|\mathscr{F}_t). \tag{4.27}$$

Applying our results to the martingale case, one immediately verifies that $L = 1$, $\tilde{a} = 0$, $K = 0$, $N = 0$, $Q^\star = P^\star = P$. Consequently, equation (4.2) for the mean-value process of the option reduces to (4.27). Moreover, the optimal hedge φ coincides with the pure hedge ξ, which satisfies $\xi \bullet \langle S, S\rangle = \langle S, V\rangle$ in accordance with (4.26).

Schweizer [42] goes beyond the martingale case. He shows that if the MVT process $\widehat{K}$ is deterministic, then the optimal hedging strategy for initial endowment v_0 contains a feedback element and is of the form

$$\varphi_t = \xi_t - (v_0 + \varphi \bullet S_{t-} - V_{t-})\tilde{\lambda}_t \tag{4.28}$$

with $\tilde{\lambda}$ from (3.35). Here, the *pure hedge coefficient* ξ is the integrand in the *Föllmer–Schweizer decomposition* of the claim, that is,

$$H = V_0 + \xi \bullet S_T + R_T,$$

where V_0 is a $\mathscr{F}_0$-measurable random variable and R denotes a martingale which is orthogonal to M^S (in the sense that $M^S R$ is a local martingale). In order to express the pure hedge coefficient similarly as in (4.26), recall that the minimal signed martingale measure Q is given by

$$\frac{dQ}{dP} := \mathscr{E}(-\hat{\lambda} \bullet M^S)_T.$$

If we define V as "Q-conditional expectation" of H in the sense of

$$V_t := E\big(H\mathscr{E}\big((-\hat{\lambda}1_{]\!]t,T]\!]}) \bullet M^S\big)_T|\mathscr{F}_t\big), \tag{4.29}$$

then the pure hedge coefficient can be written as

$$\xi_t = \frac{d\langle S, V\rangle_t}{d\langle S, S\rangle_t} = \frac{d\langle M^S, M^V\rangle_t}{d\langle M^S, M^S\rangle_t}. \tag{4.30}$$

The hedging error satisfies the equation

$$\begin{aligned} &E\big((v_0 + \varphi \bullet S_T - H)^2\big) \\ &\quad = E\big((v_0 - V_0)^2 + \mathscr{E}(\widehat{K}) \bullet \langle V - \xi \bullet S, V - \xi \bullet S\rangle_T\big)\frac{1}{\mathscr{E}(\widehat{K})_T}. \end{aligned} \tag{4.31}$$

In these formulas, all predictable covariation processes refer to the original probability measure P.

It is easy to see that (4.28)–(4.31) are special cases of our general results. To this end, recall that $L = \mathscr{E}(\widehat{K})/\mathscr{E}(\widehat{K})_T$, $P^\star = P$, and $\tilde{a} = \tilde{\lambda}$ in the case of deterministic MVT (cf. Corollaries 3.31 and 3.30). Hence

$$N^\star = -\big((1 + \Delta A^K)\tilde{a}\big) \bullet M^{S\star} = -\big((1 + \Delta\widehat{K})\tilde{\lambda}\big) \bullet M^S = -\hat{\lambda} \bullet M^S.$$

Consequently, (4.28), (4.29), (4.30), (4.31) correspond to (4.14), (4.3), (4.10), (4.23), respectively.

If the MVT process fails to be deterministic, the above formulas do not lead to the optimal hedge any more. Following Hipp [22], [44] observes that a key role in the general case is played by the variance optimal signed martingale measure $Q^\star$ and the adjustment process $\tilde{a}$. Schweizer characterizes both the adjustment process and the optimal hedging strategy in terms of backward stochastic differential equations. The use of these BSDEs in practice is complicated by their involved boundary conditions, which themselves depend on the unknown solution.

Rheinländer and Schweizer [40] show that the optimal hedging strategy φ satisfies similar equations as in the case of deterministic MVT *if S is continuous*. More specifically, it is of feedback form

$$\varphi_t = \xi_t - (v_0 + \varphi \bullet S_{t-} - V_{t-})\tilde{a}_t,$$

where $V_t := E_{Q^\star}(H|\mathscr{F}_t)$ is the martingale generated by H relative to the variance-optimal SσMM $Q^\star$ and the *pure hedge coefficient* ξ is the integrand in the Galtchouk–Kunita–Watanabe decomposition of H relative to $Q^\star$ rather than P, that is,

$$\xi_t = \frac{d\langle S, V\rangle_t^{Q^\star}}{d\langle S, S\rangle_t^{Q^\star}}.$$

This equation corresponds to our expression (4.10) because the predictable covariation does not depend on the probability measure for continuous processes.

An alternative approach in the continuous case is pursued by Gourieroux, Laurent and Pham [20] who use a new numeraire $\mathscr{E}(-\tilde{a} \bullet S)$ combined with a change of measure to transform the original semimartingale problem to a martingale problem à la Föllmer and Sondermann [19]. The task of computing $\tilde{a}$ has become a separate issue in the literature. It is tackled in a number of diffusion or jump-diffusion settings, for example, by Laurent and Pham [30], Biagini, Guasoni and Pratelli [6], Biagini and Guasoni [5], Hobson [24]. Our characterization of the adjustment process in Theorem 3.25 appears to be more suitable for direct computations than the methods available to date (cf. [11]).

The literature on discontinuous processes is more limited. Two partial results are reported by Arai [3] and Lim [33]. Arai extends the numeraire method of Gourieroux, Laurent and Pham [20] to discontinuous semimartingales assuming that $Q^\star$ is equivalent to P and shows that V in (4.3) is a $Q^\star$-martingale. However, Arai's results are hard to use for explicit computations since he does not provide a method for obtaining $\tilde{a}$.

Lim [33] uses BSDEs to compute the optimal hedge in a jump diffusion setting where asset price characteristics are adapted to a Brownian filtration. In addition he requires a certain *martingale invariance property*. He characterizes the optimal hedge explicitly at the cost of a somewhat restrictive model setup.

Finally, we want to explain another close link of our results to the formulas (4.28–4.31) of [42]. We already observed in Lemma 3.23 that the variance-optimal SσMM $Q^\star$ is the minimal SσMM relative to $P^\star$. Moreover, $\tilde{a}$ and $\hat{a}$ coincide with the processes $\tilde{\lambda}$ and $\hat{\lambda}$ in [42] or Section 3.6 relative to $P^\star$ instead of P. Consequently, equations (4.14), (4.3), (4.10) are $P^\star$-versions of the formulas (4.28), (4.29), (4.30). The change of measure $P \to P^\star$ neutralizes the effect of stochastic mean-variance tradeoff which makes the results in [42] break down. With the hedging error one has to be slightly more careful. Since $\widehat{K}^\star = A^K$, we can view (4.23) essentially as a $P^\star$-version of (4.31). We only have to replace the deterministic second factor $1/\mathscr{E}(\widehat{K})_T$ by

$$E\left(\frac{1}{\mathscr{E}(\widehat{K}^\star)_T}\right) = E\left(\frac{1}{\mathscr{E}(A^K)_T}\right) = E_{P^\star}\left(\frac{E(L_0)}{L_T}\right) = E(L_0).$$

APPENDIX

A.1. Locally square-integrable semimartingales.

DEFINITION A.1. For any special semimartingale X we define

$$\|X\|_{\mathscr{S}^2} := \|X_0\|_2 + \|\sqrt{[M^X, M^X]_T}\|_2 + \|\operatorname{var}(A^X)_T\|_2,$$

where $\operatorname{var}(A^X)$ denotes the variation process of A^X and $\|\cdot\|_2$ the L^2-norm. X is said to belong to the set $\mathscr{S}^2$ of *square-integrable semimartingales* if $\|X\|_{\mathscr{S}^2} < \infty$. The elements of the corresponding localized class $\mathscr{S}^2_{\text{loc}}$ are called *locally square-integrable semimartingales*.

LEMMA A.2. *For any semimartingale X, we have equivalence between*:

1. $X \in \mathscr{S}^2_{\text{loc}}$.
2. $X_0 \in L^2(P)$ *and X is a locally square-integrable semimartingale in the sense of* [28], II.2.27, *that is, it is a special semimartingale whose local martingale part is locally square-integrable.*
3. *X is locally in L^2 in the sense of* [15], *that is, it belongs locally to the class of processes Y with*

$$\sup\{E(Y_\tau^2) : \tau \text{ finite stopping time}\} < \infty.$$

4. *X belongs locally to the class of processes Y satisfying $E(Y_\tau^2) < \infty$ for any finite stopping time τ.*

PROOF. We refer to the time set $\mathbb{R}_+$ rather than $[0, T]$ in this proof.

$1 \Rightarrow 2$: This follows from [28], II.2.28 and from the inequality

$$E\left(\sup_{t\in\mathbb{R}_+} (Y_t - Y_0)^2\right) \le 8\|Y\|^2_{\mathscr{S}^2},$$

which holds for any semimartingale Y (cf. [39], Theorem IV.5).

$2 \Rightarrow 3$: This follows immediately from [28], II.2.28.

$3 \Rightarrow 4$: This is trivial.

$4 \Rightarrow 1$: Define a sequence of stopping times $\tau_n := \inf\{t \in \mathbb{R}_+ : |X_t| > n\} \wedge n$. Since $\sup_{t\in\mathbb{R}_+} |X_t^{\tau_n}| \leq n + |X_{\tau_n}|$ is integrable, X is a special semimartingale (cf. [28], I.4.23). Choose a localizing sequence $(\sigma_n)_{n\in\mathbb{N}}$ for the locally bounded process $\operatorname{var}(A^X)$. Then

$$\begin{aligned}\sup_{t\in\mathbb{R}_+} |(M^X)_t^{\sigma_n\wedge\tau_n}|^2 &\leq 3 \sup_{t\in\mathbb{R}_+} |X_t^{\tau_n}|^2 + 3 \sup_{t\in\mathbb{R}_+} |(A^X)_t^{\sigma_n}|^2 + 3|X_0|^2 \\ &\leq 6n^2 + 6|X_{\tau_n}|^2 + 3 \sup_{t\in\mathbb{R}_+} (\operatorname{var}(A^X)_t^{\sigma_n})^2 + 3|X_0|^2\end{aligned}$$

is integrable for any $n \in \mathbb{N}$, which yields $X \in \mathscr{S}^2_{\mathrm{loc}}$ (cf. [28], I.4.50c). $\square$

The following result on square integrability of exponential semimartingales is needed in the proof of Proposition 3.28. It extends a parallel statement for local martingales in [27], (8.27).

LEMMA A.3. *Let X be a locally square-integrable semimartingale such that $\langle M^X, M^X\rangle$ and the variation process* $\operatorname{var}(A^X)$ *are bounded. Then*

$$E\Bigl(\sup_t \mathscr{E}(X)_t^2\Bigr) < \infty.$$

PROOF. For ease of notation we prove the assertion for the time set $\mathbb{R}_+$ rather than $[0, T]$. Denote by $m \in \mathbb{R}_+$ an upper bound of $V := \langle M^X, M^X\rangle + \operatorname{var}(A^X)$. We write $Z := \mathscr{E}(X)$ and $Y_t^* := \sup_{s\in[0,t]} |Y_s|$ for any process Y. For $n \in \mathbb{N}$ define stopping times

$$\sigma_n := \inf\{t \in \mathbb{R}_+ : |Z_t| \geq n\}.$$

Fix n and set $\widetilde{Z} := Z^{\sigma_n}$.

Step 1: We show that

$$E((\widetilde{Z}^*_{\tau-})^2) \leq 3 + (12 + 3m) E\bigl((\widetilde{Z}^2_- \wedge n^2) \bullet V_{\tau-}\bigr)$$

for any predictable stopping time τ.

In view of $\widetilde{Z} = 1 + (\widetilde{Z}_- 1_{[\![0,\sigma_n]\!]}) \bullet M^X + (\widetilde{Z}_- 1_{[\![0,\sigma_n]\!]}) \bullet A^X$, we have

$$E((\widetilde{Z}^*_{\tau-})^2) \leq 3 + 3E\bigl((((\widetilde{Z}_- 1_{[\![0,\sigma_n]\!]}) \bullet M^X)^*_{\tau-})^2\bigr) + 3E\bigl((((\widetilde{Z}_- 1_{[\![0,\sigma_n]\!]}) \bullet A^X)^*_{\tau-})^2\bigr).$$

Since τ is predictable, Doob's inequality yields

$$\begin{aligned}E\bigl((((\widetilde{Z}_- 1_{[\![0,\sigma_n]\!]}) \bullet M^X)^*_{\tau-})^2\bigr) &\leq 4E\bigl((\widetilde{Z}^2_- 1_{[\![0,\sigma_n]\!]}) \bullet \langle M^X, M^X\rangle_{\tau-}\bigr) \\ &\leq 4E\bigl((\widetilde{Z}^2_- \wedge n^2) \bullet V_{\tau-}\bigr).\end{aligned}$$

For the part of finite variation we have

$$\begin{aligned}\big(\big((\widetilde{Z}_{-}1_{[\![0,\sigma_n]\!]})\bullet A^X\big)^*_{\tau-}\big)^2 &\le \big((|\widetilde{Z}_{-}|\wedge n)\bullet \operatorname{var}(A^X)_{\tau-}\big)^2\\ &\le (\widetilde{Z}^2_{-}\wedge n^2)\bullet \operatorname{var}(A^X)_{\tau-}\operatorname{var}(A^X)_{\infty}\end{aligned}$$

and hence

$$E\big(\big(\big((\widetilde{Z}_{-}1_{[\![0,\sigma_n]\!]})\bullet A^X\big)^*_{\tau-}\big)^2\big)\le mE\big((\widetilde{Z}^2_{-}\wedge n^2)\bullet V_{\tau-}\big).$$

Step 2: For $\vartheta\in\mathbb{R}_+$ define the predictable stopping time $T_\vartheta:=\inf\{t\in\mathbb{R}_+:V_t\ge\vartheta\}$ (cf. [28], I.2.13). Step 1 yields that

$$f(\vartheta):=E\big((\widetilde{Z}^*_{T_\vartheta-}\wedge n)^2\big)\le 3+(12+3m)E\big((\widetilde{Z}^2_{-}\wedge n^2)\bullet V_{T_\vartheta-}\big).$$

Since $\vartheta\mapsto T_\vartheta$ is the pathwise generalized inverse of V, we have

$$(\widetilde{Z}^2_{-}\wedge n^2)\bullet V_{T_\vartheta-}=\int_0^{V_{T_\vartheta-}}(\widetilde{Z}^2_{T_\varrho-}\wedge n^2)\,d\varrho\le\int_0^{\vartheta}(\widetilde{Z}^*_{T_\varrho-}\wedge n)^2\,d\varrho$$

and hence

$$f(\vartheta)\le 3+(12+3m)\int_0^{\vartheta}f(\varrho)\,d\varrho$$

for any $\vartheta\in\mathbb{R}_+$. By Gronwall's inequality this implies $f(\vartheta)\le 3e^{(12+3m)\vartheta}$. Since $T_{m+1}=\infty$, we have

$$E\Big(n^2\wedge\sup_{t\le\sigma_n}Z^2_t\Big)=E\big((\widetilde{Z}^*_{\infty-}\wedge n)^2\big)\le 3e^{(12+3m)(m+1)}.$$

The assertion follows now from monotone convergence. □

A.2. σ-martingales. The following facts on σ-martingales and integrability can be found, for example, in [29]. We summarize them here for the convenience of the reader.

DEFINITION A.4. A semimartingale X is called *σ-martingale* if there exists an increasing sequence $(D_n)_{n\in\mathbb{N}}$ of predictable sets such that $D_n\uparrow\Omega\times\mathbb{R}_+$ up to an evanescent set and $1_{D_n}\bullet X$ is a uniformly integrable martingale for any $n\in\mathbb{N}$.

REMARK A.5. *Uniformly integrable martingale* can be replaced by *local martingale* in the previous definition.

LEMMA A.6. *Let X be a semimartingale with differential characteristics (b,c,F,A) relative to some truncation function h. Then X is a σ-martingale if and only if $\int_{\{|x|>1\}}|x|F(dx)<\infty$ and*

$$b+\int\big(x-h(x)\big)F(dx)=0$$

hold outside some $P\otimes A$-null set.

LEMMA A.7. *X is a uniformly integrable martingale if and only if it is a σ-martingale of class* (*D*).

LEMMA A.8. *Let $P^\star \sim P$ be a probability measure with density process Z. A real-valued semimartingale X is a $P^\star$-σ-martingale if and only if XZ is a P-σ-martingale.*

LEMMA A.9. *Let X be a $\mathbb{R}^d$-valued semimartingale and let $P^\star \sim P$ be a probability measure with density process $Z = Z_0\mathscr{E}(N)$. Denote by*

$$(b^{X,N}, c^{X,N}, F^{X,N}, A) = \left(\begin{pmatrix} b^X \\ b^N \end{pmatrix}, \begin{pmatrix} c^X & c^{XN} \\ c^{NX} & c^N \end{pmatrix}, F^{X,N}, A\right)$$

differential characteristics of the $\mathbb{R}^{d+1}$-valued seminartingale (X, N) relative to some truncation function h. Then a version of the $P^\star$-differential characteristics of (X, N) is given by $(b^{X,N\star}, c^{X,N\star}, F^{X,N^\star}, A)$, where

$$b^{X,N\star} = b^{X,N} + c^{XN} + \int h(x, y)yF^{X,N}(d(x, y)),$$

$$c^{X,N\star} = c^{X,N},$$

$$\frac{dF^{X,N\star}}{dF^{X,N}}(x, y) = 1 + y.$$

LEMMA A.10. *If X is a σ-martingale and $\vartheta \in L(X)$, then $\vartheta \bullet X$ is a σ-martingale as well.*

LEMMA A.11. *Let X be a $\mathbb{R}^d$-valued semimartingale and ϑ an $\mathbb{R}^d$-valued predictable process. Then $\vartheta \in L(X)$ if and only if there exists a semimartingale Z with $Z_0 = 0$ and an increasing sequence $(D_n)_{n\in\mathbb{N}}$ of predictable sets such that $D_n \uparrow \Omega \times \mathbb{R}_+$ up to an evanescent set, $\vartheta 1_{D_n}$ is bounded, and $1_{D_n} \bullet Z = (\vartheta 1_{D_n}) \bullet X$ for any $n \in \mathbb{N}$. In this case $Z = \vartheta \bullet X$.*

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