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Walsh, Dominic Anthony

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Dominic Anthony Walsh

Department of Physics

University of Durham

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Abstract

This work is concerned with the breaking of chiral symmetry in gauge theories and the associated generation of a dynamical mass scale. We investigate this phenomenon in the context of a simple model, three dimensional QED, where the complicating factor of infinite renormalisations is absent. This model possesses an intrinsic scale, set by the coupling $[e^2] = M$, and it is the relationship between this and the dynamically generated mass scale that is of interest.

The chiral symmetry breaking mechanism is investigated using the Schwinger Dyson equations which are then truncated in a nonperturbative manner using the Ball-Chiu vertex ansatz. The complexity of the resulting coupled fermion-photon system means that the photon is initially replaced by its perturbative form. Numerical investigations of this simplified system then reveal the existence of an exponential relationship, in terms of the dimensionless parameter N, between the intrinsic and dynamical mass scales, $m \sim e^2 \exp(-cN)$. Contrary to the assertions of Appelquist *et al* the wavefunction renormalisation was found to be nonperturbative and crucial in determining this behaviour.

The sensitivity of this mechanism to the nonperturbative behaviour of the photon is investigated. A simple analysis shows it to be far stronger than previously expected. This is confirmed by a numerical analysis of the coupled photon-fermion system which suggest the relationship between the two scales in the theory is of the form $m \sim e^2 \exp(-cN^2)$. This model therefore illustrates how a large hierarchy of scales may naturally occur in a gauge theory, for instance N=3 $m/\alpha \sim 10^{-5}$.

Finally an investigation of the gauge dependence of this approach is initiated. The softening of the photon in the low momentum region is shown to amplify automatically any inadequacy of the vertex ansatz by factors of $O(\alpha/m)$ in all but the Landau gauge. It is therefore expected that any incomplete vertex form will result in the generation of a "critical gauge", ξ_c, below which chiral symmetry breaking solutions will not exist. A path of further investigation is suggested.

Acknowledgements

I would like to thank the people who have helped me through the course of study and more generally made the place brighter than it otherwise would have been. First and foremost I wish to thank my Supervisor Mike Pennington, for his cajoling, nagging, laughing, gossiping, squeaky shoes and service well above the call of duty. Thanks Mike, live long and prosper, maybe one day you'll get a secretary!

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More than these, however, I must thank my parents for their support and confidence. Far better to strive .... you may even succeed! —

This thesis is dedicated to my parents.

DECLARATION

I declare that no material in this thesis has previously been submitted for a degree at this or any other university.

The research in chapters two to four has been carried out in collaboration with Dr. M.R. Pennington and is partly summarised in the paper,

M.R. Pennington and D.Walsh,
"Masses from Nothing: A Nonperturbative investigation of QED in three dimensions." Durham Preprint DTP-90-62. To be published in Phys. Lett. B253 (1991) 246.

Also in collaboration with DR. M.R. Pennington and D.C. Curtis, the paper:

D.C. Curtis, M.R. Pennington and D.A.Walsh,
"On the gauge dependence of dynamical fermion masses", Phys. Lett. B249 (1990) 528.

which is relevant to chapter four.

Contents

Chapter 1: Introduction.

1.1 General Introduction.

The "Standard Model" is the model that currently describes the interactions of fundamental particles down to scales of the order of 10^{-16}cm. This model is a gauge theory, the matter particles are represented by fermions and forces are described in terms of the exchange of gauge bosons. The group associated with this gauge symmetry is $SU(3) \times SU(2) \times U(1)$ — the Strong nuclear force is represented by the $SU(3)$ of QCD and the Electromagnetic and Weak nuclear forces are contained in the $SU(2) \times U(1)$ Electroweak (EW) model of Salam and Weinberg.

Spontaneous symmetry breaking (SSB) plays a central role in the Standard Model. In gauge theories the force carriers are the gauge vector bosons and these are naturally massless. Such massless particles, however, cannot explain the short range nature of the Weak Nuclear force. In the Standard Model, the unified electroweak force is spontaneously broken to yield the distinct Electromagnetic and Weak forces that we observe at low energies. This is accomplished by including in the theory an elementary scalar field known as the Higgs field. At low energies it can be arranged to be energetically favourable for this field to acquire a nonzero vacuum expectation value. The effect of this vacuum expectation value, via what is known as the Higgs mechanism, is to break the gauge symmetry into two separate components. These two components are the distinct Electromagnetic and Weak forces that we observe at low energies. The attraction of the Higgs mechanism is that it also results in some of the gauge bosons aquiring a nonzero mass, via their Yukawa couplings to the Higgs vacuum expectation value. Specifically, in the Standard model the Higgs mechanism gives masses to the carriers of the Weak nuclear force, the W and Z bosons, while leaving the photon massless. A consequence of the Higgs mechanism is that these masses are naturally of the same order as the energy scale at which the symmetry breaking takes place, in the Standard Model this is of $O(100)$ GeV. This mechanism therefore explains why at low energies the weak nuclear force has effectively a very short range while the range of the electromagnetic force remains infinite.

Inspired by the long and successfull history of unification, many attempts have been made to try and incorporate the Strong and Electroweak sectors of the Standard Model into a single unified force. Such "Grand Unified Theories" need at least two stages of spontaneous symmetry breaking. Firstly the unification group G must be broken to give the Standard Model and secondly the spontaneous symmetry breaking of the Electroweak sector must take place to leave us our three distinct low energy forces. However, because of the large differences in coupling strengths this unification is not expected to become apparent below energy scales of $O(10^{15})$ GeV!

Such a large disparity between the unification (GUT) scale and the Electroweak scale results in what is known as the "Gauge Hierarchy Problem". Any "Grand Unified" theory must explain why these two scales are so vastly different. Unfortunately, if these stages of symmetry breaking are governed by elementary scalars — the Higgs mechanism in some grand form — these models become highly contrived. The breaking of the unifying group, G, down to the Standard Model is expected to take place at an energy scale $E \sim 10^{15}$ GeV and will naturally result in a vacuum expectation value for the grand Higgs field of a similar magnitude. The problem is that without extremely fine tuning the masses of the observed quarks and leptons are naturally swept up to this scale! This leaves us with the conclusion that, rather than being a natural progression, the Standard Model must be "wrung" from such theories.

This motivates the question — are Higgs fields really necessary for spontaneous symmetry breaking? Indeed are the Higgs scalars fundamental or are they merely composites? Such ideas are not new or even confined to the area of particle physics. Nambu and Jona-Lasinio [1] were the first to show that spontaneous symmetry breaking may be of an entirely dynamical nature and the BCS model is the accepted theory of Superconductivity. In both these theories the symmetry breaking is induced dynamically by a composite scalar field. However, if the Higgs fields of the Standard Model are really composite then what are they made up of and what binds these constituents together?

One attempt to implement this idea of dynamical symmetry breaking is provided by Technicolor theories [2]. In these models the $SU(3)$ of the strong interaction is embedded in a larger colour group, an "Extended Technicolor" group, $\mathcal{G}_{ETC}$. At some very high energy, $E \sim M_{ETC}$, this large group spontaneously breaks down to $SU(3)$ (QCD) and a Technicolor group, $\mathcal{G}_{TC}$.

$$\mathcal{G}_{ETC} \longrightarrow SU(3) \times \mathcal{G}_{TC} \qquad , \qquad E \lesssim M_{ETC}$$

The extra gauge bosons and fermions introduced by this construction are not observed and clearly must be "hidden" at low energies. Therefore these models arrange that below some scale, Λ_{TC}, all the additional Technifermions are confined, in the same way as quarks are confined inside hadrons. Consequently, above Λ_{TC} we expect to have a spectrum of "Techni-hadrons", of which the lightest are expected to be the technipions. Such scalar and pseudo-scalar particles effectively take the place of the Higgs fields in these models. When the Technifermions condense, the quarks and leptons acquire masses related to Λ_{TC}, induced by the Yukawa-like interations between the ordinary and Technicolor sectors. However, for these theories to work in practice there are two conditions that must be satisfied. Firstly, because the composite nature of the Higgs field is not yet apparent nor have any Techni-hadron candidates been observed, the Technicolor confinement scale Λ_{TC} must lie above the range of currently accessible energies. However, if this scale is too large it will result in unphysically large quark and lepton masses. Secondly, the scale at which the Extended Technicolor Group breaks, M_{ETC}, must lie significantly above current energies in order to avoid unobservably large flavour-changing neutral currents [3]. Consequently, it may appear that we have merely translated the problem of why quark and lepton masses are so much smaller than the vacuum expectation value of an effective Higgs field in the Electroweak sector, into the question of why there is a heirarchy between the symmetry breaking scale, M_{ETC}, and the confinement scale, Λ_{TC}, of some unknown Technicolor sector. The challenge is to show that within this type of symmetry breaking scheme such a large disparity in intrinsic scales naturally occurs. Preliminary investigations employing the Schwinger Dyson equations have suggested that this may be the case [2].

This thesis continues in the spirit of such investigations. Here we will not be concerned with the breaking of large symmetry groups but will concentrate on a simple model where the generation of a dynamical mass scale in the Schwinger Dyson formalism is more under control. Our goal is to show that a large range of mass scales is possible and perhaps even natural. The model that will be presented is not connected with any Standard or GUT Model, but this is not necessarily detrimental to the exercise. Our goal is to demonstrate in a simple context the possibility that the dynamics of more realistic models may achieve the remarkable observed range of scales quite naturally.

1.2 The Model.

The model we shall be investigating in this work is 3 dimensional QED with N flavours, or duplicates, of fermion. This section is devoted to both the motivation of what is at first sight a rather obscure choice and also the description of its important features.

Let us start with the most obvious question — Why three dimensions rather than the more obviously physical four? As is well known four dimensional QED is renormalisable, all the ultraviolet divergences of the bare theory may be absorbed into multiplicative factors, which allow us to define a consistent infrared theory via renormalisation arguments. In contrast, three dimensional QED is superrenormalisable, the theory is ultraviolet finite and no careful cancellation of divergent quantities is needed to extract "physical" results. Therefore, working in three dimensions we are able to attack the question of chiral symmetry breaking without the added complication of infinite renormalisations. However, the question of chiral symmetry breaking in three dimensional QED is far from trivial, as opposed to 2 dimensional QED [4], and a concensus on the subject has yet to be reached.

Three dimensional field theories are not unrelated to their four dimensional counterparts. There is a well known relation between a four dimensional theory at finite temperature, T, and its corresponding 3D variant. In analysing a "hot" four dimensional field theory one conventionally considers the theory after it has been rotated into Euclidean space, thereby effectively converting it into a thermodynamic system [5]. It can then be shown that for a finite temperature the excitations in the timelike direction, x^0, must be periodic or antiperiodic with period , in appropiate units, proportional to the inverse of the temperature. As viewed from within the noncompactified three space dimensions, excitations in the timelike direction then obviously form a discrete (infinite) set of scalars with excitation energy (mass) proportional to T (c.f. the Bohr Atom). Therefore, above length scales, $\xi \gg T^{-1}$, these modes have effectively decoupled and the "dynamics" of the theory is given by its three dimensional counterpart. We may therefore expect the infrared behaviour of our 3 dimensional model to be able to tell us something about the nature of hot QED4.

Also in statistical mechanics one comes naturally across lower dimensional models. When a statistical system is near a critical point, *i.e.* where one or more correlation lengths are diverging, the underlying discreteness of the system is usually unimportant and the system can be modeled by a continuous field theory. The reason that some systems may be modeled by a two or three and not four dimensional theory is that they are of an effectively lower dimensional nature. Instances where this lower dimensionality is more obvious are for example in thin films, $(2+1)$ dimensional, and models of polymers, $(1+1)$ dimensional. Implicitly lower dimensional examples include lattices that are strongly coupled in two directions, though only weakly in a third, which may be approximated by a collection of independent 2 dimensional systems. Indeed some recent theoretical explorations into high temperature superconductivity have been formulated in terms of a three dimensional gauge theory [6]. Three dimensional theories should not therefore be written off as unphysical although, in the end, they may have little to say directly about their four dimensional counterparts.

In order to appreciate what our model may say about hierarchies and chiral symmetry it is necessary to detail some of its properties [7]. The Lagrangian density for QED is given by,

$$\mathcal{L}\,[\,\overline{\psi},\psi,A^{\mu}\,] = -\frac{1}{4}F^{\mu\nu}\,F_{\mu\nu} + \overline{\psi}\,\gamma^{\mu}\,(\,\partial_{\mu} - ieA_{\mu}\,)\,\psi\;, \tag{1.1}$$

where the electromagnetic field tensor $F^{\mu\nu}$ is defined as,

$$F_{\mu\nu} = \partial_{\mu}\,A_{\nu} - \partial_{\nu}\,A_{\mu}\;. \tag{1.2}$$

This theory is invariant under U(1) gauge transformations, acting on the fundamental fields in the following way ,

$$\begin{gathered} \psi \rightarrow e^{-i\theta}\psi \quad , \quad \overline{\psi} \rightarrow e^{i\theta}\overline{\psi} \quad , \quad A^{\mu} \rightarrow A^{\mu} - \partial^{\mu}\theta \\ \theta = [\,0, 2\pi\,)\;. \end{gathered} \tag{1.3}$$

We will choose to represent our fermions using four component spinors. In terms of the Pauli spin matrices we may represent these by,

$$\gamma^0 = \begin{bmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{bmatrix} \quad , \quad \gamma^1 = \begin{bmatrix} i\sigma_1 & 0 \\ 0 & -i\sigma_1 \end{bmatrix} \quad , \quad \gamma^2 = \begin{bmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{bmatrix}\;. \tag{1.4}$$

In addition there are two other 4×4 matrices we may construct,

$$\gamma^3 = i\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \gamma^5 = i\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{1.5}$$

that anticommute with those of (1.4). If we project the fermion into its left and right hand spin components via the following,

$$\begin{aligned} \psi_L &= \mathcal{P}_L\,\psi \quad , \quad \mathcal{P}_L = \frac{1}{2}\,(\,1 - \gamma^5\,) \\ \psi_r &= \mathcal{P}_R\,\psi \quad , \quad \mathcal{P}_R = \frac{1}{2}\,(\,1 + \gamma^5\,) \\ \psi &= \psi_L + \psi_R \end{aligned} \tag{1.6}$$

then it is a simple exercise to show that the massless theory given by (1.1) is invariant under the "chiral" transformations,

$$\psi \to e^{i\alpha\gamma^3}\psi \quad , \quad \psi \to e^{i\beta\gamma^5}\psi \qquad \alpha,\beta \in [0,2\pi] \ . \tag{1.7}$$

The massless theory can be considered as that of two particles exclusively either right or left handed which may be treated independently,

$$\mathcal{L}\left[\overline{\psi},\psi\right] \to \mathcal{L}_L\left(\overline{\psi}_L,\psi_L\right) + \mathcal{L}_R\left(\overline{\psi}_R,\psi_R\right) \tag{1.8}$$

The transformations in (1.7) therefore merely express the fact that we are free to mix these two independent components. We are also free to choose independent gauge transformations for each of the right and left components. The full gauge invariance of the massless theory is then $U(1)_L \times U(1)_R$. By introducing a mass term these two components can mix,

$$m\overline{\psi}\psi \to m\left(\overline{\psi}_L\psi_R + \overline{\psi}_R\psi_L\right) \ . \tag{1.9}$$

and the independence of each spin component is broken. Chiral symmetry is explicitly broken by this term and also as a consequence the gauge transformations for each left and right component must be identical. The $U(1)_L \times U(1)_R$ symmetry is reduced to $U(1)_{L+R}$. The breaking of chiral symmetry is therefore associated with the generation of a nonzero fermion mass.

In three dimensions there is a second possible representation we may use for our fermions. In terms of the Pauli spin matrices we may associate the Dirac γ matrices with,

$$\gamma^0 = \sigma_2 \ , \quad \gamma^1 = i\sigma_3 \ , \quad \gamma^2 = i\sigma_1 \ . \tag{1.10}$$

as opposed to (1.4). However, with only 2×2 matrices available it is not possible to construct another γ matrix to anticommute with these. Therefore no generator exists to generate chiral transformations in this representation. The massless theory possesses no more symmetry than the massive. In addition there is an alternative masses term to (1.9) that preserves chiral symmetry but only at the cost of breaking parity invariance.

$$m\overline{\psi}\frac{1}{2}\left[\gamma^3,\gamma^5\right]\psi\ .$$

In this work, however, we will not avail ourselves of this possiblity, which cannot occur in four dimensions, and will concentrate solely on the conventional mass term.

A rather more important consequence of working in three dimensions is that the coupling e is naturally dimensionful, with dimension $[M]^{\frac{1}{2}}$. In a field theory a dimensionful coupling usually indicates that the theory is unrenormalisable. At each order in a perturbative expansion increasing numbers of other dimensionful quantities must be brought in to compensate for the dimensionality of the expansion parameter, e^2. In a massless theory, such as the one we are considering, there is only one other scale available and that is momentum. The expansion parameter is therefore effectively e^2/p and so increasingly higher orders in the expansion introduce stronger and stronger infrared divergences. At one time this was claimed to point to the necessity of chiral symmetry breaking in this model, by introducing a mass we introduce a scale at which this infrared divergent behaviour can be cutoff. However, if one considers this model with N identical fermions, it is possible to show within a $1/N$ expansion that chiral symmetry breaking is not necessary and that the theory is finite at each order in $1/N$. In the next section we will introduce this new perturbation theory and develop it within our model.

1.3 The 1/N expansion.

More variables usually entail greater complexity, but not always. There exists families of theories with large symmetry groups, *e.g.* SU(N), that actually become simpler as N becomes larger, more precisely these theories posses an expansion in $1/N$ [8]. Our model of QED3 with N flavours of fermion is but one example. Since each fermion flavour is identical we may freely mix these flavour in any unitary fashion,

$$\Psi(x) = \begin{Bmatrix} \psi_1(x) \\ \cdot \\ \cdot \\ \cdot \\ \psi_N(x) \end{Bmatrix} \quad , \quad \Psi(x) \rightarrow U\,\Psi(x) \quad U \in \mathrm{SU}(N)\,. \tag{1.11}$$

while still preserving the Langrangian density, $\mathcal{L}$. In contrast to a gauge symmetry this symmetry is not associated with any force. It is referred to as "Flavour" SU(N) and merely expresses the identical nature of our N fermion field components.

To get an intuitive idea as to how more fields may simplify a system consider the following. Imagine a set of N magnetic spins $s_i^\mu(x)$ undergoing statistical fluctuations, generated via a heat bath, that are coupled both to a magnetic field and to each other. The external magnetic field obviously introduces an external direction with which the individual spins tend to align themselves. The physical observable in this system is the combined spin $|S(x)|$ of all the indentical individuals spins. As we increase N this will become dominated by some average value, individual spins will tend to interfere within $|S(x)|$ leading to small fractional fluctuations of $O(1/N)$,

$$S^\mu(x) = \sum_{i=1}^{N} s_i^\mu(x) \quad , \quad \lim_{N\rightarrow\infty} S(x)\cdot S(x) \longrightarrow \overline{S}^2(x)\,(1 + O(1/N))\;. \tag{1.12}$$

In other words, even though each individual component may fluctuate wildly, the sum of squares of the N spin components is large and dominated by its average value, up to statistical fluctuations of order of $O(1/N)$. The limit $N \rightarrow \infty$ is known as the Mean Field approximation because after neglecting these frational fluctuations, the spin components effectively become coupled to some average field, a combination of the external magnetic field and their own collective field. Within selected models the higher nontrivial $1/N$ corrections to this mechanism may be calculated systematically.

Consider our model and its Lagrangian as given by (1.1). In attempting to construct a $1/N$ expansion of any model one must be careful which quantities are held constant while N is varied. We may illustrate this point with the following examples:

i) Set $eN = \alpha$, and hold α constant as N is varied. Any loop within a certain Feynman diagram obviously introduces a factor of e^2 from each vertex and at most one factor of N. When the loop is a fermion loop we have to sum over all fermion flavours leading to N identical contributions. Therefore any insertion into a tree level diagram will introduce an overall factor of, at best, α/N. In the large N limit therefore this model corresponds to the Born approximation to the theory, *all* corrections will be explicitly suppressed by at least one factor of $1/N$.

ii) In contrast, consider holding e^2 constant as usual. In this case we cannot express our theory purely in terms of a power series in $1/N$. Each fermion loop will contribute a factor of e^2N and so introduce positive powers of N into any expansion, rendering the large N limit meaningless.

There is, however, a way to define a sensible expansion and that is to set $\alpha = e^2N/8$, the factor of 8 is merely for convenience as will become clear later. With this definition fermion loop corections will introduce no factors of $1/N$ and so may be inserted into any diagram at no extra "cost". Factors of $1/N$ can be related to the insertion of photon lines as they introduce no summation over flavours needed to lift the extra explicit $1/N$ introduced by the two connecting vertices. Therefore at each order in $1/N$ there is an infinite number of corrections that may all be traced back to a "generating" diagram in e^2 perturbation theory to which any number of fermion loops have been inserted, as illustrated in Fig. 1.1.

In this sense the $1/N$ expansion is nonperturbative, at every order it sums an infinite set of standard perturbative corrections. Does this resolve our problem of infrared divergences, now that our expansion parameter $1/N$ is dimensionless? The answer to this question is yes and is linked to a nonperturbative change in

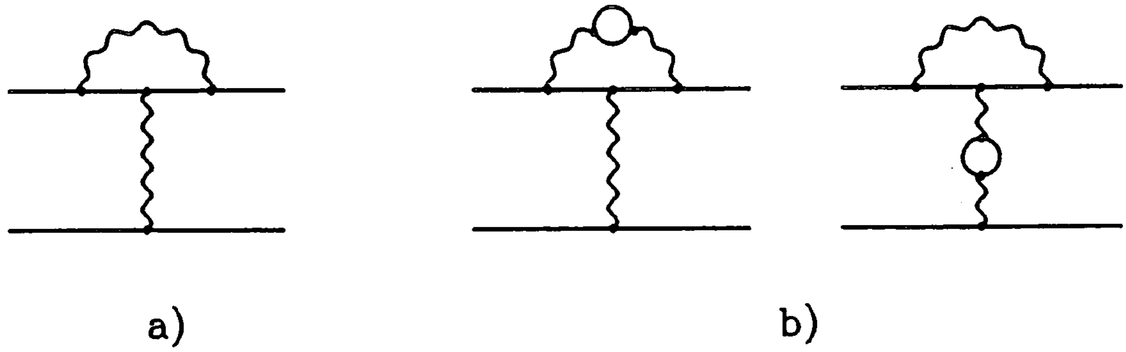

Figure 1.1 a) An e^4 diagram in e^2 perturbation theory. b) Associated diagrams at $1/N^2$ in 1/N perturbation theory.

the low momentum behaviour of the photon propagator.

It easy to see that the photon will receive corrections from fermion loops at $O(1)$ in the $1/N$ expansion and these will turn out to be essential in determining its behaviour in the infrared. That the effect of these loops is to qualitatively change the behaviour of the photon function is quite clear. The photon propagator, in a covariant gauge, may be written in the following general form,

$$\mathcal{D}_{\mu\nu}(p) = \frac{1}{\mathcal{G}(p)} \left\{ \delta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right\} + \xi \frac{p^\mu p^\nu}{p^4} \tag{1.13}$$

where ξ is the covariant gauge parameter. Gauge invariance prevents any corrections contributing to the longitudinal part of the photon. The contribution from one fermion loop is given by,

$$\begin{aligned} p^2\,\Pi_1^{\mu\nu}(p) &= -N\,e^2 \int \frac{d^3k}{(\,2\pi\,)^3}\,\mathrm{Tr}\,[\,\gamma^\mu S(k)\gamma^\nu S(k+p)\,] \\ &= \Pi(p) \left(\delta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) . \end{aligned} \tag{1.14}$$

Where the factor of N simply counts the N identical contributions from each of our fermion flavours. This integral is fairly straightforward to analyse† but we may tell one significant fact purely from dimensional arguments. Since we are

† It is not completely trivial as it is ultraviolet divergent. We will leave a discussion of this until Chapter 3 where the photon is dealt with in detail.

working in a massless theory there are two scales within $\Pi(p)$, α and p. At this order we know that α occurs explicitly only once and therefore we may write,

$$p^2\Pi_1(p) = Ne^2 pf(p/\alpha) \; . \tag{1.15}$$

Explicitly working out the integral (1.14), it is straightforward to show that $f(p/\alpha) = 1/8$. In order to calculate the full correction, however, we must sum up all possible strings of such loops corections. This is simple as each loop in a string is independent and we may write,

$$\begin{aligned} \frac{1}{\mathcal{G}(p)} &= p^2 \left[1 - \Pi_1 + \Pi_1^2 + \Pi_1^3 + \cdots\right] \\ \longrightarrow \; \mathcal{G}(p) &= p^2\left[1 + \Pi_1(p)\right] \; . \end{aligned} \tag{1.16}$$

Therefore in the infrared the photon function now has a linear dependence on p as opposed to the bare p^2 scaling,

$$\mathcal{G}(p) = p^2 + \alpha p \rightarrow \alpha p \qquad p \ll \alpha \tag{1.17}$$

where we have substituted for $\alpha = e^2N/8$, due to the dominance of fermion loops in this region.

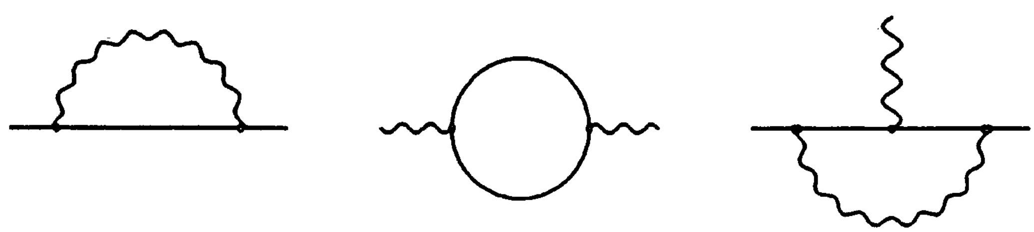

Figure 1.2 The fundamental self-energy and vertex corrections in QED.

What effect does this have on the infrared behviour of our $1/N$ expansion in comparison to that of the divergent expansion in e^2. Consider a tree level graph at any particular order, this will be finite by definition. To generate higher order correction then we may proceed by inserting the fundamental self-energy and vertex corrections illustrated in Fig. 1.2. It is beyond this work to show that these are the only type of insertions that we need to consider, a clear description of the general perturbative renormalisation proceedure in QED is given in Ref 9. The correction to the photon $\Pi(p)$, as we have just shown, is convergent so we are simply left with the fermion and vertex corrections. These are related by the Ward Identity,

$$q \cdot \Gamma\left(p, p'\right) = S^{-1}(p) - S^{-1}(p') \quad , \quad q = p - p' \; . \tag{1.18}$$

which ensures that the divergences, $\ln^i(\alpha/p)$, from these insertions will cancel leaving only infrared finite contributions. Our $1/N$ expansion will be finite at each order without the need to invoke chiral symmetry breaking. Indeed, the question is now open as to whether chiral symmetry breaking does occur in this model and under what circumstances.

In the next section we will introduce the Schwinger Dyson equations which express the dynamical content of a theory and it is these that will play the central role in determining the Chiral properties of our model.

1.4 The Schwinger Dyson Equations.

The Schwinger Dyson equations are a set of nonperturbative relations between the full Greens functions of a quantum field theory (QFT). They may be viewed as analogous to the wave equations of a classical field theory and are of interest because they may be obtained free from any approximation. In principle, they may be used to study nonperturbative features of a given model inaccessible to standard perturbative approaches. We will derive the Schwinger Dyson equation for the fermion propagator in QED in full gory detail using the partition function [10]. In order to give a more intuitive insight into what is contained in such relations, we will approach the photon Schwinger Dyson equation from a perturbative viewpoint although this approach is not strictly confined to the perturbative arena.

As in statistical mechanics, with which quantum field theory has many similarities, the physics of a quantum system is contained within the partition function, denoted by $\mathcal{Z}$. The partition function for QED is given below,

$$\mathcal{Z}\,[\,\overline{\eta},\eta,J^{\mu}\,] = N \int \mathcal{D}A^{\mu}\,\mathcal{D}\overline{\psi}\,\mathcal{D}\psi\;\exp\{\,iG\,\}$$

$$G = \int \{\,\mathcal{L}(\overline{\psi},\psi,A^{\mu}) + \overline{\eta}\psi + \overline{\psi}\eta + J^{\mu}A_{\mu}\,\}\;d^{n}x\;. \tag{1.19}$$

It is a function of the sources J^{μ}, $\overline{\psi}$, ψ which may used to perturb the system in order to determine correlation functions, amplitudes etc. The prefactor N is an overall normalisation that ensures that $Z[0]=1$, *i.e.* $\langle\, 0_{-\infty}|0_{+\infty}\,\rangle = 1$, and n is simply the dimension of spacetime. We assume the reader is aquainted with the Feynmann path integral formalism as it is beyond the scope of this work to detail this standard approach. The partition function should, by definition, be independent of the fundamental fields, so we may write,

$$\frac{\delta}{\delta A^{\mu}(x)}\mathcal{Z}[\overline{\eta},\eta,J^{\mu}] = \frac{\delta}{\delta\psi(x)}\mathcal{Z}[\overline{\eta},\eta,J^{\mu}] = \frac{\delta}{\delta\overline{\psi}(x)}\mathcal{Z}[\overline{\eta},\eta,J^{\mu}] = 0\;. \tag{1.20}$$

In other words, in defining $\mathcal{Z}$ we integrate over *all* configurations of the fundamental fields. If we assume that the integrand vanishes on the boundaries of the functional integral† then we may obviously take these differentials within the definition of $\mathcal{Z}$. This will eventually lead to relations between the full Greens functions of the theory. For example consider,

$$\int \mathcal{D}A^{\mu}\,\mathcal{D}\overline{\psi}\,\mathcal{D}\psi\;\frac{\delta}{\delta\overline{\psi}(x)}\;\exp\{\,iG\,\} = 0\;. \tag{1.21}$$

This may be simply written as,

$$\left\langle \frac{\delta S}{\delta\overline{\psi}(x)} \right\rangle + \eta(x) = 0 \tag{1.22}$$

† This is not as strightforward as it may sound since the integrand e^{iG} is oscillatory and of magnitude one for any value of G. Indeed the Feynmann path integral (1.19) is not well defined and in general some convergence prescription, such as rotating into Euclidean space, is required to ensure the integrand vanishes for extremal configurations $S \to \infty$.

where we have introduced the following notation,

$$\langle \mathcal{F}[\psi,\overline{\psi},A^\mu] \rangle = \int \mathcal{D}A^\mu \, \mathcal{D}\overline{\psi} \, \mathcal{D}\psi \; \mathcal{F}[\psi,\overline{\psi},A^\mu] \; exp\{iG\} \tag{1.23}$$

Note that in the classical limit, $\hbar \to 0$, the path integral will collapse to the saddle points given by,

$$\frac{\delta S}{\delta\psi(x)} + \overline{\eta}(x) = 0 \; . \tag{1.24}$$

Not surprisingly, this is the relation you would obtain by perturbing the classical field theory in order to deduce it's field equations in the presence of a ψ source. Returning to the quantum world, the form of the Lagrangian for QED is,

$$L\,(\overline{\psi},\psi,A^\mu) = \int d^n x \, \left\{ i\overline{\psi}\,(\not{\partial} - ie\not{A}\,)\,\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \right\} \tag{1.25}$$

from which we may immediately write (1.21) as,

$$\left\langle \eta(x) + \left(\gamma^\mu \frac{\partial}{\partial x^\mu} - m - ie\gamma^\mu A_\mu(x) \right) \psi(x) \right\rangle = 0 \tag{1.26}$$

If we use the identities below that follow directly from (1.19),

$$\begin{aligned} \langle A^\mu(x) \rangle &= \frac{\delta}{i\delta J^\mu(x)} \mathcal{Z}[\overline{\eta},\eta,J^\mu] \\ \langle \psi(x) \rangle &= \frac{\delta}{i\delta\overline{\eta}(x)} \mathcal{Z}[\overline{\eta},\eta,J^\mu] \end{aligned} \tag{1.27}$$

then we may cast (1.26) as the following differential form,

$$\left[\eta(x) + \left(i\not{\partial}_x - m - e\gamma^\mu \frac{\delta}{i\delta J^\mu(x)} \right) \frac{\delta}{i\delta\overline{\eta}(x)} \right] \mathcal{Z}\,[\eta,\overline{\eta},A^\mu] = 0 \tag{1.28}$$

As it stands this relation is of little use. If we introduce the generating functional for the connected Green's functions, $W[\overline{\eta},\eta,J^\mu]$, via the standard definition,

$$\mathcal{Z}\,[\overline{\eta},\eta,J^\mu] = \exp\{\, i\, W\,[\overline{\eta},\eta,J^\mu]\,\} \; . \tag{1.29}$$

Then differentiating (1.28) with respect to $\eta(y)$ we have,

$$\delta(x-y)\,\mathcal{Z} - \left(i\partial\!\!\!/_x - m - e\gamma^\mu \frac{\delta}{i\delta J^\mu(x)} \right) \langle\, \overline{\psi}(y)\psi(x) \,\rangle = 0 \; . \tag{1.30}$$

We have defined,

$$\begin{aligned} \langle\, \overline{\psi}(y)\psi(x) \,\rangle &= \frac{1}{i} \frac{\delta}{\delta\eta(y)} \frac{\delta}{\delta\overline{\eta}(x)}\, \mathcal{Z}\,[\eta, \overline{\eta}, J^\mu] \\ &= S(x,y;J)\,\mathcal{Z}[\eta, \overline{\eta}, J^\mu] \; . \end{aligned} \tag{1.31}$$

The function $S(x,y;J)$ describes the propagation of a fermion in the presence of our sources, generically denoted by J. Performing the differentiation with respect to J^μ by parts and cancelling an overall factor of $\mathcal{Z}$ we may write,

$$\delta(x-y) - \left(i\partial\!\!\!/ - m - e\gamma_\mu \,\langle\, A^\mu(x) \,\rangle - e\gamma^\mu \frac{\delta}{i\delta J^\mu} \right) S(x,y;J) = 0 \; . \tag{1.32}$$

Which is a relationship between the fermion propagator and the three point connected Green's function, $G_c^\mu \sim \delta S/\delta J^\mu$. If we now set all the sources to zero then $\langle\, A^\mu(x) \,\rangle$ must vanish as it is the expectation of a gauge dependent quantity. We may then rearrange (1.32) into the following suggestive form,

$$\delta^n(x-y) = \left(i\partial\!\!\!/_x - m \right) S(x,y) - ie\gamma_\mu\, G_c^\mu(x,y;x) \tag{1.33}$$

where,

$$G_c^\mu(x,y;z) = \frac{1}{i^2} \frac{\delta^3 W}{\delta\overline{\eta}(x)\delta\eta(y)\delta J^\mu(z)} \tag{1.34}$$

is the *connected* 3 point vertex. This may be uniquely written in terms of the 3 point 1 particle irreducible vertex by simply attatching fermion and photon lines as illustrated in Fig 1.3,

$$\begin{aligned} (i\partial\!\!\!/ - m)S(x,y) - ie^2 \int d^n z\, d^n x'\, d^n y'\; \gamma_\mu\, \mathcal{D}_{\mu\nu}(x,z)\, S(x,x') \\ \times \Lambda_\nu(z;x',y')\, S(y',y) = \delta^n(x-y) \end{aligned} \tag{1.35}$$

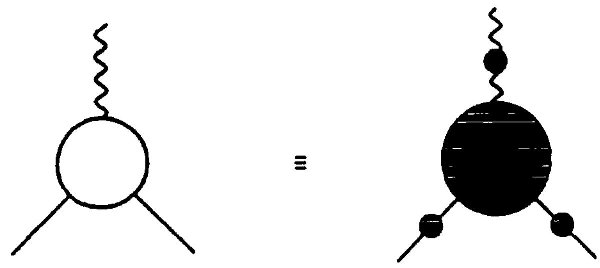

Figure 1.3 Attaching fermion and photon lines to the 3 point 1PI vertex to form the connected 3 point vertex. The o indicates a connected while ● indicates 1PI vertex.

The best way to think of this is as an operator relation for the fermion propagator,

$$\left[\,(\,i\partial\!\!\!/ - m - \mathcal{M}\,)\,S\,\right](x,y) = \delta^4(x-y) \tag{1.36}$$

where we have defined the selfmass operator,

$$\mathcal{M} = ie^2 \int d^n z\, d^n x'\; \gamma_\mu\, \mathcal{D}_{\mu\nu}(x,z)\, S(x,x')\Lambda_\nu(z;x',y)\ . \tag{1.37}$$

This may be inverted to give a relation for the full fermion propagator in terms of $\mathcal{M}$. The resulting relation is best represented diagramatically as below,

Figure 1.4 Diagrammatic form for the Fermion Schwinger Dyson equation in QED. The • indicates that the propagator line or vertex is full.

In wading through all this formal manipulation it is quite easy to loose track of what is being stated by these relations. Let us instead take a different approach, one more closely linked to the more familiar ideas of perturbation theory [9].

Consider how corrections may appear in the photon propagator. In general we may split any correction into its connected parts, the "blobs" illustrated above. These are connected in the sense that they cannot be split into disjoint parts by the cutting of a single internal photon or fermion line and may obviously occur any number of times 'along' a photon line.

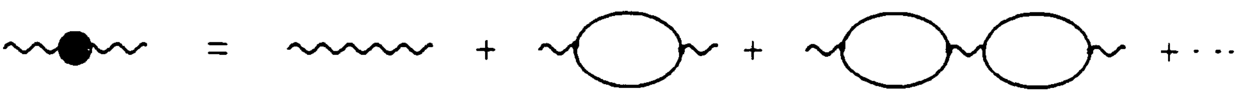

Figure 1.5 Generic corrections to the photon propagator. The open blobs are not single fermion loops but connected 2 point functions.

This set of corrections obviously forms a simple geometric series and it is not hard to convince oneself that they may be summed in the following manner,

$$\begin{aligned} \mathcal{D}^{\mu\nu} &= \mathcal{D}_0^{\mu\nu} + \mathcal{D}_0^{\mu\rho}\Pi_\rho^\mu + \mathcal{D}_0^{\mu\rho}\Pi_\rho^\sigma\mathcal{D}_0^{\sigma\delta}\Pi_\delta^\mu + \cdots \\ \rightarrow \quad \mathcal{D} &= \mathcal{D}_0\,[\,1 + \Pi\mathcal{D}\,]^{-1} \ . \end{aligned} \tag{1.38}$$

The last line in the relation above should only be considered as an indication of this mechanism and for this reason the indicies have been removed. We must now consider how we may describe what we have happily referred to as a "blob". We know, from the form of our Lagrangian, that the photon couples only to fermions and so initially, within our "blob", the photon must diassociate into two fermions.

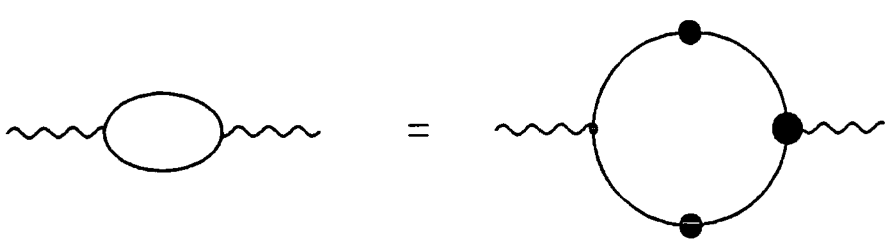

Figure 1.6 The characterisation of the general connected "blob" corections in terms of the full irreducible one particle vertices and propagators, denoted by the • symbol.

These must be then allowed to interact in all possible ways, self-interaction and with each other, before ultimately recombining to produce our outgoing photon again. In Fig 1.6 the unique decomposition of our "blob" into full 1PI vertices is illustrated. Its form is not particularly surprising although one question may spring to mind — Why is one vertex bare while the other is full?

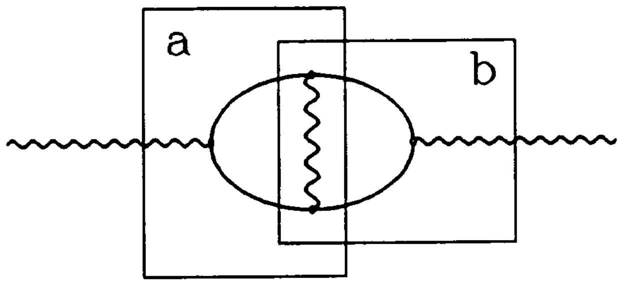

Figure 1.7 The Vertex Correction at $O(e^4)$ within our generic "blob"

This is easily answered if one considers one of the $O(e^4)$ perturbative corrections as given in Fig 1.7. This diagram occurs only once and can be considered as a correction to either vertex but clearly not both. If both the vertices in Fig 1.6 were full then we would count this diagram twice. One vertex therefore remains bare to prevent overcounting of this type. We may summarise what we have deduced about the general structure of the corrections to the photon diagramatically, as below,

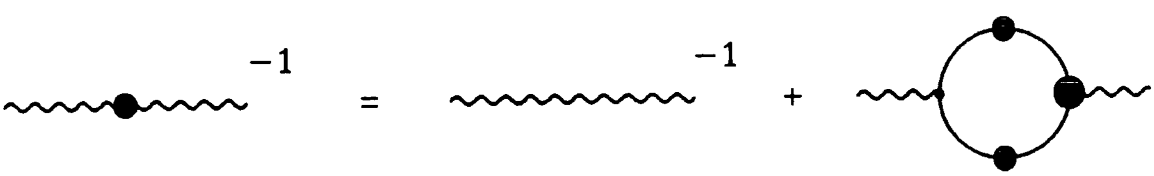

Figure 1.8 The symbolic form for the photon Schwinger Dyson equation.

This is the Schwinger Dyson equation for the photon propagator which we may have equally derived from the partition function, as we did for the fermion Schwinger Dyson equation.

Clearly we might consider repeating this proceedure for any of the infinite number of vertices or alternatively considering higher functional derivatives of some generating relation such as (1.28). The Schwinger Dyson equations therefore naturally form a hierarchy of increasing complexity. They display the general structure of any vertex which is implicitly encoded within the form of the Lagrangian of the theory. The generic form of a Schwinger Dyson equation in QED is,

$$\Gamma^n = \mathcal{R}\left[\Gamma^{n+1}, \Gamma^n, ...\Gamma^2\right] \tag{1.39}$$

and a further example, that of the three point vertex, is shown diagramatically below,

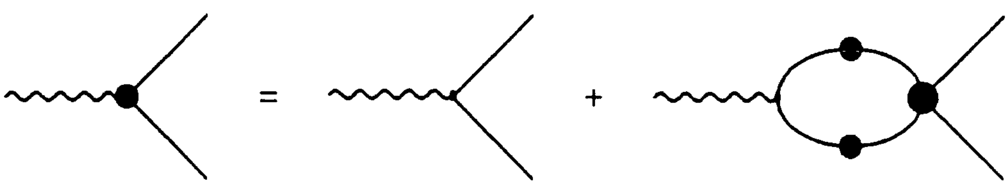

Figure 1.9 The Schwinger Dyson equations for the three point vertex in QED

For a more detailed analysis of the ideas brought up in this section we refer the reader to Ref.9& 10. In the next section we consider how the gauge symmtery of the theory constrains the form of the three point vertex.

1.5 The Ward Identity.

The Ward identity is a completely nonperturbative relation between the 3 point fermion boson vertex and the fermion propagator in QED. It is obtained by requiring the physical system, as described by the partition function, (1.19), to be gauge invariant. Only by introducing a gauge fixing term within the Lagrangian are we able to define a finite photon propagator but this explicitly breaks the gauge symmetry of $\mathcal{L}$. The Ward Identity follows from the observation that the gauge independence must be restored within $\mathcal{Z}$ † . Let us now investigate the effect of a gauge transformation on our fundamental fields $\psi, \overline{\psi}, A^{\mu}$ within $\mathcal{Z}$. Consider the infinitesimal gauge transformation, characterised by the function $\Lambda(x)$, as defined below,

$$\begin{aligned} \psi \rightarrow \psi - ie\Lambda(x)\,\psi \quad , \quad \overline{\psi} \rightarrow \overline{\psi} + ie\Lambda(x)\,\overline{\psi} \\ A^{\mu} \rightarrow A^{\mu} + \partial_{\mu}\Lambda(x)\ . \end{aligned} \tag{1.40}$$

The functional measure within the definition of $\mathcal{Z}$ is naturally invariant under this transformation,

$$\mathcal{D}f'(x) \sim \prod_{i} d\,(\,f_i + c\,) = \mathcal{D}f(x)\ . \tag{1.41}$$

The effect of Λ is simply a constant shift in the integration variable at each point in space. As the original part of the Langrangian is gauge invariant by construction, then the only "left-overs" under this transformation come from the gauge fixing and source terms,

$$\mathcal{L} \longrightarrow \mathcal{L} - \xi^{-1}\,(\,\partial_{\mu}A^{\mu}\,)\,\Box\,\Lambda + J^{\mu}\partial_{\mu}\Lambda - ie\Lambda\,(\,\overline{\eta}\psi - \overline{\psi}\eta\,) \tag{1.42}$$

We may express the invariance of $\mathcal{Z}$ as,

$$\int \mathcal{D}A_{\mu}\,\mathcal{D}\overline{\psi}_{\mu}\,\mathcal{D}\psi_{\mu}\,\widehat{O}\,exp\{\,iS\,\} = 0 \tag{1.43}$$

where,

$$\widehat{O} = \int d^n x\ \{\,-\xi^{-1}\,(\,\partial_{\mu}A^{\mu}\,)\Box\,\Lambda + J^{\mu}\partial_{\mu}\Lambda - ie\Lambda\,(\,\overline{\eta}\psi - \overline{\psi}\eta\,)\,\}\ . \tag{1.44}$$

We have expanded the exponential to first order as Λ has been chosen to be vanishing. By judicious integration by parts we may recast the operator $\widehat{O}$ as,

$$\widehat{O} = \int d^n x\,\{\,-\xi^{-1}\Box\,(\,\partial_{\mu}A^{\mu}\,) - \partial_{\mu}J^{\mu} - ie\,(\,\overline{\eta}\psi - \overline{\psi}\eta\,)\,\}\,\Lambda(x) \tag{1.45}$$

where we have assumed that $\Lambda(x)$ vanishes sufficiently fast at infinity to justify droping total derivatives within the integration over x. Since (1.43) must hold

† The integration over gauges in QED introduces ghosts into QED as in nonabelian theories such as QCD. However, in QED these do not couple to any other field and as a result only introduce an overall factor multiplying Z. Therefore $\mathcal{Z}$ defined in terms of the gauge fixed Lagrangian must also be gauge invariant.

for arbitrary $\Lambda(x)$ we may drop the integration over x within $\widehat{O}$ altogether. Also if we note the following identities,

$$\begin{aligned} \psi \rightarrow \frac{1}{i}\frac{\delta}{\delta\overline{\eta}} \quad &, \quad \overline{\psi} \rightarrow -\frac{1}{i}\frac{\delta}{\delta\eta} \\ A_\mu \rightarrow &\frac{1}{i}\frac{\delta}{\delta J^\mu} \end{aligned} \tag{1.46}$$

then we may rewrite (1.43) as the following functional differential equation for $\mathcal{Z}$,

$$\left[\frac{i}{\xi}\Box\,\partial^\mu \frac{\delta}{\delta J^\mu} - \partial^\mu J_\mu - ie\left(\overline{\eta}\frac{\delta}{\delta\overline{\eta}} - \eta\frac{\delta}{\delta\eta}\right)\right] \mathcal{Z}\left[\eta,\overline{\eta},J^\mu\right] = 0 \tag{1.47}$$

From this relation we may generate corresponding relations for the correlation functions $\left\langle \overline{\psi}(x_i)...\psi(x_j)....A^\mu(x_k)\right\rangle$. However, these are not the most convenient quantities to deal with, much more useful would be a relation between 1PI functions. The 1PI functions, $\Gamma^n\left[p_1,p_2...p_n\right]$, are generated by the effective action, $\Gamma\left[\overline{\psi}_c,\psi_c,A^\mu\right]$ which is related to the generator of connected functions, W, via the following functional Legendre transformation,

$$\Gamma\left[\overline{\psi}_c,\psi_c,A^\mu\right] = W\left[\overline{\eta},\eta,J^\mu\right] - \int d^nx\ \overline{\eta}\psi + \overline{\psi}\eta + J_\mu A_c^\mu\ . \tag{1.48}$$

In the same way as the free energy in thermodynamics is defined from the internal energy,

$$F(T,V) = U\ (T,S) - TS \tag{1.49}$$

in order to describe the system in terms of a convenient set variables, so the effective action describes the properties of our system in terms of the variables $\overline{\psi}_c,\psi_c,A_c^\mu$. These should not be confused with the fundamental fields of the quantum system and hence the subscript c, but should be thought of as expectation values of the fundamental fields, $\psi_c = \langle\,\psi\,\rangle$ *etc.*

From the definition (1.48) we may write,

$$\begin{aligned}
\frac{\delta\Gamma}{\delta A_c^\mu(x)} &= -J_\mu(x) & \frac{\delta W}{\delta J_\mu(x)} &= A_c^\mu(x) \\
\frac{\delta\Gamma}{\delta\psi_c^\mu(x)} &= \overline{\eta}(x) & \frac{\delta W}{\delta\overline{\eta}(x)} &= \psi_c(x) \\
\frac{\delta\Gamma}{\delta\overline{\psi}_c^\mu(x)} &= -\eta(x) & \frac{\delta W}{\delta\eta(x)} &= \overline{\psi}_c(x)
\end{aligned} \tag{1.50}$$

and using these we may rewrite (1.47) as a differential relation for Γ,

$$-\frac{\Box}{\xi}\partial_\mu A_c^\mu + \partial_\mu \frac{\delta\Gamma}{\delta A_c^\mu} - ie\left(\psi_c \frac{\delta\Gamma}{\delta\psi_c} - \overline{\psi}_c \frac{\delta\Gamma}{\delta\overline{\psi}_c}\right) = 0 \tag{1.51}$$

If we functionally differentiate this relation with repect to both ψ_c and $\overline{\psi}_c$ and set all the sources to zero we have,

$$\begin{aligned}
-\partial_x^\mu \frac{\delta^3\Gamma[0]}{\delta\overline{\psi}_c(x_1)\delta\psi_c(y_1)\delta A_c^\mu(x)} = \\
ie\,\delta(x-x_1)\frac{\delta^2\Gamma[0]}{\delta\overline{\psi}_c(x_1)\delta\psi_c(y_1)} - ie\,\delta(x-y_1)\frac{\delta^2\Gamma[0]}{\delta\overline{\psi}_c(x_1)\delta\psi_c(y_1)}
\end{aligned} \tag{1.52}$$

which is obviously a relation between the fermion-photon vertex and the fermion propagator. It is more convenient to work in momentum space, so let us define our fermion propagator and vertex in momentum space in the standard manner,

$$\begin{aligned}
\int dx\,dx_1\,dy_1\; e^{i(p'x_1 - py_1 - qx)}\frac{\delta^3\Gamma[0]}{\delta\overline{\psi}_c(x_1)\delta\psi_c(y_1)\delta A_c^\mu(x)} \\
= ie(2\pi)^n\delta(p'-p-q)\,\Gamma_\mu(p,p') \\
\int dx_1\,dy_1\; e^{i(p'x_1 - py_1)}\frac{\delta^2\Gamma[0]}{\delta\overline{\psi}_c(x_1)\delta\psi_c(y_1)} = (2\pi)^n\delta(p'-p)\,iS^{-1}(p)
\end{aligned} \tag{1.53}$$

Multiplying (1.52) by $\exp i\,\{p'x_1 - py_1 - qx\}$ and integrating over x_1, x and y_1 gives the familiar result,

$$q^\mu\Gamma_\mu(p,p') = S^{-1}(p) - S^{-1}(p') \quad , \quad q = p - p' \,. \tag{1.54}$$

This is the Ward-Takahashi identity which in the limit $q^\mu \to 0$ yields the original

Ward Identity,

$$\frac{\partial}{\partial p^{\mu}} S^{-1}(p) = \Gamma_{\mu}(p, p) \ . \tag{1.55}$$

We now have all the basic tools we need to embark on our investigation into the chiral properties of QED. The next chapter will be concerned with the fermion Schwinger Dyson equation, a means of determining the chiral state of our model.

Chapter 2: The Fermion Equation.

2.1 Introduction.

In this chapter we will discuss how the Schwinger Dyson equations may be used to deduce the chiral state of 3D QED. We will start by highlighting the importance of poles in the fermion propagator and go on to discuss how the Schwinger Dyson equations operate and their relation to 1/N perturbation theory. By way of illustrating the ability of perturbation theory to simplify the Schwinger Dyson equations we will present the original work on this model. This will lead us to consider the limitations of the perturbative approach, and indeed its validity, in what is primarily a nonperturbative phenomenon. The conclusions drawn from this investigation will lead us to take a fundamentaly different approach to the truncation of the Schwinger Dyson hierarchy. This new truncation is genuinely nonperturbative, no recourse to perturbation theory is required for its justification. The bulk of this chapter will be concerned with detailing this approximation and implementing it in the case of the fermion equation.

2.2 Poles, Euclidean Space and Conventions.

As was discussed in Chapter 1, Chiral Symmetry is a consequence of massless fermions. In general the propagator for a massless fermion, S(p) , may be written as,

$$S^{-1}(p) \sim \beta(p)\not{p} \tag{2.1}$$

where $\beta(p)$ is known as the wavefunction renormalisation, simply because it may be associated with a rescaling of the fermion fields $\psi \to Z^{1/2}\psi$, $S(p) \sim \left\langle \overline{\psi}\psi \right\rangle_t \to Z\, S(p)$. The wavefunction renormalisation contains all quantum corrections due to the interactions of the fermion field and all other fields of the theory. Perturbatively $\beta(p) = 1 + O(1/N)$, all quantum corrections are down by orders of the coupling with respect to the bare value of unity.

If Chiral Symmetry is broken, then there is a second variable, namely the mass, that may appear in the fermion propagator. Because "mass" transforms as a scalar under Lorentz boosts, it cannot enter in the same way as a momentum. Consequently, in Minkowski space the general form for the propagator of a massive fermion may be written as,

$$S^{-1} \sim \beta(p)\not{p} - \Sigma(p) \; . \tag{2.2}$$

The function $\Sigma(p)$ is called the mass function and contains all quantum corrections to the bare mass, m_0. Perturbatively $\Sigma(p) = m_0 + O(1/N)$.

In a free theory there are no interactions between any fields and therefore the full and the bare Lagrangians are one and the same. The appearance of a nonzero mass alters the behaviour of the fermion propagator by introducing a pole singularity at nonzero momentum, *i.e.* $p^2 = m^2$,

$$S(p) \sim \frac{\not{p} + m}{p^2 - m^2} \longrightarrow \infty \quad , \quad p^2 \to m^2 \; . \tag{2.3}$$

In an interacting theory the position of this pole will move from its bare position, at $p^2 = m_0^2$, under the influence of the other fields in the theory and in this case we can only define a physical mass from the position of the pole of the renormalised propagator,

$$S(p) \sim \frac{\beta(p)\not{p} + \Sigma(p)}{\beta^2(p)p^2 - \Sigma^2(p)} \quad , \quad m = \frac{\Sigma(m)}{\beta(m)} \; . \tag{2.4}$$

Unless there is a solution to (2.4) there is no concept of a particle being "on shell" or of having a rest mass, indeed such may be the case for quarks in QCD. This may seem like an overly complicated way of defining the mass - why is it not simply $\Sigma(p = 0)$? The physical mass is an observable and is therefore gauge independent. Both $\beta(p)$ and $\Sigma(p)$ are gauge dependent and therefore the physical mass cannot be simply $\Sigma(0)$. However, this gauge dependence is such that a mass defined by the position as the pole in the propagator (2.4) is gauge invariant. To handle this pole in any practical calculation it is usually necessary to introduce a $+i\epsilon$ into the propagator to displace it into the complex plane,

$$S^{-1} \sim \beta(p)\not{p} - \Sigma(p) + i\epsilon \ . \tag{2.5}$$

This prescription ensures that only positive energy fermions propagate forward in time and similarly only negative energy (anti-) fermions backwards. Only at the end of any calculation may the limit $\epsilon \rightarrow 0$ be taken to recover the physical result. Rather than introduce an $i\epsilon$ into only positive energy fermions propagation forward in time continue the whole theory into Euclidean space, *i.e.* "Wick" rotating the physical time coordinate t into a complex time $-i\tau$. This implicitly relies on the assumption that the theory is analytic on the complex sheet $[\,t, \tau\,]$. Given this is true, the above definition ensures that the two theories are equivalent.

Minkowski Space:
$S^{-1} \sim \beta(p)\not{p} - \Sigma(p)$.
Euclidean Space:
$S^{-1} \sim \beta(p)\not{p} + \Sigma(p)$.

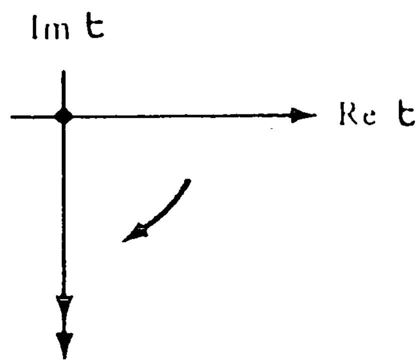

Figure 2.1 Rotating the physical time t into complex Euclidean time $-i\tau$. The $\otimes$ indicate the poles in the propragator.

After this rotation, a consistent set of conventions for calculating Feynman diagrams involving fermions are:

$$\begin{aligned} S^{-1}(p) &= -i\ \{\,\beta(p)\not{p} + \Sigma(p)\,\} \\ \Gamma^{\mu}(k,p) &= -ie(\gamma^{\mu} + O(1/N)) \\ \{\,\gamma^{\mu}, \gamma^{\nu}\,\} &= -2\delta^{\mu\nu} \quad . \end{aligned} \tag{2.6}$$

Now that we have dealt with some necessary preliminaries, let us turn our attention to the Schwinger Dyson equation for the fermion propagator.

2.3 Perturbation Theory and the Fermion Equation.

The Schwinger Dyson equation for the fermion is a self-consistency relation for the *full* fermion propagator. Symbolically we may represent it in the following manner,

Figure 2.2 Symbolic form of the Fermion Schwinger Dyson equation

The full dot • indicates that the appropriate vertex/propagator is full, as opposed to a bare or perturbative approximation. The Schwinger Dyson equation sums all the infinite quantum corrections to the bare propagator raising it to the level of a full nonperturbative propagator. If we take a greatly simplified version of the full relation, we may observe how such summations take place,

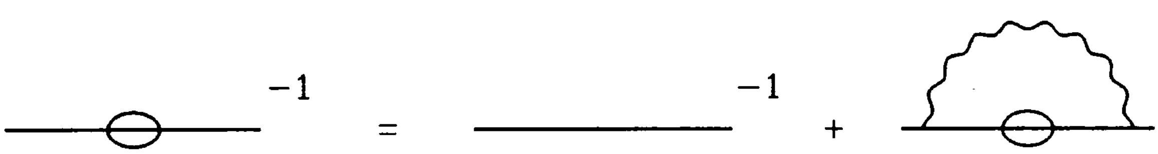

Figure 2.3 A simplified Fermion Schwinger Dyson equation.

By replacing all the full quantities on the RHS of Fig 2.2 by their bare forms, we will be able to express the "full" propagator of the LHS in terms of its perturbative form alone. Consider the perturbative corrections generated by this relation to a non-zero bare fermion mass. From the symbolic Schwinger Dyson relation, Fig. 2.3, we may write an inductive relation between the n^{th}

and $(n+1)^{th}$ order approximations. Heuristcally then,

$$m_{n+1} = m_n + e^2 \ \langle m_n \rangle . \tag{2.7}$$

We may simply iterate this relation, starting on the RHS with the bare propagator, n times to obtain the n^{th} order approximation to the fermion mass, m_n,

$$m_n = m_0 + e^2 \ \langle \ m_0 \ \rangle + e^4 \ \langle \ m_0 \ \langle \ m_0 \ \rangle \ \rangle + e^6 \ \langle \ m_0 \ \langle \ m_0 \ \langle \ m_0 \ \rangle \ \rangle \ \rangle \ \cdots \quad . \tag{2.8}$$

In terms of perturbative corrections and associated Feynman diagrams (2.8) is simply the following diagrammatic series of "Rainbow" graphs,

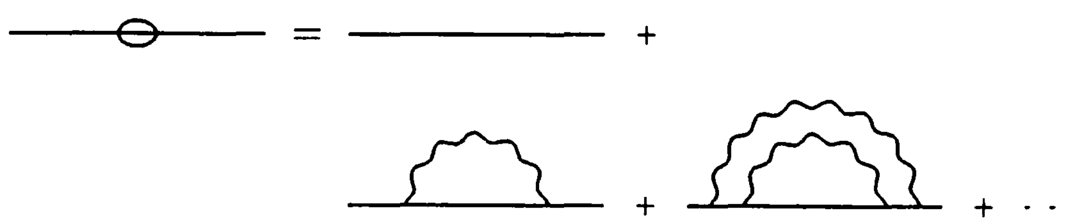

Figure 2.4 Diagrammatic representation of (2.8) in terms of "Rainbow" graphs

This approximation, which we have used as an example only, is commonly called the quenched ladder approximation [11] for the following reasons. If N is the number of fermion flavours in our theory, then in limit $N \to 0$ we may formally ignore all corrections due to fermion loops. This is the quenched limit, where there are no fermions and so, in QED, the bare photon propagator is unrenormalised. Moreover, by replacing the full vertex by its bare version we effectively resum all ladder graphs with the two fermion lines joined at one end, resulting in the aptly named rainbow corrections.

This expansion illustrates a general feature of any such perturbative calculation, to finite order, of the dynamical mass. From (2.8) we can see in the chiral limit, $m_0 \to 0$, all perturbative corrections are identically zero, therefore

a) b)

Figure 2.5 a) Ladder contribution to 2 particle scattering. b) Rainbow correction fermion selfenergy.

$m = 0$ and the chiral symmetry of the bare Lagrangian is preserved. However, this result does not hold for the solution to the Schwinger Dyson equation. The Schwinger Dyson solution has effectively summed the complete infinity of such perturbative corrections and is not constrained by any inductive relation such as (2.7). It is because the Schwinger Dyson system works in terms of infinite series of corrections that it can predict a finite mass for a chirally symmetric bare theory.

Now that we have an understanding as to how the Schwinger Dyson equations can sidestep the limitations of perturbation theory, let us turn to the original calculation for 3D QED. This was performed by Appelquist et al [7] and makes extensive use of perturbative arguments to simplify the Schwinger Dyson system to a tractable level. We will proceed by detailing the original arguments, method and results and only afterwards will we review the central assumptions used.

Let us start this analysis with the quantity we wish to calculate, the mass function $m(p)$ as defined by (2.4). Perturbatively $\beta(p) = 1 + O(1/N)$ so we may formally rewrite this relation to the first nontrivial order in 1/N as,

$$\begin{aligned} m(p) &= \frac{\Sigma(p)}{\beta(p)} = \Sigma(p) + O(1/N) \\ &= \Sigma_0(p) + O(1/N) \end{aligned} \tag{2.9}$$

However, to be consistent in orders of 1/N we must also truncate $\Sigma(p)$ to O(1) in 1/N, but as $\Sigma(p) = 0$ to all orders in 1/N, this is obviously meant in some special sense. The subscript 0 in the above referes to this truncation. In order to appreciate what we must do here, we need to study the explicit form of the Schwinger Dyson relation for $\Sigma(p)$. The Schwinger Dyson equation for the fermion propagator is given by the following,

$$-i\{\beta(p)\not{p}+\Sigma(p)\} = -i\not{p} - \int \frac{d^3k}{(2\pi)^3}(-ie\gamma^\mu)S(k)(-ie\Gamma^\nu)\mathcal{D}_{\mu\nu}(k-p). \tag{2.10}$$

As before Γ^μ is the full photon-fermion vertex and $\mathcal{D}_{\mu\nu}$ is the full photon propagator. Since to zeroth order in 1/N the $\beta(p)$ equation is the identity, $\beta(p) = 1$, we now turn our attention to the equation for $\Sigma(p)$. The relation for $\Sigma(p)$ starts at O(1/N), but we know that $\Sigma(p) \neq 0$ is unobservable at every finite order in 1/N . The explicit 1/N in the Σ relation must be compensated by some mechanism intrinsic to the operation of the Schwinger Dyson equation. If chiral symmetry breaking solutions do exist then we may ignore this explicit 1/N and apply perturbation theory to the kernel alone. This is the rather special truncation alluded to earlier. Within the kernel, to O(1) the full vertex is replaced by its bare form γ^μ but, as was discussed in Chapter 1, because of the dominance of fermion loops in the infrared the photon propagator is given by,

$$\Delta_{\mu\nu}(k,p) = \frac{1}{\mathcal{G}(p)}\left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) + \xi\frac{p_\mu p_\nu}{p^2} \quad , \quad \mathcal{G}(p) = p^2 + \alpha p. \tag{2.11}$$

The chiral symmetry breaking mechanism is expected to operate at scales of $O(\alpha)$ and for $m \ll O(\alpha)$ scales of $O(m)$ and below should not effect the symmetry breaking process, see Chapter 3 for a fuller discussion of this point. We will therefore drop all reference to the fermion mass m in the photon propagator and use the one loop corrected photon given by (2.11). After applying this perturbative truncation to the Schwinger Dyson hierarchy we are left, to O(1), with *one* equation, for $\Sigma_0(p)$. Simply tracing the original Schwinger Dyson relation with the unit spinor matrix gives us,

$$4\Sigma_0(p) = \frac{\alpha^2}{\pi^3 N}\int \frac{d^3k}{\beta^2(k)k^2+\Sigma_0^2(k)}\, Tr\left[\gamma^\mu\left(\beta(k)\not{k}-\Sigma_0(k)\right)\gamma^\nu\right]\,\Delta_{\mu\nu}(k-p). \tag{2.12}$$

For convenience we will work in the Landau gauge, $\xi = 0$, in which the photon propagator has only a transverse component. On performing the simple traces involved and setting $\beta(p) = 1$, (2.12) becomes,

$$\Sigma_0(p) = \frac{2\alpha^2}{\pi^3 N} \int \frac{d^3k\ \Sigma_0(k)}{\{\,k^2 + \Sigma_0^2(k)\,\}\,\mathcal{G}(q)} \quad , \quad q = k - p. \tag{2.13}$$

The phase space integral $\int d^3k$ can be split into two parts , a radial part $k^2 dk$ and an angular part $d\Omega$,

$$d^3k = k^2 dk\ d\cos\theta d\phi \quad , \quad k = 0 \to \infty, \cos(\theta) \to \begin{cases} +1 \\ -1 \end{cases}, \phi \to \begin{cases} +\pi \\ -\pi \end{cases} \tag{2.14}$$

and the integration over the angles θ and ϕ performed analytically.

$$\Sigma_0(p) = \frac{2\alpha^2}{\pi^3 N} \int \frac{d^3k\ \Sigma_0(k)}{k^2 + \Sigma_0^2(k)} I(\,k,p\,) \quad , \quad I(\,k,p\,) = \int \frac{d\Omega}{\mathcal{G}(q)} \quad . \tag{2.15}$$

The integral I(k,p) is an example of what will be referred to in future as an "angular integral", for obvious reasons. The evaluation of this integral is straight forward given a few tricks which we will present here using I(k,p) as a particularly simple example. The ϕ integration is obviously trivial and only results in a multiplicative factor of 2π . We are then left with the $d\cos\theta$ integration,

$$I(k,p) = \int_{-\pi}^{\pi}\int_{-1}^{1} \frac{d\phi\, d\cos(\theta)}{\mathcal{G}(q)} = 2\pi \int_{-1}^{1} \frac{d\cos(\theta)}{\mathcal{G}(q)} \quad . \tag{2.16}$$

Now notice that given $q^2 = k^2 + p^2 - 2k\,p\cos(\theta)$ we may rewrite $d\cos(\theta)$ as $-qdq/kp$. The limits of integration are then $k+p$ and $|k-p|$ (they must be symmetric in k & p). The resulting q integral is simple to evaluate,

$$I(k,p) = \frac{2\pi}{kp} \int_{|k-p|}^{k+p} \frac{q\,dq}{q^2 + \alpha q} = \frac{2\pi}{kp} \ln\left\{ \frac{k+p+\alpha}{|k-p|+\alpha} \right\} \quad . \tag{2.17}$$

Our final form for the $\Sigma(p)$ relation is then,

$$\Sigma_0(p) = \frac{4\alpha}{\pi^2 N p} \int_0^\infty \frac{k\,dk}{k^2 + {\Sigma_0}^2(k)} \Sigma_0(k) \ln\left\{ \frac{k+p+\alpha}{|k-p|+\alpha} \right\}. \tag{2.18}$$

This self-consistency relation for $\Sigma_0(p)$ does permit chiral symmetry breaking solutions, $\Sigma_0 \neq 0$, as well as the trivial $\Sigma_0 = 0$ solution, but has an interesting property that wasn't at first noticed [12]. For a theory with only a small number of fermions, $N \leq 3$, the chiral symmetry of the bare theory is broken and the fermions acquire a dynamically generated mass, but this symmetry is not broken for $N \geq 4$ and the fermions of the theory remain massless. This result may be seen analytically with a little effort. The basic structure of this relation is essentially unchanged if we linearise the log function within the integral. By this is meant that the log will be replaced by its asymptotic approximations in two cases $k > p$ and $k < p$. Note that this is also a good approximation for $k = p$ if $k, p \ll \alpha$.

$$\ln\left\{ \frac{k+p+\alpha}{|k-p|+\alpha} \right\} \longrightarrow \begin{cases} 2k/(p+\alpha) & k \leq p \\ 2p/(k+\alpha) & p \leq k \end{cases} \tag{2.19}$$

From (2.19) we see that for $p \leq \alpha$ contributions from the integration region $k > \alpha$ are rapidly damped and we may approximate this by introducing a momentum cutoff at α and dropping factors of $k \,\&\, p$ in comparison to α. The linearised form of (2.18) is then,

$$\Sigma_0(p) = \frac{8}{\pi^2 N p} \int_0^\alpha \frac{k\, dk}{k^2 + \Sigma_0^2(k)}\, \Sigma_0(k)\, \mathrm{Min}(\,k, p\,) \tag{2.20}.$$

This may easily be converted, by straightforward repeated diferentiation, into the following differential equation,

$$\frac{d}{dp}\left(p^2 \frac{d\Sigma_0(p)}{dp} \right) = -\frac{8}{\pi^2 N} \frac{p^2\, \Sigma_0(p)}{p^2 + \Sigma_0^2(p)} \quad . \tag{2.21}$$

If for $p \gg \Sigma_0(p)$ we assume that $\Sigma_0(p) \sim Ap^b$ then the condition imposed by this differential equation on b is simply,

$$b\,(\,b+1\,) = -\frac{8}{\pi^2 N} \quad . \tag{2.22}$$

with solutions,

$$b = -\frac{1}{2} \pm \frac{1}{2}\left\{ 1 - \frac{32}{\pi^2 N} \right\}^{1/2} \quad . \tag{2.23}$$

We still have a little further to go because the differential equation (2.21) has many more solutions than the nonlinear integral relation that it was derived from. In addition to satisfying the differential equation, solutions must satisfy the boundary conditions inherent to the integral relation. If we substitute (2.21) into the integral equation (2.20), then after a little integration by parts we are left with the following constraint on $\Sigma_0(p)$,

$$0 = \frac{1}{p}\left(k^2 \frac{d\,\Sigma_0(k)}{dk}\right)_{k=0} - \left(k \frac{d\,\Sigma_0(k)}{dk}\right)_{k=\alpha} - \Sigma_0(\alpha) \tag{2.24}$$

giving,

$$\left[\Sigma_0(k) + k\frac{d\,\Sigma_0(k)}{dk}\right]_{k=\alpha} = 0 \quad . \tag{2.25}$$

The condition at $k = 0$ is not strictly enforcable since at the onset of this section we assumed that $p > \Sigma_0(p)$. The condition at $k = \alpha$ is only enforcable if $b = -1$ or b is complex. From (2.23) $b = -1$ is only possible in the limit $N \to \infty$ so we are left with the conclusion that the chiral symmetry breaking solution is possible only for complex b. In other words, from (2.23), for $N \leq N_c$, where $N_c = 32/\pi^2 \simeq 3.28$.

This result explains the behaviour found by numerical studies of the full nonlinearised relation (2.18). In Fig. 2.6 we have plotted $\ln(m/\alpha)$ against the number of fermion flavours, N, which displays the extremely rapid fall of m expected with the restoration of chiral symmetry at $N \sim 3.3$.

This behaviour is also in qualitative agreement with lattice calculations peformed to date for 3D QED [13]. These calculations, however, may be called into question after a detailed analysis of the systematic finite size uncertainties inherent to the lattice approach [14]. Likewise we must also step back and review our assumptions, used in the Schwinger Dyson approach, before we place too much confidence in this interesting result.

In the previous calculation we relied heavily on perturbation theory to simplify the infinite Schwinger Dyson set, even down to a single relation! Let us review a few characteristics of the possible perturbative expansions of 3D QED. As discussed in Chapter 1, the expansion in e^2 was plagued by infrared divergences that could not be mopped up by any finite number of counterterms, and

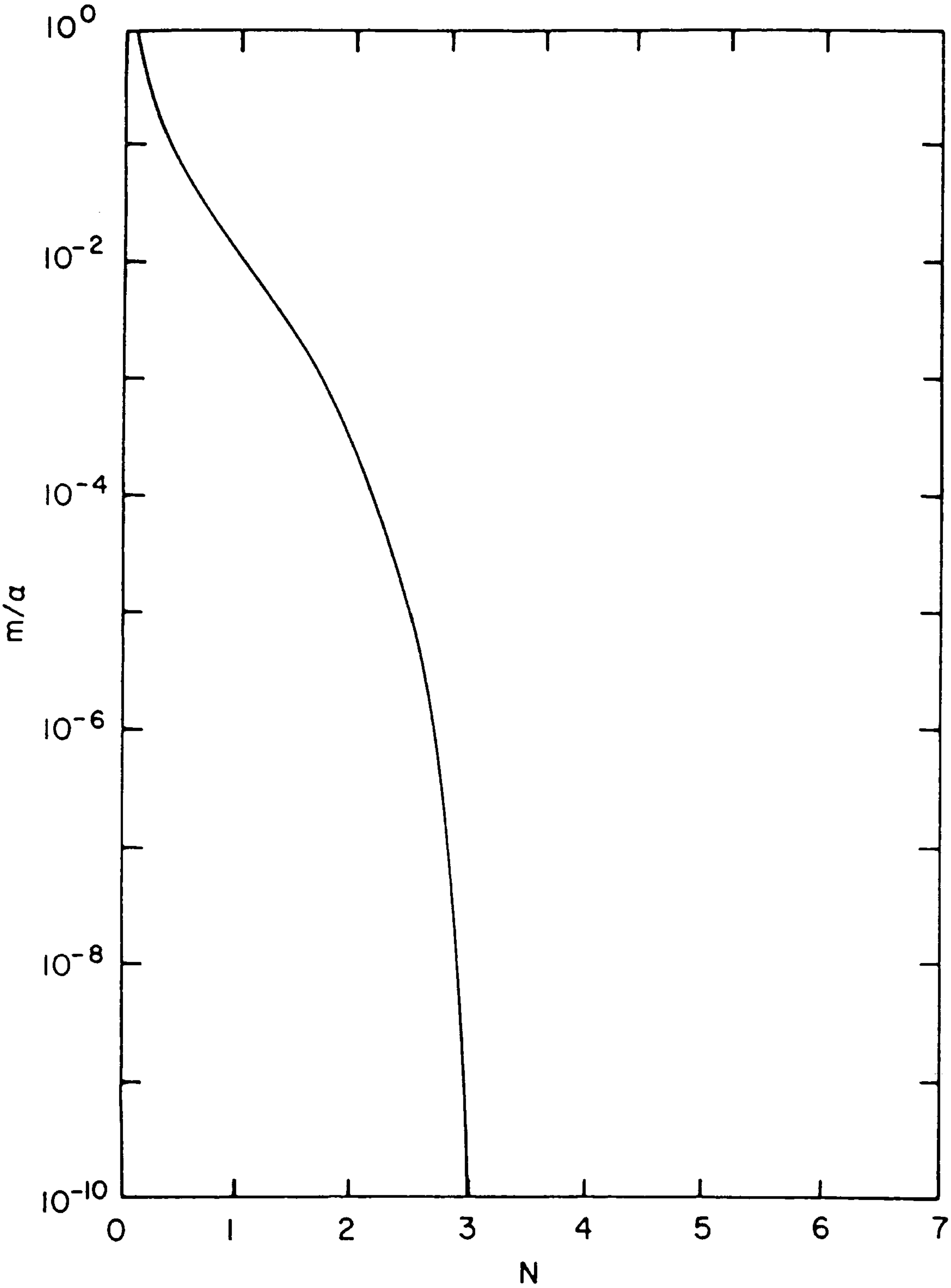

Figure 2.6 The dynamical mass, m, verses the number of fermion flavours, N, in the approximation of Appelquist et al.

while the 1/N perturbative expansion is finite order by order it is still not well defined in the infrared limit at each order. This is best seen by considering the example of the wavefunction renormalisation $\beta(p)$. To first order in 1/N we have,

$$\beta(p) = 1 + \frac{4}{3\pi^2 N}\left\{ \ln\left(\frac{p^2+m^2}{\alpha^2}\right) + b_1 \right\} \; + \; O(1/N^2) \qquad (2.26)$$

where b_1 is an N independent constant and m is the bare mass. In the limit $p \to 0$ $\beta(p)$ as defined in (2.26), is not necessarily even positive, as it must be. Consequently, we might expect the perturbative expression for $\beta(p)$ to be of the general form,

$$\beta(p) \simeq 1 + \lambda(log + b_1) + \lambda^2(\frac{log^2}{2} + c_1 log + b_2) + \cdots \quad . \qquad (2.27)$$

In this expression we are free to choose m, so let m be such that $log(m/\alpha) \sim O(\lambda)$. If we now take the limit $p \to 0$ and group terms of equal magnitude ,we have the following:

$$\beta(p) \simeq \sum_{i=0}^{\infty} \lambda^i \, \frac{log^i}{i} + \lambda \sum_{i=0}^{\infty} c_i \, \lambda^i \frac{log^{i-1}}{i} + \cdots \qquad (2.28)$$

Notice how as we take $p \to 0$ the formal ordering in 1/N is broken and corrections from vastly different orders begin to mix on equal footing. This would suggest a quite different behaviour for $\beta(p)$ in the limit $p \to 0$,

$$\lim_{p\to 0} \beta(p) = \left\{ \frac{p^2+m^2}{\alpha^2} \right\}^{\gamma} \quad , \quad \gamma = \frac{4}{3\pi^2 N} + O\left(\frac{1}{N^2}\right) \quad . \qquad (2.29)$$

This is what may be termed a renormalisation group view on the origin of the logarithms inherent to the 1/N perturbative expansion† . Since the only scale available to balance the dimensionful coupling α is the Lorentz invariant $(p^2 + m^2)^{\frac{1}{2}}$, it is natural to expect any possible power behaviour to be with respect to the dimensionless parameter $(p^2 + m^2)/\alpha^2$. In our previous calculation we

† However, for a cautionary tale concerning scaling laws and the coefficients of logs see Ref. [15].

assumed that in the Σ relation the explicit $1/N$ would be lifted dynamically in the infrared, but no thought was given to whether this may also happen in the relation for β. This perturbative result leaves our assumption that we may write $\beta(p) = 1$ up to *small* corrections in $1/N$ looking in rather bad shape. Indeed in the limit $m \to 0$, from our previous leading log example (2.29), we might expect $\beta(0) \to 0$, which is as far from the perturbative result as is possible. To summarise, we may say that the $1/N$ perturbative expansion is "infrared sensitive" by which we mean that the limit $p \to 0$ cannot be taken within all finite order perturbative result. Only in the limit of an infinite order perturbative result is the zero momentum limit always well defined. The proviso "always" is necessary because within physical quantities these divergences cancel and the perturbative expansion is well defined order by order. However in the case of the Appelquist calculation an invalid perturbative truncation was used *enroute* and this, *a priori*, invalidates the whole calculation. By truncating perturbatively we relegated $\beta(p)$ to a spectator of the chiral symmetry breaking mechanism, whereas there are signs pointing to it as an essential player.

Let us conclude by saying that the breaking of the formal $1/N$ ordering in the infrared requires us to abandon this perturbative approach to truncating the Schwinger Dyson equations and employ a wholly nonperturbative one. Consequently in simplifying the Schwinger Dyson system to a tractable level we must respect their nonperturbative nature in order to avoid inconsistencies of the type just encountered. In the following section we will expand on this theme with reference to the fermion equation.

2.4 Ansätze and the Schwinger Dyson Equations.

The Schwinger Dyson equations are hierarchical in structure and it is this property that supports their truncation. If instead of relying on perturbation theory, we truncate at some level, n, corresponding to the n-point functions, what do we do with the undetermined quantities from the level that we may think of as "below", the (n+1)-point functions? While we do not know their exact forms, we know general properties that they will possess, from their definition and the type of theory. In place of our unknown vertex functions we will substitute physically motivated ansätze that possess these general features. Note that we are implicitly assuming that the $(n+1) \to \infty$ point functions are not important to the basics of the mechanism we are studying, but are merely the fine detail.

Before we consider a specific example let us categorize the type of requirements any ansatz must fulfill.

i) Firstly in a superrenormalisable or asymtotically free theory any ansatz must have a smooth high energy limit in which the perturbative result is recovered. This is not necessarily true of an asymptotically nonfree theory, such as 4D QED where preliminary studies indicate the existence of an ultraviolet fixed point for a critical value of the coupling. Around this fixed point a new phase may be defined for QED in which the perturbatively nonrenormalisable four fermion interaction becomes renormalisable [16]. The appearance of the four point interation is due to the nonperturbative anomalous scaling of the fermion field at this point. The high energy limit of this theory will not correspond to the well known bare QED Lagrangian, the four fermi operator will be present and indeed there is speculation that the theory defined at this point is not even interacting [17]. An example of this phenomenon can be found in the 2D QED of the Schwinger Model [4], where the the fundamental fields turn out not to be the electron and photon with which the original interacting theory was defined, but totally new fields in terms of which the theory is one of free fields. An ansatz that is to be used in any investigation of this new phase should in principle be flexible enough to account for the peculiarities of this new phase. However, if the entire character of the theory changes, as in the Schwinger Model, and the fundamental fields are unknown, how this could be achieved is unclear. In the case of 4D QED this new phase is associated with an emerging four fermion interaction and its Lagrangian is expected to be closely related to that of the Jona-Lasinio model.

ii) Symmetries of the theory will also impose conditions on the vertex functions. The Ward Identity for the photon-fermion vertex in QED is a prime example of this. These will help us to rewrite parts of the undetermined vertex functions in terms of those determined by the truncated system. In this way we ensure that the numerical consistency of the closed system is automatic. Contrast this with a perturbative ansatz, a bare vertex in the fermion equation of QED. Since the vertex and fermion propagator are related by way of the Ward Identity, then it would be inconsistent to replace the full vertex with its bare form unless the truncated system returned a bare fermion. In this way the perturbative ansatz is caught out between the inconsistent and the trivial.

iii) Lastly and perhaps as equally constraining as the previous two is the expected analytic behaviour and singularity structure required of a well defined theory.

To put a bit more flesh on these ideas we will consider the example of the photon-fermion vertex in QED. Of course, this is not an idle example as it is one of the two undetermined quantities in the fermion equation. The key constraint on the vertex is the Ward-Takahashi Identity, a direct consequence of the U(1) gauge invariance.

$$q \cdot \Gamma(k,p) = (-i)^{-1}\left(S^{-1}(k) - S^{-1}(p)\right) \quad , \quad q = k - p \quad . \tag{2.30}$$

The constraint this imposes on the form of Γ^μ is easily seen if we decompose Γ^μ into components longitudinal and transverse to the momentum transfere q^μ.

$$\Gamma^\mu = \Gamma^\mu_L + \Gamma^\mu_T \longrightarrow \left\{ \begin{array}{c} q \cdot \Gamma_L = (-i)^{-1}\left(S^{-1}(k) - S^{-1}(p)\right) \\ q \cdot \Gamma_T = 0 \end{array} \right. \tag{2.31}$$

While the transverse part, Γ_T , is undetermined, a simple solution for the longitudinal part, Γ_L , is,

$$\Gamma^\mu_L = i\,\frac{q^\mu}{q^2}\,\left(S^{-1}(k) - S^{-1}(p)\right) \quad . \tag{2.32}$$

In the limit of high p and k this is well approximated by,

$$\Gamma^\mu_L \longrightarrow \frac{q^\mu\, q^\nu}{q^2}\,\gamma^\nu \tag{2.33}$$

and obviously (2.32) hasn't the correct perturbative limit. Also, we cannot use

(2.32) as an ansatz for the longitudinal vertex component because of condition (iii) above. This ansatz possess a kinematic singularity for all configurations for which $q^2 = 0$, though not necesarily $q^\mu = 0$. While a singularity at $q^2 = 0$ is not unexpected, it is in this limit that the photon approaches its mass shell, this type of singularity is prohibited. The form of this divergence is that of the effective propagation of a spin 1 vector boson. Such a behaviour cannot be generated by the normal quantum correction as summarised in Fig. 2.7.

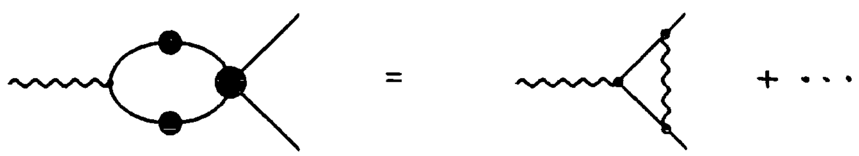

Figure 2.7 Ordinary corrections to the bare vertex.

In the full vertex this pole must be cancelled by an identical pole of the opposite sign in the transverse component. This indicates a bad decomposition of the full vertex into its two compnents, Γ^μ_L and Γ^ν_T, and so we will proceed no further with this rather naive ansatz.

Let us make a rather more serious attempt to determine Γ^μ in terms of the fermion functions $\beta(p)$ and $\Sigma(p)$. Consider the differential Ward Identity again and this time substitute the explicit form for the fermion propagator,

$$\begin{aligned}\Gamma^\mu &= i\frac{\partial}{\partial p_\mu}S^{-1}(p) \\ &= \beta(p)\gamma^\mu + 2\,\frac{\partial}{\partial p^2}\Sigma(p^2)\;\; p^\mu + 2\,\frac{\partial}{\partial p^2}\beta(p^2)\,\not{p}p^\mu \quad .\end{aligned} \tag{2.34}$$

We may generalise this relation to cases for which $k \neq p$, by simply replacing differentials by differences and single functions by averages,

$$\begin{aligned}\Gamma^\mu_{BC}(k,p) = \frac{1}{2}\left(\beta(k^2)+\beta(p^2)\right)\;\gamma^\mu + \frac{\Sigma(k^2)-\Sigma(p^2)}{k^2-p^2}\;(k+p)^\mu \\ +\frac{1}{2}\frac{\beta(k^2)-\beta(p^2)}{k^2-p^2}\;(\not{k}+\not{p})(k+p)^\mu \quad .\end{aligned} \tag{2.35}$$

In this way we generalise the zero momentum transfer result while ensuring in this limit that the differential Ward Identity is recovered smoothly. Such a form for the longitudinal part of the full vertex was first written down by Ball & Chiu [18] in their pioneering work on the analytic structure of the 3 point vertex in gauge theories. The Ward-Takahashi Identity constrains only the longitudinal part of Γ^μ, so the above result is free up to nonsingular transverse terms. Indeed another " solution " to the WT identity is,

$$\begin{aligned}\Gamma^\mu_L(k,p) = \frac{1}{2}\left(\beta(k^2)+\beta(p^2)\right)\gamma^\mu - \frac{\Sigma(k^2)-\Sigma(p^2)}{k^2-p^2}\left(\gamma^\mu \not{k} + \not{p}\gamma^\mu\right) \\ +\frac{1}{2}\frac{\beta(k^2)-\beta(p^2)}{k^2-p^2}\left(-2\not{k}\gamma^\mu\not{p} + (k^2+p^2)\gamma^\mu\right) \quad .\end{aligned} \tag{2.36}$$

which is obtained from (2.35) by the addition of a transverse piece,

$$\begin{aligned}\Gamma^\mu_T = \frac{1}{2}\frac{\beta(k^2)-\beta(p^2)}{k^2-p^2}\left((k-p)^\mu\left(\not{k}-\not{p}\right) - \gamma^\mu\not{k}\not{p} - \not{k}\not{p}\gamma^\mu - \left(k^2+p^2\right)\gamma^\mu\right) \\ - \frac{\Sigma(k^2)-\Sigma(p^2)}{k^2-p^2}\left(q^\mu + \not{q}\gamma^\mu\right) \quad .\end{aligned} \tag{2.37}$$

The components from which we may build any such transverse parts are k^μ,p^μ,γ^μ, 1,$\not{k}$,$\not{p}$, $k^\mu p^\nu \sigma_{\mu\nu}$, but it is not possible to build a vector both transverse to q^μ and symmetric in k & p from these components. Therefore, requiring that Γ^μ_L be symmetric in k and p then specifies Γ^μ_{BC} as the unique "solution" of the Ward-Takahashi identity that is free of kinematic singularities. Indeed if we try to mop up the pole singularity in our earlier example (2.32) with a transverse piece such as,

$$\Gamma^\mu_T = \left(\delta^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\gamma^\mu F(q^2) \tag{2.38}$$

$$\longrightarrow \quad \Gamma^\mu = -i\,\frac{S^{-1}(k)-S^{-1}(p)}{k^2-p^2}\gamma^\mu = \Gamma^\mu_{BC} \tag{2.39}$$

then the condition that the complete vertex be free of this pole fixes $F(q^2)$ in such a way that the full vertex is simply the Ball Chiu ansatz, (2.35). This should not come as surprise, we have only got out what we originally put in. This still leaves us with the other half of the full vertex , Γ^μ_T. While this transverse part is unknown outside perturbation theory, we may determine how it qualitatively behaves in the limit $q \to 0$ purely from its definition as transverse. Merely differentiating (2.31) yields,

$$\frac{d}{dq^\mu}(\, q \cdot \Gamma_T \,) = 0 \longrightarrow \Gamma^\mu_T + q^\nu \frac{d}{dq^\mu}\Gamma^\nu_T = 0 \tag{2.40}$$

and because Γ^μ_T is free from kinematic singularities we may take the limit $q \to 0$. Therefore we have,

$$\lim_{q^\nu \to 0} \Gamma^\mu_T \longrightarrow 0 \quad . \tag{2.41}$$

In this limit $q^\mu \to 0$, and consequently $q \to 0$, the full vertex is given by its longitudinal component alone, which is in turn completely determined by the Ward-Takahashi identity. This is really just a reiteration of the differential Ward Identity but with the emphasis being placed on the absence of kinematic singularities. However, this limit corresponds to very soft photons interacting with the full fermion and is precisely the region in which we expect the chiral symmetry breaking mechanism to operate, so we will take the longitudinal vertex Γ^μ_{BC} as our vertex ansatz. Note that this is not equivalent to saying that the transverse component is unimportant, indeed its unconstrained nature hints at its possible collusion in the preservation of gauge invariance of physical quantities. This rather vague idea will be elaborated on at some length in Chapter 4 in relation to the dynamical mass but for the moment we will stick with Γ^μ_{BC} as our vertex ansatz.

In Γ^μ_{BC} we have a truly nonperturbative ansatz which, by construction, satisfies the Ward-Takahashi Identity, itself a consequence of gauge invariance. What is more, general arguments suggest that it is the dominant part of the full vertex in the chiral symmetry breaking region $q \to 0$.

By making this replacement we have, at a stroke simplified, the complete Schwinger Dyson hierarchy down to two coupled equations for the full fermion and photon propagators.

This is a closed system which we may in principle go ahead and solve. These relations, however, are very strongly coupled and for the moment we will have

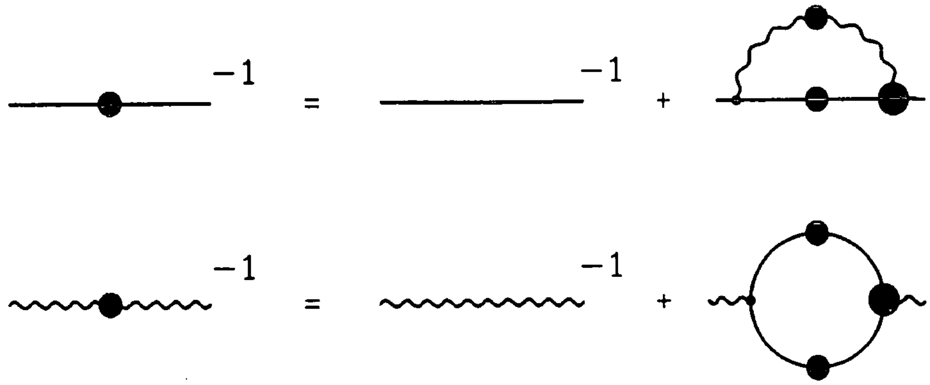

Figure 2.8 The simplified Schwinger Dyson system after truncating at the photon-vertex level

to make one further assumption and that concerns the behaviour of the photon. We will assume that the qualitative softening,

$$\mathcal{G}(p) = p^2 + \alpha p \longrightarrow \alpha p \quad , \quad p \ll \alpha \tag{2.42}$$

predicted by the first nontrivial 1/N correction is a property of the full solution. The exact details of this softening, for instance whether $\mathcal{G}(p)$ behaves as αp or $3\alpha p$, is not expected to be important, only that the softening is such that $\mathcal{G}(p)$ is linear in p. In general the form of the full correction to the photon may be written,

$$p^2\,\Pi_{\mu\nu}(p) \simeq \alpha \int d^3 k\; Tr\,[\gamma^\mu S(k)\Gamma^\nu S(k+p)]\left(\delta^{\mu\nu} - 3\frac{p^\mu p^\nu}{p^2}\right) \quad . \tag{2.43}$$

Unlike the corrections to the fermion propagator, in this correction the natural scale α enters explicitly only as a multiplicative constant. The limit $m \to 0$ is not beset by logarithms of the type encountered earlier, so for $p \gg \alpha$, we expect the result for the dynamically massive case to be equal, up to small factors of $O(m/\alpha)$, to the massless case,

$$p^2\Pi_{\mu\nu} \sim \alpha \int d^3 k\; Tr\,[\gamma^\mu S(k)\Gamma^\nu S(k+p)]_{m=0}\left(\delta^{\mu\nu} - 3\frac{p^\mu p^\nu}{p^2}\right) \tag{2.44}$$

The only dimensionful quantity left at this stage is p and therefore we expect the general form of this correction to be,

$$p^2\Pi_{\mu\nu} \sim \alpha f(p/\alpha, \beta(p))\, p \quad , \quad p \gg m \; . \tag{2.45}$$

The scale α that compensates the dimensionality of p appears via $\beta(p)$. The wavefunction renormalisation, $\beta(p)$, is confined to vary between 0 and 1 and is expected to do this smoothly, so we may expect that the dominant scaling of (2.45) will be due to the explicit factor of p. This is a very naive argument which takes no account of possible subtleties occuring at scales nearer to m that may effect m itself via a backreaction transmitted by the photon. However, for scales large in comparison to the dynamically generated mass scale this argument would indicate that the qualitative softening given by the first 1/N correction is correct. The perturbative result is expected to correct for scales $p \geq \alpha$, but the above argument extends its qualitative validity down to scales $p \gg O(m)$, which may lie far below that of α. Scales of O(m) and below are not expected to be important to the chiral symmetry breaking mechanism as it is tied to the natural scale α by the softening induced at this scale by "bare fermions".

In the following calculation we will take the first nontrivial 1/N photon as our ansatz for the full photon propagator. The validity of this replacement and the assumptions made in order to justify it may be investigated only by considering both the photon and fermion equations together. This will be carried out in Chapter 3, where the effect of dynamical fermions in the photon propagator may be investigated over the complete momentum range. Under our assumptions the Schwinger Dyson system has again been reduced to the single Schwinger Dyson relation for the fermion propagator. In the next section we will split this relation into its two coupled nonlinear equations for the fermion functions $\beta(p)$ and $\Sigma(p)$ and examine their forms.

2.5 A DYNAMICAL VERTEX.

We will now proceed to replace the full vertex and photon propagator by the appropiate ansätze as discussed in the previous section. The resulting Schwinger Dyson equation for the fermion is then,

$$-i\,\{\,\beta(p)\not p + \Sigma(p)\,\} = -i\not p - \int \frac{d^3k}{(2\pi)^3}\,(\,-ie\gamma^\mu\,)\,S(k)\,(\,-ie\,\Gamma^\nu_{BC}\,)\,\Delta_{\mu\nu}\,(\,k-p\,)\,. \tag{2.46}$$

In this way we separate the fermion equation from the rest of the Schwinger Dyson structure. From this relation (2.46) we may separate out the different spinor terms that contribute exclusively to either $\Sigma(p)$ or $\beta(p)$, but instead of working with the total wavefunction renormalisation $\beta(p)$, it is sometimes convenient to subtract off the bare behaviour and speak instead of a wavefunction renormalisation correction defined by $A(p) = \beta(p) - 1$. By tracing (2.46) with the unit spinor matrix and with $\not p$ we separate these different spinor terms and obtain the following relations for $\beta(p)$ and $\Sigma(p)$,

$$\begin{aligned} 4\,\Sigma(p) &= \frac{ie^2}{(\,2\pi\,)^3}\int d^3k\ Tr\ [\,\gamma^\mu S(k)\Gamma^\nu_{BC}(k,p)\,]\ \Delta_{\mu\nu}(k,p) \\ -4\,A(p) &= \frac{ie^2}{(\,2\pi\,)^3}\int d^3k\ Tr\ [\,\not p\gamma^\mu S(k)\Gamma^\nu_{BC}(k,p)\,]\ \Delta_{\mu\nu}(k,p) \end{aligned} \tag{2.47}$$

where $\Delta_{\mu\nu}$ is the photon ansatz,

$$\Delta_{\mu\nu}(k,p) = \frac{1}{\mathcal{G}(p)}\left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) + \xi\frac{p_\mu p_\nu}{p^4} \quad , \quad \mathcal{G}(p) = p^2 + \alpha p \quad . \tag{2.48}$$

and ξ is the covariant gauge paramter. We will work in the Landau gauge, $\xi = 0$, where the photon is purely transverse. At this stage this is simply a matter of convenience and discussions of gauge invariance will be postponed until Chapter 4. The vertex ansatze, Γ^μ_{BC}, is the Ball-Chiu longitudinal vertex as discussed in the previous section, but for convenience let us write it in the following shorthand,

$$\Gamma^\mu_{BC} = \mathcal{C}\,\gamma^\mu + \mathcal{S}\,(\not k + \not p)(k+p)^\mu + \mathcal{V}(k+p)^\mu \tag{2.49}$$

with the scalar functions $\mathcal{C}, \mathcal{S}$ and $\mathcal{V}$ given by,

$$\mathcal{C} = \frac{1}{2}\{\beta(k) + \beta(p)\}\,,\ \mathcal{S} = \frac{1}{2}\left\{\frac{\beta(k) - \beta(p)}{k^2 - p^2}\right\},\ \mathcal{V} = \frac{\Sigma(k) - \Sigma(p)}{k^2 - p^2} \tag{2.50}$$

The terms in (2.46) that contain odd and even gamma matrices contribute to $A(p)$ and $\Sigma(p)$ respectively. We will simplify the bookkeeping of this calculation if we separate these two types of terms first. Grouping together the nontrivial sources of gamma matrices, let us define two quantities K^μ_+ and K^μ_- ,

$$i\,\left[\beta^2(k)k^2 + \Sigma^2(k)\right]\; S(k)\Gamma^\mu_{BC}(k,p) \;=\; K^\mu_+ + K^\mu_- \tag{2.51}$$

where, in an obvious notation, $K^\mu_\mp$ contain odd/even numbers of gamma matrices respectively. The fermion functions are are defined by the full fermion propagator

$$S(k) = -i\,\frac{\beta(k)\not{k} - \Sigma(k)}{\beta^2(k)k^2 + \Sigma^2(k)} \tag{2.52}$$

and with the vertex ansatz written as (2.49), our two types of term are given by,

$$\begin{aligned} K^\mu_+ &= \beta(k)\mathcal{C}\not{k}\gamma^\mu + \beta(k)\mathcal{S}\not{k}(\not{k} + \not{p})(k+p)^\mu - \Sigma(k)\mathcal{V}(k+p)^\mu \\ K^\mu_- &= -\Sigma(k)\mathcal{C}\gamma^\mu - \Sigma(k)\mathcal{S}(\not{k} + \not{p})(k+p)^\mu + \beta(k)\mathcal{V}\not{k}(k+p)^\mu \quad . \end{aligned} \tag{2.53}$$

We may now perform the rather simple traces involved and after a little manipulation we find the following two relations:

$$\begin{aligned} \Sigma(p) &= \frac{\alpha}{N\pi^3}\int \frac{d^3k}{\beta^2(k)k^2 + \Sigma^2(k)}\frac{\mathcal{M}(k,p)}{q^2\mathcal{G}(q)} \\ A(p) &= -\frac{\alpha}{N\pi^3 p^2}\int \frac{d^3k}{\beta^2(k)k^2 + \Sigma^2(k)}\frac{\mathcal{H}(k,p)}{q^2\mathcal{G}(q)} \end{aligned} \tag{2.54}$$

with,

$$\begin{aligned} \mathcal{M}(k,p) &= Tr\;\left[\gamma^\mu K^\nu_-\;\right]\left(q^2\delta^{\mu\nu} - q^\mu q^\nu\right) \\ &= \;2\,\mathcal{C}\Sigma(k)q^2 + \left(4\mathcal{S}\Sigma(k) - 2\mathcal{V}\beta(k)\right)\left(p^2k^2 - (p\cdot k)^2\right) \\ \\ \mathcal{H}(k,p) &= Tr\;\left[\not{p}\gamma^\mu K^\nu_+\right]\left(q^2\delta^{\mu\nu} - q^\mu q^\nu\right) \\ &= \;2\,\mathcal{C}\beta(k)\left(k^2 - p\cdot k\right)\left(p^2 - k\cdot p\right) \\ &\quad + 2\left((k^2+p^2)\beta(k)\mathcal{S} - \mathcal{V}\Sigma(k)\right)\left(p^2k^2 - (p\cdot k)^2\right) \end{aligned} \tag{2.55}$$

2.6 ANGULAR INTEGRALS

If we were studying the coupled fermion-photon system then we could proceed no further analytically, as the full fermion function $\mathcal{G}(p)$ would be unknown. However, because of the simple nature of our photon ansatz we may proceed one step further. Our photon ansatz contains no reference to the fermion functions $\beta(p)$ & $\Sigma(p)$ and is a particularly simple function of the photon's momentum. As in Section 2.3, we may split the phase space d^3k into two parts,

$$d^3k = k^2dk\, d\cos\theta d\phi \quad , \quad k = 0 \to \infty, \cos(\theta) \to \begin{cases} +1 \\ -1 \end{cases}, \phi \to \begin{cases} +\pi \\ -\pi \end{cases} \tag{2.56}$$

Since our vertex ansatz does not explicitly depend on the photon momentum, q, then the general form of our angular integrals is the following,

$$\mathcal{A}(k,p) = \int \frac{a\,(\,k,p,\cos\theta\,)}{q^2(q^2+\alpha q)}\, d\Omega \quad , \quad d\Omega = d\phi\, d\cos(\theta) \quad . \tag{2.57}$$

Where $a\,(\,k,p,\cos\theta\,)$ represents a simple function of the vector k and p, such as $p^2k^2 - (\,p\cdot k\,)^2 = p^2k^2\sin^2\theta$. Note, that with two vectors a can only depend on one physical angle θ and so the ϕ integration is trivial, resulting in a simple factor of 2π.

When specifying our angular functions it is convenient to separate out factors of $O(\pi/kp)$.Let us define three angular functions $L(k,p), G(k,p)$ and $H(k,p)$ as,

$$\begin{aligned} L(k,p) &= \frac{kp}{2\pi}\int \frac{d\Omega}{q^2+\alpha q} \\ G(k,p) &= \frac{2kp}{\pi}\int \frac{(k^2-k\cdot p)(p^2-p\cdot k)}{q^2\,(\,q^2+\alpha q\,)}\, d\Omega \\ F(k,p) &= \frac{kp}{\pi}\int \frac{p^2k^2-(p\cdot k)^2}{q^2\,(\,q^2+\alpha q\,)}\, d\Omega \end{aligned} \tag{2.58}$$

We have encountered the function $L(k,p)$ before when we were discussing the Appelquist calculation in Section 2.3. Following that analysis we may make a change of variables and replace $d\Omega$ with $2\pi/kp\; q\,dq$ and introduce new limits $q_+ = k+p$ and $q_- = |k-p|$. The function $L(k,p)$ is then simple to calculate, yielding the familiar result.

$$L(k,p) = \int_{q-}^{q+} \frac{q\,dq}{q^2+\alpha q} = \ln\left\{ \frac{k+p+\alpha}{|k-p|+\alpha} \right\} \quad . \tag{2.59}$$

Similarly if we also use the identity $p\cdot k \;=\; -\frac{1}{2}\left\{\, q^2 - (k^2+p^2) \,\right\}$ we may rewrite $G(k,p)$ and $H(k,p)$ in terms of q alone as,

$$\begin{aligned} G(k,p) &= \int_{q-}^{q+} q^4 - (k^2-p^2)^2 \; \frac{q\,dq}{q^2\,(\,q^2+\alpha q\,)} \\ F(k,p) &= \int_{q-}^{q+} 2(p^2+k^2)q^2 - q^4 - (p^2-k^2)^2 \; \frac{q\,dq}{q^2\,(\,q^2+\alpha q\,)} \end{aligned} \tag{2.60}$$

Upon performing these standard integrals we obtain the final forms for these angular functions,

$$\begin{aligned} G(k,p) &= \left(\alpha^2 - \frac{(k^2-p^2)^2}{\alpha^2} \right) \ln\left\{ \frac{k+p+\alpha}{|k-p|+\alpha} \right\} + \frac{(k^2-p^2)^2}{\alpha^2} \ln\left| \frac{k+p}{k-p} \right| \\ &\qquad + 2kp - 2\alpha\ \mathrm{Min}(k,p) \left(1 + \frac{|k^2-p^2|}{\alpha^2} \right) \\[2ex] F(k,p) &= \left(2(k^2+p^2) - \alpha^2 - \frac{(p^2-k^2)^2}{\alpha^2} \right) \ln\left\{ \frac{k+p+\alpha}{|k-p|+\alpha} \right\} - 2kp \\ &\qquad + 2\alpha\ \mathrm{Min}(k,p) \left(1 - \frac{|k^2-p^2|}{\alpha^2} \right) + \frac{(k^2-p^2)^2}{\alpha^2} \ln\left| \frac{k+p}{k-p} \right| \end{aligned} \tag{2.61}$$

While these functions look complicated this complexity only hides simple asymptotic scaling behaviours. Each function is symmeteric in k & p and it is in the regions $k \gg p$ and $p \gg k$ that each simplifies enormously. If we define $Q = \mathrm{Max}(k,p)$ and $q = \mathrm{Min}(k,p)$, then expanding in factors of q/Q yields the simple asymptotic forms given below,

$$\begin{aligned} L(k,p) &= \frac{2\,q}{Q+\alpha} + \frac{2}{3}\frac{q^3}{(Q+\alpha)^3} + O\left(\frac{q^5}{(Q+\alpha)^5} \right) \\ G(k,p) &= \frac{8\,\alpha}{3}\,\frac{q^3}{(Q+\alpha)^2} + O\left(\frac{\alpha\, q^5}{(Q+\alpha)^4} \right) \\ F(k,p) &= \frac{16}{3}\,\frac{q^3}{Q+\alpha} + O\left(\frac{q^5}{(Q+\alpha)^3} \right) \end{aligned} \tag{2.62}$$

2.7 The Coupled System.

Given the analytic forms for the angular functions we may now piece together our final relations for the two fermions functions,

$$
\begin{aligned}
\Sigma(p) &= \frac{\alpha}{\pi^2 N p} \int \frac{k\,dk}{\beta^2(k)k^2 + \Sigma^2(k)} \,\langle\, \mathcal{M}(k,p) \,\rangle \\
A(p) &= -\frac{\alpha}{\pi^2 N p^3} \int \frac{k\,dk}{\beta^2(k)k^2 + \Sigma^2(k)} \,\langle\, \mathcal{H}(k,p) \,\rangle
\end{aligned}
\tag{2.63}
$$

$$
\begin{aligned}
\langle\, \mathcal{M}(k,p) \,\rangle &= 2\,(\,\beta(k) + \beta(p)\,)\,\Sigma(k)\,L(k,p) + \\
&\qquad \{\, \Sigma(p) - \Sigma(k) + A(k)\Sigma(p) - \Sigma(k)A(p) \,\}\, \frac{F(k,p)}{k^2 - p^2} \\
\langle\, \mathcal{H}(k,p) \,\rangle &= \frac{(\,\beta(k) + \beta(p)\,)}{2}\,\beta(k)\,G(k,p) + \frac{F(k,p)}{k^2 - p^2} \\
&\qquad \times \left\{ \frac{(\,k^2 + p^2\,)}{2}\beta(k)\,(\,\beta(k) - \beta(p)\,) + \Sigma(k)\,(\,\Sigma(k) - \Sigma(p)\,) \right\} .
\end{aligned}
\tag{2.64}
$$

Let us compare and contrast these coupled relations with the single relation of Appelquist.

$$
\Sigma_0(p) = \frac{4\alpha}{\pi^2 N p} \int_0^\infty \frac{k\,dk}{k^2 + {\Sigma_0}^2(k)}\,\Sigma_0(k)\,\ln\left\{ \frac{k + p + \alpha}{|k - p| + \alpha} \right\}. \tag{2.65}
$$

The first major difference is, of course, that we have made no assumption as to the possible behaviour of the wavefunction renormalisation $\beta(p)$ and because of this we are forced to analyse two coupled equations instead of one. The way these two equations are coupled, through factors such as $(\beta^2(k)k^2 + \Sigma^2(k))^{-1}$ and $\beta(k)\,(\,\beta(k) - \beta(p)\,)$, is highly nontrivial and greatly increases the work necessary to solve them. However, there is a reward for such an increase in workload and it is that we may now investigate the behaviour of $\beta(p)$ across the complete momentum range. As a result, our ideas on the possible asymptotic behaviour of the 1/N perturbative series for $\beta(p)$ may be easily verified or discounted. We will be able to see directly whether the explicit 1/N factor in the relation for $\beta(p)$ is dynamically lifted or not.

Up to this point we have insisted that our vertex ansatz must satisfy the Ward -Takahashi identity. This requirement of gauge invariance is obviously essential if we wish to continue our calculations into other gauges and obtain gauge invariant results. However, we may ask how important are the specifics of our vertex ansatz to the *qualitative* operation of the chiral symmetry breaking mechanism. It is not clear that the details essential to preserving gauge invariance are required for the breaking of the chiral symmetry of the bare Lagrangian. We will make simplifications to our present vertex ansatz and compare the effect of these in the final solutions in order to gain some qualitative understanding as to the importance of the vertex in the symmetry breaking mechanism. When we come to investigate the behaviour of our approximation in other gauges, we will be able to perform the same simplifications to the vertex ansatz, in an effort to gain a more general understanding of the importance of the vertex in the chiral symmetry breaking mechanism.

So how do we go about solving our two relations for $\Sigma(p)$ and $\beta(p)$? The analysis used to determine the critical behaviour of the Appelquist relation is not open to us here. In the analysis of Section 2.3, after linearising the angular functions, we were able to convert the nonlinear integral equation into a linear differential equation. Boundary conditions, determined from the original integral relation, were then used to reduce the overcomplete set of solutions to the differential equation down to the single solution to the integral equation. The reason this approach cannot be used is quite simple. The relations (2.64) contain explicit occurences of the momentum p and unknown functions of it, such as $\beta(p)$, that cannot simply be factored out of the integrals. Therefore, even after linearising the angular functions L,G and H, subsequent differentiations will not terminate to form a differential system of finite order.

Consequently, because of the complicated nature of our relations, we are forced into a wholly numerical analysis. The methods and problems associated with solving such relations will be discussed in the following section. This discussion may be skipped by the reader if he/she has no intention of solving such relations themselves.

2.8 Numerical Procedures.

The first point that should be emphasised in this section is that far from popular belief the computer is not a perfect calculating device. The source of this imperfection is fundamental. A physical computer can only deal in finite rational numbers. As an example, consider a digital computer which represents any number as a binary sequence of 1's and 0's. If the number of binary units allocated to store any particular number is N, then the maximum number that it is possible to represent is of $O(2^N)$ and at most any number will be represented to an accuracy of the order of 1 part in 2^N. As a practical consequence of this, consider calculating the angular function $G(k,p)$ in the case $p \gg k$. In dimensionless units, individual terms in the general expression for G, are of the order of kp/α^2, k^4/α^4 or p^4/α^4, but we know that for $k \gg p$ large cancellations occur and the final result is actually of $O(p^3/(\alpha^2 k))$. Therefore to get the correct answer numerically each intermediate term must be held to at least this *absolute* accuracy. If this is not possible and the intermediate terms have been "rounded-off", then the final numerical answer will be of the order of the lowest significant bit of the largest intermediate term. When this is larger than the true answer the computer obviously has no hope of getting the right answer. However, in this case we may perform these cancellations analytically and in order to avoid meaningless answers it is the asymptotic expressions, (2.62), that must be used.

As a second consequence of such roundings and truncations, but now rather of errors propagating still greater errors, consider the following example. Suppose we wish to solve numerically a differential equation whose possible solutions have two possible asymptotic behaviours e^{-x} and e^{x}. Suppose further that the boundary conditions at $x = 0$ are such that the "physical" e^{-x} behaviour is selected. We may then ask for the value of this solution at some nonzero value of x, for example $x = 10$. By application of some iterative algorithm the numerical solution may be traced out step by step until $x = 10$ is reached. It is inevitable that in this process small errors will be made, perhaps due to the step interval being made too large or because of very large cancellations being inadequately carried out in intermediate calculations. The generation of a finite error may be thought of as a mixing of our "pure" e^{-x} solution with a trace of the the unphysical e^{x} solution. However, due to the latter's very strong asymptotic behaviour, this initially negligible component will rapidly grow to dominate the behaviour of the numerical solution. This is illustrated in Fig. 2.9.

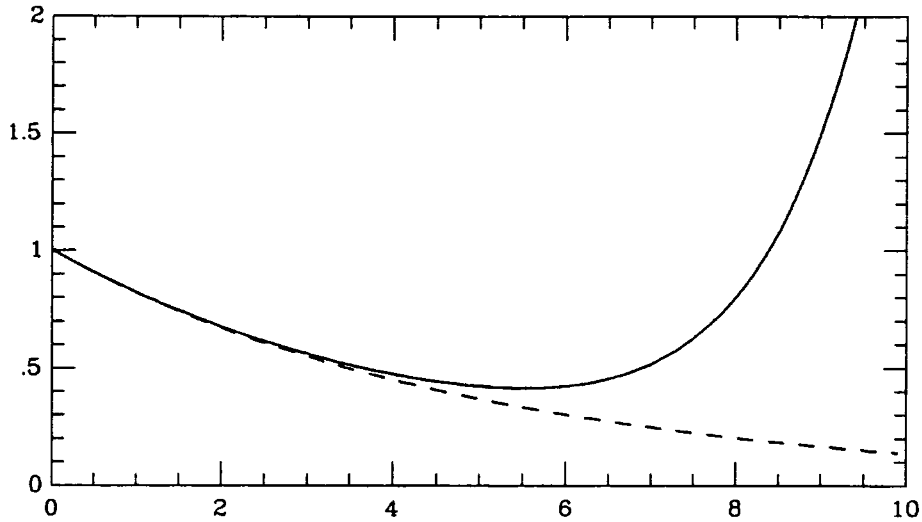

Figure 2.9 Illustration of a numerical instability

While the instability of this method of solution to small perturbations around the correct result would not be of any importance in a perfect machine, in any real machine this characteristic is eventually fatal. This is a rather elementary point, but it is one which will be of direct relevance to one of the two methods we will dicuss for solving our coupled relations.

Before we become embroiled in the nitty gritty details of these two methods let us consider what we are asking of them. The physical processes involved are characterized by two scales, the natural scale α and the dynamically generated mass scale m and it is important that all processes in the range $m \rightarrow \alpha$ are well represented. Given that we may be studying cases in which m is many orders of magnitude smaller than α, it is inevitable that we will run into numerical accuracy problems. In particular we will not be able to study the $m \rightarrow 0$ limit directly, but only by extrapolation of "small" m results. Let us now consider in detail two possible methods for solving our integral relations for $\beta(p)$ and $\Sigma(p)$.

2.9 An Iterative Solution.

There is a well known iterative method for finding an eigenvector of a general matrix. If we take a random vector R and act on this n times with the matrix in question, $\widehat{O}$, then in the limit $n \to \infty$ the final result will be dominated by the eigenvector l_i with the largest eigenvalue λ_i,

$$\lim_{n\to\infty} \widehat{O}^n R \longrightarrow \lambda_i^n l_i \quad , \quad \widehat{O} l_i = \lambda_i l_i . \tag{2.66}$$

In a similar way we may iterate the operation defined by the Schwinger Dyson relations for $\beta(p)$ and $\Sigma(p)$ on two sets of trial solutions B_n and S_n. We will have found a solution to the relations (2.64) if the trial solutions tend to forms invariant under this operation.

$$\left.\begin{array}{ll}\Sigma & = \mathcal{F}_\sigma[\Sigma,\beta] \\ \beta & = \mathcal{F}_b[\Sigma,\beta]\end{array}\right\} \longrightarrow \left\{\begin{array}{ll} S_{n+1} & = \mathcal{F}_\sigma[S_n, B_n] \\ B_{n+1} & = \mathcal{F}_b[S_n, B_n]\end{array}\right. \tag{2.67}$$

In placing any confidence in this method, we are implicitly assuming that our series of solutions are attracted to the true solution, let alone that they will converge to it in some finite, hopefully small, number of iterations. To start this proceedure we need a pair of trial solutions that have both the correct behaviour and he correct magnitude. We expect for $N \approx 1$ that the dynamically generated mass will be of $O(\alpha)$ as there is no parameter available in this case for the generation of any large hierarchy. As a quick check let us take $\beta(p) = 1$ and $\Sigma(p) = \Sigma(0)$ in the relation for $A(p)$. We may perform the integral over k very simply in the limit $p \to 0$, where we may linearize the angular function G. This yields a simple relation between $\Sigma(0)$, $\beta(0)$ and N,

$$\begin{aligned} A(0) &\simeq -\frac{1}{\pi^2 N p^3} \int_0^\alpha \frac{k\,dk}{k^2+\Sigma^2(0)} \, \frac{8}{3} \, \frac{p^3}{(k+\alpha)^3} \\ &\simeq -\frac{8}{3\pi^2 N} \int_0^\alpha \frac{k\,dk}{k^2+\Sigma^2(0)} \\ A(0) &\simeq -\frac{8}{3\pi^2 N} \ln\left(\frac{\alpha^2}{\Sigma^2(o)}\right) \end{aligned} \tag{2.68}$$

Therefore for $\Sigma(0) \sim O(\alpha)$, $A(0)$ is expected to be small and the Appelquist result qualitatively correct. Notice, however, that a large N extrapolation of this result contradicts Appelquist's findings and suggests that,

$$\Sigma(0) \geq \alpha \ \exp \left\{ -\frac{3\pi^2}{4} N \right\} \tag{2.69}$$

in order to ensure $\beta(0)$ is positive. A similarly simplistic analysis of the relation for $\Sigma(p)$ leads to the relation,

$$\ln \left\{ \frac{\Sigma(0)}{\alpha} \right\} \sim -N \quad . \tag{2.70}$$

So for small N we expect the dynamical mass scale to be close to the natural scale of the theory α and the Appelquist solution to be a good approximation to the full solution. Therefore, as a first guess in the small N limit, to start our series of trial solutions, we will take the solution to the far simpler Appelquist relation and set $\beta(p) = 1$. If this behaviour is borne out by the final results of our analysis of the full nonlinear Schwinger Dyson relations, then it leads to an intersting conclusion. In the infrared, if we consider the dynamical mass m to be "bare" in origin, then in this region the 1/N perturbative results results are in good agreement with the true solutions in the limit of $N \to 0$ as opposed to $1/N \to 0$! This is the another indication that there are some highly nonperturbative processes at work within these truncated Schwinger Dyson relations. Once we have a solution for one value of N then it will be a simple matter to apply the same procedure to obtain solutions at other values of N.

It is only when we come to implement this iterative procedure on a computer, that we come across a serious problem alluded to earlier in this section. While the series of trial solutions do appear to be converging, they become unstable in the infrared region and rapidly degenerate into nonsense before any reasonable accuracy is achieved. This behaviour is illustrated in Fig. 2.10.

The trial solutions for $A(p)$ and $\Sigma(p)$ appear to be close to convergence when both start to develop slight gradients in the previously flat region $p \ll \Sigma(0)$, in Fig 2.10 trial solutions are shown for every third iteration. As can be seen these gradients rapidly increase until the positivity of both Σ and β is violated at the

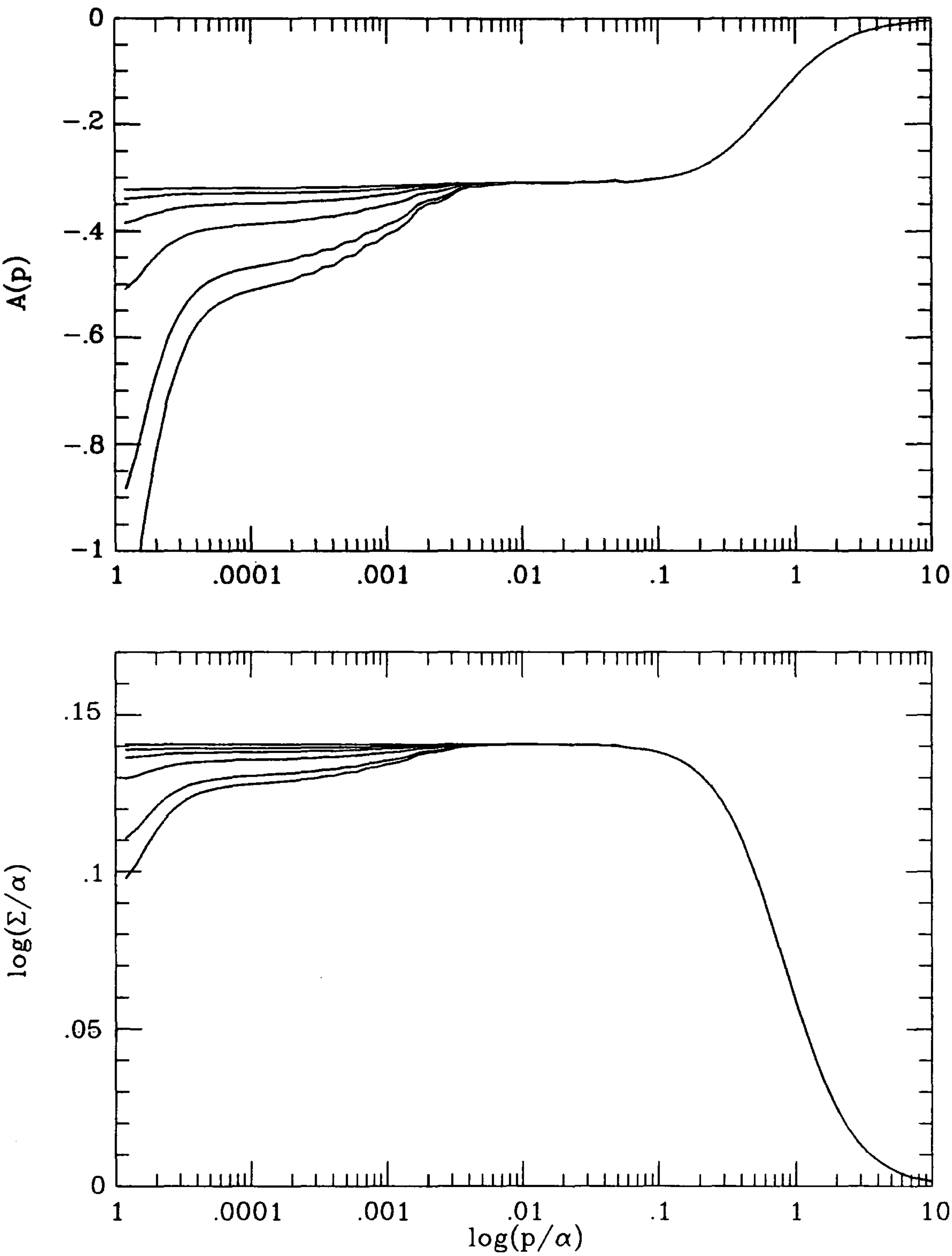

Figure 2.10 The degeneration of the series of trial solutions.

infrared edge of the momentum grid. Once this has taken place their further degeneration is unimportant as they are now clearly unphysical.

In order to understand this behaviour, reminiscient of our earlier example, we need to consider some of the peculiarities of the low momentum region. The functions $A(p)$ and $\Sigma(p)$ are represented on a logarithmic grid, to ensure that the integrals present in each Schwinger Dyson relation are accurately discretised. Consider calculating the "gradient" of Σ, $\Delta\Sigma$,

$$\Delta\Sigma = \frac{\Sigma(k) - \Sigma(p)}{k - p} \tag{2.71}$$

and how errors introduced by discretising the system influence the behaviour of this term. We may define three quantities on our grid Σ_i^g , Σ_i^p and ϵ_i via the following relation,

$$\Sigma_i^g = \Sigma_i^p + \epsilon_i \quad . \tag{2.72}$$

The superscript g refers to the actual numerically calculated result and the supercript p refers to the result obtained if all numerical errors ϵ_i were zero. The gradient obtained on the grid may then be written as,

$$\Delta\Sigma^g = \Delta\Sigma^p + \Delta\epsilon \quad . \tag{2.73}$$

The grid function Σ^g is trying to represent the "perfect" result Σ^p but if the behaviour of this is such that $\Delta\Sigma^p \sim 0$ then $\Delta\Sigma^g$ will be dominated by the error term $\Delta\epsilon$. In the worst possible case any error in Σ^g will be scaled by factors up to the inverse of the grid spacing $\delta k \sim k$. Therefore if $\Sigma' \to 0$ as $p \to 0$, as seems likely from the behaviour of the trial solutions, the term $\Delta\Sigma$ will become increasingly inaccurately defined on our grid in this limit. Before we can link this trend to the observed behaviour of our series of trial solutions it is necessary that we discover some bootstrap mechanism in our coupled relations by which these errors may be amplified.

This mechanism operates in the low momentum region $p \ll m$, as can be seen from the Fig 2.10 the region $p \gg m$ is relatively insensitive to the low momentum behaviour. So let us study the relation for $\Sigma(p_0)$ and $\beta(p_0)$ where p_0 is the lowest value represented on our momentum grid. Consider the relation for Σ first. After linearising both the angular functions and dropping irrelevant numerical factors we have,

$$\Sigma(p_0) \sim \int \frac{k\,dk}{\beta^2(k)k^2 + \Sigma^2(k)}\,\mathcal{E}(k,p)$$

$$\mathcal{E}(k,p) = \quad (\beta(p)+\beta(k))\,\Sigma(k) + \{\,\Sigma(k)\ \Delta\beta - \beta(k)\,\Delta\Sigma\,\}\ \frac{p^2}{(k+p)} \tag{2.74}$$

At the onset of the instability both Σ and β are flat enough that we may neglect the second gradient term, it is damped by an effective factor of p_0^2/k^2. The value of $\Sigma(p_0)$ is then linearly related to $\beta(p_0)$. We may write this as,

$$\Sigma(p_0) = \overline{\Sigma} + \sigma\,\beta(p_0) \quad , \tag{2.75}$$

where initially both $\overline{\Sigma}$ and σ are both of $O(m)$. If we linearise the relation for $\beta(p_0)$ in the same manner we have:

$$A(p_0) \sim -\int \frac{k\,dk}{\beta^2(k)k^2 + \Sigma^2(k)}\,\mathcal{W}(k,p)$$

$$\mathcal{W}(k,p) = (\beta(k)+\beta(p))\beta(k) + \frac{k^2+p^2}{k+p}\,\frac{\beta(k)}{2}\Delta\beta + \frac{\Sigma(k)}{k+p}\Delta\Sigma \tag{2.76}$$

Since the region $p \ll m$ is initially the only region effected, let us concentrate on the contribution to the integral from this region alone. In this region all terms except the $\Delta\Sigma$ term are explicitly damped by factors of k^2/Σ^2, from the measure $k\,dk/(\beta^2(k)k^2 + \Sigma^2(k))$. This term is clearly involved in the bootstrap, as the magnitude of its contribution is determined directly by the form of Σ. Let us introduce a small gradient into Σ such that in the region $p_0 \le p \le \mu$ $\Sigma(p)$ is well described by the following,

$$\Sigma(p) = \Sigma(p_0)(1 + \epsilon\,\ln{(p/p_0)}\,) \quad . \tag{2.77}$$

where μ is some scale far below m. We may easily work out the effect of this small slope on $\beta(p_0)$,

$$\begin{aligned}
\beta_\Delta(p_0) &\sim -\int_{p_0}^{\mu} \frac{k\,dk}{\Sigma^2(k)}\,\Sigma(k)\,\frac{(\Sigma(k)-\Sigma(p))}{k^2-p^2} \\
&\sim -\int_{p_0}^{\mu} \frac{dk^2}{k^2}\left(1+\frac{p^2}{k^2}\right)\,\epsilon\ln(k/p) \\
&\sim -\epsilon\ln^2(\mu/p_0)
\end{aligned} \tag{2.78}$$

This gradient then obviously pulls $\beta(p_0)$ negative and through (2.75), $\Sigma(p_0)$ towards zero, but now under the influence of a stronger gradient.

$$\Sigma(k) = \Sigma(p_0)(1+\epsilon\ln^2(k/p_0)) \tag{2.79}$$

This in turn contributes an even larger negative amount to $\beta(p_0)$ and the run-away train is launched. These logarithms are characteristic of this degeneration. If instead we assumed intially a power-law behaviour, p^{-n}, then under the action of the β relation, (2.76), which raises negative powers of k in Σ into positive powers of logarithms, a similar logarithmic scenario would unfold. Our simple iterative method is fatally flawed† and without some knowledge of the behaviour of the solutions we cannot repair it. In the next section we will consider a more "intelligent" algorithm which is able to sidestep this instability.

2.10 MINIMISATION

In contrast to the blind generation of trial solutions, as exemplified by the previous method, consider now a more directed search for the solutions of our coupled system. Before we may direct any search we must have some measure of how close any trial solution is to the true answer. One obvious answer is to define a χ^2 between the trial solution and true answer, but since it is the answer we seek we are forced into a far less satisfactory definition. We can only define

† Note that for the central vertex there are no gradient terms and therefore no problems associated with numerical instability.

a χ^2 between a trial solution and its "daughter".

$$\chi^2 = \lim_{l\to\infty} \frac{1}{l} \int \left(\frac{f^{n+1}(x) - f^n(x)}{f^{n+1}(x) + f^n(x)} \right)^2 dx \quad , \quad f^{n+1} = \mathcal{F}[\, f^n \,] \tag{2.80}$$

This single value tells us, in an average sense, how close our trial solution is to its daughter. While it is the case that the above is positive definite and identically zero only for the true solution, it is by no means a reliable pointer to the true answer. If we reduce the scale of the problem to that of an arithmetic series, the previous statement will be clear. To say that the n^{th} component of a series is within 1% of the value of its sum, to date, is no guarantee that this truncated sum is within 1% of the full sum. In the same way a trial solution and its daughter may be very close together but very far from the true solution, they are merely sitting in a local minimum of the χ^2 potential. We may, of course, increase the complexity of our χ^2 defintion, but this flaw is general and while there is little that can be done about it, we should be aware of the limitations of any χ^2 definition.

There is a well known method for solving linear integral equations of the Fredholm or Volterra type. The generic form of such a relation is,

$$f(y) = \int k(x,y)\, f(x)\; dx \tag{2.81}$$

where the kernel $k(x,y)$ is independent of f. To solve such a problem numerically let us approximate our solution by a truncated series of orthogonal polynomials multiplied by, as yet, unknown coefficients. In principle we may now perform the x integration analytically, using these known polynomials in place of f, leaving a linear matrix problem. This procedure is summarized by:

$$f(x) = \sum_{i=1}^{N} a_i\, P_i(x) \quad , \quad a_j = \frac{1}{h_j} K_{ji}\, a_i \tag{2.82}$$

with

$$K_{mn} = \int k(x,y)\, P_n(x)\, P_m(y)\, dx\, dy \quad , \quad \int P_i(x)\, P_j(x)\, dx = h_i\, \delta_{ij} \tag{2.83}$$

We may then cast the task of finding the a_i as a minimisation problem by defining a χ^2 between the a_i and their daughters a'_i,

$$\chi^2 = \frac{1}{N}\sum_{i=1}^{N}\left\{\frac{a'_i - a_i}{a'_i + a_i}\right\}^2 \quad . \tag{2.84}$$

and simply let a minimisation routine do the hard work. It is straightforward to extend this approach to our current problem. As opposed to a linear integral equation we now have two nonlinear coupled algebraic relations. The nonlinearity of the numerator in each simply ensures that our final matrix problem will be nonlinear in the function coefficients in the same way. The presence of the denominator term $\beta^2(k)k^2 + \Sigma^2(k)$, however, requires that we actually expand the complexity of the system. Originally we had two primary functions $\Sigma(p)$ and $\beta(p)$ whose polynomial coefficients we would vary to find their solutions. Since $(\beta^2(k)k^2 + \Sigma^2(k))^{-1}$ cannot be simply expressed in terms of $\Sigma(k)$ and $\beta(k)$ the we are forced to define another primary function $D(k)$,

$$D(k) = \left\{\,\beta^2(k)k^2 + \Sigma^2(k)\,\right\}^{-1} \tag{2.85}$$
$$= \sum_i \, d_i P^i(k) \tag{2.86}$$

and represent it also by a truncated series of polynomials. The relation this new function must satisfy is that the d_i are such that $D(k)$, numerically defined by (2.85), satisfies the constraint, (2.86), that defines it. So now we have three primary functions that together must satisfy three nonlinear coupled algebraic relations.

In order to represent the functions $\Sigma(p)$,$\beta(p)$ and $D(p)$ by a series of polynomials over the complete range $p = 0 \to \infty$ the following change of variables was performed.

$$p = p_0\left\{\frac{1-z}{z+1}\right\} \longrightarrow \left\{\begin{array}{ll} z \to -1\,, & p^{-1} \sim (z+1) \\ z \to +1\,, & p \sim (1-z) \end{array}\right. \tag{2.87}$$

The range $p = 0, \infty$ is mapped by this transformation onto the region $z = [-1, 1]$. The scale p_0 determines the “midpoint” of this transformation, $p =$

$p_0 \to z = 0$, and should be chosen so that all the functions are well "centered" on the range $z = -1 \to 1$. At large momenta, $p \gg \alpha$ we expect that our functions have a simple power law decay, namely

$$A(p),\ D(p),\ \Sigma(p) \sim \frac{1}{p^2} \quad , \quad p \gg \alpha \, . \tag{2.88}$$

and for small momenta are approximately constant. The behaviour of our functions with z is then expected to be,

$$A(p),\ D(p),\ \Sigma(p) \longrightarrow \begin{cases} \sim (z+1)^2, & z \to -1 \\ \sim \text{constant}, & z \to 1 \end{cases} \tag{2.89}$$

The polynomials chosen to express $\Sigma(k)$ and $D(k)$ were the Jacobi polynomials, $P_n^{(\alpha,\beta)}(z)$, that satisfy,

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(z) P_m^{(\alpha,\beta)}(z)\,(1-z)^\alpha\,(1+z)^\beta\; dz = C(\alpha,\beta,n)\,\delta_{mn} \tag{2.90}$$

leading us to define their two series expansions as,

$$\begin{aligned} \Sigma(k) &= (1+z)^2 \sum_{i=1}^{J} \sigma_i\ P_i^{(2,0)}(z) \\ D(k) &= (1+z)^2 \sum_{i=1}^{J} d_i\ P_i^{(2,0)}(z) \quad . \end{aligned} \tag{2.91}$$

which ensures their expected $k \to \infty$ behaviour of k^{-2}. The third primary function $A(p)$ was found to be best represented in terms of the Legendre polynomials $P_n^{(0,0)} = L_n$. Although $A(p)$ does fall eventually as p^{-2} this occurs at scales of $O(\alpha)$, as opposed to $O(m)$ which is the case for both Σ and D.

$$A(p) = \sum_{i=1}^{J} a_i\ L_i(z) \tag{2.92}$$

Substituting these expressions into the Schwinger Dyson relations for $A(p)$ and $\Sigma(p)$, we may peform the final radial integration, though unfortunately not analytically. In contrast to the simple linear example given earlier we did not attempt to project out the polynomial coefficients of the daughter solutions, this is not strictly necessary. The daughter functions alone are sufficient. The detailed forms of these relations are very lengthy and equally opaque, so we will only outline their general form here. If we represent a generic solution coefficient by f_i then their general form is

$$f(x) = \sum_i K_i\ f_i + \sum_{ij} K_{ij}\ f_i f_j + \sum_{ijk} K_{ijk}\ f_i f_j f_k \quad . \tag{2.93}$$

Using these algebraic relations and the definitions (2.91) & (2.92), we may construct the two versions of the solutions Σ, β and D, initial and daughter. We may then calculate a χ^2 between these two versions, defined on a set of appropiate momentum values, p_i, that are deemed to cover the physically important momentum ranges.

$$\chi^2 = \frac{1}{N} \left[\sum_{i=1}^{N_p} C_a \left(\frac{A'(p_i) - A(p_i)}{A'(p_i) + A(p_i)} \right)^2 + C_d \left(\frac{D'(p_i) - D(p_i)}{D'(p_i) + D(p_i)} \right)^2 + \cdots \right] \quad . \tag{2.94}$$

N_p is the number of points on our grid and f' is the daughter function of f. The coefficients C_a, C_d and C_σ are used to ensure that if for example $A(p)$ was very sensitive to small variations in Σ, then C_σ would be set much greater than C_a.

The pratical implementation of this method proceeds as follows. First trial solutions for A, Σ and D were used to determine the initial values for the polynomial coefficients and the midpoint scale, p_0. In practice between 8 and 12 polynomials were needed for each primary function, resulting in a parameter space of very high dimension, 24 D - 36 D. This enormously increased the scope for local minima and therefore the difficulty of finding the true solution. In order to construct the algebraic relations K_{ij} *etc.* approximately 1500-5000 complicated integrals needed to be peformed numerically and stored so that the results could be called on at each minimising operation. Moreover, because a small change in function "shape" doesn't necessarily correspond to a small change in polynomial coefficients, moving from one value of N to another could only be achieved in very small steps. All these factors together ensured that obtaining a solution was extremely time consuming. Indeed it is a credit to the writer†, of the minimisation program used that it didn't get stuck in local minima *too* often. However, solutions could be obtained to approximately 1% accuracy in the range N=1 to 3. For higher values of N the functions A, Σ and D could not be accurately represented on the grid of z values. For $N \sim 1$, the dynamical mass m is close to α so only a small range in p needed to be accurately represented and each solution could be well centered, Fig 2.11a.

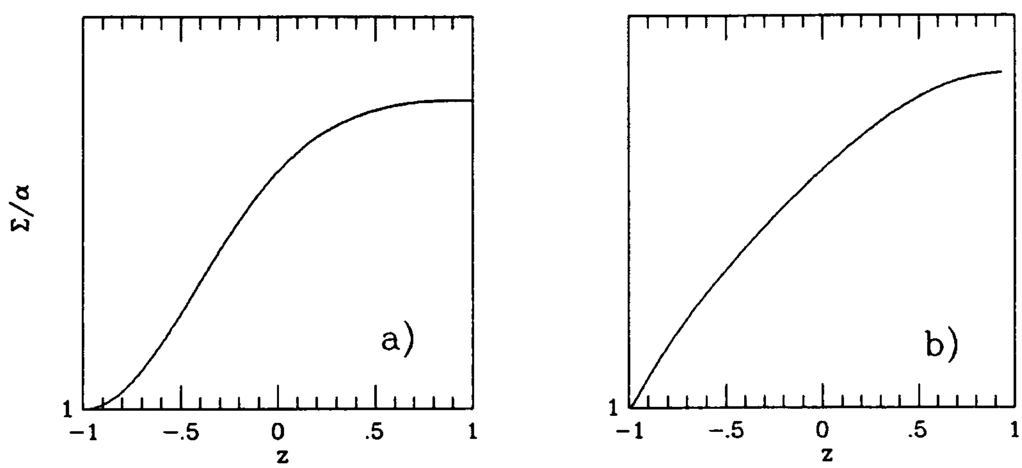

Figure 2.11 a) A well centered fit. b) Badly centered fit.

However, as N increases and m falls the important region $p = O(m) \to O(\alpha)$ becomes increasingly hard to represent as $m \to -1$ and $\alpha \to +1$. In Fig 2.11b it can be seen that the asypmtotic regions $p < m$ and $p > \alpha$ have been squashed out to the limits $z = \pm 1$, where the polynomial expansion is expected to be inaccurate. The behaviour of $A(p)$ in particular was such that it varied evenly over the complete range $p = m \to \alpha$, before entering into its aymptotic behaviour $A(p) \sim p^{-2}$. Therefore as N increased, the effort required to find a solution increased and at the same time the reliability of any such answer decreased. This is the reason why this approach had to be abandoned. However, by N=3 the forms of our solutions up to this point had conformed to an expected behaviour that allowed us to understand and tame the instabilities of the previous iterative method.

† Fred James, Cern library Minimisation package MINUIT 1990.

2.11 THE ITERATIVE METHOD REVISITED.

The mechanism by which the iterative method of Section 2.9 was rendered useless was the amplification of gradients in the low momentum region. The solutions found using the minimisation method were flat in this region to the accuracy achievable. Rather then considering unphysical inverse power law or logarithmic behaviour in this region, that are both inconsistent and unstable, let us consider scalings of the form,

$$\begin{aligned} \Sigma(p) &= \Sigma(p_0) + \sigma_1(p/m) + \sigma_2(p/m)^2 + \cdots \quad p \ll m \\ \beta(p) &= \beta(p_0) + b_1(p/m) + b_2(p/m)^2 + \cdots \quad p \ll m \, . \end{aligned} \tag{2.95}$$

We will follow the analysis of Section 2.9 by linearising all angular functions and dropping irrelvant multiplicative factors. The resulting relations are (as given in (2.74)& (2.76)),

$$A(p_0) \sim -\int \frac{k\,dk}{\beta^2(k)k^2 + \Sigma^2(k)} \mathcal{W}(k,p)$$

$$\Sigma(p_0) \sim \int \frac{k\,dk}{\beta^2(k)k^2 + \Sigma^2(k)} \mathcal{E}(k,p)$$

$$\mathcal{W}(k,p) = (\beta(k) + \beta(p))\beta(k) + \frac{k^2+p^2}{k+p}\frac{\beta(k)}{2}\Delta\beta + \frac{\Sigma(k)}{k+p}\Delta\Sigma$$

$$\mathcal{E}(k,p) = \quad (\beta(p) + \beta(k))\,\Sigma(k) + \{\,\Sigma(k)\;\Delta\beta - \beta(k)\,\Delta\Sigma\,\}\,\frac{p^2}{(k+p)} \tag{2.96}$$

Let us now split the integration region into two regions , $0 \le k \le \mu$ and $\mu \le k \le \alpha$ where as before μ is some scale far below m where our infrared scalings (2.95) are expected to end. Consider the contributions from the high k region, β_μ^α and Σ_μ^α. In this region the dependence on p is weak and we may Taylor expand all dependence on p about p=0. It is a simple matter to check that, to first nontrivial order in p, these contributions are linearly dependent on p. In the intermediate region the kernels in (2.96) are damped, initially by factors of k^2/m^2 from the measure and then, once $k \le p$, by factors of k^n/p^2 coming from the angular functions. These damping factors ensure that the contributions from this region are, to first order, proportional to p/m. Therefore both functions are constant up to factors of the order of p/m. In a similar manner any scaling $p^n, n \geq 1$ is consistent, but any expansion must contain a term proportional to p^2, as this is automatically introduced by the region $\mu \leq k \leq \alpha$. In principle, in a detailed analysis of the linearised relations, we might determine the values of the coefficients in (2.95), but fortunately this is unnecessary.

Consider again the series of trial solutions Fig 2.10 that illustrate the infrared instability of the algorithm to date. Notice how the physically important region $p \geq O(m)$ is relatively immune to the eruptions taking place in the lower momentum regions. The reason for this has been stated already in the analysis just performed. To effect the region $p \sim O(m)$, the low momentum behaviour must be extreme enough to lift the damping factors introduced by the " measure " k^2/m^2, as well as those introduced by the angular functions, of $O(k^n/p^n)$. Even in the most extremely distorted solution of Fig. 2.10, this does not happen to any discernable extent. We may view the exact behaviour of the low momentum region as really rather cosmetic and of no real importance to our calculation of m. We will therefore break the bootstrap mechanism in the most basic manner, by artificially setting $\Sigma(p) = \Sigma(0)$ for all $p \leq \mu$. To ensure that the extent of this flattening is not constraining the behaviour of the two solutions, the range μ must be varied in order to test the dependence of the final result with this parameter. In practice this dependence was very weak, of the order of 0.1% for an obviously constraining range and significantly smaller, $\sim O(0.01\%)$ for the ranges used in practice $\mu \sim O(m/100)$. Once the system was stabilised in this manner there was no trouble in obtaining solutions for $N = 1-8$, to an accuracy of the order of 0.02%. In principle there was no obstacle to expanding our grid and proceeding to yet higher values of N, but this was deemed unnecessary. By this stage a definite asymptotic large N behaviour could readily be deduced and so the numerical investigation was brought to an end. In the next section we will discuss in detail these results and the mechanisms that they imply.

2.12 RESULTS.

In Fig 2.12 the dynamical mass m(p) is plotted as a function of momentum for varying values of N. As can be seen no critical behaviour was found for $N \sim 3$, but for the moment we will postone a discussion of this fact and concentrate on the general properties of m(p). At low momentum $p \ll O(m)$, there is very little development of m(p). As discussed in the previous section it is to all intents and purposes flat. Consequently to the physics in this region, the source of the fermion mass is obscured. The low energy effective theory is that of QED with a nonzero "bare" mass, the chiral symmetry of the high energy theory $p \gg \alpha$ is hidden. At large momenta $p \gg \alpha$ the dynamical mass is falling as $1/p^2$. This is easily seen by considering the equation for $\Sigma(p)$. In this region all corrections to the vertex are down by explicit factors of $1/p$ and the integral is dominated by the contributions coming from the region $0 \geq k \geq p$. To a good approximation, we may linearise both angular functions F and H, leaving the high momentum form for the $\Sigma(p)$ relation as,

$$\Sigma(p) \simeq \frac{\alpha}{Np} \int_0^\alpha \frac{k\,dk}{\beta^2(k)k^2 + \Sigma^2(k)} \{\cdots\} \; \frac{k^3}{p+\alpha} \quad , \quad p \gg \alpha \tag{2.97}$$

which has a solution,

$$\Sigma(p) = \Sigma(p_0)\, \frac{p_0(p_0+\alpha)}{p(p+\alpha)} \quad , \quad p, p_0 \gg \alpha \tag{2.98}$$

Physically what is happening is the following: in the infrared a nonzero vacuum condensate $\langle\, \overline{\psi}\psi \,\rangle$ is generated and it is through this that the previously massless fermions acquire a mass. At higher scales the fermions are still aware of this condensate but its influence is damped by factors of p^{-1}. This dependence may be more systematically investigated in the language of the operator product expansion [19]. This states that a product of operators acting at the same or very close space-time points factorise onto a series of local operators. For example

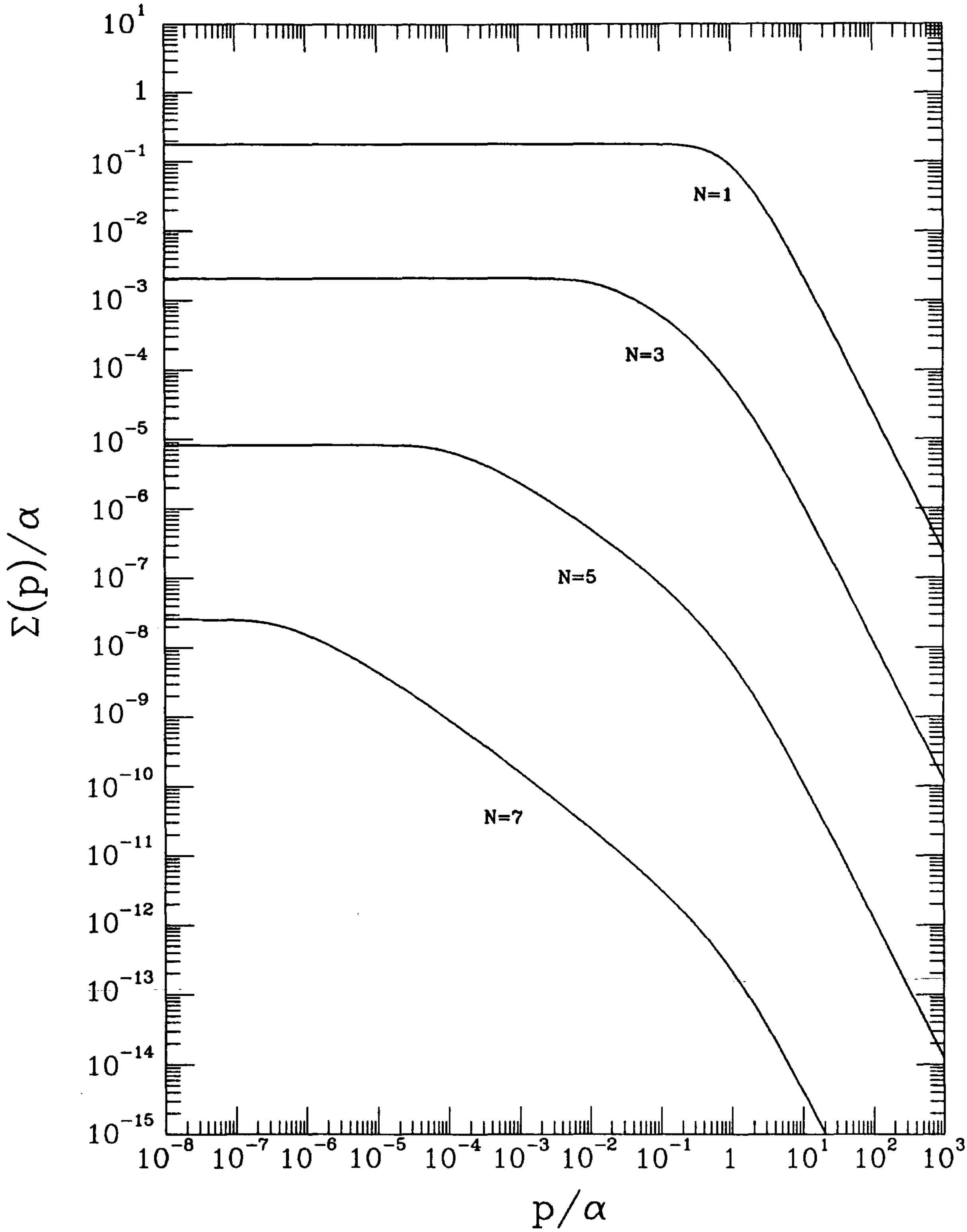

Figure 2.12 The dynamical mass function $m(p)$ plotted against p on logarithmic axes for N=1, 3, 5, 7. The vertex used was Ball-Chiu longitudinal form.

consider the fermion propagator,

$$\int d^n x\, e^{-iq\cdot x} \, \langle 0 | T \, [\overline{\psi}(x)\psi(0)] \, | 0 \rangle_{x\to 0} = \sum_n \langle 0 | O_n | 0 \rangle \quad . \tag{2.99}$$

This relation may be diagrammatically represented in Fig 2.13.

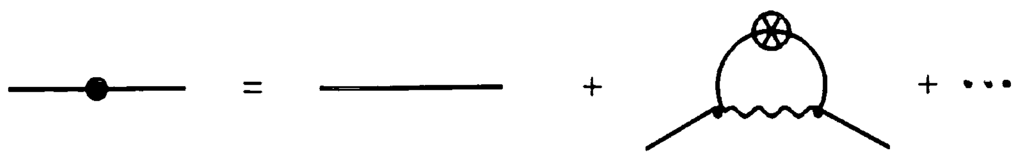

Figure 2.13 Diagramatic representation of the first two terms of the operator product expansion of the full propagator. The $\otimes$ represents the insertion of the chiral condensate, $\langle \overline{\psi}\psi \rangle$.

The second term represents the interaction of the bare fermion with the vacuum condensate $\langle \overline{\psi}\psi \rangle$. The essence of the operator product expansion is that the asymptotic short distance behaviour of a product of operators is contained in the *coefficients* of a set of well defined local operators. Obviously, as the dimension of the local operator increases this must be compensated for by additional powers of inverse momentum. Since the ultraviolet limit of 3D QED is trivial, we expect no logarithms to upset the canonical scaling of any operator† . Therefore to first nontrivial order in $1/p$, we need consider the interaction of the fermion with the vacuum condensate alone. For $p \gg \alpha$ we may write the propagator as,

$$S(p) \;\sim\; c_1(p) \, \langle 0 | 1 | 0 \rangle + c_2(p) \, \langle 0 | \overline{\psi}\psi | 0 \rangle + \cdots \tag{2.100}$$

The resulting mass function is then asymptotically given by,

$$m(p) = m_0 + p^2 \, c_2(p) \, \langle 0 | \overline{\psi}\psi | 0 \rangle . \quad p/\alpha \to \infty \tag{2.101}$$

Simply on dimensional grounds we expect $p^2 \, c_2(p) \sim \alpha/p^2$, because we expect

† This of course is not true of a model of QED3 plus a four fermi interaction, which though renormalisable in 1/N is not superrenormalisable or asymtotically free. This additional interaction is thought to herald a new nonperturbative phase with an associated ultraviolet fixed point and highly nontrivial dynamics. See Ref 20

the canonical scaling law for $\langle\, \overline{\psi}\psi \,\rangle$ to be accurate in this region. Therefore, in the chiral limit $m_0 \to 0$, we have,

$$m(p) = \langle 0|\,\overline{\psi}\psi\,|0\,\rangle \frac{\alpha}{p^2} \quad , \quad p \gg \alpha \,. \tag{2.102}$$

The asymptotic behaviour of the mass function is independent of any perturbation theory, but is in fact simply due to the scaling properties of our superrenormalisable model.

The behaviour of the dynamical mass, however, was not what prompted this investigation. The motivation for that was the belief that the wavefunction renormalisation $\beta(p)$ could not be treated perturbatively and perturbative arguments in 1/N had to be abandoned in general. The argument against a perturbative β went as follows. The perturbative result for β is not well defined in the infrared. The appearance of a logarithmic factor indicates for scales very much smaller than α that the $O(1/N)$ perturbative correction is not small and the formal 1/N ordering is broken. Consequently when studying the chiral symmetry breaking mechanism, especially in the region of any critical behaviour, where the dynamical scale is far smaller than the natural scale α , we expect the naive perturbative approximations to be misleading. In section 2.3 an example was presented in which the higher order corrections could be summed, resulting in a power law behaviour,

$$\beta(p) \sim \left\{ \frac{p}{\alpha} \right\}^{\gamma} \quad , \quad \gamma \sim O(1/N) \,. \tag{2.103}$$

which is what we might expect to happen here. The actual behaviour of $\beta(p)$ found from our truncated Schwinger Dyson equations, which implicitly resume an infinite number of perturbative corrections, is not as this example might lead us to expect. The results for N=1, 3, 5, 7 are given in Fig 2.14. Notice for momentum scales greater than m $A(p)$,the wavefunction renormalisation correction, is well ordered in 1/N but for $p \leq O(m)$ their behaviour is such that $\beta(p) = 1 + A(p)$ is of $O(1/N)$. Let us consider each region in turn, $p \gg \alpha$ the asymptotically high momentum region, $m < p < \alpha$ the transition region and $p \leq \alpha$ the infrared region.

In the asymptotic region $p \gg \alpha$ the main contribution to $A(p)$ is due to the self-interation of the fermion via "hard", perturbative photons. The form of this correction is $A(p) \sim 1/(Np^2)$, which is the behaviour also found in our numerical solutions. This is not surprising since for large p the Schwinger Dyson relation for $\beta(p)$ is dominated by the perturbative regime $k \geq p$. Any effect due to the interaction of our fermions with a vacuum condensate will be proportional to $c_2(p)$ and is therefore down by p^{-2} compared to these perturbative corrections. Therefore the form for $\beta(p)$ in this region is well approximated by,

$$\beta(p) = 1 - \frac{c}{N}\left(\frac{\alpha}{p}\right)^2 \quad , \quad p \gg \alpha \; . \tag{2.104}$$

In the transition region $m \leq p \leq \alpha$, while the 1/N ordering is clearly still in operation, the actual numerical value of the wavefunction correction is not small. As opposed to the powerlaw behaviour expected from our naive example of section 2.3 we see that the development of $A(p)$ in this region is still basically logarithmic. Let us try and analyse the behaviour of the Schwinger Dyson equation for $\beta(p)$, in an attempt to understand this. If we linearise our angular functions, then in the transition region we have,

$$A(p) \sim -\frac{1}{N}\int \frac{k\,dk}{\beta^2(k)k^2 + \Sigma^2(k)}\left(\theta(k-p) + \theta(p-k)\frac{k^3}{p^3}\right)\mathcal{W}(k,p)$$

$$\mathcal{W}(k,p) = (\beta(k) + \beta(p))\beta(k) + \frac{k^2+p^2}{k+p}\,\beta(k)\,\Delta\beta + \frac{\Sigma(k)}{k+p}\,\Delta\Sigma \tag{2.105}$$

were $\theta(x)$ is the standard step function. The region $0 \leq k \leq p$ is rapidly damped by the factor k^3/p^3 which is not lifted by the smooth behaviour of the two solutions, so we may drop this region. For a sufficiently large value of p, then the $\Sigma^2(k)$ factor in the denominator is irrelevant so we may neglect it too. In this last approximation we are assuming there exits a value of p such that $m \ll p \ll \alpha$ and so are implicitly confining our investigation to the large N

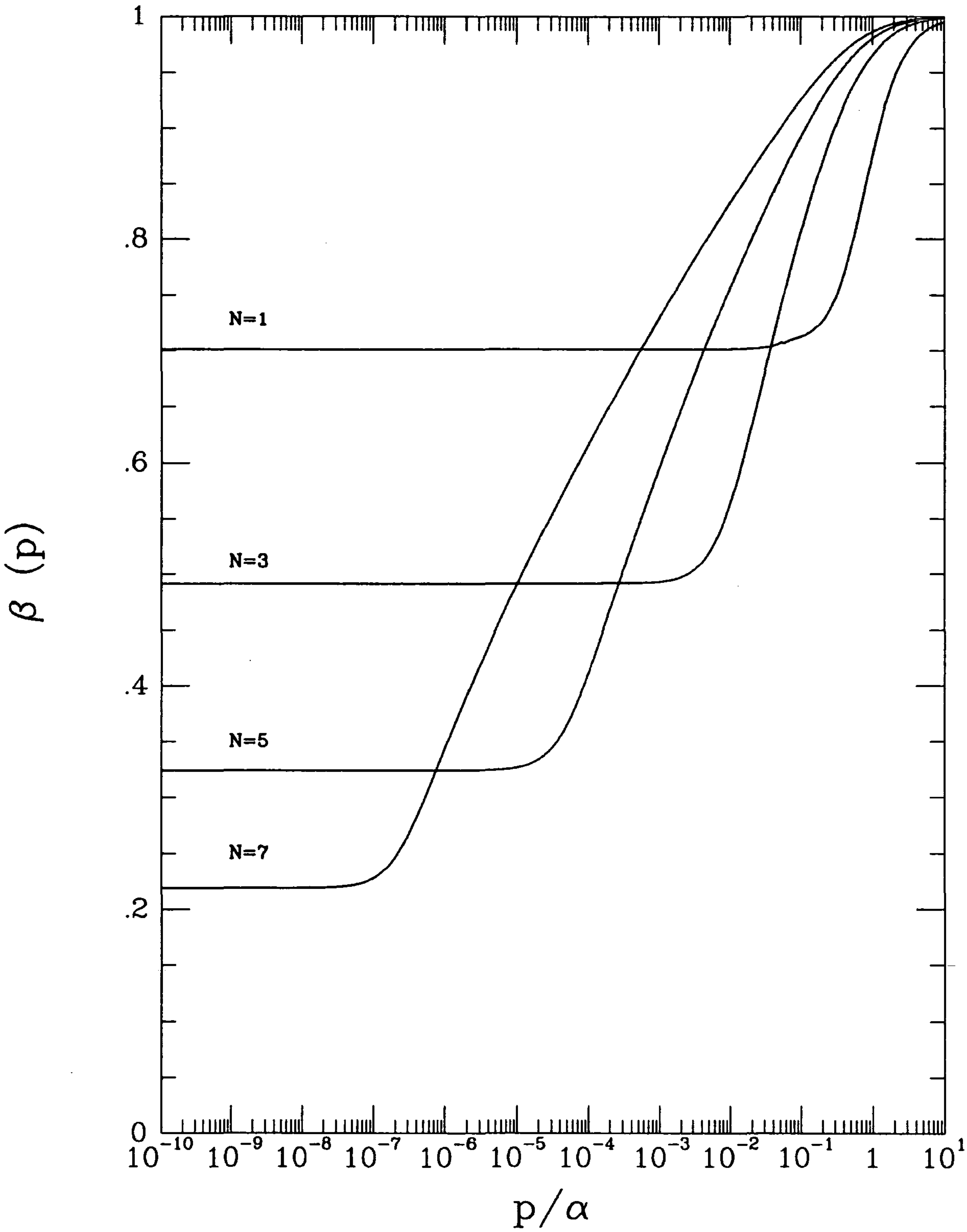

Figure 2.14 The Wavefunction Renormalisation $\beta(p)$ plotted against log(p/α) for the N values 1,3,5,7. The vertex used was the Ball-Chiu longitudinal form.

behaviour of $A(p)$, where m/α is vanishing. In this region the relation for $A(p)$ may be approximated by the following,

$$A(p) \sim -\frac{1}{N} \int_{\ln(p/\alpha)}^{0} d\ln(p/\alpha)\, \mathcal{W}(k,p)$$

$$\mathcal{W} = \frac{\beta(k)+\beta(p)}{\beta(k)} + \frac{k^2+p^2}{k^2-p^2}\frac{\beta(k)-\beta(p)}{\beta(k)} + 2\,m(k)\,\frac{m(k)-m(p)}{k^2-p^2} \tag{2.106}$$

From the behaviour of our solutions we know that the integrand above is smooth, so we expect the scaling of $\beta(p)$ to be determined by the logarithmic measure,

$$A(p) \sim \ -\frac{1}{N}\ln\left(\frac{\alpha}{p}\right) \quad , \quad m \leq p \leq \alpha. \tag{2.107}$$

Though this is very similar to the perturbative result it cannot be considered as such, because in the first place it is not the result of any single perturbative correction and secondly it is not numerically small. This analysis also lets us understand the behaviour for $p \leq m$. If we again split our integration range but this time into 3 parts, it is easy to see that the dynamical mass acts effectively as an infrared cutoff to the above scaling. The region $p \leq k \leq m$ is damped by factors of k^2/m^2 and the region $0 \leq k \leq p$ is further damped by factors of k^3/p^3. Therefore we expect the development of $A(p)$ to stop at a value of the order of

$$A(p) \sim -\frac{1}{N}\ln\left(\frac{\alpha}{m}\right) \quad , \quad p \ll m. \tag{2.108}$$

This result suggests that the chiral symmetry of the bare Lagrangian must always be broken, since $A(p)$ must be greater than -1 in order to preserve the positivity of $\beta(p)$. From the observed behaviour of $\beta(0)$ we can see that the explicit factor of 1/N has been lifted dynamically and this implies, that even for moderate values of N, the dynamically generated scale m must be many orders of magnitude smaller than the natural scale α. Our expectation that $\beta(p)$ could not be treated perturbatively is justified, even if our naive model of section 2.3 was not. Now we may ask ourselves how this behaviour effects the chiral symmetry breaking mechanism and can it possibly remove the critical behaviour found by Appelquist?

We may understand how the wavefunction renormalisation may lift the critical behaviour in N if we consider a small p approximation much as we did when we analysed Appelquist's original equation. After linearizing the angular functions and dropping the irrelevant $k \leq p$ region and irrelevant numerical factors we have

$$\Sigma(p) \sim \frac{1}{N} \int_p^\alpha \frac{k\, dk}{\beta^2(k)k^2 + \Sigma^2(k)} \mathcal{F}(k,p)$$

with

$$\mathcal{F}(k,p) = (\,\beta(k) + \beta(p)\,)\,\Sigma(k) + (\;\Sigma(k)\,\Delta\beta + \beta(k)\,\Delta\Sigma\;)\;\frac{p^2}{k+p}\;. \qquad (2.109)$$

As before, for sufficiently small momentum we may drop the "gradient" terms and are left again with an integral which is dominated by the $m \leq k \leq \alpha$ region,

$$\Sigma(0) \sim \frac{1}{N} \int \frac{k\, dk}{\beta^2(k)k^2 + \Sigma^2(k)} \,(\beta(k) + \beta(p))\,\Sigma(k) \quad . \qquad (2.110)$$

Imagine factoring out the dependence of $\beta(p)$, by replacing it by some constant "average" $\overline{\beta}$. The effect of this is to effectively change N to $\overline{\beta}N$. Therefore as N increases, β falls so as to ensure that this effective value of N remains safely below any critical number. This finite dynamical mass in return is generated in such a way as to ensure that β is finite and positive. Therefore, the action of the wavefunction renormalisation is crucial, it "supports" the chiral symmetry breaking and removes the possibility of any critical behaviour.

How then is the dynamically generated mass scale linked to the number of fermion flavours? The approximate relation (2.69) suggests that

$$m \geq m_0\, exp\,(\,-N/N_0\,) \qquad (2.111)$$

and this is exactly the behaviour found by the numerical analysis of the two Schwinger Dyson equations as is shown in Fig. 2.15. The exact values for the constants N_0 & m_0 are,

$$\begin{aligned} m_0 &= exp(2.72) \simeq 15.8 \\ N_0 &= 2.67^{-1} \simeq \frac{3}{8} \end{aligned} \tag{2.112}$$

We may draw an analogy between this behaviour and another nonperturbative process. The instanton is a solution of a classical field theory which governs the tunneling of the ground state in the quantum theory between inequivalent vacua [21]. In general the tunneling probability between two vacua θ_1 & θ_2 is of the form,

$$\begin{aligned} |\langle\theta_1||\theta_2\rangle|^2 &\sim exp\left\{-\frac{1}{g^2}\right\} \\ &\not\simeq 1+g^2+g^4+\cdots \quad . \end{aligned} \tag{2.113}$$

The coupling in our model is played by the parameter $1/N$ and so the dynamical mass relation (2.111) is of this type. Both these mechanisms are genuinely nonperturbative as is illustrated by the nontrivial way in which the coupling enters in to each expression.

These results and our simple analysis of the Schwinger Dyson relations display a pleasing consistency, but we may ask how sensitive this qualitative behaviour is to the details of our vertex ansatz. We may make one simplification very easily and that is to set the angular integral $H(k,p)=0$, and so effectively remove all "gradient" parts from the vetex, leaving the simple "central" term,

$$\Gamma^\mu = \frac{1}{2}\left\{\beta(k)+\beta(p)\right\}\gamma^\mu \quad . \tag{2.114}$$

The results from this further truncation are given in Figs. 2.16 & 2.17 and are qualitatively identical to those of its big brother Γ^μ_{BC}. The dynamical mass displays the same exponential fall with N and as before $\beta(p)$ displays the same fall that is required to lift any possible critical behaviour. While the replacement of the Ball-Chiu ansatz with its simpler "central" version has altered the large N scaling coefficient N_0 and the logarithmic scaling is now less good at small

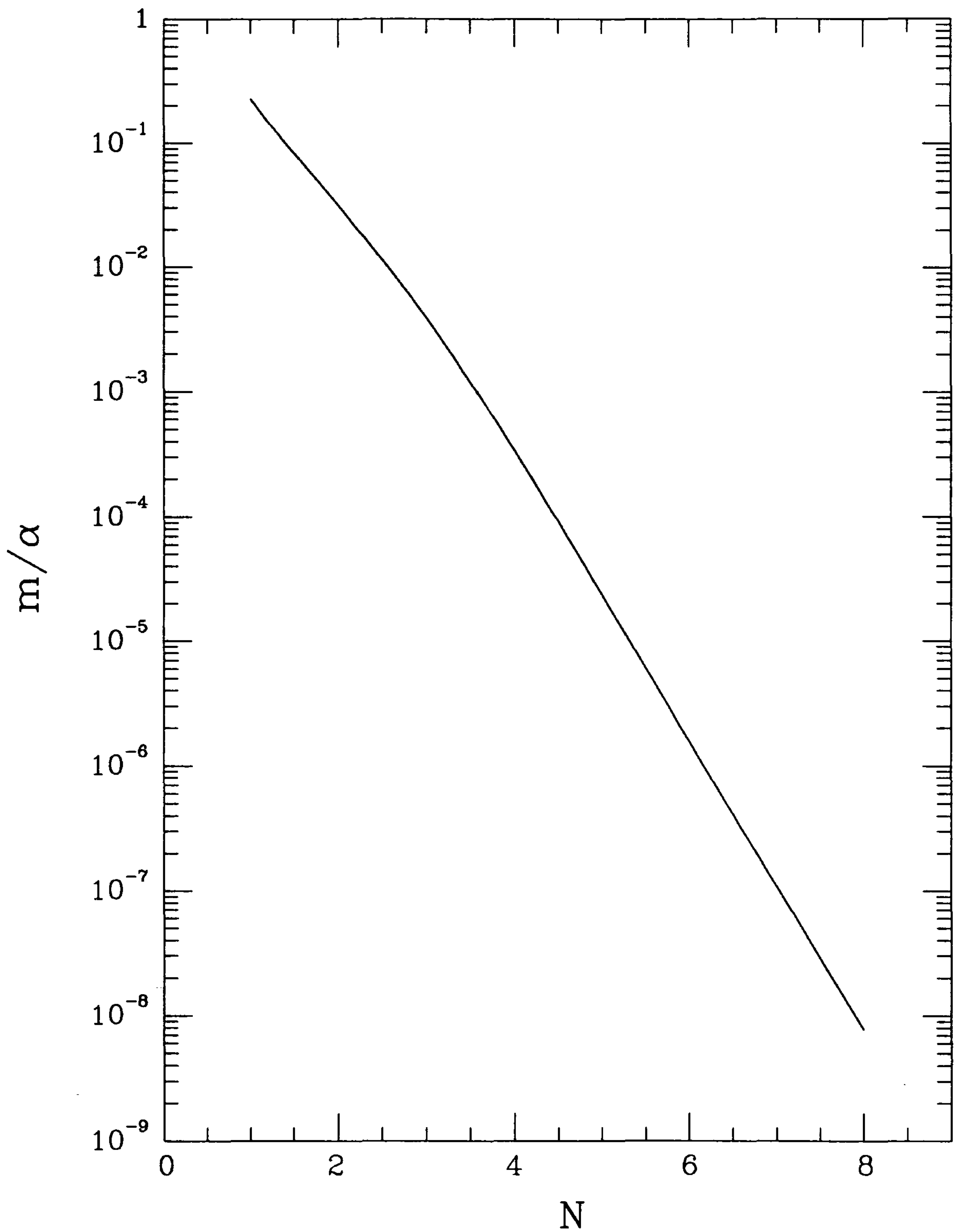

Figure 2.15 The dynamical Mass as determined by the pole in the fermion propagator against N, for the range N=1,8. The vertex used was the full Ball-Chiu ansatz

N, these are mere details. Perhaps not surprisingly these results are also in agreement with an early calculation by Pennington & Webb [22] in which a bare vertex was used, the simplest possible ansatz. We must therefore conclude that the basic mechanism, by which the chiral symmetry of the bare Lagrangian is broken, seems to be very insensitive to the details of the vertex, but is driven by the softening of the photon, the source of the logarithms that are so characteristic of this model.

This is obviously the next step that we need to check - how reliable is the photon form we have used? Before we embark on this new investigation let us summarize the main points of this rather lengthy chapter.

2.13 Conclusions.

The first and most important point to note is that the 1/N perturbative series is in general not reliable, order by order, in the infrared. This is illustrated by the unphysical behaviour of the first order result for $\beta(p)$. The breaking of chiral symmetry is primarily a low momentum effect and therefore any truncations of the Schwinger Dyson system based on perturbation theory are, *a priori*, incorrect.

If the perturbative avenue is closed to us, we must approach the truncation of the Schwinger Dyson hierarchy in a truly nonperturbative fashion. We may accomplish this by truncating, at some specified level this hierarchy and replacing the undetermined quantities of this truncated system by physically motivated ansätze. In this procedure, symmetry constraints such as the Ward-Takahashi identity for the photon-fermion vertex are our guiding lights. Indeed we may solve the Ward-Takahashi identity under general assumptions to give what is expected to be the dominant part of the vertex in the infrared region. By replacing the full vertex with this nonperturbative approximation we simplify Schwinger Dyson hierarchy, while ensuring numerical consistency of the truncated system. Because of the complexity of the two coupled relations for the fermion and photon propagators, we were forced to simplify further. The photon propagator in the fermion equation was replaced by its perturbative one loop version. Again

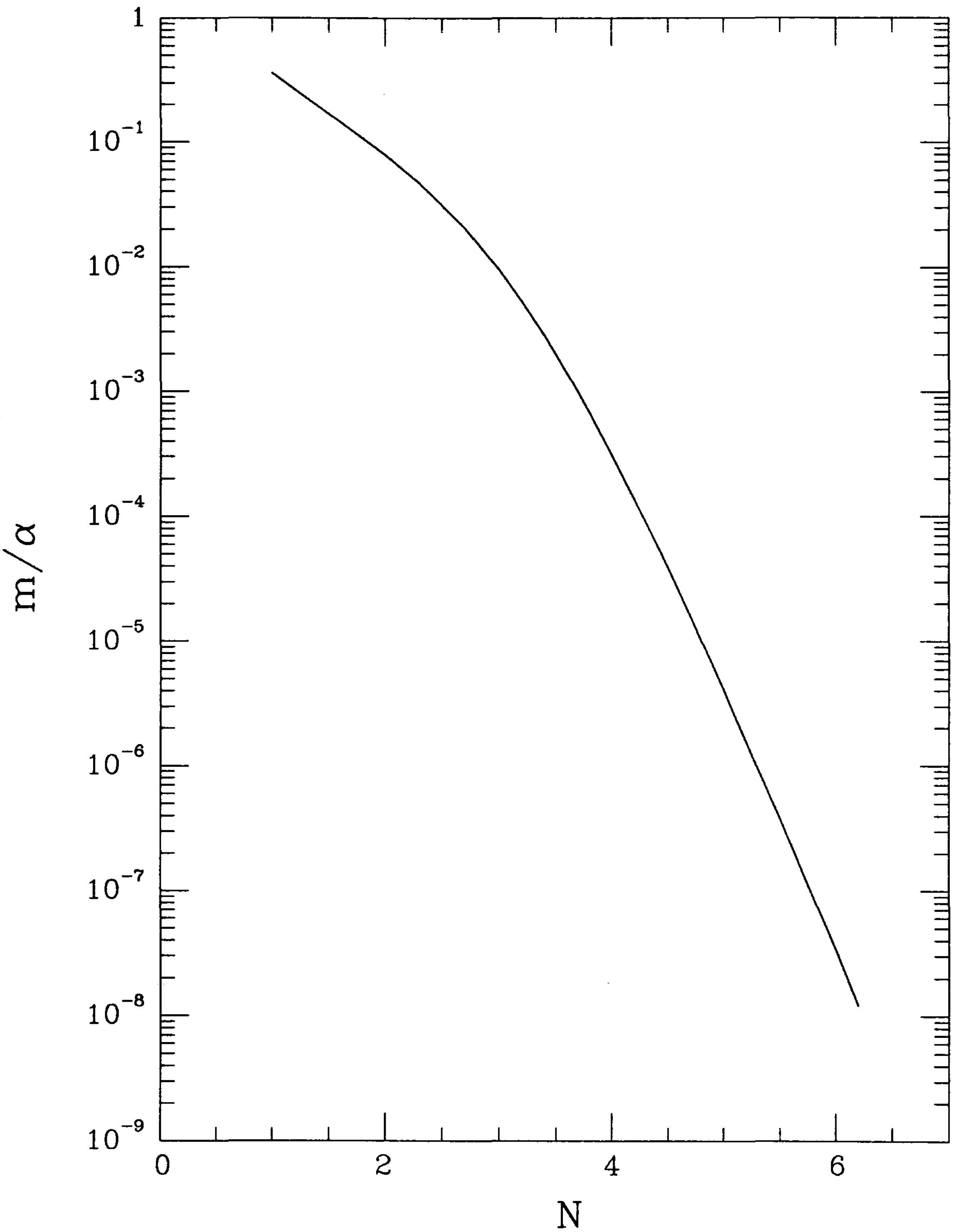

Figure 2.16 The dynamical mass versus N, for the range N=1,6. The vertex used was the "central" vertex, $\Gamma = \frac{1}{2}\{\,\beta(p) + \beta(k)\,\}$.

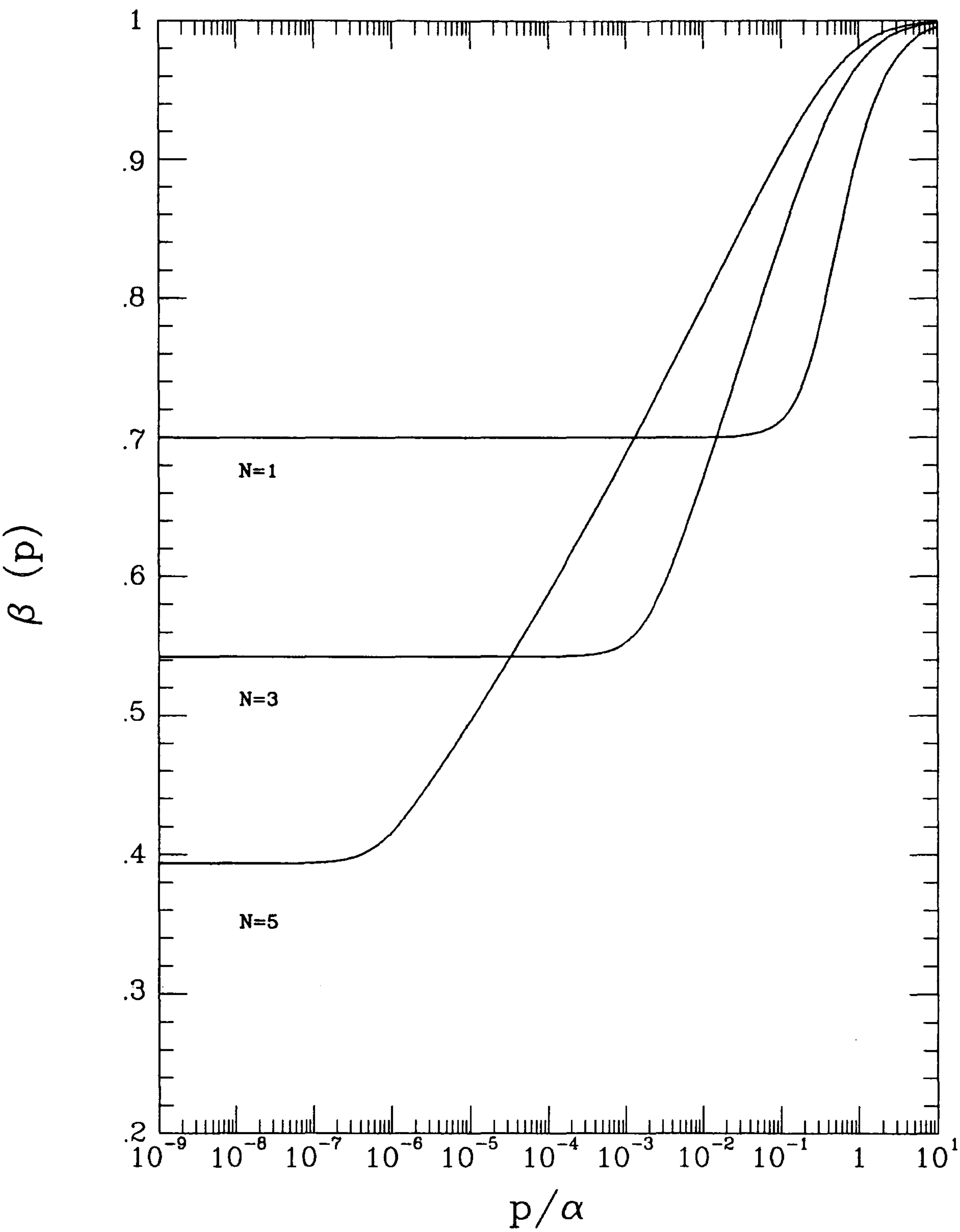

Figure 2.17 The wavefunction renormalisation $\beta(p)$ versus log(p/α) for N=1,3,5. The vertex is the "central" form.

general arguments suggested that the softening of the full photon is qualitatively well described by this form. Now only the fermion relation remains, which may be split into two coupled relations for $\beta(p)$ and $\Sigma(p)$, in which both are treated on an equal footing.

These equations were investigated numerically and the results indicate that chiral symmetry is broken for all values of N. Clearly the critical behaviour found by Appelquist *et al* is not in evidence. The observed exponential dependence of the dynamical mass on the number of fermion flavours is found not to be sensitive to the details of the vertex ansatz used. Indeed it is the nonperturbative inclusion of the wavefunction renormalisation, $\beta(p)$, that pushes the critical value of N to infinity. In contrast to naive perturbative expectations, $\beta(p)$ is not merely a spectator of the symmetry breaking process. The softening of the photon propagator in the infrared, by effectively massless fermions, raises it to a crucial member of the process. It is the influence of the wavefunction renormalisation that supports the chiral symmetry breaking and ensures that the effective value of N remains safely below any critical value. Therefore we expect that the chiral symmetry of the bare Lagarangian to be broken for all values of N , as long as the behaviour of the photon propagator contains no surprises. The possible importance of dynamical fermions in the photon and its nonperturbative form is what will be dealt with in the following chapter.

Chapter 3: The Photon.

3.1 Introduction.

The conclusions and mechanisms of the previous chapter emphasise the importance of the photon in the chiral symmetry breaking mechanism of this model. It is precisely the form of the photon's softening that ensures that the chiral symmetry of the bare Lagrangian is always broken and also, through the introduction of infrared logarithms, ensures that the resulting hierarchy of scales are exponentially separated with repect to the number of fermion flavours, N. Consequently, the use of a wholly perturbative ansatz in this important role may be viewed as the weak link in our previous analysis and in need of further study. There are two quite separate points that stem from this observation and these need to be cleared up before we may take the conclusions of the previous chapter too seriously. These two points are:

i) The previous calculation, along with many others, was very willing to ignore the dynamical fermion mass within its photon ansatz, *i.e.* the fermion was treated as essentially massless in loops within the photon propagator. This ansatz was then used in relations to predict a *nonzero* dynamical fermion mass. *A priori*, this is inconsistent but in its support the following argument was presented. The chiral symmetry breaking mechanism is securely tied to the natural scale α and therefore we may safely ignore any effects at scales of $O(m)$ and below. We expect that the inclusion of such effects would lead only to relative corrective factors of $O(m/\alpha)$ and therefore in the interesting region of any possible critical behaviour, or large N limit, where $m \to 0$ these may safely be ignored. However, we have found no critical behaviour and cannot rely on this type of intuitive argument without further justification.

ii) At a more fundamental level, while we expect the perturbative photon ansatz to be qualitatively correct down to scales p, such that $m \ll p \ll \alpha$, we have no real understanding of its nonperturbative nature, 1/N approximations have been misleading in the past and may be again. In order to investigate the photon's nonperturbative behaviour across all momentum scales, we will carry out a detailed study of its Schwinger Dyson equation. Using the results of this study, we will be able to determine whether the perturbative ansatz was consistent, or whether we are required to study the two coupled photon and fermion Schwinger Dyson equations together in all their complexity.

The first of these is by far the simplest, so we will start with the question of a second, dynamical, scale in the photon.

3.2 The Hard Photon.

How may the inclusion of the fermion mass in the photon correction influence its own generation in the chiral symmetry breaking mechanism? The assumption of our previous calculation was that the inclusion of this nonzero mass would lead only to an $O(m/\alpha)$ correction to the dynamical mass,

$$m \longrightarrow m\left\{1+O\left(\frac{m}{\alpha}\right)\right\} . \tag{3.1}$$

As a preliminary to a full investigation into the behaviour of the photon in the fermion equation, let us consider our 1/N perturbative ansatz but now generated by fermions possessing a nonzero bare mass, m_0 [23]. The calculation is fairly straightforward extension of that of chapter 1 and in the Landau gauge yields the rather more complicated looking correction below,

$$\Pi_m(p) = \frac{2\alpha}{\pi p^2}\left\{2m + \frac{p^2-4m^2}{p}\sin^{-1}\sqrt{\frac{p^2}{p^2+4m^2}}\right\} \tag{3.2}$$

with the photon function defined by,

$$\mathcal{G}_m(p) = p^2\left[1+\Pi_m(p)\right] . \tag{3.3}$$

The subscript m is used in an obvious way to refer to the inclusion of a nonzero bare fermion mass in the photon correction, in order to distinguish it from the massless result , $p^2 + \alpha p$, which we will refer to in future as $\mathcal{G}_0(p)$. For momenta large in comparison to the fermion mass m_0, as expected, (3.3) is well approximated by the massless result $\mathcal{G}_0$, up to factors of $O(m/p)$. However, in the new region, $p \leq m$, the photon changes character, as it did for $p \sim \alpha$, and regains its bare scaling behaviour.

$$\mathcal{G}_m = \left\{ 1 + \frac{4\alpha}{3\pi m_0} \right\} p^2 + O\left(\frac{p^3}{m_0} \right) , \quad p \ll m_0 . \tag{3.4}$$

The reason for this change in behaviour is quite simple. At these low energies the photon cannot interact with real massive fermions, these modes require far too much energy and have long since decoupled, but only with virtual fermions. In this example these virtual interactions lead only to a multiplicative correction to the photon's bare scaling that acts to lift the extra factor of p and so ensure continuity at $p \sim O(m)$. Even given the explicit factor of α/m, the photon's behaviour at low momentum, as given in (3.2), is strikingly different in comparison to the massless result in this limit.

$$\mathcal{G}_0(p)/\mathcal{G}_m(p) \simeq m_0/p \quad , \quad q \ll m \ll \alpha . \tag{3.5}$$

Though in itself this is an interesting fact, the behaviour of the photon in isolation is not the central question. Rather we should be asking, how does this change in behaviour of the photon effect the dynamical mass generation? This hardening of the photon suggests that, although it occurs in the previously unimportant region, $p < m$, its effect is unlikely to lead to insignificant $O(m/\alpha)$ corrections, as was naively assumed.

We may investigate the importance of this new behaviour by taking (3.3) as our ansatz in a second analysis of the fermion's Schwinger Dyson equation. Since it is only the qualitative effect of our "massive" photon ansatz that we seek, in this calculation it is not necessary to employ the full Ball-Chiu vertex ansatz. As was discussed in Section 1.12 , the far simpler "central vertex",

$$\Gamma^\mu(k,p) = \frac{1}{2} \left(\beta(k) + \beta(p) \right) \gamma^\mu \tag{3.6}$$

reproduces the qualitative behaviour of the Ball-Chiu system for large N and is entirely sufficient for our purposes. For the same reason the mass in the photon correction, (3.2), need not be determined precisely by the pole in the fermion propagator, but may be approximated by the zero momentum value of the dynamical mass function,

$$\mathcal{G}_m(p) \longrightarrow \mathcal{G}_{m_0}(p) \quad , \quad m_0 = \frac{\Sigma(0)}{\beta(0)} \simeq m \ . \tag{3.7}$$

By this association we hope to mimic the effect of a dynamical fermion mass scale in the photon. The calculation proceeds in exactly the same way as that of Sections 2.5,6 but neglecting the gradient terms, until we come to the angular functions. The forms of these are, of course, determined by $\mathcal{G}(p)$. We may write the two coupled relations for $A(p)$ and $\Sigma(p)$ as,

$$\Sigma(p) = \frac{2\alpha}{\pi^2 N p} \int \frac{k\,dk}{\beta^2(k)k^2 + \Sigma^2(k)} \left(\beta(k) + \beta(p)\right) \Sigma(k)\, I_\sigma(k,p)$$

$$A(p) = -\frac{\alpha}{2\pi^2 N p^3} \int \frac{k\,dk}{\beta^2(k)k^2 + \Sigma^2(k)} \left(\beta(k) + \beta(p)\right) \beta(k)\, I_a(k,p) \tag{3.8}$$

and for a general photon function, the angular integrals $I_\sigma(k,p)$ and $I_a(k,p)$ are defined as in Eqn. 2.58 by,

$$\begin{aligned} I_\sigma(k,p) &= \frac{kp}{2\pi} \int \frac{d\Omega}{\mathcal{G}(q)} \\ I_a(k,p) &= \frac{2kp}{\pi} \int \frac{(k^2 - k\cdot p)(p^2 - p\cdot k)}{q^2 \mathcal{G}(q)}\, d\Omega \ . \end{aligned} \tag{3.9}$$

In the zero mass limit, $m_0 \to 0$, the angular functions $I_\sigma(k,p)$ and $I_a(k,p)$ are simply the angular functions $L(k,p)$ and $G(k,p)$ of our original analysis, Section 2.6. However, for a complicated, or unknown, photon function we may only be able to calculate these angular functions analytically in the two asymptotic regions $p \gg k$ and $k \gg p$, where functions involving q may be simply Taylor expanded. Both functions are symmetric in k and p, so in these two regions we may expand in powers of k/p and p/k respectively. This results in the two general asymptotic forms,

$$
\begin{aligned}
I_\sigma(q,Q) &= 2\,\frac{Q\,q}{\mathcal{G}(Q)} \\
I_a(k,p) &= \frac{16}{3}\,\frac{Q\,q^3}{\mathcal{G}^2(Q)}\left\{\mathcal{G}(Q) - Q^2\,\frac{d}{d\,Q^2}\,\mathcal{G}(q)\right\}
\end{aligned}
\tag{3.10}
$$

where

$$
Q = \mathrm{Max}(k,p) \quad \& \quad q = \mathrm{Min}(k,p)\ .
$$

Furthermore we may exploit our knowedge of the explicit form of $\mathcal{G}_m(p)$ to also Taylor expand in powers of q/m and m/q in the other two asymptotic regions $q \ll m$ and $q \gg m$. In the region $q \gg m$, we recover the "massless" functions $L(k,p)$ & $G(k,p)$ up to small factors of $O(m/q)$, as indeed we must. However, in the region $q \ll m$, as expected from the change in behaviour of $\mathcal{G}_m(p)$, these functions change character significantly,

$$
I_\sigma(k,p) = F^{-1}\,\ln\left|\frac{k+p}{k-p}\right|
$$

$$
I_a(k,p) = \frac{\alpha}{30\pi m_o^3}\,F^{-2}\,\left\{q_+^4 - q_-^4 - 4q_+^2 q_-^2\,\ln\left|\frac{k+p}{k-p}\right|\right\}\ .
$$

where F and $q_\pm$ are defined as,

$$
F^{-1} = \left\{1 + \frac{4\alpha}{3\pi m_0}\right\}\quad,\quad q_\pm = k \pm p\ .
$$

The qualitative changes involved are well illustrated by comparing the asymtotic, $q \ll Q \ll m \ll \alpha$, forms of these functions with those of their massless counterparts, $L(k,p)$ & $G(k,p)$,

$$
\begin{aligned}
I_\sigma(k,p) &= 2\,\frac{m}{Q}\,\frac{q}{\alpha} \quad \leftrightarrow \quad L(k,p) = 2\,\frac{q}{Q+\alpha} \\
I_a(k,p) &= \frac{2\pi}{5}\,\frac{Q}{m}\,\frac{q^3}{\alpha} \quad \leftrightarrow \quad G(k,p) = \frac{8}{3}\,\frac{q^3}{\alpha}
\end{aligned}
\tag{3.11}
$$

It is worth mentioning that although in the Landau gauge we may naively expect I_a to be zero since in this region as $\mathcal{G}_m \sim q^2$, this scaling is true only to first nontrivial order in p/m and $I_a(k,p)$ is therefore is not identically zero. This new asymptotic form is suppressed by a factor of Q/m in comparison to it's "massless" version $G(k,p)$ and so we would expect the value of $|A(p)|$ to be correspondingly smaller for the same value of N. Also in this region the contributions to $\Sigma(p)$ will be enhanced by an additional factor of m/Q in I_σ and even though this occurs in the relatively unimportant region $k \leq m$, the strength of this enhancment suggests that the mass function itself will significantly increase. These variations are at first sight reminiscent of a rescaling of N, but to discover the exact effect of these alterations we will need to solve the two coupled Schwinger Dyson relations with this rather more complicated photon ansatz. Computationally this is quite straightforward, as there are no "gradient" terms to worry us and we may apply the simple iterative algorithm of Section 2.9 without having to consider any stability factors.

As can be seen from Fig 2.1, the exponential relation between the dynamical mass and the number of fermion flavours has been preserved. Indeed up to a shift in N the observed large N behaviour of both systems is identical. If we write,

$$m = m_0 \; exp(-N/N_0) \tag{3.12}$$

then the constants m_0 and N_0 resulting from each photon ansatz are given below.

Photon	$\ln(m_0)$	N_0^{-1}
Massless	9.18	-4.76
Massive	11.4	-4.78

and the effective shift in N, is then simply

$$\delta N = \frac{11.4 - 9.18}{4.7} = 0.49 \tag{3.13}$$

Consider the qualitative effect of the extra factor of m/k in I_σ on the value

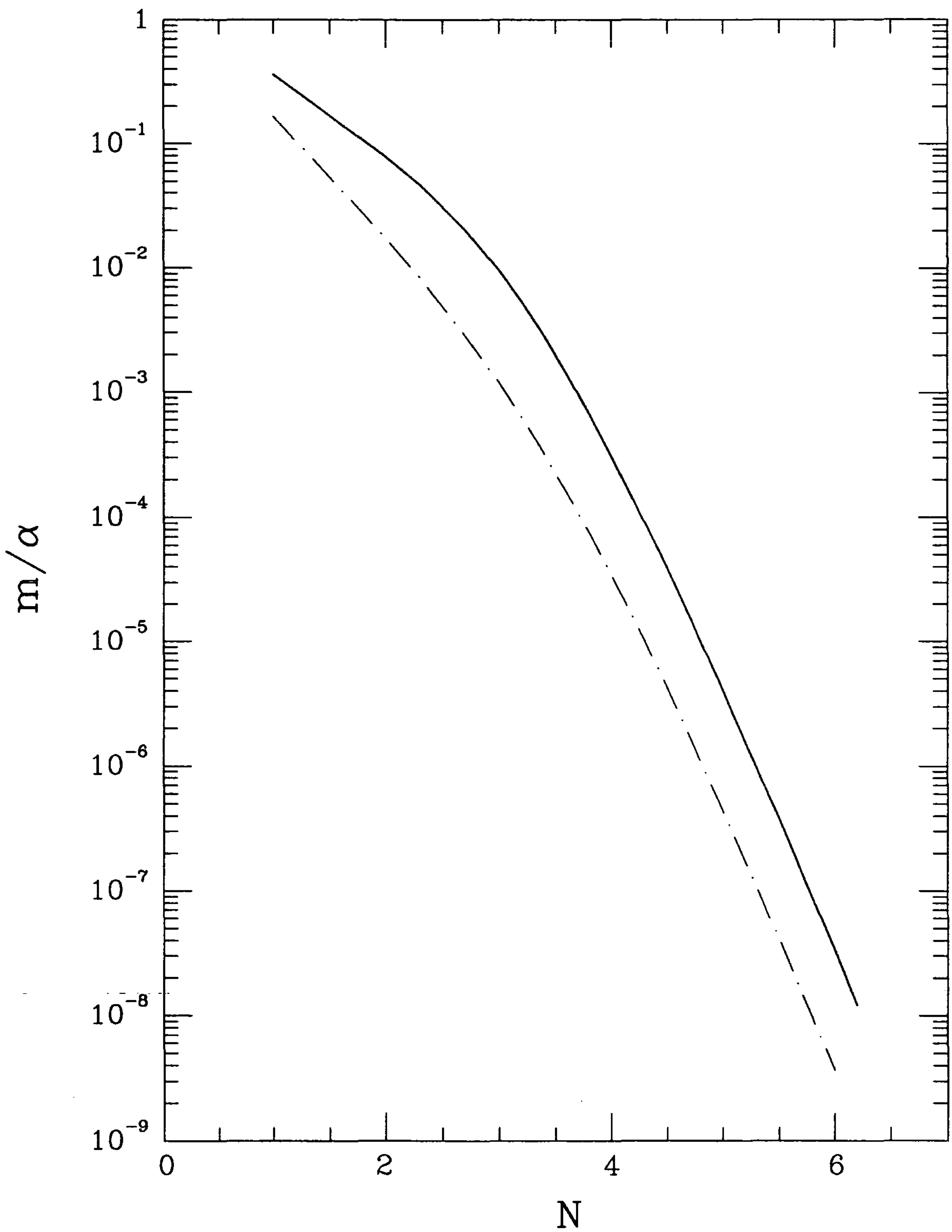

Figure 3.1 Developement of the Dynamical Mass m with number of fermion flavours, N, for both the "massive" — and "massless" -·- photon ansätze

of $\Sigma(0)$,

$$
\begin{aligned}
\Sigma(0) &\sim \int \frac{k\,dk}{k^2+m_0^2}\, m_0 \left(\frac{m_0}{k}\,\theta\,(m_0-k)+\theta\,(k-m_0)\right) \\
&\sim \{\, a \ln\{\, \alpha/m\,\} + c\,m/2\,\}
\end{aligned}
\tag{3.14}
$$

where the massless result under the same simplistic approximations would be identical except that the term $c\,m/2$ would be simply replaced by $c\,m$. In contrast, the angular function for $A(p)$, in (3.11), effectively damps the contributions from the low q region leading to a reduced value of $|A(p)|$ and this is indeed what is observed, see Fig 3.2. For our new photon, $|A(p)|$, as expected, is considerably smaller compared to the "massless" result. Therefore *qualitatively* all is well and good, but we have not shown how the extra factors introduced in this region result in a simple shift in N, for $N \gg 1$, and not some more general rescaling, $N \to f(N)$.

Let us investigate a simplified version of the relation for $A(p)$, in each of the two cases. Firstly we will linearise the angular integral I_a, as we may in the limit $p \to 0$ and proceed to drop all multiplicative constants. We will also assume that we may approximate the "massive" photon's smooth behaviour at $p \sim O(m)$ by an abrupt change at $p = m$. By ignoring the smooth interpolation of the physical photon between these two different behaviours we are effectively only ignoring another irrelevant constant. Consider the relation for $A(p)$ for the massive ansatz first,

$$
N\,A(p) \sim -\int \frac{k\,dk}{\beta^2(k)k^2+\Sigma^2(k)}\,(\beta(p)+\beta(k))\,\beta(k)\;\mathcal{A}(k,p)
$$

$$
\mathcal{A}(k,p) = \theta(p-k)\,\frac{k^4}{mp^3} + \theta(m-k)\,\frac{k}{m} + \theta(\alpha-k)
\tag{3.15}
$$

If we differentiate (3.15) with respect to p we have ,

$$
\begin{aligned}
N\,\frac{\partial}{\partial p}A(p) &\sim \int_0^p \frac{k\,dk}{\beta^2(k)k^2+\Sigma^2(k)}\,(\beta(p)+\beta(k))\,\beta(k)\,\frac{k^4}{p^4 m} \\
&\quad - \left\{\frac{\partial}{\partial p}A(p)\right\} \int \frac{k\,dk}{\beta^2(k)k^2+\Sigma^2(k)}\,\beta(k)\mathcal{A}(k,p)
\end{aligned}
\tag{3.16}
$$

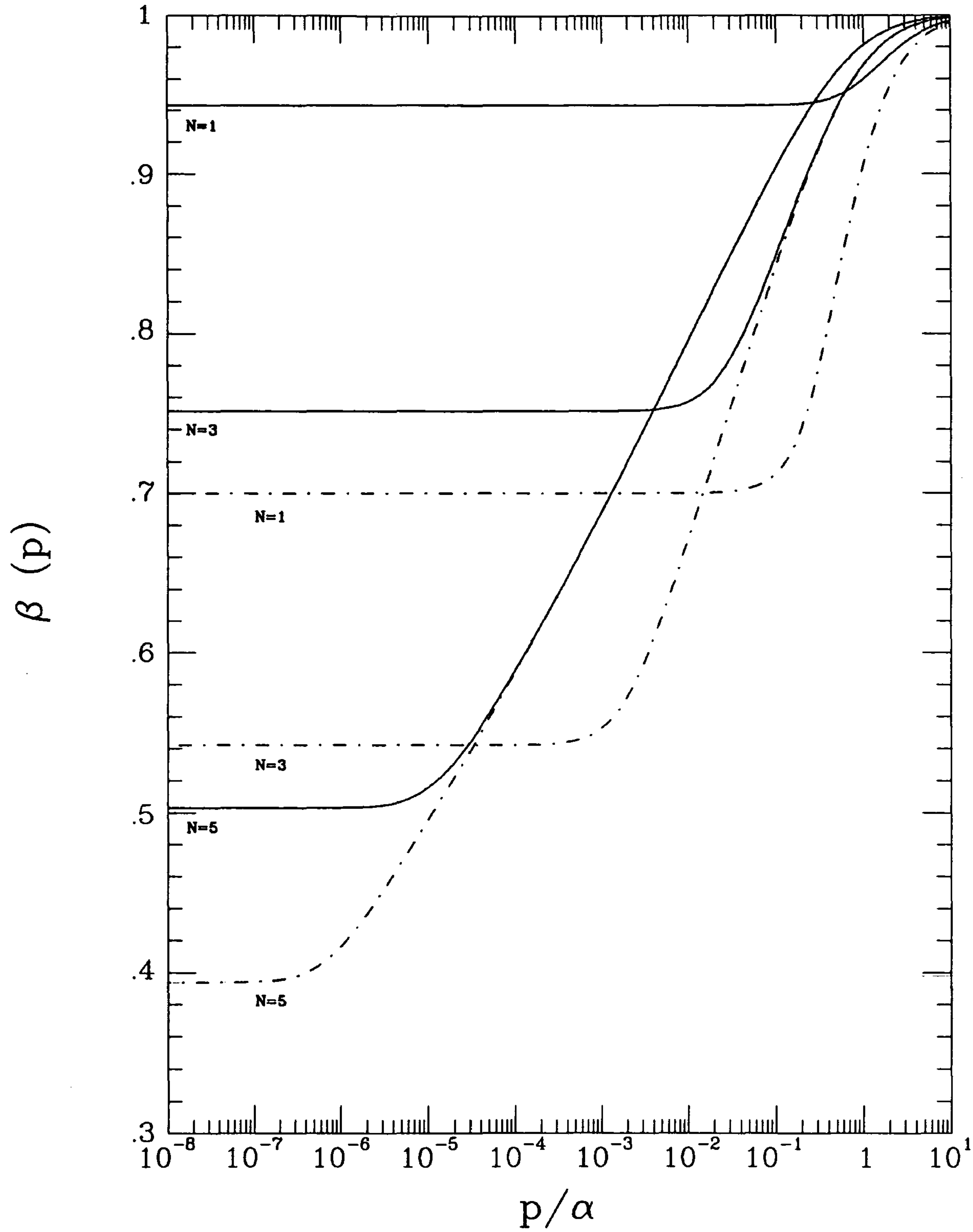

Figure 3.2 Developement of the Wavefunction renormalisation $\beta(p)$ with N for: a) The Massless Photon --- and b) The Massive Photon – . The full vertex being approximated by its "central" term in each case.

which upon simple rearangement yields the following relation,

$$\left[N + \int \frac{A(k,p)\, k\, dk}{\beta^2(k)k^2 + \Sigma^2(k)}\, \beta(k)\right] \frac{\partial}{\partial p} A(p) \sim \int_0^p \frac{(\beta(p) + \beta(k))\, \beta(k)}{\beta^2(k)k^2 + \Sigma^2(k)} \frac{k^5\, dk}{p^4 m} \tag{3.17}$$

Since all integrals involved are positive definite this implies that,

$$\frac{\partial}{\partial p} A(p) \gtrsim 0 \quad , \quad p \leq m \, . \tag{3.18}$$

It is a simple exercise to show that this is not altered by the lack of the k/m factor in the massless case, the analysis only depending on the positive definitness of the kernel. Moreover, a simple extension leads to the conclusion that (3.18) actually holds for all p. A similar reasoning approach to the relation for $\Sigma(p)$ leads to the condition,

$$\frac{\partial}{\partial p} \Sigma(p) \lesssim 0 \quad , \quad p \leq m \, . \tag{3.19}$$

Let us return to (3.15) and write the relation for $A(0)$ as,

$$\kappa\, N\, A(0) = \int \frac{k\, dk}{\beta^2(k)k^2 + \Sigma^2(k)}\, (\beta(0) + \beta(k))\, \beta(k)\, \mathcal{A}(k,p)$$

$$\mathcal{A}(k,p) = \theta(m-k)\, \frac{k}{m} + \theta(\alpha - k) \tag{3.20}$$

Where the factor κ contains all the numerical factors we have dropped and for $m \ll \alpha$ is expected to be weakly N dependent. Using the inequalities (3.18) and (3.19) we may place an upper bound on the value of $A(0)$,

$$\begin{aligned} \kappa\, N\, A(0) &\lesssim -2 \int \frac{k\, dk}{k^2 + m^2}\, \left(\theta(m-k)\, \frac{k}{m} + \theta(k - m)\right) \\ &= -\ \ln\left(\frac{\alpha^2 + m^2}{m^2}\right) \end{aligned} \tag{3.21}$$

For the massless case the factor k/m is simply absent leading to the similar result,

$$\kappa\, N\, A(0) \lesssim -\left\{ \ln 2 + \ln\left(\frac{\alpha^2 + m^2}{m^2}\right) \right\} \, . \tag{3.22}$$

If we now also introduce the physical constraint that $\beta(p)$ must be positive, $A(p) \geq -1$, then (3.21) and (3.22) yield simple lower bounds for the dynamically generated mass m in each case,

$$\frac{\alpha^2 + m^2}{m^2} \leq exp\,(\,\kappa N + \delta\,)\ .$$

Where the constant δ for the massless ansatz is zero and in the massive case is equal to $\ln 2$. This simple example illustrates how correctly incorporating the nonzero fermion mass into the photon introduces no new mechanisms, but leads only to a constant shift in the parameter N, and shows how the wavefunction renormalisation prevents the occurance of any critical behaviour. Only in a very perverse sense may the resulting enhancement of the dynamical mass be considered as a small perturbative correction,

$$m \sim \exp\,(\,-N/N_0\,) \rightarrow \exp\left\{-N\left(1+\frac{\delta}{N}\right)/N_0\right\}\ . \tag{3.23}$$

The very strong N dependence of the Schwinger Dyson system obviates many of our simple $1/N$ perturbative intuitions so let us now move out of the realm of a perturbative ansatz and consider the nonperturbative Schwinger Dyson equation for the photon.

3.3 The "Central" Photon.

In Section 2.4 of the previous chapter we saw how truncating at the photon-fermion vertex level naturally leaves us with the coupled fermion-photon system as in Fig. 3.3. In our original analysis the complexity of the coupled system forced us to truncate further and use a perturbative ansatz for the photon, simply throwing the photon relation away. In this section we will seek to redress the balance by analysing the photon equation in detail, in order to check the consistency of our perturbative ansatz and if necessary with the aim of coupling the two relations for the photon and fermion.

Let us start by discussing some of the properties of the full photon correction Fig. 3.4. Simple gauge invariance arguments demand that this correction be

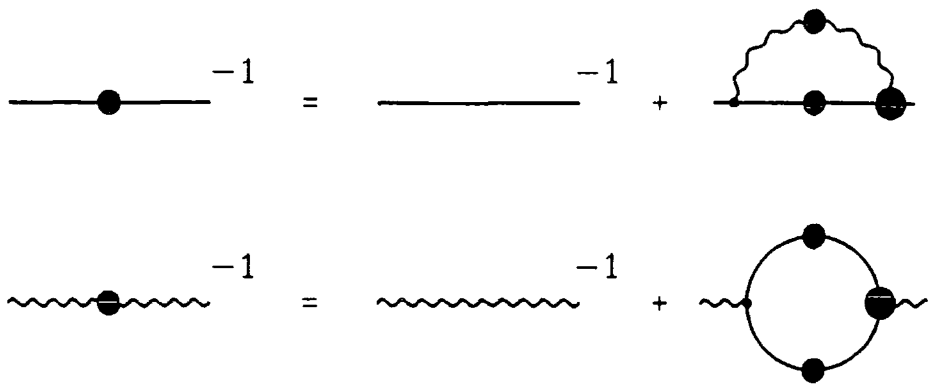

Figure 3.3 The truncated fermion-photon system obtained by simply replacing the full vertex by an ansatz.

purely transverse and in general the full photon may be written,

$$\mathcal{D}_{\mu\nu}(p) = \frac{1}{\mathcal{G}(p)} \left\{ \delta_{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right\} + \xi \frac{p^\mu p^\nu}{p^4} \ . \tag{3.24}$$

The explicitly gauge dependent longitudinal term receives no renormalisation corrections.

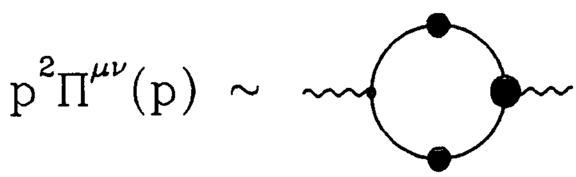

figure 3.4 The Full Photon Correction. The • indicate full verticies/propagators.

Let us consider the form of this longitudinal component, by projecting with the tensor $p^\mu p^\nu / p^2$,

$$p^2 \Pi^L_{\mu\nu}(p) \simeq \frac{1}{N} \int d^3k \; Tr \; \left[\not{p} S(k+p) \; (p \cdot \Gamma) \; S(k) \right] \ . \tag{3.25}$$

Using the Ward Identity we may rewrite this as,

$$p^2 \Pi^L_{\mu\nu}(p) \simeq \frac{1}{N} \int d^3k \; Tr \; \left[\not{p} (\, S(k) - S(k+p) \,) \; \right] \ . \tag{3.26}$$

Naively one might expect that this component is trivially zero, since we may simply shift the integration variable and cancel one component against the other for every value of k. However, as it stands this correction is not well defined. In terms of an ultraviolet cutoff it is linearly divergent and a simple shift in the integration variable for only part of the integrand is not allowed. Clearly, a momentum cutoff does not respect the gauge invariance of our theory and is therefore not a reliable regulator in this case. If instead one employs a regulator that does respect gauge invariance, such as Dimensional Regularisation or Pauli-Villars, then this contribution is indeed identically zero and the full correction purely transverse. Therefore the Ward Identity, itself a consequence of Gauge Invariance, ensures that this invariance is preserved at the level of the photon propagator.

However, there is a "real" divergence within the transverse component of the photon correction. We may discover the form of this ultraviolet divergence by simply considering the perturbative result,

$$\Pi_0^{\mu\nu}(p) \sim \alpha \int d^3k \; Tr \left[\gamma^\mu \frac{\not{k}}{k^2} \gamma^\nu \frac{(\not{k} + \not{p})}{k^2} \right] . \tag{3.27}$$

The divergent term is constrained to the term proportional to $\delta^{\mu\nu}$,

$$\Pi_0^{\mu\nu} \sim \alpha \, \Lambda \, \delta^{\mu\nu} . \tag{3.28}$$

and in this case there is no possiblity of any cancellations occurring to reduce this to a finite answer, as in $\Pi^L_{\mu\nu}(p)$. However, if we assume that gauge invariance is preserved then under a rigorous regularisation and renormalisation programme this contribution will yield a term such that the total correction is indeed transverse and finite. Rather than worry about the details of such a procedure we may simply project out the component of $p^\mu p^\nu / p^2$ using the tensor,

$$\mathcal{P}^{\mu\nu} = \delta^{\mu\nu} - 3 \, \frac{p^\mu p^\nu}{p^2} . \tag{3.29}$$

to find $\Pi(p)$. The value of this component is finite and will be invariant under the renormalisation procedure. Consequently, we need not carry out any such renormalisations explicitly. The action of $\mathcal{P}^{\mu\nu}$ on $\Pi_{\mu\nu}$ is simply,

$$\Pi_{\mu\nu} = \Pi(p)\left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) \longrightarrow \mathcal{P}^{\mu\nu}\,\Pi_{\mu\nu} = 2\,\Pi(p)\ . \tag{3.30}$$

and the full form the photon correction is given by,

$$\Pi_{\mu\nu} = (-)\,N\int\frac{d^3k}{(2\pi)^3}\,Tr\,\left[\,(-ie\gamma^\mu)\,S(k)\,(-ie\Gamma^\nu)\,S(q)\,\right]\ . \tag{3.31}$$

The factor of N appears because we have summed over all fermion flavours, the contributions from each being obviously identical. Projecting with the tensor $\mathcal{P}^{\mu\nu}$ we may write the photon correction $\Pi(p)$ as,

$$\Pi(p) = -\frac{Ne^2}{2(2\pi)^3}\int\frac{k^2\,dk}{\beta^2(k)k^2+\Sigma^2(k)}\int\frac{\mathcal{M}_{\mu\nu}\mathcal{P}^{\mu\nu}}{\beta^2(q)q^2+\Sigma^2(q)}\,d\Omega$$

with,

$$\mathcal{M}^{\mu\nu} = Tr\,\left[\gamma^\rho(\beta(k)\not{k}-\Sigma(k))\,\Gamma^\mu(\beta(q)\not{q}-\Sigma(q))\right]\quad,\quad q=k-p \tag{3.32}$$

splitting the phase space integral into magnitude and angular components as usual.

Before undertaking the calculation of the photon function from this relation using the full Ball-Chiu form let us study a simpler example, that of the "central" vertex. This is not only helpful in introducing both notation and method without becoming bogged down in opaque details, but by using this vertex we will be able to judge the consistency of the previous calculation that employed the"massive" photon ansatz. If the fermion functions produced by that analysis show in (3.32) that the use of the perturbative ansatz was inadequate, then we will be forced to solve the coupled photon-fermion system in its entirety.

Let us collect the vector functions formed by the product $\mathcal{P}^{\mu\nu}\,\mathcal{M}_{\mu\nu}$ and define a kernel $\mathcal{T}_c$ as,

$$-4\mathcal{T}_c = \mathcal{P}_{\mu\nu}\,Tr\,\left[\gamma^\mu\,\left(\beta(k)\,\not{k}-\Sigma(k)\right)\,\gamma^\nu\,\left(\beta(q)\,\not{q}-\Sigma(q)\right)\,\right]\,\mathcal{C} \tag{3.33}$$

with

$$\mathcal{C} = \frac{1}{2}\,\{\beta(q)+\beta(p)\}\ . \tag{3.34}$$

The gamma matrix traces involved are straightforward so we will just present the result,

$$\mathcal{T}_c = 2\,\beta(k)\beta(q)\mathcal{C}\left\{k^2 + 2\,k\cdot p - 3\,\frac{(p\cdot k)^2}{p^2}\right\}. \tag{3.35}$$

In analogy to all our previous calculations we will rewrite the photon function relation as,

$$\Pi(p) = -\frac{2\alpha}{\pi^2}\int \frac{k^2\,dk}{\beta^2(k)k^2 + \Sigma^2(k)}\,\mathcal{A}_c(k,p)$$

defining the angular function $\mathcal{A}_c$ from the kernel (3.35) by,

$$\mathcal{A}_c(k,p) = \int \frac{\mathcal{T}_c\,dz}{\beta^2(q)q^2 + \Sigma^2(q)} \tag{3.36}$$

where $z = \cos\theta$ and θ is the angle between q and k as usual. Obviously the analytic forms of the fermion functions $\beta(p)$ and $\Sigma(p)$ are unknown, so this integration over z will have to be performed numerically in most cases, but not all. In the regions where $p \gg k$ or $k \gg p$ then we may Taylor expand (3.36) in terms of the ratios k/p or p/k respectively. The integration over z then becomes trivial and we may obtain a good approximations to $\mathcal{A}_c$ in both these regions.

We may define an expansion for each fermion function, say $\beta(q)$, in the following manner,

$$\beta(q) = \beta(L) + \beta_1\,\delta + \beta_2\,\delta^2 + \beta_3\,\delta^3 + \cdots \tag{3.37}$$

with

$$l = \text{Min}\,(k,p) \quad , \quad L = \text{Max}\,(k,p) \quad \& \quad \delta = \frac{l}{L}$$

$$\beta_i(L) = \frac{\partial}{\partial \ln q}\,\beta(q)\Big|_{q=L}$$

for which we will only need the first three terms of such an expansion in all the examples to come. Using this definition, the Taylor expansion for the denominator term in (3.36) may be written,

$$\left(\beta^2(q)q^2 + \Sigma^2(q)\right)^{-1} = d_0 + d_1\,\delta\,z + \delta^2\,\left(d_2 + d_3\,z^2\right) + \cdots$$

where the coefficients d_i are functions of L only and are,

$$\begin{aligned}
d_0 =& \ \{\,\beta^2(L)L^2 + \Sigma^2(L)\,\} \\
d_1 =& \ 2\,d_0^2\,\{\,\beta(L)\,(\,\beta(L) + \beta_1(L)\,)\;L^2 + \Sigma(L)\,\Sigma_1(L)\,\} \\
d_2 =& \ -\,\frac{d_1}{2} \\
d_3 =& \ d_0^2\Big\{\Sigma(L)\,(\,2\,\Sigma_1(L) - \Sigma_2(L)\,) - \Sigma^2(L) + \\
& \quad L^2\;(\,\beta_1^2(L) - 2\,\beta(L)\beta_1(L) - \beta(L)\beta_2(L)\,)\Big\} + d_1^2/d_0\;.
\end{aligned} \tag{3.38}$$

We may employ the above approximations to calculate the contribution from this central term to the full angular integral in the asypmtotic regions for which $\delta \ll 1$. Note that in contrast to the example of the fermion equation's angular functions, these are not symmetric in k and p, but rather in k and q. Therefore,

$$\begin{aligned}
A_c =& \ p^2\Big\{-\frac{4}{5}\,d_0\;(\,\beta^2(3\,\beta_1 + \beta_2) + 2\,\beta_1^2\,\beta)\,) \\
& \quad + \frac{8}{3}\,d_1\,\beta^2\left(\beta + \frac{3}{5}\,\beta_1\right) - \frac{16}{15}\,d_3\,\beta^3\; + O(\delta)\Big\} \qquad k \gg p \\
A_c =& \ \frac{4}{3}\,k^2\Big\{-\,d_0\;(\,2\,\beta + \beta(k)\,)\;\beta(k)\;\beta_1 \\
& \quad + d_1\,\beta(k)\,\beta\;(\,\beta(k) + \beta\,) + O(\delta)\Big\} \qquad p \gg k.
\end{aligned} \tag{3.39}$$

Since in the region $k \gg p$, $q \to k$, this function has a particularly simple form,

$$A_c \sim p^2\,f(k) \quad , \quad p \ll k \tag{3.40}$$

where the function f is completely independent of the value of p, as expected from the form of (3.35). Also for k large , $f(k) \sim k^{-2}$, which ensures that the full photon function is convergent, as it must be. The corresponding approximation in the region $p \gg k$, is not so simple as the function $f(p)$ is now dependent on the values of $\Sigma(k)$ and $\beta(k)$, though the explicit k^2 dependence is expected to be the dominant scaling,

$$\mathcal{A}_c \sim k^2 f(p, \beta(k), \Sigma(k)) \quad , \quad k \ll p \, . \tag{3.41}$$

We are now ready to calculate the photon correction for this simple "central" vertex. We will use the fermion functions obtained in our previous "massive" analysis as input and so will be able to check the consistency of that purely perturbative ansatz.

Both the "massive" ansatz used in the previous chapter and the photon function derived from this simplified photon Schwinger Dyson equation are plotted on a log-log scale in Fig 2.5 for $N = 3$. It is clear that the qualitative softening of the perturbative ansatz is well reproduced in the "central" photon but to compare their differences more closely the ratio of this "central" photon function $\mathcal{G}_c(p)$ and the "massive " photon $\mathcal{G}_m(p)$ is shown in Fig 2.6.

While the variation of this ratio is not perturbative, *i.e.* $1 + O(\epsilon)$, $\epsilon \ll 1$ it is really only the qualitative behaviour we are interested in at this point and that is well reproduced. Therefore, if we had studied the coupled photon-fermion Schwinger Dyson equations, for this simple vertex, we would expect to get results that were qualitatively the same as those of our "massive" analysis of Section 3.2. This qualitative agreement between our "full" photon function and the perturbative result should not come as a complete surprise. The "massive" ansatz was computed using a bare vertex for which the "central" vertex is a mere embellishment and assuming a well behaved $\beta(p)$ then shouldn't both be identical up to factors of O(1)? We will postpone a discussion of what is and what is not qualitatively important in the photon until we have performed the same analysis using the more realistic Ball-Chiu form.

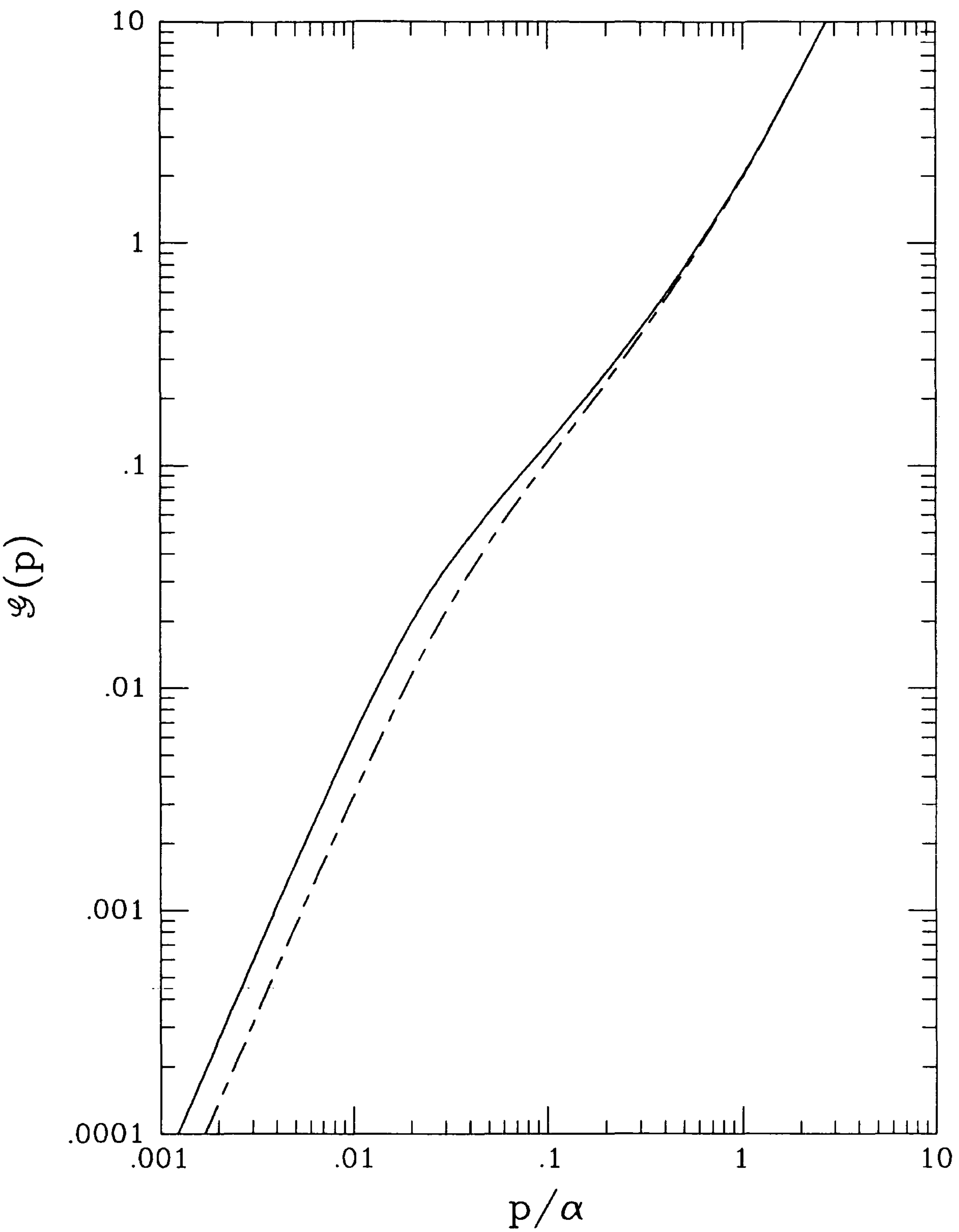

Figure 3.5 The "central" and "Massive" Photon functions for N=3. Both have been constructed using the fermion functions obtained by the "hard" photon analysis of Sec 3.2. In each Schwinger Dyson relation the full vertex has been replaced by its "central" term only.

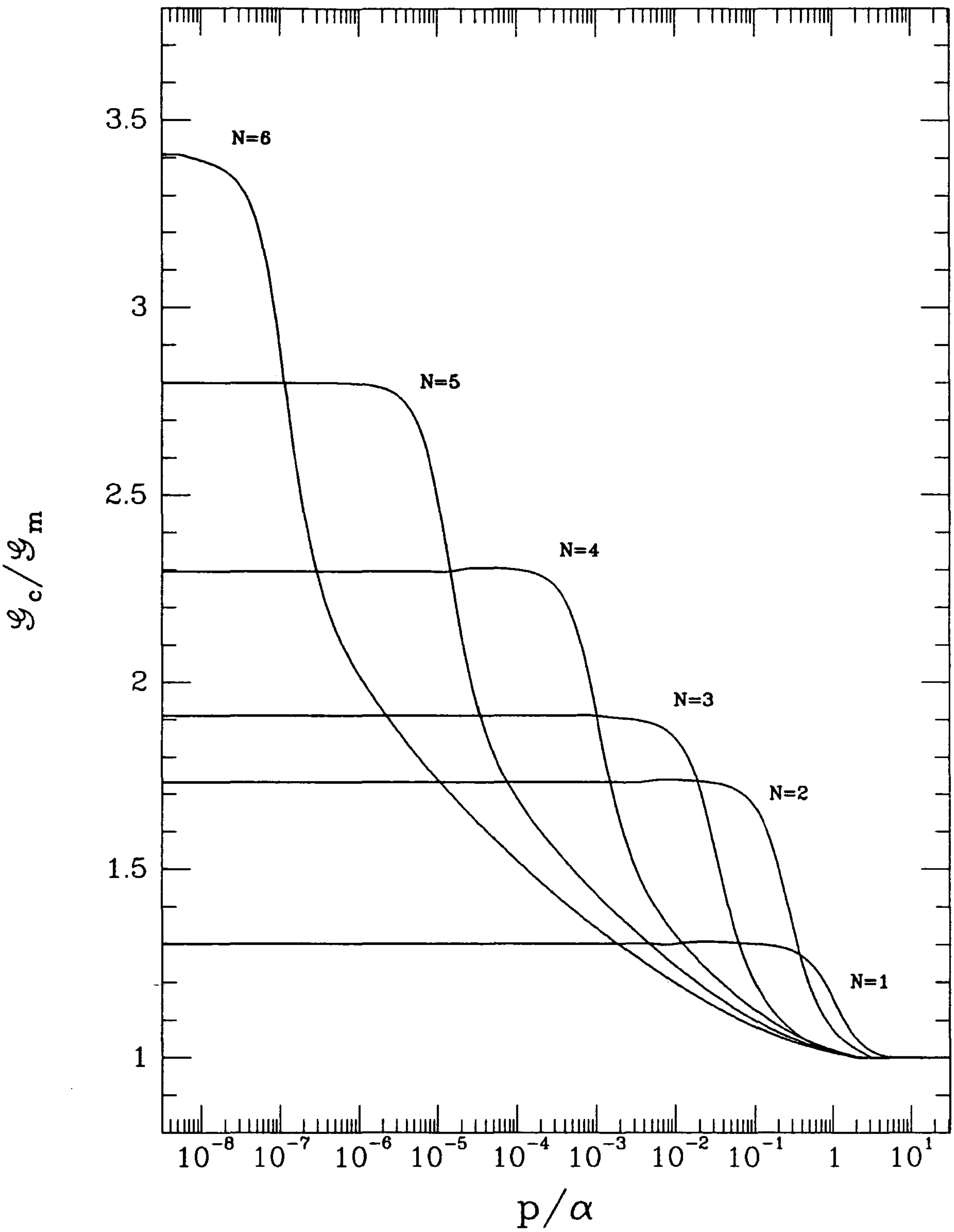

Figure 3.6 The ratio of the two photon functions "central" over "massive" showing the qualitative aggreement across all scales and for of N=1,6

3.4 THE "FULL" PHOTON.

The Ball-Chiu vertex is far more complicated than its central term. The "gradient" pieces introduce new spinor forms and, via finite differences, introduces the overall variation of the fermion functions directly. While in Chapter 2.12 it was shown that, under the simplification of a "massless" photon, these two vertices produce the same large N scaling for the dynamical mass, the quantitative differences are far from trivial. Comparing the "massive" and "full" functions in Fig 2.6 suggests that the residual N dependence, even after factoring out some of the dependence on m, is far from mild. Just as the Ball-Chiu ansatz in the fermion equation suggested that the dependence of the dynamical mass, m, on N was exponential for all N, so will it also here suggest a similarly simple N dependence for the low p scaling constant in (3.4)? Or will the simple p^{-2} scaling for $p \ll m$ be completely altered by the inclusion of these new terms.

Let us replace the full vertex in (3.32) by the Ball-Chiu form using the following shorthand,

$$\Gamma^{\mu}_{BC} = \mathcal{C}\,\Gamma^{\mu} + \mathcal{S}\,(\not{k} + \not{q})\,(k+q)^{\mu} + \mathcal{V}\,(k+q)^{\mu} \; . \tag{3.42}$$

where $\mathcal{S}$ and $\mathcal{V}$ are given by,

$$\mathcal{V} = \frac{\Sigma(q) - \Sigma(k)}{q^2 - k^2} \quad , \quad \mathcal{S} = \frac{1}{2}\,\frac{\beta(q) - \beta(k)}{q^2 - k^2} \; . \tag{3.43}$$

In the same way as we defined $\mathcal{T}_c$, let us construct the corresponding kernels $\mathcal{T}_s$ and $\mathcal{T}_v$,

$$\begin{aligned} -4\mathcal{T}_s &= \mathcal{P}_{\mu\nu}\,Tr\,\left[\gamma^{\mu}\,\left(\beta(k)\not{k} - \Sigma(k)\right)\,(\not{k} + \not{q})(k+q)^{\mu}\,\left(\beta(q)\not{q} - \Sigma(q)\right)\,\right]\,\mathcal{S} \\ -4\mathcal{T}_v &= \mathcal{P}_{\mu\nu}\,Tr\,\left[\gamma^{\mu}\,\left(\beta(k)\not{k} - \Sigma(k)\right)\,(k+q)^{\mu}\,\left(\beta(q)\not{q} - \Sigma(q)\right)\,\right]\,\mathcal{V} \; . \end{aligned} \tag{3.44}$$

After evaluating the gamma matrix traces these become,

$$\begin{aligned}
\mathcal{T}_s =&\beta(k)\beta(q)\mathcal{S}\left\{4k^4 + 2p^2k\cdot p + 2(k\cdot p)^2 + \right.\\
&\qquad\qquad \left. 4k^2p\cdot k - \frac{12}{p^2}\left(k^2(p\cdot k)^2 - (p\cdot k)^3\right)\right\}\\
&+\Sigma(k)\Sigma(q)\mathcal{S}\left\{4k^2 - 2p^2 + 8p\cdot k - 12\frac{(p\cdot k)^2}{p^2}\right\} \qquad (3.45)\\
\mathcal{T}_v =&2\beta(k)\beta(q)\mathcal{V}\left\{\frac{3}{p^2}(k\cdot p)^2 - k^2 - p\cdot k\right\}\\
&+2\beta(q)\Sigma(k)\mathcal{V}\left\{\frac{3}{p^2}(k\cdot p)^2 - k^2 + p^2 - 3p\cdot k\right\}.
\end{aligned}$$

In a completely analogous way to the previous central case we will define two more angular functions $\mathcal{A}_s$ and $\mathcal{A}_v$ via,

$$\mathcal{A}_\# = \int \frac{\mathcal{T}_\#\, dz}{\beta^2(q)q^2 + \Sigma^2(q)} \qquad (3.46)$$

For exactly the same reasons as for the central function $\mathcal{A}_c$ we are only able to perform the integration over z analytically in the asymptotic regions $k \gg p$ and $p \gg k$, in all the cases that fall in between it must be calculated numerically. Indeed even the complexity of the asymptotic approximations necessitated the use of the algebraic manipulation package SMP to determine them. We will not present them here (they may be found in Appendix I),because of their length and unrevealing nature, but will illustrate their general forms.

In the limit of $p \ll k$ the general form of (3.40) is reproduced, as it must since p falls smoothly out of all the kernels. The behaviour at the other extreme, however, is quite different from that of (3.41) and perhaps not what we might have expected,

$$\mathcal{A}_\# \sim d_0\ \{h_1(p) - h_1(k)\}\ \Sigma(k)h_2 + k^2\cdots \quad , \quad k \ll p \qquad (3.47)$$

where $h_{1(2)} = \Sigma(p)\,(\,\beta(p)\,)$. Therefore, the gradient terms enter, but at a higher level than we might have expected, the cancellations that occured in the central function which explicitly lowered the most significant terms to $O(k^2/p^2)$ do not occur here. Consequently, the influence of the gradient terms cannot be considered as a mere perturbation on any dominant, "central" behaviour.

To determine the photon function $\mathcal{G}(p)$ we need the two fermion functions $\Sigma(p)$ and $\beta(p)$ and to be consistent these should come from an extension of the "massive" analysis of Section 3.2, employing the full Ball-Chiu ansatz. However, this calculation has not been performed at this stage and in its place we will take the fermion functions of Chapter 2. These were determined from the system containing the full vertex but a "massless" photon. We will return later to any possible inconsistency of this replacement when we come to discuss and identify what are the determining factors in the photon's behaviour.

The calculation of $\mathcal{G}(p)$ is computationally straightforward though in the region $p \ll m$ it is prone to numerical errors that arise naturally from dealing with discretised solutions for $\Sigma(p)$ and $\beta(p)$. The photon correction $\Pi(p)$ was calculated for the $N = 2, 4, 6$ and 8, the ratio of this "full" correction and its perturbative approximation being given in Fig. 2.7. Let us review its general properties and compare them with those of the "central" and perturbative versions.

In the high momentum region $p \gg \alpha$ both the mass scale, m, and the natural scale α, are lost and the photon has the standard bare behaviour $\mathcal{G}(p) \sim p^2$. As we lower p towards and finally below α, but let us assume well above the dynamical scale m, the photon is softened by effectively massless fermions,

$$\mathcal{G}(p) \simeq \alpha p \quad , \quad m \ll p \ll \alpha \, . \tag{3.48}$$

This is what we expect from the rather simple scaling arguments put forward in Section 2.4. However, as we lower p still further the dynamical scale m does make itself felt. In comparison to the "central" correction this Ball-Chiu result displays a significant enhancement at scales $p \sim O(m)$, though the absence of this behaviour from the " central" and perturbative forms is not entirely surprising. The "gradient" terms introduced by the full Ball-Chiu ansatz are far more sensitive to the changes of scaling occuring at the approach to the dynamical

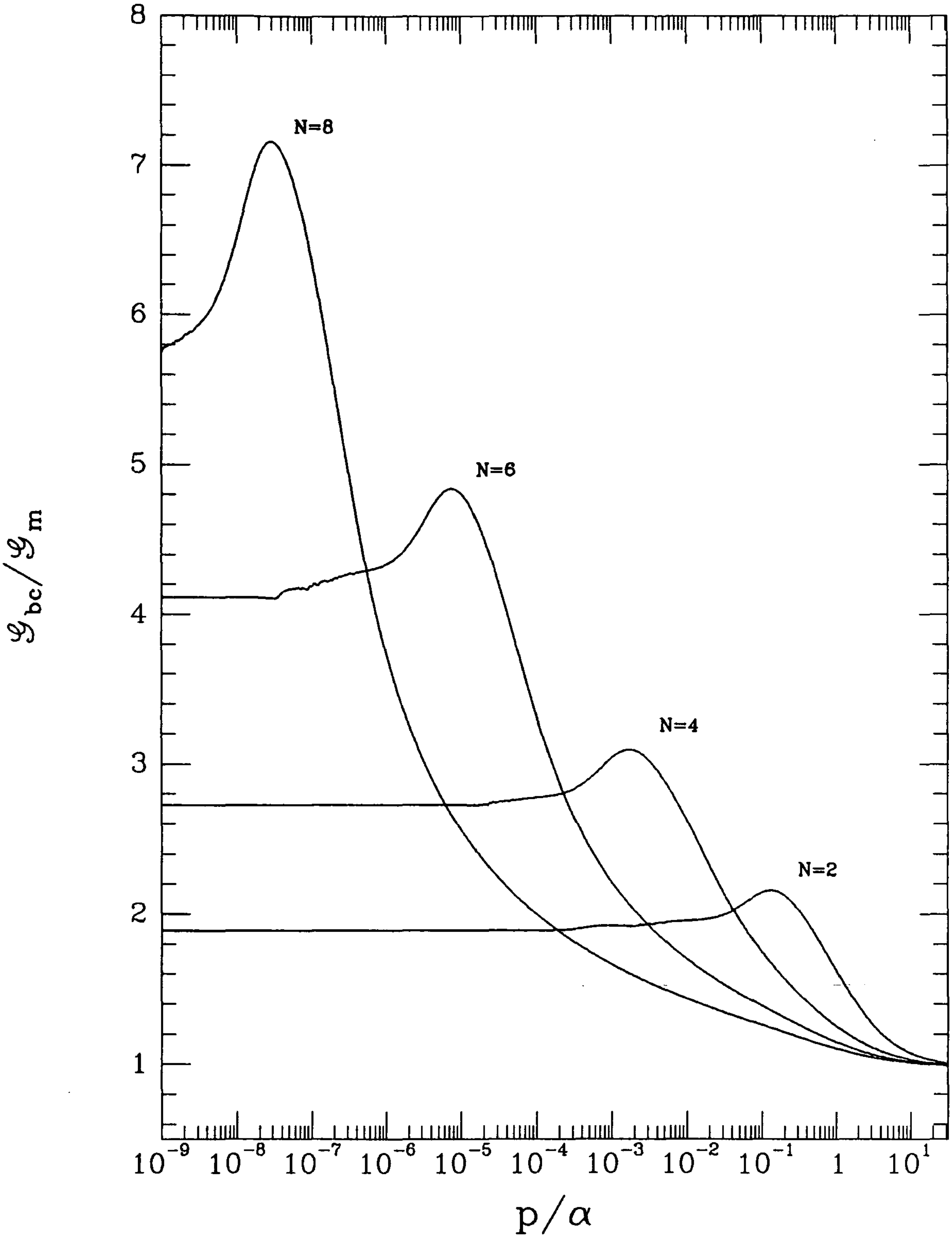

Figure 3.7 The ratio of the "Full" photon correction determined using the Ball-Chiu ansatz over the perturbative ("massive") result, showing the qualitative agreement across all scales and the N values 2,4,6,8.

mass scale than the simple scalar vertex term. Physically, at this momentum scale the photon may disassociate into two nearly on-shell fermions and it is this process, in conjunction with the dynamical generation of a fermion mass, that we expect is being signalled by this resonance-like behaviour. Contrary to a simple scattering amplitude this is an example of a destructive interference as the photon propagator is proportional to $\mathcal{G}^{-1}(p)$ and not $\mathcal{G}(p)$.

As we finally lower p below m we observe that the fermions do decouple and the photon returns to its bare scaling behaviour,

$$\mathcal{G} \simeq \mathcal{F} p^2 \quad , \quad p \ll m \,. \tag{3.49}$$

The factor $\mathcal{F}$ contains the effects of the photon's now virtual interactions with the massive fermions and is constant up to factors of $O(\,p/m\,)$.

All our analyses have so far produced results with the same qualitative features, leading to the conclusion that the photon contains no surprises and the use of a "massive" ansatz would seem to be consistent if not entirely quantitatively accurate. It is time that we returned to the analytic form of the photon correction in order to determine why, in this example, $1/N$ perturbation theory is accurate? Indeed we may reverse this question and ask — could perhaprs higher $1/N$ corrections upset these simple scalings?

Let us write the relation for the photon correction $\Pi(p)$ as,

$$\Pi(p) \simeq \alpha \int \frac{k^2\, dk}{\beta^2(k)k^2 + \Sigma^2(k)}\, \mathcal{A} \,. \tag{3.50}$$

and in place of its general form in terms of angular integrals let us replace $\mathcal{A}$ by its analytic approximations in the two regions, $p \geq k$ and $k \geq p$, $\mathcal{A}_l$ and $\mathcal{A}_h$ respectively,

$$\mathcal{A}(p) \sim \; \theta(k-p)\, \mathcal{A}_h + \theta(p-k)\, \mathcal{A}_l \,. \tag{3.51}$$

The full forms of $\mathcal{A}_l$ and $\mathcal{A}_h$ are given in Appendix 1 and, as mentioned before, are very lengthy. However, most of this detail is superfluous and not responsible for the qualitative features illustrated by our numerical results. Consider the expansion of the denominator in the cases where $k \gg p$, that defines the coefficients d_i,

$$\left(\beta^2(k)k^2 + \Sigma^2(k)\right) = d_0 + \delta\, d_1\, z + \delta^2\, \left(\, d_2 + d_3\, z^2\,\right) + \cdots \tag{3.52}$$

Let us approximate all these coefficents by d_0 , this is reasonable if we assume that,

$$\left.\begin{array}{ll} \Sigma_i(p) & \sim \Sigma(p) \\ \beta_i(p) & \sim O(1) \end{array}\right\} \longrightarrow\ d_i \sim O(d_0)\ . \tag{3.53}$$

These requirements are clearly satisfied in each asymptotic region,

$$\Sigma(p)\ \sim \left\{\begin{array}{c} p^{-2} \\ \text{constant} \end{array}\right.,\ \beta(p)\ \sim \left\{\begin{array}{cc} 1 + c\,\ln(p/\alpha) & p \gtrsim m \\ \text{constant} & p \lesssim m \end{array}\right. \tag{3.54}$$

Therefore, in the region $k \leq m$ the gradient terms will be down by implicit factors of k/m whereas in the region $k \geq m$ all the terms containing the dynamical mass will be overtaken by explicit factors of k/m. If we make these same approximations to the asymptotic relations for the angular integrals given in Appendix 1, we are left with the general forms,

$$\begin{array}{ll} p \gg k\ :\ \ \mathcal{A}_l \simeq k^2\, d_0(p) \left\{\, f_1 + f_2 \dfrac{m^2}{p^2}\, \right\} + m\,(\,m(k) - m\,)\, f_3\, d_0(p) \\ k \gg p\ :\ \ \mathcal{A}_h \simeq p^2\, d_0(k) \left\{\, f_1' + f_2' \dfrac{m^2}{k^2}\, \right\}\ . \end{array} \tag{3.55}$$

The function f_1 originates from the central vertex whereas both f_2 and f_3 come from the gradient terms contained within the Ball-Chiu vertex form. All three functions are expected to be smoothly varying and of $O(\leq N)$. Let us examine the case for which $m \ll p \ll \alpha$ when we may write,

$$\begin{aligned} \Pi_l(p) &\sim \frac{\alpha}{p^2} \int\limits_0^p \frac{k^2\, dk}{k^2 + \Sigma^2(k)} \left(k^2 + \frac{m^2(k)}{p^2} \right) + m(k)(\,m(k) - m(p)\,) \\ &\quad + \alpha p^2 \int\limits_p^\infty \frac{dk}{k^2 + \Sigma^2(k)} \left\{\, f_1' + f_2' \frac{m^2}{k^2} \right\} \\ &\sim \frac{\alpha}{p^2} \int\limits_0^p k^2\, dk \left(1 + \frac{m^2(k)}{k^2} \right) + \alpha\, p^2 \int\limits_p^\infty \frac{dk}{k^2} \left(1 + \frac{m^2(k)}{k^2} \right) \\ &\sim\ \alpha p\ +\ O(\,\alpha m\,) \qquad\quad , \qquad\quad p \gg m\ . \end{aligned} \tag{3.56}$$

Notice how the behaviour of the mass is largely irrelevant, even if we chose the worst possible case for the potentially troublesome term $m(k) - m(p)$, constant and of $O(m)$. This insensitivity is also a feature of the $p \ll m$ case where,

$$\begin{aligned} \Pi(p) &\sim \frac{\alpha}{m^2}\left\{\left(1+\frac{m^2}{p^2}\right)p^5 + p^3\right\} + \frac{\alpha}{m}p^2 \\ &\sim \frac{\alpha}{m}p^2\left(1+O(p/m)\right) . \end{aligned} \tag{3.57}$$

The mass function here is acting as little more than a dynamically generated infrared cutoff for the softening effects of the fermion contributions. The basic scaling properties of $\mathcal{G}(p)$ are a consequence of the dimensionality of the phase-space integral alone. We will return to this point and contrast with the properties of the corresponding four dimensional correction after we have discussed the origin of the uniformly increasing discrepancy of $O(N)$ between the "full" and perturbative forms, see Fig. 3.7.

For simplicity, consider the simpler "central" correction in comparison to the perturbative correction as illustrated by the ratio given in Fig 3.6. Notice how the discrepancy smoothly increases as we lower p until at low momentum scales, $p \leq O(m)$, it is of $O(N)$. Contrast this behaviour with what we might expect from a naive $1/N$ analysis. We have calculated the first $1/N$ perturbative correction and we would expect the general i^{th} order correction to come in with well ordered damping factors of $O(1/N^i)$, indeed nothing explicit in the form of the photon correction suggests that anything to the contrary is to be expected,

$$\mathcal{G}(p) = \mathcal{G}_m(p) + \frac{1}{N}\mathcal{G}_1(p) + \frac{1}{N^2}\mathcal{G}_2(p) + \cdots . \tag{3.58}$$

Where the scaling behaviour of each of the $\mathcal{G}_i(p)$ will be determined, as in the "full" results, by the tight dimensional contraints on the form of the correction, $\Pi(p)$. Therefore as N increases we would expect our "full" result to agree with its perturbative version with ever increasing accuracy. On the contrary, rather than the higher order corrections terms leading to smaller and smaller corrections there appears to be a nonperturbative generation of an overall factor of N. This is not an unfamiliar phenomenom, it previously occured in the wavefunction renormalisation correction. In that scenario it raised the significance of $A(p)$ from a perturbative $O(1/N)$ correction to an $O(1)$ nonperturbative effect responsible for supporting the dynamical generation of fermion mass across all values of the "coupling", $1/N$. This similarity is not at all coincidental and may be understood in terms of a well known approximation due to Mandelstam[24].

Assume for the time being that our fermions are essentially massless. Then the Mandelstam approximation states that we may effectively cancel the renormalisation factor associated with the full vertex with that of one of the full fermions. We may only do this in a gauge theory, where the renormalisation factor associated with the fermion propagator is linked to that of the vertex through a Ward-Takahashi or Slavnov-Taylor relation. In QED the differential Ward identity requires that,

$$\Gamma^\mu = \frac{\partial}{\partial p} S^{-1}(p) \sim \beta(p)\,\gamma^\mu + \cdots \ . \tag{3.59}$$

and although this is obviously true only for $p = k$, in addition to ignoring the complicating gradient term as given in Section 2.6, it captures a qualitative dependence on β. The condition that the fermions be essentially massless ensures that this cancellation is not obstructed by any mass factors.

Let us then employ this approximation in a simplified calculation for $\Pi(p)$,

$$\begin{aligned} \Pi(p) &= \frac{\alpha}{(2\pi)^3} \int d^3k\, \mathcal{P}^{\mu\nu}\, Tr\,\left[\gamma^\mu S(k)\,\Gamma^\nu\, S(q-p)\right] \\ &\sim \alpha \int d^3k\, \mathcal{P}^{\mu\nu}\, Tr\,\left[\gamma^\mu S_0(k)\,\gamma^\mu S_0(q-k)\right] \ . \end{aligned} \tag{3.60}$$

$S_0(p)$ is the bare fermion propagator $\sim 1/\not{k}$. Inserting the appropriate asymptotic approximations for $\mathcal{A}$, (3.40) and (3.41), in the appropriate regions yields,

$$\Pi(p) \sim \alpha \int \frac{k^2\,dk}{k^2 + m^2(k)} \frac{1}{\beta(k)} \left\{ \frac{p^2}{k^2}\,\theta(k-p) + \frac{k^2}{p^2}\,\theta(p-k) \right\} \ .$$

The dominant contributions to $\Pi(p)$ occur for $k \sim O(p)$ and so we expect the naive perturbative, $\beta(p) = 1$, result to be effectively scaled by a factor of $\overline{\beta^{-1}}(p)$,

$$\Pi(p) \sim \alpha\, p\, \overline{\beta^{-1}}(p) \quad , \quad p \gg m \; . \tag{3.61}$$

where $\overline{\beta^{-1}}(p)$ is some local average of $\beta^{-1}(p)$. If the corrections to $\beta(p)$ were well ordered in $1/N$ then we would expect these to generate similarly well ordered corrections to $\Pi(p)$. However, we know that the corrections to $\beta(p)$ are not of $O(1/N)$ but rather for large N and small momenta are of $O(1)$ leading to a behaviour for $\beta(p)$ closer to $\beta(p) \sim O(1/N)$ than $\beta(p) \sim 1 + O(1/N)$. We would therefore expect for large N an enhancement in the photon correction of $O(N)$ in comparison to the perturbative result. If we consider the corrections generated by the central vertex and Ball-Chiu vertices we see that this qualitaively describes the features of the numerical results, Fig. 2.5 and Fig 2.7.

Now let us contrast this rather simple behaviour of our 3D photon with a similar calculatation of it's 4D counterpart. We would expect the scaling of the angular integral $\mathcal{A}$ to be essentially the same as in 3D because the dimensional factors in $d\Omega$ only result in simple constants for $k \gg p$ †,

$$\mathcal{A} \sim \theta(k-p)\,\frac{p^2}{k^2}\, f_1(k) + \theta(p-k)\,\frac{k^2}{p^2}\, f_2(p) \; . \tag{3.62}$$

The functions f_i will depend on β and Σ and keeping only the leading log terms we may write a general f_i as,

$$f_i(k) \simeq 1 + c_1\, e^2 \,\ln \frac{\Lambda^2}{k^2} + c_2\, e^4 \,\ln^2 \frac{\Lambda^2}{k^2} + c_3\, e^6 \,\ln^3 \frac{\Lambda^2}{k^2} + \cdots \; . \tag{3.63}$$

where Λ is the ultraviolet cutoff introduced to regulate the theory. In contrast to the 3 dimensional model no constraints are imposed on the scaling behaviour of $\mathcal{G}(p)$ via the dimensionality of spacetime. In 4D the coupling is dimensionless and, as remarked earlier, the dimensionality of α is effectively transferred to the magnitude part of the phase space integration, $\alpha\, k^2\, dk \to e^2\, k^2\, dk^2$. Let us

† Assuming no cancellations due to symmetry. This will be true for $\mathcal{G}$ but turns out not to be true for $\beta(p)$ as shall be discussed in the following chapter.

consider the effect of this on $\Pi(p)$, concentrating on the ultraviolet contributions only,

$$\begin{aligned} \Pi(p) &\sim e^2 \int_p^\infty \frac{k^2\,dk^2}{k^2+m^2}\,\frac{p^2}{k^2}\,f_1(k) \\ &\sim e^2 \int_p^\infty d\ln\frac{\Lambda^2}{k^2}\,f_1(k) \end{aligned} \tag{3.64}$$

Consider introducing an approximation to f_1, (3.63), that has been truncated to some finite order in e^2. The action of the measure in (3.64) is simply to boost each term in f_1 by a factor of $e^2\ln$. Returning Π to the fermion equation would result in a form for f_1 of one order higher than the original. This inductive generation of higher order corrections is simply perturbation theory in disguise. This boosting of corrections will also take place in the three dimensional version, but because of dimensional constraints we knew these all had the same basic scaling properties. This is not the case in 4D as is obvious from (3.63) ,we expect each ascending order to scale with every increasing powers of $\ln(\Lambda^2/k^2)$ and we will therefore need to consider the dominant corrections at all orders before we may deduce anything of the photon's nonperturbative behaviour. To summarise we may say that in contrast to a renormalisable theory the dimensionality of the coupling in QED3 constrains the basic scaling properties of the photon to the following simple form,

$$\mathcal{G}(p) \sim \beta^{-1}(p)\theta(p-m)\left\{\,p^2+\alpha p\,\right\} + \theta(m-p)\,\beta^{-1}(p)\,\frac{\alpha}{m}\,p^2 \tag{3.65}$$

In this section we have found that the full photon function is expected to have the general form given by the first order perturbative result, Eqn. 3.2, up to factors of $O(N)$. We have also seen in Section 3.2 how corrections expected to be of $O(m/\alpha)$ actually turned out to be of $O(1)$ via the exponention so characteristic of this model. We therefore cannot ignore the enhancement in $\mathcal{G}(p)$ in the same region in the hope that it is merely an unimportant effect. The following section will detail a numerical investigation of the coupled photon-fermion system in order to discover if indeed it is a small effect.

3.5 The coupled Fermion-Photon System.

The appearence of another factor of N within our system cannot be treated lightly. Naively we might think that this factor will make little difference compared with the observed exponential scaling,

$$N^{-1}\exp\{N/N_0\}\sim\exp\{N/N_0\}\ . \tag{3.66}$$

Alternatively, because this discrepancy is only of $O(N)$ for momenta $p\sim O(m)$, one might think that within the Schwinger Dyson equation for the fermion it will only alter higher order coefficients, leaving the basic exponential behaviour unchanged. However, as has been shown explicitly in previous chapters, naively small factors at these scales cannot be simply discounted. We are therefore led to the conclusion that we must study the coupled photon fermion system. Let us remind ourselves of the main points of the original mechanism:

i) As predicted by the first order perturbative result, the wavefunction renormalisation $\beta(p)$ develops logarithmically towards zero.

ii) However, before $\beta(p)$ can be driven less than zero a dynamical mass is generated. This effectively cuts off the period of logarithmic scaling and thereafter $\beta(p)$ is approximately constant.

Therefore we are able write,

$$m\gtrsim\exp\{-N/N_0\} \tag{3.67}$$

which tends increasingly towards an equality in the large N limit. The mass depends exponentially on N because of the logarithmic scaling of $A(p)$. The fact that the exponent is linear in N is because the factor outside the integral for $A(p)$ is simply N^{-1}. The danger is that since $\beta(p)$ appears to be so closely linked with the form of the exponential scaling any overall factors, though naively small, may also be exponentiated leading to significant changes.

In the last section we performed all the work necessary to calculate the photon function $\mathcal{G}(p)$ from any set of fermion functions. Therefore, in principle we may go ahead and solve the coupled photon-fermion system for the full Ball-Chiu ansatz. However, to date, this has not been possible due to the extraordinary sensitivity of the system to the lower momentum scaling of the fermion function $\Sigma(p)$. This is the same problem that occured in Section 2.9 where small gradients in $\Sigma(p)$ induced instability. In that case we could simply flatten $\Sigma(p)$ artificially because the sensivity of the relevant scale, $p \sim O(m)$, to this preceedure was shown to be negligible. However, including the full photon within our analysis will effectively boost the importance of this region, as in Section 3.2. The end result is that in this low momentum region $\Sigma(p)$ must have the form,

$$\Sigma(p) = \Sigma_0 + \sigma_2 \frac{p^2}{m^2} + \cdots \quad p \ll m \,. \tag{3.68}$$

Where the influence of the low momentum scaling coefficient, σ_2, on the wave-function renormalisation $A(p)$ is of $O(1)$. This form of scaling cannot be maintained numerically. In order to correctly calculate $A(m/10)$ to $O(1\%)$ then $\Sigma(m/10)$ must be known to an accuracy of the order of 0.01%. As we lower p this discrepency obviously becomes worse. However, simply flattening $\Sigma(p)$ for $p \ll m$ is far too strong a condition and causes a significant underestimate of the value of $A(p)$. This problem has yet to be resolved and so by necessity we are forced into considering the simple "central" vertex.

The variation of the dynamical mass with N is illustrated in Fig 3.8. The lack of data is due to the shear amount of computing time required to solve even this simplified coupled system. The discrepancy between our investigations employing various perturbative ansätze and the results of this investigation, allowing the photon to become a dynamical member of the mechanism, is obviously enormous. Indeed the plummeting of the dynamical mass is uncomfortably close to what we would expect of a critical system with a critical number of fermions $N_c \sim 3.5$. Numerically we are unable to follow the downward path of the dynamical mass in order to decide this matter conclusively and in this we are in a position akin to that of lattice calculations on this question.

However, consider the development of the wavefunction renormalisation correction, $A(p)$, with N as illustrated by Fig 3.9. In this simple system the only

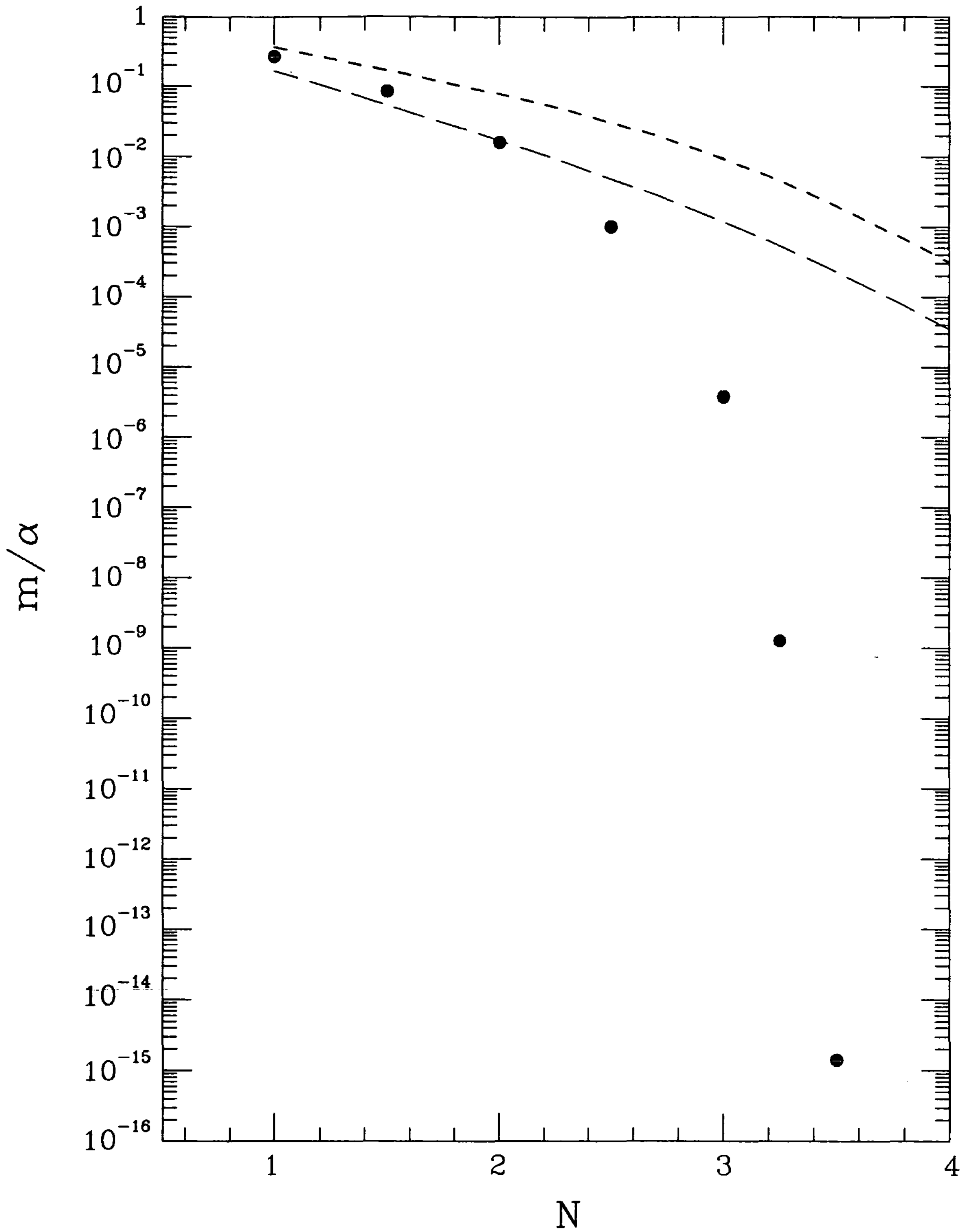

Figure 3.8 The variation of the dynamical mass m with N employing the central vertex. i) • the coupled Photon - Fermion system ii) -.- the massive photon iii) –.– the massless photon.

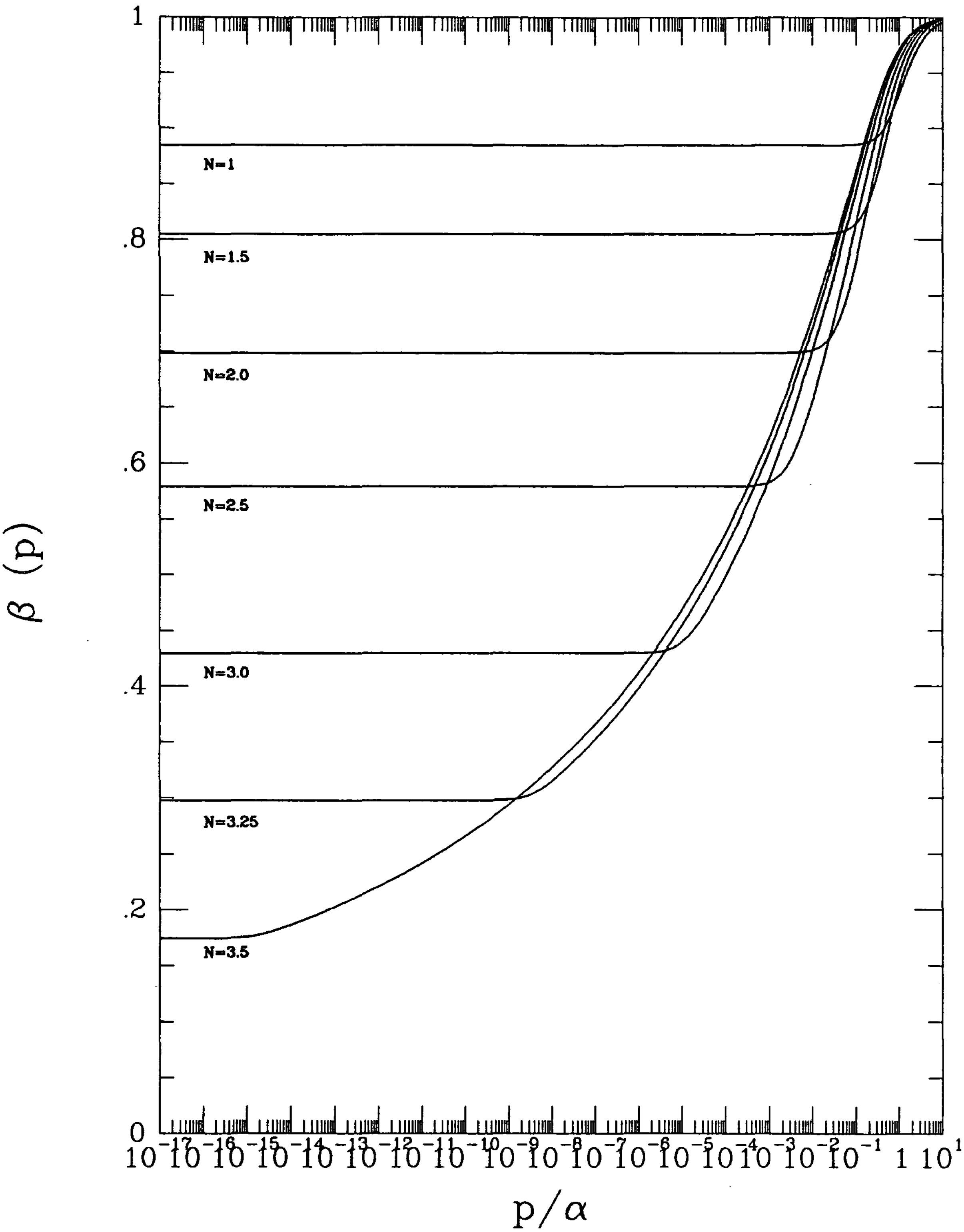

Figure 3.9 The variation of the Wavefunction renormalisation correction $A(p)$ with momentum for N= 1, 1.5, 2, 2.5, 3, 3.25, 3.5

way we know by which a critical point may appear is if the explicit factor of $1/N$ multiplying the equation for $\Sigma(p)$ is not lifted by $\beta(p)$. However, from the trend in Fig 3.9 the role of $\beta(p)$ in this system appears to be the same as in our previous analyses. This is illustrated most clearly in Fig 3.10, where we have plotted $\beta(m)$ against $\ln(m/\alpha)$ for each of our previous calculations. While the development of m with N is clearly very different in the coupled case, the behaviour of $\beta(p)$ with $\ln(m/\alpha)$ is qualitatively the same. Indeed it is quite unnatural from the trend of this data to expect the development of $\beta(m)$ to "freeze" and so permit a critical behaviour in the fermion equation. We therefore conclude that the apparent critical behaviour in our system is the result of a very strong rescaling of the N.

Can we understand this behaviour in some simple minded approximation of this highly convoluted system? The answer to this question is yes and the argument proceeds as a simple development of that in Section 3.2. Let us believe the Mandelstam approximation of Section 3.4 and approximate the photon function $\mathcal{G}(p)$ by,

$$\mathcal{G}(p) \simeq \beta^{-1}(p)\left\{\theta(p-m)\alpha\, p \;+\; \theta(m-p)\,\frac{\alpha}{m}\,p^2\right\}\,. \tag{3.69}$$

This approximation qualitatively expresses the link between the observed enhancement of $\mathcal{G}(p)$ and the strong nonperturbative behaviour of $\beta(p)$ in the low momentum region. Now let us look at the dependence of $\beta(0)$ on the dynamical mass. Following the analysis of Section 3.2 we may write,

$$\begin{aligned}\beta(0) \sim 1 - \frac{\alpha}{N}\int_{\alpha}^{\infty}\frac{dk}{k}\frac{1}{k+\alpha} \\ -\frac{1}{N}\int_{0}^{\alpha} k\,dk\,\left(\beta(k)+\beta(0)\right)\left\{\theta(m-k)\frac{k}{m^3}+\theta(k-m)\frac{1}{k^2}\right\}\end{aligned} \tag{3.70}$$

due to the fact that $\mathcal{G}(p)$ introduces an effective factor of $\beta(p)$. We may characterise the solution of this relation by,

$$A(0) \sim\; -\frac{1}{N}\left\{\beta_0\ln\left(\frac{\alpha}{m}\right)+\beta_0+c\right\}$$

Let us for the moment assume that the developement of $\beta(0)$ with N does freeze

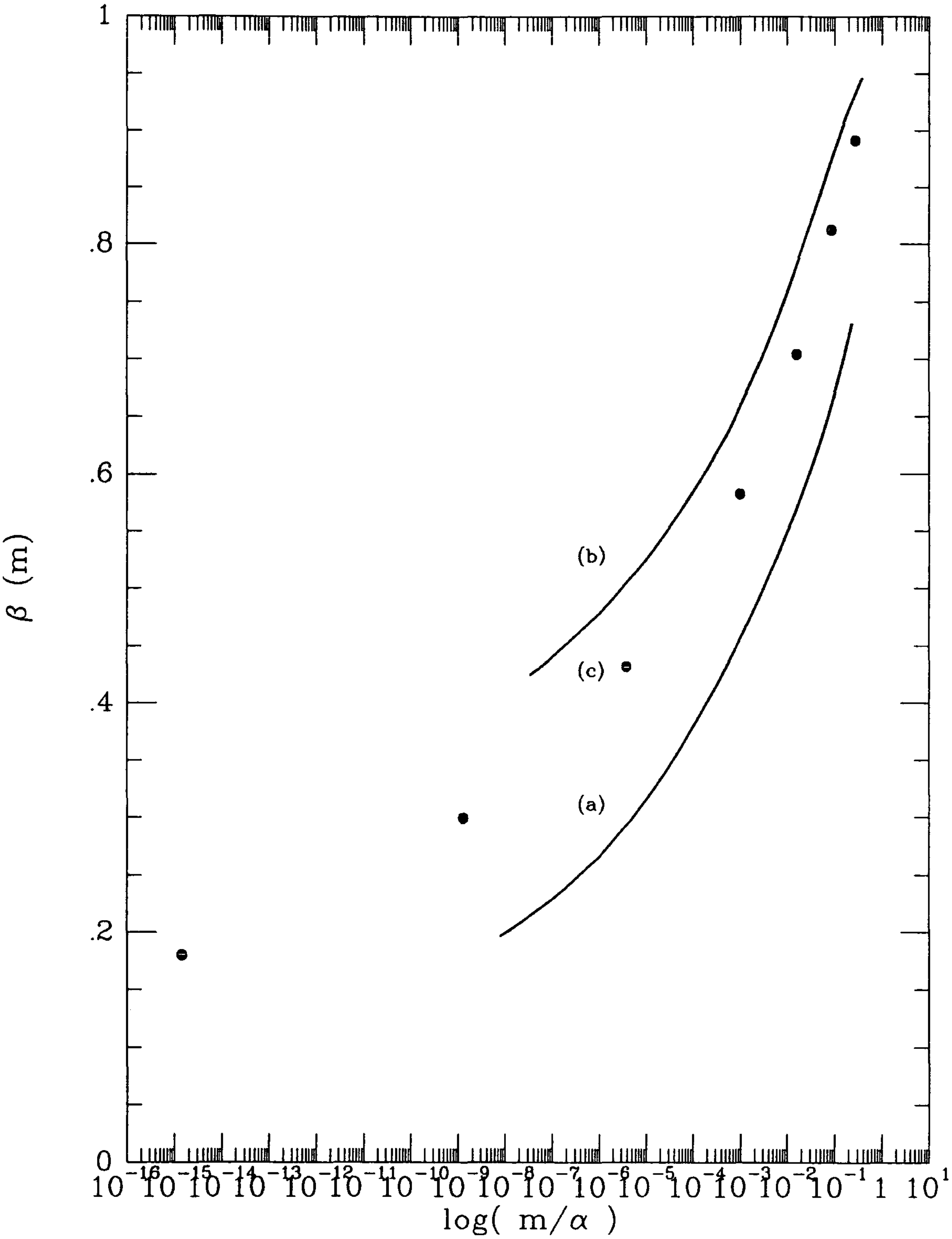

Figure 3.10 The variation of the wavefunction renormalistion $\beta(p)$ with $log(m/\alpha)$ for a) The Ball-Chiu system. b) The "Hard" Photon system c) The coupled Photon - Fermion system.

at some constant value. Then demanding that $\beta(0)$ is positive we may write,

$$-\kappa N \lesssim -\beta(0) \ln\left(\frac{\alpha}{m}\right) + c \tag{3.71}$$

This suggests that m would have an exponential development with N as usual, but we know this to be false. If $\beta(p)$ freezes then the Schwinger Dyson equation for $\Sigma(p)$ cannot support a nonzero mass across all values of N and a critical point will appear, in exactly the same way and for the same reasons as those in Appelquist's original analysis. The assumption that β may freeze is inconsistent. However, consider the alternative that $\beta(0) \to 1/N$ as suggested by Fig 3.9. Then (3.71) suggests that the dynamical mass could fall as quickly as,

$$m \gtrsim \exp\left\{ -N^2/N_0^2 \right\} . \tag{3.72}$$

The behaviour of $\beta(0)$ lifts the explicit factor of $1/N$ in front of the relation for $\Sigma(p)$ leaving it effectively "scaleless" with repect to N. As N increases then the relation for $\Sigma(p)$ edges closer and closer to the critical point but will never reach it for any finite value of N, much as under renormalisation a theory will never exactly reach the renormalisation group fixed points but tend infinitesimally closer and closer.

Can we test this hypothesis from our result for m as displayed by Fig 3.8? Naively we may have thought that since $m \ll \alpha$ we would reach the large N scaling regime quickly, indeed it has been suggested by some [25] that the large N and small m limits bring identical simplifications within the fermion Schwinger Dyson kernel. However, as hinted at by our analysis of the relation for $\beta(p)$ the large N limit is not governed by m but rather only dominates as $\beta(p) \to 0$. Unfortunately, we have not been able to amass sufficient data to confirm or reject this hypothesis. However, analytically the signs point to the large N behaviour remaining exponential but scaling as e^{-N^2} rather than e^{-N}. Let us summarise what we have discovered in this chapter.

3.6 Summary and conclusions.

Let us summarise the contents of this chapter in the order in which they they appeared. Firstly we have shown that the dynamical mass scale in the photon is important. Indeed below this scale the photon regains it's perturbative scaling as the massive fermions decouple. This hardening boosts the importance of the low momentum region leading to $O(1)$ as opposed to the $O(m/\alpha)$ corrections expected from the hand-waving arguments of Chapter 2.

The nonperturbative behaviour of the photon was investigated using its Schwinger Dyson equation. The scaling of the photon was found to be well given by its perturbative result once the fermion mass scale had been correctly incorporated. This could be understood via dimensional analysis of the photon correction $\Pi(p)$. However, numerical computation of the photon correction indicated a systematic discrepacy of $O(N)$ between the perturbative and full forms. This could be understood as the result of the nonperturbative large N behaviour of the wavefunction renormalisation by employing the Mandelstam approximation.

The effective occurance of an overall factor of N meant that the photon could not be simply replaced by a perturbative ansatz and therefore an analysis of a simplified coupled photon-fermion system was performed. The results of this investigation displayed a totally different behaviour of m with N compared to the simple fermion system results. Our intution that the photon may have an important effect on the chiral symmetry breaking mechanism was confirmed. However, the developement of the dynamical mass with N was far too rapid for a large N behaviour to be extrapolated. A simple analysis suggested that although the dynamical mass falls extremely rapidly the system is still held above the Appelquist critical point via the intervention of the wavefunction renormalisation. The development of m with N is expected to be of the form,

$$m \simeq \alpha \exp\left\{ -\frac{N^2}{N_0^2} \right\} \quad , \quad N \gg 1 . \tag{3.73}$$

This is not expected to alter with full analysis including the complete Ball-Chiu ansatz though obviously more work is needed to make these ideas more concrete.

As a final conclusion we may say that a full account of the behaviour of the photon is crucial to a correct description of the chiral symmetry breaking mechanism in this model.

Chapter 4: Gauge Invariance

4.1 INTRODUCTION.

In this chapter we will address the question of gauge invariance in our truncated Schwinger Dyson system. In contrast to perturbation theory the nonperturbative operation of gauge invariance is only slowly being unveiled. However, we do expect our Landau gauge results to be correct and the question of gauge invariance is to show that similar results are obtained in other gauges. We will show how the gauge fixing term within the Lagrangian introduces new contributions that are completely determined by the Ward Identity. This result, which is free from any approximations, will show that our truncated system will be dominated by these new gauge terms in any gauge other than the Landau gauge, where they are simply absent. Short analytic and numerical investigations will reveal that nonzero dynamical masses will not be generated in an arbitrary gauge but only for gauges $\xi \geq \xi_c$, where they will display an approximately linear dependence on the covariant gauge parameter. This is clearly unphysical, the fermion mass is observable and therefore must be independent of gauge. In order to understand this behaviour we will review some of the more recent results of work on the 4 dimensional variant of QED. These will indicate that it is the softening of our photon in the infrared that introduces such a pathological gauge dependence and only through the action of the transverse component of the full 3 point vertex may it be removed. Although this component is unspecified by symmetry arguments a short perturbative calculation will show that at least to first order it has the scaling required to remove this extreme dependence on the gauge. We will conclude with a short discussion on some of the general properties of the $1/N$ expansion in 3 dimensional QED as it pertains to the expansion in e^2 in the 4 dimensional quenched theory.

4.2 Gauge contributions and the transverse vertex.

In a gauge theory it is necessary in any calculation to restrict oneself to one gauge alone, in order to prevent overcounting of gauge equivalent contributions/configurations. This is usually accomplished by including within the Lagrangian of the model a gauge fixing term such as,

$$\mathcal{L}_{\mathrm{gf}} = \frac{1}{2\xi} \int d^n x \, (\partial \cdot A)^2 \ . \tag{4.1}$$

Such a term explicitly breaks the gauge invariance of the Lagrangian although it can be shown that all physical observables will be unaffected by this addition. The gauge is determined by the parameter ξ called the covariant gauge parameter, $\xi = 0$ and $\xi = 1$ corresponding to the Landau and Feynman gauges respectively. More complicated gauge fixing terms, that may introduce quartic terms for the gauge field $A^\mu(x)$ or have nonlocal characteristics, are possible but the conventional term above simply adds an extra ξ dependent term to the photon propagator as follows,

$$\mathcal{D}_{\mu\nu}(p) = \mathcal{G}^{-1}(p) \left\{ \delta_{\mu\nu} - \frac{p_\nu p_\nu}{p^2} \right\} + \xi \, \frac{p_\nu p_\nu}{p^4} \ . \tag{4.2}$$

As explained in Chapter 3 this new gauge dependent term receives no quantum corrections at any order because of the constraints imposed by the Ward Identity.

Let us now consider the fermion equation in an arbitrary gauge. Schematically we may display the explicit gauge dependence by writing,

$$S^{-1}(p) = S_{\mathrm{L}}^{-1}(p) - S_{\mathrm{g}}^{-1}(p) \ . \tag{4.3}$$

where $S^{-1}(p)$ is the full propagtor, $S_L^{-1}(p)$ are the Landau terms and $S_g^{-1}(p)$ the new terms introduced by the longitudinal component of the photon. The form of this last component is exactly the same as those of our previous calculations and so we may immediately write,

$$S_g^{-1}(p) = \xi \int \frac{d^3k}{(2\pi)^3} \, (-ie\gamma^\mu) \, S(k) (-ie\Gamma^\nu) \, \frac{q^\mu q^\nu}{q^4} \, . \tag{4.4}$$

where $q = k - p$ is the photon loop momentum. If for convenience we write,

$$S_g^{-1}(p) = -i \, \{ A_g(p) \not{p} + \Sigma_g(p) \} \tag{4.5}$$

then the above becomes simply,

$$A_g \not{p} + \Sigma_g(p) = -i\xi \int \frac{d^3k}{(2\pi)^3} \frac{\not{q}}{q^4} S(k) (q \cdot \Gamma) \, . \tag{4.6}$$

The product $q \cdot \Gamma$ is obviously determined via the Ward Identity to be,

$$-i \, q \cdot \Gamma(k,p) = S^{-1}(k) - S^{-1}(p) \quad , \quad q = k - p \, . \tag{4.7}$$

so the gauge contribution (4.6) may be written entirely in terms of the two fermion functions $\Sigma(p)$ and $\beta(p)$ without approximation. This simplification is not confined to this low level of the Schwinger Dyson hierarchy, a similar reduction in the "level" of vertex functions occurring within the explicitly gauge dependent terms is expected to operate at all levels. The Ward Identity constrains the 3 point vertex but similar relations constrain all vertices involving external photon lines. Consider the "generating" relation for the Ward Identity Eqn.(1.52)of chapter one,

$$\begin{aligned} -\partial_x \frac{\delta^3\Gamma[0]}{\delta\overline{\psi}(x_1)\,\delta\psi(y_1)\,\delta A^\mu(x)} &= ie\,\delta(x - x_1) \frac{\delta^2\Gamma[0]}{\delta\overline{\psi}(x_1)\,\delta\psi(y_1)} \\ &\quad - ie\,\delta(x - y_1) \frac{\delta^2\Gamma[0]}{\delta\overline{\psi}(x_1)\,\delta\psi(y_1)} \end{aligned} \tag{4.8}$$

where $\Gamma[0]$ is the generating functional for all n-point connected vertex functions, in the limit where all the sources have been set to zero. Obviously we may functionally differentiate this relation with respect to $\overline{\psi}$, ψ and A^μ to produce relations between the higher point functions of the form,

$$
\begin{aligned}
q_1^\sigma\, \Gamma^{(m,n-m)}_{\sigma,\rho,\ldots,i,j,\ldots}\left[\, q_1, q_2, \ldots, q^m, p_1, p_2, \ldots, p_{n-m}\,\right] = \\
\sum_{\rho\ldots,i\ldots} C^{(m-1,n-m+1)}_{\rho,\kappa,\ldots,i,j,\ldots}\, \Gamma^{(m-1,n-m+1)}_{\rho,\kappa\ldots,i,j\ldots}\left[\, q_2, \ldots, q^{m-1}, p_1, \ldots, p_{n-m+1}\,\right] \, .
\end{aligned}
\tag{4.9}
$$

Where $\Gamma^{(m,n-m)}$ is a general n-point function, with m photon and $n-m$ fermion lines attatched, and $C^{(m,n-m)}$ is an (anti-)symmetrising factor. We therefore expect the explicitly gauge dependent terms in higher level Schwinger Dyson relations to simplify by the same basic mechanism as that of (4.6), the fermion equation. At the present time, however, this is little more than an interesting curiosity as we have enough trouble handling the highly simplified systems of our previous chapters.

Let us return to our gauge term in the fermion equation and perform the now familiar sequence of operations necessary to reduce it to two integral forms for $A_g(p)$ and $\Sigma_g(p)$. Inserting (4.7) into (4.4) we have,

$$
A_g(p)\, \not{p} + \Sigma_g(p) \;=\; \frac{e^2\xi}{(2\pi)^3} \int \frac{d^3k}{q^4}\, \not{q}\, \left\{\, 1 - S(k)\, S^{-1}(p)\,\right\} \, . \tag{4.10}
$$

After tracing this relation with $\not{p}$ and 1 to split it into its two components for A_g and Σ_g we have,

$$
\begin{aligned}
4\,\Sigma_g(p) \;&=\; \mathcal{I}\left\{\, \mathrm{Tr}\left[\, \not{q}\, (\, \beta(k)\not{k} - \Sigma(k)\,)\, (\, \beta(p)\not{p} + \Sigma(p)\,)\,\right]\,\right\} \\
4p^2\, A_g(p) \;&=\; \mathcal{I}\Big\{ \mathrm{Tr}\Big[\not{p}\not{q} \{ \beta^2(k) k^2 + \Sigma^2(k) + \\
&\qquad (\, \beta(k)\not{k} - \Sigma(k)\,)\, (\, \beta(p)\not{p} + \Sigma(p)\,)\, \} \Big] \Big\}
\end{aligned}
\tag{4.11}
$$

where,

$$
\mathcal{I}\{\, F(k,p)\,\} \;=\; \frac{\alpha\xi}{\pi^3 N} \int \frac{d^3k}{\beta^2(k)k^2 + \Sigma^2(k)}\, \frac{1}{q^4}\, F(k,p) \tag{4.12}
$$

Note that within the γ matrix traces the ordering of $S^{-1}(k)$ and $S^{-1}(p)$ in (4.10) is important. Performing these traces then (4.11) becomes,

$$\begin{aligned}\Sigma_g(p) &= \mathcal{I}\,\{\,\beta(p)\Sigma(k)\,\langle\, p\cdot q\,\rangle - \beta(k)\Sigma(p)\,\langle\, k\cdot q\,\rangle\,\}\\ p^2 A_g(p) &= \mathcal{I}\,\{\beta(k)(k^2\beta(k)\,\langle p\cdot q\rangle - p^2\beta(p)\,\langle q\cdot k\rangle) + \Sigma(k)(\Sigma(k)-\Sigma(p))\,\langle p\cdot q\rangle\}\end{aligned} \tag{4.13}$$

The two angular integrals involved are of the form,

$$\langle\, l\cdot q\,\rangle = \int \frac{l\cdot q}{q^4}\,d\Omega \tag{4.14}$$

and are easily evaluated to be,

$$\begin{aligned}\langle\, p\cdot q\,\rangle &= \frac{\pi}{kp}\left\{\frac{2kp}{k^2-p^2} - \ln\left|\frac{k+p}{k-p}\right|\right\}\\ \langle\, k\cdot q\,\rangle &= \frac{\pi}{kp}\left\{\frac{2kp}{k^2-p^2} + \ln\left|\frac{k+p}{k-p}\right|\right\}\,.\end{aligned} \tag{4.15}$$

The final forms for $A_g(p)$ and $\Sigma_g(p)$ are therefore found to be,

$$\begin{aligned}\Sigma_g &= -\frac{\alpha\xi}{\pi^2 Np}\int \frac{k\,dk}{\beta^2(k)k^2+\Sigma^2(k)}\,\mathcal{M}_g(k,p)\\ A_g &= -\frac{\alpha\xi}{\pi^2 Np^3}\int \frac{k\,dk}{\beta^2(k)k^2+\Sigma^2(k)}\,\mathcal{A}_g(k,p)\end{aligned}$$

with,

$$\begin{aligned}\mathcal{M}_g(k,p) &= (\,\beta(p)\Sigma(k)+\beta(k)\Sigma(p)\,)\,\ln\left|\frac{k+p}{k-p}\right|\\ &\qquad -(\,\beta(p)\Sigma(k)-\beta(k)\Sigma(p)\,)\,\frac{2kp}{k^2-p^2}\\ \mathcal{A}_g(k,p) &= \{\,\beta(k)\,(\,k^2\beta(k)+p^2\beta(p)\,\}+\Sigma(k)\,(\,\Sigma(k)-\Sigma(p)\,)\,\}\,\ln\left|\frac{k+p}{k-p}\right|\\ &\qquad -\{\,\Sigma(k)\,(\,\Sigma(k)-\Sigma(p)\,)+\beta(k)\,(\,k^2\beta(k)-p^2\beta(p)\,)\,\}\,\frac{2kp}{k^2-p^2}\,.\end{aligned} \tag{4.16}$$

Notice that the natural scale α that previously occupied such a central position within the Landau gauge angular integrals is completely absent. The scale α enters the Landau angular integrals only via the photon function $\mathcal{G}(p) \simeq q^2 + \alpha q$ which is dynamically softened by fermion loops in the infrared. The gauge term in (4.2) by contrast remains "hard" and so the intrinsic scale is naturally absent.

Therefore the functions in the above can only scale with k/p or p/k rather than $p/(k+\alpha)$ or $k/(p+\alpha)$ as was previously the case. This rather simple observation, however, has very serious implications for our system outside of the Landau gauge, $\xi = 0$.

Let us investigate the dependence on ξ of the "central" system where we replace the full vertex with the following,

$$\Gamma^\mu = \frac{1}{2}\left\{\beta(p) + \beta(k)\right\}\gamma^\mu \, . \tag{4.17}$$

together with the simple "massless" perturbative ansatz for a pair of widely spaced values of $N = 1, \; 5$. Under these assumptions (4.16) reduces to the following,

$$\begin{aligned} \mathcal{M}_g(k,p) &= (\,\beta(p) + \beta(k)\,)\;\Sigma(k)\,\ln\left|\frac{k+p}{k-p}\right| \\ \mathcal{A}_g(k,p) &= (\,\beta(p) + \beta(k)\,)\;\beta(k)k^2\left\{\ln\left|\frac{k+p}{k-p}\right| - \frac{2kp}{k^2-p^2}\right\} \, . \end{aligned} \tag{4.18}$$

In Fig 4.1 the wavefunction renormalisation and mass functions are shown for N=1 and various values of the covariant gauge parameter, ξ. The dependence on ξ illustrated by these is obviously extremely strong. The gauge dependences of $A(p)$ and $\Sigma(p)$ must be related in such a way as to leave the dynamical mass of the fermion invariant. This is clearly not the case as is shown in Fig. 4.2. Moreover, in the region $p \ll m$ and $\xi \to \xi_c$ the gauge terms pieces appear to dominate completely the behaviours of both $\beta(p)$ and $\Sigma(p)$. For sufficiently large ξ then $\beta(p)$ is no longer bounded above by unity and similarly for a large and negative ξ $\;\Sigma(p)$ and $\beta(p)$ are no longer positive. If we make the comparison with a forced harmonic oscillator then the characteristics of the original system are being completely obscured by the external forcing motion.

Consider simplifying yet further to the case of a bare vertex,

$$\Gamma^\mu(k,p) = \gamma^\mu \tag{4.19}$$

this is equivalent to setting the $(\beta(k) + \beta(p))$ factors to unity in (4.18). Then after replacing the angular integrals by their asymptotic forms in the appropriate regions, the two relations for $\beta(p)$ and $\Sigma(p)$ reduce to the following linearized relations,

$$\begin{aligned}\beta(p) &= 1 + \frac{8}{3\pi^2 N}\int \frac{k\,dk\,\beta(k)}{\beta^2(k)k^2+\Sigma^2(k)}\left\{\frac{k^3}{p^3}\left(-1+\xi\frac{\alpha}{p}\right)\theta(p-k)\right.\\ &\qquad\qquad\left. + \left(1+\xi\frac{\alpha}{k}\right)\theta(k-p)\right\}\\ \Sigma(p) &= \frac{8}{\pi^2 N}\int \frac{k\,dk\,\Sigma(k)}{\beta^2(k)k^2+\Sigma^2(k)}\left\{\frac{k}{p}\left(1+\xi\frac{\alpha}{2p}\right)\theta(p-k)\right.\\ &\qquad\qquad\left. + \left(1+\xi\frac{\alpha}{2k}\right)\theta(k-p)\right\}\end{aligned} \tag{4.20}$$

This analysis was originally performed by Kondo and Nakatani† but where they restricted themselves to the Landau gauge we will concentrate specifically on gauge dependence. Notice the explicit appearance of the factors of α/p multiplying ξ that are a result of the discrepancy between $\mathcal{G}(p)$ and p^2. These two integral relations may be simply converted into the following differential equations for $\Sigma(p)$ and $\beta(p)$,

$$\begin{aligned}p^2\frac{d^2\,\Sigma(p)}{dp^2} + \left(\frac{1+2\xi\frac{\alpha}{p}}{1+\xi\frac{\alpha}{p}}\right)p\frac{d\,\Sigma(p)}{dp} + \left(1+\xi\frac{\alpha}{p}\right)\frac{\kappa\Sigma(p)}{\beta^2(p)+\Sigma^2(p)/p^2} &= 0\\ p^2\frac{d^2\,\beta(p)}{dp^2} + \left(\frac{12-20\xi\frac{\alpha}{p}}{3-4\xi\frac{\alpha}{p}}\right)p\frac{d\beta(p)}{dp} - \left(1-\frac{4}{3}\frac{\alpha}{p}\right)\frac{\kappa\beta(p)}{\beta^2(p)+\Sigma^2(p)/p^2} &= 0\ .\end{aligned} \tag{4.21}$$

where $\kappa = 8/(\pi^2 N)$. If we expand $\Sigma(p)$ and $\beta(p)$ in powers of p for $p \ll m$ then we may use the above to specify the first scaling coefficients,

$$\begin{aligned}\Sigma(k) &= \sigma_0 + \sigma_1 p + \cdots & \beta(p) &= b_0 + b_1 p + \cdots\\ \sigma_1 &= -\frac{1}{3}\frac{\alpha}{\Sigma(0)}\kappa\,\xi & b_1 &= -\frac{4}{15}\frac{\alpha}{\Sigma^2(0)}\kappa\beta(0)\,\xi\ .\end{aligned} \tag{4.22}$$

Factors of α/m are expected to lead to the dominance of the gauge terms in the infrared region. Notice how the behaviour illustrated in Fig 4.2 for $p \ll m$ and $\xi \to \xi_c$ suggest this pathological dependence on ξ.

† Note that the one of the gauge coefficients in the relation for $\beta(p)$ in Ref. 26 is incorrect.

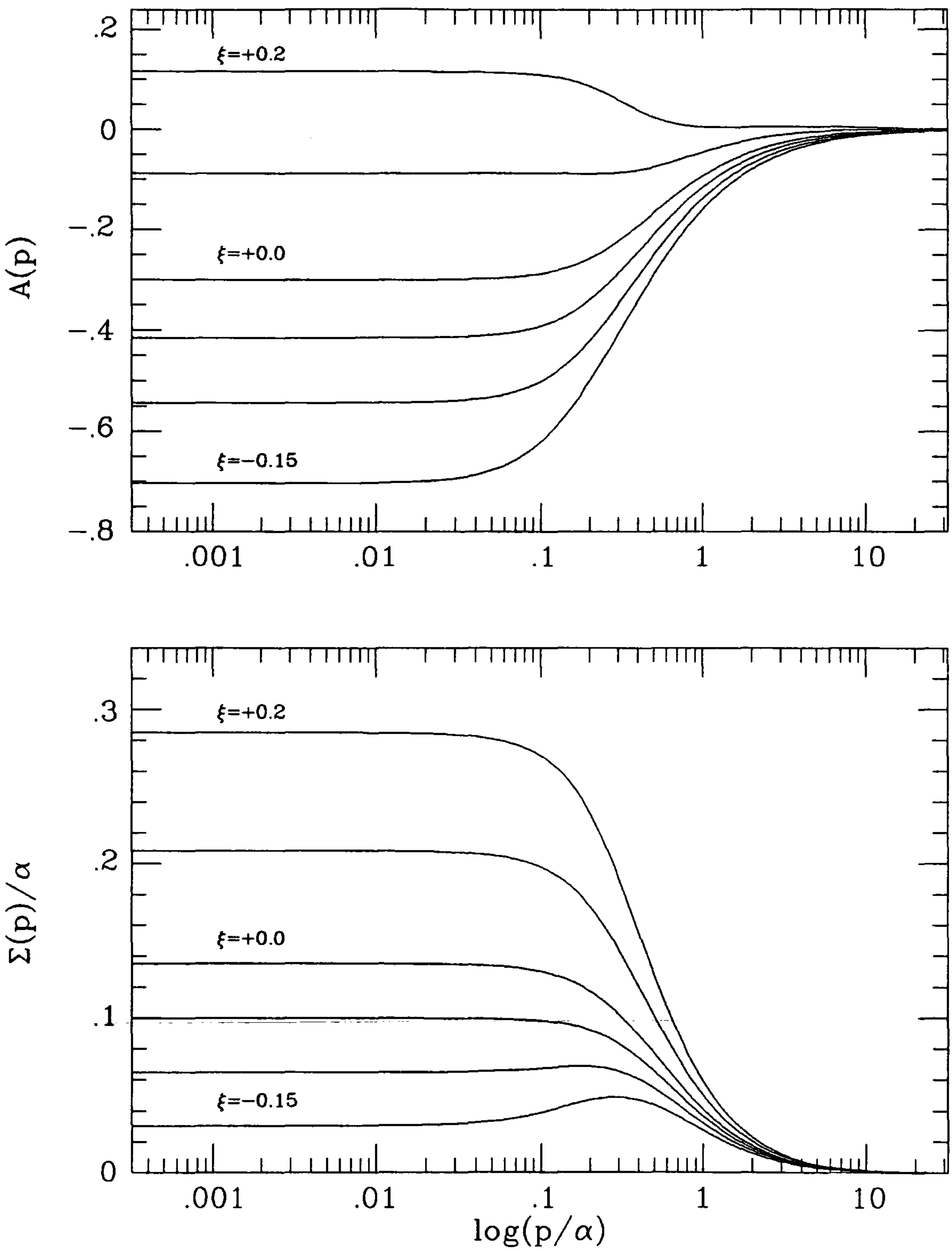

Figure 4.1 The fermion functions $A(p)$ and $\Sigma(p)$ for N=1 for varying values of the gauge parameter $\xi = 0.2$, 0.1, 0.0, -0.5, -0.1, -0.15. In this example full vertex has been replaced by it's "central" part alone. No solution was possible for $\xi \lesssim -0.151$.

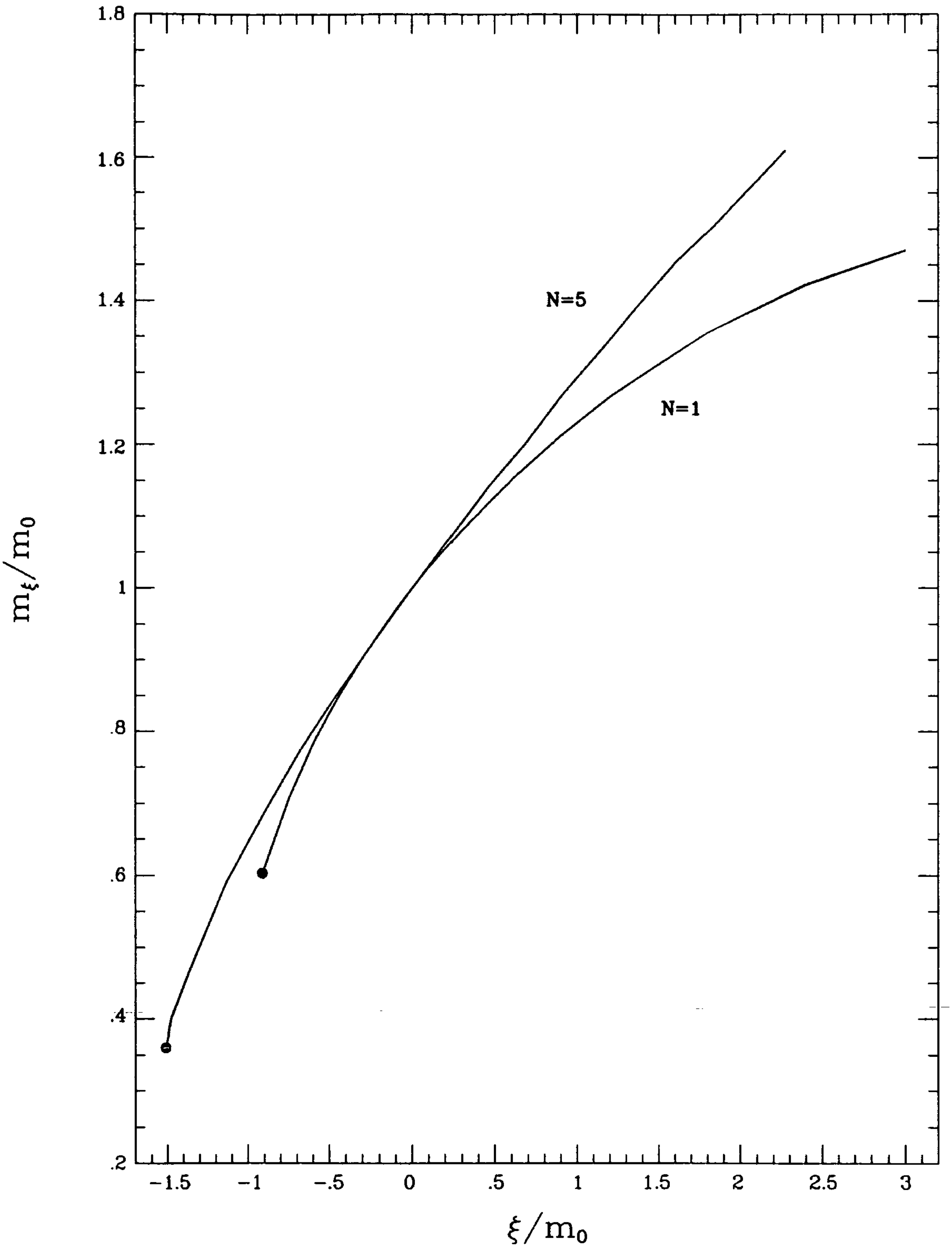

Figure 4.2 The ratio of the mass in a general gauge m_ξ over the Landau value m_0 versus the ratio ξ/m_0 for the "central" system. The • indicate the position of the "critical gauge" ξ_c below which no solutions are possible.

Since we have dropped the transverse component of the full 3 point function in our analysis we cannot expect complete gauge independence for the masses we calculate. However, one might expect the Ball-Chiu ansatz to be closely related to the question of gauge invariance through the Ward Identity and possibly to be able to relieve the gauge dependence of the central system. In this section we will proceed by analysing the full Ball-Chiu system in a general gauge, in order to see whether or not the additional gradient terms indeed soften the pathological gauge dependence illustrated in Fig 4.2. Because of the complexity of this system very little headway may be made without making rather severe approximations. However, the question we have set ourselves does not require a detailed analysis of each and every individual term as it is primarily a question of competing scalings. We may replace our original question with this one: Will these additional gradient terms be able to generate dynamically factors of α/p etc. in order to counter directly the forcing gauge terms?

In a familiar manner we will approximate the photon function by patching it's asymptotic scaling behaviours together at $p = m$, as in Section 3.2,

$$\mathcal{G}(p) \sim \left(p^2 + \alpha p\right) \theta(p-m) + \frac{\alpha}{m} p^2 \, \theta(m-p) \; . \tag{4.23}$$

In addition we will choose to approximate the scaling of the two fermion functions in the low $p \leq m$ region by,

$$\Sigma(p) = \sigma + \sigma_1 \frac{p^\gamma}{m^{\gamma-1}} + \cdots \quad , \quad p \ll m \; . \tag{4.24}$$

where γ is a constant. We have ignored the scaling of $\beta(p)$ in the interest of simplicity as the scaling of $\beta(p)$ is always multiplied by factors of p/m in comparison to $\Sigma(p)$ and are therefore of little importance in this region. In time honoured fashion we will also replace all angular integrals by their asymptotic scaling behaviours in each appropiate region and introduce a cutoff at α.

Let us consider the resulting relation for $A(p)$ with $p \ll m$, specifically the contribution from the region $m \le k \le \alpha$ which we will denote by $A_t(p)$,

$$
\begin{aligned}
A_t(p) \sim \frac{\alpha}{Np^3} \int_m^\alpha \frac{dk}{k} \; & \beta_k \,(\, \beta_k + \beta_p \,)\, \frac{p^3}{\alpha} + \{\, k^2 \beta_k \nabla\beta + \Sigma_k \nabla\Sigma \,\}\, \frac{p^3}{\alpha k^2} \\
& + \xi \left\{ k^2 \beta_k^2 \frac{p^3}{k^3} - p^2 \beta_k \beta_p \frac{p}{k} + \Sigma_k \,(\, \Sigma_k - \Sigma_p \,)\, \frac{p^3}{k} \right\} .
\end{aligned}
\tag{4.25}
$$

where we have introduced the following conventions for brevity,

$$
\nabla f = \frac{f(k) - f(p)}{k - p} = \frac{f(k) - f(p)}{k} + O(p/k) \quad \text{with} \quad f_k = f(k) \,, \tag{4.26}
$$

and dropped p in comparison to k. If we use the fact that $\beta(p) \sim 1$ then we may write,

$$
\begin{aligned}
A_t(p) &\sim -\frac{1}{N} \int_m^\alpha \frac{dk}{k} \; 1 + \frac{\Sigma_k \,(\, \Sigma_k - \Sigma_p \,)}{k^2} + \xi \left\{ \frac{\alpha}{k} + \Sigma_k \,(\, \Sigma_k - \Sigma_p \,)\, \frac{\alpha}{k^3} \right\} \\
&\sim -\frac{1}{N} \left\{ \ln\left(\frac{\alpha}{m}\right) + c_1 + (c_2 + \frac{\alpha}{m})\,\xi \right\} .
\end{aligned}
\tag{4.27}
$$

where the constants c_1 and c_2 are expected to be of $O(1)$. Notice how the gradient terms in addition to the simple central terms are also shifted relative to each other by the softening of the photon function, removing the possibility of cancellation between them in this region. From the region $p \le k \le m$ in a similar manner the contribution to $A(p)$, denoted by $A_m(p)$, may be characterised by,

$$
A_m(p) \sim -\frac{1}{N} \left\{ c + \frac{\alpha}{m} \left(\frac{m}{\alpha} + \xi \right) (\gamma - 1)^{-1} \right\} . \tag{4.28}
$$

The factor of $(\gamma - 1)^{-1}$ comes from the scaling of $\Sigma(p)$. This leads to the condition that $\gamma > 1$ for a finite $\beta(0)$. Finally in the lowest region $0 \le k \le p$ the contributions, denoted by $A_l(p)$, are of the following form,

$$
A_l(p) \sim -\frac{1}{N} \left\{ \frac{p^3}{m^3} + \frac{m}{\alpha} \left(\frac{p^{\gamma-1}}{m^{\gamma-2}} \right) + \xi \left(1 + \frac{p^2}{m^2} \right) \frac{p^{\gamma-1}}{m^{\gamma-2}} \right\} . \tag{4.29}
$$

where again the gauge terms are boosted by factors proportional to $\mathcal{G}(m)/m^2$.

Collecting the dominant terms together we may write,

$$A(p) \sim -\frac{1}{N}\left\{\ln\left(\frac{\alpha}{m}\right) + \xi\,\frac{\alpha}{m} + c + \cdots\right\} \quad p \ll m\;. \tag{4.30}$$

A similar analysis for $\Sigma(0)$ leads to the result,

$$\Sigma(p) \sim \frac{1}{N}\left\{m\ln\left(\frac{\alpha}{m}\right) + \alpha\left(\xi + \frac{m}{\alpha}\right)(\gamma-1)^{-1} + \cdots\right\} \quad p \ll m\;. \tag{4.31}$$

where the dominant contributions come from the region $k > p$. If we compare the gauge and Landau contributions to $A(p)$ and $\Sigma(p)$ in an arbitrary gauge, then we see that the gauge contributions are enhanced by a factor of $\mathcal{G}(m)/m^2$, which is of the order of α/m because of the strong softening effect of fermion loop corrections as summed in the $1/N$ expansion. This extremely strong enhancement effectively removes any possibility of cancellation between the gauge and Landau pieces and therefore we feel justified in assuming that the Ball-Chiu vertex components will be insufficient to remove this artificial critical behaviour. In the complete system each component in the gauge sector must be partnered by terms in the Landau sector in such a way as to guarantee gauge invariance of the dynamical mass. By ignoring the transverse component of the full vertex some of these gauge terms are obviously freed and eventually dominate the system in a general gauge. To summarize, consider stepping-off in the gauge parameter from the Landau, $\xi = 0$, result for a particular N, then (4.30) suggests that below the "critical gauge",

$$\xi < -\xi_c \quad , \quad \xi_c \sim \frac{\alpha}{m} \tag{4.32}$$

the positivity of $\beta(p)$ will be violated and similarly for $\xi \gg \xi_c$ $\beta(p)$ will no longer be bounded above by unity. The transverse component in the Landau gauge, $\xi = 0$, is well behaved and indeed some calculations have been performed to check this [27]. Clearly in a general gauge the situation is more complicated, the above behaviour is meaningless and bears no relation to what must occur in the physical system.

This artificially induced critical behaviour is clearly a consequence of our incomplete treatment of corrections at each order in $1/N$.As stressed in Chapter 2 a truncated system of Schwinger Dyson equations sums infinite subsets of perturbative corrections, in contrast to perturbation theory in which a finite number of orders are treated completely.

This is both the strength and weakness of the Schwinger Dyson approach, for although it introduces the problem of gauge invariance in 3 dimensions and the related question of multiplicative renormalisability in 4 dimensions, without this "weakness" we could not have predicted nonzero dynamical masses from a chirally symmetric bare theory. In the next section we will review some of the more recent work performed on the 4 dimensional quenched version of QED which is of relevance to this problem.

4.3 Gauge Invariance and the Transverse vertex.

This section will be devoted to trying to understand how, in a completely nonperturbative calculation, the extreme gauge dependence predicted for our truncated system could be removed. To aid us in this we will make a brief foray into some of the more recent work performed in the area of the quenched ($N = 0$) 4 dimensional version of our model. We will only sketch the general arguments here and refer the more inquisitive reader to the pioneering paper by Brown & Dorey[28] and more recently the comprehensive analysis of Curtis and Pennington[29].

In simplifying our Schwinger Dyson system we have attempted to include most of the physical aspects of the 3 point vertex by using the Ball-Chiu ansatz. However, it may be shown that in at least one other case this is not sufficient [30]. Consider choosing a vertex ansatz for a quenched, $N = 0$, version of QED. Then it is a straightforward procedure to deduce the coupled integral relations for the two fermion functions, $\beta(p)$ and $\Sigma(p)$, of our "test" fermion. Since we are working in 4 dimensions it will be necessary to introduce an ultraviolet cutoff, Λ, in order to regularise the theory. We may consider rewriting these relations in terms of renormalised quantities or alternatively investigating the behaviour of the solutions as the cutoff is removed and in both these ways we hope to extract the low energy characteristics of the model. However, consider a different approach which is directly concerned with the ultraviolet consistency of the truncated system. Given our integral relations $\mathcal{F}$, let us expand our solutions for $p \to \Lambda$ in terms of the coupling, e^2, so that:

$$\left.\begin{aligned} \Sigma(p) &= \Sigma_0 + e^2\,\mathcal{F}_\sigma\,[\,\Sigma,\beta,\Gamma\,] \\ \beta(p) &= 1 + e^2\,\mathcal{F}_a\,[\,\Sigma,\beta,\Gamma\,] \end{aligned}\right\} \to \left\{\begin{aligned} \Sigma(p) &= \Sigma_0 + \sigma_1 e^2 \ln\tfrac{\Lambda^2}{p^2} + \cdots \\ \beta(p) &= 1 + a_1 e^2 \ln\tfrac{\Lambda^2}{p^2} + \cdots \end{aligned}\right. \tag{4.33}$$

where we have kept the bare fermion mass m_0 nonzero — otherwise all the perturbative corrections to $\Sigma(p)$ as defined above would vanish identically. Note that this perturbation theory need not agree with standard perturbation theory. However, in the real system they will agree exactly to every order and it is this requirement that gives us a unique insight into the nonperturbative behaviour of the vertex. For any given vertex ansatz to be physically consistent it must not only satisfy the Ward Identity but also must result in fermions functions that are multiplicatively renormalisable. By this we mean that it must be possible to factor the cutoff dependence of the solutions (4.33) in such a way to leave their form invariant. For example,

$$\Sigma(p,\Lambda) = Z(\Lambda/\mu)\,\Sigma_r(p,\mu) \tag{4.34}$$

introducing the physical renormalisation scale μ in place of the cutoff Λ. Only in this way may we define a consistent ultraviolet theory. This last condition will introduce relations between the expansion coefficients σ_i and b_i and it was these that enabled Curtis and Pennington to derive a possible form for the previously unknown transverse component of the full 3 point vertex,

$$\Gamma^\mu_T(k,p) \sim \frac{1}{2}\,(\,\beta(k) - \beta(p)\,)\; t^\mu_6(k,p)\;. \tag{4.35}$$

The function $t^\mu_6(k,p)$ is a tensor transverse to the momentum transfer $q = k - p$. This form when added to the longitudinal vertex as specified by Ball & Chiu was shown to result in fermions functions that were multiplicatively renormalisable to every order in e^2 at both the leading and subleading logarithm level. One may grasp an intuitive understanding of how such a term may dynamically introduce factors of ξ if one remembers that for $\mathcal{G}(p) = p^2$, then to first order,

$$\beta(p) = 1 + \xi\, e^2 \ln \frac{\Lambda}{p} + O(e^4) \tag{4.36}$$

and therefore the difference factor in (4.35) competes on an equal footing with the explicitly gauge dependent contribution. Surely such a term is present in the Ball-Chiu ansatz? Indeed it is and a simple calculation shows, that at least perturbatively, it does much to soften the gauge invariance of the system[30]. The question now becomes this: might we expect that such a term to be able to lift the gauge dependence in unquenched QED3? The link between 3 dimensional QED and 4 dimensional quenched QED has been made in the past to support possible critical behaviour in the 3 dimensional model[25]. The link between these two theories is not trivial however, and the question of whether we may simply take the Curtis-Pennington form and use it in an analysis of unquenched QED3 is unclear. In the next section we will investigate both the properties of the transverse component and any possible association between the 4D quenched and 3D unquenched theories.

4.4 The Transverse Vertex in QED3.

Let us put to one side any general arguments we might make for the behaviour of the transverse component and instead settle for a simple though instructive perturbative calculation. We will calculate the first order correction to the vertex and, after calculating $\beta(p)$, compare the transverse part of this with the corresponding Curtis/Pennington form to the same order.

The only correction to first order to the vertex is shown below, from which we may immediately write,

$$\Gamma^{\mu}_{(2)}(k,p) = \int \frac{d^3k}{(2\pi)^3}\, (\,-ie\gamma^{\rho}\,)\, S(k-p)(\,-ie\gamma^{\nu}\,)\, S(k+p')(\,-ie\gamma^{\sigma}\,)\; \Delta_{\mu\nu}(k)\ . \tag{4.37}$$

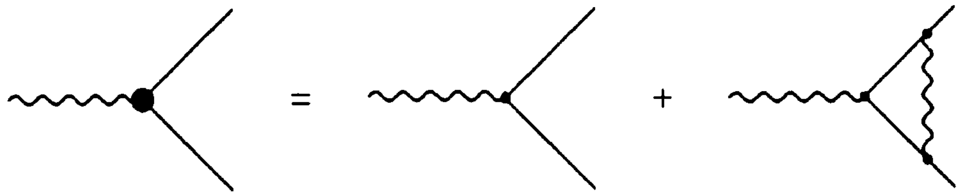

Figure 4.3 First order correction to the bare 3 point vertex.

In order to extract the constant as well as leading logarithmic term it was necessary to choose p and p' parallel, with the convention that if both vectors are aligned then p and p' have the same sign, consequently $q = p+p'$. Expanding the γ matrix components we may write,

$$\Gamma^{\mu}_{(2)} = -\frac{ie^2}{(2\pi)^3}\int \frac{k^2\,dk}{k^2+\alpha k}\int \frac{\mathcal{M}^{\nu}\,d\Omega}{(k-p)^2\,(k+p')^2} \tag{4.38}$$

with,

$$\begin{aligned}\mathcal{M}^{\nu} =&\gamma^{\nu}\left\{\left(\frac{k^2+pp'}{k^2}\right) I_l - \left(k^2-pp'\right) I\right\}\\ &-2\frac{\not p p^{\nu}}{p^2}\left\{\frac{3k^2-pp'}{2k^2} I_l - \frac{1}{2}\left(k^2+pp'\right) I + \left(pp'-p^2\right) I'\right\}.\end{aligned} \tag{4.39}$$

working in the Landau gauge. The angular integrals in the above may be straightforwardly calculated to be,

$$I(k,p) = \int \frac{d\Omega}{(k-p)^2\,(k+p)^2} = \frac{2\pi}{qk(k^2+pp')}\left\{\mathcal{L}+\mathcal{L}'\right\}$$

$$\begin{aligned}I_l(k,p) &= p^{-2}\int \frac{(k\cdot p)^2}{(k-p)^2\,(k+p)^2}\,d\Omega\\ &= \frac{\pi}{2qk(k^2+pp')}\left\{\frac{(k^2+p^2)^2}{p^2}\mathcal{L} + \frac{(k^2+p'^2)^2}{p'^2}\mathcal{L}'\right\} - \frac{\pi}{pp'}\end{aligned}$$

$$
\begin{aligned}
I'(k,p) &= p^{-2}\int \frac{(k\cdot p)}{(k-p)^2\,(k+p)^2}\,d\Omega \\
&= \frac{\pi}{qkp(k^2+pp')}\left\{\frac{(k^2+p^2)^2}{p^2}\mathcal{L}-\frac{(k^2+p'^2)^2}{p'^2}\mathcal{L}'\right\}
\end{aligned}
\tag{4.40}
$$

where

$$
\mathcal{L}=\ln\left|\frac{k+p}{k-p}\right| \quad , \quad \mathcal{L}'=\ln\left|\frac{k+p'}{k-p'}\right| \; . \tag{4.41}
$$

Using these, $\Gamma^{\mu}_{(2)}$ reduces to the form,

$$
\Gamma^{\mu}_{(2)} = -\frac{ie^3}{(2\pi)^3}\int \frac{k^2\,dk}{k^2+\alpha k}\left\{A\gamma^{\mu}+B\frac{\not p p^{\mu}}{p^2}\right\} \tag{4.42}
$$

with

$$
\begin{aligned}
A =& \frac{\pi}{kq}\left\{\frac{(k^2-p^2)^2}{2k^2p^2}\mathcal{L}+\frac{(k^2-p'^2)^2}{2k^2p'^2}\mathcal{L}'\right\}-\frac{\pi\,(k^2+pp')}{k^2\,pp'} \\
B =& \frac{\pi}{kq}\left\{\frac{(3k^2+p^2)(p^2-k^2)}{2k^2p^2}\mathcal{L}+\frac{(3k^2+p'^2)(p'^2-k^2)}{2k^2p'^2}\mathcal{L}'\right\} \\
&-\frac{\pi\,(3k^2-pp')}{k^2pp'}
\end{aligned}
\tag{4.43}
$$

After performing the radial integration over k this in turn reduces to the following rather innocuous looking result,

$$
\frac{\Gamma^{\mu}_{(2)}}{(-ie)} = -\frac{8}{3\pi^2N}\left\{\gamma^{\mu}\left(\frac{1}{3}+\frac{1}{(p+p')}\left(p\ln\frac{\alpha}{p}+p'\ln\frac{\alpha}{p'}\right)\right)-\frac{\not p p^{\mu}}{p^2}\right\} \; . \tag{4.44}
$$

A simple calculation for the wavefunction renormalisation yields the first order result,

$$
\beta(p) = 1-\frac{8}{3\pi^2N}\left\{\ln\frac{\alpha}{p}-\frac{2}{3}\right\}+O(1/N^2)\; . \tag{4.45}
$$

If we reconstruct the Ball-Chiu ansatz to first order using this form for $\beta(p)$ and remove it from the complete vertex (4.44), we are left with the following transverse vertex component,

$$\Gamma_T^\mu = -\frac{8}{3\pi^2 N}\left\{1+\frac{1}{2}\left(\frac{k+p}{k-p}\right)\ln\frac{p}{k}\right\}\left(\delta^{\mu\nu}-\frac{p^\mu p^\nu}{p^2}\right)\gamma^\nu + O(1/N^2)\,. \quad (4.46)$$

Alternatively, if we substitute (4.45) into the Curtis-Pennington transverse vertex (4.35) we obtain the transverse form below,

$$\Gamma_T^\mu = -\frac{4}{3\pi^2 N}\left(\frac{k+p}{k-p}\right)\ln\left(\frac{p}{k}\right)\left(\delta^{\mu\nu}-\frac{p^\mu p^\nu}{p^2}\right)\gamma^\nu + O(1/N^2)\,. \quad (4.47)$$

These forms are obviously identical apart from the constant factor in (4.46), that we were at such pains to extract. This factor, however, is essential in ensuring that,

$$\lim_{k\to p}\Gamma_T^\mu(k,p)\to 0 \quad (4.48)$$

at this order, which is a consequence of the Ward Identity. In the Curtis-Pennington form this is mopped up within the transverse tensor t_6 which contains necessary factors of $m(k)$ and $m(p)$ that ensure that it vanishes in this limit. Should we be surprised by this agreement at the logarithmic level and may we expect this agreement at higher orders, perhaps to leading-logarithms?

In comparing the O(1/N) results for the standard perturbation theory and the Curtis-Pennington ansatz we are really only comparing the two results for the transverse part of the first order correction, Fig 4.3. In the 4 dimensional quenched theory this term is effectively,

$$e^2\int^\Lambda \frac{d^4k}{k^2\mathcal{G}(k)}f(k,p) \sim e^2\int^\Lambda \frac{dk}{k}f(k,p) \quad (4.49)$$

where the function $f(k,p)$ is approximately constant for $k \gg p$. Similarly in three dimensions we have,

$$\frac{\alpha}{N}\int^\alpha \frac{d^3k}{k^2\mathcal{G}(k)}f'(k,p) \sim \frac{1}{N}\int^\alpha \frac{dk}{k}f'(k,p) \quad (4.50)$$

where $f'(k,p)$ is equal to $f(k,p)$ up to an overall factor in the limit $k \gg p$. Therefore the softening of the 3 dimensional photon results in equivalent logarithmic corrections and motivates the association,

$$(\alpha, 1/N) \leftrightarrow (\Lambda, e^2) \tag{4.51}$$

Note that we may only make an association between the QED3 corrections in the *Landau* gauge and quenched QED4 corrections, if we may make it at all. Since the generation of logarithmic terms in both perturbative series is due to photon loops may we expect this analogy to carry to all orders? If indeed this is true then they will only agree at the leading logarithm level. This is rather a simple point. In addition to possible configurations for which the 4 dimensional model may generate sub-leading logarithmic terms the 3 dimensional model generates extra subleading terms in the following way,

$$g \int_p \frac{d^n k}{k^2 \mathcal{G}(k)} \to \begin{cases} \frac{1}{N} \ln \frac{\alpha}{p} + \frac{c}{N} & n = 3 \\ e^2 \ln \frac{\Lambda}{p} & n = 4 \end{cases} \qquad k \gg p \, . \tag{4.52}$$

Leading logarithmic corrections to the fermion propagator mass may be associated with graphs that have effectively independent photon lines, the rainbow corrections of Section 2.3. In this case a rainbow correction to order m would have the general form

$$\prod_{j=1,m} \left\{ g \int_p \frac{d^n k_j}{k_j^2 \mathcal{G}(k_j)} \right\} \sim g^m \ln^m \left(\frac{\Lambda}{p} \right) \, . \tag{4.53}$$

and we might expect our association to hold to all orders, as the dominant corrections at higher orders are simple products of the first order result. Since we expect the behaviour of our model to be dominated by the these leading log corrections should not 3 dimensional QED display a critical behaviour such as its quenched 4 dimensional counterpart?

The simple answer to this question is no. The argument of the above has placed great emphasis on the scaling of the photon to link the dominant perturbative corrections at each order. However, we have neglected to mention whether the dimension of space time may enter in a slightly less nontrivial manner. Consider the wavefunction renormalisation to $O(e^2)$ in the 4D theory,

$$\beta_4(p) = 1 - \frac{e^2}{4\pi}\xi \ln \frac{\Lambda}{p} + O(e^4) . \tag{4.54}$$

Notice how to this order the correction only contains a ξ dependent term. This is because in the Landau gauge the scaling of the photon function, $\mathcal{G}(p) = p^2$, ensures that the correction is identically zero at this order. In contrast, compare the same result in 3 dimensions in the relevant Landau gauge,

$$\beta_3(p) \sim 1 - \frac{8}{3\pi^2 N} \ln \frac{\alpha}{p} + O(1/N^2) \tag{4.55}$$

In this case the softening of the photon does produce logarithmic corrections as expected, but within the Landau term there are no cancellations, simply because $\mathcal{G}(p)$ now effectively scales as p and not p^2. Therefore even to first order the leading terms of both series disagree and the simple association (4.51) cannot hold.

4.5 Further Work.

In this chapter we have investigated the behaviour of a simplified fermion system in a general gauge. It was found that the softening of the photon renormalistaion function greatly amplified the inadeqacy of our vertex ansatz, to such an extend that for any gauge $\xi \lesssim O(m/\alpha)$ no chiral symmetry breaking solutions were possible. This result is expected to hold for any ansatz other than the complete vertex, this reason for this is quite simple. The gauge terms within the fermion Schwinger Dyson equation are completely determined by the Ward Identity and within the full system each is matched with a corresponding term in the Landau section. Only by the cooperation of these terms is the gauge invariance in the dynamical mass ensured. Problems obviously arise when one uses an incomplete vertex ansatz in the Landau section, the photon renormalisation will lead to dominance of any unmatched gauge terms. Therefore the only way this gauge problem can be resolved is by including the transverse component of the full vertex in addition to the Ball-Chiu ansatz in any investigation of the chiral symmetry breaking . Let us speculate on what form and properties this undetermined component may possess.

In Section 4.2 we made a simple analysis of the behaviour of the fermion equation including the Ball-Chiu vertex and came to the conclusion that the resulting system would also be dominated by the gauge sector. Indeed that analysis suggested that to combat this artificial gauge behaviour the transverse component would have to have a form such as the following,

$$\Gamma^{\mu}_{T}(k,p) \simeq \frac{\mathcal{G}(p)}{q^2}\,(\,\beta(k) - \beta(p)\,)\; t^{\mu} \tag{4.56}$$

where t^{μ} is simply a transverse tensor. The factor $\mathcal{G}/q^2$ would produce relative factors of α/k or α/p that would enable this component to compete on the same footing as the gauge terms. However, there are two related reasons why this rather crude artifact cannot, in fact, be correct,

i) The explicit factor $\mathcal{G}/q^2$ in (4.56) does not occur to first order in perturbation theory as is explicitly shown in Section 4.4. For such a term to occur it must be generated nonperturbatively and from the form of (4.46) this type of multiplicative factor seems highly unlikely.

ii) Also consider what would happen if we were to introduce such a term into the fermion equation. In a general gauge this transverse component would effectively combine with the gauge terms and perhaps we could hope to "tune" such a term in order to regain gauge invariance† . (We ignore the trivial case of "inventing" a Γ^{μ}_{T} to explicitly remove the gauge terms.) However it is quite straightforward to see that it is not possible to do so and still to hope to satisfy perturbation theory in the limit $p \to \alpha$. In the Landau gauge the explicitly gauge dependent terms in the fermion equation are simply absent. The form of the transverse component, (4.56), is such that it would dominate the behaviour of the system in the infrared,

† Such ideas have been applied to four dimensional QED but remain unconvincing, see Ref. 31.

though this is really not the crucial point. Since this component does not have the correct pertubative limit then the resulting fermion functions also will not. However, in this region we know that perturbation theory must apply and therefore the use of this form would be incorrect.

The operation of gauge invariance in our system is expected to operate at a far more subtle level that (4.56) would suggest. Indeed the power law behaviours needed to interpolate between the gauge and Landau sections in the fermion equation do not need to be introduced by such crude factors. We knew, however, this must be true as gauge invariance is guaranteed in perturbation theory where such factors are absent. In this respect the gauge mechanism takes care of itself.

Consider the first order perturbative form for $\beta(p)$ in the limit $p \gg \alpha$. Simply from dimensional arguments we may write,

$$\beta(p) \simeq 1 + (\, a_1 + b_1 \, \xi \,) \, \frac{\alpha}{Np} + \cdots \qquad p \gg \alpha. \tag{4.57}$$

where the factors a_1 and b_1 are simple constants. Now, as we lower p towards and finally below α we know the photon renormalisation function $\mathcal{G}$ is softened from $p^2 \to \alpha p$. This in turn alters the scaling of the gauge independent (Landau) term from α/p to $\ln(\alpha/p)$. However, the gauge dependent term is due to the longitudinal component of the photon and this receives no corrections, remaining "hard", due to gauge invariance. The scaling of this term therefore remains the same accross all momenta and allowing us to write,

$$\beta(p) \simeq 1 + \frac{1}{N} \left(a_1 \, \ln\left(\frac{\alpha}{p}\right) + b_1 \, \frac{\alpha}{p} + \cdots \right) \qquad p \lesssim \alpha \, . \tag{4.58}$$

If, for example we consider, the rainbow diagrams then we know that at the i^{th} order in the perturbative expansion for β will contain the terms $log^i(\alpha/p)$ and $\xi^i \, \alpha^i/p^i$ and presumable all combinations in between,

$$\beta(p) = 1 + \frac{1}{N} \left(a_1 \ln\left(\frac{\alpha}{p}\right) + \xi \, b_1 \, \frac{\alpha}{p} + c_1 \right) \\ + \frac{1}{N^2} \left(a_2 \ln^2\left(\frac{\alpha}{p}\right) + \xi \, b_2 \, \frac{\alpha^2}{p^2} + d_2 \, \xi \, \frac{\alpha}{p} \, \ln\left(\frac{\alpha}{p}\right) + c_2 \right) + \cdots \tag{4.59}$$

It is interesting to note that with a conventional gauge fixing term the perturbative $1/N$ expansion in an arbitrary gauge is beset by all the infrared problems of its predecessor, the expansion in e^2. However, we know from the perturbative series in the Landau gauge that this infrared problem is merely a mirage and does not signal the death of our theory.

The Ball-Chiu and indeed any ansatz apart from the trivial bare vertex will have all these power scalings available, (4.59), no explicit factor of $\mathcal{G}(q)/q^2$ is necessary to generate them. Within the fermion Schwinger Dyson equation the relative factors of α/p and α/k introduced by the gauge sector will perform an increasingly complicated mixing of these terms at each order in $1/N$. Indeed it would be instructive to perform a short pertubative calculation of the corrections to a bare mass to $O(1/N^2)$ so that the cancelation of these gauge terms could be observed explicitly. Such a calculation has been performed by Nash [32] in the context of Appelquist's perturbative approach to the Schwinger Dyson equations†. However this calculation was performed in a nonlocal gauge, chosen in such a way that the photon could be written as,

$$\mathcal{D}(p)_{\mu\nu} = \frac{1}{\mathcal{G}(p)} \left\{ \delta_{\mu\nu} - (1-\xi) \frac{p^\mu p^\nu}{p^2} \right\} . \tag{4.60}$$

and so avoids confronting any powerlaw scalings. Within this formalism, Nash was able to show explicitly that all gauge dependence cancels in the expression for the dynamical mass, as is guaranteed by perturbation theory. Consider repeating the calculation of Section 4.4 for the $O(1/N)$ vertex in an arbitrary gauge. We may extract the leading terms of this form for any configuration of p and p' by simply partial fractioning the denominator in (4.38), but in general we will only be able to determine the constant terms for exceptional configurations, *i.e.* p and p' parallel or $p, p' = 0$ *etc.* . It would be interesting to compare the transverse component of this result with that obtained by substituting the first order form for $\beta(p)$ into the Curtis-Pennington form to see if the agreement of leading terms is preserved in a general gauge and for a general configuration of external momenta p, p'. In addition, if we were to calculate the first order

† We will leave a discussion of this calculation as it applies to the chiral symmetry breaking mechanism to the next, concluding, Chapter.

correction to the photon function then we would able to construct the second order results for Σ, assuming a non-zero bare mass, and β. In doing so we would be able to explicitly follow the operation of the relative factors of α/p and α/k in the fermion equation and their role in ensuring the gauge independence of the mass.

Unfortunately, as instructive as such a perturbative calculation may be it will not be of any direct use in our investigation of chiral symmetry breaking . We have seen that pertubative results cannot be used in the Schwinger Dyson equations as they operate outside the areas in which such results are accurate, only an all orders result can be any use. A mechanism by which we may constrain Γ_T^μ is contained in the work of Curtis and Pennington on quenched QED4 [29] and in principle it should be applicable here. Firstly note that while Curtis and Pennington required that any vertex ansatz must produce multiplicatively renormalisable fermion functions we cannot pursue this option, multiplicative renormalisability is trivially guaranteed in a superrenormalisable theory. In its place, however, we may require that the dynamical mass function $\Sigma(p)/\beta(p)$ be gauge invariant. From (4.59) we expect higher orders in $1/N$ to introduce ever higher orders in $\xi\,\alpha/p$. Therefore, instead of considering the leading logarithmic terms we should investigate the leading power dependences, $(\xi\,\alpha/Np)^i$, at each order and require our vertex ansatz to be such that these terms cancel between β and Σ. We expect that this will lead to constraints on the form of Γ_T^μ in much the same way as Multiplicative Renormalisability did in Ref [29]. It should be stressed that the above is no more than a sketch. This calculation has not been attempted to date and furthermore promises to be extremely involved. In place of the single variable, β, and associated Schwinger Dyson equation of the quenched theory the above will require the analysis of the three variables, β, Σ and $\mathcal{G}$, and their associated Schwinger Dyson relations! However, at the present time we believe such a calculation to be possible and likely to constrain Γ_t^μ to a form not unlike the Curtis-Pennington transverse vertex.

The transverse parts of the vertex are not expected to important in the Landau gauge, extra terms such as the Curtis-Pennington form will not change our results qualitatively [27]. However, the introduction of power terms in a general gauge ensures that the chiral symmetry breaking mechanism is far more complicated. Unfortunately we must close this chapter on this rather inconclusive note. At the moment we can only outline a further avenue of investigation and obviously much more work is needed before we may say anything more concrete on the subject of gauge invariance in this model, other than there is a problem in gauges other than the Landau gauge.

Chapter 5: Summary, Lattices and Conclusions.

5.1 Introduction.

This chapter is intended to serve two purposes. Firstly we will summarise the results and conclusions of the past three chapters and then we will move on to discuss the results of lattice investigations into the possibility of chiral symmetry breaking in this model. Such a discussion has, up to this point, been notable by its absence. However, it is important that we compare our results with those from the lattice as the lattice approach attempts nothing less than the simulation of the full theory. It is therefore, in principle, a benchmark against which we may test our continuum approximations.

5.2 Summary.

The interest in QED3 was sparked by the original work of Appelquist *et al* [7]. They proposed that QED3 is a model in which the question of chiral symmetry breaking could be addressed straightforwardly, while at the same time being nontrivial. They proceeded to analyse the Schwinger Dyson equations in a perturbative fashion and were able to show that chiral symmetry breaking was indeed broken. However, this was claimed to take place only for three or less fermion flavours; for four or more the chiral symmetry of the bare lagrangian was preserved. Connections were quickly drawn between this phenomenon and a similar critical behaviour expected in four dimensional QED. Indeed lattice investigations in to QED3 and 4 tended to support this connection. This agreement was therefore considered as a success for both the lattice and the continuum Schwinger Dyson approaches.

However, it was not long before the perturbative approach to the Schwinger Dyson equations was questioned [22, 33]. Indeed if one "freed" the wavefunction renormalisation, $\beta(p)$, and treated it in the same nonperturbative manner as the mass function, $\Sigma(p)$, then quite a different scenario unfolded. The critical behaviour of Appelquist [12] disappeared completely.† In three dimensions the removal of Appelquist's critical value of N is accomplished by the nonperturbative behaviour of the wavefunction renormalisation, $\beta(p)$. Contrary to Appelquist's assumption, $\beta(p) = 1 + O(1/N)$, $\beta(p)$ was found to have an extremely strong N dependence and for low momenta to be of $O(1/N)$. It was precisely this nonperturbtive behaviour that lifted the $1/N$ in front of the Schwinger Dyson relation for $\Sigma(p)$ and consequently the critical behaviour with N. The resulting behaviour of the dynamical mass m with N was found to be,

$$m = \alpha \exp\{-cN/N_0\} \qquad , \qquad N \gg 1. \tag{5.1}$$

A further calculation also confirmed this qualitative behaviour for the rather more physical vertex ansatz, $\Gamma^\mu = \beta(k)\gamma^\mu$ [33]. Indeed it was concluded that perturbative truncations within the Schwinger Dyson equations were misleading. This is because in the infrared region such series become increasingly badly ordered, as illustrated by the series for $\beta(p)$. This is not an unknown phenomenon. In QCD calculations before one may hope to investigate infrared properties one often has to sum "leading" logarithms *to every order.*

After abandoning perturbation theory, however, we are directly confronted with how we are choose a consistent vertex ansatz. In the two analyses mentioned above both vertices failed to satisfy the Ward-Takahashi Identity, which surely must be a key requirement of any vertex. In Chapter 2 we investigated the existence of chiral symmetry breaking solutions using the Ball-Chiu ansatz[18]. This is a truely nonperturbative ansatz, a simple function of the two fermion functions β and Σ, that satisfies the Ward Identity by construction. The results of this investigation were found to reproduce the qualitative behaviour of the two earlier analyses, chiral symmetry was found to be broken for all N. Again the wavefunction renormalisation, β, acted to "support" the chiral symmetry breaking above the Appelquist critical point. Therefore we concluded that higher

† As a matter of interest it should be noted that in similar calculation for QED4 the original critical behaviour of the quenched system was found to be preserved [34]. It is therefore not clear whether the connection between the two cases is real or merely an illusion.

order corrections to the vertex, as contained in the the Ball-Chiu ansatz, do not appear to influence this mechanism qualitatively.

Before we proceed to summarise Chapter 3, however, we should mention a calculation due to Nash[32]. In this calculation the analysis of Appelquist is extended to $O(1/N)$ and the resulting system is found to exhibit the same critical behaviour as the original. This is in contradiction to our results, the reason, however, is quite easy to understand. Both Appelquist and Nash's calculations are really just perturbative calculations which are then exploited over the complete momentum range. Such an extension cannot be justified and surely not in the critical region where the characteristic scale of the problem m is vanishing. In the calculation of Nash the wavefunction renormalisation does enter but simply in a perturbative guise and is immediately cancelled against vertex corrections, implicitly assuming that the characteristic scale in the problem is α. The fact that this "higher order" kernel results in the same critical behaviour as the original up to relative factors of $O(1/N)$ is therefore not surprising. In contrast, we have explicitly shown that the behaviour of β is not consistent with perturbative expectations and that this critical behaviour is solely an artifact of throwing away what turns out to be a crucial dynamical variable.

In Chapter 3 we investigated the possible importance of the photon in the chiral symmetry breaking mechanism. All the calculations up to this point had treated the photon perturbatively, there was no route by which the behaviour of the fermions could influence the photon. In this sense the photon was constrained in the same way as the wavefunction renormalisation in the perturbative calculations of Nash and Appelquist. However, due to the difficulty expected in analysing the coupled fermion-photon Schwinger Dyson system, the investigation into the importance of the photon proceeded in a step by step manner:

i) Firstly the dynamical mass was introduced into the perturbative photon and the fermion equation solved with the simple "central" vertex ansatz, $\Gamma^\mu = (\beta(p) + \beta(k))\gamma^\mu$. This rather ad hoc formulation was designed to test the sensitivity of the system to the low energy behaviour of the photon. Indeed, contrary to naive expectations of a relative correction of $O(m/\alpha)$, the inclusion of the dynamical mass scale, albeit in a very crude manner, was found to have a large effect. The basic mechanism, however, remained unchanged. The wavefunction renormalisation continued to "support" the system above the Appelquist critical point, resulting in the dynamical mass retaining its exponential dependence on N. Indeed in the large N limit the effect of this new photon appears to be merely to introduce a relative shift in N, $N \to N + c$.

ii) Secondly, having determined the extent of the dynamical mass' sensitivity to the behaviour of the photon, the accuracy of the purely perturbative ansatz became of considerable importance. By analysing the Schwinger Dyson equation for the photon we were able to investigate the behaviour of the "full" photon. The results of this investigation illustrated that although the scaling behaviour given by the perturbative result was indeed well replicated by the "full" solutions, there appeared to be a consistent disparity of $O(N)$ between them at low momenta. By considering the Mandelstam approximation [24] this could be explained as a side-effect of the nonperturbative behaviour of $\beta(p)$ at low momenta. Such an overall factor of N, however, threatened to undermine the previous analyses in which purely perturbative photon ansätze were used. One could not assume that because this factor only became apparent at low momenta it would simply result in relative factors of $O(1/N)$. The sensitivity of the symmetry breaking mechanism to the behaviour of the photon had already been demonstrated.

iii) Therefore a third analysis, of the coupled photon and fermion Schwinger Dyson equations was performed for a simple vertex ansatz. Although the numerical results are far from complete it was evident that the dependence of m with N had undergone a dramatic change. Indeed, a simple analytic analysis suggests that the true dependence is of the form,

$$m = \alpha \exp\left\{ -c\,N^2/N_0^2 \right\} \quad , \qquad N \gg 1. \tag{5.2}$$

Unfortunately, the numerical data did not extend far enough to test this hypothesis or extract the large N scaling constant, N_0. Although the fall of m with N is dramatic, qualitatively the variation of β with $\ln(m/\alpha)$ is as before. We therefore expect it to continue supporting the broken symmetry phase for all N.

To summarise this chapter we may say that it is essential that, in addition to the wavefunction renormalisation, the photon is also be treated nonperturbatively before a true picture of the chiral symmetry breaking mechanism may be formed.

So far we have only considered working in the Landau gauge and the approximations have been tailored to this gauge. In Chapter 4 a start was made on the question of gauge invariance in this approach. By analysing a simple system we were able to show, using these same approximations, not only that the dynamical mass was very strongly gauge dependent, but also that chiral symmetry breaking was not expected to occur in all gauges! This is a very serious problem and indeed it is not expected to be qualitatively altered by the use of the Ball-Chiu vertex, an ansatz which satisfies the Ward Identity. The source of this behaviour is softening of the transverse component of the photon in the infrared. This amplifies any inadequacy of our vertex ansatz by factors of $O(\alpha/m)$ outside of the Landau gauge. Indeed, it is expected that this artificial behaviour is removed only by including the undetermined transverse component in addition to the Ball-Chiu ansatz, to form the full vertex of the model. By making a slight detour into related work in four dimensions we were able to sketch a route by which this unknown component may be constrained, but surely much work is required to complete such a calculation. However, if, as seems likely, this transverse component resembles the Curtis-Pennington form [29] then we may expect the key mechanism discovered in the Landau gauge to be preserved in all gauges. Indeed the whole behaviour of the system turns on the relation of $\beta(0)$ to m in the infrared which is dominated by the "central" part of the Ball-Chiu vertex. Introducing a transverse vertex such as the Curtis-Pennington form will not alter our results in the Landau gauge† . Given that this component results

† This has been explicitly checked for the original calculation of chapter 2 using a perturbative

in the gauge independence of our masses then this relation between m and N is expected to be preserved in all gauges.

5.3 Lattice Results.

In this section we will discuss primarily the results and method set out by Dagotto, Kocič and Kogut [13] in what is to our knowledge the most recent investigation into chiral symmetry breaking in this model.

In contrast to our continuum Schwinger Dyson approach, on the lattice the question of chiral symmetry breaking cannot be directly addressed, massless (bare) fermions will not "fit" onto a finite lattice. Rather the behaviour of the model and more precisely the chiral condensate, $\langle \overline{\psi}\psi \rangle$, is measured as a function of small bare masses. The condensate, $\langle \overline{\psi}\psi \rangle$, is the order parameter associated with the chiral phase transition, it is nonzero only when chiral symmetry has been broken. The proceedure by which these finite bare mass, m_0, results are extrapolated to the $m_0 = 0$ system is crucial, all finite size and statistical errors should be under control. In addition, if the smallest test mass is of $O(m_0)$ and the statistical accuracy of the data is of $O(1\%)$ then it seems unlikely that this approach would be able to distinguish between a theory where the chiral symmetry was broken to give masses of $O(0.01m_0)$ and a theory in which chiral symmetry remained unbroken. In their analysis Dagotto *et al* use test masses of $O(0.01)$ and lattice sizes of 8^3 and 10^3. In Fig 5.1 we have plotted their results for $N = 1, 2, 3$ in addition to the values of $\langle \overline{\psi}\psi \rangle$ obtained from our coupled analysis of Section 3.5.

From their results for $\langle \overline{\psi}\psi \rangle$ it appears that the extrapolation of m_0 to zero is not under control. In addition, our results would suggest that such lattice investigations would have trouble detecting chiral symmetry breaking for $N = 2$ let alone for $N = 3$ and higher. We may only conclude that the sizes of the lattices were not large enough to reach the continuum limit and/or the magnitudes of test masses were small enough to take the $m_0 = 0$ limit. Unfortunately, this

---

photon ansatz [27]. The behaviour of the coupled system is also expected to be insensitive to the details of the vertex *when working in the Landau gauge.*

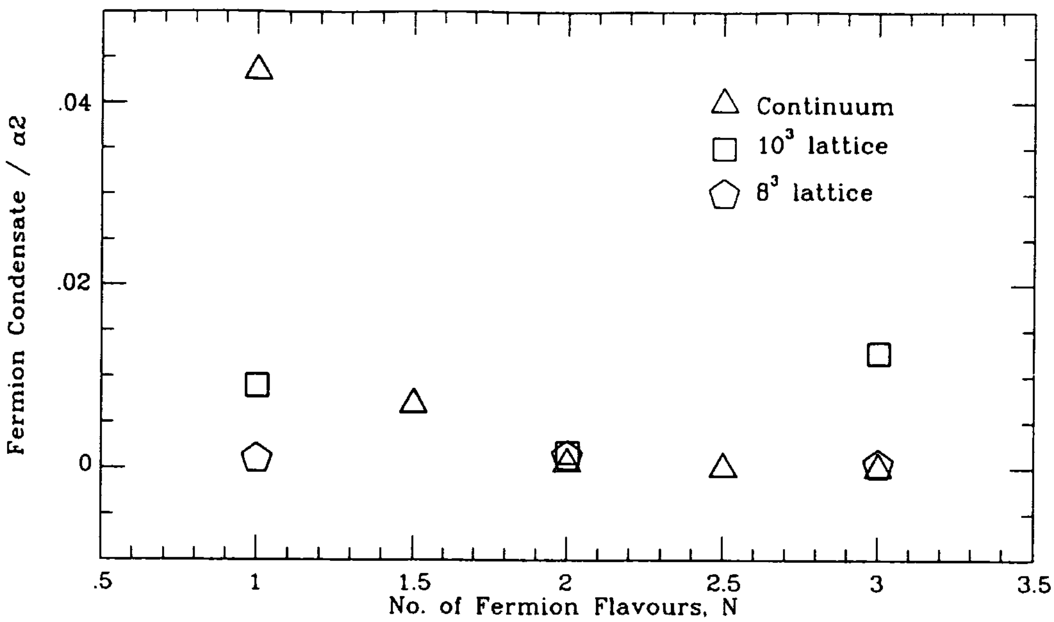

Figure 5.1 The values of $\langle \overline{\psi}\psi \rangle$ for the lattice calculation of Dagotto èt al and those obtained from the coupled Schwinger Dyson analysis of section 3.5

analysis seems not to be bettered significantly until either lattice algorithms or computing power significantly increase. Indeed the kind of exponential decrease in the dynamical mass with N predicted by our Schwinger Dyson analysis would presumably need exponential increases in lattice sizes! Therefore we must conclude that lattice investigations are not at a state where they may distinguish between the critical behaviour of Appelquist and the exponential behaviour expected from our investigations.

However, before we leave the lattice there is one point we should take note of. The chiral condensate for $N = 1$ we would expect to be the best determined of the unquenched data points, masses would presumably be close to unity. The results from the lattice investigations show a very large discrepacy between the 8^3 and 10^3 data. Indeed, Kogut and Hands [14] have recently made an extensive investigation into the finite size effects in the quenched, $N = 0$, version of this model. We cannot draw any direct conclusions from this investigation because in the quenched case the photon remains unrenormalised and this is to a logarithmically confining potential between "test" fermions. Consequenly, this system is expected to be far more sensitive to finite size effects than our quenched model. As an investigation of finite size effects, however, this analysis is very extensive, lattices ranging from 8^3 to 80^3 were used to test the sensitivity of $\langle \overline{\psi}\psi \rangle$ to the size of the lattice. The general conclusion of this study was that finite size effects tended to artificially suppress the chiral condensate and even lattices of 80^3 were not able to reach the continuum limit! We expect future analyses of the unquenched theory will also have to contend with this problem, though it expected not to manifest itself in such a severe form.

The finite size of a lattice means that momenta are naturally cutoff below a scale, ϵ, related to the "physical size" of the lattice. As pointed out by Kondo and Nakatani [26] introducing such an infrared cutoff into the Schwinger Dyson equations leads directly to the appearance of a critical value of N. Let us return to Section 3.2, linearising the relation for $\beta(p)$ and introducing an infrared cutoff ϵ we may write, (eqn. 3.21):

$$\kappa N\, A(0) \lesssim -\ln\left\{ \frac{\alpha^2+m^2}{\epsilon^2+m^2} \right\} . \tag{5.3}$$

Enforcing the inequality $\beta(0) \geq 0$ then constrains the behaviour of m with N,

$$\frac{\epsilon^2+m^2}{\alpha^2} \lesssim \exp\{ -\kappa N \} \tag{5.4}$$

Therefore as $N \rightarrow N_c \sim k^{-1}\ln(\alpha/\epsilon)$ the dynamical mass is forced to vanish. In their paper, Kondo and Nakatani show this rigorously for a bare and the simple vertex, $\Gamma^\mu = \beta(k)\gamma^\mu$. However, this behaviour is expected to be a general characteristic of the model. By introducing an infrared cutoff the logarithmic development of $\beta(p)$ with N is "frozen", as m falls below κ. Consequently $1/N$ factor in the relation for $\Sigma(p)$ will then be incompletely lifted, resulting in the reappearance of an Appelquist-like critical behaviour.

To summarise this section we may say that the lattice approach has some way to go before it may accurately determine the chiral properties of this model.

5.4 Final Conclusions.

This work has been concerned with the dynamical generation of mass. The path of our investigation has taken many turns, but at each turn we have been guided by the same underlying principle. We have attempted to investigate the chiral symmetry breaking in this model in as nonperturbative a manner as possible. Indeed, we found to do otherwise artificially constrained the behaviour of the system. The final result of our analysis is that in this model chiral symmetry is broken for all N, resulting in exponentially small fermion masses. Therefore, this model illustrates how a hierarchy of scales may be generated naturally.

APPENDIX I: The Photon Angular Integrals

In this Appendix we will list the analytic approximations to the angular integrals $\mathcal{A}_{\#}$ of Chapter 3. The integrals are defined by the generic relation,

$$\mathcal{A}_{\#} = \int \frac{\mathcal{T}_{\#}\, dz}{\beta^2(q)q^2 + \Sigma^2(q)} \, . \tag{I.1}$$

where $z = \cos\theta$ and θ is the angle between by k and q. The angular functions, $\mathcal{T}_{\#}$, are given by,

$$\begin{aligned}
\mathcal{T}_c =& \beta(k)\beta(q)\mathcal{C}\left\{ k^2 + 2k\cdot p - \frac{3}{p^2}(p\cdot k)^2 \right\} \\
\mathcal{T}_{s1} =& \beta(k)\beta(q)\mathcal{S}\left\{ 4k^4 + 2p^2k\cdot p + 2(k\cdot p)^2 \right. \\
& \left. + 4k^2p\cdot k - \frac{12}{p^2}(k^2(p\cdot k)^2 - (p\cdot k)^3) \right\} \\
\mathcal{T}_{s2} =& \Sigma(k)\Sigma(q)\mathcal{S}\left\{ 4k^2 - 2p^2 + 8p\cdot k - 12\frac{(p\cdot k)^2}{p^2} \right\} \\
\mathcal{T}_{v1} =& 2\beta(k)\beta(q)\mathcal{V}\left\{ \frac{3}{p^2}(k\cdot p)^2 - k^2 - p\cdot k \right\} \\
\mathcal{T}_{v2} =& 2\beta(q)\Sigma(k)\mathcal{V}\left\{ \frac{3}{p^2}(k\cdot p)^2 - k^2 + p^2 - 3p\cdot k \right\} .
\end{aligned} \tag{I.2}$$

The vertex functions $\mathcal{C}$ etc. are simply,

$$\begin{aligned}
\mathcal{C} = \frac{1}{2}\{\beta_k + \beta_p\} \quad , \quad \mathcal{S} = \frac{1}{2}\frac{\beta_k - \beta_p}{k^2 - p^2} \\
\mathcal{V} = \frac{\Sigma_k - \Sigma_p}{k^2 - p^2}
\end{aligned} \tag{I.3}$$

Where we have introduced the shorthand $f(k) \to f_k$ for brevity.

In the asymptotic region $k \gg p$ then $q \to k$ and the momentum scale p factors out,

$\underline{k \gg p\ :}$

$$\mathcal{T}_c = p^2\beta_k\left\{ -\frac{4}{5}d_0\left(\beta_k(\beta_2 + 3\beta_1) + \frac{2}{3}\beta_1^2\right) + 8\beta_k d_1\left(\frac{\beta_k}{3} + \frac{\beta_1}{5}\right) - \frac{16}{15}\beta_k^2 d_3 \right\}$$

$$
\begin{aligned}
\mathcal{T}_{s1} &= p^2\beta_k\Bigg\{ d_0\left(\beta_k\left(\frac{5}{3}\beta_1-\frac{2}{3}\beta_2\right)-\frac{4}{5}\beta_1\left(\beta_1-\frac{2}{3}\beta_2\right)\right) \\
&\qquad + d_1\left(4\beta_k\left(\beta_1+\frac{1}{5}\beta_2\right)+\frac{8}{5}\beta_1^2\right)-\frac{8}{15}\beta_1\beta_k d_3\Bigg\} \\
\mathcal{T}_{s2} &= \frac{p^2}{k^2}\Sigma_k\Bigg\{-\frac{d_0}{5}\left(\frac{4}{3}\Sigma_1(\beta_1+\beta_2)-\Sigma_k(\beta_1+2\beta_2)-\frac{4}{3}\beta_1\Sigma_2\right) \\
&\qquad + \frac{4}{5}d_1\left(\beta_1\left(\Sigma_k+\frac{2}{3}\Sigma_1\right)+\frac{1}{3}\beta_2\Sigma_k\right)-\frac{8}{15}\beta_1\Sigma_k d_3\Bigg\} \\
\mathcal{T}_{v1} &= \frac{p^2}{k^2}\beta_k\Bigg\{\frac{d_0}{5}\left(-2\Sigma_1\left(\Sigma_1-\frac{4}{3}\Sigma_2+\frac{\Sigma_k}{3}\right)+\frac{1}{3}\Sigma_k\Sigma_2\right) \\
&\qquad -\frac{2}{15}d_1\left(\Sigma_1(4\Sigma_1+\Sigma_k)+2\Sigma_2\Sigma_k\right)+\frac{8}{15}d_3\Sigma_1\Sigma_k\Bigg\} \\
\mathcal{T}_{v2} &= \frac{p^2}{k^2}\Sigma_k\Bigg\{\frac{d_0}{15}\left(\Sigma_1(14\beta_1+4\beta_2+4\beta_1+8\beta_k)+11\Sigma_2\beta_k\right) \\
&\qquad -\frac{d_1}{15}\left(\Sigma_1(8\beta_1+22\beta_k)+4\beta_k\Sigma_2\right)+\frac{8}{15}\beta_k\Sigma_1 d_3\Bigg\}
\end{aligned}
\tag{I.4}
$$

Where the f_i are defined as,

$$
f_i = l\frac{\partial}{\partial l}f(l)\bigg|_{l=Q} \quad , \quad Q = \mathrm{Max(k,p)}\ . \tag{I.5}
$$

In the other asymptotic region, $p \gg k$, the smaller momenta, k, generally does not factor out because $q \to p$ leaving two relevant momenta within the angular integrals. In this region we also see the gradient terms entering strongly in both $\mathcal{T}_{s2}$ and $\mathcal{T}_{v2}$.

$\underline{p \gg k\ :}$

$$
\mathcal{T}_c = \frac{4}{3}k^2\beta_k\{-d_0\beta_1(\beta_k+2\beta_p)+d_1\beta_p(\beta_k+\beta_p)\}
$$

$$\mathcal{T}_{s1} = \frac{2}{3}k^2\beta_k\left\{d_0\left((\beta_p-\beta_1)(\beta_p-\beta_k)+2\left(\beta_p-\beta_k-\frac{\beta_1}{2}\right)\beta_p\right)\right.$$
$$\left.+\,d_1\beta_p\,(\,\beta_p-\beta_k\,)\right\}$$

$$\mathcal{T}_{v1} = \frac{4}{3}\frac{k_2}{p^2}\beta_k\left\{d_0\,(\,\Sigma_1\,(\,2\Sigma_p-\Sigma_k\,)+2\Sigma_p\,(\,\Sigma_k-\Sigma_p\,)\,)\right.$$
$$\left.+\,d_1\Sigma_p\,(\,\Sigma_k-\Sigma_p\,)\right\}$$

$$\mathcal{T}_{s2} = 2d_0\Sigma_k\Sigma_p\,(\,\beta_k-\beta_p\,)+$$
$$\frac{k^2}{p^2}\Sigma_k\left\{+\frac{2}{3}d_1\,(\,(\beta_1-2\beta_k)\Sigma_p+(\beta_p-\beta_k)\Sigma_1+2\beta_p\Sigma_p\,)\right.$$
$$+\frac{d_0}{3}\,(\,\beta_p\,(\,8\Sigma_p-5\Sigma_1-\Sigma_2\,)+(5\beta_k-2\beta_1)\Sigma_1+\beta_k\Sigma_2\,)$$
$$\left.-\,d_0\frac{\Sigma_p}{3}\,(\,5\beta_1+\beta_2+8\beta_k\,)+2\Sigma_p\,(\,\beta_k-\beta_p\,)\left(d_2+\frac{2}{3}d_3\right)\right\}$$

$$\mathcal{T}_{v2} = 4\beta_p\Sigma_k d_0\,(\,\Sigma_p-\Sigma_k\,)+$$
$$\frac{k^2}{p^2}\left\{\frac{4}{3}\Sigma_k d_1\,(\,\beta_p\,(\,\Sigma_k-\Sigma_1-\Sigma_p\,)+\beta_1\,(\,\Sigma_k-\Sigma_p\,)\,)\right.$$
$$+\,d_0\Sigma_k\left((\Sigma_k-\Sigma_p)\left(\frac{8}{3}\beta_p-\frac{2}{3}\beta_2-2\beta_1\right)+\right.$$
$$\left.\beta_p\left(\frac{8}{3}\Sigma_k+2\Sigma_1+\frac{2}{3}\Sigma_2\right)+\frac{4}{3}\beta_1\Sigma_1\right)$$
$$\left.+\,4\beta_p\Sigma_k\,(\,\Sigma_p-\Sigma_k\,)\left(d_2+\frac{d_3}{3}\right)\right\}. \tag{I.6}$$

The denominator expansion coefficients, d_i, are found to be,

$$\begin{aligned}
d_0 &= \left\{\,\beta^2(Q)Q^2+\Sigma^2(Q)\,\right\}\\
d_1 &= 2\,d_0^2\,\left\{\,\beta(Q)\,(\,\beta(Q)+\beta_1(Q)\,)\;Q^2+\Sigma(Q)\,\Sigma_1(Q)\,\right\}\\
d_2 &= -\,\frac{d_1}{2}\\
d_3 &= d_0^2\left\{\Sigma(Q)\,(\,2\,\Sigma_1(Q)-\Sigma_2(Q)\,)-\Sigma^2(Q)+\right.\\
&\qquad\left.Q^2\;(\,\beta_1^2(Q)-2\,\beta(Q)\beta_1(Q)-\beta(Q)\beta_2(Q)\,)\right\}+d_1^2/d_0\,.
\end{aligned} \tag{I.7}$$

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