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Research Article

The Distribution of the Interval between Events of a Cox Process with Shot Noise Intensity

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Applying piecewise deterministic Markov processes theory, the probability generating function of a Cox process, incorporating with shot noise process as the claim intensity, is obtained. We also derive the Laplace transform of the distribution of the shot noise process at claim jump times, using stationary assumption of the shot noise process at any times. Based on this Laplace transform and from the probability generating function of a Cox process with shot noise intensity, we obtain the distribution of the interval of a Cox process with shot noise intensity for insurance claims and its moments, that is, mean and variance.

1. Introduction

In insurance modeling, the Poisson process has been used as a claim arrival process. Extensive discussion of the Poisson process, from both applied and theoretical viewpoints, can be found in [1–6]. However there has been a significant volume of literature that questions the suitability of the Poisson process in insurance modeling [7, 8]. From a practical point of view, there is no doubt that the insurance industry needs a more suitable claim arrival process than the Poisson process that has deterministic intensity.

As an alternative point process to generate the claim arrivals, we can employ a Cox process or a doubly stochastic Poisson process [9–15]. An important book on Cox processes is the book by Bening and Korolev [16], where the applications in both insurance and finance are discussed. A Cox process provides us with the flexibility to allow the intensity not only to depend on time but also to be a stochastic process. Dassios and Jang [17] demonstrated how a Cox process with shot noise intensity could be used in the pricing of catastrophe reinsurance and derivatives.

It is important to measure the time interval between the claims in insurance. Thus in this paper, we examine the distribution of the interval of a Cox process with shot noise intensity for insurance claims. The result of this paper can be used or easily modified in computer science/telecommunications modeling, electrical engineering, and queueing theory.

We start by defining the quantity of interest; this is a doubly stochastic (with a shot-noise intensity) point process of claim arrivals. Then, we derive the probability generating function of a Cox process with shot noise intensity using piecewise deterministic Markov processes (PDMPs) theory, for which see the appendix. The piecewise deterministic Markov processes theory is a powerful mathematical tool for examining nondiffusion models. For details, we refer the reader to [17–25]. In Section 3, we derive the Laplace transform of the distribution of the shot noise process at claim times, using stationary assumption of the shot noise process at any times. Using this Laplace transform within the probability generating function of a Cox process with shot noise intensity, we derive the distribution between events of a Cox process with shot noise intensity. These can be insurance claims for examples. We also derive the first two moments of this distribution. Section 4 contains some concluding remarks .

2. A Cox process and the shot noise process

A Cox process (or a doubly stochastic Poisson process) can be viewed as a two-step randomisation procedure. A process λ_t is used to generate another process N_t by acting as its intensity. That is, N_t is a Poisson process conditional on λ_t which itself is a stochastic process (if λ_t is deterministic then N_t is a Poisson process). Many alternative definitions of a doubly stochastic Poisson process can be given. We will offer the one adopted by Brémaud [15].

Definition 2.1. Let (Ω, F, P) be a probability space with information structure given by $F = \{\Im_t,\ t \in [0,T]\}$. Let N_t be a point process adapted to F. Let λ_t be a nonnegative process adapted to F such that

$$\int_0^t \lambda_s\, ds < \infty \text{ almost surely (no explosions).} \tag{2.1}$$

If for all $0 \le t_1 \le t_2$ and $u \in \Re$

$$E\{e^{iu(N_{t_2}-N_{t_1})} \mid \Im^{\lambda}_{t_2}\} = \exp\left\{(e^{iu}-1)\int_{t_1}^{t_2}\lambda_s\, ds\right\} \tag{2.2}$$

then N_t is called a $\Im_t$-doubly stochastic Poisson process with intensity, λ_t where $\Im^{\lambda}_t$ is the σ-algebra generated by λ up to time t, that is, $\Im^{\lambda}_t = \sigma\{\lambda_s;\ s \le t\}$.

Equation (2.2) gives us

$$\Pr\left\{N_{t_2} - N_{t_1} = k \mid \lambda_s;\ t_1 \le s \le t_2\right\} = \frac{\exp\left(-\int_{t_1}^{t_2}\lambda_s\, ds\right)\left(\int_{t_1}^{t_2}\lambda_s\, ds\right)^k}{k!}, \tag{2.3}$$

$$\Pr\left\{\tau_2 > t \mid \lambda_s;\ t_1 \le s \le t_2\right\} = \Pr\left\{N_{t_2} - N_{t_1} = 0 \mid \lambda_s;\ t_1 \le s \le t_2\right\} = \exp\left(-\int_{t_1}^{t_2}\lambda_s\, ds\right), \tag{2.4}$$

where $\tau_k = \inf\{t > 0 : N_t = k\}$. Therefore from (2.4), we can easily find that

$$\Pr(\tau_2 \le t) = E\left\{\lambda_{t_2} \exp\left(-\int_{t_1}^{t_2} \lambda_s \, ds\right)\right\}. \tag{2.5}$$

If we consider the process $\Lambda_t = \int_0^t \lambda_s \, ds$ (*the aggregated process*), then from (2.3) we can also easily find that

$$E\left(\theta^{N_{t_2}-N_{t_1}}\right) = E\left\{e^{-(1-\theta)(\Lambda_{t_2}-\Lambda_{t_1})}\right\}, \tag{2.6}$$

where θ is a constant between 0 and 1. Equation (2.6) suggests that the problem of finding the distribution of N_t, the point process, is equivalent to the problem of finding the distribution of Λ_t, the aggregated process. It means that we just have to find the probability generating function (p.g.f.) of N_t to retrieve the moment generating function (m.g.f.) of Λ_t and vice versa.

One of the processes that can be used to measure the impact of primary events is the shot noise process [26–28]. The shot noise process is particularly useful within the claim arrival process as it measures the frequency, magnitude, and time period needed to determine the effect of primary events. As time passes, the shot noise process decreases as more and more claims are settled. This decrease continues until another event occurs which will result in a positive jump in the shot noise process. Therefore the shot noise process can be used as the parameter of doubly stochastic Poisson process to measure the number of claims due to primary events, that is, we will use it as a claim intensity function to generate the Cox process. We will adopt the shot noise process used by Cox and Isham [26]:

$$\lambda_t = \lambda_0 e^{-\delta t} + \sum_{i=1}^{M_t} Y_i e^{-\delta(t-S_i)}, \tag{2.7}$$

where

(i) λ_0 is initial value of λ_t;

(ii) $\{Y_i\}_{i=1,2,\ldots}$ is a sequence of independent and identically distributed random variables with distribution function $G(y)$ $(y > 0)$, where $E(Y_i) = \mu_1$;

(iii) $\{S_i\}_{i=1,2,\ldots}$ is the sequence representing the event times of a Poisson process M_t with constant intensity ρ;

(iv) δ is rate of exponential decay.

We assume that the Poisson process M_t and the sequences $\{Y_i\}_{i=1,2,\ldots}$ are independent of each other. Figure 1 is the graph illustrating shot noise process. Figure 2 is the graph illustrating a Cox process with shot noise intensity.

The generator of the process $(\Lambda_t, \lambda_t, t)$ acting on a function $f(\Lambda, \lambda, t)$ belonging to its domain is given by

$$\mathcal{A} f(\Lambda, \lambda, t) = \frac{\partial f}{\partial t} + \lambda \frac{\partial f}{\partial \Lambda} - \delta\lambda \frac{\partial f}{\partial \lambda} + \rho\left[\int_0^\infty f(\Lambda, \lambda + y, t)\, dG(y) - f(\Lambda, \lambda, t)\right]. \tag{2.8}$$

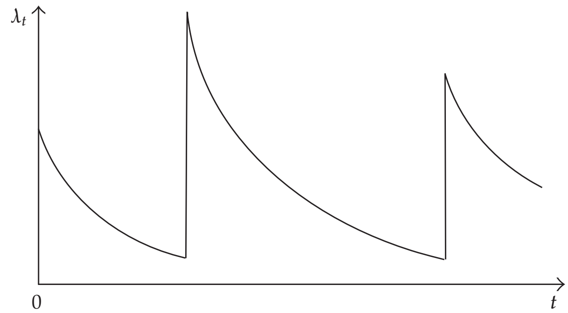

Figure 1: Graph illustrating shot noise process.

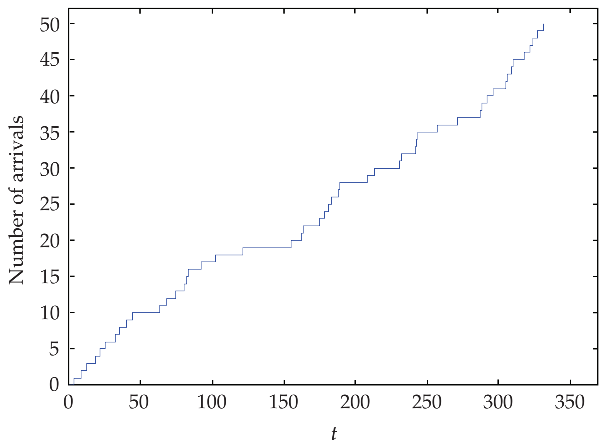

Figure 2: Graph illustrating a Cox process with shot noise process.

For $f(\Lambda, \lambda, t)$ to belong to the domain of the generator A, it is sufficient that $f(\Lambda, \lambda, t)$ is differentiable with respect to Λ, λ, t for all Λ, λ, t and that $|\int_0^\infty f(\cdot, \lambda + y, \cdot) dG(y) - f(\cdot, \lambda, \cdot)| < \infty$.

Let us find a suitable martingale in order to derive the probability generating function (p.g.f.) of N_t at time t.

Theorem 2.2. *Let us assume that Λ_t and λ_t evolve up to a fixed time t^*. Considering constants k_1 and k_2 are such that $k_1 \geq 0$ and $k_2 \geq -k_1 e^{-\delta t^*}$,*

$$\exp\left(- k_1 \delta \Lambda_t \right) \exp\left\{ - \left(k_1 + k_2 e^{\delta t} \right) \lambda_t \right\} \exp\left[\rho \int_0^t \left\{ 1 - \widehat{g}\left(k_1 + k_2 e^{\delta s} \right) \right\} ds \right] \tag{2.9}$$

is a martingale, where $\widehat{g}(u) = \int_0^t e^{-uy} dG(y)$ and $t > 0$.

Proof. Define $W_t = \delta \Lambda_t + \lambda_t$ and $Z_t = \lambda_t e^{\delta t}$, then the generator of the process (W_t, Z_t, t) acting on a function $f(w, z, t)$ is given by

$$A f(w, z, t) = \frac{\partial f}{\partial t} + \rho \left[\int_0^\infty f\left(w + y, z + y e^{\delta t}, t \right) dG(y) - f(w, z, t) \right], \tag{2.10}$$

and $f(w,z,t)$ has to satisfy $Af = 0$ for $f(W_t, Z_t, t)$ to be a martingale. We try a solution of the form $e^{-k_1 w} e^{-k_2 z} h(t)$, where $h(t)$ is a differentiable function. Then we get the following equation:

$$h'(t) - \rho\left[1 - \widehat{g}\left(k_1 + k_2 e^{\delta t}\right)\right] h(t) = 0. \tag{2.11}$$

$e^{-k_1 w} e^{-k_2 z} h(t)$ belongs to the domain of the generator because of our choice of k_1, k_2; the function is bounded for all $t \le t^*$ and our process evolves up to time t^* only. Solving (2.11)

$$h(t) = K e^{\rho \int_0^t \{1 - \widehat{g}(k_1 + k_2 e^{\delta s})\} ds}, \tag{2.12}$$

where K is an arbitrary constant. Therefore

$$e^{-k_1 W_t} e^{-k_2 Z_t} e^{\rho \int_0^t \{1 - \widehat{g}(k_1 + k_2 e^{\delta s})\} ds} \tag{2.13}$$

is a martingale and hence the result follows. □

Corollary 2.3. *Let $\nu_1 \ge 0$, $\nu_2 \ge 0$, $\nu \ge 0$, $0 \le \theta \le 1$, and t_1, t_2 be fixed times. Then*

$$\begin{aligned} E\left\{ e^{-\nu_1 (\Lambda_{t_2} - \Lambda_{t_1})} e^{-\nu_2 \lambda_{t_2}} \mid \Lambda_{t_1}, \lambda_{t_1} \right\} &= \exp\left[-\left\{ \frac{\nu_1}{\delta} + \left(\nu_2 - \frac{\nu_1}{\delta} \right) e^{-\delta(t_2 - t_1)} \right\} \lambda_{t_1} \right] \\ &\quad \times \exp\left[-\rho \int_0^{t_2 - t_1} \left[1 - \widehat{g}\left\{ \frac{\nu_1}{\delta} + \left(\nu_2 - \frac{\nu_1}{\delta} \right) e^{-\delta s} \right\} \right] ds \right], \end{aligned} \tag{2.14}$$

$$\begin{aligned} E\left\{ \theta^{(N_{t_2} - N_{t_1})} e^{-\nu \lambda_{t_2}} \mid N_{t_1}, \lambda_{t_1} \right\} &= \exp\left[-\left\{ \frac{1-\theta}{\delta} + \left(\nu - \frac{1-\theta}{\delta} \right) e^{-\delta(t_2 - t_1)} \right\} \lambda_{t_1} \right] \\ &\quad \times \exp\left[-\rho \int_0^{t_2 - t_1} \left[1 - \widehat{g}\left\{ \frac{1-\theta}{\delta} + \left(\nu - \frac{1-\theta}{\delta} \right) e^{-\delta s} \right\} \right] ds \right]. \end{aligned} \tag{2.15}$$

Proof. We set $k_1 = \nu_1/\delta$, $k_2 = (\nu_2 - \nu_1/\delta) e^{-\delta t_2}$, $t^* \ge t_2$ in Theorem 2.2 and (2.14) follows immediately. Equation (2.15) follows from (2.14) and (2.6). □

Now we can easily derive the probability generating function (p.g.f.) of N_t and the Laplace transform of λ_t using Corollary 2.3.

Corollary 2.4. *The probability generating function of N_t is given by*

$$\begin{aligned} E\left\{ \theta^{(N_{t_2} - N_{t_1})} \mid \lambda_{t_1} \right\} &= \exp\left[-\frac{1-\theta}{\delta} \left\{ 1 - e^{-\delta(t_2 - t_1)} \right\} \lambda_{t_1} \right] \\ &\quad \times \exp\left[-\rho \int_0^{t_2 - t_1} \left[1 - \widehat{g}\left\{ \frac{1-\theta}{\delta} \left(1 - e^{-\delta s} \right) \right\} \right] ds \right], \end{aligned} \tag{2.16}$$

the Laplace transform of the distribution of λ_t is given by

$$E\{e^{-\nu\lambda_t} \mid \lambda_0\} = \exp\left(-\nu\lambda_0 e^{-\delta t}\right)\exp\left[-\rho\int_0^t \{1-\widehat{g}(\nu e^{-\delta s})\}ds\right] \tag{2.17}$$

and if λ_t is asymptotic (stationary), it is given by

$$E(e^{-\nu\lambda_t}) = \exp\left[-\rho\int_0^\infty \{1-\widehat{g}(\nu e^{-\delta s})\}ds\right] \tag{2.18}$$

which can also be written as

$$E(e^{-\nu\lambda_t}) = \exp\left\{-\frac{\rho}{\delta}\int_0^\nu \widehat{\mathbf{G}}(u)du\right\}, \tag{2.19}$$

where $\widehat{\mathbf{G}}(u) = (1-\widehat{g}(u))/u$.

Proof. If we set $\nu = 0$ in (2.15) then (2.16) follows. Equation (2.17) follows if we either set $\nu_1 = 0$ in (2.14) or set $\theta = 1$ in (2.15). Let $t \to \infty$ in (2.17) and the result follows immediately. □

Theorem 2.2, Corollaries 2.3 and 2.4 can be found in [17, 19], but they have been included here for completeness and for comparison purposes.

If we differentiate (2.17) and (2.19) with respect to ν and put $\nu = 0$, we can easily obtain the first moments of λ_t, that is,

$$E(\lambda_t \mid \lambda_0) = \frac{\mu_1\rho}{\delta} + \left(\lambda_0 - \frac{\mu_1\rho}{\delta}\right)e^{-\delta t}, \tag{2.20}$$

$$E(\lambda_t) = \frac{\mu_1\rho}{\delta}. \tag{2.21}$$

The higher moments can be obtained by differentiating them further, that is,

$$\begin{aligned}\operatorname{Var}(\lambda_t \mid \lambda_0) &= (1-e^{-2\delta t})\frac{\mu_2\rho}{2\delta},\\ \operatorname{Var}(\lambda_t) &= \frac{\mu_2\rho}{2\delta},\end{aligned} \tag{2.22}$$

where $\mu_2 = E(Y^2) = \int_0^\infty y^2 dG(y)$.

3. The distribution of the interval between events of a Cox process with shot noise intensity and its moment

Let us examine the Laplace transform of the distribution of the shot noise intensity at claim times. To do so, let us denote the time of the nth claim of N_t by τ_n and denote the value of λ_t, when N_t takes the value n for the first time by λ_{τ_n}. Since a claim occurs at time τ, this implies that the intensity at claim times, λ_τ, should be higher than the intensity at any times

λ_t. Therefore the distribution of λ_τ should not be the same as the distribution of λ_t, which will be clear from Theorem 3.2.

Let us start with the following lemma in order to obtain the Laplace transform of the distribution of the shot noise intensity at claim times. We assume that the claims and jumps (or primary events) in shot noise intensity do not occur at the same time.

Lemma 3.1. *Let N_t be a Cox process with shot noise intensity λ_t. Let A be the generator of the process λ_t and suppose that $f(\lambda)$ is a function belonging to its domain and furthermore that it satisfies*

$$\lim_{t\to\infty} E\left\{ f(\lambda_t)\exp\left(-\int_0^t \lambda_s\, ds\right) \mid \lambda_0 \right\} = 0. \tag{3.1}$$

If $h(\lambda)$ is such that

$$\lambda\{h(\lambda) - f(\lambda)\} + A f(\lambda) = 0 \tag{3.2}$$

then

$$E\{h(\lambda_{\tau_1}) \mid \lambda_0\} = f(\lambda_0). \tag{3.3}$$

Proof. From (3.2)

$$f(\lambda_t) + \int_0^t [\lambda_s\{h(\lambda_s) - f(\lambda_s)\}]\, ds \tag{3.4}$$

is a martingale and since τ_1^t is a stopping time, where $\Pr(\tau_1 \le s) = \Pr(N_s > 0)$ and N_s is λ_s-measurable, we have

$$E f\{(\lambda_{\tau_1^t} \mid \lambda_0)\} + E\left[\int_0^{\tau_1^t} [\lambda_s\{h(\lambda_s) - f(\lambda_s)\}]\, ds \mid \lambda_0\right] = f(\lambda_0). \tag{3.5}$$

Conditioning on the realisation λ_v, $0 \le v \le t$, τ_1^t is distributed with density

$$\lambda_r \exp\left(-\int_0^r \lambda_u\, du\right) \tag{3.6}$$

on $(0,t)$ and a mass $\exp(-\int_0^t \lambda_u\, du)$ at t. Hence,

$$E\{f(\lambda_{\tau_1^t}) \mid \lambda_v,\ 0 \le v \le t\} = \int_0^t \left\{ f(\lambda_r)\lambda_r \exp\left(-\int_0^r \lambda_u\, du \right) \right\} dr + f(\lambda_t) \exp\left(-\int_0^t \lambda_u\, du \right), \tag{3.7}$$

$$\begin{aligned} E&\left[\int_0^{\tau_1^t} \lambda_s\{h(\lambda_s) - f(\lambda_s)\} ds \mid \lambda_v,\ 0 \le v \le t\right] \\ &= \int_0^t \int_0^r \lambda_s\{h(\lambda_s) - f(\lambda_s)\} ds\, \lambda_r \exp\left(-\int_0^r \lambda_u\, du \right) dr \\ &\quad + \int_0^t \lambda_s\{h(\lambda_s) - f(\lambda_s)\} ds \exp\left(-\int_0^t \lambda_u\, du \right). \end{aligned} \tag{3.8}$$

Changing the order of integration on the first term of this, it becomes

$$\begin{aligned} &= \int_0^t \int_s^t \lambda_r \exp\left(-\int_0^r \lambda_u\, du \right) dr \lambda_s\{h(\lambda_s) - f(\lambda_s)\} ds \\ &\quad + \int_0^t \lambda_s\{h(\lambda_s) - f(\lambda_s)\} ds \exp\left(-\int_0^t \lambda_u\, du \right) \\ &= \int_0^t \left\{ \exp\left(-\int_0^s \lambda_u\, du \right) - \exp\left(-\int_0^t \lambda_u\, du \right) \right\} \lambda_s\{h(\lambda_s) - f(\lambda_s)\} ds \\ &\quad + \int_0^t \lambda_s\{h(\lambda_s) - f(\lambda_s)\} ds \exp\left(-\int_0^t \lambda_u\, du \right) \\ &= \int_0^t \exp\left(-\int_0^s \lambda_u\, du \right) \lambda_s\{h(\lambda_s) - f(\lambda_s)\} ds. \end{aligned} \tag{3.9}$$

Adding (3.7) and (3.9), we notice that more terms cancel and we get

$$\begin{aligned} E&\left\{ f(\lambda_{\tau_1^t}) + \int_0^{\tau_1^t} \lambda_s\{h(\lambda_s) - f(\lambda_s)\} ds \mid \lambda_v,\ 0 \le v \le t \right\} \\ &= \int_0^t \exp\left(-\int_0^s \lambda_u\, du \right) \lambda_s h(\lambda_s) ds + f(\lambda_t) \exp\left(-\int_0^t \lambda_u\, du \right) \\ &= E\{h(\lambda_{\tau_1}) 1_{\{\tau_1 \le t\}} \mid \lambda_v,\ 0 \le v \le t\} + f(\lambda_t) \exp\left(-\int_0^t \lambda_u\, du \right), \end{aligned} \tag{3.10}$$

and hence

$$E\left\{f(\lambda_{\tau_1^t}) + \int_0^{\tau_1^t} \lambda_s\{h(\lambda_s) - f(\lambda_s)\}ds \mid \lambda_0\right\} \\ = E\left\{(h(\lambda_{\tau_1})1_{\{\tau_1\le t\}}) + f(\lambda_t)\exp\left(-\int_0^t \lambda_u\, du\right) \mid \lambda_0\right\}. \tag{3.11}$$

From (3.5), we then have

$$E\left\{(h(\lambda_{\tau_1})1_{\{\tau_1\le t\}}) + f(\lambda_t)\exp\left(-\int_0^t \lambda_u\, du\right) \mid \lambda_0\right\} = f(\lambda_0) \tag{3.12}$$

and setting $t \to \infty$, we get (3.3). □

Assuming that the shot noise process λ_t is stationary, let us derive the Laplace transform of the distribution of the shot noise process at claim times, λ_τ.

Theorem 3.2. *If the shot noise process λ_t is stationary, the Laplace transform of the distribution of the shot noise process at claim times is given by*

$$E(e^{-\nu\lambda_{\tau_i}}) = \frac{\widehat{\mathbf{G}}(\nu)}{\mu_1}\cdot \exp\left\{-\frac{\rho}{\delta}\int_0^\nu \widehat{\mathbf{G}}(u)du\right\}, \tag{3.13}$$

where $\widehat{\mathbf{G}}(u) = (1-\widehat{g}(u))/u$ and $\widehat{g}(u) = \int_0^t e^{-uy}dG(y)$.

Proof. From Lemma 3.1, which implies that if $f(\lambda)$ and $h(\lambda)$ are such that

$$\lambda\{h(\lambda) - f(\lambda)\} - \delta\lambda f'(\lambda) + \rho\left\{\int_0^\infty f(\lambda+y)dG(y) - f(\lambda)\right\} = 0 \tag{3.14}$$

and (3.1) is satisfied, we have

$$E\{h(\lambda_{\tau_{i+1}}) \mid \lambda_{\tau_i}\} = f(\lambda_{\tau_i}) \tag{3.15}$$

by starting the process from τ_i. Employing $f(\lambda) = \{\lambda - \widehat{g}'(\nu)/(1-\widehat{g}(\nu))\}e^{-\nu\lambda}$, the function $f(\lambda)$ clearly satisfies (3.1) and substituting into (3.14), then we have

$$\lambda\left\{h(\lambda) - \lambda e^{-\nu\lambda} + \frac{\widehat{g'}(\nu)}{1-\widehat{g}(\nu)}e^{-\nu\lambda}\right\} + \delta\nu\lambda\left\{\lambda - \frac{\widehat{g'}(\nu)}{1-\widehat{g}(\nu)}\right\}e^{-\nu\lambda} - \delta\lambda e^{-\nu\lambda} = -\rho\lambda e^{-\nu\lambda}\{\widehat{g}(\nu) - 1\}. \tag{3.16}$$

Divide by λ and simplify then we have

$$h(\lambda) = \lambda e^{-\nu\lambda}(1-\delta\nu) + \delta e^{-\nu\lambda} - (1-\delta\nu)\frac{\widehat{g'}(\nu)}{1-\widehat{g}(\nu)}e^{-\nu\lambda} + \rho e^{-\nu\lambda}\{1-\widehat{g}(\nu)\}. \tag{3.17}$$

From (3.15), it is given that

$$E\{h(\lambda_{\tau_{i+1}})\} = E\left[E\{h(\lambda_{\tau_{i+1}}) \mid \lambda_{\tau_i}\}\right] = E\{f(\lambda_{\tau_i})\}. \tag{3.18}$$

So put (3.17) into (3.18), then

$$\begin{aligned} E\Bigg[&\lambda_{\tau_{i+1}} \exp(-\nu\lambda_{\tau_{i+1}})(1-\delta\nu) + \delta \exp(-\nu\lambda_{\tau_{i+1}}) - (1-\delta\nu)\frac{\widehat{g'}(\nu)}{1-\widehat{g}(\nu)} \\ &\times \exp(-\nu\lambda_{\tau_{i+1}}) + \rho\exp(-\nu\lambda_{\tau_{i+1}})\{1-\widehat{g}(\nu)\}\Bigg] \\ &= E\left\{\lambda_{\tau_i}\exp(-\nu\lambda_{\tau_{i+1}}) - \frac{\widehat{g'}(\nu)}{1-\widehat{g}(\nu)}\exp(-\nu\lambda_{\tau_{i+1}})\right\}. \end{aligned} \tag{3.19}$$

When the process λ_t is stationary, $\lambda_{\tau_{i+1}}$, and λ_{τ_i} have the same distribution whose Laplace transform we denote by $H(\nu) = E(e^{-\nu\lambda_{\tau_i}})$. Therefore from (3.19), we have

$$-(1-\delta\nu)H'(\nu) - (1-\delta\nu)\frac{\widehat{g'}(\nu)}{1-\widehat{g}(\nu)}H(\nu) + \left[\delta+\rho\{1-\widehat{g}(\nu)\}\right]H(\nu) = -H'(\nu) - \frac{\widehat{g'}(\nu)}{1-\widehat{g}(\nu)}H(\nu). \tag{3.20}$$

Divide both sides of (3.20) by $\delta\nu$, then we have

$$H'(\nu) + \frac{\widehat{g'}(\nu)}{1-\widehat{g}(\nu)}H(\nu) + \left\{\frac{1}{\nu} + \frac{\rho}{\delta}\frac{1-\widehat{g}(\nu)}{\nu}\right\}H(\nu) = 0. \tag{3.21}$$

Solving (3.21), subject to

$$H(0) = 1 \tag{3.22}$$

then the Laplace transform of a distribution of the shot noise process at claim times is given by

$$H(\nu) = K\left(\frac{1-\widehat{g}(\nu)}{\nu}\right)\exp\left\{-\frac{\rho}{\delta}\int_0^\nu \widehat{\mathbf{G}}(u)\,du\right\}, \tag{3.23}$$

where K is a constant. Therefore from (3.22), $K = 1/\mu_1$ and

$$H(\nu) = \frac{1}{\mu_1}\frac{1-\widehat{g}(\nu)}{\nu}\cdot\exp\left\{-\frac{\rho}{\delta}\int_0^\nu \widehat{\mathbf{G}}(u)\,du\right\} = \frac{\widehat{\mathbf{G}}(\nu)}{\mu_1}\cdot\exp\left\{-\frac{\rho}{\delta}\int_0^\nu \widehat{\mathbf{G}}(u)\,du\right\}. \tag{3.24}$$

□

Equation (3.24) provides us with an interesting result. The distribution defined by the Laplace transform (3.24) (and (3.13)) is the same as the distribution of two random variables;

one having the stationary distribution of λ_t (see Corollary 2.4) and the other having density $\mathbf{G}(y)/\mu_1$, where $\mathbf{G}(y) = 1-G(y)$. Comparing it with the distribution of the shot noise process, λ_t at any times, we can easily find that

$$\frac{\widehat{\mathbf{G}}(\nu)}{\mu_1}\cdot \exp\left\{-\frac{\rho}{\delta}\int_0^{\nu}\widehat{\mathbf{G}}(u)du\right\} > \exp\left\{-\frac{\rho}{\delta}\int_0^{\nu}\widehat{\mathbf{G}}(u)du\right\}. \tag{3.25}$$

It is therefore the case that λ_τ is stochastically larger than λ_t. In other words, the intensity at claim times is higher than the intensity at any times.

Now let us derive the distribution of the interval of a Cox process with shot noise intensity for insurance claims using Theorem 3.2.

Corollary 3.3. *Assume that 0 is the time at which a claim of N_t has occurred and the stationary of λ_t has been achieved. Then the tail of the distribution of the interval of a Cox process with shot noise intensity is given by*

$$\Pr(\tau > t) = \frac{\widehat{\mathbf{G}}(1/\delta - (1/\delta)e^{-\delta t})}{\mu_1}\exp\left\{-\frac{\rho}{\delta}\int_0^{t}\widehat{\mathbf{G}}\left(\frac{1}{\delta}-\frac{1}{\delta}e^{-\delta s}\right)ds\right\}. \tag{3.26}$$

Proof. From (2.16), the probability generating function of N_t is given by

$$E(\theta^{N_t} \mid \lambda_0) = \exp\left\{-\frac{1-\theta}{\delta}(1-e^{-\delta t})\lambda_0\right\}\exp\left[-\rho\int_0^{t}\left[1-\hat{g}\left\{\frac{1-\theta}{\delta}(1-e^{-\delta s})\right\}\right]ds\right]. \tag{3.27}$$

Set $\theta = 0$ in (3.27) and take expectation, then the tail of the distribution of τ is given by

$$\Pr(\tau > t) = \exp\left[-\rho\int_0^{t}\left\{1-\hat{g}\left(\frac{1-e^{\delta s}}{\delta}\right)\right\}ds\right]E\left[\exp\left\{-\frac{(1-e^{-\delta t})}{\delta}\lambda_0\right\}\right]. \tag{3.28}$$

Substitute (3.13) into (3.28), then the result follows immediately as 0 is the time at which a claim has occurred and λ_t is stationary. □

Corollary 3.4. *The expectation and variance of the interval between claims are given by*

$$E(\tau) = \int_0^{\infty}\Pr(\tau > t)dt = \frac{\delta}{\mu_1\rho}, \tag{3.29}$$

$$Var(\tau) = 2\int_0^{\infty}\left[u\frac{\widehat{\mathbf{G}}(1/\delta - (1/\delta)e^{-\delta u})}{\mu_1}\exp\left\{-\frac{\rho}{\delta}\int_0^{u}\widehat{\mathbf{G}}\left(\frac{1}{\delta}-\frac{1}{\delta}e^{-\delta s}\right)ds\right\}\right]du - \left(\frac{\delta}{\mu_1\rho}\right)^2. \tag{3.30}$$

Proof. Integrate (3.26), then (3.29) follows. (3.30) is obtained from

$$E(\tau^2) = \int_0^\infty t^2 f(t)dt = 2\int_0^\infty \left[u \frac{\widehat{\mathbf{G}}(1/\delta - (1/\delta)e^{-\delta u})}{\mu_1} \exp\left\{ -\frac{\rho}{\delta}\int_0^u \widehat{\mathbf{G}}\left(\frac{1}{\delta} - \frac{1}{\delta}e^{-\delta s}\right)ds\right\}\right]du. \tag{3.31}$$

□

An interesting result we can find from (3.29) and (2.21) is that the expected interval between claims is the inverse of the expected number of claims, where the number of claims follows a Cox process with shot noise intensity, which is also the case for a Poisson process.

4. Conclusion

We started with deriving the probability generating function of a Cox process with shot noise intensity, employing piecewise deterministic Markov processes theory. It was necessary to obtain the distribution of the shot noise process at claim times as it is not the same as the distribution of the shot noise process at any times. Assuming that the shot noise process is stationary, we derived the distribution of the interval of a Cox process with shot noise intensity for insurance claims and its moments from its probability generating function. The result of this paper can be used or easily modified in computer science/telecommunications modeling, electrical engineering, and queueing theory as an alternative counting process to a Poisson process.

Appendix

This appendix explains the basic definition of a piecewise deterministic Markov process (PDMP) that is adopted from [20]. A detailed discussion can also be found in [18, 24].

PDMP is a Markov process X_t with two components (η_t, ξ_t), where η_t takes values in a discrete set K and given $\eta_t = n \in K$, ξ_t takes values in an open set $M_n \subset \Re^{d(n)}$ for some function $d : K \to N$. The state space of X_t is equal to $E = \{(n, z) : n \in K, z \in M_n\}$. We further assume that for every point $x = (n, z) \in E$, there is a unique, deterministic integral curve $\phi_n(t, z) \subset M_n$, determined by a differential operator χ_n on $\Re^{d(n)}$, such that $z \in \phi_n(t, z)$. If for some $t_0 \in \Re^+$, $X_{t_0} = (n_0, z_0) \in E$, then ξ_t, where $t \geq t_0$ follows $\phi_{n_0}(t, z_0)$ until either $t = T_0$, some random time with hazard rate of function ρ or until $\xi_t = \partial M_{n_0}$, the boundary of M_{n_0}. In both cases, the process X_t jumps, according to a Markov transition measure Q on E, to a point $(n_1, z_1) \in E$. ξ_t again follows the deterministic path ϕ_{n_1} till a random time T_1 (independent of T_0) or till $\xi_t = \partial M_{n_1}$, and so forth. The jump times T_i are assumed to satisfy the following condition:

$$\forall t > 0, \quad E\left(\sum_i I(T_i \leq t)\right) < \infty. \tag{A.1}$$

The stochastic calculus that will enable us to analyse various models rests on the notion of (extended) generator A of X_t. Let Γ denotes the set of boundary points of E, $\Gamma = \{(n, z) : n \in K, z \in \partial M_n\}$, and let A be an operator acting on measurable functions $f : E \cup \Gamma \to \Re$ satisfying the following.

(i) The function $t \to f(n, \phi_n(t, z))$ is absolutely continuous for $t \in [0, t(n, z)]$ for all $(n, z) \in E$.

(ii) For all $x \in \Gamma$, $f(x) = \int_E f(y)Q(x;dy)$ (boundary condition).

(iii) For all $t \geq 0$, $E\{\sum_{T_i \leq t} |f(X_{T_i}) - f(X_{T_i-})|\} < \infty$.

Hence, the set of measurable functions satisfying (i), (ii), and (iii) form a subset of the domain of the extended generator A, denoted by $D(A)$. Now, for piecewise deterministic Markov processes, we can explicitly calculate A by [18, Theorem 5.5]

$$\forall f \in D(A): \quad Af(x) = \chi f(x) + \rho(x)\int_E \{f(y) - f(x)\}Q(x;dy). \tag{A.2}$$

In some cases, it is important to have time t as an explicit component of the PDMP. In those cases A can be decomposed as $\partial/\partial t + A_t$, where A_t is given by (A.2) with possibly time-dependent coefficients.

An application of Dynkin's formula provides us with the following important result (martingales will always be with respect to the natural filtration $\sigma\{X_s : s \leq t\}$).

(a) If for all t, $f(\cdot, t)$ belongs to the domain of A_t and $(\partial/\partial t)f(x,t) + A_t f(x,t) = 0$, then process $f(X_t, t)$ is a martingale.

(b) If f belongs to the domain of A and $Af(x) = 0$, then $f(X_t)$ is a martingale.

The generator of the process X_t acting on a function $f(X_t)$ belonging to its domain as described above is also given by

$$Af(X_t) = \lim_{h\downarrow 0} \frac{E\{f(X_{t+h}) \mid X_t = x\} - f(X_t)}{h}. \tag{A.3}$$

In other words, $Af(X_t)$ is the expected increment of the process X_t between t and $t+h$, given the history of X_t at time t. From this interpretation the following inversion formula is plausible, that is,

$$E[f(X_{t+h}) \mid X_t = x] - f(X_t) = \int_0^h E\{Af(X_s)\}ds \tag{A.4}$$

which is Dynkin's formula.

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