Lie algebras: infinite generalizations and deformations

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LIE ALGEBRAS:
Infinite Generalizations and Deformations

by

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Abstract

There are many applications of Lie algebras to theoretical physics. This thesis is a study of some new mathematical structures which also are applicable to current physical ideas. The structures studied are Lie algebras of infinite dimension and the deformations of Lie algebras known as quantum algebras. The approach is algebraic, although physical applications are indicated.

Chapter 1

The mathematics of finite and infinite dimensional Lie algebras is reviewed, together with an indication of well established uses in physics. The terms and notation used in the rest of the thesis are introduced.

Chapter 2

Explicit examples of new infinite dimensional algebras of a type related to the algebras of conformal transformations on arbitrary genus Riemann surfaces are given. The relationship of these algebras to the Virasoro algebra is discussed.

Chapter 3

The sine algebra is introduced and its relationship to the Moyal bracket discussed. The finite Lie algebras are given in a trigonometric basis. The many applications of the Moyal algebra are reviewed.

Chapter 4

An original proof of the uniqueness of the Moyal algebra is presented. It is shown that the Moyal bracket is the most general Lie bracket of functions of two variables, and thus that the underlying associative star product is unique. It follows that all 2-index Lie algebras correspond to the Moyal algebra in some basis.

Chapter 5

Quantum deformations of Lie algebras, or quantum algebras, are introduced. The many deformations of $su(2)$ are described and the associativity conditions are discussed. Some new higher dimensional and infinite dimensional quantum algebras are given.

Chapter 6

Quantum groups are discussed as groups of transformations of the quantum plane. Higher dimensional quantum groups and quantum supergroups are also described.

Preface

This thesis is the result of work carried out in the Department of Mathematical Sciences at the University of Durham, between October 1987 and September 1990, under the supervision of Dr. D. B. Fairlie. No part of it has been previously submitted for any degree, either in this or any other university.

No claim of originality is made for the material presented in chapter 1, which consists of a review of well established work, or for the review parts of chapter 5. The rest of the material in the thesis is original.

The work in chapter 2 was done in collaboration with David Fairlie and Jean Nuyts and has been published.[1]

The work in chapter 3 was done in collaboration with David Fairlie and Cosmas Zachos and has been published.[2, 3] In particular, sections 3.6, 3.7, 3.8 contain the work of the author.

The proof given in chapter 4 is the work of the author, and has been accepted for publication.[4]

Chapter 5 consists partly of a review of quantum algebras, with some new examples. The viewpoint and the new examples are the result of discussions with David Fairlie.

The work in chapter 6 was done in collaboration with Ed Corrigan, David Fairlie and Ryu Sasaki and has been published.[5]

Acknowledgements

First and foremost I wish to thank David Fairlie, my supervisor. Much of the work described in this thesis was done in collaboration with him, a collaboration I have very much enjoyed. Many thanks also go to Cosmas Zachos for stimulating discussions and for making it possible for me to attend the Argonne Quantum Group Workshop. I have gained much from collaboration and discussions with Jean Nuyts, Ed Corrigan and Ryu Sasaki, and from the other staff and students in the Durham Mathematics department, in particular Hisham Zainuddin. I am grateful to the University of Durham Research Studentship fund for financial support for the last three years.

I would also like to thank my parents for their support, and my friends in Durham and elsewhere for making my time here so enjoyable. A long list of names seems inappropriate; they are the people with whom I shared homes, those who provided tea and friendship, and those to whom I escaped Durham when it all got too much. And special thanks to Becky for her encouragement in the painful final weeks.

Contents

1. Introduction

The ancient astronomers held sacred the perfect geometrical shapes of the circle and the sphere, and today scientists still believe that symmetry is fundamental to the physical laws governing the universe. Symmetries are described mathematically by group theory; the groups associated with continuous symmetries, such as those of the circle and sphere, are called Lie groups. The symmetry group of the circle is the group of rotations in two dimensions, SO(2), which may be represented by 2×2 matrices of the form

$$A(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} .$$

These matrices are closed under multiplication, which corresponds to composition of rotations, as

$$A(\theta)A(\phi) = A(\theta + \phi) .$$

A more useful mathematical structure is the Lie algebra of infinitesimal rotations. In two dimensions this is $so(2)$, generated by the matrix

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .$$

The Lie group corresponding to a Lie algebra may be obtained by exponentiating the generators of the algebra; here,

$$A(\theta) = e^{\theta\epsilon} .$$

The symmetry group of the sphere is SO(3), and is non-Abelian — that is, the same two rotations performed in a different order may have a different result. The corresponding Lie algebra of infinitesimal rotations in three dimensions is $so(3)$. This is the algebra satisfied by the angular momentum operators, and has long been used in physics. More recently hadrons were seen to have the equivalent algebra $su(2)$ as an internal symmetry, called isospin. This led to a breakthrough in the understanding of hadronic physics; the introduction of quarks and an $su(3)$ symmetry and the formulation of the theory of quantum chromodynamics. This generalization of $su(2)$ to $su(3)$ is one example of a new mathematical idea shedding light on a well established physical problem, and shows the fundamental importance of Lie algebras for theoretical physics.

The 'standard model' of particle physics is firmly based on the idea that the interactions between particles are governed by Lie algebras, specifically the algebra $su(2) \times u(1)$ for the electromagnetic and weak forces, and $su(3)$ for the strong force. The theory of relativity is simply a statement that space-time has an $so(3,1)$ symmetry, which is the symmetry generated by Lorentz transformations. Lie algebras are also of great importance to quantum theory, where the phase space variables satisfy the Heisenberg algebra, and Dirac's quantization procedure involves replacing Poisson brackets by commutators. Quantum mechanics may also be formulated in terms of another Lie bracket, the Moyal bracket, which is discussed in chapter 3.

The major questions in particle theory today are those of unification, of the existence of some theory predicting the standard model with the correct experimental results for its parameters, and of possible theories unifying the standard model and gravity. Many of the theories put forward, if not all of them, are based on some new larger symmetry given by some Lie algebra. The simplest candidate for a 'grand unified theory' has the symmetry group $su(5)$, as this is the smallest simple Lie algebra that contains $su(3) \times su(2) \times u(1)$ as a subalgebra. The theory of supersymmetry posits a new symmetry between fermions and bosons, and is described by Lie superalgebras. Making this symmetry local requires the introduction of spin-2 fields, and there was a hope that this 'supergravity' theory would describe the gravitational force as well as the standard model. However, its predictions do not fit with experimental data.

Superstring theories and conformal field theories have been introduced as 'theories of everything', and they have as a symmetry an infinite dimensional Lie algebra, the Virasoro algebra. There are many different string theories; one example is the heterotic string, which has gauge group $E_8 \times E_8$, a very large symmetry which contains that of the standard model. There are also intriguing correspondences between rational conformal field theories and Lie algebras.

New infinite Lie algebras are arising in many areas of theoretical physics. There are the conformal algebras on higher genus Riemann surfaces; infinite generalizations of the classical finite Lie algebras used in $su(\infty)$ Yang-Mills theories, which have connections to string theory; the conformal algebras of higher spin, whose infinite limit is a Lie algebra; and more.

Other ideas suggest that the symmetries of the real world are not described by a Lie algebra, but by a deformation of one. These structures which describe perturbed symmetries are commonly referred to as 'quantum groups', more accurately as quantum algebras or Yang-Baxter algebras, and come into the mathematician's category of Hopf algebras. They possess one or more deformation parameters, in some limit of which they reduce to a Lie algebra — this is often thought of as the classical limit of some quantum mechanical structure. Quantum groups have already found applications in two dimensional solvable models, anisotropic spin chains, three dimensional Chern-Simons theory, rational conformal field theories and fractional statistics.

1.1 Finite Dimensional Lie Algebras

A Lie algebra L is an algebra over a field F whose product, denoted by [,], is bilinear, antisymmetric and satisfies the Jacobi identity,

$$\begin{aligned} [x+y,z] &= [x,z]+[y,z] \ , \\ [ax,y] &= a[x,y] \ , \\ [x,y] &= -[y,x] \ , \\ [[x,y],z]+[[y,z],x]+[[z,x],y] &= 0 \ , \end{aligned} \tag{1.1}$$

where $a \in F$ and $x, y, z \in L$.[6–9]

Given an associative algebra it is possible to build a Lie algebra from the same set of elements but with a new product, commutation, the commutator [,] being given in terms of the associative product $*$ by

$$[x,y] = x*y - y*x \ . \tag{1.2}$$

It is straightforward to verify the axioms (1.1) for an algebra built in this way.

A Lie algebra may be defined by the multiplication table of its basis elements under the bracket operation. For a d dimensional algebra with a basis τ^a, where $a \in \{1,\ldots,d\}$, these 'commutation relations' may be written⋆

$$[\tau^a, \tau^b] = f_c^{a,b}\tau^c \ ,$$

where the $f_c^{a,b}$ are the 'structure constants', members of the field over which the algebra is taken. The d matrices of dimension d defined by $(\mathrm{ad}\,\tau^a)_{c,b} = f_c^{a,b}$ form a basis for

⋆ repeated indices imply summation throughout this thesis, unless otherwise stated the 'adjoint representation' of the algebra. A 'representation' of a Lie algebra is a set of matrices, such as these, which satisfy the commutation relations, so that

$$[\mathrm{ad}\,\tau^a, \mathrm{ad}\,\tau^b] = \mathrm{ad}([\tau^a, \tau^b]) \ .$$

The associative product underlying the commutator for a matrix representation is simply the matrix product. Under this product the matrices will generate an associative algebra, which will have the dimension the same as or higher than the original Lie algebra, called the 'enveloping associative algebra' of the Lie algebra.

It is possible to define a metric, a bilinear, basis invariant form, for a Lie algebra, the 'Killing form', in terms of the adjoint representation and the matrix product,

$$\langle x, y \rangle = \mathrm{Tr}(\mathrm{ad}\,x \;\; \mathrm{ad}\,y) \ .$$

A 'simple' Lie algebra is one with no proper ideals, a 'semi-simple' Lie algebra has no Abelian ideals. Thus simplicity implies semi-simplicity. An equivalent definition of semi-simplicity is that the Killing form is non-degenerate. The semi-simple Lie algebras are the ones of particular interest to the physicist, usually over the complex or real numbers. Any semi-simple Lie algebra may be written as the direct sum of its ideals, *i.e.* as a direct sum of simple Lie algebras.

An important example is the algebra of 2×2 matrices over the complex numbers, for which a convenient basis is that of Hermitian matrices

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ , \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ ,$$

which satisfy†

$$\begin{aligned} \mathbb{1}\mathbb{1} &= \mathbb{1} \ , \\ \mathbb{1}\sigma_j = \sigma_j \mathbb{1} &= \sigma_j \ , \\ \sigma_j \sigma_k &= \delta_{jk} \mathbb{1} + i\epsilon_{jk}{}^{l} \sigma_l \ , \end{aligned}$$

and so are closed under matrix product, and all 2×2 matrices may be defined as linear combinations of these four. This is a four (complex) dimensional associative algebra. Now

† δ and ϵ are the Kronecker and Levi-Civita symbols respectively

under commutation this is a basis for the Lie algebra of 2×2 matrices over the complex numbers,

$$[\mathbb{1}, \mathbb{1}] = 0 \ ,$$
$$[\mathbb{1}, \sigma_j] = 0 \ ,$$
$$[\sigma_j, \sigma_k] = 2i\epsilon_{jk}{}^l \sigma_l \ ,$$

called $gl(2, \mathbb{C})$, or $u(2, \mathbb{C})$. This is clearly a direct sum of two algebras, one spanned by the σ matrices, consisting of all traceless 2×2 matrices, $sl(2, \mathbb{C})$ or $su(2, \mathbb{C})$, and the other the one dimensional Abelian algebra $u(1, \mathbb{C})$ represented by multiples of the identity. These algebras may be generalized to N dimensional matrices, giving $gl(N, \mathbb{C}) \equiv u(1, \mathbb{C}) \times sl(N, \mathbb{C})$.

The unique three dimensional complex Lie algebra is $sl(2, \mathbb{C})$; this has two distinct real forms, *i.e.* real Lie algebras for which it is the complexification. The three dimensional real Lie algebras are $sl(2, \mathbb{R})$, real traceless 2×2 matrices, and $su(2)$, skew-Hermitian traceless 2×2 matrices. Inclusion of the identity element leads to $gl(2, \mathbb{R})$ and $u(2)$. A convenient basis for $su(2)$ is $\tau_j = -\frac{i}{2}\sigma_j$, so that

$$[\tau_j, \tau_k] = \epsilon_{jk}{}^l \tau_l \ ,$$

and the algebra is closed over the reals. This is the 'compact' real form of $sl(2, \mathbb{C})$, the unique real form with negative definite Killing form — equivalently, its corresponding group is compact. For every complex Lie algebra there is precisely one compact real form.

This example may also be used to illustrate a change of basis. Any three linearly independent complex combinations of the σ matrices may be used as a basis for the algebra $sl(2, \mathbb{C})$, such as

$$\sigma_+ = \sigma_1 + i\sigma_2 \ , \quad \sigma_- = \sigma_1 - i\sigma_2 \ , \quad \sigma_3 \ ,$$

which satisfy the algebra

$$[\sigma_3, \sigma_\pm] = \pm 2\sigma_\pm \ ,$$
$$[\sigma_+, \sigma_-] = 4\sigma_3 \ .$$

This new basis is the standard basis, or Cartan-Weyl basis for $sl(2)$, which is diagonal, in that σ_3 acts on a basis element to give something proportional to that element. As the structure constants are real, this is a basis for both $sl(2, \mathbb{C})$ and $sl(2, \mathbb{R})$.

Any Lie algebra over the complex numbers has a basis of this form, with real structure constants, so that *one* of its real forms also has this basis. In fact, Chevalley has shown that this basis may be written with integral structure constants.[10] In general the basis is built by taking a maximal mutually commuting subalgebra, the Cartan subalgebra, H, whose dimension, n, is the 'rank' of the algebra. As these elements commute, their adjoint representations $\operatorname{ad} h_j$, $h_j \in H$, are simultaneously diagonalizable over $\mathbb{C}$. Then there is a basis for the other elements, e_α, such that

$$[h, e_\alpha] = \alpha(h) e_\alpha \ ,$$

where α is called a root. The commutators between different e's may be calculated using the Jacobi identity,

$$[h, [e_\alpha, e_\beta]] + [e_\alpha, [e_\beta, h]] + [e_\beta, [h, e_\alpha]] = 0 \ ,$$

so that

$$[h, [e_\alpha, e_\beta]] = (\alpha(h) + \beta(h))[e_\alpha, e_\beta] \ ,$$

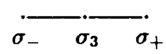
which means that $[e_\alpha, e_\beta] = e_{\alpha+\beta}$ is also a basis element, or $\beta = -\alpha$ and $[e_\alpha, e_\beta] \in H$, or $[e_\alpha, e_\beta] = 0$. The roots can be thought of as vectors in some space, Σ, with the Cartan elements lying at the origin, and then commutation of two basis elements has result proportional to the vector sum of the corresponding roots. If the vector sum lies outside the root space, the structure constant must be zero and the elements commute.

Thus the form of this basis for a general algebra is:

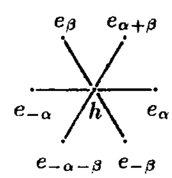
$$\begin{aligned}
[e_\alpha, e_\beta] &= \begin{cases} N_{\alpha\beta} e_{\alpha+\beta} & \text{if } \alpha+\beta \in \Sigma \\ \langle e_\alpha, e_{-\alpha} \rangle h_\alpha & \text{if } \alpha + \beta = 0 \\ 0 & \text{otherwise} \end{cases} \\
[h, e_\alpha] &= \alpha(h) e_\alpha \\
[h_j, h_k] &= 0
\end{aligned}$$

As any complex Lie algebra has a basis which may be constructed in this way, a classification of allowable root spaces amounts to a classification of finite simple Lie algebras over the complex numbers. The constraints on the root spaces come from the Jacobi identity, and are quite severe — in particular the only allowed angles between roots are integer multiples of $\pi/12$. This classification was achieved by Killing and Cartan before the end of the 19^{th} century.[11, 12] There are two useful constructions from the root diagram, the Cartan matrix and the Dynkin diagram,[13] whose classification in turn is equivalent to that of complex simple Lie algebras.

The root diagram for Σ has the Cartan subalgebra at the origin and a vector for each root, such as for $sl(2)$:



The algebra of $sl(3)$ is eight dimensional, with a two dimensional Cartan subalgebra. Its root diagram is:



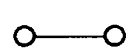
The root space may be generated by its 'simple' roots, of which there are n, the rank of the algebra. This is the dimension of the root space. For example, for $sl(3)$ above the simple roots are α and β. The Cartan matrix, A, for any root space is defined as

$$A_{jk} = \frac{2\langle \alpha_j, \alpha_k \rangle}{\langle \alpha_j, \alpha_j \rangle} ,$$

where α_j is a simple root, so for $sl(3)$ this is

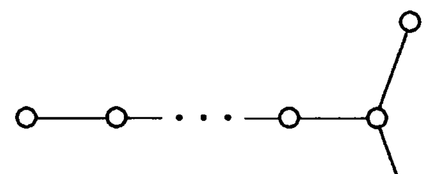
$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} .$$

For every Cartan matrix there is a corresponding Dynkin diagram, which consists of a vertex for each simple root α_j, with a weight $\langle \alpha_j, \alpha_j \rangle$, with $A_{jk}A_{kj}$ lines between vertices α_j and α_k. In fact there are only two possible weights, these are denoted by white and black blobs, and the number of lines may only be $0, 1, 2, 3$, these numbers coming from the (few) allowed angles between the root vectors. The Dynkin diagram for $sl(3)$ is:

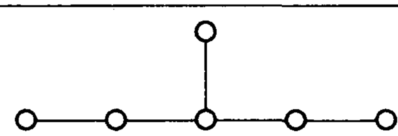
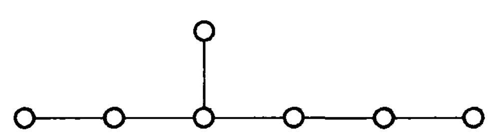
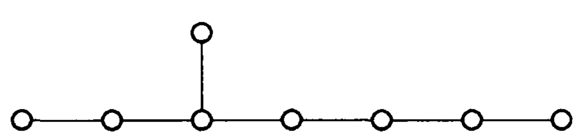
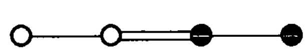
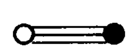


The conditions on allowable root spaces translate to allowable Cartan matrices and Dynkin diagrams, which may then be classified. The result of the classification in terms of Dynkin diagrams is given in the following tables; there are four infinite series and five exceptional cases, all labelled by their rank, n. Also given are the fundamental matrix representations, dimension N, the physicist's names for the algebras and their dimensions.

The classical algebras:

	matrices, (N)	dimension	Dynkin diagram
A_n	traceless, $sl(n+1)$	$n(n+2) = N^2 - 1$	
B_n	antisym, $so(2n+1)$	$n(2n+1) = \frac{1}{2}N(N-1)$	
C_n	symplectic,* $sp(2n)$	$n(2n+1) = \frac{1}{2}N(N+1)$	
D_n	antisym, $so(2n)$	$n(2n-1) = \frac{1}{2}N(N-1)$	 (n blobs)

The exceptional algebras:

	dimension	Dynkin diagram
E_6	78	
E_7	133	
E_8	248	
F_4	52	
G_2	14	

Matrix representations have been found for all the exceptional algebras. Notice that D_1 is the one dimensional Abelian algebra, $A_1 \cong B_1 \cong C_1$, $B_2 \cong C_2$, $D_2 \cong A_1 \times A_1$ and $D_3 \cong A_3$.

$\star$ A symplectic matrix M satisfies $M^T J + JM = 0$ where J is block diagonal with blocks $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

For every complex Lie algebra there is a unique compact real form, for $sl(N)$ this is $su(N)$.* These are of interest as they correspond to unitary groups, or algebras of Hermitian matrices. The compact form of $sp(N)$ is $usp(N)$, represented by Hermitian symplectic matrices. There are non-compact forms of $so(N)$, written $so(p,q)$, $p+q=N$, which have metric with signature (p,q). In a similar way, the non-compact real form of $su(2)$, $sl(2,\mathbb{R})$ is sometimes written $su(1,1)$.

There is another useful basis for $gl(N)$, the 'physicist's basis', defined in terms of the matrices E_{jk}, which are zero but for a 1 in the j,k position. These satisfy the algebra

$$[E_{jk}, E_{lm}] = \delta_{kl} E_{jm} - \delta_{jm} E_{lk} \ .$$

There is a corresponding basis for $sl(N)$, taking $H_j = E_{j,j} - E_{j+1,j+1}$ as the (traceless) Cartan elements.

A Casimir operator is an operator which commutes with every element of the algebra. It is not actually an element of the Lie algebra, but of the enveloping associative algebra, as it is defined in terms of products of the elements of the Lie algebra. For example, in $su(2)$ there is a second order Casimir operator defined in terms of the σ matrices,

$$C_2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 \ .$$

A Lie superalgebra is an algebra with two types of operators, usually referred to as bosonic and fermionic. The bosonic operators B_i satisfy an ordinary Lie algebra. As well as these there are fermionic operators, which commute with a bosonic operator to regain a fermionic operator. The fermionic operators themselves satisfy anticommutation relations; the anticommutator $\{\ ,\ \}$ is a symmetric product given in terms of the underlying associative product by $\{A,B\} = A*B + B*A$ (see (1.2)). Two fermionic operators yield a bosonic operator on anticommutation, so the general relations are

$$\begin{aligned} [B_i, B_j] &= c_{ij}^k B_k \ , \\ [B_i, F_j] &= d_{ij}^k F_k \ , \\ \{F_i, F_j\} &= f_{ij}^k B_k \ . \end{aligned}$$

$\star$ In the physics literature, and indeed elsewhere in this thesis, $su(N)$ is often used to mean its complexification, A_{N-1}.

A superalgebra must satisfy several Jacobi-type identities,

$$\begin{aligned}
[[B_i, B_j], B_k] + [[B_j, B_k], B_i] + [[B_k, B_i], B_j] &= 0 \ , \\
[\{F_i, F_j\}, B_k] + [\{F_j, F_k\}, B_i] + [\{F_k, F_i\}, B_j] &= 0 \ , \\
\{[B_i, F_j], F_k\} + \{[B_j, F_k], F_i\} + \{[B_k, F_i], F_j\} &= 0 \ ,
\end{aligned}$$

which follow from the associativity of the underlying product.

The simplest example of a Lie superalgebra is $su(1|1)$, which has four generators, two bosonic, B and I, and two fermionic, $A_\pm$, satisfying

$$\begin{aligned}
[B, I] &= 0 \ , \\
[B, A_\pm] &= \pm A_\pm \ , \\
[I, A_\pm] &= 0 \ , \\
\{A_+, A_-\} &= B \ .
\end{aligned}$$

1.2 Infinite Dimensional Lie Algebras

Symmetry groups of algebraic or geometric objects of physical interest are not necessarily finite dimensional. This inspires the study of infinite dimensional Lie algebras, which have an infinite number of independent operators. These may only be represented by matrices of infinite dimension, or by differential operators.

Indeed, one starting point for infinite dimensional algebras is the consideration of an algebra of differential operators. The algebra of vector fields on the circle, Diff(S^1) is the Virasoro algebra.[14, 15] Here the infinite dimensional group is that of smooth one-to-one maps $S^1 \to S^1$, with group multiplication defined by composition. The representation of the corresponding Lie algebra obtained by considering the action of infinitesimal elements on functions of z is given by the differential operators

$$L_m = z^{1-m} \frac{\mathrm{d}}{\mathrm{d}z} \ .$$

These are closed under commutation, satisfying the algebra

$$[L_m, L_n] = (m - n) L_{m+n} \ . \tag{1.3}$$

The Jacobi identity follows from the associativity of the differential operators. The algebra

(1.3) is the centreless version of the Virasoro algebra,

$$[L_m, L_n] = (m-n)L_{m+n} + \tfrac{c}{12}m(m^2-1)\delta_{m+n,0} \ . \tag{1.4}$$

This is the algebra of the conformal group in one or two dimensions. The central element c satisfies $[L_m, c] = 0$, and in most physical applications this is a constant real number. The central extension comes from Dirac's quantization procedure, where the Poisson bracket describing the classical system is replaced by a commutator, and an extra term of order $\hbar$ is introduced on the right hand side — this is the central term. The Jacobi identity requires that it has the form given in (1.4).

This algebra is $\mathbb{Z}$-graded, with each grade one dimensional, *i.e.* the basis elements, or 'generators', have an integer index and commute to give something labelled by the sum of their indices.

Another important family of infinite dimensional Lie algebras are the affine Lie algebras, or Kac-Moody algebras,[16–18] also reviewed in [15]. Given a d dimensional Lie algebra with commutation relations

$$[\tau^a, \tau^b] = f_c^{a,b}\tau^c \ ,$$

a second index may be introduced, this time an integer, and the resulting 'Kac-Moody' algebra

$$[\tau_m^a, \tau_n^b] = f_c^{a,b}\tau_{m+n}^c \ ,$$

satisfies the axioms for a Lie algebra, and is infinite dimensional. These algebras are also '$\mathbb{Z}$-graded', with d dimensional grades. As for the Virasoro algebra, there is a central extension,

$$[\tau_m^a, \tau_n^b] = f_c^{a,b}\tau_{m+n}^c + km\delta^{ab}\delta_{m+n,0} \ .$$

Infinite dimensional Lie algebras remain unclassified. It is difficult even to classify the simplest type of such algebras,

$$[L_m, L_n] = C_{m,n}L_{m+n} \ , \tag{1.5}$$

$\mathbb{Z}$-graded algebras with one dimensional grades. The Jacobi identity imposes a condition on the structure constants $C_{m,n}$, explicitly

$$C_{m,n}C_{m+n,p} + C_{n,p}C_{n+p,m} + C_{p,m}C_{p+m,n} = 0 \ .$$

The algebra (1.3) is of the form of (1.5), and so one solution for $C_{m,n}$ is simply $m-n$. There is a further solution,

$$C_{m,n} = \frac{\alpha_m \alpha_n}{\alpha_{m+n}}(m-n) \ ,$$

the algebra here is equivalent to the Virasoro algebra up to a normalization of the generators, as the arbitrary function α_m can be absorbed by redefining L_m to L_m/α_m.

Kaplansky[19] has shown that this is the only solution of (1.5) with certain of the structure constants non-vanishing, specifically $C_{0,1}, C_{1,-1}, C_{2,-1}, C_{-2,1}$. Fairlie, Nuyts and I[1] conjectured that this is the only solution for $C_{m,n}$ a ratio of multinomials in m, n. In fact Mathieu[20] has classified simple $\mathbb{Z}$-graded Lie algebras of growth ≤ 1, which includes the case discussed here, and has some other results for those of finite growth.

One interesting approach to the classification of Lie algebras is that of a presentation. A presentation of an algebra is the definition of its generators in terms of certain initial generators, and a few conditions on their commutators. This has been done for the Virasoro algebra.[21, 22] In this case all the L's may be defined in terms of just two generators, L_3 and L_{-2}. The imposition of the Jacobi identity then requires the structure constants to be $m-n$ as long as six extra conditions are imposed.

There is in fact another solution of (1.5) for the structure constants,[2]

$$C_{m,n} = \sin\tfrac{2\pi}{3}(m-n) \ ,$$

which avoids the hypotheses of the above proofs, and does not satisfy the extra conditions in the presentation. The corresponding algebra is

$$[K_m, K_n] = \tfrac{2}{\sqrt{3}} \sin\tfrac{2\pi}{3}(m-n)K_{m+n} + c\tfrac{m+1}{3}\delta_{m+n,0} \ .$$

This algebra is isomorphic to the $su(2)$ Kac-Moody algebra through the identification

$$K_{3m} = \tau^0_m \ , \quad K_{3m+1} = \tau^-_m \ , \quad K_{3m-1} = \tau^+_m \ .$$

The τ's are graded, but with three generators at each level. Each level is conceptually the slat of an open Venetian blind; on closing the blind the generators spread out evenly, so there is now one at each level, and the levels are closer together — this corresponds to the K's.

Kac[18] was aware that the $su(2)$ Kac-Moody algebra could be written as a $\mathbb{Z}$-graded algebra; that the structure constants could be written as an analytic function was discovered by Fairlie, Zachos and myself.[2] The infinite dimensional Cartan subalgebra consists of the generators K_{3m}, and there are $su(2)$ subalgebras generated by $K_{3m+1}, K_{3m-1}, -K_0 + cm$, for all m. Note also the parity automorphism $K_m \to -K_{-m}$, $c \to -c$. There is a presentation for this algebra in terms of K_5, K_{-2}, K_{-3}.[23]

Infinite dimensional Lie algebras may be supersymmetrized, for example the Virasoro algebra has two possible supersymmetric generalizations, the Ramond and the Neveu-Schwarz algebras,[24, 25] given by

$$
\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \tfrac{c}{12}(m^3 - m)\delta_{m+n,0} \ , \\
[L_m, F_a] &= (\tfrac{1}{2}m - a)F_{m+a} \ , \\
\{F_a, F_b\} &= 2L_{a+b} + \tfrac{c}{12}(4a^2 - 1)\delta_{a+b,0} \ ,
\end{aligned}
$$

where the indices a, b run over the integers for the Ramond case and the half-integers in the Neveu-Schwarz case.

2. Generalized-graded Algebras

New infinite dimensional Lie algebras may be discovered by relaxation of the property of $\mathbb{Z}$-grading held by the algebras discussed in the introduction. An alternative, weaker, condition on the commutator is that of 'generalized-grading'. Generalized-graded algebras are similar to the Virasoro algebra as they have generators indexed by an integer, with one generator for each integer. However, the commutator of any two generators is proportional to not just one other, but a combination of r others, of general form

$$[N_m, N_n] = \sum_{k=0}^{r-1} C^k(m,n) N_{m+n-r+1+2k} \ . \tag{2.1}$$

The work described here was done in collaboration with David Fairlie and Jean Nuyts.[1] We constructed examples of generalized-graded algebras under certain conditions on the $C^k(m,n)$, with the aim of a classification of these algebras, working from a completely algebraic point of view. We conjectured that all such algebras may be expressed in terms of linear combinations of Virasoro generators, and so either they are equivalent to the full Virasoro algebra, or they are an infinite subalgebra of it. Evidence for this conjecture is given in the final section of this chapter.

2.1 Introduction

Generalized-graded algebras were first introduced by Krichever and Novikov,[26,27] as an algebraic extension of conformal invariance on an arbitrary genus Riemann surface, with applications to string theory and theories of solitons. This leads to algebras of the form (2.1) with $r = 3g + 1$, where g is the genus of the surface. The work of Krichever and Novikov has been taken up by many others.[28–34]

The algebras discussed by Krichever and Novikov have a complicated form, with structure constants given in terms of elliptic functions, so at first sight it is surprising to find that it is possible to express their generators, at least in the torus case, as linear combinations of the generators of the Virasoro algebra,[33] and vice-versa, implying that this algebra is actually the Virasoro algebra in a different basis.

The case of generalized-grading with $r = 1$,

$$[N_m, N_n] = C(m,n) N_{m+n} \ ,$$

is equivalent to $\mathbb{Z}$-grading, so the Virasoro algebra is a generalized-graded algebra, with

$$C(m,n) = (m-n) \ .$$

This algebra describes conformal invariance on the sphere (genus 0). This was the starting point of Krichever and Novikov in their generalization. The uniqueness of this solution to the structure constants is discussed in chapter 1.

A generalized-graded algebra with r terms on the right hand side in the commutation relation (2.1) will be described as an r-term algebra. For $r > 1$ there are many different possible solutions of the Jacobi identities. Determining these is an extremely difficult non-linear problem — the number of such identities which arise in the r-term case is $2r - 1$.

Within the conjecture that the operators N_m may be re-expressed as a finite sum of contiguous even or odd Virasoro generators L_m,

$$N_m = \sum_{k=0}^{r-1} a^k(m) L_{m-r+1+2k} \ , \tag{2.2}$$

the problem may be transmuted to a simpler, though still difficult, form. The conditions for N_m to generate a generalized-graded algebra are still $2r - 1$ equations, but the r unknown functions $a^k(m)$ depend on one variable rather than two. If the relationship (2.2) can be inverted to express the L's in terms of a sum (possibly semi-infinite) of the N's then the algebra is equivalent to the full Virasoro algebra, otherwise it is an infinite sub-algebra. These algebras may have central terms. These will be automatically induced by a suitable representation of the Virasoro generators.

These algebras may be categorized according to the degree of m in the $a^k(m)$. The constraints for a closed algebra become more complicated as this increases. There is a convenient representation of the Virasoro generators,

$$L_m = z^{1-m}\frac{\mathrm{d}}{\mathrm{d}z} .$$

Within this representation (2.2) becomes a power series in z,

$$N_m = \sum_{k=0}^{r-1} a^k(m) z^{r-1-2k} z^{1-m}\frac{\mathrm{d}}{\mathrm{d}z} . \tag{2.3}$$

There are no constraints for the case of constant $a^k(m)$. However for the cases of $a^k(m)$ linear and quadratic in m, there are eigenvalue conditions for the parameters, the solutions of which exhibit remarkable regular integral sequences.

2.2 Constant Coefficients

The generalized-graded algebras for which the $a^k(m)$ are independent of m correspond to the Virasoro case — the structure constants $C^k(m,n)$ are just $(m-n)$ up to multiplicative constants. Indeed it is clear from the following theorem that all algebras of this form may be expressed as a sum of L's.

Theorem 1. *If the coefficients $a^k(m)$ in (2.2) do not depend on m then the N_m form an algebra satisfying (2.1) with $C^k(m,n) = (m-n)a^k$.*

Proof: If the $a^k(m)$ are constants, (2.3) becomes

$$\begin{aligned} N_m &= \sum_{k=0}^{r-1} a^k z^{r-1-2k} z^{1-m}\frac{\mathrm{d}}{\mathrm{d}z} \\ &= g(z) z^{1-m}\frac{\mathrm{d}}{\mathrm{d}z} , \end{aligned}$$

defining $g(z)$ as a power series. When forming the commutator $[N_m, N_n]$ terms symmetric in m, n cancel, leaving only the part where z^{1-n} is differentiated.

$$\begin{aligned} [N_m, N_n] &= (m-n)\left(g(z)\right)^2 z^{1-m-n}\frac{\mathrm{d}}{\mathrm{d}z} \\ &= (m-n)\sum_{k=0}^{r-1} a^k g(z) z^{1-(m+n-r+1+2k)}\frac{\mathrm{d}}{\mathrm{d}z} \\ &= (m-n)\sum_{k=0}^{r-1} a^k N_{m+n-r+1+2k} . \end{aligned}$$

Comparing with (2.1) gives the result.

Alternatively, if $p(z)$ is an elliptic or hyperelliptic function of z satisfying $(p'(z))^2 = g(p(z))$ where $g(p)$ is a polynomial of finite degree in p, then

$$N_m = p'(z)(p(z))^{1-m-r+1}\frac{\mathrm{d}}{\mathrm{d}z} \tag{2.4}$$

$$= g(p)p^{1-m-r+1}\frac{\mathrm{d}}{\mathrm{d}p} . \tag{2.5}$$

Commuting (2.4) gives a sum of N's. If $g(p)$ is a polynomial in p^2, the N_m satisfy (2.1). Putting $L_m = p^{1-m}\frac{\mathrm{d}}{\mathrm{d}p}$ makes (2.5) equivalent to (2.2), with constant coefficients. Nuyts and Platten[35] have found a presentation for these algebras.

2.3 Linear Coefficients

If the $a^k(m)$ depend on m the problem is much harder. When forming the commutator, $g(z)$ is a function of m so there are more antisymmetric terms and the above proof for the constant case no longer applies. Under the assumption that N_m is a sum of L's, the Jacobi identities are automatically satisfied. The constraints on the $a^k(m)$ come from the closure requirement; the result of commuting two such sums of L's must be expressible as a sum of N's:

$$\begin{array}{ccc} [N,N] & \stackrel{?}{=} & \Sigma N \\ \downarrow & & \uparrow \\ [\Sigma L, \Sigma L] & \rightarrow & \Sigma L \end{array}$$

Searching for solutions to this problem with the help of the computer algebra package REDUCE led to the following ansatz for algebras with linear $a^k(m)$.

$$N_m = \left((m+f)z + (m-f)\tfrac{1}{z}\right)\left(z+\tfrac{1}{z}\right)^{r-2} z^{1-m}\frac{\mathrm{d}}{\mathrm{d}z} , \tag{2.6}$$

which gives a closed algebra for certain values of the parameter f. Further experiments with REDUCE indicate that all such algebras may be expressed as (2.6) or as a generalization given in a later section. Note that this is invariant up to sign under a 'parity automorphism' $z \mapsto -\frac{1}{z}$, $m \mapsto -m$. This is the remnant of the automorphisms of the Virasoro algebra generated by $L_m \mapsto \pm\frac{1}{\lambda}L_{\lambda m}$. The original algebra studied by Krichever and Novikov has a different normalization for the N's. In their papers[26, 27] the normalization is chosen so that the end structure constants are given by

$$C^{r-1}(m,n) = (m-n) ,$$
$$C^{0}(m,n) = (m-n)\frac{\alpha(m)\alpha(n)}{\alpha(m+n-r+1)} .$$

The conditions of parity invariance are then

$$N_{-m} = \mp\alpha(-m)N_m \quad \text{and} \quad \alpha(m)\alpha(-m) = \mp 1 .$$

These equations imply that the general structure of $\alpha(m)$ is given by

$$\alpha(m) = iA^m \prod_j \frac{m-\beta_j}{m+\beta_j} ,$$

where the β_j and A are constants. The ansatz (2.6) corresponds to the case where there is only one factor in the product. The merit of this ansatz is that the $2r-1$ conditions for closure reduce to $r+1$ linear equations for r unknowns, giving a single consistency requirement. The commutator may be calculated,

$$[N_m, N_n] = (m-n)\Big\{\big((m+f)(n+f)z + (m-f)(n-f)\tfrac{1}{z}\big)\left(z+\tfrac{1}{z}\right) - 4f(f-1)\Big\} \times \left(z+\tfrac{1}{z}\right)^{2r-4} z^{1-m-n}\frac{\mathrm{d}}{\mathrm{d}z} ,$$

or, as a power series in z,

$$[N_m, N_n] = (m-n)\sum_{k=0}^{r}\begin{pmatrix} \binom{r-1}{k}(m+f)(n+f) \\ +\binom{r-1}{k-1}(m-f)(n-f) \\ -\binom{r-2}{k-1}4f(f-1) \end{pmatrix} z^{r-2k}\left(z+\tfrac{1}{z}\right)^{r-2} z^{1-m-n}\frac{\mathrm{d}}{\mathrm{d}z} . \tag{2.7}$$

Equation (2.1) may be also be expressed as a power series in z, using the same ansatz for N_m (2.6), and rearranging the summation

$$[N_m, N_n] = \sum_{k=0}^{r}\begin{pmatrix} C^k(m,n)(m+n-r+2k+1+f) \\ +\,C^{k-1}(m,n)(m+n-r+2k-1-f) \end{pmatrix} z^{r-2k}\left(z+\tfrac{1}{z}\right)^{r-2} z^{1-m-n}\frac{\mathrm{d}}{\mathrm{d}z} , \tag{2.8}$$

where $C^{-1}(m,n) = C^{r}(m,n) = 0$.

The set of $r+1$ equations obtained by equating coefficients in (2.7) and (2.8) may be written as a determinant which must be zero for all m, n for a closed algebra. This determinantal equation may be solved; the general case may be illustrated by explicit calculation of the case $r=5$. First remove the factor of $(m-n)$, and define

$$\begin{aligned} \lambda =&(m+f)(n+f)\ , \\ \mu =&(m-f)(n-f)\ , \\ \nu =&-4f(f-1)\ , \end{aligned} \tag{2.9}$$

which will be treated as arbitrary parameters in the first stage of the analysis. Then for $r=5$ the determinant is:

$$\begin{vmatrix} m+n-4+f & 0 & 0 & 0 & 0 & \binom{4}{0}\lambda \\ m+n-4-f & m+n-2+f & 0 & 0 & 0 & \binom{4}{1}\lambda+\binom{4}{0}\mu+\binom{3}{0}\nu \\ 0 & m+n-2-f & m+n+f & 0 & 0 & \binom{4}{2}\lambda+\binom{4}{1}\mu+\binom{3}{1}\nu \\ 0 & 0 & m+n-f & m+n+2+f & 0 & \binom{4}{3}\lambda+\binom{4}{2}\mu+\binom{3}{2}\nu \\ 0 & 0 & 0 & m+n+2-f & m+n+4+f & \binom{4}{4}\lambda+\binom{4}{3}\mu+\binom{3}{3}\nu \\ 0 & 0 & 0 & 0 & m+n+4-f & \binom{4}{4}\mu \end{vmatrix}$$

This may be reduced by row and column operations. Put $R1' = R1 - R2 + R3 - R4 + \cdots$ Then it is evident that the combinatorial factors in the last column cancel and that $f=0$ is an eigenvalue. Then perform $C5' = C5 + C4$, $C4' = C4 + C3$, *etc.*, giving:

$$\begin{vmatrix} 2f & 0 & 0 & 0 & 0 & 0 \\ m+n-4-f & 2m+2n-6 & m+n-2+f & 0 & 0 & \binom{4}{1}\lambda+\binom{4}{0}\mu+\binom{3}{0}\nu \\ 0 & m+n-2-f & 2m+2n-2 & m+n+f & 0 & \binom{4}{2}\lambda+\binom{4}{1}\mu+\binom{3}{1}\nu \\ 0 & 0 & m+n-f & 2m+2n+2 & m+n+2+f & \binom{4}{3}\lambda+\binom{4}{2}\mu+\binom{3}{2}\nu \\ 0 & 0 & 0 & m+n+2-f & 2m+2n+6 & \binom{4}{4}\lambda+\binom{4}{3}\mu+\binom{3}{3}\nu \\ 0 & 0 & 0 & 0 & m+n+4-f & \binom{4}{4}\mu \end{vmatrix}$$

And finally $R5' = R5 - R6$, $R4' = R4 - R5'$, ..., yielding:

$$\begin{vmatrix} 2f & 0 & 0 & 0 & 0 & 0 \\ m+n-4-f & m+n-4+f & 0 & 0 & 0 & \binom{3}{0}\lambda \\ 0 & m+n-2-f & m+n-2+f & 0 & 0 & \binom{3}{1}\lambda+\binom{3}{0}\mu+\binom{2}{0}\nu \\ 0 & 0 & m+n-f & m+n+f & 0 & \binom{3}{2}\lambda+\binom{3}{1}\mu+\binom{2}{1}\nu \\ 0 & 0 & 0 & m+n+2-f & m+n+2+f & \binom{3}{3}\lambda+\binom{3}{2}\mu+\binom{2}{2}\nu \\ 0 & 0 & 0 & 0 & m+n+4-f & \binom{3}{3}\mu \end{vmatrix}$$

This is $2f$ times a 5×5 sub-determinant which is equivalent to the determinant in the

4-term case with f shifted to $f+1$. Thus the 5-term matrix has a zero at $f=0$, and also at zeros of the 4-term case shifted by one. It is clear that the same row and column operations will work for the r-term case, which has zeros at $f=0, f'+1$ where the f' are the zeros of the $(r-1)$-term case. The 3-term determinant has a single zero at $f=0$, completing the inductive proof that the r-term case has zeros at $f=0,1,\dots,r-3$, regardless of λ, μ, and ν. It should be noted when $f=0$ this reduces to the constant coefficient case. Explicit calculation of the 3-term determinant with λ, μ, and ν as in (2.9) show that the case with $r=3$ has extra zeros at $f=1,2$. These zeros do not carry over in the induction above as the shift in f removes them.

This family may be extended by noting that the only properties of the binomial coefficients $\binom{n}{i}$ used in the above proof are that

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}=0 \qquad \text{and} \qquad \binom{n-1}{i-1}+\binom{n-1}{i}=\binom{n}{i}\ .$$

There are other sets of coefficients that satisfy these conditions — those in the expansion of $\left(z+\frac{1}{z}\right)^{n-q}\left(z-\frac{1}{z}\right)^q$, which we shall refer to as $\binom{n}{i}^{(q)}$. When $q=0$ these reduce to the ordinary binomial coefficients. Note that

$$\binom{n}{n-i}^{(q)}=(-1)^q\binom{n}{i}^{(q)}\ ,$$

i.e. the parity automorphism still holds, and also that

$$\binom{n-1}{i-1}^{(q-1)}-\binom{n-1}{i}^{(q-1)}=\binom{n}{i}^{(q)}\ .$$

This allows a new ansatz,

$$N_m=\left((m+f)z+(-1)^p(m-f)\tfrac{1}{z}\right)\left(z+\tfrac{1}{z}\right)^{r-2-q}\left(z-\tfrac{1}{z}\right)^q z^{1-m}\frac{\mathrm{d}}{\mathrm{d}z}\ , \tag{2.10}$$

which has three parameters, r, q ($q\le r-2$) and p ($p=0,1$), and there is a very similar inductive proof that the corresponding algebra is closed for the following values of f:

$$f=\begin{cases}0 & \text{if } p=0, q=r-2\\ 0,1,2 & \text{if } p=0, q=r-3\\ 0,1,\dots,r-3-q & \text{if } p=0, q>r-3\\ 0,1,\dots,q-1 & \text{if } p=1\end{cases}$$

The solution (2.10) has positive parity if $r-1-q+p$ is even.

Abandoning the parity requirement allows more general solutions with two free parameters, *e.g.* in the four term case,

$$N_m = (m+f)L_{m+3} + c(m+f+\tfrac{2}{3})L_{m+1} + \tfrac{c^2}{3}(m+f+\tfrac{4}{3})L_{m-1} + \tfrac{c^3}{27}(m+f+2)L_{m-3} \; .$$

This is simply a transformation of the previous solution, of the form

$$L_m \mapsto 3c^{m/2} L_{m+f+2} \; .$$

The parity requirement is tantamount to the imposition of unitarity or anti-unitarity. Thus the eigenvalue condition plays a similar rôle to the restrictions on the c-number for unitary representations of the Virasoro algebra found by Belavin, Polyakov and Zamolodchikov[36] and Friedan, Qiu and Shenker.[37]

2.4 Quadratic Coefficients

This form of construction generalizes to second order in m. Again there is a basic ansatz which respects the parity operation:

$$N_m = \left((m^2+am+b)z^2 + 2(m^2+c) + (m^2-am+b)\tfrac{1}{z^2}\right)\left(z+\tfrac{1}{z}\right)^{r-3} z^{1-m}\frac{\mathrm{d}}{\mathrm{d}z} \; . \quad (2.11)$$

In this case there are two eigenvalue equations, as there are $r+2$ linear equations for the r structure constants. For the simplest example, $c=b$, one if the eigenvalue equations is solved automatically, and the ansatz reduces to

$$N_m = \left((m^2+am+b)z + (m^2-am+b)\tfrac{1}{z}\right)\left(z+\tfrac{1}{z}\right)^{r-2} z^{1-m}\frac{\mathrm{d}}{\mathrm{d}z} \; , \quad (2.12)$$

with only one eigenvalue equation remaining. Note that if a is zero the ansatz (2.12) is trivial as the $C^k(m,n)$ are constant, since the N_m can be renormalized by dividing by the factor (m^2+b). Also if b is zero it reduces to the linear case by dividing N_m by m and with a replaced by f, so it is no surprise that for this case a must satisfy the same conditions as f. For the general case (2.11), the linear equations which must be solved for the structure constants take the form

$$\begin{pmatrix} F(m+n-r+1+2j)C^j(m,n) \\ +H(m+n-r-1+2j)C^{j-1}(m,n) \\ +G(m+n-r-3+2j)C^{j-2}(m,n) \end{pmatrix} = (m-n)\sum_{i=0}^{4}\binom{r-3}{j-i}R_i \ ,$$

where

$$\begin{aligned} F(m) &= m^2+am+b \ , \\ H(m) &= 2(m^2+c) \ , \\ G(m) &= m^2-am+b = F(-m) \ , \\ R_0 &= F(m)F(n) \ , \\ R_1 &= F(m)H(n)+F(n)H(m)+\tfrac{2}{(m-n)}(H(m)F(n)-H(n)F(m)) \ , \\ R_2 &= H(m)H(n)+F(m)G(n)+F(n)G(m)+\tfrac{4}{(m-n)}(F(n)G(m)-F(m)G(n)) \ , \\ R_3 &= H(m)G(n)+H(n)G(m)+\tfrac{2}{(m-n)}(H(n)G(m)-H(m)G(n)) \ , \\ R_4 &= G(m)G(n) \ , \end{aligned}$$

and where $j=0,\ldots,r+1$ and $C^{-2}(m,n)=C^{-1}(m,n)=C^r(m,n)=C^{r+1}(m,n)=0$.

The conditions for these equations to admit a non-trivial solution are that the following $(r+2)\times(r+1)$ matrix is of rank r for all m,n,

$$\begin{pmatrix} F(s-r+1) & 0 & 0 & \cdots & 0 & \sum_{i=0}^4\binom{r-3}{0-i}R_i \\ H(s-r+1) & F(s-r+3) & 0 & \cdots & 0 & \sum_{i=0}^4\binom{r-3}{1-i}R_i \\ G(s-r+1) & H(s-r+3) & F(s-r+5) & \cdots & 0 & \sum_{i=0}^4\binom{r-3}{2-i}R_i \\ 0 & G(s-r+3) & H(s-r+5) & \cdots & 0 & \sum_{i=0}^4\binom{r-3}{3-i}R_i \\ 0 & 0 & G(s-r+5) & \cdots & 0 & \sum_{i=0}^4\binom{r-3}{4-i}R_i \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & G(s+r-1) & \sum_{i=0}^4\binom{r-3}{r+1-i}R_i \end{pmatrix} ,$$

where $s=m+n$. This problem may be solved by using row and column operations to introduce two rows of zeroes, which puts constraints on a,b,c.

It proves convenient to define the functions

$$\begin{aligned} \Delta(k,x) &= kG(x-2)+H(x)-kG(x) \ , \\ \Phi(k) &= F(x)-\Delta(k,x+2k)+G(x+2k) \ , \\ \Theta(k,x) &= \Delta(k,x)-2G(x) \ . \end{aligned}$$

Note that Φ is independent of x, and may be written as

$$\Phi(k) = 4k^2 - 4k - 4ak + 2b - 2c \ , \tag{2.13}$$

and also that $\Delta(0, x) = H(x)$.

Now subtract each row from the one preceding it, starting at the bottom of the matrix, and work upwards:

$$\begin{pmatrix} \Phi(0) & -\Phi(0) & \cdots & \pm\Phi(0) & 0 \\ H(s-r+1) - G(s-r+1) & \Phi(0) & \cdots & \mp\Phi(0) & \sum_{i=0}^{4} \binom{r-4}{0-i} R_i \\ G(s-r+1) & H(s-r+3) - G(s-r+3) & \cdots & \pm\Phi(0) & \sum_{i=0}^{4} \binom{r-4}{1-i} R_i \\ 0 & G(s-r+3) & \cdots & \mp\Phi(0) & \sum_{i=0}^{4} \binom{r-4}{2-i} R_i \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & G(s+r-1) & \sum_{i=0}^{4} \binom{r-4}{r-i} R_i \end{pmatrix}$$

Next add columns, starting with the penultimate one and adding the one before, working from right to left. Then repeat the previous operation of subtracting rows, this time stopping at the second row:

$$\begin{pmatrix} \Phi(0) & 0 & 0 & \cdots & 0 & 0 \\ \Theta(0, s-r+1) & \Phi(1) & -\Phi(1) & \cdots & \pm\Phi(1) & 0 \\ G(s-r+1) & \begin{matrix}\Delta(1, s-r+3) \\ -G(s-r+3)\end{matrix} & \Phi(1) & \cdots & \mp\Phi(1) & \sum_{i=0}^{4} \binom{r-5}{0-i} R_i \\ 0 & G(s-r+3) & \begin{matrix}\Delta(1, s-r+5) \\ -G(s-r+5)\end{matrix} & \cdots & \pm\Phi(1) & \sum_{i=0}^{4} \binom{r-5}{1-i} R_i \\ 0 & 0 & G(s-r+5) & \cdots & \mp\Phi(1) & \sum_{i=0}^{4} \binom{r-5}{2-i} R_i \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & G(s+r-1) & \sum_{i=0}^{4} \binom{r-5}{r-1-i} R_i \end{pmatrix}$$

Repeat the previous operations r times, each time operating on one fewer row and column. The top part of the resulting matrix is then:

$$\begin{pmatrix} \Phi(0) & 0 & 0 & 0 & \cdots & 0 \\ \Theta(0, s-r+1) & \Phi(1) & 0 & 0 & \cdots & 0 \\ G(s-r+1) & \Theta(1, s-r+3) & \Phi(2) & 0 & \cdots & 0 \\ 0 & G(s-r+3) & \Theta(2, s-r+5) & \Phi(3) & \cdots & 0 \\ 0 & 0 & G(s-r+5) & \Theta(3, s-r+7) & \cdots & 0 \\ 0 & 0 & 0 & G(s-r+7) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix}$$

and the bottom part is:

$$
\begin{pmatrix}
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & \Phi(r-3) & 0 & 0 & R_0 - R_1 + R_2 - R_3 + R_4 \\
0 & \cdots & \Theta(r-3, s+r-5) & \Phi(r-2) & 0 & R_1 - 2R_2 + 3R_3 - 4R_4 \\
0 & \cdots & G(s+r-5) & \Theta(r-2, s+r-3) & \Phi(r-1) & R_2 - 3R_3 + 6R_4 \\
0 & \cdots & 0 & G(s+r-3) & \Theta(r-1, s+r-1) & R_3 - 4R_4 \\
0 & \cdots & 0 & 0 & G(s+r-1) & R_4
\end{pmatrix}
$$

All the solutions (*i.e.* values of c, a and b which give this matrix rank r, corresponding to closed algebras) yet found are such that two of the Φ's are zero, which may be labelled $\Phi(k)$ and $\Phi(k+l)$. If none of the Φ's are zero it is easy to prove that $r = 4$. For $r > 4$ the possible algebras (2.11) all fit into a parameterization of c, a, b in terms of k, l, and j, given by:

$$
\left.\begin{array}{l}
c = b - 2k(k+l) \\[2ex]
a = 2k + l - 1 \\[2ex]
b = \begin{cases} (k+j)(k+l-j-1) & l \text{ even} \\ \text{arbitrary} & l \text{ odd} \end{cases}
\end{array}\right\}
\quad
\begin{array}{ll}
\text{where} & \begin{cases} k = 0, \ldots, r-5 \\ l = 1, \ldots, r-4-k \end{cases} \\[4ex]
\text{or} & \begin{cases} k = 0: & l = r-3^*, r-2^\dagger \\ k = 1: & l = r-4^* \\ k = 2: & l = r-4^\ddagger \end{cases}
\end{array}
$$

$$
\text{and where} \quad j = 0, \ldots, \tfrac{1}{2}l - 1 \quad (l \text{ even}) ,
$$

the marks *, †, ‡ indicating special cases which will be explained later.

If $\Phi(k) = 0$ for some $k \leq r-5$, then row $k+1$ is a linear combination of the previous rows, assuming that k is the smallest zero of Φ. Row $k+l+1$ is also a linear combination of previous rows, provided that $\Phi(k+l) = 0$ (where $k+l \leq r-4$) and that the determinant made from rows $(k+2), \ldots, (k+l+1)$ and columns $(k+1), \ldots, (k+l)$ is zero. For example, the top two rows are all zero if $\Phi(0) = 0$, $\Phi(1) = 0$ and $\Theta(0, s-r+1) = 0$, the determinant in this case being just one element.

Now, if $\Phi(k) = \Phi(k+l) = 0$, then a, c and the other Φ's are fixed,

$$
\begin{aligned}
a &= 2k + l - 1 \ , \\
c &= b - 2k(k+l) \ , \\
\Phi(k+j) &= 4j(j-l) \ .
\end{aligned}
$$

The size of the determinant is $l \times l$, and it is of the form:

$$
\begin{vmatrix}
\Theta(k,x) & \Phi(k+1) & 0 & \cdots & 0 & 0 \\
G(x) & \Theta(k+1,x+2) & \Phi(k+2) & \cdots & 0 & 0 \\
0 & G(x+2) & \Theta(k+2,x+4) & \cdots & 0 & 0 \\
0 & 0 & G(x+4) & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & G(x+2l-4) & \Theta(k+l-1,x+2l-2)
\end{vmatrix}
$$

If l is odd then the above determinant is automatically zero (independently of b). By elementary row and column operations the determinant may be rewritten as an antisymmetric one of odd dimension and hence vanishes. Otherwise, for even l, the determinant vanishes if

$$
b = (k+j)(k+l-j-1) \qquad \text{for} \quad j = 0, 1, \ldots, \tfrac{1}{2}l - 1 \ . \tag{2.14}
$$

The $l = 6$ case serves as an example of the general principle. Two further substitutions will simplify the expressions,

$$
b' = b - k(k+l-1) \qquad \text{and} \qquad u = 2(k-x) - 3 \ .
$$

The determinant becomes:

$$
\begin{vmatrix}
5(u-3) & -20 & 0 & 0 & 0 & 0 \\
\frac{1}{4}(u-13)(u-3)+b' & 3u+17 & -32 & 0 & 0 & 0 \\
0 & \frac{1}{4}(u-9)(u+1)+b' & u+33 & -36 & 0 & 0 \\
0 & 0 & \frac{1}{4}(u-5)(u+5)+b' & -(u-33) & -32 & 0 \\
0 & 0 & 0 & \frac{1}{4}(u-1)(u+9)+b' & -(3u-17) & -20 \\
0 & 0 & 0 & 0 & \frac{1}{4}(u+3)(u+13)+b' & -5(u+3)
\end{vmatrix}
$$

All the lower diagonal elements may be made proportional to b' by column operations. When this is done the top left element turns out to be zero, so that if $b' = 0$, the determinant is zero. The determinant may then be restored to a similar continuant form to the above by row operations. Repeat the column operations on the last four columns, so that the lower diagonal entries have a factor $(b'-4)$, and again a zero appears on the diagonal (in the $(3,3)$ position). Restore the continuant form again, by row operations on the last three rows. Finally repeat the column and row operations on the last two rows and columns, giving:

$$\begin{vmatrix} 0 & -20 & 0 & 0 & 0 & 0 \\ b' & -(u-13) & -32 & 0 & 0 & 0 \\ 0 & \frac{1}{4}(u-13)(u+3)+b' & 0 & -36 & 0 & 0 \\ 0 & 0 & b'-4 & -\frac{3}{2}(u-7) & -32 & 0 \\ 0 & 0 & 0 & \frac{1}{4}(u-9)(u+1)+b' & 0 & -20 \\ 0 & 0 & 0 & 0 & b'-6 & -\frac{15}{8}(u-1) \end{vmatrix}$$

Manifestly the value of the determinant is $14400b'(b'-4)(b'-6)$, with zeroes as given by (2.14).

There are two identities relating the R_i,

$$R_0 - R_1 + R_2 - R_3 + R_4 = \Phi(0)\Phi(1) \ , \tag{2.15}$$

$$R_1 - 2R_2 + 3R_3 - 4R_4 = 2\Phi(0)((a-2)(m+n) - \Phi(1)) \ , \tag{2.16}$$

which introduce extra solutions for closed algebras. If $\Phi(0) = 0$ or $\Phi(1) = 0$, (2.15) implies that there is another zero in the last column, giving an additional value of l for $k = 0, 1$, marked *. If $\Phi(0) = 0$ (or $\Phi(1) = 0$ and $a = 2$) then (2.16) implies that there are two extra zeros in the last column, so there is yet another value of l permitted if $k = 0$, marked †. If $\Phi(2) = 0$ and $\Phi(r-2) = 0$, and $r > 5$ there is also a solution. In this case, the $\Phi(2)$ serves a dual purpose; it allows the third row to be made zero, and allows us to make two extra zeroes in the last column, in the $(r-2)$ and $(r-1)$ rows. These rows are

$$\begin{pmatrix} 0 & \cdots & \Phi(r-3) & 0 & 0 & R_0 - R_1 + R_2 - R_3 + R_4 \\ 0 & \cdots & \Theta(r-3, s+r-5) & \Phi(r-2) & 0 & R_1 - 2R_2 + 3R_3 - 4R_4 \end{pmatrix} .$$

But if $\Phi(2) = 0$, then $R_1 - 2R_2 + 3R_3 - 4R_4 = \frac{1}{2}\Phi(0)\Phi(1)(s-4)$, and the values of the Φ and Θ may be calculated to give

$$\begin{pmatrix} 0 & \cdots & -4(r-5) & 0 & 0 & \Phi(0)\Phi(1) \\ 0 & \cdots & -2(r-5)(s-4) & 0 & 0 & \frac{1}{2}\Phi(0)\Phi(1)(s-4) \end{pmatrix} ,$$

showing that adding the required multiple of column $r-2$ to the last column makes the two entries shown zero. Thus the determinant of the same form as above with $k = 2$ and $l = r - 2$ is the only other condition. These solutions are marked ‡.

In the same way as for the linear case, the matrices for $r = 3$ and $r = 4$ are special, and their eigenvalues must be calculated explicitly.

The results are classified in the following table.

	c	a	b	k	l
$r=3$	b	0	free	0	1•
		1	0		2•
		2	free		3•
$r=4$	b	0	free	0	1•
		1	0		2•
	$b-2a$	2	free	1	1•
		3	2		2•
		4	5		3•
	$b-4a+4$	4	3	2	1•
$r=5$	b	0	free	0	1
		1	0		2*
		2	free		3†
	$b-2a$	2	free	1	1*
$r=6$	b	0	free	0	1
		1	0		2
		2	free		3*
		3	0, 2		4†
	$b-2a$	2	free	1	1
		3	2		2*
	$b-4a+4$	5	6	2	2‡
$r=7$	b	0	free	0	1
		1	0		2
		2	free		3
		3	0, 2		4*
		4	free		5†
	$b-2a$	2	free	1	1
		3	2		2
		4	free		3*
	$b-4a+4$	4	free	2	1
		6	free		3‡

•The $r=3,4$ cases are special.
*Using (2.15).
†Using (2.15) and (2.16).
‡See text.

It is possible to make a similar generalization to (2.10) by replacing some of the factors of $\left(z+\frac{1}{z}\right)$ in (2.11) by $\left(z-\frac{1}{z}\right)$.

2.5 Discussion

In this chapter I have given various examples of generalized-graded algebras. The solutions found are surprisingly simple, belying the tortuous route to their discovery, and perhaps indicates a deeper explanation. The fact that the determinants possess any zeroes for all m, n is due to highly non-trivial cancellations.

The expression (2.2) gives the N's as a finite sum of L's,

$$N_m = \sum_{k=0}^{r-1} a^k(m) L_{m-r+1+2k} \ .$$

If the $a^k(m)$ are non-zero, this expression can be inverted to give L_m as a semi-infinite sum of N's,

$$L_m = \sum_{k=0}^{\infty} b^k(m) N_{m+r-1-2k} \ ,$$

and this gives a representation of L_m in terms of pseudo-differential operators,[38] and the algebra of N's is in fact a transformation of the original Virasoro algebra. For the linear solutions the $a^k(m)$ have zeroes, which prevent this inversion for all L's and thus they are genuinely different.* Similar results hold for the quadratic case, when b is such that the quadratic $F(m)$ factorizes. Note that this always happens when b is not arbitrary.

Returning to the problem of whether the generators of any generalized-graded algebra can be expressed in terms of a finite sum of Virasoro generators, some evidence for this hypothesis comes from an examination of the structure constants for the second highest term. In Krichever and Novikov's normalization the leading constant, $C^{r-1}(m,n)$ is simply $(m-n)$. The Jacobi identity may be calculated, and the result is a sum of N's ranging from $N_{m+n+p-2r+2}$ to $N_{m+n+p+2r-2}$. The coefficient of the highest N cancels automatically. The coefficient of the next highest term is linear in the structure constant $C^{r-2}(m,n)$, and is, explicitly:

$$\left[((m-n)C^{r-2}(m+n+r-1,p) + (m+n+r-3-p)C^{r-2}(m,n)\right] + \text{cyclic} = 0 \ . \quad (2.17)$$

This must hold for all values of m, n, p. It is possible to solve for $C^{r-2}(m,n)$ in terms of the function of one variable $C^{r-2}(m, 1-r)$ by looking at the special case of (2.17) with

* The matrix of the set of linear equations (2.2) from $m = -\infty$ to a given m is lower triangular: If the diagonal contains any zeroes, it is singular.

$p = 1 - r$, with the result

$$C^{r-2}(m,n) = \frac{1}{2}\Big\{(m-n)C^{r-2}(m+n+r-1, 1-r) \\ - (m-n-2)C^{r-2}(m, 1-r) - (m-n+2)C^{r-2}(n, 1-r)\Big\} .$$

The remarkable property of this solution of the special case $p = 1 - r$ is that it is actually a solution of (2.17) for all m, n, p. Furthermore if one looks for a representation for N_m of the form of (2.2) the coefficients are given by

$$a^{r-1}(m) = 1 ,$$
$$a^{r-2}(m) = C^{r-1}(m, 1-r) + (m+r)a^{r-2}(1-r) .$$

Of course this is only a first order identification, but it holds promise that similar connections may be deduced for further terms in the expansion. This conjecture is supported by computer experiments using REDUCE.

3. The Sine Algebra

In October 1988 David Fairlie, Cosmas Zachos and I introduced the sine algebra.[2] This is an infinite dimensional Lie algebra, which is graded in a similar fashion to the Virasoro algebra, additively and with one generator in each grade; but each generator is indexed by a 2-vector $\boldsymbol{m} = (m_1, m_2)$ where the m_i may be integer, real or even complex. Algebras which are indexed in this way are referred to as 2-index algebras; the analogue of $\mathbb{Z}$-grading for these algebras is $\mathbb{Z}^2$-grading.

The sine algebra specifies an enormous symmetry. In this chapter I shall demonstrate that it contains all the classical finite dimensional Lie algebras,[3] and many other Lie algebras, either as special cases or as subalgebras. This provides a very useful 'egalitarian' basis for Lie algebras, one in which all the generators appear on the same footing, rather than being split into Cartan elements and roots. This has the advantage of only having one general commutation relation to work with, and the form of the relation is such that there is a straightforward way of taking the infinite limit of the classical algebras, yielding $su(\infty)$, $so(\infty)$ and $usp(\infty)$.

3.1 Introduction

The most general form of the algebra is given by

$$[K_{\boldsymbol{m}+\boldsymbol{b}}, K_{\boldsymbol{n}+\boldsymbol{b}}] = r \sin \kappa(\boldsymbol{m} \times \boldsymbol{n})\ K_{\boldsymbol{m}+\boldsymbol{n}+\boldsymbol{b}} + \boldsymbol{a}.\boldsymbol{m}\ \delta_{\boldsymbol{m}+\boldsymbol{n},0}\ ,$$

where $\boldsymbol{m} \times \boldsymbol{n} = \epsilon^{ij} m_i n_j = m_1 n_2 - m_2 n_1$ and $\boldsymbol{a}, \boldsymbol{b}$ are arbitrary 2-vectors. Unless otherwise stated, the m_i are taken to be integers, so that the generators lie on a square lattice. Also $\boldsymbol{b}$ is taken to be zero — if it is any lattice vector then the generators may be redefined to eliminate it. The resulting $\mathbb{Z}^2$-graded algebra is

$$[K_{\boldsymbol{m}}, K_{\boldsymbol{n}}] = r \sin \kappa(\boldsymbol{m} \times \boldsymbol{n})\ K_{\boldsymbol{m}+\boldsymbol{n}} + \boldsymbol{a}.\boldsymbol{m}\ \delta_{\boldsymbol{m}+\boldsymbol{n},0}\ . \tag{3.1}$$

Of course, r may be absorbed into the normalization of the generators.

The algebra may only have a supersymmetric extension in the case where $\boldsymbol{a}$ vanishes, *i.e.* there is no central extension. In this case the superalgebra is:

$$\begin{aligned}
[K_{\boldsymbol{m}}, K_{\boldsymbol{n}}] &= r \sin \kappa(\boldsymbol{m} \times \boldsymbol{n})\ K_{\boldsymbol{m}+\boldsymbol{n}} \ , \\
[K_{\boldsymbol{m}}, F_{\boldsymbol{n}}] &= r \sin \kappa(\boldsymbol{m} \times \boldsymbol{n})\ F_{\boldsymbol{m}+\boldsymbol{n}} \ , \\
\{F_{\boldsymbol{m}}, F_{\boldsymbol{n}}\} &= s \cos \kappa(\boldsymbol{m} \times \boldsymbol{n})\ K_{\boldsymbol{m}+\boldsymbol{n}} \ .
\end{aligned} \tag{3.2}$$

One of the most interesting special cases of this algebra is the limit algebra where $r = \frac{1}{\kappa}$ and $\kappa \to 0$ in (3.1), labelling the generators L,

$$[L_{\boldsymbol{m}}, L_{\boldsymbol{n}}] = (\boldsymbol{m} \times \boldsymbol{n})\ L_{\boldsymbol{m}+\boldsymbol{n}} + \boldsymbol{a}.\boldsymbol{m}\ \delta_{\boldsymbol{m}+\boldsymbol{n},\boldsymbol{0}} \ . \tag{3.3}$$

As in the above case, there is only a supersymmetric extension for the case with no central extension,[39] the result being the $\kappa \to 0$ limit of (3.2) with $r = \frac{1}{\kappa}$ and $s = 1$:

$$\begin{aligned}
[L_{\boldsymbol{m}}, L_{\boldsymbol{n}}] &= \boldsymbol{m} \times \boldsymbol{n}\ L_{\boldsymbol{m}+\boldsymbol{n}} \ , \\
[L_{\boldsymbol{m}}, G_{\boldsymbol{n}}] &= \boldsymbol{m} \times \boldsymbol{n}\ G_{\boldsymbol{m}+\boldsymbol{n}} \ , \\
\{G_{\boldsymbol{m}}, G_{\boldsymbol{n}}\} &= L_{\boldsymbol{m}+\boldsymbol{n}} \ .
\end{aligned}$$

3.2 The Moyal Bracket

The Poisson bracket of two differentiable functions f and g,

$$\{f, g\}_{\text{Poisson}} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x} \ ,$$

is well known. This is a Lie bracket, as it is antisymmetric and satisfies the Jacobi identity

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0 \ .$$

The Moyal bracket, or sine bracket, is a deformation of the Poisson bracket involving a parameter κ; the bracket becoming the Poisson bracket in the limit as $\kappa \to 0$. It is also a Lie bracket.

There are various ways of writing the Moyal bracket of two functions f and g; in terms of a formal operator,

$$\{f,g\}_{\text{Moyal}} = \lim_{\boldsymbol{x}\to\boldsymbol{x}'} \tfrac{1}{\kappa}\sin(\kappa\nabla\times\nabla')f(\boldsymbol{x})g(\boldsymbol{x}') ,$$

$\boldsymbol{x}$ a 2-vector, or its expansion,

$$\{f,g\}_{\text{Moyal}} = \sum_{s=0}^{\infty}\frac{(-1)^s\kappa^{2s}}{(2s+1)!}\sum_{j=0}^{2s+1}(-1)^j\binom{2s+1}{j}(\partial_x^j\partial_y^{2s+1-j}f(x,y))(\partial_x^{2s+1-j}\partial_y^j g(x,y)) ,$$

or as a generalized convolution,

$$\{f,g\}_{\text{Moyal}} = \tfrac{1}{4\pi^2\kappa}\int d\boldsymbol{x}'d\boldsymbol{x}''f(\boldsymbol{x}')g(\boldsymbol{x}'')\sin\kappa\left(\boldsymbol{x}\times\boldsymbol{x}'+\boldsymbol{x}'\times\boldsymbol{x}''+\boldsymbol{x}''\times\boldsymbol{x}\right) .$$

The associative product underlying this Lie bracket is the 'star product', or exponential bracket,

$$f\star g = \lim_{\boldsymbol{x}'\to\boldsymbol{x}}\exp(\kappa\nabla\times\nabla')f(\boldsymbol{x})g(\boldsymbol{x}') ,$$

which is a deformation of the ordinary product. Just as the antisymmetrization gives the Moyal, or sine, bracket, there is a symmetrization giving a symmetric, anti-commutator like bracket, the cosine bracket. This was introduced by Baker,[40] and in his paper he shows that the sine and cosine brackets together form a Lie superalgebra, satisfying the super-Jacobi identities.

The Moyal bracket was introduced by Moyal[41] in an alternative formulation of quantum mechanics in terms of Wigner distribution functions,[42] f, which are statistical distributions on phase space. This work was continued by Baker[40] and Fairlie.[43] Baker proved that the equations

$$\{f,H\}_{\text{Moyal}} = \hbar\frac{\partial f}{\partial t} ,$$
$$\{f,f\}_{\text{cosine}} = af ,$$

require the existence of a wave function ψ which satisfies the Schrödinger equation, and in terms of which f may be expressed as

$$f = \int\bar{\psi}(x-y,t)\psi(x+y,t)e^{ipy/\hbar}dy .$$

In this context the parameter κ is proportional to $\hbar$, so in the classical limit the Poisson bracket is obtained.

A derivation δ satisfies $\delta(fg) = (\delta f)g + f(\delta g)$. The Poisson bracket $\{f, h\}$ for a given h acts as a derivation for the Poisson bracket and for the ordinary product, so

$$\{\{f,g\},h\} = \{\{f,h\},g\} + \{f,\{g,h\}\} , \qquad \{fg,h\} = \{f,h\}g + f\{g,h\} ,$$

It follows from the Jacobi and super-Jacobi identities that the Moyal bracket acts as a derivation for itself and also for the cosine bracket.

3.3 Algebras of Modes

The Poisson and Moyal brackets are Lie brackets of functions. By specifying a basis for the functions, a Lie algebra of generators under commutation is obtained. Given a bracket of two functions, define generators L_f such that $[L_f, g] = \{f, g\}$, which satisfy the algebra

$$[L_f, L_g] = L_{\{f,g\}} . \tag{3.4}$$

For the Poisson bracket

$$L_f = \frac{\partial f}{\partial x}\frac{\partial}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial}{\partial x} . \tag{3.5}$$

Now any choice of a complete set of functions as a basis for f, g may be substituted into the above to give a 2-index algebra. In the torus basis $e_{\boldsymbol{m}}(\boldsymbol{x}) = -e^{i\boldsymbol{m}.\boldsymbol{x}}$, for $f = e_{\boldsymbol{m}}$ in (3.5)

$$L_{\boldsymbol{m}} \equiv L_{e_m} = -ie^{i\boldsymbol{m}.\boldsymbol{x}}\ \boldsymbol{m} \times \frac{\partial}{\partial \boldsymbol{x}} ,$$

and with $g = e_{\boldsymbol{m}}$ (3.4) becomes

$$[L_{\boldsymbol{m}}, L_{\boldsymbol{n}}] = (\boldsymbol{m} \times \boldsymbol{n})\ L_{\boldsymbol{m}+\boldsymbol{n}} ,$$

which is the centreless version of (3.3)— the Poisson algebra in the torus basis.

This is effectively a Fourier transformation, in the sense that, for general f,

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{m}} f_{\boldsymbol{m}} e_{\boldsymbol{m}}(\boldsymbol{x}) ,$$
$$L_f = \sum_{\boldsymbol{m}} f_{\boldsymbol{m}} L_{\boldsymbol{m}} ,$$

and so any other f, g may be expanded in terms of the $e_{\boldsymbol{m}}$ to give a linear combination of the $L_{\boldsymbol{m}}$. A different choice of the basis $e_{\boldsymbol{m}}(\boldsymbol{x})$ will lead to different structure constants, for example the choice $e_{\boldsymbol{m}}(\boldsymbol{x}) = x^{m_1+1}y^{m_2+1}$, the plane, leads to the algebra

$$[L_{\boldsymbol{m}}, L_{\boldsymbol{n}}] = \big((m_1+1)(n_2+1) - (m_2+1)(n_1+1)\big) L_{\boldsymbol{m}+\boldsymbol{n}} \ .$$

This algebra is the Poisson algebra in the plane basis, equivalent to (3.3) up to a change of basis.

The same procedure may be applied to the Moyal bracket. The basis-independent differential operator realization of K_f corresponding to (3.5) is[44]

$$K_f = \tfrac{1}{2i\kappa} f(x + i\kappa\tfrac{\partial}{\partial y}, y - i\kappa\tfrac{\partial}{\partial x}) \ .$$

In the torus basis, this becomes

$$\begin{aligned} K_{\boldsymbol{m}} &= \tfrac{i}{2\kappa} \exp(im_1 x + \kappa m_2 \tfrac{\partial}{\partial x} + im_2 y - \kappa m_1 \tfrac{\partial}{\partial y}) \\ &= \tfrac{i}{2\kappa} \exp(i\boldsymbol{m}.\boldsymbol{x}) \exp(-\kappa \boldsymbol{m} \times \tfrac{\partial}{\partial \boldsymbol{x}}) \ , \end{aligned}$$

somewhat analogous to the one-variable realization found by Hoppe,[45]

$$K_{\boldsymbol{m}} = \tfrac{2i}{\kappa} \exp(\sqrt{2\kappa}(m_1 \frac{\mathrm{d}}{\mathrm{d}x} + im_2 x)) \ .$$

The $K_{\boldsymbol{m}}$'s defined here satisfy the centreless sine algebra (3.1),[44]

$$[K_{\boldsymbol{m}}, K_{\boldsymbol{n}}] = r \sin \kappa(\boldsymbol{m} \times \boldsymbol{n}) \ K_{\boldsymbol{m}+\boldsymbol{n}} \ ,$$

the Moyal algebra on the torus. Again, a different set of basis functions will lead to an apparently different algebra.

Thus these 2-index algebras may be thought of as algebras of the modes in the expansion of the algebra of derivatives of continuous coordinates on manifolds, the expansion being taken over the basis functions for that manifold.

The similarity in form of the Poisson algebra in the torus basis, (3.3), to the Virasoro algebra, the algebra of diffeomorphisms of the circle, Diff(S^1), is manifest. It is intriguing to note that the Virasoro structure constants may be replaced by a sine function, just as they may for the Poisson algebra, obtaining the $su(2)$ Kac-Moody algebra (see chapter 1).

The algebra (3.3) constitutes the algebra of infinitesimal area-preserving diffeomorphisms of the torus, $\mathrm{SDiff}_0(T^2)$.[46, 47] This may be seen by considering the generator L_f, which transforms (x, y) to $(x - \frac{\partial f}{\partial y}, y + \frac{\partial f}{\partial x})$. Infinitesimally, this is a canonical transformation generated by f which preserves the area element $dxdy$. These infinitesimal transformations do not depend on the global properties of the surface, so the algebra of area-preserving diffeomorphisms of any 2-surface is related to (3.3) through a change of basis.

3.4 The Finite Algebras

One of the most interesting features of the sine algebra is that it contains, as special cases, all the finite classical simple Lie algebras. The result of this is a trigonometric basis for these finite algebras, which has many advantages over the more commonly used Cartan-Weyl basis. The particular cases of (3.1) are those which are centreless and have $\kappa = 2\pi/N$ for some integer N. This choice imposes a modulo-N arithmetic on the structure constants, and so generators differing by N in either index may be consistently identified. This means that $K_{\boldsymbol{m}+N\boldsymbol{a}}$ is identified with $K_{\boldsymbol{m}}$ for all integral 2-vectors $\mathbf{a}$. There remain N^2 distinct generators, $\mathcal{K}_{\boldsymbol{m}}$, $m_1, m_2 = 0, \ldots, N-1$, lying on a toroidal integer lattice. The generators $\mathcal{K}_{\boldsymbol{m}}$ may equivalently be thought of as lattice averages,

$$\mathcal{K}_{\boldsymbol{m}} = \lim_{n \to \infty} \frac{1}{4n^2} \sum_{j_1, j_2 = -n}^{n} K_{\boldsymbol{m}+N\boldsymbol{j}} .$$

The resulting finite algebra is, absorbing normalization,

$$[\mathcal{K}_{\boldsymbol{m}}, \mathcal{K}_{\boldsymbol{n}}] = \sin \tfrac{2\pi}{N}(\boldsymbol{m} \times \boldsymbol{n})\, \mathcal{K}_{\boldsymbol{m}+\boldsymbol{n}} . \tag{3.6}$$

This is just the algebra of $u(N)$ for N odd, or $u(N/2)^4$ when N is even, as will be proven in a later section.

First, note some interesting properties of (3.6). The generator $\mathcal{K}_{0,0}$ factors out of the algebra, as it commutes with the other $N^2 - 1$ and cannot result as a commutator of any of them. So this is a $u(1)$. Each of the sets of N generators $\{\mathcal{K}_{m,0}\}$, $\{\mathcal{K}_{0,m}\}$, $\{\mathcal{K}_{m,N-m}\}$ has mutually commuting elements. For N odd these are the maximal such sets, and so may be taken as Cartan subalgebras. For N even the maximal mutually commuting sets are of size $2N$: $\{\mathcal{K}_{m,0}, \mathcal{K}_{m,N/2}\}$, $\{\mathcal{K}_{0,m}, \mathcal{K}_{N/2,m}\}$, $\{\mathcal{K}_{m,N-m}, \mathcal{K}_{m,N/2-m}\}$; and the generators

$\mathcal{K}_{0,N/2}, \mathcal{K}_{N/2,0}, \mathcal{K}_{N/2,N/2}$ factor out like $\mathcal{K}_{0,0}$, making four $u(1)$'s in all. For all N the whole algebra may be generated by repeated commutation of $\mathcal{K}_{0,1}$ and $\mathcal{K}_{1,0}$.

For $N = 2^p M$, where M is odd, there are manifest subalgebras consisting of generators $\mathcal{K}_{2^r \boldsymbol{j}}$ where $r \leq p$, and the indices $\boldsymbol{j}$ are taken modulo-$(2^{p-r}M)$. These subalgebras are equivalent to (3.6) with the new $N = 2^{p-r}M$.

Other finite special cases of (3.1) occur when m_1 and m_2 repeat with different periods, *i.e.* the generators are identified

$$K_{m_1+sNa_1, m_2+tNa_2} \equiv K_{m_1,m_2} \ ,$$

for arbitrary a_1, a_2. It is sufficient to consider $t = 1$, as the structure constant is unchanged by the replacement

$$(m_1, tm_2) \to (tm_1, m_2) \quad \forall m_1, m_2 \ .$$

This produces a consistent closed algebra, specifically s mutually commuting copies of the finite algebra (3.6). The copies are generated by

$$\sum_{p=0}^{s-1} e^{2\pi i pq/s} K_{m_1+pN, m_2} \ ,$$

where $q = 0, ..., s-1$ labels each copy.

It is also worth noting that there are algebra automorphisms of (3.1), (3.3) and (3.6) induced by modular transformations of the index vectors

$$\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \to \begin{pmatrix} m_1' \\ m_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} ,$$

where a, b, c, d are integers with $ad - bc = 1$, and $\boldsymbol{a}$ undergoes the inverse transformation.

3.5 The Trigonometric Basis for $su(N)$

Introduce the two unitary matrices[48]

$$g = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \omega & 0 & \dots & 0 \\ 0 & 0 & \omega^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \omega^{N-1} \end{pmatrix} , \quad h = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix} , \quad g^N = h^N = \mathbb{1} \ ,$$

where ω is an N'th root of unity with period no smaller than N, such as $e^{2\pi i/N}$. Then define the $N \times N$ matrices

$$J_{(m_1,m_2)} = \omega^{m_1 m_2/2} g^{m_1} h^{m_2} \;, \tag{3.7}$$

which satisfy

$$J^\dagger_{(m_1,m_2)} = J_{(-m_1,-m_2)} \;; \quad \mathrm{Tr}\, J_{(m_1,m_2)} = 0 \text{ except for } m_1 = m_2 = 0 \bmod N \;. \tag{3.8}$$

These matrices are all linearly independent, and they are traceless except for $J_{(0,0)}$ which is the identity, and so these N^2 matrices form a basis for $u(N, \mathbb{C})$.

By virtue of the identity

$$hg = \omega g h \;,$$

these matrices close under mere multiplication:

$$J_{\boldsymbol{m}} J_{\boldsymbol{n}} = \omega^{\boldsymbol{n}\times\boldsymbol{m}/2} J_{\boldsymbol{m}+\boldsymbol{n}} \;. \tag{3.9}$$

and so satisfy the algebra

$$[J_{\boldsymbol{m}}, J_{\boldsymbol{n}}] = (\omega^{\boldsymbol{n}\times\boldsymbol{m}/2} - \omega^{-\boldsymbol{n}\times\boldsymbol{m}/2}) J_{\boldsymbol{m}+\boldsymbol{n}} \;. \tag{3.10}$$

If $\omega = e^{2\pi i/N}$ (3.10) becomes

$$[J_{\boldsymbol{m}}, J_{\boldsymbol{n}}] = -2i \sin \tfrac{\pi}{N}(\boldsymbol{m} \times \boldsymbol{n}) J_{\boldsymbol{m}+\boldsymbol{n}} \;, \tag{3.11}$$

which has an apparent period of $2N$. However, due to the symmetry

$$J_{\boldsymbol{m}+N\boldsymbol{a}} = (-1)^{(m_1+1)a_2+(m_2+1)a_1} J_{\boldsymbol{m}} \;,$$

only indices in the fundamental $N \times N$ cell need be considered, these N^2 distinct operators forming a trigonometric basis for $u(N)$.

For odd N, $\omega = e^{4\pi i/N}$ has period N, and the structure constant of (3.10) reduces to $-2i \sin \frac{2\pi}{N} \boldsymbol{m} \times \boldsymbol{n}$. Thus for odd N we see that (3.6) is the algebra of $u(N)$.

The matrices J were first introduced by Weyl[48] as a basis for $su(N)$, and have been used in various areas of mathematical physics.[49–57] That it is possible to write the commutation relations in this basis with structure functions given by a simple function was discovered by Fairlie, Zachos and myself.[2] This appearance of N in an analytic way allows the consideration of the large N limit of $su(N)$. As N increases, the fundamental $N \times N$ cell covers the entire index lattice; the operators $\mathcal{K}$ are supplanted by the K's and, in turn, since $\kappa \to 0$, by the operators L of (3.3). More directly, it is immediately evident that, as $N \to \infty$, the $su(N)$ algebra (3.11) goes over to the Poisson algebra through the identification:

$$\frac{iN}{2\pi} J_{\boldsymbol{m}} \to L_{\boldsymbol{m}} \ .$$

An identification of this type was first noted by Hoppe[58, 59] in the context of membrane physics: He connected the infinite N limit of the $su(N)$ algebra in a special basis to that of $\mathrm{SDiff}_0(S^2)$, *i.e.* the infinitesimal area preserving diffeomorphisms of the sphere.

As they are closed under the matrix product, the J's defined in (3.7) also satisfy the fermionic algebra

$$\{J_{\boldsymbol{m}}, J_{\boldsymbol{n}}\} = (\omega^{\boldsymbol{n}\times\boldsymbol{m}/2} + \omega^{-\boldsymbol{n}\times\boldsymbol{m}/2}) J_{\boldsymbol{m}+\boldsymbol{n}} \ ,$$

so the same matrices may represent both bosonic and fermionic operators in the corresponding finite case of the superalgebra (3.2).

This is a realization of the superalgebra which goes back to Weyl[48] and his correspondence rule,

$$K_{\boldsymbol{m}} = \frac{1}{2i\kappa} e^{i(2\kappa m_1 P + m_2 X)} = \frac{1}{i\kappa} F_{\boldsymbol{m}} \ ,$$

where (X, P) are canonically conjugate quantum variables with $[X, P] = i$. Using the Baker-Campbell-Haussdorf expansion, the product is

$$K_{\boldsymbol{m}} K_{\boldsymbol{n}} = \frac{1}{2i\kappa} e^{i\kappa(m_1 n_2 - m_2 n_1)} K_{\boldsymbol{m}+\boldsymbol{n}} \ , \tag{3.12}$$

and therefore

$$K_{\boldsymbol{m}} K_{\boldsymbol{n}} = \frac{1}{2i\kappa} e^{2i\kappa(m_1 n_2 - m_2 n_1)} K_{\boldsymbol{n}} K_{\boldsymbol{m}} \ . \tag{3.13}$$

3.6 The Algebras $so(N)$ and $usp(N)$

In this section, the other classical Lie algebras are exhibited in a 'trigonometrical' basis analogous to that of $su(N)$. Since these can fit as subalgebras in $su(N)$, they can be extracted from it, and hence $so(\infty)$ and $usp(\infty)$ out of the limit $su(\infty)$. The analysis is done in terms of the matrices J introduced above which satisfy (3.11), and form a basis for $su(N)$ for all N.

The subalgebras may be written as combinations of J's which close on themselves. Those of most interest are:

$$J_{m_1,m_2} - (-)^a J_{m_1,-m_2} \quad \begin{cases} a = 0 & so(N) \\ a = m_1, N \text{ even} & usp(N) \\ a = m_2, N \text{ even} & so(N) \\ a = m_1 + m_2, N = 4M & usp(N) \\ a = m_1 + m_2, N = 4M + 2 & so(N) \end{cases}$$

$$J_{m_1,m_2} - (-)^a J_{m_2,m_1} \quad \begin{cases} a = 0 & so(N) \\ a = m_1 + m_2, N \text{ even} & so(N) \end{cases}$$

As an example, consider the second case, with $a = 0$, N odd. Denoting

$$J_{[m_1,m_2]} = J_{m_1,m_2} - J_{m_2,m_1} \ . \tag{3.14}$$

The number of generators of these algebras is $\frac{1}{2}N(N-1)$. The commutation relations are:

$$[J_{[m_1,m_2]}, J_{[n_1,n_2]}] = -2i \begin{pmatrix} \sin \frac{\pi}{N}(m_1 n_2 - m_2 n_1) J_{[m_1+n_1, m_2+n_2]} \\ - \sin \frac{\pi}{N}(m_1 n_1 - m_2 n_2) J_{[m_1+n_2, m_2+n_1]} \end{pmatrix} .$$

The other subalgebras satisfy similar relations.

It is possible to take the large N limits of these subalgebras of $su(N)$, resulting in the infinite algebras $so(\infty)$ and $usp(\infty)$. These are infinite subalgebras of the Poisson algebra, $su(\infty)$.

3.7 Basis Change for $su(N)$

To analyse the structure of these algebras further the transformation from the above trigonometric basis to the more usual Cartan-Weyl basis is given. This will show that the full algebra in the N even case is $u(N/2)^4$.

The Cartan-Weyl basis for $su(N)$ has the following commutation relations, in the usual notation (see chapter 1):

$$[e_\alpha, e_\beta] = \begin{cases} N_{\alpha\beta} e_{\alpha+\beta} & \text{if } \alpha+\beta \in \Sigma \\ (e_\alpha, e_{-\alpha}) h_\alpha & \text{if } \alpha+\beta = 0 \\ 0 & \text{otherwise} \end{cases} \tag{3.15}$$

$$[h_i, e_\alpha] = \alpha(h) e_\alpha \tag{3.16}$$

$$[h_i, h_j] = 0 \ . \tag{3.17}$$

The cases N odd and even differ slightly in detail, although the principle of finding the transformation is the same. In the case of N odd, the combinations of $\mathcal{K}$'s which give this basis are:

$$E_q^p = \sum_{j=0}^{N-1} \omega^{(2j-q)p} \mathcal{K}_{j,q-j} \ , \quad \omega^N = 1 \ ,$$

where $q = 0$ and $p = 1, \ldots, \frac{1}{2}(N-1)$ for the Cartan subalgebra, and $q = 1, \ldots, N-1$ and $p = 0, \ldots, N-1$, for the remaining generators

This may be shown by checking the commutation relations as follows:

$$\begin{aligned}
[E_{q_1}^{p_1}, E_{q_2}^{p_2}] &= \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} \omega^{(2j-q_1)p_1 + (2k-q_2)p_2} [\mathcal{K}_{j,q_1-j}, \mathcal{K}_{k,q_2-k}] \\
&= \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} \omega^{(2j-q_1)p_1 + (2k-q_2)p_2} \sin \tfrac{2\pi}{N}(jq_2 - kq_1) \mathcal{K}_{j+k,q_1+q_2-j-k} \\
&= \frac{1}{2i} \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} \omega^{(2j-q_1)p_1 + (2k-q_2)p_2} (\omega^{jq_2-kq_1} - \omega^{-jq_2+kq_1}) \mathcal{K}_{j+k,q_1+q_2-j-k} \ .
\end{aligned}$$

Putting $s = j + k$ and $q = q_1 + q_2$ and then resumming, using the invariance modulo N:

$$
\begin{aligned}
[E^{p_1}_{q_1}, E^{p_2}_{q_2}] &= \frac{1}{2i}\left(\sum_{s=0}^{N-1}\sum_{k=0}^{s} + \sum_{s=N}^{2N-2}\sum_{\substack{k=\\ s-N+1}}^{N-1}\right)\omega^{(2s-2k-q_1)p_1+(2k-q_2)p_2}(\omega^{sq_2-kq} - \omega^{-sq_2+kq})\mathcal{K}_{s,q-s} \\
&= \frac{1}{2i}\sum_{s=0}^{N-1}\omega^{(2s-q_1)p_1-q_2p_2}\sum_{k=0}^{N-1}(\omega^{2k(p_2-p_1)-kq+sq_2} - \omega^{2k(p_2-p_1)+kq-sq_2})\mathcal{K}_{s,q-s} \,.
\end{aligned}
$$

Using $\sum\limits_{k=0}^{N-1}\omega^{ak} = N\delta_{a,0}$, and then comparing with the expression for E^p_q,

$$
\begin{aligned}
[E^{p_1}_{q_1}, E^{p_2}_{q_2}] &= \frac{1}{2i}\sum_{s=0}^{N-1}\omega^{(2s-q_1)p_1-q_2p_2}N(\omega^{sq_2}\delta_{2(p_2-p_1)-q,0} - \omega^{-sq_2}\delta_{2(p_2-p_1)+q,0})\mathcal{K}_{s,q-s} \\
&= \frac{N}{2i}\left(E^{p_1+q_2/2}_{q_1+q_2}\delta_{2(p_2-p_1)-q,0} - E^{p_1-q_2/2}_{q_1+q_2}\delta_{2(p_2-p_1)+q,0}\right) \,,
\end{aligned}
$$

where the $q_2/2$ may be defined as an integer (as N may be added to the power of ω to ensure that this power is even), so the halving of the index is defined by:

$$
\frac{q}{2} = \begin{cases} q/2 & q \text{ even} \\ (q+N)/2 & q \text{ odd} \end{cases} \,.
$$

And so

$$
[E^{p_1}_{q_1}, E^{p_2}_{q_2}] = \begin{cases} \pm\frac{N}{2i}E^{p_1\pm q_2/2}_{q_1+q_2} & \text{if } 2(p_2-p_1) = \pm(q_1+q_2) \\ \frac{N}{2i}\left(E^{p_1+q_2/2}_0 - E^{p_1-q_2/2}_0\right) & \text{if } q_1+q_2 = 0 = p_2-p_1 \\ 0 & \text{otherwise} \end{cases}
$$

which corresponds to (3.15).

$$
[E^{p_1}_0, E^{p_2}_q] = \begin{cases} \pm\frac{N}{2i}E^{p_2}_q & \text{if } 2(p_2-p_1) = \pm q \\ 0 & \text{otherwise} \end{cases}
$$

showing the basis is diagonal (3.16), and

$$
[E^{p_1}_0, E^{p_2}_0] = 0 \,,
$$

as required.

For the case N even, the situation is more complicated. This time four of the $\mathcal{K}$'s disconnect into $u(1)$'s, $\mathcal{K}_{0,0}, \mathcal{K}_{\frac{N}{2},0}, \mathcal{K}_{0,\frac{N}{2}}$ and $\mathcal{K}_{\frac{N}{2},\frac{N}{2}}$. This leaves N^2-4 generators, which span four commuting $su(N/2)$'s. There are slight differences between the cases (a) $N \equiv 0 \bmod 4$ and (b) $N \equiv 2 \bmod 4$, but the principle of construction is the same. As before, the Cartan subalgebra is spanned by the elements whose indices sum to 0 mod $N/2$. Note that there are $2N-4$ such operators, after excluding the four $u(1)$'s.

For (a), $N \equiv 0 \bmod 4$, the generators in the Cartan-Weyl basis are:

$$E_q^{s,p} = \sum_{a=0}^{1}\sum_{j=0}^{N-1} \omega^{pj}(-1)^{s(j+a)+a(j+1)}\mathcal{K}_{j,q-j+a\frac{N}{2}}\ , \tag{3.18}$$

and for (b), $N \equiv 2 \bmod 4$,

$$E_q^{s,p} = \sum_{a=0}^{1}\sum_{j=0}^{N-1} \omega^{pj}(-1)^{s(j+1)+(a+1)(j+1+p)}\mathcal{K}_{j,q-j+a\frac{N}{2}}\ ,$$

where the q labels the sum of the indices, $q = 0, \ldots, \frac{N}{2}-1$, and s, p take the values $s = 0, 1, p = 0, \ldots, N-1$. Then for case (a) the elements $E_q^{s,p}$ for $s = 0, 1$; $q+p$ even, odd span the four commuting $su(N/2)$'s. For case (b), the splitting is into those with $s+q$ even, odd and p even, odd.

That the above combinations are the generators of $su(N/2)^4$ in the Cartan-Weyl basis may be shown by checking the commutation relations in a similar fashion to the N odd case above. Commuting two of the E's gives an expression which may be re-summed so that the coefficients of the $\mathcal{K}$'s can be read off. The re-summed expression for (a), with $a = a_1 + a_2$, $j = j_1 + j_2$, $q = q_1 + q_2$ and $p_- = p_1 - p_2$ is

$$[E_{q_1}^{s_1,p_1}, E_{q_2}^{s_2,p_2}] = \\ \sum_{a=0}^{1}\sum_{j=0}^{N-1}\sum_{a_1=0}^{1}\sum_{j_1=0}^{N-1} \omega^{j_1 p_- + p_2 j}\sin\tfrac{2\pi}{N}(qj_1 - jq_1)(-1)^{(s_1+s_2)(a_1+j_1)+s_2(a+j)+a(j+1)}\mathcal{K}_{j,q-j+\frac{N}{2}a}\ ,$$

The separation into $s = 0, 1$ is evident as the only dependence of the coefficient of the $\mathcal{K}$ on a_1 is of the form $(-1)^{(s_1+s_2)a_1}$, so if $s_1 \neq s_2$ then the two terms in that sum exactly cancel. When $s_1 = s_2$, the coefficient indexed by a, j becomes:

$$
\begin{aligned}
&(-1)^{s_1(a+j)+a(j+1)}\omega^{p_2 j}\tfrac{1}{i}\sum_{j_1=0}^{N-1}\omega^{j_1 p_-}\left(\omega^{j_1 q - j q_1} - \omega^{-j_1 q + j q_1}\right) \\
&= (-1)^{s_1(a+j)+a(j+1)}\omega^{p_2 j}\tfrac{N}{i}\left(\omega^{-jq_1}\delta_{p_-+q,0} - \omega^{jq_1}\delta_{p_--q,0}\right) .
\end{aligned}
$$

The δ's are both zero if p_1+q_1 and p_2+q_2 have different parity, showing the overall split into four commuting subspaces. This coefficient may be compared that in (3.18), and the commutator rewritten as

$$
[E_{q_1}^{s_1,p_1}, E_{q_2}^{s_2,p_2}] = \tfrac{N}{i}\delta_{s_1,s_2}\left(E_{q_1+q_2}^{s_1,p_2-q_1}\delta_{p_1-p_2+q_1+q_2,0} - E_{q_1+q_2}^{s_1,p_2+q_1}\delta_{p_1-p_2-(q_1+q_2),0}\right) .
$$

Similarly for (b),

$$
\begin{aligned}
&[E_{q_1}^{s_1,p_1}, E_{q_2}^{s_2,p_2}] = \\
&\sum_{a=0}^{1}\sum_{j=0}^{N-1}\sum_{a_1=0}^{1}\sum_{j_1=0}^{N-1}\omega^{j_1 p_- + p_2 j}(-1)^{(s_1+s_2)(j_1+1)+a_1 p_- + a(j+p_2+1)+j(s_2+1)+p_-} \\
&\qquad\qquad\sin\tfrac{2\pi}{N}(qj_1 - jq_1)\mathcal{K}_{j,q-j+\frac{N}{2}a} .
\end{aligned}
$$

This time, the coefficient of a_1 in the exponent is p_-, so the space splits into p even, odd. The coefficient of the $\mathcal{K}$ here is

$$
\begin{aligned}
&(-1)^{s_1+s_2+(s_2+a+1)j+a(p_1+1)}\omega^{p_2 j}\tfrac{1}{i}\sum_{j_1=0}^{N-1}(-1)^{j_1(s_1+s_2)}\omega^{j_1 p_-}\left(\omega^{j_1 q - jq_1} - \omega^{-j_1 q + jq_1}\right) \\
&= (-1)^{s_1+s_2+(s_2+a+1)j+a(p_1+1)}\omega^{p_2 j}\tfrac{N}{i}\left(\omega^{-jq_1}\delta_{p_-+q,(s_1+s_2)\frac{N}{2}} - \omega^{jq_1}\delta_{p_--q,(s_1+s_2)\frac{N}{2}}\right) \\
&= \tfrac{N}{i}(-1)^{(s_1+q_1+p_1+1)+(j+1)(s_2+q_1)+(a+1)(j+1+p_1)}\begin{pmatrix}\omega^{(p_2-q_1(1+\frac{N}{2}))j}\delta_{p_-+q,(s_1+s_2)\frac{N}{2}} \\ -\,\omega^{(p_2+q_1(1+\frac{N}{2}))j}\delta_{p_--q,(s_1+s_2)\frac{N}{2}}\end{pmatrix} .
\end{aligned}
$$

Both δ functions are zero if s_1+q_1 and s_2+q_2 have different parity. In performing the above manipulations the fact that $\omega^{\frac{N}{2}} = -1$ has been used. Thus, for $p_1 \equiv p_2 \bmod 2$,

$$
\begin{aligned}
&[E_{q_1}^{s_1,p_1}, E_{q_2}^{s_2,p_2}] = \\
&\quad\tfrac{N}{i}(-1)^{s_1+q_1+p_1+1}\left(E_q^{s,p_2-q_1(1+\frac{N}{2})}\delta_{p_-+q,(s_1+s_2)\frac{N}{2}} - E_q^{s,p_2+q_1(1+\frac{N}{2})}\delta_{p_--q,(s_1+s_2)\frac{N}{2}}\right) ,
\end{aligned}
$$

where $q \equiv q_1 + q_2 \bmod N/2$, and $s \equiv q + q_1 + q_2 \bmod 2$, so that $s + q \equiv s_1 + q_1 \equiv s_2 + q_2 \bmod 2$.

3.8 Basis Change for $so(N)$ and $usp(N)$

In a similar way, the subalgebras may be transformed to their Cartan-Weyl basis. As an example, consider (3.14). It is convenient to label the generators by $q = m_1 + m_2$, the sum of the indices. Those with $q = 0 \bmod N$ all mutually commute, and this is taken as the Cartan subalgebra. Forming the Cartan-Weyl basis amounts to simultaneously diagonalizing the matrices which are the elements of the Cartan subalgebra in the adjoint representation, *i.e.* the matrices of structure constants on commutation of h with the e_α's;

$$[h, e_\alpha] = \sum_\beta M_{\alpha\beta} e_\beta \ . \tag{3.19}$$

These matrices are block diagonal, with a block for each $q = 1, \ldots, N-1$, of size $r = \frac{1}{2}(N-1)$. The blocks are all of the form, independent of q, of

$$M_r = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix} .$$

Thus the combinations of the generators with a given q proportional to the eigenvectors of M_r are diagonal.

The characteristic polynomial, P_r, of M_r is given (up to sign) by the recurrence relation

$$P_r = \lambda P_{r-1} - P_{r-2} \ ,$$

with $P_0 = 1$ and $P_1 = \lambda - 1$. This may be solved by writing $\lambda = 2\cos\phi$, then

$$\begin{aligned} P_r &= \cos r\phi - \tfrac{1-\cos\phi}{\sin\phi} \sin r\phi \\ &= \tfrac{1}{\sin\phi}(\sin(r+1)\phi - \sin r\phi) \\ &= \tfrac{2}{\sin\phi} \cos(r+\tfrac{1}{2})\phi \ \sin\tfrac{\phi}{2} \ . \end{aligned}$$

This vanishes when $\phi = \phi^k = (2k-1)\pi/N$, $k = 1, \ldots, N$.

Defining

$$P_j^k = \frac{1}{\sin\phi^k}(\sin(j+1)\phi^k - \sin j\phi^k) ,$$

so that $P_r^k = P_r = 0$. Then the eigenvector of M_r corresponding to the eigenvalue $\lambda^k = 2\cos\phi^k$ is

$$(P_0^k, P_1^k, \ldots, P_{r-1}^k) .$$

Now it is clear that the combinations of generators which diagonalize the basis are:

$$S_q^k = \sum_{j=0}^{r-1} P_j^k J_{[\frac{1}{2}(q+2j-1),\frac{1}{2}(q-2j+1)]} ,$$

The Cartan elements are $H^\alpha = J_{[\alpha,-\alpha]}$.

Working out the commutation relations in this basis,

$$\begin{aligned}
[H^\alpha, S_q^k] &= [J_{[\alpha,N-\alpha]}, \sum_{j=0}^{r-1} P_j^k J_{[\frac{1}{2}(q+2j-1),\frac{1}{2}(q-2j+1)]}] \\
&= \sum_{j=0}^{r-1} \frac{1}{\sin\phi^k}(\sin(j+1)\phi^k - \sin j\phi^k)[J_{[\alpha,-\alpha]}, J_{[\frac{1}{2}(q+2j-1),\frac{1}{2}(q-2j+1)]}] \\
&= \frac{-2i\sin\frac{\pi}{N}\alpha q}{\sin\phi^k} \sum_{j=0}^{r-1} \left(\sin(j+1)\phi^k - \sin j\phi^k\right) \\
&\qquad \left(J_{[\frac{1}{2}(q+2j-1)+\alpha,\frac{1}{2}(q-2j+1)-\alpha]} - J_{[\frac{1}{2}(q-2j+1)+\alpha,\frac{1}{2}(q+2j-1)-\alpha]}\right) .
\end{aligned}$$

Now consider the coefficient of $J_{[\frac{1}{2}(q+2l-1),\frac{1}{2}(q-2l+1)]}$. This is

$$\frac{-2i\sin\frac{\pi}{N}\alpha q}{\sin\phi^k}\left(\sin(l-\alpha)\phi^k - \sin(l-\alpha-1)\phi^k + \sin(l+\alpha)\phi^k - \sin(l+\alpha-1)\phi^k\right)$$

$$= \frac{-4i\sin\frac{\pi}{N}\alpha q\cos\alpha\phi^k}{\sin\phi^k}\left(\sin l\phi^k - \sin(l-1)\phi^k\right) .$$

So

$$\begin{aligned}
[H^\alpha, S_q^k] &= \frac{-4i\sin\frac{\pi}{N}\alpha q\cos\alpha\phi^k}{\sin\phi^k}\sum_{l=0}^{r-1} P_l^k J_{[\frac{1}{2}(q+2l-1),\frac{1}{2}(q-2l+1)]} \\
&= \frac{-4i\sin\frac{\pi}{N}\alpha q\cos\alpha\phi^k}{\sin\phi^k} S_q^k ,
\end{aligned}$$

showing that this is the diagonal basis.

A similar identification of some subalgebras was made by Pope and Romans.[60] They introduce a basis for $so(N)$ and another for $usp(N)$ which are extensions of the Weyl[48] basis to make the identification. Furthermore, they identify the infinite limits of these subalgebras with the group of diffeomorphisms acting on two dimensional manifolds with different topology from the torus; in the one case a Klein bottle, in the other a projective plane.

3.9 Casimir Invariants

The construction of Casimir invariants is modelled upon that for the finite algebras discussed by Patera and Zassenhaus.[53] The quadratic Casimir is

$$\sum_{\boldsymbol{m}} K_{\boldsymbol{m}} K_{-\boldsymbol{m}} \ .$$

There are in general two Casimir invariants of each degree above the quadratic. They are the real and imaginary parts of

$$\sum_{\boldsymbol{m},\boldsymbol{n}} e^{i\kappa \boldsymbol{m}\times\boldsymbol{n}} \; K_{\boldsymbol{m}} K_{\boldsymbol{n}} K_{-\boldsymbol{m}-\boldsymbol{n}} \ , \ \cdots \ ,$$

$$\sum_{\boldsymbol{m}_1,\boldsymbol{m}_2,\ldots,\boldsymbol{m}_r} \left(\prod_{\alpha<\beta} e^{i\kappa(\boldsymbol{m}_\alpha \times \boldsymbol{m}_\beta)} \right) K_{\boldsymbol{m}_1} K_{\boldsymbol{m}_2} \ldots K_{\boldsymbol{m}_r} K_{-\boldsymbol{m}_1-\boldsymbol{m}_2-\cdots-\boldsymbol{m}_r} \ .$$

Taking the imaginary part, a generic coefficient will be of the form

$$\sin(\kappa(\boldsymbol{m}\times\boldsymbol{n} + \boldsymbol{m}\times\boldsymbol{p} + \boldsymbol{n}\times\boldsymbol{p} + \cdots)) \ .$$

By use of the addition formula for sines, this will always be reducible to terms with a typical $\sin\kappa(\boldsymbol{m}\times\boldsymbol{n})$ factor. Whenever the remaining factor in such a term is symmetric in $\boldsymbol{m}$ and $\boldsymbol{n}$, after use of the commutation relations to make $K_{\boldsymbol{m}}$ and $K_{\boldsymbol{n}}$ adjacent, it is easy to see that this contribution to the Casimir may be reduced to one of one degree lower. For example, in the case of the cubic,

$$\sum_{\boldsymbol{m},\boldsymbol{n}} \sin(\kappa(\boldsymbol{m}\times\boldsymbol{n})) K_{\boldsymbol{m}} K_{\boldsymbol{n}} K_{-\boldsymbol{m}-\boldsymbol{n}} = \sum_{\boldsymbol{m},\boldsymbol{n}} \sin^2(\kappa(\boldsymbol{m}\times\boldsymbol{n})) K_{\boldsymbol{m}+\boldsymbol{n}} K_{-\boldsymbol{m}-\boldsymbol{n}} \ .$$

Re-summing over $\boldsymbol{m}+\boldsymbol{n}$ and $\boldsymbol{m}-\boldsymbol{n}$ the right-hand side diverges, without an infinite renormalization of $K_{\boldsymbol{m}}$. Such a renormalization, however, would make the cosine-like contributions vanish.

The Casimirs of the Poisson algebra (3.3) follow by a $\kappa \to 0$ limiting procedure, again there are apparently two for each degree, one of which can be reduced in degree as above, with a divergent result.

In the matrix representation given by the J's, the Casimir operators are all proportional to the identity, as the sum of the indices of each of their terms is zero.

3.10 Triangular Lattices

It is possible to realize similar infinite algebras on other lattices. In this section results for triangular lattices are given.

It is convenient to choose a system of barycentric coordinates, and index K by three integers m_1, m_2, m_3 where $m_1 + m_2 + m_3 = 0$. (Barycentric coordinates measure the perpendicular distances of any point from the edges of the fundamental reference triangle, as in the Dalitz plot.)

For this case, the relations are

$$[K_{\boldsymbol{m}}, K_{\boldsymbol{n}}] = \sin(\kappa \boldsymbol{u}.(\boldsymbol{m} \times \boldsymbol{n}))\ K_{\boldsymbol{m}+\boldsymbol{n}}\ ,$$

where $\kappa = \pi/N$, and $\boldsymbol{u}$ is the vector $(1,1,1)$. As before, finite algebras are found by identifying generators at lattice points equivalent modulo N in each index. When $N \equiv 0 \bmod 3$ the generators whose indices are congruent modulo N all disconnect into $u(1)$'s. This leaves a hexagonal lattice, and the algebras obtained are $u(N/3)^6$. When $N \not\equiv 0 \bmod 3$, the fundamental lattice vectors of points reduced mod$N$ contains only one disconnected member, $(0,0,0)$, and the remaining $N^2 - 1$ points are associated with generators which close on $su(N)$. This situation is parallel with that for the square lattice.

3.11 Applications

There has been a lot of interest in 2-index algebras and their physical applications. The apparently different algebras appearing in the literature are all equivalent to the Moyal algebra or its special case, the Poisson algebra, but in different bases. In chapter 4 I shall show that all 2-index algebras are the Moyal algebra in some basis. The applications may be summarized in terms of the basis used.

As mentioned above, on the torus the Moyal algebra is the sine algebra (3.1), which contains all the classical finite Lie algebras as special cases. The representation theory of this algebra is treated by Floratos.[61, 62]

Saveliev and Vershik[63–65] have studied this algebra and some of its applications. They use a formalism intermediate between a bracket and a 2-index algebra, and Fourier transform just one variable, giving $\mathbb{Z}$-graded algebras with the elements of each grade indexed by a continuous parameter. They also transform the Poisson algebra to the Cartan-Weyl basis. In my notation, the Cartan-Weyl basis for the infinite algebra of L's given by (3.3) is

$$E_q^p = \sum_j e^{ipj} L_{\frac{1}{2}(q+j),\frac{1}{2}(q-j)} \ .$$

This result may be obtained by checking the commutation relation

$$[E_{q_1}^{p_1}, E_{q_2}^{p_2}] = \tfrac{1}{2}(q_1 + q_2)\delta'(p_1 - p_2)E_{q_1+q_2}^{\frac{1}{2}(p_1+p_2)} - \tfrac{1}{2}(q_1 - q_2)\delta(p_1 - p_2)\frac{\partial}{\partial p_1}E_{q_1+q_2}^{\frac{1}{2}(p_1+p_2)} \ . \quad (3.20)$$

In Saveliev and Vershik's notation,

$$X_q(f) = \int_{-\infty}^{\infty} f(p)E_q^p dp \ .$$

Thus, multiplying equation (3.20) by $f(p_1)g(p_2)$ and performing this integral transform yields precisely their equation for the $su(\infty)$ commutator

$$[X_{q_1}(f), X_{q_2}(g)] = X_{q_1+q_2}(q_2 f'(p_1 + p_2)g(p_1 + p_2) - q_1 f(p_1 + p_2)g'(p_1 + p_2)) \ .$$

They also consider the non-linear equations associated with these algebras, in particular the generalization of the Liouville equation, the 'heavenly equation'.

On the torus, the Poisson bracket algebra is in its simplest form, and has also been studied by Novikov *et al.*[66, 67] It first appeared in the work of Arnold.[46, 47] As has been shown, the Poisson algebra on the torus may be thought of as $su(\infty)$. Floratos, Iliopoulos and Tiktopoulos[68] utilized Hoppe's identification[58, 59] of $\mathrm{SDiff}_0(S^2)$ with $su(N)$ to take the limit of $su(N)$ gauge theory. In this context an intriguing connection to strings emerges,[3, 44, 69, 70] as there is a correspondence between the classical vacuum states of the resulting $su(\infty)$ Yang-Mills theory and the configurations of the classical string in terms of the quadratic Schild-Eguchi[71–74] action density. This is also a symmetry of toroidal membranes,[75, 76] which provides an explanation for the connection between membrane theories and $su(\infty)$ Yang-Mills theory. It is possible to build another gauge invariant theory by replacing the Poisson bracket by the Moyal bracket, but it is not clear what system this represents.

On the sphere, basis functions $Y_{m_1 m_2}(x, y)$, there are also applications in membrane physics[45, 58, 59, 77–81] and in atmospheric physics.[82, 83]

Bender and Dunne[84] study quantum mechanical systems, finding exact solutions of the Heisenberg operator differential equations by finding a quantum analogue of the classical action-angle variable. Their operators satisfy the Moyal algebra in a basis for the plane, $x^{m_1+1}y^{m_2+1}$. The Poisson algebra on the plane was studied by Fuks,[85] as the algebra of Hamiltonian vector fields. This is also the algebra of conserved currents of the Kadomtsev-Petviashvili equation, as investigated by Case and Monge.[86, 87]

Other algebras of interest are the Zamolodchikov W_N-algebras,[88] the conformal algebras of spin $\leq N$, which are not Lie algebras, as they are not closed under commutation, but are non-linear. Work of Bakas,[89, 90] and Bilal,[91] demonstrates that the infinite limit of these algebras *is* a Lie algebra, explicitly, the Poisson algebra on the cylinder, basis functions $e_{\boldsymbol{m}}(\boldsymbol{x}) = e^{m_1 x_1} x_2^{m_2-1}$, so that

$$[L_{\boldsymbol{m}}, L_{\boldsymbol{n}}] = ((n_2 - 1)m_1 - (m_2 - 1)n_1)L_{(m_1+n_1, m_2+n_2-2)} \,.$$

This basis for the algebra has a manifest Virasoro subalgebra, the generators $L_{(m,2)}$.

Further work by Pope *et al.*[92–94] finds a deformation of this which has central extensions at all spins; this is also a Lie algebra, as shown by Fairlie and Nuyts,[95] the Moyal algebra on the cone, basis functions $e^{m_1 x/y} y^{2(m_2+1)}$. In their paper, Fairlie and Nuyts have three 2-index algebras, the first is a renormalization of that of Pope *et al.*, the other two are the Moyal algebra on the cylinder and with hypergeometric basis functions.

The allowable central extensions in these cases depend on the particular basis, so for physical applications this choice is of great importance. However, the fundamental algebra and its underlying associative product is the same in all these varied applications.

4. The Uniqueness of the Moyal Algebra

As discussed in the previous chapter, the Moyal bracket has intriguing connections with quantum mechanics, and its algebras of modes appear in many varied areas of theoretical physics. In this chapter I present a proof of the uniqueness of the Moyal bracket, by proving that all Lie brackets of functions which satisfy the Jacobi identity may be transformed to the Moyal bracket. I show that any 2-index algebra may be written in terms of some Lie bracket of functions, and thus that all 2-index algebras are locally equivalent to the sine algebra, or one of its subalgebras.[4]

4.1 Introduction

There is a detailed paper by Bayen *et al.*,[96] which includes a statement that the Moyal bracket is the only deformation of the Poisson bracket which may be used for a phase space formulation of quantum mechanics. The main assumption they make is that their posited bracket is a function of the Poisson bracket. A more recent paper by Arveson[97] gives a more direct proof that the only function of iterated Poisson brackets which satisfies the Jacobi identities is the Moyal bracket.

I will consider a more general bracket. The form of brackets most easily dealt with is that of a sum of derivatives of the functions, such as, for the Moyal case

$$\{f,g\}_{\text{Moyal}} = \sum_{s=0}^{\infty} \frac{(-1)^s \kappa^{2s}}{(2s+1)!} \sum_{j=0}^{2s+1} (-1)^j \binom{2s+1}{j} (\partial_x^{2s+1-j}\partial_y^j f(x,y))(\partial_x^j \partial_y^{2s+1-j} g(x,y)) \,, \tag{4.1}$$

as given in the previous chapter which in the limit $\kappa \to 0$ becomes the Poisson bracket

$$\{f,g\}_{\text{Poisson}} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x} \,. \tag{4.2}$$

The most general possible bracket of this form is

$$\{f,g\} = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \kappa^{r+s-2} \sum_{j=0}^{r}\sum_{k=0}^{s} b_{rj,sk} (\partial_x^j \partial_y^{r-j} f)(\partial_x^k \partial_y^{s-k} g) \,, \tag{4.3}$$

where the b's are arbitrary constants.

As discussed in the previous chapter for the particular cases of Moyal and Poisson, from any bracket algebra many seemingly different 2-index algebras may be written down by choosing a basis for the functions. Conversely, for any 2-index algebra there is a corresponding algebra of functions under the operation of some bracket. This has also been shown by Dorfman and Gelfand;[98] here I give a simpler argument with a slight restriction on the structure constants. The general 2-index algebra is

$$[L_{\boldsymbol{m}}, L_{\boldsymbol{n}}] = C^{\boldsymbol{p}}_{\boldsymbol{mn}} L_{\boldsymbol{p}} \ , \tag{4.4}$$

where $\boldsymbol{m}$ *etc.* are integral 2-vectors and summation is implied.

First rewrite (4.4) as an algebra of functions. Let $e_{\boldsymbol{m}}(\boldsymbol{x})$ be a basis of functions, and define $L_{e_m} = L_{\boldsymbol{m}}$. Then any L_f is defined through the composition $f(\boldsymbol{x}) = \sum_{\boldsymbol{m}} f_{\boldsymbol{m}} e_{\boldsymbol{m}}(\boldsymbol{x})$ to be $\sum_{\boldsymbol{m}} f_{\boldsymbol{m}} L_{e_m}$. The commutation relations become

$$[L_{e_m}, L_{e_n}] = C^{\boldsymbol{p}}_{\boldsymbol{mn}} L_{\boldsymbol{p}} = L_{C^{p}_{mn} e_p} \ .$$

This is equivalent to a bracket algebra of the above type if there is some bracket such that $\{e_{\boldsymbol{m}}, e_{\boldsymbol{n}}\} = C^{\boldsymbol{p}}_{\boldsymbol{mn}} e_{\boldsymbol{p}}$. Now if e's are powers, $e_{\boldsymbol{m}}(\boldsymbol{x}) = x_1^{m_1+s_1} x_2^{m_2+s_2}$, for sufficiently large integers s_1, s_2, and the structure constants $C^{\boldsymbol{p}}_{\boldsymbol{mn}}$ tend to zero as $\boldsymbol{p} - \boldsymbol{m} - \boldsymbol{n}$ tends to infinity, then it is possible to write

$$\sum_{\boldsymbol{p}} C^{\boldsymbol{p}}_{\boldsymbol{mn}} e_{\boldsymbol{p}} = \sum_{r,j} b_{r,j} \partial_x^j e_{\boldsymbol{m}} \partial_x^{r-j} e_{\boldsymbol{n}} \ ,$$

This expression is a polynomial in x_1, x_2, and by equating coefficients the b's may be determined in terms of the (given) C's, and thus the algebra may be written as a bracket algebra of the form (4.3).

Here I provide a straightforward proof that all 2-index infinite Lie algebras correspond to the Moyal algebra (or its special case, the Poisson algebra) in some basis. This is done by showing that any bracket algebra satisfying the Jacobi identities may be transformed to the Moyal bracket algebra. This also means that the only allowable associative product is that corresponding to the Moyal bracket, the exponential bracket,* or star product, which in the limit becomes the ordinary product. This basis-independent formulation of 2-index algebras in terms of brackets does not allow the discussion of central extensions, the existence and form of which depend on the basis used.

⋆ which is an associative product, not a Lie bracket

4.2 Case $r = s$

First consider a bracket of the following form:

$$\{f, g\} = \sum_{r=1}^{\infty} \kappa^{r-1} \sum_{j=0}^{r} \sum_{k=0}^{r} b_{rjk} (\partial_x^j \partial_y^{r-j} f)(\partial_x^k \partial_y^{r-k} g) \; . \tag{4.5}$$

When $\kappa = 0$ this reduces to the Poisson bracket, antisymmetry forcing $b_{100} = b_{111} = 0$, $b_{110} = -b_{101}$. Here there is an overall normalization, b_{101}, which may be set to 1. In general, antisymmetry requires $b_{rjk} = -b_{rkj}$, and there will be one overall normalization, corresponding to a choice of b_{101}. The parameter κ may be absorbed into the b's, and henceforth this is done.

It is also necessary to factor out by transformations of the independent variables x, y, such as

$$\begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} .$$

This transformation changes the b's but does not indicate a genuinely different bracket. The coefficient of $(\partial_y f \partial_x g - \partial_x f \partial_y g)$ transforms from b_{101} to $(ad - bc)b_{101}$ so $ad - bc$ is chosen to make the new b_{101}=1. This is merely a choice of overall normalization. There is still some freedom; that of transformations of the above type with $ad - bc = 1$, which leave this choice unaffected. Given a bracket of the above form, with arbitrary b's, a, b, c, d may be chosen to transform the b_2's to a simple form. Under the above change of variables they are mapped

$$\begin{pmatrix} b_{201} \\ b_{202} \\ b_{212} \end{pmatrix} \rightarrow \begin{pmatrix} b'_{201} \\ b'_{202} \\ b'_{212} \end{pmatrix} = \begin{pmatrix} d^2 & 2bd & b^2 \\ cd & bc + ad & ab \\ c^2 & 2ac & a^2 \end{pmatrix} \begin{pmatrix} b_{201} \\ b_{202} \\ b_{212} \end{pmatrix} .$$

Note that this mapping preserves $b_{202}^2 - b_{201} b_{212} = \eta^2$. The matrix

$$\begin{pmatrix} -\frac{b_{201}}{2\eta} & 1 \\ \frac{b_{201} b_{212}}{2\eta(b_{202}+\eta)} & -\frac{b_{202}+\eta}{b_{201}} \end{pmatrix} \begin{pmatrix} \frac{1}{b} & 0 \\ 0 & b \end{pmatrix} \quad \text{for } b_{201}, \eta \neq 0$$

makes $b'_{201} = b'_{212} = 0, b'_{202} = \eta$. When $b_{201} = 0$ the corresponding matrix is

$$\begin{pmatrix} 0 & 1 \\ -1 & -\frac{b_{212}}{2b_{202}} \end{pmatrix} \begin{pmatrix} \frac{1}{b} & 0 \\ 0 & b \end{pmatrix}$$

When $\eta = 0$ there are certain subtleties, but detailed analysis shows that this is equivalent to the case $b'_{201} = b'_{212} = 0$, $b'_{202} = \eta = 0$. Thus any bracket of the form (4.5) can be transformed to this case with general η.

Further constraints on the b's are given by the Jacobi identity

$$\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0 \,. \tag{4.6}$$

$$\{\{f,g\},h\} = \sum_{t=1}^{\infty}\sum_{l=0}^{t}\sum_{m=0}^{t} b_{tlm}(\partial_x^l \partial_y^{t-l}\{f,g\}_q)(\partial_x^m \partial_y^{t-m} h)$$

$$= \sum_{t=1}^{\infty}\sum_{l=0}^{t}\sum_{m=0}^{t}\sum_{r=1}^{\infty}\sum_{j=0}^{r}\sum_{k=0}^{r}\sum_{n=0}^{l}\sum_{p=0}^{t-l} \binom{l}{n}\binom{t-l}{p} b_{tlm} b_{rjk} (\partial_x^{j+n}\partial_y^{r-j+p} f)(\partial_x^{k+l-n}\partial_y^{r-k+t-l-p} g)(\partial_x^m \partial_y^{t-m} h) \,.$$

The Jacobi identity requires that this expression plus cyclic permutations in f, g, h must be zero. This must be true for arbitrary functions f, g, h, so each derivative of these functions must have coefficient zero. Let the coefficient of

$$(\partial_x^{f_x}\partial_y^{f_y} f)(\partial_x^{g_x}\partial_y^{g_y} g)(\partial_x^{h_x}\partial_y^{h_y} h)$$

be $T(f_x, f_y, g_x, g_y, h_x, h_y)$, and then $T + \mathrm{cyclic}(f, g, h) = 0$.

$$T(f_x, f_y, g_x, g_y, h_x, h_y) = \sum_{t,l,m,r,j,k,n,p} \binom{l}{n}\binom{t-l}{p} b_{tlm} b_{rjk} \,,$$

summing under the constraints:

$$\begin{array}{ll} f_x = j + n & 0 \le l, m, j, k, n, p \\ f_y = r - j + p & 1 \le t, r \\ g_x = k + l - n & l, m \le t \\ g_y = r - k + t - l - p & j, k \le r \\ h_x = m & n \le l \\ h_y = t - m & p \le t - l \end{array}$$

which may be rewritten as

$$
\begin{aligned}
m &= h_x \\
t &= h_x + h_y \\
r &= (f_x + f_y + g_x + g_y - h_x - h_y)/2 \\
j &= f_x + g_x - k - l \\
p &= r + t - g_y - k - l \\
n &= k + l - g_x
\end{aligned}
\qquad
\begin{gathered}
0, r - g_y \le k \le r, g_x \\
0 \le l \le t \\
g_x, f_x + g_x - r \le k + l \le f_x + g_x, r + t - g_y
\end{gathered}
$$

from which any T can be calculated by summing over allowed k, l.

Now the Jacobi identities are

$$CT(f_x, f_y, g_x, g_y, h_x, h_y) = T(f_x, f_y, g_x, g_y, h_x, h_y) + \text{cyclic}(f, g, h) = 0 \ .$$

The b's can be determined by looking at certain of these, in particular:*

$$
\begin{aligned}
CT(\beta, \alpha - \beta, \gamma, \alpha - \gamma, 1, 1) &= 2b_{201}b_{\alpha-1,\beta,\gamma} - 2b_{212}b_{\alpha-1,\beta-1,\gamma-1} \\
&\quad + (2\gamma - \alpha)b_{\alpha,\beta,\gamma} + (2\beta - \alpha)b_{\alpha,\beta,\gamma} \\
&= 0 \\
CT(\zeta, \alpha - \zeta, \alpha - \zeta + 1, \zeta - 1, 0, 2) &= \zeta b_{\alpha,\zeta,\alpha-\zeta} - b_{201}b_{\alpha-1,\zeta,\alpha-\zeta} - 2b_{202}b_{\alpha-1,\zeta-1,\alpha-\zeta} \\
&\quad - b_{201}b_{\alpha-1,\zeta-1,\alpha-\zeta+1} + (\alpha - \zeta + 1)b_{\alpha,\zeta-1,\alpha-\zeta+1} \\
&= 0 \\
CT(\alpha, 0, 0, \alpha - 1, 1, 2) &= b_{312}b_{\alpha-2,0,\alpha-2} + \tfrac{1}{2}\alpha(\alpha - 1)b_{\alpha,0,\alpha} \\
&\quad - b_{\alpha-1,0,\alpha-2}b_{212} - (\alpha - 1)b_{\alpha-1,0,\alpha-1}b_{202} \\
&= 0
\end{aligned}
$$

which may be solved to give the following recurrence relations for the b's:

$$
\begin{aligned}
b_{\alpha\beta\gamma} &= \tfrac{1}{\alpha-\beta-\gamma}(b_{201}b_{\alpha-1,\beta,\gamma} - b_{212}b_{\alpha-1,\beta-1,\gamma-1}) \qquad \text{for} \ \ \alpha \neq \beta + \gamma \ , \\
b_{\alpha,\zeta,\alpha-\zeta} &= \tfrac{1}{\zeta}((\zeta - 1 - \alpha)b_{\alpha,\zeta-1,\alpha-\zeta+1} + b_{201}b_{\alpha-1,\zeta,\alpha-\zeta} \\
&\qquad + 2b_{202}b_{\alpha-1,\zeta-1,\alpha-\zeta} + b_{201}b_{\alpha-1,\zeta-1,\alpha-\zeta+1}) \qquad \text{for} \ \ \zeta \neq 0 \ ,
\end{aligned}
$$

and, using $b_{312} = -3b_{303} + b_{212}b_{201} + 2b_{202}^2$ and $b_{\alpha0\alpha-1} = b_{201}b_{\alpha-1,0,\alpha-1}$,

$$b_{\alpha,0,\alpha} = \tfrac{2}{\alpha(\alpha-1)}((3b_{303} - 2b_{202}^2)b_{\alpha-2,0,\alpha-2} + (\alpha - 1)b_{202}b_{\alpha-1,0,\alpha-1}) \ .$$

There is a unique solution to these recurrence relations* in closed form in the case

* The calculation of the T's and the solution of the recurrence relations are given in appendices.

$b_{201} = b_{212} = 0$ and $b_{202} = \eta$:

$$b_{\alpha,\zeta,\alpha-\zeta} = \frac{(\eta-\rho)^{\zeta}(\eta+\rho)^{\alpha-\zeta} - (\eta+\rho)^{\zeta}(\eta-\rho)^{\alpha-\zeta}}{2\rho\zeta!(\alpha-\zeta)!} ,$$

where $\rho = \sqrt{6b_{303} - 3\eta^2}$ and $b_{\alpha\beta\gamma} = 0$ if $\alpha \neq \beta + \gamma$.

If $\eta = 0$ this is the Moyal bracket, where $b_{\alpha,\zeta,\alpha-\zeta} = \rho^{\alpha-1}/\zeta!(\alpha-\zeta)!$ if α is odd, zero otherwise, so

$$\{f,g\}_{\text{Moyal}} = \sum_{s=0}^{\infty} \frac{(-1)^s \kappa^{2s}}{(2s+1)!} \sum_{j=0}^{2s+1} (-1)^j \binom{2s+1}{j} (\partial_x^j \partial_y^{2s+1-j} f)(\partial_x^{2s+1-j} \partial_y^j g) ,$$

where $s = (r-1)/2$, $\kappa = i\rho$.

In fact the η dependence may be removed as there is a convolution of the arbitrary functions that transforms this from any η to $\eta = 0$. This corresponds to a change of origin for the independent variables, where $f(x,y)$ is replaced by a convolution:

$$\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(\frac{i(x-x')(y-y')}{\eta}\right) f(x',y') dx' dy' .$$

The iteration of this expression with a second parameter η' just reproduces the same formula with parameter $\eta + \eta'$.

This convolution takes any bracket of the above type to Moyal form, and so all brackets of the form (4.5) which satisfy the Jacobi identities are equivalent to the Moyal bracket.

4.3 General Case

There is a more general bracket than that considered above, one in which the functions f, g are not always of the same order in derivatives, that is r and s are not necessarily the same in the following expression:

$$\{f,g\} = \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \kappa^{r+s-2} \sum_{j=0}^{r} \sum_{k=0}^{s} b_{rj,sk} (\partial_x^j \partial_y^{r-j} f)(\partial_x^k \partial_y^{s-k} g) . \tag{4.7}$$

The analysis of the previous section may be applied to this more general case, and in a similar way produces recurrence relations defining the b's in terms of those at the lowest level. In particular, all the b's with $r \neq s$ are defined in terms of $b_{00,10}$ and $b_{00,11}$. But there are transformations which take the general bracket above to one with these particular b's vanishing, and so the Jacobi identities imply that all the b's with $r \neq s$ must vanish.

The $\kappa = 0$ case of this bracket is

$$\{f, g\}_0 = b_{00,10}(f\partial_y g - g\partial_y f) + b_{00,11}(f\partial_x g - g\partial_x f) + b_{10,11}(\partial_y f\partial_x g - \partial_y g\partial_x f) ,$$

the last term of this being the Poisson bracket. If the basis functions are multiplied by a factor and simultaneously the coordinates are changed:

$$\begin{aligned} f(x,y) &\rightarrow \exp(b_{00,10}x - b_{00,11}y)F(x,y) \\ g(x,y) &\rightarrow \exp(b_{00,10}x - b_{00,11}y)G(x,y) \\ (x,y) &\rightarrow (X,Y) = \tfrac{1}{2\sqrt{2}}\left(\tfrac{1}{b_{00,11}}e^{-2b_{00,11}y}, \tfrac{1}{b_{00,10}}e^{2b_{00,10}x}\right) , \end{aligned}$$

this reduces to the Poisson bracket,

$$\{f, g\}_0 = b_{10,11}(\partial_Y F\partial_X G - \partial_Y G\partial_X F) .$$

Thus all brackets of the form (4.7) may be transformed to a bracket with $b_{00,10} = b_{00,11} = 0$, and any such bracket satisfying the Jacobi identity must be of the form of (4.5), and so these, too, are all equivalent to Moyal.

4.4 Discussion

Almost all of the many Lie algebras discussed in the physics literature are somehow encompassed by the Moyal bracket. All the classical finite Lie algebras are special cases of the sine algebra, or indeed of the Moyal algebra on any manifold. The exceptional algebras have not been found explicitly, although they are clearly contained as subalgebras of sufficiently large members of the A_n series. It may be possible to find a more tasteful expression of them in a trigonometric basis, as part of some infinite series, as the exceptional Lie algebras may be thought of as being part of E, F and G series, the lower cases being isomorphic to classical algebras, and the higher cases being infinite algebras.

The Virasoro algebra is present as a subalgebra of the Poisson algebra, and in a similar way the $su(2)$ Kac-Moody algebra in its $\mathbb{Z}$-graded form (discussed in chapter 1) is a subalgebra of the Moyal algebra. As for the higher Kac-Moody algebras, they may be constructed by affinizing the finite Lie algebras in this basis, introducing a third index on the K's.

As discussed in [2], there are Lie algebras whose generators carry more than two indices, so that they lie on a d dimensional integer lattice, $\boldsymbol{m} = (m_1, \ldots, m_d)$, satisfying

$$[K_{\boldsymbol{m}}, K_{\boldsymbol{n}}] = r \sin \det(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{d-2}, \boldsymbol{m}, \boldsymbol{n}) K_{\boldsymbol{m}+\boldsymbol{n}} + \boldsymbol{a}.\boldsymbol{m}\delta_{\boldsymbol{m}+\boldsymbol{n},\boldsymbol{0}} \ , \tag{4.8}$$

where the $\boldsymbol{x}_p$ are arbitrary d-vectors. However, a rotation in d dimensions brings $\boldsymbol{x}_1$ to the form $(x, 0, \ldots, 0)$, and thus reduces the d dimensional lattice algebra to a stack of $(d-1)$ dimensional ones, with the first component of the generators labelling the position in the stack, and not appearing in the structure constants. This process may be repeated until the algebra becomes the sine algebra with $d-2$ extra inert indices, which merely add under commutation.

These extra indices are just the same as those introduced in the formulation of Kac-Moody algebras. Bakas and Kiritsis[99] discuss the algebra W_∞^∞, associated with symplectic diffeomorphisms in four dimensions, and find the algebra (4.8) with $d = 4$, the generators indexed by two 2-vectors,

$$[L^j_{\boldsymbol{m}}, L^k_{\boldsymbol{m}}] = \sin \tfrac{\pi}{N}(\boldsymbol{m} \times \boldsymbol{n}) L^{j+k}_{\boldsymbol{m}+\boldsymbol{n}} \ .$$

An interesting algebra is introduced by Savvidy,[100] the general algebra of vector fields on the torus, Vect(T^2). The generators carry three indices, a 2-vector $\boldsymbol{m}$, over which the algebra is $\mathbb{Z}^2$-graded, and also an index taking the values 1 or 2. The commutation relations are

$$\begin{aligned}
[L^1_{\boldsymbol{m}}, L^1_{\boldsymbol{n}}] &= (m_1 - n_1) L^1_{\boldsymbol{m}+\boldsymbol{n}} \ , \\
[L^2_{\boldsymbol{m}}, L^2_{\boldsymbol{n}}] &= (m_2 - n_2) L^2_{\boldsymbol{m}+\boldsymbol{n}} \ , \\
[L^1_{\boldsymbol{m}}, L^2_{\boldsymbol{n}}] &= m_2 L^1_{\boldsymbol{m}+\boldsymbol{n}} - n_1 L^2_{\boldsymbol{m}+\boldsymbol{n}} \ .
\end{aligned}$$

This was generalized by David Fairlie and myself to the 3-index algebra

$$\begin{aligned}
[L^j_{\boldsymbol{m}}, L^j_{\boldsymbol{n}}] &= (m_j - n_j) L^j_{\boldsymbol{m}+\boldsymbol{n}} \ , \\
[L^j_{\boldsymbol{m}}, L^k_{\boldsymbol{n}}] &= m_k L^j_{\boldsymbol{m}+\boldsymbol{n}} - n_j L^k_{\boldsymbol{m}+\boldsymbol{n}} \ .
\end{aligned}$$

It is not clear how such algebras fit into a bracket formalism.

The exponential bracket, or star product of two functions of two variables is

$$f \star g = \lim_{\boldsymbol{x}' \to \boldsymbol{x}} \exp(\kappa \nabla \times \nabla') f(\boldsymbol{x}) g(\boldsymbol{x}') ,$$

and may be generalized in a straightforward way to functions of r variables by[101]

$$f \star g = \sum_{n=0}^{\infty} \sum_{s=0}^{n} \sum_{j=1}^{r} \frac{1}{(n-s)!s!} \kappa^n \frac{\partial^n f}{\partial x_j^{n-s} \partial y_j^s} \frac{\partial^n g}{\partial x_j^s \partial y_j^{n-s}} .$$

By antisymmetrizing and choosing a set of basis functions, infinite dimensional Lie algebras with an arbitrary number of indices may be built. It is also possible to consider a generalized product of more than two functions of an arbitrary number of variables.[101]

5. Quantum Algebras

Some of the many applications of Lie algebras to physics have been discussed in the previous chapters, and it is evident that these mathematical structures are of importance in our understanding of the physical world. However, as yet there is no genuine unified theory based on a symmetry described by a Lie algebra. The generalizations to infinite dimensional algebras indicated in this thesis are one basis for new unifying theories; another is the introduction of quantum algebras.

Quantum algebras, variously referred to as Yang-Baxter algebras, quantum universal enveloping algebras or quantum groups, are algebraic structures which describe perturbed symmetries, one source of such perturbations being quantum corrections to some classical structure. They have one or more parameters, in some limit of which the quantum algebra becomes a Lie algebra; this may be thought of as the classical limit of the algebra. In this sense quantum algebras are deformations of Lie algebras.

These structures have appeared in many areas of mathematical physics, amongst others; two dimensional solvable models; anisotropic spin chains; three dimensional Chern-Simons theory; rational conformal field theories; and non-standard quantum statistics. There are several good reviews of the subject.[102–105]

From the point of view of the mathematician these algebras are Hopf algebras, which are bialgebras, and have a coproduct, counit and antipode as well as the ordinary product. I shall give a brief definition of these structures, taken from [105], but in most quantum algebras described in later sections I will not give the coproducts explicitly.

5.1 Introduction

The first quantum deformations of Lie algebras were studied by Kulish and Reshetikhin,[106] who found a deformation of $su(2)$,

$$[J_0, J_\pm] = \pm J_\pm \ , \qquad [J_+, J_-] = \frac{\sinh(2\hbar J_0)}{2\sinh\hbar} \ ,$$

which retrieves $su(2)$ in the limit $\hbar \to 0$. Equivalently, in terms of a parameter q, $q = e^\hbar$, the deformed commutator is

$$[J_+, J_-] = \frac{q^{2J_0} - q^{-2J_0}}{2(q - q^{-1})} \ ,$$

which may be written more simply in terms of 'q-deformations', where the q-deformation of x is defined by the Chebyshev polynomial of the second kind,

$$[x]_q \equiv \frac{q^x - q^{-x}}{q - q^{-1}} ,$$

and then the algebra becomes

$$[J_0, J_\pm] = \pm J_\pm , \qquad [J_+, J_-] = \tfrac{1}{2}[2J_0]_q . \tag{5.1}$$

Drinfeld[107] and Jimbo[108, 109] have also studied this deformation, and the $su(N)$ case, with reference to their applications as solutions to the Yang-Baxter factorization equations in the S-matrices of two dimensional solvable models. Another application of this deformation of $su(2)$ is in the theory of spin chains, where certain Hamiltonians are invariant under this algebra.[110]

Many alternative deformations of Lie algebras, in particular of $su(2)$, have been introduced. The example above is unusual in that is involves the exponential of one of the generators. Almost all of the other deformations of interest are of order no more than quadratic in the generators. One exception is the algebra of Sklyanin,[111]

$$\begin{aligned} &[S_0, S_3] = 0 , \qquad [S_+, S_-] = 4S_0S_3 , \qquad [S_3, S_\pm] = \pm(S_0S_\pm + S_\pm S_0) , \\ &[S_0, S_\pm] = \pm\tanh^2 u(S_\pm S_3 + S_3S_\pm) , \qquad S_0^2 - S_3^2 \tanh^2 u = 4\sinh^2 u , \end{aligned}$$

also studied by Macfarlane.[112]

A coproduct, Δ, is an algebra homomorphism $V \to V \otimes V$, where V is the vector space spanned by the generators. For the algebra (5.1) one possible coproduct is

$$\Delta(J_0) = J_0 \otimes \mathbb{1} + \mathbb{1} \otimes J_0 , \qquad \Delta(J_\pm) = J_\pm \otimes q^{J_0} + q^{-J_0} \otimes J_\pm .$$

Note that the $\Delta(J)$ also satisfy the algebra (5.1), and that the coproduct is coassociative but not cocommutative.

A counit, ϵ, is an algebra homomorphism which reverses the effect of the above comultiplication. This satisfies

$$m\big((\epsilon \otimes \mathbb{1})\Delta(J_a)\big) = m\big((\mathbb{1} \otimes \epsilon)\Delta(J_a)\big) = J_a ,$$

where m is the multiplication map $m(a \otimes b) \equiv ab$ For the example above, it is $\epsilon(J_a) = 0$, $\epsilon(\mathbb{1}) = \mathbb{1}$.

Finally, an antipode, S, is an algebra antihomomorphism, $S(J_aJ_b) = S(J_b)S(J_a)$, satisfying

$$\sigma\big(\Delta(S(J_a))\big) = (S\otimes S)\Delta(J_a)\ , \qquad m\big((S\otimes \mathbb{1})\Delta(J_a)\big) = m\big((\mathbb{1}\otimes S)\Delta(J_a)\big) = \epsilon(J_a)\ ,$$

where σ is the permutation map $\sigma(a\otimes b) \equiv b\otimes a$. Here, $S(J_\pm) = -q^{\pm 1}J_\pm$, $S(J_0) = -J_0$.

5.2 Deformers and Representations

Curtright and Zachos[113] shed considerable light on the field of quantum algebras through their definition of 'deformers'. They provide a set of simple invertible functionals which transform between the a Lie algebra and any of its deformations, and hence between any two deformations of a given algebra. Substituting any representation of the Lie algebra into these functionals consequently produces a representation of the quantum algebra, and thus there is a correspondence between the representations of quantum algebras and their undeformed counterparts. Through these deformers comultiplication rules can be deduced from those of the classical algebra.

However, there are certain special values of the deformation parameters for which the above deformers are *not* invertible, usually when q is some root of unity. These special cases of quantum algebras and their representations have to be treated differently.

As an example, consider the Lie algebra $su(2)$ in the Cartan-Weyl basis,

$$[j_0, j_\pm] = \pm j_\pm\ , \qquad [j_+, j_-] = j_0\ , \tag{5.2}$$

and its deformation (5.1)

$$[J_0, J_\pm] = \pm J_\pm\ , \qquad [J_+, J_-] = \tfrac{1}{2}[2J_0]_q\ .$$

The Casimir invariant of (5.2) is

$$C \equiv 2j_+j_- + j_0(j_0-1) = 2j_-j_+ + j_0(j_0+1) \equiv j(j+1)\ ,$$

defining the operator j, and the corresponding invariant for (5.1) is

$$C_q \equiv 2J_+J_- + [J_0]_q[J_0-1]_q = 2J_-J_+ + [J_0]_q[J_0+1]_q \equiv [j]_q[j+1]_q\ .$$

The deformers between these algebras give the generators of the deformed algebra in terms of functionals, Q, of those of the Lie algebra, and for the case of Hermitian operators[113]

$$J_0 = Q_0(j_0) = j_0\,, \qquad J_+ = Q_+(j_0, j) = \sqrt{\frac{[j_0+j]_q[j_0-1-j]_q}{(j_0+j)(j_0-1-j)}}\,j_+\,, \qquad J_- = Q_+^\dagger(j_0, j)\,.$$

Now representations of (5.1) may be found by substituting representations of $su(2)$ into the deformers. The fundamental representation of $su(2)$ is also a representation of the quantum version — this is often the case — and so

$$J_0 = \tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,, \qquad J_+ = \tfrac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\,, J_- = J_+^\dagger\,,$$

satisfies (5.1). Another representation of $su(2)$ is given by

$$j_0 = \tfrac{1}{2}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}\,, \qquad j_+ = \tfrac{1}{\sqrt{2}}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}\,,$$

which maps to

$$J_0 = j_0\,, \qquad J_+ = \tfrac{1}{\sqrt{2}}\begin{pmatrix} 0 & \sqrt{[3]_q} & 0 & 0 \\ 0 & 0 & [2]_q & 0 \\ 0 & 0 & 0 & \sqrt{[3]_q} \\ 0 & 0 & 0 & 0 \end{pmatrix}\,.$$

In the special case $q = e^{2\pi i/3}$, $[3]_q = 0$, and the **4** representation above reduces to $\mathbf{1} \oplus \mathbf{2} \oplus \mathbf{1}$. This type of reduction always takes place for q a root of unity with period smaller than the dimensionality of the representation.

5.3 Quommutator Algebras

One way of deforming a Lie algebra is to introduce one or more parameters into the left hand side of the commutation relations, replacing the commutator by a q-commutator, or 'quommutator',*

$$[Q_j, Q_k]_{q_{jk}} = q_{jk} Q_j Q_k - q_{jk}^{-1} Q_k Q_j \ , \tag{5.3}$$

where the q_{jk} are some parameters, real or complex numbers. The quommutator is made antisymmetric,

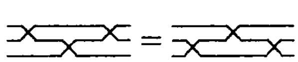
$$[Q_j, Q_k]_{q_{jk}} = -[Q_k, Q_j]_{q_{kj}} \ ,$$

by requiring $q_{kj} = q_{jk}^{-1}$, and in the limit $q_{jk} \to 1$ this becomes the ordinary commutator.

The quommutator is defined in terms of a product (5.3), the condition that this product is associative puts a constraint on the possible algebras, just as the Jacobi identity dictates possible Lie algebras. A sufficient, though not necessary condition for the associativity of the underlying algebra is that the 'braiding relation' is satisfied. Starting with a product of three generators, there are two distinct ways of turning the order around; by swapping the first pair using the quommutation relations, then the second pair, then the first pair once more; or the second pair, the first, and the second; as shown in the following diagram:

$$\begin{tikzcd}
 & YXZ \arrow[r] & YZX \arrow[dr] & \\
XYZ \arrow[ur] \arrow[dr] & & & ZYX \\
 & XZY \arrow[r] & ZXY \arrow[ur] &
\end{tikzcd}$$

The condition that the results of the two processes are equal for any three generators X, Y, Z is the braiding relation, so-called as it corresponds to the equivalence of the two braids:



In the $q_{jk} \to 1$ limit this reduces to the Jacobi identity.

So a quommutator algebra may be written down starting with any Lie algebra in any basis, and all that is required is that the braiding relation holds. This gives conditions on the allowable values of q_{jk}. In a different basis, the same Lie algebra may give a different quommutator algebra. Thus there are many deformations of even the simplest Lie algebra, $su(2)$.

* some authors use the less symmetric definition $[Q_j, Q_k]_{q_{jk}} = Q_j Q_k - q_{jk} Q_k Q_j$

One basis for $su(2)$ is the cyclically symmetric basis,

$$[X,Y]=Z\ , \qquad [Y,Z]=X\ , \qquad [Z,X]=Y\ .$$

A parameter may be introduced into each commutation relation, giving

$$\begin{aligned}
[X,Y]_q &= qXY - \tfrac{1}{q}YX = Z\ ,\\
[Y,Z]_r &= rYZ - \tfrac{1}{r}ZY = X\ ,\\
[Z,X]_p &= pZX - \tfrac{1}{p}XZ = Y\ .
\end{aligned} \tag{5.4}$$

The braiding relation may be calculated,

$$\frac{p^2}{r^2q^2}ZYX + \frac{p^2}{r^2q}Z^2 - \frac{p}{r^2}Y^2 + \frac{1}{r}X^2 = \frac{p^2}{r^2q^2}ZYX + \frac{p^2}{q^2r}X^2 - \frac{p}{q^2}Y^2 + \frac{1}{q}Z^2\ .$$

If X^2, Y^2, Z^2 are independent, this requires $p^2 = q^2 = r^2$, and so there is one free parameter remaining. This algebra has been studied in detail, and irreducible representations of arbitrary size have been found, by Fairlie.[114] It has also been studied by Odesskii.[115]

On the other hand, if $\frac{1}{r}X^2 = \frac{1}{p}Y^2 = \frac{1}{q}Z^2$, so there are some further quadratic relations between the generators, the braiding relation is satisfied for all p, q, r. There is a representation for which this is true of this three parameter algebra, given in terms of the σ matrices by

$$X = -\tfrac{i\sqrt{r}}{2}\sigma_1\ , \qquad Y = -\tfrac{i\sqrt{p}}{2}\sigma_2\ , \qquad Z = -\tfrac{i\sqrt{q}}{2}\sigma_3\ .$$

However, it is possible to prove that this is the *only* irreducible representation of this algebra.

In the Cartan-Weyl basis, the commutation relations for $su(2)$ are

$$[J_0, J_\pm] = \pm J_\pm\ , \qquad [J_+, J_-] = J_0\ ,$$

and again, parameters may be introduced into each commutator, the braiding relation this time imposing only one condition, so that there is a two parameter quantum algebra[114]

$$\begin{aligned}
[J_0, J_+]_r &= rJ_0J_+ - \tfrac{1}{r}J_+J_0 = J_+\ ,\\
[J_0, J_+]_{1/r} &= \tfrac{1}{r}J_0J_- - rJ_-J_0 = -J_-\ ,\\
[J_+, J_-]_{1/s} &= \tfrac{1}{s}J_+J_- - sJ_-J_+ = J_0\ .
\end{aligned}$$

Special cases of this have been studied by Witten,[116] with $r = \sqrt{s}$, and Woronowicz,[117] with $r = s^2$. Again, its representations have been found by Fairlie.[114]

5.4 Quadratic Algebras

The most general form for a quadratic algebra is given by

$$Q_j Q_k = S_{jk}^{lm} Q_l Q_m + C_{jk}^r Q_r \ , \tag{5.5}$$

summation implied. The special case of this with $S_{jk}^{lm} = \delta_k^l \delta_j^m$ is

$$Q_j Q_k - Q_k Q_j = C_{jk}^r Q_r \ ,$$

a Lie algebra with structure constants C_{jk}^r.

The braiding relations for such an algebra may be calculated:

$$\begin{aligned} Q_m Q_n Q_p &= S_{mn}^{ab} Q_a Q_b Q_p + C_{mn}^r Q_r Q_p \\ &= S_{mn}^{ab} S_{bp}^{cl} Q_a Q_c Q_l + C_{mn}^r Q_r Q_p + S_{mn}^{ab} C_{bp}^s Q_a Q_s \\ &= S_{mn}^{ab} S_{bp}^{cl} S_{ac}^{jk} Q_j Q_k Q_l + C_{mn}^r Q_r Q_p + S_{mn}^{ab} C_{bp}^s Q_a Q_s + S_{mn}^{ab} S_{bp}^{cl} C_{ac}^t Q_t Q_l \end{aligned}$$

$$\begin{aligned} Q_m Q_n Q_p &= S_{np}^{ab} Q_m Q_a Q_b + C_{np}^r Q_m Q_r \\ &= S_{np}^{ab} S_{ma}^{jc} Q_j Q_c Q_b + C_{np}^r Q_m Q_r + S_{np}^{ab} C_{ma}^s Q_s Q_b \\ &= S_{np}^{ab} S_{ma}^{jc} S_{cb}^{kl} Q_j Q_k Q_l + C_{np}^r Q_m Q_r + S_{np}^{ab} C_{ma}^s Q_s Q_b + S_{np}^{ab} S_{ma}^{jc} C_{cb}^t Q_j Q_t \end{aligned}$$

These two expressions must be equal. The condition that the cubic terms in cancel is the Yang-Baxter equation,[118, 119]

$$S_{mn}^{ab} S_{bp}^{cl} S_{ac}^{jk} = S_{np}^{ab} S_{ma}^{jc} S_{cb}^{kl} \tag{5.6}$$,

which is satisfied for the Lie algebra case $S_{jk}^{lm} = \delta_k^l \delta_j^m$. There are also conditions from the quadratic terms, giving conditions on C, which in the Lie algebra case reduce to the Jacobi identity.

It is possible to simplify notation, and avoid the many indices in the above expressions. Let V be the vector space in which the Q's live. Then the Yang-Baxter equation is an equation in the space $V \otimes V \otimes V$. The operator S acts on $V \otimes V$, and so we denote S

acting on the α and β copies of V in the triple space $V \otimes V \otimes V$ by $S_{\alpha\beta}$. The Yang-Baxter equation may now be written more transparently as

$$S_{12}S_{23}S_{12} = S_{23}S_{12}S_{23} \ .$$

A quommutator algebra,

$$[Q_j, Q_k]_{q_{jk}} = C^r_{jk} Q_r \ ,$$

is equivalent to the general algebra (5.5) with $S^{lm}_{jk} = \delta^l_k \delta^m_j q^2_{kj}$ (no summation implied), and the Yang-Baxter equation is satisfied automatically. There are other solutions, such another deformation of $su(2)$ studied by Witten[116]

$$[E_0, E_+]_p = E_+ \ , \qquad [E_0, E_-]_{1/p} = -E_- \ , \qquad [E_+, E_-] = E_0 - (p - \tfrac{1}{p})E_0^2 \ .$$

The problem of classifying algebras of this type is a very difficult one.

5.5 Higher Quantum Algebras

The generalization of the original deformation of Kulish and Reshetikhin to quantum $su(N)$ for general N, and their affine versions, was done by Drinfeld[107] and Jimbo.[108, 109] I repeat here the deformations of $su(n+1)$ in the notation of Pasquier and Saleur[110], with generators $E^\alpha, F^\alpha, q^{\pm H^\alpha/2}$, where $\alpha = 1, \ldots, n$. The relations are

$$\begin{aligned}
q^{H^\alpha/2} E^\beta q^{-H^\alpha/2} &= q^{a_{\alpha\beta}/2} E^\beta \ , \\
q^{H^\alpha/2} F^\beta q^{-H^\alpha/2} &= q^{-a_{\alpha\beta}/2} F^\beta \ , \\
[E^\alpha, F^\beta] &= \delta_{\alpha\beta} [H^\alpha]_q \ , \\
[E^\alpha, E^\beta] &= 0 \quad \text{if } a_{\alpha\beta} = 0 \ , \\
[F^\alpha, F^\beta] &= 0 \quad \text{if } a_{\alpha\beta} = 0 \ , \\
E^{\alpha^2} E^\beta - (q + q^{-1}) E^\alpha E^\beta E^\alpha + E^\beta E^{\alpha^2} &= 0 \quad \text{if } a_{\alpha\beta} = -1 \ , \\
F^{\alpha^2} F^\beta - (q + q^{-1}) F^\alpha F^\beta F^\alpha + F^\beta F^{\alpha^2} &= 0 \quad \text{if } a_{\alpha\beta} = -1 \ ,
\end{aligned}$$

where $a_{\alpha\beta}$ denotes the elements of the Cartan matrix. Quatum versions of $so(N)$ and $sp(N)$ were introduced by Reshetikhin.[120]

There is an alternative deformation of $su(N)$ for arbitrary N in terms of the physicist's basis,

$$[E_{jk}, E_{mn}] = \delta_{km} E_{jn} - \delta_{jn} E_{mk} \ .$$

This has a differential operator representation,

$$E_{jk} = x_j \frac{\partial}{\partial x_k} \ .$$

One interesting property of this basis is that the structure constants do not involve N. The indices are taken over $0 \leq j, k < N$, and by letting N tend to infinity this produces an alternative basis for $su(\infty)$.

This may be deformed to[121]

$$[E_{jk}, E_{mn}]_{q_{jkmn}} = q_{jkmn} \delta_{km} E_{jn} - q_{jkmn}^{-1} \delta_{jn} E_{mk} \ ,$$

which satisfies the braiding relation when

$$q_{jkmn} = q^{g_{km} - g_{kn} + g_{jn} - g_{jm}} \ ,$$

where g_{jk} is antisymmetric. At first sight the number of q parameters is simply $\frac{1}{2}N(N-1)$, but some of them never appear in the relations, and the actual number is $\frac{1}{2}(N-1)(N-2)$. The $su(2)$ case is undeformed, the $su(3)$ has one deformation parameter, and so on. This is an interesting algebra, as the cases for $N > 2$ all have subalgebras of (undeformed) $su(2) \times u(1)$, which has possible implications for an understanding of the standard model in terms of a quantum algebra.

This deformed algebra may be represented in same way as the Lie algebra, by considering the parameters to lie in a quantum space and replacing the derivatives with quantum derivatives. These ideas are discussed further in the next chapter. The same procedure may be carried out for the alternative representation of $su(2)^\star$ given by

$$J_+ = \frac{\partial}{\partial x} \ , \qquad J_0 = x \frac{\partial}{\partial x} \ , \qquad J_- = x^2 \frac{\partial}{\partial x} \ ,$$

resulting in the deformation of $su(2)$ studied by Woronowicz and mentioned above. The

$\star$ more accurately, $su(1,1)$

corresponding basis for $su(N)$, generated by the N^2-1 operators

$$\frac{\partial}{\partial x_j}\ , \qquad x_j\left(x_s\frac{\partial}{\partial x_s}\right)\ , \qquad x_j\frac{\partial}{\partial x_k}\ ,$$

may also be deformed.

5.6 Infinite Quantum Algebras

It is interesting to ask whether there are quantum analogues of infinite dimensional algebras, in particular the Virasoro algebra and the sine algebra. Consider a graded quommutator algebra satisfying the relation

$$[Q_j, Q_k]_{q_{jk}} = C_{jk}Q_{j+k}\ , \tag{5.7}$$

where j, k may be integers or integral 2-vectors, satisfying $C_{kj} = -C_{jk}$ and $q_{kj} = q_{jk}^{-1}$.

The conditions from the quadratic terms in the braiding relation on $Q_mQ_nQ_p$ is

$$\begin{aligned}&q_{nm}C_{mn}Q_{m+n}Q_p + q_{nm}^2q_{pm}C_{mp}Q_nQ_{m+p} + q_{nm}^2q_{pm}^2q_{pn}C_{np}Q_{n+p}Q_m\\ &\qquad = q_{pn}C_{np}Q_mQ_{n+p} + q_{pn}^2q_{pm}C_{mp}Q_{m+p}Q_n + q_{pn}^2q_{pm}^2q_{nm}C_{mn}Q_pQ_{m+n}\ ,\end{aligned}$$

no summation implied, which may be re-expressed using (5.7) to give

$$\begin{aligned}&q_{nm}C_{mn}Q_{m+n}Q_p + q_{nm}^2q_{pm}C_{mp}(q_{m+p,n}^2Q_{m+p}Q_n + q_{m+p,n}C_{n,m+p}Q_{m+n+p})\\ &\quad + q_{nm}^2q_{pm}^2q_{pn}C_{np}Q_{n+p}Q_m\\ &\qquad = q_{pn}C_{np}(q_{n+p,m}^2Q_{n+p}Q_m + q_{n+p,m}C_{m,n+p}Q_{m+n+p}) + q_{pn}^2q_{pm}C_{mp}Q_{m+p}Q_n\\ &\qquad\qquad + q_{pn}^2q_{pm}^2q_{nm}C_{mn}(q_{m+n,p}^2Q_{m+n}Q_p + q_{m+n,p}C_{p,m+n}Q_{m+n+p})\ .\end{aligned}$$

The terms quadratic in the generators give the condition

$$q_{mp}q_{np} = q_{m+n,p} \qquad \text{and} \qquad q_{pm}q_{pn} = q_{p,m+n}\ , \tag{5.8}$$

and if this condition is satisfied the coefficient of the linear term simply reduces to the Jacobi identity on the structure constants,

$$C_{mn}C_{m+n,p} + C_{np}C_{n+p,m} + C_{pm}C_{p+m,n} = 0\ .$$

If the indices are integers, the only solutions to (5.8) are $q_{jk} = (\pm 1)^{j+k}$, unless extra conditions are imposed on the Q's, so there is no genuine quantum analogue of the Virasoro algebra. The two cases corresponding to plus or minus are the Virasoro algebra and its Neveu-Schwarz supersymmetrization.

The algebra

$$[L_m, L_n]_{q^{n-m}} = [m-n]_r L_{m+n} \ ,$$

put forward by Curtright and Zachos[113] may be represented by

$$L_m = x^{-m}\frac{(q^{2x\partial}-1)}{q-q^{-1}} \ ,$$

but only satisfies the braiding relation if extra quadratic conditions are imposed on the generators.[122]

If the indices are 2-vectors there is an alternative solution to (5.8),

$$q_{\boldsymbol{mn}} = q^{\boldsymbol{m}\times\boldsymbol{n}} \ ,$$

giving a two parameter quantum Moyal algebra, discovered by Chand Devchand, David Fairlie, Tony Sudbery and myself, which supports a linear central extension,

$$[Q_{\boldsymbol{m}}, Q_{\boldsymbol{n}}]_{q^{\boldsymbol{m}\times\boldsymbol{n}}} = q^{\boldsymbol{m}\times\boldsymbol{n}}Q_{\boldsymbol{m}}Q_{\boldsymbol{n}} - q^{\boldsymbol{n}\times\boldsymbol{m}}Q_{\boldsymbol{n}}Q_{\boldsymbol{m}} = [\boldsymbol{m}\times\boldsymbol{n}]_p Q_{\boldsymbol{m}+\boldsymbol{n}} + \boldsymbol{a}.\boldsymbol{m}\delta_{\boldsymbol{m}+\boldsymbol{n},\boldsymbol{0}} \ .$$

This may be represented in terms of the K's of the sine algebra, and elements g, h such that $gh = qhg^{\star}$ by

$$Q_{\boldsymbol{m}} = g^{m_1}h^{m_2}\otimes K_{\boldsymbol{m}} \ .$$

This algebra has finite cases when p and q are both roots of unity, which are versions of quantum $su(N)$. The finite algebras only close in the case where both parameters are roots of unity, so this is not a deformation with a general q parameter.

$\star$ for example, the matrices discussed in chapter 3

6. Quantum Groups

An alternative approach to deformed symmetries is to consider the deformations of a Lie group, rather than the algebra. The correspondence between quantum algebras and quantum groups is not as simple as the exponential function relating Lie algebras and Lie groups; this has been studied by Woronowicz[117] and Sudbery.[123] A more straightforward idea of quantum groups is that of matrix groups which act as transformations of, in the simplest case, a deformed, or quantum, plane. This deformation manifests itself through non-commutativity of the coordinates, and thus the elements of the transformation matrices must themselves be non-commutative, obeying sets of bilinear product relations. In some limit of the parameters a Lie group is obtained. Quantum groups arise in quantum inverse scattering theory and as representations of transfer matrices in statistical mechanics. There are several useful reviews of this approach.[103, 124–127]

This chapter is based on work done in collaboration with Ed Corrigan, David Fairlie and Ryu Sasaki,[5] developing ideas arising principally from the viewpoint of Manin.[128] His starting point is to define a quantum group as effecting linear transformations upon a space whose elements, or coordinates, are non-commutative. The conditions for such a mapping to be an endomorphism constitute the quantum group relations. The group $GL_q(2)$ is studied in detail, and its dual introduced. This is generalized to the quantum supergroup $GL_q(1|1)$, and to higher quantum groups and supergroups.

6.1 The Quantum Plane

Manin introduces what he calls the quantum plane $R_q[2,0]$, whose elements are pairs $\boldsymbol{x} = (x, y)$, where the components x, y are assumed to satisfy the algebraic relation

$$xy = q^{-1} yx \ , \tag{6.1}$$

where q is a complex number. Clearly, in the $q \to 1$ limit the classical plane is retrieved. The components neither commute nor anticommute unless $q = \pm 1$ respectively. A Grassmannian quantum plane $R_q[0,2]$ dual to the (x, y) plane is also introduced, with elements $\boldsymbol{\xi} = (\xi, \eta)$ which are required to satisfy

$$\xi^2 = 0 \ , \qquad \eta^2 = 0 \ , \qquad \xi\eta + q\eta\xi = 0 \ . \tag{6.2}$$

A quantum matrix,

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_q(2) \ ,$$

effects simultaneously linear transformations of the quantum plane and its dual,

$$\boldsymbol{x}' = M\boldsymbol{x} \in \mathrm{R}_q[2,0] \ ,$$
$$\boldsymbol{\xi}' = M\boldsymbol{\xi} \in \mathrm{R}_q[0,2] \ .$$

The condition that the images $\boldsymbol{x}', \boldsymbol{\xi}'$ lie in the appropriate planes, *i.e.* their components satisfy (6.1) and (6.2) imposes restrictions upon M, giving the $\mathrm{GL}_q(2)$ relations*

$$\begin{aligned} ab &= q^{-1}ba \ , & cd &= q^{-1}dc \ , \\ ac &= q^{-1}ca \ , & bc &= cb \ , \\ bd &= q^{-1}db \ , & ad - da &= (q^{-1} - q)bc \ . \end{aligned} \tag{6.3}$$

It is relatively simple to show that these quantum matrices possess a comultiplication, counit and antipode. A suitable comultiplication is simply given by

$$\Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} a & b \\ c & d \end{pmatrix} ,$$

with corresponding counit and antipode just the ordinary matrix identity and inverse,

$$\epsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \qquad S \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ad - bc)^{-1} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} .$$

Using the relations it is easy to show that $\mathrm{Det}_q M = ad - q^{-1}bc$ commutes with all the elements a, b, c, d and thus may be considered as a number, the 'quantum determinant'. The choice $\mathrm{Det}_q M = 1$ restricts the quantum 'group' to $\mathrm{SL}_q(2)$ by analogy with the classical restriction to the special linear group. Because $\mathrm{Det}_q M$ commutes with the elements of M there exists an inverse

$$M^{-1} = (\mathrm{Det}_q M)^{-1} \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix} ,$$

which is both a left and right inverse for M. Note that M^{-1} is a member of $\mathrm{GL}_{q^{-1}}(2)$ rather than $\mathrm{GL}_q(2)$, and thus $\mathrm{GL}_q(2)$ is not strictly speaking a group. Furthermore, it is

---

⋆ The elements of M are supposed to commute with x, y, ξ, η.

clear that if

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad M' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \mathrm{GL}_q(2) ,$$

and (a, b, c, d) pairwise commute with (a', b', c', d') then MM' and $M'M$ are both $\mathrm{GL}_q(2)$ matrices. Also

$$\mathrm{Det}_q(MM') = \mathrm{Det}_q(M'M) = (\mathrm{Det}_q M)(\mathrm{Det}_q M') ,$$

reinforcing the identification with a determinant.

The relations (6.3) may be expressed in terms of an R-matrix[103, 124–126]

$$R_{ijkl} M_{km} M_{ln} = M_{jl} M_{ik} R_{klmn} , \tag{6.4}$$

where R_{ijkl} is a matrix, whose explicit form is given by

$$\begin{array}{cc} & (k,l) \\ (i,j) & \begin{pmatrix} q^{-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q^{-1} - q & 1 & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix} \end{array} , \tag{6.5}$$

where the rows are all pairs $(i, j), i, j = 1, 2$ in natural order, and similarly the columns are pairs (k, l). The expression (6.4) may be expressed in the tensor product form introduced in the previous chapter,

$$R_{12} M_1 M_2 = M_2 M_1 R_{12} ,$$

and the R-matrix satisfies the Yang-Baxter relation, in a slightly different form,

$$R_{12}(x) R_{13}(xy) R_{23}(y) = R_{23}(y) R_{13}(xy) R_{12}(x) , \tag{6.6}$$

which for $x, y = 0$ is a sufficient condition for the associativity of the quantum matrices.

Expression (6.5) is a member of a general class of R-matrices, each labelled by an additional parameter x, and each associated with one of the classical affine Lie algebras.[107, 127]

An explicit form of the R-matrices for the classical series is given by Jimbo. For $\hat{A}_n$ it is

$$
\begin{aligned}
R(x) = (q^{-1} - xq) \sum E_{\alpha\alpha} \otimes E_{\alpha\alpha} + (1 - x) \sum_{\alpha \neq \beta} E_{\alpha\alpha} \otimes E_{\beta\beta} \\
+ (q^{-1} - q) \left(\sum_{\alpha<\beta} + x \sum_{\alpha>\beta} \right) E_{\alpha\beta} \otimes E_{\beta\alpha} \ .
\end{aligned}
\tag{6.7}
$$

In this expression, the indices i, j, k, l have been suppressed for the sake of clarity. The i, jth element of the matrix $E_{\alpha\beta}$ is given by

$$
(E_{\alpha\beta})_{ij} = \delta_{i\alpha}\, \delta_{j\beta} \ .
$$

For $\hat{A}_1$, and $x = 0$, the matrix (6.5) is recovered.

A curious property[5] of 2×2 quantum groups is that if $M \in \mathrm{GL}_q(2)$ then $M^n \in \mathrm{GL}_{q^n}(2)$, where the product is the ordinary matrix product, not the comultiplication which preserves the relations (6.3). It is interesting but appears neither to generalize nor to fit into a proper algebraic scheme.*

6.2 Quantum Supergroups

Returning to the quantum plane (x, y) and its dual (ξ, η), suppose there is a linear transformation $\hat{M}$ which maps the plane into its dual and vice-versa, *i.e.*

$$
\begin{aligned}
\boldsymbol{\xi}' &= \hat{M} \boldsymbol{x} \ , \\
\boldsymbol{x}' &= \hat{M} \boldsymbol{\xi} \ ,
\end{aligned}
$$

and again impose the quantum plane conditions upon (ξ', η') and (x', y'). If the elements of $\hat{M}$ are designated by

$$
\hat{M} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} ,
$$

then the constraints are ten in number;

$$
\begin{aligned}
&\alpha\beta + q\beta\alpha = 0 \ , & &\alpha\delta + \delta\alpha = 0 \ , \\
&\alpha\gamma + q\gamma\alpha = 0 \ , & &\beta\gamma + \gamma\beta + (q - q^{-1})\delta\alpha = 0 \ , \\
&\beta\delta + q\delta\beta = 0 \ , & &\alpha^2 = \beta^2 = \gamma^2 = \delta^2 = 0 \ . \\
&\gamma\delta + q\delta\gamma = 0 \ , & &
\end{aligned}
\tag{6.8}
$$

These relations may be considered as a deformation of a Grassmann algebra on four

* This observation has also been noted by Zumino.[129, 130]

elements $(\alpha, \beta, \gamma, \delta)$. As with the quantum matrix, they may be expressed in terms of an $\hat{R}$-matrix in the form (6.4)

$$\hat{R}\hat{M}\hat{M} = -\hat{M}\hat{M}\hat{R}\,,$$

$$\text{where} \quad \hat{R} = \begin{pmatrix} q+q^{-1} & 0 & 0 & 0 \\ 0 & 2 & q-q^{-1} & 0 \\ 0 & -(q-q^{-1}) & 2 & 0 \\ 0 & 0 & 0 & q+q^{-1} \end{pmatrix}.$$

Note that in the classical limit (*i.e.* $q \longrightarrow 1$) $\hat{R}$ becomes twice the identity matrix. This matrix $\hat{R}$ is (6.7) evaluated at $x = -1$. Notice also that although the algebra (6.8) is an associative algebra of the matrix elements of $\hat{M}$, $\hat{R}$ does not satisfy the Yang-Baxter equation (6.6), thus demonstrating that the Yang-Baxter relation is not a necessary condition for associativity.

Since $\hat{M}$ is entirely Grassmannian, an inverse proper cannot exist. However, the analogue of left and right adjugate matrices can be constructed, giving

$$\begin{aligned} \begin{pmatrix} q\delta & \beta \\ -\gamma & -q^{-1}\alpha \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} &= (\beta\gamma + q\delta\alpha)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} -q^{-1}\delta & \beta \\ -\gamma & q\alpha \end{pmatrix} &= (\gamma\beta + q\delta\alpha)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{6.9}$$

The combination $\beta\gamma + q\delta\alpha$ may be thought of as a left quantum determinant and Δ_L and $\gamma\beta + q\delta\alpha$ as a right quantum determinant Δ_R. The expressions, Δ_L, Δ_R satisfy the relation

$$\Delta_L \begin{pmatrix} -q^{-1}\delta & \beta \\ -\gamma & q\alpha \end{pmatrix} = \begin{pmatrix} q\delta & \beta \\ -\gamma & -q^{-1}\alpha \end{pmatrix} \Delta_R\,,$$

which is a consequence of (6.9) and associativity.

In a similar manner one can construct the quantum analogue of GL(1|1), $\mathrm{GL}_q(1|1)$, the group of linear transformations acting upon a quantum superplane with one bosonic and one fermionic coordinate.† Consider a quantum superplane and its dual,

$$\begin{pmatrix} x \\ \xi \end{pmatrix}, \quad \begin{pmatrix} \eta \\ y \end{pmatrix},$$

† The convention of Roman (Greek) script for bosonic (fermionic) quantities is used.

satisfying:

$$
\begin{aligned} x\xi - q^{-1}\xi x &= 0 \\ \xi^2 &= 0 \end{aligned} \qquad \text{and} \qquad \begin{aligned} \eta^2 &= 0 \\ \eta y - q y \eta &= 0 \end{aligned} \tag{6.10}
$$

Define a $\mathrm{GL}_q(1|1)$ matrix

$$
\mathcal{M} = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix} ,
$$

and require

$$
\mathcal{M}\begin{pmatrix} x \\ \xi \end{pmatrix} = \begin{pmatrix} x' \\ \xi' \end{pmatrix} , \qquad \mathcal{M}\begin{pmatrix} \eta \\ y \end{pmatrix} = \begin{pmatrix} \eta' \\ y' \end{pmatrix} ,
$$

and impose (6.10) once again on the transformed variables. It is assumed that β and γ anticommute with ξ and η. Then eight relations are obtained

$$
\begin{aligned}
a\beta &= q^{-1}\beta a , & \beta^2 &= 0 , \\
a\gamma &= q^{-1}\gamma a , & \gamma^2 &= 0 , \\
d\beta &= q^{-1}\beta d , & \beta\gamma + \gamma\beta &= 0 , \\
d\gamma &= q^{-1}\gamma d , & ad - da + q^{-1}\beta\gamma + q\gamma\beta &= 0 .
\end{aligned}
$$

In this case the left and right inverses may be defined and are equal

$$
\mathcal{M}_L^{-1} = \begin{pmatrix} \Delta_1^{-1} d & -\Delta_1^{-1} q^{-1}\beta \\ -\Delta_2^{-1} q^{-1}\gamma & \Delta_2^{-1} a \end{pmatrix} = \begin{pmatrix} d\Delta_1^{-1} & -q\beta\Delta_2^{-1} \\ -q\gamma\Delta_1^{-1} & a\Delta_2^{-1} \end{pmatrix} = \mathcal{M}_R^{-1} ,
$$

where $\Delta_1 = ad - q\beta\gamma$ and $\Delta_2 = da - q\gamma\beta$. The theorems in section 2 also apply to $\mathrm{GL}_q(1|1)$. In particular, if $\mathcal{M} \in \mathrm{GL}_q(1|1)$ then $\mathcal{M}^n \in \mathrm{GL}_{q^n}(1|1)$. Similar results may be deduced for the dual matrix

$$
\widehat{\mathcal{M}} = \begin{pmatrix} \alpha & b \\ c & \delta \end{pmatrix} .
$$

which transforms the superplane into its dual. Quantum supergroups have also been studied by Manin[131] and others.[132, 133]

6.3 Higher Quantum Groups

It is obviously desirable to extend the analysis to the quantum analogues of linear transformations in higher dimensional spaces. Consider first $\mathrm{GL}_q(N)$. Instead of the quantum 2-plane, take a vector

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} \in \mathrm{R}_q[N,0] ,$$

and impose the relations

$$x_i x_j - q^{-1} x_j x_i = 0 \quad \text{for} \quad i < j . \tag{6.11}$$

Adjoin a dual quantum space

$$\boldsymbol{\xi} = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_N \end{pmatrix} \in \mathrm{R}_q[0,N] ,$$

with the relations

$$\xi_i^2 = 0 , \quad \xi_i \xi_j + q \xi_j \xi_i = 0 \quad \text{for} \quad i < j . \tag{6.12}$$

The relations (6.11) , (6.12) can be written in the form

$$\boldsymbol{x}^T G_{kl} \boldsymbol{x} = 0 , \quad \boldsymbol{\xi}^T F_{kl} \boldsymbol{\xi} = 0 ,$$

where G_{kl} is a matrix whose entries are all zero except for the klth and the lkth, *i.e.*

$$(G_{kl})_{rs} = \frac{\sqrt{q}}{\sqrt{q+q^{-1}}} \delta_{rk}\delta_{sl} - \frac{\sqrt{q^{-1}}}{\sqrt{q+q^{-1}}} \delta_{rl}\delta_{sk} , \qquad k < l ,$$

Similarly

$$(F_{kl})_{rs} = \frac{\sqrt{q^{-1}}}{\sqrt{q+q^{-1}}} \delta_{rk}\delta_{sl} + \frac{\sqrt{q}}{\sqrt{q+q^{-1}}} \delta_{rl}\delta_{sk} , \qquad k < l ,$$

$$(F_{kk})_{rs} = \delta_{rk}\delta_{sk}.$$

Now

$$(G_{ij})_{rs}(F_{kl})_{rs} = \mathrm{Tr}(G_{ij}F_{kl}^T) = 0 \tag{6.13}$$

by construction. This enables us to write the quantum matrix condition very succinctly. Suppose the matrix of linear transformations is given by M, *i.e.*

$$\boldsymbol{x}' = M\boldsymbol{x} \ , \quad \boldsymbol{\xi}' = M\boldsymbol{\xi} \ . \tag{6.14}$$

Then, $\boldsymbol{x}^T M^T G_{ij} M \boldsymbol{x} = 0$ implies that $M^T G_{ij} M$ is a linear combinations of G's, *i.e.*

$$M^T G_{ij} M = \sum_{k,l} A_{ijkl} G_{kl} \ ,$$

and similarly $\boldsymbol{\xi}^T M^T F_{ij} M \boldsymbol{\xi} = 0$ implies

$$M^T F_{ij} M = \sum_{k,l} B_{ijkl} F_{kl} \ .$$

Due to orthogonality (6.13) there are sets of relations

$$\mathrm{Tr}(M^T G_{ij} M F_{kl}^T) = 0 \ , \quad \mathrm{Tr}(M^T F_{ij} M G_{kl}^T) = 0 \ . \tag{6.15}$$

The number of relations of the first kind is simply the number of independent G's, $\frac{1}{2}N(N-1)$ multiplied by the number of independent F's, $\frac{1}{2}N(N+1)$ giving $\frac{1}{4}(N^4 - N^2)$, and similarly for the second kind, resulting in $\frac{1}{2}N^2(N^2-1)$ relations, the full set for $\mathrm{GL}_q(N)$. Notice that the relations (6.15) imply also that M^T is a quantum matrix, as it satisfies the same bilinear algebra. In fact, the dual (Grassmannian) plane can be dispensed with in setting up the quantum group conditions, by simply taking

$$\boldsymbol{x}' = M\boldsymbol{x} \in \mathrm{R}_q[N,0] \ ,$$
$$\boldsymbol{x}'' = M^T\boldsymbol{x} \in \mathrm{R}_q[N,0] \ .$$

In the classical case $q = 1$, and G_{ij} spans the space of antisymmetric matrices, while F_{ij} spans the symmetric ones. The quantum case of G_{ij} is referred to as q-antisymmetric, and F as q-symmetric matrices.

It is now relatively easy to construct an R-matrix, and to exhibit these relations in the form of equations (6.4).

Define the $N^2 \times N^2$ matrix

$$R(\mu,\nu) = \mu \sum_{i,j} (G_{ij})^T G_{ij} + \nu \sum_{k,l} (F_{kl})^T F_{kl} \ , \tag{6.16}$$

where μ and ν are arbitrary parameters. Then, on account of the orthogonality relations (6.13) together with the additional orthonormality conditions

$$\mathrm{Tr}(G_{ij}^T G_{kl}) = \delta_{ik}\delta_{jl} \ , \tag{6.17}$$

$$\mathrm{Tr}(F_{ij}^T F_{kl}) = \delta_{ik}\delta_{jl} \ , \tag{6.18}$$

the equation (6.16), written with explicit indices as

$$R_{st,uv}(\mu,\nu) = \mu \sum_{i,j} (G_{ij})_{ts}(G_{ij})_{uv} + \nu \sum_{i,j} (F_{ij})_{ts}(F_{ij})_{uv} \ ,$$

is just the eigenvalue expansion of an $N^2 \times N^2$ matrix with two degenerate eigenvalues with degeneracies $\frac{1}{2}N(N-1)$ and $\frac{1}{2}N(N+1)$. The sets of quantities $(G_{ij})_{st}$, $(F_{ij})_{st}$ are eigenvectors in the sense that:

$$\begin{aligned} (G_{ij})_{ts} R_{st,uv} &= \mu (G_{ij})_{uv} \\ (F_{ij})_{ts} R_{st,uv} &= \nu (F_{ij})_{uv} \end{aligned} \qquad \text{or} \qquad \begin{aligned} G_{ij}^T R &= \mu G_{ij} \\ F_{ij}^T R &= \nu F_{ij} \end{aligned}$$

Imposing the conditions

$$R_{\sigma\tau pq}(\mu,\nu) M_{pu} M_{qv} = M_{\tau q} M_{\sigma p} R_{pquv}(\mu,\nu) \ , \tag{6.19}$$

produces a set of equations whose content is just (6.15), as may be readily derived by taking the trace of (6.19) with $(G_{ij})^T F_{kl}$ and $(F_{ij})^T G_{kl}$. The orthogonality properties (6.13), (6.17), (6.18) ensure that G_{ij} and F_{ij} are eigenvectors of R, and since the eigenvalues differ, the equations (6.15) are a consequence of (6.19). Note, however, that the relations (6.19) are not all necessarily independent, while (6.15) are, by construction. No further conditions result from taking the trace of (6.19) with the combinations $(F_{ij})^T F_{kl}$ and $(G_{ij})^T G_{kl}$. It is easy to see that (6.16) gives $R(x)$, (6.7) for $\mu = -q + xq^{-1}$, $\nu = q^{-1} - xq$.

The extension for the dual Grassmann matrix $\hat{M}$ is very much the same. Postulate a similar ansatz for the $\hat{R}$-matrix, but with different eigenvalues, μ, ν. Then impose

$$\hat{R}_{\sigma\tau pq}(\mu,\nu)\hat{M}_{pu}\hat{M}_{qv} = -\hat{M}_{\tau q}\hat{M}_{\sigma p}\hat{R}_{pquv}(\mu,\nu) \ .$$

The eigenvalues of $\hat{R}$ are $\pm(q+q^{-1})$, *i.e.* $x=-1$ in (6.7). This fact has the consequence that this time the matrix elements of this relation which do not vanish are those of the trace with $G_{ij}(G_{kl})^T$ and $F_{ij}(F_{kl})^T$, while those with a mixed G and F are automatically satisfied, thus giving $\frac{1}{2}N^2(N^2+1)$ independent relations for the quantum Grassmann group

$$\mathrm{Tr}(\hat{M}^T G_{ij}\hat{M}G^T{}_{kl}) = 0 \ , \tag{6.20}$$

$$\mathrm{Tr}(\hat{M}^T F_{ij}\hat{M}F^T{}_{kl}) = 0 \ . \tag{6.21}$$

These provide the generalization of (6.8) to arbitrary N.

This generalization gives the class of R-matrices associated with the Lie groups of the $\hat{A}_n$ series.[127, 107] It is also possible to enquire about the corresponding extension to other series, *e.g.* the C_n series. What must be done is to obtain the C_n, *i.e.* the Sp($2n$) series is to adjoin to the quantum plane conditions (6.14) an additional symplectic requirement,

$$M^T\epsilon M = \lambda\epsilon \ ,$$

where ϵ is an $N\times N$ matrix ($N=2n$), with non-vanishing elements only for $i+j=N+1$, *i.e.* on the anti-diagonal, where they are

$$q^N,\ldots,q^{\frac{N}{2}+1},q^{\frac{N}{2}-1},\ldots,q,1 \ .$$

The quantum "group" condition can be writtem in a form analogous to (6.15) after redefining matrices

$$G'_{ij} = G_{ij} - \frac{2\,\mathrm{Tr}(\epsilon^T G_{ij})}{Nq^N}\epsilon \ ,$$

$$F'_{ij} = F_{ij} - \frac{2\,\mathrm{Tr}(\epsilon^T F_{ij})}{Nq^N}\epsilon \ ,$$

so that they are orthogonal to ϵ in the sense of (6.20), (6.21). Then the quantum conditions can be written as

$$
\begin{aligned}
\mathrm{Tr}(M^T G'_{ij} M {F'_{kl}}^T) = \mathrm{Tr}(M^T F'_{ij} M {G'_{kl}}^T) = 0 \ , \\
\mathrm{Tr}(M^T G'_{ij} M \epsilon^T) = \mathrm{Tr}(M^T F'_{ij} M \epsilon^T) = 0 \ , \\
\mathrm{Tr}(M^T \epsilon M {G'_{ij}}^T) = \mathrm{Tr}(M^T \epsilon M {F'_{ij}}^T) = 0 \ .
\end{aligned}
\tag{6.22}
$$

The number of such relations is

$$
\tfrac{1}{2} N(N-1)(N^2+N+2) - 2 \ . \tag{6.23}
$$

For $N = 2$ this gives 6, as before, and for $N = 4$ it gives 130, a number which agrees with computer calculations in REDUCE, using the Sp(2) R-matrix of Jimbo to define quantum group conditions via (6.4).

In an analogous fashion the dual group relations can be found by replacing (6.22) by

$$
\mathrm{Tr}(\hat{M}^T G'_{ij} \hat{M} {G'_{kl}}^T) = \mathrm{Tr}(\hat{M}^T F'_{ij} \hat{M} {F'_{kl}}^T) = \mathrm{Tr}(\hat{M}^T \epsilon^T \hat{M} \epsilon) = 0 \ ,
$$

the number of relations being

$$
\tfrac{1}{2} N^4 - \tfrac{1}{2} N^2 + N + 2 \ . \tag{6.24}
$$

As is to be expected, this is complementary to the previous calculation; the sum of (6.23) and (6.24) is N^4.

6.4 Further Generalizations

The assumptions made for the quantum hyperplane conditions (6.11) and (6.12) need not be the only viable structures. In fact, there is a natural generalization of the Clifford sequence. Start with a quantum plane (x, y) and its dual (ξ, η). Then construct the quantum matrix M and its dual $\hat{M}$. Now view the elements of M as constituting the coordinates a, b, c, d in a quantum hyperplane, with $\hat{M}$ furnishing the dual coordinates, and take the relations (6.3) and (6.8) as those to be preserved by linear transformations $M', \hat{M}'$ acting upon the quantum hyperplanes. This leads to conditions on the 16 elements of M' and those of $\hat{M}'$, which in turn can be thought of as the requirements for a 16 dimensional hyperplane, subject to a linear transformation M'' *etc.* This sequence will generate a quantum Clifford sequence.

This approach to quantum groups raises the obvious question of the representation of the elements of the quantum plane, and of the quantum matrix itself, by finite dimensional matrices whose elements themselves commute. That such representations do exist with q an nth root of unity is demonstrated by setting $x = g$ and $y = h$, where g, h are the $n \times n$ matrices discussed in chapter 3, given by

$$g = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \omega & 0 & \cdots & 0 \\ 0 & 0 & \omega^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \omega^{n-1} \end{pmatrix} , \quad h = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix} , g^n = h^n = I, \ \omega^n = 1 \ .$$

It is easy to verify that $gh = \omega^{-1} hg$, the quantum plane condition.

It is difficult to find representations of M with $\mathrm{Det}_q(M) \neq 0$ and $q \neq \pm 1$. A specific example for A_1 is

$$M = \begin{pmatrix} g^2 & \omega h \\ \omega h^4 & g^4(1 + h^5) \end{pmatrix} ,$$

where $n = 6$ and $q = \omega^2$. Others have been found by Floratos.[134]

It is also possible to create infinite dimensional representations, such as

$$M = \begin{pmatrix} e^{ia(p_1+p_2)} & e^{ia(x_1+p_2)} \\ e^{ia(p_1+x_2)} & e^{ia(x_1+x_2)} \end{pmatrix} ,$$

where p_1, p_2 and x_1, x_2 satisfy the commutation rules appropriate to canonically conjugate variables and $q = e^{-ia^2}$, though this has $\mathrm{Det}_q(M)$ vanishing. An alternative example, found by Weyers,[135] with unit determinant, is

$$M = \begin{pmatrix} e^{i\alpha x} & e^{i\beta p} \\ e^{i\gamma p} & e^{-i\alpha x} + e^{i((\beta+\gamma)p - \alpha x - \alpha\beta)} \end{pmatrix} ,$$

where $q = e^{i\alpha\beta}$ and $\alpha\beta = \alpha\gamma + 2\pi n$ for some integer n.

For A_n quantum groups a representation of the quantum hyperplane (6.11) is given by:

$$\begin{aligned}
x_1 &= x \otimes x \otimes x \otimes \cdots \otimes x \\
x_2 &= y \otimes x^2 \otimes x^2 \otimes \cdots \otimes x^2 \\
x_3 &= x \otimes y \otimes x^2 \otimes \cdots \otimes x^2 \\
\vdots &= \qquad\quad \vdots \\
x_{n+1} &= x \otimes x \otimes \cdots \otimes x \otimes y.
\end{aligned}$$

Manin[131] and Sudbery[136] have introduced quantum groups of dimension greater than two with more than one q parameter.

6.5 The q-derivative

The introduction of the quantum plane naturally leads to the question of whether there is some quantum analogue of derivatives. There is; and the consequent q-calculus and q-analysis have been studied.[137–139] Here I shall give a brief outline of the ideas. A useful starting point is the Leibniz rule for the q-derivative ${}_qD_x$[139]

$${}_qD_x(f(x)g(x)) = ({}_qD_x f(x))g(x) + f(qx)({}_qD_x g(x)) \ .$$

This, with the condition that

$${}_qD_x(g(x)f(x)) = {}_qD_x(f(x)g(x)) \ ,$$

defines the q-derivative to be

$${}_qD_x \ f(x) = \frac{f(qx) - f(x)}{x(q-1)} \ .$$

This satisfies, for example,

$$\begin{aligned}
{}_qD_x \ x^n &= {}_q[n]x^{n-1} \qquad \text{where} \quad {}_q[n] = \frac{1-q^n}{1-q} \ , \\
{}_qD_x \ \exp_q(\mu x) &= \mu \exp_q(\mu x) \ ,
\end{aligned}$$

where the q-analogue of the exponential is given by[139, 140]

$$\exp_q(x) = \sum_{n=0}^{\infty} \frac{1}{{}_q[n]!} x^n \ .$$

It is the replacement of the ordinary derivative by this quantum derivative that leads to the quantum analogues of $su(N)$ in the physicist's basis discussed in chapter 5. The quantum groups that correspond to these quantum algebras are the multiparameter deformations of GL(N) studied by Manin[131] and Sudbery.[136] In a similar way, the quantum group of Woronowicz[117] corresponds to the algebra of the quantum differential operators

$${}_qD_x\ , \qquad x\ {}_qD_x\ , \qquad x^2\ {}_qD_x\ .$$

Appendices

A1 Calculation of T's

$T(\beta, \alpha-\beta, \gamma, \alpha-\gamma, 1, 1)$:

$m = 1, t = 2, r = \alpha - 1$

$0, \gamma - 1 \leq k \leq \alpha - 1, \gamma \Rightarrow k = \gamma - 1 \ (\gamma \neq 0)$ or $k = \gamma \ (\gamma \neq \alpha)$

$0 \leq l \leq 2$

$\gamma, \beta + \gamma + 1 - \alpha \leq k + l \leq \beta + \gamma, \gamma + 1 \Rightarrow k + l = \gamma \ (\beta \neq \alpha)$ or $\gamma + 1 \ (\beta \neq 0)$

$$k, l = \begin{cases} \gamma - 1, 1 & \Rightarrow j = \beta, p = 1, n = 0 \Rightarrow 0 \\ \gamma - 1, 2 & \Rightarrow j = \beta - 1, p = 0, n = 1 \Rightarrow 2b_{221} b_{\alpha-1,\beta-1,\gamma-1} \\ \gamma, 0 & \Rightarrow j = \beta, p = 1, n = 0 \Rightarrow 2b_{201} b_{\alpha-1,\beta,\gamma} \\ \gamma, 1 & \Rightarrow j = \beta - 1, p = 0, n = 1 \Rightarrow 0 \end{cases}$$

$$T(\beta, \alpha-\beta, \gamma, \alpha-\gamma, 1, 1) = 2b_{201} b_{\alpha-1,\beta,\gamma} - 2b_{212} b_{\alpha-1,\beta-1,\gamma-1}$$

where $b_{\alpha\beta\gamma} = 0$ for $\beta, \gamma < 0$ or $\beta, \gamma > \alpha$.

$T(\gamma, \alpha-\gamma, 1, 1, \beta, \alpha-\beta)$:

$m = \beta, t = \alpha, r = 1$

$0, 0 \leq k \leq 1, 1 \Rightarrow k = 0$ or $k = 1$

$0 \leq l \leq \alpha$

$1, \gamma \leq k + l \leq \alpha, \gamma + 1 \Rightarrow k + l = \gamma$ or $\gamma + 1$

$$k, l = \begin{cases} 0, \gamma & \Rightarrow j = 1, p = \alpha - \gamma, n = \gamma - 1 \Rightarrow \gamma b_{\alpha\gamma\beta} b_{110} \\ 0, \gamma + 1 & \Rightarrow j = 0 \Rightarrow 0 \\ 1, \gamma - 1 & \Rightarrow j = 1 \Rightarrow 0 \\ 1, \gamma & \Rightarrow j = 0, p = \alpha - \gamma - 1, n = \gamma \Rightarrow (\alpha - \gamma) b_{101} b_{\alpha,\gamma,\beta} \end{cases}$$

$$T(\gamma, \alpha-\gamma, 1, 1, \beta, \alpha-\beta) = (2\gamma - \alpha) b_{\alpha,\beta,\gamma}$$

and, interchanging β and γ, f and g,

$$T(1, 1, \beta, \alpha-\beta, \gamma, \alpha-\gamma) = (2\beta - \alpha) b_{\alpha,\beta,\gamma}$$

$T(\zeta, \alpha-\zeta, \alpha-\zeta+1, \zeta-1, 0, 2)$:

$m = 0, t = 2, r = \alpha - 1$

$0, \alpha - \zeta \leq k \leq \alpha - 1, \alpha - \zeta + 1 \Rightarrow k = \alpha - \zeta$ or $k = \alpha - \zeta + 1$

$0 \leq l \leq 2$

$2, \alpha - \zeta + 1 \leq k + l \leq \alpha + 1, \alpha - \zeta + 2 \Rightarrow k + l = \alpha - \zeta + 1$ or $\alpha - \zeta + 2$

$$k, l = \begin{cases} \alpha - \zeta, 1 & \Rightarrow j = \zeta, p = 1, n = 0 \Rightarrow b_{210} b_{\alpha-1,\zeta,\alpha-\zeta} \\ \alpha - \zeta, 2 & \Rightarrow j = \zeta - 1, p = 0, n = 1 \Rightarrow 2b_{220} b_{\alpha-1,\zeta-1,\alpha-\zeta} \\ \alpha - \zeta + 1, 0 & \Rightarrow m = l \Rightarrow 0 \\ \alpha - \zeta + 1, 1 & \Rightarrow j = \zeta - 1, p = 0, n = 1 \Rightarrow b_{210} b_{\alpha-1,\zeta-1,\alpha-\zeta+1} \end{cases}$$

$$T(\zeta, \alpha-\zeta, \alpha-\zeta+1, \zeta-1, 0, 2) = -b_{201} b_{\alpha-1,\zeta,\alpha-\zeta} - 2b_{202} b_{\alpha-1,\zeta-1,\alpha-\zeta} - b_{201} b_{\alpha-1,\zeta-1,\alpha-\zeta+1}$$

$T(\alpha-\zeta+1, \zeta-1, 0, 2, \zeta, \alpha-\zeta)$:

$m = \zeta, t = \alpha, r = 1$

$0, -1 \leq k \leq 0, 1 \Rightarrow k = 0$

$0 \leq l \leq \alpha$

$0, \alpha - \zeta \leq k + l \leq \alpha - 1, \alpha - \zeta + 1 \Rightarrow l = k + l = \alpha - \zeta$ or $\alpha - \zeta + 1$

$$k, l = \begin{cases} 0, \alpha - \zeta & \Rightarrow j = 1, p = \zeta - 1, n = \alpha - \zeta \Rightarrow \zeta b_{110} b_{\alpha,\alpha-\zeta,\zeta} \\ 0, \alpha - \zeta + 1 & \Rightarrow j = 0 = k \Rightarrow 0 \end{cases}$$

$$T(\alpha-\zeta+1, \zeta-1, 0, 2, \zeta, \alpha-\zeta) = \zeta b_{\alpha,\zeta,\alpha-\zeta}$$

$T(0, 2, \zeta, \alpha-\zeta, \alpha-\zeta+1, \zeta-1)$:

$m = \alpha - \zeta + 1, t = \alpha, r = 1$

$0, \zeta + 1 - \alpha \leq k \leq 1, \zeta \Rightarrow k = 0, 1$

$0 \leq l \leq \alpha$

$\zeta, \zeta - 1 \leq k + l \leq \zeta, \zeta + 1 \Rightarrow k + l = \zeta$

$$k, l = \begin{cases} 0, \zeta & \Rightarrow j = 0 = k \Rightarrow 0 \\ 1, \zeta - 1 & \Rightarrow j = 0, p = 1, n = 0 \Rightarrow (\alpha - \zeta + 1) b_{101} b_{\alpha,\zeta-1,\alpha-\zeta+1} \end{cases}$$

$$T(0, 2, \zeta, \alpha-\zeta, \alpha-\zeta+1, \zeta-1) = (\alpha - \zeta + 1) b_{\alpha,\zeta-1,\alpha-\zeta+1}$$

$T(\alpha, 0, 0, \alpha-1, 1, 2)$:

$m = 1, t = 3, r = \alpha - 2$

$0, -1 \leq k \leq \alpha - 2, 0 \Rightarrow k = 0$

$0 \leq l \leq 3$

$0, 2 \leq k + l \leq \alpha, 2 \Rightarrow l = k + l = 2$

$$k, l = \{\, 0, 2 \quad \Rightarrow j = \alpha - 2, p = 0, n = 2 \Rightarrow b_{\alpha-2,\alpha-2,0} b_{321}$$

$$T(\alpha, 0, 0, \alpha-1, 1, 2) = b_{312} b_{\alpha-2,0,\alpha-2}$$

$T(0, \alpha-1, 1, 2, \alpha, 0)$:

$m = \alpha, t = \alpha, r = 1$

$0, -1 \leq k \leq 1, 1 \Rightarrow k = 0, 1$

$0 \leq l \leq \alpha$

$1, 0 \leq k + l \leq 1, \alpha - 1 \Rightarrow k + l = 1$

$$k, l = \begin{cases} 0, 1 & \Rightarrow j = 0 = k \Rightarrow 0 \\ 1, 0 & \Rightarrow j = 0, p = \alpha - 2, n = 0 \Rightarrow \frac{1}{2}\alpha(\alpha - 1) b_{\alpha,0,\alpha} b_{101} \end{cases}$$

$$T(0, \alpha-1, 1, 2, \alpha, 0) = \tfrac{1}{2}\alpha(\alpha-1) b_{\alpha,0,\alpha}$$

$T(1, 2, \alpha, 0, 0, \alpha-1)$:

$m = 0, t = \alpha - 1, r = 2$

$0, 2 \leq k \leq 2, \alpha \Rightarrow k = 2$

$0 \leq l \leq \alpha - 1$

$\alpha - 1, \alpha \leq k + l \leq \alpha + 1, \alpha + 1 \Rightarrow k + l = \alpha, \alpha + 1$

$$k, l = \begin{cases} 2, \alpha - 2 & \Rightarrow j = 1, p = 1, n = 0 \Rightarrow b_{\alpha-1,\alpha-2,0} b_{212} \\ 2, \alpha - 1 & \Rightarrow j = 0, p = 0, n = 1 \Rightarrow (\alpha - 1) b_{\alpha-1,\alpha-1,0} b_{202} \end{cases}$$

$$T(1, 2, \alpha, 0, 0, \alpha-1) = -b_{\alpha-1,0,\alpha-2} b_{212} - (\alpha-1) b_{\alpha-1,0,\alpha-1} b_{202}$$

A2 Solution of Recurrence Relations

Since $b_{201} = b_{212} = 0$, $b_{202} = \eta$, the recurrence relations become

$$\begin{aligned} b_{\alpha\beta\gamma} &= 0 \qquad \text{for} \quad \alpha \neq \beta + \gamma \ , \\ \zeta b_{\alpha,\zeta,\alpha-\zeta} &= -(\alpha - \zeta + 1) b_{\alpha,\zeta-1,\alpha-\zeta+1} + 2\eta b_{\alpha-1,\zeta-1,\alpha-\zeta} \ , \\ \alpha(\alpha-1) b_{\alpha 0 \alpha} &= 2(3b_{303} - 2b_{202}^2) b_{\alpha-2,0,\alpha-2} + 2b_{202}(\alpha - 1) b_{\alpha-1,0,\alpha-1} \ . \end{aligned}$$

It is possible to rewrite the last of these as

$$b'_\alpha - 2\eta b'_{\alpha-1} - 2(3\xi - 2\eta^2) b'\alpha - 2 = 0 \ ,$$

by defining $b'_\alpha = \alpha!\, b_{\alpha 0 \alpha}$, $b_{303} = \xi$, which may be solved by putting $b'_\alpha = A\lambda_1^\alpha + B\lambda_2^\alpha$ in the normal way. The solution is:

$$b'_\alpha = \frac{1}{2\rho}((\eta + \rho)^\alpha - (\eta - \rho)^\alpha)) \ ,$$

where $\rho = \sqrt{6\xi - 3\eta^2}$.

Putting

$$b'_{\alpha,\zeta} = \zeta!(\alpha - \zeta)! b_{\alpha,\zeta,\alpha-\zeta} \ ,$$

the other recurrence relation becomes

$$b'_{\alpha,\zeta} - 2\eta b'_{\alpha-1,\zeta-1} + b'_{\alpha,\zeta-1} = 0 \ ,$$

which has solution

$$b'_{\alpha,\zeta} = \sum_{k=0}^{\zeta} \binom{\zeta}{k} (-1)^{\zeta-k} (2\eta)^k b'_{\alpha-k,0} \ .$$

This may be seen by considering the following diagram:

$$\begin{array}{ccccccccc}
\alpha,\zeta & \rightarrow & \alpha,\zeta-1 & \rightarrow & \alpha,\zeta-2 & \rightarrow & \cdots & \rightarrow & \alpha,0 \\
 & \searrow & & \searrow & & \searrow & \ddots & \searrow & \\
 & & \alpha-1,\zeta-1 & \rightarrow & \alpha-1,\zeta-2 & \rightarrow & \cdots & \rightarrow & \alpha-1,0 \\
 & & & \searrow & & \searrow & \ddots & \searrow & \\
 & & & & \alpha-2,\zeta-2 & \rightarrow & \cdots & \rightarrow & \alpha-2,0 \\
 & & & & & \searrow & \ddots & \searrow & \\
 & & & & & & \ddots & \ddots & \vdots \\
 & & & & & & & \searrow & \\
 & & & & & & & & \alpha-\zeta,0
\end{array}$$

Each horizontal brings in a factor -1, each diagonal 2η, so the coefficient of $b'_{\alpha-k,0}$ is $(-1)^{\zeta-k}(2\eta)^k$, and the number of ways of getting there is given by a binomial coeffecient.

These may be put together to give the final result

$$b_{\alpha,\zeta,\alpha-\zeta} = \frac{(\eta-\rho)^{\zeta}(\eta+\rho)^{\alpha-\zeta} - (\eta+\rho)^{\zeta}(\eta-\rho)^{\alpha-\zeta}}{2\rho\zeta!(\alpha-\zeta)!} \; .$$

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