Quasi-integrable models in (2+1) dimensions

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by

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A thesis presented for the degree
of Doctor of Philosophy at the
university of Durham

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April 1992

ABSTRACT

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Recently σ-models have received a lot of attention for many reasons. One interesting aspect of the CP^n sigma models is the fact they are the simplest Lorentz invariant models which possess topologically stable (minimum of the action) solutions in (2+0) dimensions. Unfortunately, it appears that Lorentz covariance and integrability are incompatable in (2+1) dimensions.

In the literature a few integrable models were constructed in (2+1) dimensions at the expense of Lorentz invariance (*e.g.* modified chiral model,...). An alternative way to proceed is to retain Lorentz invariance and relax the property of integrability by replacing it with a new property of quasi-integrability.

Zakrzewski and others have constructed an example of such quasi-integrable models. Their example is based on the CP^1 model modified by the addition of two stabilising terms (the first called the "Skyrme-like" term and the second the "potential-like" term) to the basic Lagrangian. In this thesis we have addressed the following relevant questions: How unique is this model? What are the properties of its static structures (skyrmions)? Is it possible to generalise this model? Is quasi-integrabilty, as a property, shared by all CP^n models, or it is only restricted to the CP^1 model?

It turns out that the first stabilising term (*i.e* the Skyrme-like term) is only unique for CP^1 model and this uniqueness does not survive the generalisations to larger coset spaces, say, CP^2. The second stabilising term is not unique. By taking advantage of this observation, *i.e* arbitrariness of the potential term, a generalisation of Zakrzewski's model has become possible. Most important of all is the fact that all the CP^n models are quasi-integrable provided one incurs the size instabilities of their soliton solutions.

DECLARATION

The work presented in this thesis was carried out in the Department of Mathematical Sciences at the University of Durham between October 1989 and April 1992. This material has not been submitted previously for any degree in this or any other university.

No claim of originality is made for the first three chapters; the work in chapters 4, 5 and 6 is claimed as original, except where the authors have been specifically acknowledged in the text.

Most of the work in chapter 4 and 5 has been published in two papers by the author in collaboration with B.Piette, W.J.Zakrzewski, J.M.Izquierdo and J.A. de Azcárraga[1] [2] ; while the work in chapter 6, undertaken by the author with B. Piette and W. Zakrzewski, is available as a Durham University preprint[3] , which has been submitted for publication to Nonlinearity.

ACKNOWLEDGEMENTS

There is a number of people to whom I owe a great deal and would like to thank for their contributions towards the completion of this thesis:

I am indebted to my supervisor Wojtek Zakrzewski, for his continuous support and encouragment, true friendship and guidance during the time of my research studies. I would also like to thank him for his constructive suggestions in improving the form of this thesis, and also his patience whilst reading the manuscript and correcting my mistakes.

Many thanks to all the staff of the Mathematical Sciences Department for the help and advice they have given me in the past years. I would like to acknowledge Jan Govaerts for his sheer friendship and very useful discussions and suggestions on the substance of this thesis. I would also acknowledge useful discussions with Bernard Piette and Paul Sutcliffe. In addition, I have enjoyed stimulating conversations with David Hind and Richard Hall.

Financial support was partially provided by the Arab Internatial Student Aid, whom I acknowledge with thanks. I would like to thank Dr. Armitage and Dr. Richardson for their support during the Gulf crisis.

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I. INTRODUCTION

It is unanimously agreed that non-Abelian gauge theories are at the heart of particle dynamics. For example, the electroweak forces are described by the $U(1) \times SU(2)_L$ gauge theory. Likewise strong interactions, though there are some fundamental difficulties yet to be resolved, are best described by the $SU(3)_C$ gauge theory. To get a taste of the technical difficulties associated with the non-Abelian gauge theories let us consider the example of a $SU(2)$ gauge theory in (3+1) dimensions which is defined by the following Lagrangian

$$L = \frac{1}{4} tr(F \wedge F), \tag{1.1}$$

with F being the coefficient of a Lie algebra-valued curvature two form, which is given in terms of the one form connection A as

$$F = D(A) = dA + A \wedge A. \tag{1.2}$$

Most of the quantities we are interested in, after making a Wick rotation, are given in terms of functional integrals of the form

$$\int D[A_\mu] \exp^{-\int d^4x L(A_\mu)} O(A_\mu), \tag{1.3}$$

with $O(A_\mu)$ being some function of the fields A_μ. Unfortunately in most cases we are not able to compute analytically the above integrals, which prevents us from making any further progress with these theories at least at the nonperturbative level. However, one approach to the evaluation of the above integrals is to resort to numerical simulations; which is part of the reason for the strong interest in lattice field theory.

On the other hand one can try the so called quantum fluctuations approach which is based on an expansion around the stationary points of the Euclideanised action of the theory and then quantum perturbation theory of the resulting effective theory. Therefore, if we are to take this option seriously, we have to determine first all the stationary points of the action. But these are just the solutions of the Euclidean equations of motion. Thus, if we apply the variational principle to the Lagrangian given by (1.1), we get

$$D^*F = 0, \tag{1.4}$$

or in terms of local coordinate indices,

$$D_\mu F_{\mu\nu} = \partial_\mu F_{\mu\nu} + [A_\mu, F_{\mu\nu}] = 0, \tag{1.5}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \tag{1.6}$$

However, these are highly nonlinear second order equations in A_μ. Taking advantage of the Bianchi identity *i.e.*

$$DF = 0, \tag{1.7}$$

one can show that a subclass of solutions of the equations of motion (1.5) is provided by the solutions to the first order equations

$$F = \pm^* F \tag{1.8}$$

which are known as the "self-duality"(antiself-duality) equations. These equations stem from a Bogomolny bound on the energy density of the model. In fact the Bogomolny condition reads

$$L = \pm Q, \tag{1.9}$$

where Q is given by

$$Q = tr(F \wedge^* F). \tag{1.10}$$

When we integrate the charge density (1.10) over all of space-time we obtain the (unnormalised) topological charge of the given field configuration.

The most interesting solutions of the "self-duality" equations are those for which the action is finite as it is only for these solutions that perturbation theory makes sense. All finite action solutions of (1.8) have been determined by Atiyah and et al[4] [5] [6] . In the case of the plus (minus) sign in (1.8), the corresponding finite energy solutions are called instantons (anti-instantons). Because these solutions are absolute minima of the action, they are stable under fluctuations.

Calculating fluctuations around the instanton solutions of the equations of motion has turned out to be a hard problem. Hence, due to the complexity of non-Abelian gauge theories in 4 dimensions, people started to look at models in lower dimensions which share many features with the 4 dimensional theories. Many of such models have been constructed in (2+0) dimensions; here I am going to list some of them, such as: The $O(n)$ nonlinear σ-models, the principal chiral models $U(n)$, the projective space CP^n models and their non-Abelian generalisations *i.e.* Grassmannian models $G_{nm}(\mathbb{C})$.

All the above mentioned models are in many respects similar to the four dimensional non-Abelian theories. At the classical level they share with the non-abelian 4 dimensional theories the topological nontriviality of the space of solutions; they are geometrical in origin and they possess conformal invariance. At the quantum level, it is the dynamical mass generation and asymptotic freedom that are common aspects to both classes of theories.

The importance of low dimensional σ-models extends to many diverse disciplines. Since the low dimensional σ-models, from the point of view of performing computations, are easier to handle, they have been a test-ground for many ideas in particle physics. Progress in string theories reveals that the origin of many properties of strings and superstrings is also very much attached to the two dimensional nature of their world sheet. In fact string theorists have shown that at low energy, the physics of strings can be described by effective σ-models. Furthermore, σ-models have been used to trace the implications of nonlinearity in field theories *e.g.* the existence of extended structures (monopoles, skyrmions, vortices,....) and their scattering patterns. In addition, σ-models provide many examples of harmonic maps which are mathematically interesting in their own right. Moreover, they also provide examples of integrable systems in $(2+0)$ dimensions since one can show that their nonlinear equations of motion are the compatibility conditions for a Lax-pair containing a free parameter. Therefore they posses an infinite number of conservation laws which generate an infinite dimensional Lie (Kac-Moody) algebra. For further examples of low dimensional σ- models one can consult Zakrzewski[7] .

In this thesis we will put more emphasis on the CP^n models since they are the simplest of all the nonlinear σ-models. In chapter three we will discuss all (2+0) dimensional instanton and anti-instanton solutions that had been explicitly constructed[8] . Since the CP^n models are not integrable in $(2+1)$ dimensions, Din and Zakrzewski[9] [10] considered the static extended structures of the CP^1 model as slowly moving objects in $(2+1)$

dimensions, *i.e.* a system of these lumps, during its time evolution, can be approximated by a sequence of static configurations for which the parameters are appropriately modified. The main advantage of this collective coordinates approximation, lies in the fact that it truncates the infinite number of degrees of freedom of the solution space of the model to a finite number corresponding to the parameters of the solution. Din and Zakrzewski have argued that the classical dynamics of such structures can be described by the geodesic motion on the Kähler manifold of the parameters of the extended structures solutions. In fact, a similar approach was adopted by Manton in the study of the dynamics of monopoles[11] . This suggests that extended structures in (2+0) dimensions are reasonable candidates for being soliton-like objects in (2+1) dimensions. To check various nonstatic properties of the extended structures is a highly nontrivial task; in most cases one has to resort to numerical work, or to drastic and sometimes not very reliable approximations.

For this purpose Zakrzewski and others[12] conducted further studies of the same model by performing many numerical simulations of the full field equations. Their studies revealed that the time evolution of static lumps in (2+1) dimensions is very much like that of solitons in $(1+1)$ dimensions. When the lumps are sent towards each other, they scatter at $90°$ to the original direction and shrink rapidly after their scattering, until they become too spiky, so that the numerical procedure breaks down. In addition, the studies conducted[13] have disclosed that the CP^1 lumps are not stable under perturbations, in the sense that if they are subjected to a perturbation (*e.g.* squashing, scattering effects......), they become either very broad or more and more point-like (spiky). However, one can attribute this behaviour of CP^1 lumps to the conformal invariance of the model in (2+0) dimensions, which makes the evolution operator insensitive to the size of CP^1 lumps or, in simpler terms, the CP^n models as they stand do lack an intrinsic scale.

Independently Leese[14] has used the collective coordinates approximation in studying the evolution of the static CP^1 lumps. This approximation is very good for small velocities, but it is less clear how good it is for higher velocities. The results of refs.[10,11] are pretty much in agreement with each other, thus supporting the use of the collective coordinates approximation approach to study the main features of the scattering properties of lumps.

Had the CP^1 lumps shown size stability under perturbations, one could have claimed that some of the properties of solitons may not be restricted to the integrable models in which they arise; in fact they may arise in many models whether they are integrable or not. Consequently we are required to adopt a finer division among the models in terms of their integrability *i.e.* we have basically three categories of models, those which are integrable, quasi-integrable and not-integrable.

So if we can overcome the lumps instability observed in the first simulations, then the modified CP^1 model could become an example of a quasi-integrable model in (2+1) dimensions. A similar situation was faced in the original soliton model of the proton[15] in (3+1) dimensions. However, there is much arbitrariness in the choice of the various terms used in the description of a proton treated as soliton. The original suggestion was based on the ideas of Skyrme[16] . It involved $SU(2)$ (and later a $SU(3)$) σ-model which, in addition to the usual term $tr\partial_\mu U\partial^\mu U^{-1}$, had an additional "Skyrme" term involving four derivatives. The additional term has the form

$$L_{Sky} = \frac{1}{32e^2}tr([U^{-1}\partial_\mu U, U^{-1}\partial_\nu U][U^{-1}\partial^\mu U, U^{-1}\partial^\nu U]), \tag{1.11}$$

where U is an $SU(2)$ valued matrix and e^2 is a coupling constant. This term, it was argued, is uniquely determined by various conditions imposed on the model, such as positivity of the Hamiltonian, Lorentz invariance, etc...

Zakrzewski and others[17] , inspired by the old idea of Skyrme, suggested a modification of the CP^1 model by adding a new additional term made up of four derivatives, which they called the Skyrme-like term. They also added a potential term. The Lagrangian of the modified CP^1 model, when the equivalent formulation of CP^1 model in terms of the variables of the $O(3)$ model is used (this is explained in chapter three), is given by

$$\begin{aligned} L = {} & \partial_\mu\vec{\phi}\cdot\partial^\mu\vec{\phi} - \theta_1[(\partial_\mu\vec{\phi}\cdot\partial^\mu\vec{\phi})^2 - (\partial_\mu\vec{\phi}\cdot\partial_\nu\vec{\phi})(\partial^\mu\vec{\phi}\cdot\partial^\nu\vec{\phi})] \\ & - \theta_2(1+\phi^3)^4. \end{aligned} \tag{1.12}$$

Indeed, all the results obtained[17][18] have shown that the additional terms stabilise the solitons, and, at the same time, have little effect on the dynamics of the scattering; namely, in ref. [18] it was shown that the scattering at 90° observed in the original CP^1 model is reproduced in the modified model when the velocities of the incoming solitons are above a critical value V_{cr}. This is partly due to the complicated forces acting on a system of two solitons as a result of the introduction of the additional terms. These forces are mainly repulsive; if one places two solitons some distance apart after some rearrangement, they start moving away from each other. Moreover, the size of solitons is fixed; so in the simulations their sizes tend to oscillate around their correct values. However, apart from that, most of the solitonic properties of the extended structures are not seriously modified by the addition of the stabilising terms.

At this stage it is very important to understand the geometrical origin of the Skyrme-like terms, to be able to answer the following three questions. How many candidates are there for the Skyrme-like term in $(2+1)$ dimensions ? Are they equivalent ? And how do the Skyrme-like terms generalise to larger cosets *i.e.* CP^n for $n > 1$, or even to larger group manifolds *i.e.* $SU(n), n > 2$? The fourth chapter provides the answers to these questions, and reports on our investigations on the possibility of constructing further additional terms other than Skyrme terms, which could be of topological nature, *e.g.* the WZW term and the Hopf term.

At this stage a natural question arises: can one construct a quasi-integrable modified CP^1 model which admits k displaced static Skyrmions ? At first sight, the answer to this question is far from obvious. However, in chapter five we construct classes of modified CP^1 models which possess k displaced static Skyrmions. Since these configurations are expected to have solitonic nature in (2+1) dimensions we also investigate their topological stability.

Motivated by the two incentives to show that all CP^n models are potentially quasi-integrable, and to shed more light on the scattering properties of the interacting CP^n solitons we consider the CP^2 model and investigate its soliton scattering. This is the topic addressed in chapter six. First, we study the solitonic properties of the static solutions of the CP^2 model in $(2+1)$ dimensions. We also show that, as in the CP^1 model, in head-on scatterings solitons scatter at 90° and undergo a shift along their trajectories. In addition, we also modify the CP^2 model by the addition of two further terms to the basic Lagrangian and investigate the effects of the additional terms on the behaviour of the CP^2 lumps. The first additional term is an analogue of the CP^1 Hopf term, which in the CP^2 case is not locally a total divergence and gives a nonvanishing contribution to the equations of motion. We show that the Hopf term has a subtle rotational effect on the CP^2 lumps, *i.e.* it rotates the different parts of the extended objects unequally. The second term is one of the possible candidates for a Skyrme-like term in CP^2 spaces. We compute its contribution to the equations of motions and show that, as in the CP^1 model, it fixes the size of the CP^2 lumps.

Chapter seven summarises the main results of this thesis and indicates some of the topics which could be studied further to broaden and deepen our knowledge of this interesting area of mathematical physics.

II. GEOMETRY OF SIGMA-MODELS

2.1 Structures on complex manifolds

Since all the sigma models we are going to encounter in this thesis are of complex nature, *i.e.* based on Kähler manifolds, it becomes very compelling to understand their structure. Thus it is useful to include this introductory section to describe the fundamental tools we can use to understand the geometry of complex manifolds. Clearly for further discussion one can always consult the vast literature on this subject[19] [20] [21] .

To define complex manifolds one first needs to set up the definition of holomorphic functions, or in a larger context, of holomorphic maps.

Definition [2.1] :
Let $f : \mathbb{C}^m \longrightarrow \mathbb{C}$; then $f = (f_1 + if_2)$ is said to holomorphic if it satisfies the Cauchy-Riemann relations for each $z^\mu = x^\mu + iy^\mu$ *i.e.*

$$\frac{\partial f_1}{\partial x^\mu} = \frac{\partial f_2}{\partial y^\mu} \qquad \frac{\partial f_2}{\partial x^\mu} = \frac{-\partial f_1}{\partial y^\mu}. \tag{2.1}$$

As a straightforward generalisation a map

$$(f^1, f^2, ..., f^\lambda, ..., f^n) : \mathbb{C}^m \longrightarrow \mathbb{C}^n$$

is called holomorphic if and only if each component f^λ $(1 \leq \lambda \leq n)$ is holomorphic.

Having defined holomorphicity of a function f, one can define complex manifolds. M is called a complex manifold of complex dimension m $(dim_{\mathbb{C}} M = m)$ if the following three axioms are satisfied :

(1) M is a topological space,

(2) M is provided with a family of pairs $\{(U_i, \phi_i)\}$, where U_i is a family of open sets which cover M and ϕ_i is a homeomorphism from U_i to an open set of $\mathbb{C}^m$,

(3) Given U_i, U_j such that $U_i \cap U_j \neq \phi$ then the map $\psi_{ji} = \phi_j \circ \phi_i$ from $\phi_i(U_i \cap U_j)$ to $\phi_j(U_i \cap U_j)$ is holomorphic.

It is worth emphasising that complex manifolds have the liberty to admit more than one complex structure. In fact if $T_1(U_i, \phi_i)$ and $T_2(U_i, \phi_i)$ are two atlases of M (an atlas on M is a family of coordinate systems $(U_i, \phi_i)_{i\in I}$ with the properties that the family $(U_i, \phi_i)_{i\in I}$ covers M, and any two coordinate systems in the family are smoothly related) of M then if $T_1 \cap T_2$ is another atlas which satisfies all the axioms of complex manifolds then they are said to define equivalent complex structures.

Good examples of complex manifolds are the complex projective spaces CP^n and their non-Abelian generalisations, the Grassmannian manifolds $G_{nm}(\mathbb{C})$. The CP^n spaces are defined as the quotient of $\mathbb{C}^{n+1}$ by the equivalence relation $\sim$, which states that $w \sim z$ if there exists a non vanishing λ such that

$$w = \lambda z. \tag{2.2}$$

More precisely if a point $z \in \mathbb{C}^{n+1}$ has the coordinates $(z^\circ, z^1, ..., z^n)$, then by the equivalence relation $\sim$ all points given by $(\lambda z^\circ, \lambda z^1, ..., \lambda z^n)$ belong to the same equivalence class. But these equivalence classes are the straight lines passing through the origin provided that $z \neq 0$. Therefore one can write the CP^n manifolds as

$$CP^n = \frac{(\mathbb{C}^{n+1} - 0)}{\sim}. \tag{2.3}$$

For the Grassmannian manifolds $G_{nm}(\mathbb{C})$, we proceed as follows; let $M_{nm}(\mathbb{C})$ be the set of complex $n \times m$ matrices of rank $m \leq n$. Take $A, B \in M_{nm}(\mathbb{C})$ and define an equivalence relation $\sim$ by $A \sim B$ if there exists $g \in GL(n, \mathbb{C})$ such that $B = gA$. Then the $G_{nm}(\mathbb{C})$ is identified with $M_{nm}(\mathbb{C})/GL(n, \mathbb{C})$, or in different terms, the $G_{nm}(\mathbb{C})$ spaces are sets of m-dimensional subspaces of $\mathbb{C}^{n+1}$ (note that $CP^n = G_{n1}(\mathbb{C})$).

Attached to every point $P \in M$ we have a tangent space $T_p(M)$ which is spanned by $2m$ vectors

$$(\frac{\partial}{\partial x^{\mu_1}}, ..., \frac{\partial}{\partial x^{\mu_m}}, \frac{\partial}{\partial y^{\mu_1}}, ..., \frac{\partial}{\partial y^{\mu_m}}), \tag{2.4}$$

where $z^\mu = x^\mu + iy^\mu$ are the coordinates of P in the chart (U, ϕ). Similarily, the cotangent space $T^*_p(M)$ is spanned by

$$(dx^{\mu_1}, ..., dx^{\mu_m}, dy^{\mu_1}, ..., dy^{\mu_m}). \tag{2.5}$$

However, it is more convenient to work with complex orthonormal basis for both $T_p(M)$

and $T_p^*(M)$. Thus for the tangent space attached to P they take the form

$$\frac{\partial}{\partial z^\mu} = \frac{1}{2}(\frac{\partial}{\partial x^\mu} - i\frac{\partial}{\partial y^\mu}) \qquad \frac{\partial}{\partial \bar{z}^\mu} = \frac{1}{2}(\frac{\partial}{\partial x^\mu} + i\frac{\partial}{\partial y^\mu}), \tag{2.6}$$

whereas for the cotangent space at P, the basis read

$$dz^\mu = dx^\mu + idy^\mu \qquad d\bar{z}^\mu = dx^\mu - idy^\mu. \tag{2.7}$$

A very important object defined on complex manifolds is the almost complex structure $\mathbb{J}$, which is a linear map

$$\mathbb{J}_p : T_pM \longrightarrow T_pM$$

such that

$$\mathbb{J}_p\frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial y^\mu} \qquad \mathbb{J}\frac{\partial}{\partial y^\mu} = -\frac{\partial}{\partial x^\mu}. \tag{2.8}$$

Hence $\mathbb{J}_p$ is a real tensor field of type $(1,1)$ with $\mathbb{J}^2 = -1$. In our complex basis it takes the form,

$$\mathbb{J}_p = \begin{pmatrix} i\mathbb{I} & 0 \\ 0 & -i\mathbb{I} \end{pmatrix}, \tag{2.9}$$

and, consequently, the tangent space $T_P^{\mathbb{C}}M$ of complex manifolds is split into two disjoint vector spaces

$$T_p^{\mathbb{C}}M = T_p^+ \oplus T_p^-. \tag{2.10}$$

It is worth emphasising that any complex manifold with $dim_{\mathbb{C}} = m$ locally admits a tensor $\mathbb{J}$ which squares to $-\mathbb{I}$. However, $\mathbb{J}$ may be patched across charts and defined globally only on complex manifolds[19] [20] .

Other very interesting structures that one can define on complex manifolds are differential forms. For example a (r,s)-form ω, in the basis given by (2.7), is written as (for further details see[19] [20])

$$\omega = \frac{1}{r!s!}\omega_{\mu_1.....\mu_r\nu_1........\nu_s}\ dz^{\mu_1} \wedge .... \wedge dz^{\mu_r} \wedge \overline{dz}^{\nu_1} \wedge .... \wedge \overline{dz}^{\nu_s}, \tag{2.11}$$

where the components are totally antisymmetric in μ and ν separately.

So far we have been handling complex manifolds only as topological spaces. In fact one can assign a certain geometry to our manifold by choosing a metric g. Take, say, $z = (x + iy)$ and $w = (u + iv) \in T_p(M)$ and define g so that

$$g(z,w) = g_p(x,u) - g_p(y,v) + i[g_p(x,v) + g_p(y,u)]. \tag{2.12}$$

Then the components of the metric tensor in the complex basis (2.6) take the form:

$$\begin{aligned} g_{\mu\nu} &= g_p(\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial z^\nu}) \\ g_{\mu\overline{\nu}} &= g_p(\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial \overline{z}^\nu}) \\ g_{\overline{\mu}\nu} &= g_p(\frac{\partial}{\partial \overline{z}^\mu}, \frac{\partial}{\partial z^\nu}) \\ g_{\overline{\mu}\overline{\nu}} &= g_p(\frac{\partial}{\partial z^{\overline{\mu}}}, \frac{\partial}{\partial z^{\overline{\nu}}}) \end{aligned} \tag{2.13}$$

with

$$g_{\mu\nu} = g_{\nu\mu} \quad , \quad g_{\overline{\mu}\overline{\nu}} = g_{\overline{\nu}\overline{\mu}} \quad , \quad \overline{g_{\mu\overline{\nu}}} = g_{\overline{\mu}\nu} \quad , \quad \overline{g_{\mu\nu}} = g_{\overline{\mu}\overline{\nu}}. \tag{2.14}$$

If a Riemmanian metric g of a complex manifold M satisfies

$$g_p(JX, JY) = g_p(X,Y) \tag{2.15}$$

at each point $P \in M$, and for each $X, Y \in T_pM^{\mathbb{C}}$, then g is said to be a hermitian metric and the pair (M, g) is called a hermitian manifold. It is easy to show that for a hermitian metric g the components $g_{\mu\nu}$ and $g_{\overline{\mu}\overline{\nu}}$ vanish; thus the hermitian metric form is given by

$$g = g_{\mu\overline{\nu}} \quad dz^\mu \wedge dz^{\overline{\nu}} + g_{\overline{\mu}\nu} \quad dz^{\overline{\mu}} \wedge dz^\nu. \tag{2.16}$$

Of all hermitian manifolds we are particularly interested in a subclass called Kähler manifolds. To define them, let us define first the so-called Kähler form Ω as

$$\Omega_p(X,Y) = g_p(JX, Y). \tag{2.17}$$

From this definition it follows that

(i) Ω is antisymmetric *i.e.* $\Omega(X,Y) = -\Omega(Y,X)$,

(ii) Ω is invariant under the action of J *i.e.*

$$\Omega(JX, JY) = \Omega(X, Y), \tag{2.18}$$

(iii) Ω is a two form of type $(1,1)$ with the components given by

$$\Omega_{\mu\nu} = \Omega_{\overline{\mu\nu}} = 0 \qquad \Omega_{\mu\overline{\nu}} = -\Omega_{\overline{\nu}\mu} = ig_{\mu\overline{\nu}}. \tag{2.19}$$

Thus Ω can be written as

$$\begin{aligned} \Omega &= -J_{\mu\overline{\nu}} \quad dz^{\mu} \wedge dz^{\overline{\nu}}, \\ J_{\mu\overline{\nu}} &= -ig_{\mu\overline{\nu}}. \end{aligned} \tag{2.20}$$

Then we have the following definition.

Definition [2.3]

A Kähler manifold is a hermitian manifold (M, g) whose Kähler form Ω is closed ($d\Omega = 0$). The metric g is called a Kähler metric of M.

In fact one can show that a hermitian manifold (M, g) is a Kähler manifold if and only if the almost complex structure J satisfies

$$\nabla_{\mu} J = 0, \tag{2.21}$$

where ∇_{μ} is the Levi-Civita connection associated with g. The condition of the Kähler form Ω is satisfied provided the following equations hold:

$$\frac{\partial g_{\mu\overline{\nu}}}{\partial z^{\lambda}} = \frac{\partial g_{\lambda\overline{\nu}}}{\partial z^{\mu}} \qquad \frac{\partial g_{\mu\overline{\nu}}}{\partial \overline{z}^{\lambda}} = \frac{\partial g_{\mu\overline{\lambda}}}{\partial \overline{z}^{\nu}} \tag{2.22}$$

Suppose a hermitian metric g is given in a chart U_i by

$$g_{\mu\overline{\nu}} = \partial_{\mu}\partial_{\nu} K_i. \tag{2.23}$$

Then this metric satisfies (2.22), and hence the pair (M, g) defines a Kähler manifold. Conversly, it can be shown that any Kähler metric is expressed locally as (2.23)[19][20] ; the function K_i is called the Kähler potential of the Kähler metric.

Let us close this section with a few useful facts[19][20] .

(*a*) S^2 is the only sphere which admits a complex structure, since $S^2 \cong CP^1$, it is actually a Kähler manifold.

(*b*) A product of two odd-dimensional spheres $S^{(2m+1)} \times S^{(2n+1)}$ always admits a complex structure, but this complex structure does not admit a Kähler metric.

(*c*) Any complex submanifold of a Kähler manifold is Kähler.

(*d*) A compact Kähler manifold with a vanishing first Chern class $C_1(m) = 0$ is called Calabi-Yau manifold which has a great relevance in superstring compactification.

2.2 Fiber bundles

It is helpful to introduce the notion of fibre bundles. Fiber bundles have been used in the formulation of gauge theories and σ-models in physics[22] [23] . A fibre bundle, intuitively, looks locally like a product $I\!R^N \times I\!R^M$ but these two spaces may be glued together globally in a non-trivial way. Tangent bundles are familiar examples of fibre bundles given by the product of the base manifold with its tangent space. A formal definition of fibre bundles and related concepts is as follows.

Definition [2.1]:

A fibre bundle $F \rightarrow E \rightarrow X$ is a treble manifold E, X and F with a smooth projection map π from E onto X. At each point $x \in X$, the set $\pi^{-1}(x) = F_x$ is diffeomorphic to F and is called the fibre at x. The space E is called the total space and, for simplicity, it will be interchangeably referred to as the bundle itself. A section of the bundle E is a map $S : X \rightarrow E$ such that $\pi \circ S = id_x$, and id_x is the identity map on X. A trivial bundle is a bundle whose total space is the product $X \times F$ (globally) with a natural projection map onto X. A bundle is therefore always trivial (locally); if one considers the cover U_a of X then on each U_a, there exists the commutative diagram.

$$\begin{array}{ccc} \pi^{-1}(U_a) & \xleftarrow{S_a} & U_a \times F \\ \downarrow & & \downarrow{\scriptstyle Pr_{U_a}} \\ U_a & \xleftarrow{id_x} & U_a \end{array}$$

where Pr_{U_a} is the natural projection onto U_a and S_a is the local trivialisation of E given by the diffeomorphism $U_a \times F \rightarrow \pi^{-1}(U_a)$. This trivialisation serves as a local chart for the bundle; thus a section $S(x)$ of E may be given a pair of coordinates, namely that of the base space X and of the fibre F, using the local trivialisation map

$$S(x) = S_a(x, f) = (x, f), \tag{2.24}$$

where f are the fibre coordinates and $x \in U_a$. Note that the coordinatization depends on the local chart U_a. In order to see the coordinatization in other (overlapping) charts and for the bundle to be glued together from such local charts, we consider two neighbourhoods U_a and U_b at x. On the overlap $U_a \cap U_b$, one can define the transition functions

$\Omega_{ab} : U_a \cap U_b \to Diff(F)$ such that

$$\Omega_{ab} = S_a^{-1} \circ S_b. \tag{2.25}$$

In most physical examples[22] [21] , $Diff(F)$ is realised by a much smaller symmetry group G through the monomorphism $\rho : G \to Diff(F)$

$$\Omega_{ab} = \rho \circ t_{ab}, \tag{2.26}$$

where $t_{ab} : U_a \cap U_b \to G$ obeys the relations

$$\begin{aligned} t_{aa} &= e && \forall x \in U_a \\ t_{ab}t_{ba} &= e && \forall x \in U_a \cap U_b \\ t_{ab}t_{bc}t_{ca} &= e && \forall x \in U_a \cap U_b \cap U_c \end{aligned} \tag{2.27}$$

and where e is the identity in G. In this case the bundle E is called a G-bundle and the group G is said to be the structure group of E. Note that a change of local charts from that determined by U_a to that of U_b, results in a change of the coordinatization of the section S as

$$S_b(x, f) = S_a(x, f)\Omega_{ab}(x).$$

or,

$$S_b(x, f) = (x, t_{ab}^{-1}(x) f). \tag{2.28}$$

Equipped with the above definitions, one has the following theorem.

Theorem [2.1] :

Given spaces X,F, a covering U_a and transition functions t_{ab} satisfying (2.27), there exists a G-bundle $F \to E \to X$ over X determined up to an isomorphism.

Isomorphic G-bundles over X form an equivalence class. The following proposition gives a condition when an isomorphism between G-bundles can be established.

Proposition [2.2] :

Let t_{ab} and t'_{ab} be two sets of transition functions defined on the covering U_a of X. They define isomorphic G-bundles over X if and only if there exists functions $\lambda_a : U_a \to G$ such that

$$t'_{ab} = \lambda_a t_{ab} \lambda_b. \tag{2.29}$$

From the theorem (2.1) and the proposition (2.2), the isomorphic G-bundles over X fall into the same set of equivalence classes irrespective of their fibres. Thus it is sufficient to consider one representative of each fibre space F to demonstrate the necessary properties of the G bundle. One fibre space which has a natural group action from both left and right, is the group G itself. The G-bundle that has G as its fibre is called the principal G- bundle $(G \to P \to X)$, where its total space now is denoted by P. The group G is often called the gauge group. The local section σ is trivialised by

$$\sigma(x) = (x, g_a), \qquad x \in U_a \qquad g_a \in G. \tag{2.30}$$

The natural right G-action on the bundle P is defined by

$$r_g \sigma = (x, g_a g) \qquad x \in U_a \quad , \quad g \in G. \tag{2.31}$$

This gives an automorphism of P which maps each fibre into itself

$$\begin{array}{ccccc} G & \longrightarrow & P & \xrightarrow{r_g} & P \\ & & \searrow & & \swarrow \\ & & & x & \end{array}$$

Such automorphisms are called gauge transformations of P.

Associated with a bundle P are various other G-bundles built out of different fibre spaces, depending on the structure that is required of F. The construction of such bundles is given by first defining a right G-action on $P \times F$ by

$$T_g(p, \psi) = (r_g p, U(g^{-1})\psi) \qquad g \in G \quad , \quad p \in P, \tag{2.32}$$

where ψ is a function of $x = \pi(p)$ taking values in F and $U(g^{-1})$ is the representation of g^{-1} on F. The bundle E associated with P with fibre F then has the total space taken to be the quotient of $P \times F$ with respect to the G action (2.33) *i.e.*

$$E = \frac{P \times F}{G} = P \times_G F. \tag{2.33}$$

The projection map π_E from E to X is obtained by the following commutative diagram

$$\begin{array}{ccc} P \times F & \xrightarrow{Pr_p} & P \\ \downarrow{\scriptstyle \chi} & & \downarrow{\scriptstyle \pi} \\ E & \xrightarrow{\pi_E} & X \end{array}$$

where χ is the projective map of $P \times F$ to the set of equivalence classes under (2.32) and thus

$$\pi_E(\chi(p, \psi)) = \pi(p). \tag{2.34}$$

A section Ψ of E is given by.

$$\Psi(x) = [\sigma(x), \psi(x)], \tag{2.35}$$

where $[.,.]$ is the equivalence class of $P \times F$ under the equivalence relation

$$[\sigma(x), \Psi(x)] = [r_g \sigma(x), U^{-1}\Psi]. \tag{2.36}$$

Under the change of local sections $\sigma_b(x) = \sigma_a \Omega_{ab}$ for $x \in U_a \cap U_b$, one finds

$$\begin{aligned} \Psi(x) &= [\sigma_a, \Psi_a] = [\sigma_a, \Psi_b] \\ &= [\sigma_a \Omega_{ab}, \Psi_b] \\ &= [\sigma_a, \Omega_{ab}\Psi_b], \end{aligned} \tag{2.37}$$

where Ψ_a and Ψ_b are functions on U_a, U_b respectively. Hence the function ψ obeys the relation

$$\Psi_a(x) = \Omega_{ab}\Psi_b \tag{2.38}$$

To close this section, we shall briefly look into the notion of lifting of structures on fibre bundles. A lifting is to be considered if an object or a specific property defined on the base space are to be extended or generalised to a corresponding object or a property defined on the whole bundle . A good example for lifting objects from the base manifold to the bundle E is the group action itself. Consider the bundle $F \to E \to X$ where X has a G-action τ_G defined on it. The action τ_G is said to be lifted to E if there exists a G-action $\tau_G^\uparrow$ on E such that the following diagram commutes.

$$\begin{array}{ccc} G \times E & \xrightarrow{T_G^\uparrow} & E \\ \Big\downarrow{\scriptstyle (id_G;\pi)} & & \Big\downarrow{\scriptstyle \pi_E} \\ G \times X & \xrightarrow{T_G} & X \end{array}$$

2.3 Sigma Models And Their Topology

[2.3.1] General Aspects Of Sigma Models

A sigma model[24] [22] uses a set of fields ϕ_i which map a $(d+1)$ dimensional pseudo-Riemannian space-time manifold with a signature $(1,-1,-1,....,-1)$ to a Riemannian manifold M endowed with a metric g. The action describing these models take the form

$$\frac{1}{4}\int d^dx dt \quad g_{ij}\partial_\mu\phi^i\partial^\mu\phi^j, \tag{2.39}$$

where $\phi^i(i = 1,....,n)$ and g_{ij} are the coordinates and the metric on M respectively. The Greek indices range over $0,1,2,.....,d$ and they are associated with the space-time coordinates. Minimising the above action using the variational principle leads to the equations of motion which have the form.

$$\partial_\mu\partial^\mu\phi^i + \Gamma^i_{jk}\partial_\mu\phi^j\partial^\mu\phi^k = 0. \tag{2.40}$$

Note that these equations are nonlinear due to the presence of the quadratic term in the derivatives of the fields. The coefficients of this term are the usual Christoffel symbols (Γ^i_{jk}), showing that in σ-models nonlinearities arise purely from the curvature of the target manifold M. As we shall see later it is this type of model that leads to soliton solutions in two dimensions. Moreover these models possess Lorentzian invariance *i.e.* they are invariant under the action of $SO(d,1)$ on the space-time manifold.

Sometimes it is more convenient to use an alternative formulation of these models which is based on taking a free field theory, containing m independent fields ($m > n$), and then using Lagrange multipliers to impose $m - n$ constraints, so as to restrict the fields to lie on M. A typical example of the last procedure is the $O(3)$ sigma model, in which case M is just the two sphere S^2. It is common to use the fields $\vec{\phi} = (\phi_1, \phi_2, \phi_3)$, with the constraint $\vec{\phi} \cdot \vec{\phi} = 1$. Introducing Lagrange multiplier λ, the relevant Lagrangian becomes

$$L = \frac{1}{4}\partial_\mu \vec{\phi} \cdot \partial^\mu \vec{\phi} - \lambda(\vec{\phi} \cdot \vec{\phi} - 1). \tag{2.41}$$

The equations of motion are given by

$$\partial_\mu \partial^\mu \vec{\phi} + (\partial_\mu \vec{\phi} \cdot \partial_\mu \vec{\phi})\vec{\phi} = 0, \quad \vec{\phi} \cdot \vec{\phi} = 1. \tag{2.42}$$

It is convenient to consider an alternative setting for σ-models[25] *i.e.* describe them in a more abstract and geometrical approach. The virtue of resorting to the geometrical approach lies in the fact this approach leads to a coset description of σ-models. To show how this idea works let us choose M to be a homogeneous space, with continuous group of symmetry G acting transitively on it. Let $m_\circ$ be a base point and let us then considers the little group H (isotropy group) of $m_\circ$, defined by

$$H = \{h \in G : hm_\circ = m_\circ\} \tag{2.43}$$

Suppose that g_1 and $g_2 \in G$ and have the same action on $m_\circ$; then

$$g_1(m_\circ) = g_2(m_\circ),$$

and so

$$g_1^{-1} g_2(m_\circ) = m_\circ,$$

which shows that

$$g_1^{-1} g_2 \in H.$$

Hence g_1, g_2 belong to the same left coset of G with respect to H. Clearly, the converse is also true, namely every two elements in the same left coset have the same action on $m_\circ$.

But as G acts transitively then any point of M may be obtained by the action of some left coset. Hence we can make the following identification between points of M and cosets of G

$$M \cong G/H = \{gH : g \in G\}. \tag{2.44}$$

Note that as M is homogeneous, this construction is independent of the choice of m_o. To give examples of sigma models which are defined on homogeneous spaces, let us choose first the $O(n)$ sigma models which are defined as

$$O(n) \approx SO(n)/SO(n-1). \tag{2.45}$$

Similarly CP^n models are defined as

$$CP^n \approx SU(n+1)/(SU(n) \times U(1)), \tag{2.46}$$

and the Grassmannian models have the following coset description.

$$G_{nm} = SU(m+n)/(SU(m) \times SU(n) \times U(1)). \tag{2.47}$$

At this stage a very intriguing question arises: Given a Lagrangian L, can we tell whether the theory described by L admits solitons? Unfortunately, it is difficult to answer this question in general but luckily there exists a theorem due to Derrick[26] which sometimes can be of some help. In the next few paragraphs we discuss this theorem.

Consider a nonlinear field theory, living in $(d+1)$ dimensional Minkowski space-time whose Lagrangian is

$$L = \frac{1}{4}\ g_{ij}\partial_\mu\phi^i\partial^\mu\phi^j - V[\phi]. \tag{2.48}$$

Of course, one has always to ensure that the energy density is positive definite, which shows that $V[\phi]$ must be nonnegative. The energy E may be split into two parts:

$$E[\phi] = E_1[\phi] + E_2[\phi] \tag{2.49}$$

with

$$\begin{aligned} E_1[\phi] &= \frac{1}{4}\int d^dx\ g_{ij}\partial_\mu\phi^i\partial^\mu\phi^j \\ E_2[\phi] &= \int d^dx\ V[\phi], \end{aligned} \tag{2.50}$$

where in the last equations the summation convention over μ is performed with Euclidean metric.

As far as solitons are concerned one can always consider first static structures *i.e.* $\phi(x^1, x^2, ..., x^d)$ (*i.e.* independent of time). The moving solitons are then obtained by Lorentz boost of the static fields. Derrick[26] observed that the difference in the scaling behaviour of E_1, E_2 constrains the theories which possess such structures. To see this consider a one parameter family of configurations

$$\phi_\lambda(\underline{r}) = \phi(\lambda \underline{r}) \tag{2.51}$$

and observe that the energy associated with these configurations responds to this rescaling as follows:

$$E_\lambda = \lambda^{2-d} E_1[\phi] + \lambda^{-d} E_2[\phi]. \tag{2.52}$$

However; ϕ_λ is a static solution if it extremises E, namely if it makes E_λ stationary with respect to the variations of λ, *i.e.* $\frac{dE_\lambda}{d\lambda}|_{\lambda=1} = 0$. But

$$\frac{dE_\lambda}{d\lambda} = (2-d)\lambda^{1-d} E_1[\phi] - d\lambda^{-d-1} E_2[\phi]. \tag{2.53}$$

Hence, we see that the following cases can occur, depending on the dimensionality of the spatial coordinates.

$$\begin{aligned} d = 1: &\quad \frac{dE_\lambda}{d\lambda}|_{\lambda=1} = 0 \quad \leftrightarrow E_1[\phi] = E_2[\phi] \\ d = 2: &\quad \frac{dE_\lambda}{d\lambda}|_{\lambda=1} = 0 \quad \leftrightarrow E_2[\phi] = 0 \\ d > 2: &\quad \frac{dE_\lambda}{d\lambda}|_{\lambda=1} = 0 \quad \leftrightarrow E_2 < 0. \end{aligned} \tag{2.54}$$

The implications of the first equation is that, for solitons to exist in one spatial dimension there must be a potential term. The second relation suggests a very opposing conclusion to the first, namely if σ-models contain a potential term in two spatial dimensions then there is no chance for solitonic solutions to exist. In spatial dimensions higher than 2, say 3, solutions cannot exist unless one modifies (2.49). Let us discuss the different ways of overcoming the constraints of Derrick's theorem in three spatial dimensions. The first way is to redefine what is meant by static solutions, by considering configurations which have explicit time dependence but whose energy is independent of time *e.g.* Q-balls which were originally introduced by Coleman[27] . Another possibility is to allow for the inclusion of gauge fields *e.g.* a Yang-Mills gauge field coupled to scalar Higgs fields; these are the theories in which magnetic monopoles occur.

The third way is to add higher-derivative terms to the Lagrangian. This is the idea behind the Skyrme model[16] , in which the additional term contributes to the potential energy. Consequently the scaling behaviour of the total potential energy is changed as

$$E_\lambda = \lambda^{2-d} E_1[U] + \lambda^{-d} E_2[U] + \lambda^{4-d} E_{Sky}[U], \tag{2.55}$$

so that for $d = 3$:

$$\frac{dE}{d\lambda} = -\lambda^{-2} E_1[U] - 3\lambda^{-4} E_2[U] + E_{Sky}[U]. \tag{2.56}$$

If we set E_2 to zero (potential), then there is a possibility for soliton solutions; in fact such a solution does exist and is called the Skyrmion.

[2.3.2] Topological Aspects Of Sigma-Models

So far we have been concerned mainly with the properties of the target manifold M. Remarkably, one can extract a lot of information about possible solitonic solutions by a careful study of the global topology of M. For this purpose one needs some results about fundamental groups (homotopy groups) of topological spaces.

For the sake of clarity let us show intuitively that homotopy classes of maps form a group. First, let us start by defining homotopy between two maps say f and g. Roughly speaking, if two maps are continuously deformable into each other then they are said to be homotopic. More precisely, if X, Y are two manifolds and f, g are two continuous maps from X to Y, then f, g are homotopic if there exists a continuous map

$h : X \times [0,1] \to Y$ such that for all $x \in X$

$$\begin{aligned} h(x,0) &= f(x) \\ h(x,1) &= g(x). \end{aligned} \tag{2.57}$$

Taking advantage of this definition one can classify all continuous maps between two spaces X, Y into homotopy classes *e.g.* $f \sim g \in [f]$. Then one can define a group law between the homotopy classes such that

$$[f + g] = [f] + [g] = [h] \tag{2.58}$$

It is not difficult to show that the binary operation defined by (2.58) and the set of all homotopy classes form a group.

If in constructing homotopy groups one chooses the first topological space X to be either S^1 or any other generalisation of S^1 *e.g.* S^p, then the homotopy groups are denoted by $\pi_1, \pi_2, ..., \pi_p$ respectively. The first two homotopy groups do have a nice geometrical interpretation. First , $\pi_o(Y)$; counts the number of disjoint pieces of Y. Hence if $\pi_o = 0$, this means that Y is connected. Secondly, $\pi_1(Y)$ classifies the set of loops in Y; in fact if $\pi_1(Y) = 0$ then Y is said to be simply connected.

In order to fit the homotopy theory into the context of σ-models, one should observe that the constraint of finite energy configurations amounts to compactifying the spatial degrees of freedom from $\mathbb{R}^d$ to S^d by identifying all points at spatial infinity. Therefore, since each and every possible field configuration may be thought of as a map

$$\phi : S^d \to M, \tag{2.59}$$

such configurations are classified by the homotopy group $\pi_d(M)$, which also implies that time evolution of the field configurations does not allow any transition or tuneling between different topological sectors. If $\pi_d(M) \neq 0$, the model is said to be topologically nontrivial. It is worth mentioning that topologically nontrivial theories are the ones enjoying topological stability, in the sense that configurations cannot evolve out of their topological sectors, hence solutions cannot decay into the vacuum.

Some of the results which are very useful to us in homotopy theory are,

$$\begin{aligned} &\pi_n(S^n) = \mathbb{Z} \\ &\pi_n(S^m) = 0. \qquad (m > n). \end{aligned} \tag{2.60}$$

The first result shows that the $O(3)$-sigma model in two dimensions is topologically nontrivial. Another useful result is

$$\pi_2(G) = 0, \tag{2.61}$$

with G being a Lie-group which indicates that chiral models are topologically trivial. Two other useful results which are relevant in working out the homotopy groups of Grassmannian models are

$$\begin{aligned} \pi_n(X \times Y) &= \pi_n(X) \oplus \pi_n(Y) \\ \pi_2(X/Y) &= \pi_1(Y); \qquad \pi_o(X) \Leftrightarrow \pi_1(X) = 0. \end{aligned} \tag{2.62}$$

Therefore

$$\begin{aligned} \pi_2(G_{nm}) &= \pi_1(SU(m) \times SU(n) \times U(1)) \\ &= \mathbb{Z} \end{aligned} \tag{2.63}$$

In fact, for any compact Kähler symmetric space M, one can easily show that $\pi_2(M) = \mathbb{Z}$, *i.e.* all such models are topologically nontrivial.

In some other exotic cases the structure of the homotopy group is more complicated (in this thesis we have restricted ourselves only to Abelian homotopy groups , but in general homotopy groups are not necessarily Abelian). A good example of such exotic spaces is the coset space defined as

$$F_n = \frac{SU(n+1)}{U(1) \times U(1) \times \ldots\ldots\ldots\ldots \times U(1)} \tag{2.64}$$

with N copies of $U(1)$ in the denominator of (2.64). To determine the form of the second homotopy group of F_n one should take the advantage of (2.62); having done this one can easily show the following result,

$$\pi_2(F_n) = \mathbb{Z} \oplus \mathbb{Z} \oplus \ldots\ldots\ldots\ldots \oplus \mathbb{Z} \qquad (n\, times). \tag{2.65}$$

Note that in the examples of nontrivial topology, the homotopy group is isomorphic either to $\mathbb{Z}$ or to the direct sum of, say N copies of $\mathbb{Z}$. An elegant interpretation of the above result is given by the so called topological charge which is perceived as an integer label of the homotopy classes. In the case when we have more than one copy of $\mathbb{Z}$, we need more than one topological charge to label completely the homotopy classes (topological sectors).

It is worth saying a few words about ways of constructing Q (topological charge) for any Kähler model in (2 + 1) dimensions. Closed forms ($d\alpha = 0$) are divided into cohomology classes, with two elements belonging to the same class if and only if they differ by an exact form. Therefore the set of cohomology classes is written as

$$H^p(M) = \frac{Z^p(M)}{B^p(M)} \tag{2.66}$$

with

Z^p : the set of closed p-forms

B^p :the set of exact p-forms.

However, one can define a natural binary operation on the above cohomology classes, which is the addition since the sum of two closed forms is a closed form and the sum of two exact forms is an exact form. Then $H^p(M)$ is a group and it is called the p^{th} order cohomology group of M.

Despite the fact that cohomology is defined locally via the exterior derivative, it contains some information about the global topology of M. In fact a theorem due to Hurewicz establishes the connection between cohomology groups and the topology of manifolds. In its simplest version this theorem states that if M is both connected and simply connected with the lowest nonzero homotopy group $\pi_n(M)$, then $\pi_n(M) = H^n(M)$ and all the lower cohomology groups are zero. In order to define the topological charge one needs the notion of a pullback mapping.

Let (X, Y) be two differentiable manifolds, g a real valued function on Y, ϕ a mapping from X to Y, ω a 1-form living on $T^*_{\phi(p)}(Y)$ and $v \in T_p(M)$. Then ϕ induces the following maps

$$\begin{aligned} \phi_* : T_p(X) \to T_{\phi(p)}(Y) \quad &; \qquad (\phi_* V)_g = V_g(\phi(p)) \\ \phi^* : T^*_{\phi(p)}(Y) \to T^*_p(M)(X) \quad &; \quad < \phi^* \omega, V > = < \omega, \phi_* V > \end{aligned} \tag{2.67}$$

The second map ϕ^* is called the pullback mapping, with the property that it takes closed forms on Y to closed forms on X; therefore one can consider it as a map between cohomology classes. In connection with our σ-models we have defined the fields ϕ as maps from S^2 (look at (2.59)), hence each field configuration defines a pullback mapping

$$\phi^* : H^2(M) \to H^2(S^2). \tag{2.68}$$

On the other hand we have a natural closed form on all Kähler manifolds, which is the

Kähler form Ω defined in (2.17). Taking advantage of ω the topological charge is defined as

$$Q = c^{-1} \int_{S^2} \phi^*(\Omega), \tag{2.69}$$

where c is a normalisation constant so chosen that Q takes integer values. For Q to be a topological charge we have to show that Q is invariant under field deformations. To do so we have to resort to a theorem in differential geometry which states that if ϕ_1, ϕ_2 are homotopic *i.e.* $(\phi_1, \phi_2) \in [\phi]$, their pullbacks ϕ_1^*, ϕ_2^* are homotopic *i.e.* $\phi_1^* = \phi_2^*$. Furthermore, in our definition of Q there exists another feature which shows the topological nature of Q, due to the fact that the domain of integral in Q is defined on a class of homeomorphic geometries. For example consider S^2 and a squashed S^2 *i.e.* $\widetilde{S}^2$, and define

$$f : \widetilde{S^2} \to S^2.$$

Then

$$Q = \int_{S^2} \phi^*(\omega) = \int_{\widetilde{S}^2} f^*(\phi^*(\omega)) = \int_{\widetilde{S}^2} f^* \circ \phi^*(\omega). \tag{2.70}$$

But throughout their time evolution, the fields ϕ must remain in the same homotopy class. Thus ϕ^* is unchanged which is equivalent to the invariance of Q. Note that Q is different from the conserved quantities obtained by the continuous symmetries of the Lagrangian. In fact the construction of Q depends only on the compactification of the spatial dimensions into a two sphere.

[2.3.3] Bogomolny Bounds

As the previous sections suggest static structures in topologically nontrivial models in $(2+1)$ dimensions contain a great deal of information about the solitonic behaviour of these theories in $(2+1)$ dimensions. Hence it is very intriguing to construct these configurations explicitly. In fact, Bogomolny[28] suggested a general technique for the construction of such solutions. To see how this method works let us consider a scalar field theory in $(1+1)$ dimensions given by the Lagrangian

$$L = \frac{1}{4}\partial_\mu \phi \partial^\mu \phi - V[\phi] \tag{2.71}$$

The total energy for a static solution in this model is

$$\begin{aligned} E &= \int_{-\infty}^{\infty} (\frac{1}{4}\phi_x^2 + V[\phi])dx \\ &= \int_{-\infty}^{\infty} (\frac{1}{2}\phi_x - \sqrt{V[\phi]})^2 + \phi_x\sqrt{V[\phi]})dx \\ &= \int_{-\infty}^{\infty} (\frac{1}{2}\phi_x - \sqrt{V[\phi]})^2 dx + \int_{\phi_-}^{\phi_+} \sqrt{V[\phi]}d\phi. \end{aligned} \tag{2.72}$$

This leads to

$$E \geq \int_{\phi_-}^{\phi^+} \sqrt{V[\phi]}d\phi, \tag{2.73}$$

with the equality holding if and only if

$$\frac{1}{2} \quad \phi_x = \sqrt{V[\phi]}. \tag{2.74}$$

The lower bound on E is known as the Bogomolny bound, and the condition for equality is the Bogomolny equation. For example, in the case of the Sine-Gordon equation one has

$$E \geq \frac{1}{\sqrt{2}} \int_0^{2\pi} \sqrt{1 - cos(\phi)}d\phi = 4 \tag{2.75}$$

with equality if and only if

$$\phi_x - \sqrt{2(1 - cos(\phi))} = 0 \tag{2.76}$$

The solutions of the last equation are called kink (anti-kink) solutions. The advantage of Bogomolny equations (B.E.) over the equations of motion, lies in the fact that BE are first order, whereas the equations of motion are second order (in fact "self-duality" equations are examples of BE).

Let us try to extend this idea to Kähler models in $(2+1)$ dimensions. But before doing so, let us observe the following identity

$$\int g_{\alpha\overline{\beta}}(\partial_i U^\alpha \pm i\epsilon_{ij}\partial_j U^\alpha) \quad (\partial_i \overline{U}^\beta \mp i\epsilon_{ik}\partial_k \overline{U}^\beta) \quad dx^2 \quad \geq 0. \tag{2.77}$$

But for static solutions

$$E = \frac{1}{4}\int g_{\alpha\overline{\beta}} \ \partial_i U^\alpha \ \partial_i \overline{U}^\beta \ d^2x \tag{2.78}$$

and

$$cQ = i\int g_{\alpha\overline{\beta}} \ \epsilon_{ij}\partial_i U^\alpha \ \partial_j \overline{U}^\beta \ d^2x; \tag{2.79}$$

hence (2.77) reads

$$E \geq \frac{1}{4}|cQ|, \tag{2.80}$$

with equality holding if and only if

$$\partial_i U^\alpha \pm i\epsilon_{ij}\partial_j U^\alpha = 0 \tag{2.81}$$

By analogy with "self-duality" in Yang-Mills, the situation in (2.81) is sometimes referred to as "self-duality". It does have a much simpler form in holomorphic and anti-holomorphic coordinates *i.e.* $x_+ = x + iy$, $x_- = x - iy$ respectively. In fact it takes the form

$$\begin{aligned} \partial_- U^\alpha &= 0, \\ \partial_+ \overline{U}^\alpha &= 0, \end{aligned} \tag{2.82}$$

where $\partial_+ = \frac{\partial}{\partial x_+}$. The corresponding solutions are called instantons and anti-instantons.

To sum up, for models with nontrivial topology one can construct a lower bound on the potential energy in a given topological sector. It is usually proportional to the topological charge. Provided the bound can be attained, the corresponding static solutions arise as solutions of the first order B.E. For instance this is the situation in the $(2+1)$ dimensional Kähler models (static solutions are known as instantons). However, there exist also the so-called frustrated models, such as the Skyrme model, in which the bound cannot be attained. In these models one must either resort to approximations or conduct numerical simulations.

III. CP^N MODELS

3.1 Formulation of the CP^n Models and Properties of Their Solutions

In chapter one we stated that the modified CP^1 model[17] is based on the CP^1 model; in chapter six we will discuss the CP^2 model; hence we need a deeper understanding of CP^n models. These models in two dimensions are typical examples of Kähler manifolds. They were first discussed by Eichenherr[29] , Cremmer and Scherk[30] , Golo and Perelomov[31] , and d'Adda et al[32] . In chapter two, CP^n manifolds were defined by equation (2.3) and the $n+1$-dimensional complex vector field Z^α, where $\alpha = 0, 1, .., n$, are the coordinates parametrising this space. These coordinates are subject to the constraint

$$|Z|^2 = 1. \tag{3.1}$$

Two such fields are equivalent if they are related by a phase:

$$Z'_\alpha = Z_\alpha exp^{i\Lambda(x,y)}. \tag{3.2}$$

Hence, the theory is required to be $U(1)$ gauge invariant. The covariant derivative is given by

$$D_\mu = \partial_\mu - Z^\dagger \cdot \partial_\mu Z \tag{3.3}$$

and the Lagrangian describing the model takes the form

$$\begin{aligned} L &= (D_\mu Z)^\dagger \cdot (D_\mu Z) \\ &= \partial_\mu Z^\dagger \cdot \partial_\mu Z + (Z^\dagger \cdot \partial_\mu Z)^2, \end{aligned} \tag{3.4}$$

with the action S defined by

$$S = \int d^2 x L. \tag{3.5}$$

To have a deeper understanding of CP^n models, one ought to understand the geometrical implications of both the Lagrangian (3.4) and the constraint (3.1). For a start it is evident that the basic Lagrangian depends on the parametrisation of the target manifold *i.e.* it is not reparametrisation invariant. In order to see this, first rewrite the Lagrangian as

$$L_{\circ} = \delta_{ij}(D_\mu Z_i)^+(D^\mu Z_j). \tag{3.6}$$

Then, recall that the sphere S^{2n+1} and the product space $U(1) \times CP^n$ are, locally, isomorphic. The covariant derivatives are one forms on the cotangent space of S^{2n+1}. Since the δ_{ij} is not the intrinsic metric on S^{2n+1} then the Lagrangian $L_{\circ}$ is not invariant under reparametrisations either. In other words, if $Z_i = f_i(\phi^i)$ and $Z'_i = f'_i(\phi^i)$ are two different parametrisations, then the Lagrangians take the forms, respectively,

$$L_{\circ} = g_{ij}\partial_\mu\phi^i\partial^\mu\phi^j, \tag{3.7}$$

and

$$L_{\circ} = g'_{ij}\partial_\mu\phi'^i\partial^\mu\phi'^j, \tag{3.8}$$

where

$$g'_{ij} \neq \frac{\partial\phi^l}{\partial\phi'^i}\frac{\partial\phi^m}{\partial\phi'^j}g_{nm}, \tag{3.9}$$

which implies that g_{ij} does not transform as a second rank tensor under the diffeomorphism group of the target manifold.

Let us discuss the meaning of these results. Let M^2 be a two dimensional space-time manifold and $M_Z^{\mathbb{C}}$ be another complex manifold (*i.e.* target manifold). Then our fields are mappings from M^2 to $M_Z^{\mathbb{C}}$:

$$Z_i : M^2 \to M_Z^{\mathbb{C}}.$$

But as the fields are constrained to lie on the unit sphere S^{2n+1}, we are more interested in a submanifold of $M_Z^{\mathbb{C}}$ which is S^{2n+1}. So the whole picture reduces to the immersion of the unit sphere S^{2n+1} into the nonlinear space $M_Z^{\mathbb{C}}$ endowed with the metric g. Therefore choosing a particular parametrisation is equivalent to choosing the metric g on $M_Z^{\mathbb{C}}$. To clarify and illuminate the above remark, let us consider the CP^1 model as an example and show how the nonflat metric affects the curvature of the sphere. One parametrisation which satisfies the constraint (3.1), is given as

$$Z = \begin{pmatrix} e^{i\chi}\phi_1 \\ e^{i\chi}(\phi_2 + i\phi_3) \end{pmatrix} \tag{3.10}$$

and

$$\vec{\phi}\cdot\vec{\phi} = 1. \tag{3.11}$$

In this parametrisation of the CP^1 model the Lagrangian takes the form

$$\begin{aligned} L_o &= \vec{\phi}_\mu \cdot \vec{\phi}_\mu - (\phi_2\phi_{3\mu} - \phi_3\phi_{2\mu})^2 \\ L_o &= g_{ij}\partial_\mu\phi^i\partial^\mu\phi^j, \end{aligned} \tag{3.12}$$

with g_{ij} given by

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (1-\phi_3^2) & \phi_2\phi_3 \\ 0 & \phi_2\phi_3 & (1-\phi_2^2) \end{pmatrix}. \tag{3.13}$$

Here, g_{ij} is the metric on the three dimensional space parametrised by the vector $\vec{\phi}$. The constraint $\vec{\phi}\cdot\vec{\phi} = 1$ is the immersion of S^2 into the nonlinear space parametrised by $\vec{\phi}$ and endowed with the metric g. In the remainder of this chapter we will refer to this space as M^3_ϕ. Furthermore the metric g induces a metric on M_2, namely,

$$g \rightarrow g_{ij}\partial_\mu\phi^i\partial_\nu\phi^j \tag{3.14}$$

and another metric on S^2 (target manifold). But to determine the form of the induced metric on S^2, let us parametrise it by

$$\phi_1 = cos\theta_2, \qquad \phi_2 = sin\theta_2 cos\theta_1, \qquad \phi_3 = sin\theta_1 sin\theta_2. \tag{3.15}$$

Then the induced metric on S^2 reads

$$g_{ij}\partial_{\theta^\alpha}\phi^i\partial_{\theta^\beta}\phi^j. \tag{3.16}$$

At this stage it is very useful to compare the immersion of S^2 into $\mathbb{R}^3$ to the immersion of the sphere S^2 into M^3_ϕ. Obviously the immersion of the unit sphere into $\mathbb{R}^3$ is given by the standard spherical polar coordinates

$$x = sin\theta cos\psi, \qquad y = sin\theta sin\psi, \qquad z = cos\theta, \tag{3.17}$$

and the induced metric on S^2 is obtained from the line element given by

$$(ds)^2 = sin^2\theta d^2\theta + d^2\psi, \tag{3.18}$$

the corresponding scalar Ricci curvature R has the value

$$R = -2, \tag{3.19}$$

which is calculated using

$$\begin{aligned} \Gamma^{\rho}_{\mu\nu} &= \frac{1}{2} g^{\rho\lambda}[\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} + \partial_\lambda g_{\mu\nu}], \\ R^{\lambda}_{\mu\nu\rho} &= \partial_\rho \Gamma^{\lambda}_{\mu\nu} + \Gamma^{\eta}_{\mu\nu}\Gamma^{\lambda}_{\rho\eta} - (\mu \leftrightarrow \nu), \\ R_{\mu\nu} &= R^{\lambda}_{\mu\lambda\nu} \qquad R = g^{\mu\nu}R_{\mu\nu}. \end{aligned} \tag{3.20}$$

In our case the embedding of S^2 into the space M^3_ϕ is simply given by

$$\phi_1 = cos\psi, \qquad \phi_2 = sin\psi cos\theta, \qquad \phi_3 = sin\psi sin\theta, \tag{3.21}$$

the line element in S^2 is

$$\begin{aligned} ds^2 &= d\phi_1^2 + (1 - \phi_3^2)d\phi_2^2 + (1 - \phi_2^2)d\phi_3^2 + \phi_2\phi_3 d\phi_2 d\phi_3 + \phi_2\phi_3 d\phi_3 d\phi_2 \\ ds^2 &= \frac{1}{4}[sin^2\psi d\theta^2 + d(2\psi)^2], \end{aligned} \tag{3.22}$$

and the Ricci curvature R is

$$R = -16. \tag{3.23}$$

After elucidating the geometrical meaning of the parametrisation (3.10) of the CP^1 model, an intriguing question arises: can one make a transformation which euclideanises the above metric ? In order to answer this question, let us first try to put the Lagrangian in a more convenient form by taking advantage of the following relations

$$\vec{\phi}\cdot\vec{\phi}=1 \qquad \vec{\phi}\cdot\vec{\phi}_\mu=0 \qquad (\vec{\phi}\cdot\vec{\phi}_\mu)^2=0. \tag{3.24}$$

A few lines of algebra shows that L takes the new form

$$L=\phi_{1\mu}^2+\phi_1^2(\phi_{1\mu}^2+\phi_{2\mu}^2+\phi_{3\mu}^2). \tag{3.25}$$

The new form of the Lagrangian is more suggestive, in the sense that it implies that the transformation we are looking for should involve ϕ_1 as the very basic object. So, let us try the transformation

$$\Psi_1=1-2\phi_1^2, \qquad \Psi_2=2\phi_1\phi_2, \qquad \Psi_3=2\phi_1\phi_3, \tag{3.26}$$

and note that $\vec{\Psi}$ still parametrise S^2 since $\vec{\Psi}\cdot\vec{\Psi}=1$. After a long but straightforward calculation one can show that the Lagrangian takes the new form

$$L=\frac{1}{4}\partial_\mu\vec{\Psi}\cdot\partial^\mu\vec{\Psi}. \tag{3.27}$$

So, indeed the transformation (3.26) linearises the metric on the background nonlinear space. Therefore one can draw the conclusion that we have at our disposal two independent parametrisations of the CP^1 model which produce two inequivalent Lagrangians.

The equations of motion resulting from applying the variational principle to the Lagrangian (3.4) read

$$D_\mu D_\mu Z+(D_\mu Z)^\dagger\cdot(D_\mu Z)Z=0, \qquad Z^+Z=1. \tag{3.28}$$

On the other hand the form of the CP^1 equations of motion with respect to the parametrisation given by (3.12), after decoupling, are

$$\begin{aligned}
&\phi_{1\nu\nu}-2\phi_1B_\nu^2+\phi_1\vec{\phi}_\nu\cdot\vec{\phi}_\nu=0\\
&\phi_{2\nu\nu}-2\phi_2B_\nu^2+\phi_2\vec{\phi}_\nu\cdot\vec{\phi}_\nu+2\frac{(\phi_1^2\phi_{3\nu}-\phi_1\phi_3\phi_{1\nu})}{\phi_1^2}B_\nu^2=0\\
&\phi_{3\nu\nu}-2\phi_3B_\nu^2+\phi_3\vec{\phi}_\nu\cdot\vec{\phi}_\nu+2\frac{(\phi_1^2\phi_{2\nu}-\phi_1\phi_2\phi_{1\nu})}{\phi_1^2}B_\nu^2=0,
\end{aligned} \tag{3.29}$$

where

$$B_\nu=(\phi_2\phi_{3\nu}-\phi_3\phi_{2\nu}), \tag{3.30}$$

together with $\vec{\phi}\cdot\vec{\phi}=1$.

The solutions to these equations, which have finite action and energy, are the required solutions of the model. More precisely, imposing the condition that the action is finite means that the base space of the model is compactified from E^2 to S^2 as we have argued in chapter two.

As was first suggested by Din and Zakrzewski[33] , it is more convenient to use holomorphic and antiholomorphic coordinates. Then the Lagrangian has the form

$$L = 2[|D_+Z|^2 + |D_-Z|^2], \tag{3.31}$$

with

$$D_\pm = \partial_\pm - Z^\dagger \cdot \partial_\pm Z \tag{3.32}$$

where $\partial_+ = \frac{\partial}{\partial x_+}$, and the equations of motion become

$$\begin{aligned} D_-D_+Z + Z|D_+Z|^2 &= 0 \\ D_+D_-Z + Z|D_-Z|^2 &= 0. \end{aligned} \tag{3.33}$$

It is useful to introduce the quantity q defined by

$$q = 2[|D_+Z|^2 - |D_-Z|^2] \tag{3.34}$$

which, as we shall see later on, is the topological charge density.

Having written the equations in the above form, it is clear from (3.33) that there exists a subclass of equations called the self-duality equations

$$D_\pm Z = 0, \tag{3.35}$$

which correspond to the condition

$$L = \pm q. \tag{3.36}$$

The solutions of $D_-Z = 0$ are the instanton solutions of the CP^n models (anti-instantons being the solutions of $D_+Z = 0$). In fact d'Adda et al.[32] have shown that the general instanton solution has the form

$$Z_\alpha = \frac{f_\alpha(x_+)}{|f_\alpha(x_+)|}, \tag{3.37}$$

where f_α are polynomials in x_+ of the form

$$f_\alpha(x_+) = \lambda_\alpha \prod_{i=1}^{k} (x_+ - a_\alpha^i), \tag{3.38}$$

with λ_α and a_α^i complex constants. The general anti-instanton solution is obtained by complex conjugation. The degree of the polynomial corresponds to the instanton number, and the action for such solutions is

$$S = 2\pi k. \tag{3.39}$$

For example, in CP^1 model, choosing the polynomials f_α as

$$f_\alpha(x_+) = (1, x_+) \tag{3.40}$$

and using (3.37) results in the configuration

$$Z_\alpha = \frac{(1, x_+)}{\sqrt{1 + |x_+|^2}}. \tag{3.41}$$

It is then a straightforward task to show that the value of the action associated with (3.40) is

$$S = 2\pi, \tag{3.42}$$

which shows that the instanton number, which is the topological charge for the configuration (3.40), is one. Thus this configuration is an example of a one instanton solution of CP^1.

Using the well-known result from pure mathematics that $\pi_2(CP^n) = \mathbb{Z}$, d'Adda et al. concluded that classical field configurations fall into homotopy classes labelled by an integer winding number Q. Din and Zakrzewski[34] exhibited a simple expression for Q

as follows

$$Q = \int d^2x J^\circ, \tag{3.43}$$

with

$$J^\circ = -\frac{i}{2\pi}\epsilon^{\mu\lambda}[(D_\mu Z)^\dagger \cdot (D_\lambda Z)]. \tag{3.44}$$

written in terms of complex variables $x_\pm$ this becomes

$$Q = \frac{1}{2\pi}\int d^2x[2(|D_+Z|^2 - |D_-Z|^2)] = \frac{1}{2\pi}\int d^2xq. \tag{3.45}$$

Given this definition we see that if Z is an instanton (anti-instanton) field, then the topological charge is positive (negative) and Q gives directly the instanton number. Furthermore, one can use equations (3.31) , (3.34) to show that the action of a general field obeys the inequality

$$S \geq |2\pi Q|; \tag{3.46}$$

with the equality holding only if Z is either an instanton or anti-instanton solution. Consequently the instanton (anti-instanton) solutions are absolute minima of the action, and therefore they are topologically stable and have definite positive (negative) integer topological charge.

Are there any other finite action solutions? In other words, do there exist finite action solutions which are not solutions of self-duality (or antiself-duality)? Din and Zakrzewski[35] studied this problem and exhibited some of these. Their method of deriving them relied on some results from two-dimensional Euclidean $O(n)$ non-linear σ-models.

The $O(n)$ models[36] [37] , as we have argued in chapter two, are defined by the coset description (2.46) which is, globally, equivalent to S^{n-1}. They are parametrised by n-component fields $\phi^i = \phi^i(x,y)$, where $i = 1, 2, ..., n$, subject to the constraint

$$\phi^i \cdot \phi^i = 1. \tag{3.47}$$

Of course the fields $\vec{\phi}$ are functions of the spatial coordinates of the Euclidean space E^2.

The Lagrangian for these models is defined to be

$$L = \partial_\mu \vec{\phi} \cdot \partial_\mu \vec{\phi} \tag{3.48}$$

and gives rise to the equations of motions

$$\partial_\mu \partial_\mu \vec{\phi} + (\partial_\mu \vec{\phi} \cdot \partial_\mu \vec{\phi})\vec{\phi} = 0; \quad \vec{\phi} \cdot \vec{\phi} = 1. \tag{3.49}$$

Again the base space E^2 is compactified by requiring the solutions to correspond to finite action.

In (2+0) dimensions the $O(3)$ sigma model (or S^2 model) has a special role to play since it is the simplest Lorentz invariant σ-model in (2+0) dimensions, which possess static solutions which could be interpreted as solitons. However, the fact that the $O(3)$ model is the only model among all the other $O(n)$ models which possesses stable static structures in (2+0) dimensions is attributed to the observation that, of all $O(n)$ nonlinear sigma models, it is only the $O(3)$ model which is topologically nontrivial in (2+0) dimensions (see chapter two). The nontrivial topology of the $O(3)$ model stems from the boundary conditions on the vector field $\vec{\phi} = (\phi^1, \phi^2, \phi^3)$ at spatial infinity, which is equivalent to imposing a one-point compactification of E^2 into a two sphere S^2. Hence, there are distinct topological sectors, labelled by the winding number of the map $\vec{\phi}$ from S^2 to the target manifold which happens to be another S^2. As we have shown in chapter two this winding number can be interpreted as an integer-valued topological charge and may be expressed as the integral of a topological charge density over all space:

$$Q = \frac{1}{8\pi} \int \epsilon_{ij} \vec{\phi} \cdot (\vec{\phi} \times \vec{\phi}) d^2x. \tag{3.50}$$

Belavin and Polyakov[8] , and Woo[38] have constructed, explicitly, the static solutions of the $O(3)$ sigma model and have shown that a general k instanton solution has the form:

$$W = \lambda \frac{\prod_{i=1}^{k} (x_+ - a_i)}{\prod_{i=1}^{k} (x_+ - b_i)}, \tag{3.51}$$

where W is a complex field related to $\vec{\phi}$ by the relation

$$W = \frac{\phi^1 + i\phi^2}{1 + \phi^3}, \tag{3.52}$$

and λ, a_i, b_i are complex constants such that $a_i \neq b_i$ for all i and j. Anti-instanton solutions are given by the complex conjugate of the field W, and the instanton number of the solution (3.51) is k. Finally the action S for this solution is

$$S = 8\pi k. \tag{3.53}$$

In order to have a better feeling for the physical interpretation of the parameters in solution (3.51), let us concentrate on the general one instanton solution which is given as

$$W = \lambda \frac{x_+ - a}{x_- - b}, \tag{3.54}$$

where a, b and λ are arbitrary complex numbers. The appearance of 6 real parameters in this solution is a reflection of the conformal invariance of the two-dimensional model (under such a transformation the values of these parameters change but the general form of (3.54) remains the same). It is easy to calculate the energy density E corresponding to the static solution (3.54):

$$E = \frac{8|\lambda|^2|a-b|^2}{(|x_+ - b|^2 + |x_+ - a|^2)^2} \tag{3.55}$$

Hence, the instanton has a bell-like shape, with its position and size respectively determined by

$$\frac{a|\lambda|^2 + b}{|\lambda|^2 + 1}, \qquad \frac{|\lambda|^2|a-b|^2}{(|\lambda|^2+1)^2}.$$

Its total energy is $E_t = 2\pi$ and so is independent of the instanton's position (translational invariance) or its size (conformal invariance).

Returning to CP^n models, we first observe that the CP^1 model is equivalent to the $O(3)$ model. In fact, we can establish the following mapping between the two fields of the models

$$\phi^i = Z^\dagger_\alpha \sigma^i_{\alpha\beta} Z_\beta \quad (\alpha, \beta = 1,2) \quad (i = 1,2,3), \tag{3.56}$$

where σ^i are the Pauli matrices. Here again $\vec{\phi} \cdot \vec{\phi} = Z^\dagger \cdot Z = 1$. Then a very simple calculation shows that

$$\partial_\mu \vec{\phi} \cdot \partial_\mu \vec{\phi} = 4(D_\mu Z)^\dagger \cdot (D_\mu Z). \tag{3.57}$$

Hence the two models are classically equivalent.

There exists another formulation of the CP^1 model; instead of using the fields $\vec{\phi}$ we express the dependence on $\vec{\phi}$ in terms of its stereographic projection onto the complex plane W. The $\vec{\phi}$ field is then related to W by

$$\phi^1 = \frac{W + W^*}{1 + |W|^2}, \quad \phi^2 = i\frac{W - W^*}{1 + |W|^2}, \quad \phi^3 = \frac{1 - |W|^2}{1 + |W|^2}. \tag{3.58}$$

The Lagrangian, in the new formulation of the model, is

$$L = \frac{\partial_\mu W \partial^\mu W^*}{(1 + |W|^2)^2}, \tag{3.59}$$

and the new equations of motion are

$$\partial_\mu \partial^\mu W - 2\frac{W^* \partial_\mu W \partial^\mu W}{1 + |W|^2} = 0. \tag{3.60}$$

The topological charge Q is now given by

$$Q = \int \frac{\partial_x W \partial_y W^* - \partial_y W \partial_x W^*}{(1 + |W|^2)^2} d^2x, \tag{3.61}$$

and the Noether charge, due to the $U(1)$ rigid symmetry, is obtained as

$$K = i\alpha \int \frac{W \partial_t W^* - W^* \partial_t W}{(1 + |W|^2)^2} d^2x. \tag{3.62}$$

Returing to the non-instanton finite action solutions, we reproduce an argument made by Din and Zakrzewski[35] which they used to prove the existence of noninstanton solutions. First we consider the energy momentum tensor which has the form

$$J_{\mu\nu} = -\delta_{\mu\nu}(D_\lambda Z)^\dagger \cdot (D_\lambda Z) + (D_\mu Z)^\dagger \cdot (D_\nu Z) + (D_\mu Z) \cdot (D_\nu Z)^\dagger. \tag{3.63}$$

This tensor is conserved and so it satisfies

$$\partial_\mu J_{\mu\nu} = 0. \tag{3.64}$$

Rewriting this conservation law in terms of complex coordinates we obtain

$$\partial_-[D_+Z \cdot D_-Z^\dagger] = 0 \tag{3.65}$$

which indicates that $D_+Z \cdot D_-Z^\dagger$ is a function of x_+ only. Imposing the condition of the finiteness of the action gives $|D_\mu Z| \to 0$ as $|x| \to \infty$, which implies that

$$D_+ Z \cdot D_- Z^\dagger = 0. \tag{3.66}$$

In the CP^1 case this implies that

$$D_- Z = 0, \tag{3.67}$$

or

$$D_+ Z = 0. \tag{3.68}$$

However, for CP^n $(n > 1)$ this does not have to be the case. In fact Din and Zakrzewski showed how to exploit this fact to find new solutions of the equations of motion.

Another (gauge invariant) reformulation of the CP^n models was introduced independently by Sasaki[39] and Zakrzewski[40] . In this case an $n \times n$ projection matrix $\mathbb{P}$ is introduced. This matrix is defined by

$$\mathbb{P} = ZZ^\dagger, \tag{3.69}$$

and so it possesses the following properties:

$$\mathbb{P} = \mathbb{P}^\dagger = \mathbb{P}^2 \qquad (\Leftrightarrow Z^\dagger \cdot Z = 1). \tag{3.70}$$

The Lagrangian of the model in this formulation takes the form

$$L = \frac{1}{2} tr(\partial_\mu \mathbb{P} \partial_\mu \mathbb{P}), \tag{3.71}$$

and the equations of motion become

$$[\partial_\mu \partial_\mu \mathbb{P}, \mathbb{P}] = 0, \tag{3.72}$$

or, when written in terms of complex coordinates,

$$[\partial_+ \partial_- \mathbb{P}, \mathbb{P}] = 0. \tag{3.73}$$

The self-duality equations take the form

$$\partial_- I\!P \cdot I\!P = 0, \qquad I\!P \cdot \partial_- I\!P = 0, \tag{3.74}$$

or equivalently

$$I\!P \cdot \partial_+ I\!P = 0, \qquad \partial_+ I\!P \cdot I\!P = 0. \tag{3.75}$$

In chapter one we have argued that, in quasi-integrable models stable static structures in (2+0) dimensions (*e.g.* instantons solutions and anti-instanton solutions of CP^1) are reasonable candidates for solitons in (2+1) dimensions. Subsequently, numerical simulations showed that their scattering properties are more subtle than the scattering properties of the solitons of integrable models in (1+1) dimensions. Therefore, one may wonder whether general solutions (*i.e.* those which are neither instantonic or anti-instantonic) might reveal even a more subtle scattering pattern. Of course the only way to find out is to first find these solutions, investigate their topological stability and then perform the numerical simulations to explore their scattering behaviour.

Thus, for completeness, we present here the general form of these solutions. We follow Din and Zakrzewski[40] . First of all, note the following property that these solutions must satisfy

$$A^m_{ij} = (D^i_- Z_\alpha)^\dagger + \cdot (D^j_+ Z_\alpha) = 0 \quad ; m = i + j \geq 1. \tag{3.76}$$

Thus for a given solution Z_α, we can construct two orthogonal subspaces of $\mathbb{C}^n$ defined by

$$\begin{aligned} H &= [D^i_- Z_\alpha, i = 1, 2, ...] \\ H' &= [D^i_+ Z_\alpha, i = 1, 2, ...] \end{aligned} \tag{3.77}$$

Let the dimension of these two spaces be k, m respectively, and for an obvious reason consider the case $k + m = n - 1$. The spaces H_k, H'_m are spanned by the first k and m vectors of (3.77).

Next define the space $\widehat{H}_k = H_k \cup Z$, and let $f \in \mathbb{C}^n$ such that $f \in \widehat{H}_k$ where

$$f^+_\alpha \cdot D^i_- Z_\alpha = \omega \delta^{ij} \quad i = 0, 1, ..., k. \tag{3.78}$$

Then one can show that the set

$$f_\alpha, \partial_+ f_\alpha, ..., \partial^k_+ f_\alpha \tag{3.79}$$

spans the space $\widehat{H}_k$. Din and Zakrzewski then showed that the general finite action solution can be expressed in terms of the analytic vector f by

$$Z = \frac{\widehat{Z}^{(k)}}{|\widehat{Z}^{(k)}|} \tag{3.80}$$

where $k = 0, 1, ..., n-1$ and

$$\widehat{Z}^{(k)} = \partial_+^k f - \sum_{i,j=0}^{k-1} \partial_+^i f (M_{i,j}^{(k)})^{-1} \partial_+ M_{j,k-1}^{(k)} \tag{3.81}$$

and where the matrix $M^{(k)}$ is given by

$$M_{i,j}^{(k)} = \overline{\partial}_+^i f \cdot \partial_+^j f \qquad i, j = 0, 1, ..., k-1. \tag{3.82}$$

In equation (3.70) taking $k = 0$ corresponds to the instanton solutions; $k = n - 1$ results in the anti-instanton solutions. However, for any other choice of k within the specified range, new classes of solutions are obtained.

There exists an alternative construction of this general solution. This new construction is based on the use of the Gramm-Schmidt orthonormalisation procedure, we will describe this construction using Zakrzewski's notation[40] although making certain identifications in the two formulations can be used to prove their equivalence. In addition we will reproduce Sasaki's proof[39] that the expression obtained by the second method solves the CP^n equations of motion in their projector formulation. The construction of these solutions starts by considering a vector field $g \in \mathbb{C} - \{0\}$, and an operator P_+, which is defined by its action on g as

$$P_+ g = \partial_+ g - \frac{g(g^\dagger \cdot \partial_+ g)}{|g|^2}. \tag{3.83}$$

Its repeated action is defined by

$$P_+^k g = P_+(P_+^{k-1} g), \tag{3.84}$$

where

$$P_+^0 g = g. \tag{3.85}$$

Let us add a further operator, P_-; this operator is like P_+ except that the differentiation is performed with respect to x_- instead of x_+. It is not difficult to show that $P_+ P_- g \sim g$ and so $P_\pm$ seem to behave like a pair of raising and lowering operators. In their construction Din and Zakrzewski observed that

(1) $(P_+^k f)^\dagger \cdot P_+^l f = 0 \quad if\, l \neq k$

(2) $\partial_-(P_+^k f) = -P_+^{k-1} f \frac{|P_+^k f|^2}{|P_+^{k-1}|^2}$,

(3) $\partial_+(\frac{P_+^{k-1} f}{|P_+^{k-1} f|^2}) = \frac{P_+^k f}{|P_+^{k-1} f|^2}$,

(4) $P_+^n f = 0$.

These orthogonality properties show that the $P_+^k f$ vectors can be thought of as being obtained by the Gramm-Schmidt orthogonalisation of the sequence of vectors given by (3.79) and when normalised, as shown by Sasaki[39] , they provide solutions of the CP^n Euler-Lagrange equations. To see this denote by

$$e_1, e_2, \ldots., e_n \tag{3.86}$$

the set of vectors obtained from (3.79) by the Gramm-Schmidt orthonormalisation method. Then take the j^{th} element of the sequence and consider

$$\mathbb{P} = e_j e_j^\dagger. \tag{3.87}$$

Also consider another projector

$$\mathbb{Q} = \sum_{k=1}^{j-1} e_k e_k^\dagger, \tag{3.88}$$

which satisfies the equations

$$\partial_- \mathbb{Q} \cdot \mathbb{Q} = 0. \tag{3.89}$$

Moreover,

$$\partial_-(\mathbb{P} + \mathbb{Q}) \cdot (\mathbb{P} + \mathbb{Q}) = 0. \tag{3.90}$$

Using the properties of the $e's$ one can show that

$$\partial_- \mathbb{P} \cdot \mathbb{Q} = 0, \tag{3.91}$$

as well as

$$\begin{aligned} \partial_- \mathbb{P} \cdot \mathbb{P} + \partial_- \mathbb{Q} \cdot \mathbb{P} = 0, \\ \mathbb{P} \cdot \partial_+ \mathbb{Q} = \partial_+ \mathbb{Q}, \\ \partial_- \mathbb{Q} \cdot \mathbb{P} = \partial_- \mathbb{Q}. \end{aligned} \tag{3.92}$$

Then putting the last two equations of (3.92) into the first one of the same set of equations leads to

$$\partial_- \mathbb{P} \cdot \mathbb{P} + \partial_- \mathbb{Q} = 0. \tag{3.93}$$

Taking the hermitian conjugate of this equation gives

$$\mathbb{P} \cdot \partial_+ \mathbb{P} + \partial_+ \mathbb{Q} = 0, \tag{3.94}$$

and finally, if the combination $\partial_+(3.93) - \partial_-(3.94)$ is considered, it is found that

$$[\partial_+ \partial_- \mathbb{P}, \mathbb{P}] = 0 \tag{3.95}$$

which is the required equation (3.72). This completes the proof that

$$Z = \frac{P_+^k f}{|P_+^k f|} \tag{3.96}$$

is a genuine solution of the CP^n model: $k = 0$ correspond to instanton solutions, $k = n-1$ to anti-instantons solutions and any other choice of k gives new noninstanton solutions.

Let us end this chapter by reporting Din and Zakrzewski investigation of the properties of non-instanton solutions. In fact, Din and Zakrzewski showed that any solution which is neither instanton or anti-instanton in nature is necessarily unstable[33] . To see this, we start with a certain solution Z such that

$$D_\pm Z \neq 0, \tag{3.97}$$

and then consider a small complex fluctuation ϕ about Z of the form

$$Z' = \sqrt{1 - |\phi|^2} Z + \phi; \qquad Z^\dagger \cdot \phi = 0. \tag{3.98}$$

The action for this new field Z' is

$$S' = 2\int (|D'_+Z'|^2 + |D'_-Z'|^2)d^2x. \tag{3.99}$$

However, since the quantity

$$Q' = 2(|D'_+Z'|^2 - |D'_-Z'|^2) \tag{3.100}$$

is a topological invariant, we must have $Q' = Q = 2(|D_+Z|^2 - |D_-Z|^2)$. Hence the action can be rewritten as

$$S' = \int Qd^2x + 4\int |D'_-Z'|^2 d^2x. \tag{3.101}$$

To second order in ϕ it can be shown that

$$\begin{aligned} |D'_-Z'|^2 &= |D_-Z|^2 + |D_+\phi|^2 - |\phi|^2|D_-Z|^2 - |Z^*\cdot D_-\phi + \phi^*\cdot D_-Z|^2 \\ &+ (D_-Z)^*\cdot D_-\phi + (D_-\phi)^*\cdot D_-Z. \end{aligned} \tag{3.102}$$

Thus

$$S' = S + 4\int V(\phi)d^2x, \tag{3.103}$$

where S is the action associated with Z and $V(\phi)$ is given by the expression

$$V(\phi) = |D_-\phi|^2 - |\phi|^2|D_-Z|^2 - |Z^*\cdot D_-\phi + \phi^*\cdot D_-Z|^2. \tag{3.104}$$

Then if we choose

$$\phi = \epsilon D_+Z \tag{3.105}$$

where ϵ is a small complex number, we find that

$$D_-\phi = \epsilon D_-D_+Z = -\epsilon|D_+Z|^2 Z, \tag{3.106}$$

and

$$\phi^*\cdot D_-Z = 0, \tag{3.107}$$

and so $V(\phi)$ becomes

$$\begin{aligned} V(\phi) &= |\epsilon|^2|D_+Z|^2 - |\epsilon|^2|D_+Z|^2|D_-Z|^2 - |\epsilon|^2|D_+Z|^2 \\ &= -|\epsilon|^2|D_-Z|^2|D_+Z|^2. \end{aligned} \tag{3.108}$$

The last equation implies that

$$S' < S. \tag{3.109}$$

Therefore the solution Z does not correspond to a minimum of the action: it is unstable under fluctuations, and is, in fact, a saddle point of the action.

Thus we see that any solution of the CP^n models which is neither instanton nor anti-instanton is necessarily unstable. One should note that what is meant by stability in this context, is the topological stability *i.e.* the impossibility of the reduction of the value of the action by an addition of a small fluctuation. However, this type of stability should not be confused with the size stability discussed in later chapters.

IV. SKYRME TERMS AND TOPOLOGICAL TERMS IN SIGMA MODELS

4.1 Introduction

As we showed in chapter two, recently it has become clear that many field theories possess classical solutions[41] [42] [43] describing various extended structures. Some of these structures are stable with respect to small perturbations; often such stability is guaranteed by the topological properties of these theories.

There are many examples of such structures. They range from kinks and antikinks in some simple dynamical systems to monopoles of non-abelian gauge theories. In addition, as we have argued in chapter one, in ref.[15] it was shown that a proton can be described by such an extended structure. A model has been proposed (Skyrme model[16]) in which properties of such an approximation to the proton were studied. They were found to be in a good agreement with experimental values[44] . Most of these comparisons refer to the static properties of the proton; the model has been much less successful in reproducing proton's scattering properties. This may be partly due to the fact that most of the applications were based on the collective coordinates approach to the proton treated as a soliton of the proposed model. To test this approximation in more detail one has to study the full evolution of an extended structure in field theory. Unfortunately, this cannot be done analytically and numerical procedures require too much computing power to be a practical proposition at present.

Of course, one can study this approximation in some lower dimensional models. As the proton of the Skyrme model and the monopoles arise as classical solutions of theories in $(3+1)$ dimensions which are difficult to handle, we may look for simpler models in $(1+1)$ or $(2+1)$ dimensions. Of these the $(1+1)$ dimensional models are easier to deal with, but are too simple to study these problems in sufficient generality. In $(1+1)$ dimensions there is no scattering angle and, moreover, many models are integrable and, as such, guarantee the solitonic nature of their extended solutions. On the other hand the Skyrme model of the proton is not integrable and so a more realistic model in which its properties should be studied would correspond to a model in $(2+1)$ dimensions.

As was discussed in chapter one, the modified CP^1 model is an example of a nonintegrable model whose extended structures (skyrmions) show solitonic scattering properties in (2+1) dimensions together with the release of a little bit of radiation during the scattering. Therefore this model can be thought of as a sort of (2+1) dimensional analogue of the proton model in (3+1) dimensions.

However, it is not clear why we should consider only the $O(3)$ sigma model, or whether there are further possibilities if we try to generalise the original $SU(2)$ to larger groups. Thus, in this chapter we shall study this problem and look at the various possibilities which arise (in different dimensions, but concentrating on the (2+1) dimensions) for different choices of the target manifold. In particular we are going to consider various $SU(n)$ groups and some interesting coset spaces such as the CP^n spaces. Most of our results can be generalised to the case of the more general Grassmannian spaces $G_{nm}(\mathbb{C})$.

In the next section various σ models which have been used in this context are introduced and a discussion of the additional terms that can be added to the conventional Lagrangian terms is included. A discussion of what conditions these terms have to satisfy, and what would be their effects on the equations of motion is given. The discussion puts more emphasis on "Skyrme-like" terms but mentions also some topological terms, such as the WZW term or the Hopf term. The following two sections scrutinize the problem and look in detail at the terms that can be used when the target manifold is $SU(2)$ or CP^1. Section four generalises the discussion to more general spaces.

4.2 σ models and the additional terms

The original model of a proton[15][16] involves a standard $U(n)$ σ-model with the Lagrangian

$$L_\circ = \frac{1}{2} \; tr\partial_\mu U \partial^\mu U, \qquad U^{-1} = U^\dagger, \tag{4.1}$$

to which an additional term (called the "Skyrme term") given by (1.11) was added in order to stabilise the solitonic solutions of the classical equations of motion. The model was considered in (3+1) dimensions. Its static solutions fall into disjoint classes characterised by the value of the integer valued topological index

$$B = \frac{1}{24\pi^2}\epsilon^{ijk} \int d^3x \; tr(U^{-1}\partial_i U U^{-1}\partial_j U U^{-1}\partial_k U). \tag{4.2}$$

The simplest nontrivial solution which corresponds to $B = 1$ is called the "Skyrmion", and since B is interpreted as the baryon number, this Skyrmion solution of the static equations of motion is taken as providing us with a description of a proton.

The original Lagrangian (4.1) (in (3+1) dimensions) has no stable static or even nonstatic solutions, as for any given field configuration $U(\underline{r})$ we can always decrease its energy by replacing it by $U(\lambda\underline{r})$, where $\lambda > 1$. However, as (4.1) involves four derivatives, it scales under $\underline{r} \rightarrow \lambda\underline{r}$ in the opposite way to (4.1) and this property stabilises the Skyrmions. The above scaling argument shows why we need a term containing more than 3 derivatives, in addition to (4.1). Not all of them are possible, as we want the additional terms to posses most of the symmetries of the original Lagrangian (4.1). Hence the additional terms should be Lorentz and $SU(n)$ invariant. Moreover, they should not alter the equations of motion too much; thus we need terms which are, at most, of the second order in time derivatives. In addition, we want the total Hamiltonian to be well defined and non-negative.

It was shown in the earliest papers on the Skyrme model[16][15] , that (4.2) is a unique term which satisfies all these conditions in the $SU(2)$ case. However, as we will see below, this is not the case for $SU(n)$ when $n > 2$. We will demonstrate this point in section 4.

Of course, the four derivative Skyrme term is not the only term we can add to (4.1) to have a physically relevant theory. We could add also further terms involving either more derivatives (say, six) or terms which are topological in nature. The addition of terms involving more derivatives may seem a bit arbitrary at first, and against the spirit of simplicity; however the appearance of such terms can be justified in a low energy effective theory approximation to QCD as studied in detail by the Oxford group[45] . However, as the results of such studies are not very encouraging, we will not consider such terms in this thesis.

The topological terms are more important. In fact Witten has shown[15] that, if we want the soliton of the model to represent a fermion, the Lagrangian of the Skyrme model should be supplemented by an appropriate Wess-Zumino (WZW) term. Such a term provides a contribution to the equations of motion of the full theory but vanishes for its static extended solutions. In the latter case, it allows the solitons of the theory to be quantised as fermions and it is responsible for the imposition of the fermionic statistics on these solitons.

We are also interested in field theories defined on appropriate coset spaces and, in particular, in CP^n and other Grassmannian models obtained *e.g.* from the $U(n)$ σ models by the reduction $U = \gamma U^{-1}$, where γ is a complex number of modulus 1. In fact such reductions were introduced and used in Uhlenbeck's approach to the construction of classical solutions of the $U(n)$ σ models in (2+0) dimensions[46]. In this chapter we shall consider first the case of CP^n models; the generalisation to more general Grassmannian models is relatively easy. To consider such models we take $U = \beta(1 - 2\mathbb{P})$, where β is a constant matrix such that the matrix $U \in SU(n)$ and $\mathbb{P}$ is a projector of rank 1 and so it can be written as $\mathbb{P} = ZZ^\dagger$ since, of course, $Z^\dagger Z = 1$. Then, the natural description of the reduced model involves $D_\mu Z$ and $D_\mu Z^\dagger$, where D stands for the covariant derivative. The conventional term in the Lagrangian for these models is given by equation (3.4).

Of course the modified CP^1 model, discussed earlier, can be obtained by this method. However, the question then arises as to the uniqueness of the "Skyrme-term" added to this model and its generalisation when one goes beyond the the CP^1 model. This problem will be considered in the next sections.

We can also look at further topological and nontopological terms that can be added to the Lagrangian of the coset space models. Ignoring terms involving more derivatives, the obvious candidate is the Hopf term[47]. This term is purely topological in nature only in the CP^1 case, and in this case, it does not contribute to the equations of motion (it would contribute to the equations for CP^n, with higher values of n); for static solitons it would be responsible for their quantisations as fermions, bosons or anyons[47]. We shall discuss the role of this term further in section 4 of this chapter, where we will study more general coset space models.

4.3 $SU(2)$ or CP^1 models

To construct G-invariant models we use the left invariant forms on G, which can be written as[48]

$$\theta = U^{-1}\, dU \quad = \omega \circ X \tag{4.3}$$

where X generate the Lie algebra of G. For $SU(2)$, we take X hermitian, $X = \vec{\sigma}$ and $\theta^\dagger = -\theta$. Selecting the third axis we find

$$\omega = A\sigma^3, \qquad A = \frac{1}{2}tr(\sigma^3\theta) = Z^\dagger dZ = -A^\dagger \tag{4.4}$$

where the complex 2-vector $Z = (z_1, z_2)$ parametrises the $SU(2)$ elements by

$$U = \begin{pmatrix} z_1 & -z_2^* \\ z_2 & z_1^* \end{pmatrix} \tag{4.5}$$

with $Z^\dagger Z = 1$, which guarantee that $U^{-1} = U^\dagger$ and $det U = 1$. The action of $U(1)$ on U

or Z is given by

$$U \mapsto U\begin{pmatrix} e^{i\phi} & 0 \\ o & e^{-i\phi} \end{pmatrix}, \qquad Z \mapsto Ze^{i\phi}. \tag{4.6}$$

Thus, on account of the unimodularity condition, the $SU(2)$ elements may be parametrised by

$$Z = \frac{(1, W)}{\sqrt{1+|W|^2}} e^{i\phi} \tag{4.7}$$

and the elements of the S^2 coset by $\frac{(1,W)}{\sqrt{1+|W|^2}}$ in terms of one complex parameter W. The one-form (4.4) defines a $U(1)$ connection for the $SU(2)(U(1), S^2)$ Hopf bundle, which can be used to define the covariant differential

$$DZ = dZ - AZ \qquad DZ^* = dZ^* + AZ^*, \tag{4.8}$$

since, under a local $U(1)$ phase transformation, $Z \mapsto Ze^{i\phi}, A \mapsto A + id\phi$ so that $DZ \mapsto e^{i\phi}DZ$. The covariant derivative also fulfills the obvious condition

$$Z^\dagger DZ = 0. \tag{4.9}$$

In terms of U, (4.8) and (4.9) read

$$DU = dU - UA\sigma^3 \qquad (DU)^\dagger = dU^\dagger + \sigma^3 U^\dagger A \tag{4.10}$$

and

$$tr(\sigma^3 U^\dagger DU) = 0 \tag{4.11}$$

respectively. The curvature $F = D(A)(= dA)$ is given by the well known two-form

$$F = dZ^* \wedge dZ \tag{4.12}$$

and can be written as $F = DZ^* \wedge DZ$. This two-form has the same expression both on $SU(2)$ and the coset S^2. All the above expressions are defined on the group manifold $SU(2)$; they are also defined on the coset $\frac{SU(2)}{U(1)} = S^2$ if they do not depend on ϕ *i.e.* if they are gauge invariant.

Let us consider now $SU(2)$ as the target space for the map $\phi : M \to SU(2)$ where M is some $(d+1)$ dimensional manifold parametrised by x^μ, $\mu = 0, 1, 2, ..., d$. Then the forms on $SU(2)$ induce forms on M, $dZ \mapsto \partial_\mu Z dx^\mu$. In particular, we find that

$$A = A_\mu \, dx^\mu, \qquad A_\mu = Z^\dagger \partial_\mu Z \equiv Z^\dagger Z_\mu, \tag{4.13}$$

$$F = \frac{1}{2} F_{\mu\nu} \, dx^\mu \wedge dx^\nu, \qquad F_{\mu\nu} = Z_\mu^\dagger Z_\nu - Z_\nu^\dagger Z_\mu. \tag{4.14}$$

We now turn our attention to the definition of the Lagrangian on M. The standard term necessarily involves the metric on M, which we shall assume to be flat. Using ϕ^* to denote the form on M induced by the forms on the group, the standard term is given by

$$tr[\phi^*(\theta) \wedge (*\phi^*(\theta))], \tag{4.15}$$

where $*$ before ϕ^* is the Hodge operator which includes the Lorentzian metric on M. In coordinates, this gives rise to the standard term in the action

$$S_\circ = -\frac{1}{2} \int tr[(U^{-1}\partial_\mu U)(U^{-1}\partial^\mu U)] d^{d+1}x \; = \int L_\circ d^{d+1}x,$$

$$L_\circ = \frac{1}{2} tr[\partial_\mu U^{-1} \partial^\mu U]. \tag{4.16}$$

For $SU(2)$, (4.16) gives

$$L_\circ = Z_\mu^* Z^\mu. \tag{4.17}$$

Another possible term is the (gauge invariant) term obtained from

$$\widetilde{L}_\circ = \phi^*(DZ^\dagger) \wedge (*\phi^*(DZ)). \tag{4.18}$$

In coordinates it has the form

$$\widetilde{L}_\circ = (D_\mu Z)^\dagger (D^\mu Z) = Z_\mu^\dagger Z^\mu - A_\mu^\dagger A^\mu. \tag{4.19}$$

We are also interested in σ fields taking values in CP^n spaces *i.e.* when the fields U are considered as fields on coset spaces $\frac{U(n)}{U(n-1)\times U(1)}$ which replace the group as a target manifold. In fact, for topological reasons, many static extended structures in (2+1) coset models are stable, while they are not stable when considered as fields on the group manifold. Clearly, only the $U(1)$ gauge invariant Lagrangian on $SU(2)$ leads to well defined terms on the coset space. This implies the replacing of dZ by DZ, which effectively corresponds to subtraction of the $U(1)$ part from $U^{-1}dU$. Thus (4.18) is replaced by $\widetilde{L}_o$ in (4.19) which with Z given by (4.7) leads to

$$\widetilde{L}_o = \frac{W_\mu^* W^\mu}{(1+|W|^2)^2}, \tag{4.20}$$

where $W_\mu = \frac{\partial W}{\partial x^\mu}$, and

$$A_\mu = i\phi_\mu + \frac{W^* W_\mu - W W_\mu^*}{2(1+|W|^2)}, \tag{4.21}$$

and so that $\widetilde{L}_o$ corresponds to L_o given by (4.18), from which $A_\mu^\dagger A^\mu$ has been subtracted.

There is an alternative way of implementing the Hopf projection $SU(2) \mapsto S^2$. Using the projector $ZZ^\dagger$ an alternative parametrisation for the elements of the coset is given by

$$U = i(I\!I - 2ZZ^\dagger), \tag{4.22}$$

which amounts to defining coset elements U by the condition $U = -U^\dagger$. They clearly depend on two parameters: we may set $Z \in S^2 = CP^1$ as

$$Z = \frac{(1, W)}{\sqrt{1+|W|^2}}, \tag{4.23}$$

or we may take $|z_1|^2$, and the relative phase of z_1 and z_2 as the independent parameters. The advantage of (4.22) and (4.23) is that they generalise to more general complex Grassmannian models $G_{nm}(\mathbb{C})$. Using (4.23) in (4.18) leads directly to the coset Lagrangian (4.19).

Let us consider which terms could be added to the action based on (4.16). The best way to list all the possible terms is to start defining them from the invariant forms on the $SU(2)$ manifold. The terms which do not depend on the metric M are topological. The two possible terms are (*a*) WZW- like term. This term is defined from the three form on M

$$L_{WZW} = tr(\phi^*(\theta) \wedge \phi^*(\theta) \wedge \phi^*(\theta)) \tag{4.24}$$

which comes, in fact, from the only nonzero form $tr(\theta \wedge \theta \wedge \theta)$ on $SU(2)$ which can be obtained by taking the trace of products of the forms (4.3).

(*b*) Hopf term. This term was introduced by Wilczek and Zee[47] , and it is defined by the Hopf invariant integral. Let ψ be a map $\psi : S^3 \rightarrow S^2$, then the area element ω on S^2 induces a form $\psi^*(\omega)$ which is exact on S^3. If α is the potential form for $\psi^*(\omega)$, $d\alpha = \psi^*(\omega)$, the Hopf invariant of the map ψ is defined by

$$H(\psi) = \int_{S^3} \alpha \wedge \psi^*(\omega) = \int_{S^3} h. \tag{4.25}$$

It is easy to check, *e.g.* by using (3.58) that the area element on S^2 is given by the two form F (*i.e.* equation (4.12)). With ψ being the map of the compactified three-dimensional space-time on CP^1, it is clear that h is given by

$$\begin{aligned} h &= \epsilon^{\mu\nu\rho}(Z^\dagger \partial_\mu Z)(\partial_\nu Z^\dagger \partial_\rho Z) \\ &= \epsilon^{\mu\nu\rho}(Z^\dagger \partial_\mu Z)(D_\nu Z^\dagger D_\rho Z). \end{aligned} \tag{4.26}$$

It can also be written in the form $h = A_\mu J^\mu$, where $J^\mu = \epsilon^{\mu\nu\rho}\partial_\nu A_\rho$ is a topological current.

The metric M is introduced by using the Hodge operator. This gives rise to various possible forms of the Skyrme terms. In the $SU(2)$ case the standard form of this term is given by

$$tr[\phi^*(\theta^\dagger \wedge \theta) \wedge (*\phi^*(\theta^\dagger \wedge \theta))]. \tag{4.27}$$

On M this expression can be written as

$$L_1 = \frac{1}{2} tr([\theta^\dagger_\mu, \theta_\nu][\theta^{\mu\dagger}, \theta^\nu]). \tag{4.28}$$

One may also consider a term of the form

$$L_2 = -(tr(\theta^\dagger_\mu \theta^\mu))^2 + tr(\theta^\dagger_\mu \theta_\nu) tr(\theta^{\mu\dagger}\theta^\nu), \tag{4.29}$$

but for $SU(2)$, $L_1 = L_2$. This can be proved easily by writing $\theta_\mu = U^{-1}\partial_\mu U = i\vec{\sigma}V_\mu$ with V_μ real and making use of $tr(\theta^\dagger_\mu \theta^\mu) = 2V_\mu V^\mu$.

The other simple term one may think of is the one defined on M by the form $\phi^*(F) \wedge (*\phi^*(F))$. In coordinates, this term is given by

$$L_3 = F_{\mu\nu} F^{\mu\nu}. \tag{4.30}$$

Finally we could also written the term

$$L_4 = \epsilon^{\mu\nu\gamma}(D_\mu Z^\dagger D^\rho Z + c.c)\epsilon_{\rho\sigma\gamma}(D_\nu Z^\dagger D^\sigma Z + c.c). \tag{4.31}$$

Next we look at possible terms defined on the CP^1 coset. The terms (4.31) and (4.30), being gauge invariant, are already defined on S^2 ; the terms (4.28) and (4.29) are not. Nevertheless, replacing ∂_μ by D_μ in $\theta_\mu = U^{-1}\partial_\mu U$ results in (4.28) and (4.29) being defined on S^2. There is another way of proceeding, however, which produces the same effect: the same result is obtained, if, when computing (4.28) and (4.29), the parametrisation (4.22) and (4.23) of CP^1 is used for $U^{-1}\partial_\mu U$. This is based on the observation that, with U given by (4.22) and (4.23)

$$\begin{aligned} tr(\partial_\mu U^{-1}\partial_\nu U) &= (\partial_\mu Z^\dagger \partial_\nu Z) + (\partial_\nu Z^\dagger \partial_\mu Z) + 2(Z^\dagger \partial_\mu Z)(Z^\dagger \partial_\nu Z) \\ &= ((D_\mu Z^\dagger D_\nu Z) + c.c). \end{aligned} \tag{4.32}$$

Next we observe that on S^2 all these terms (4.28)-(4.31) give rise to the same Skyrme term for the action. In particular, the expression (4.30) is given by

$$F^{\mu\nu}F_{\mu\nu} = \frac{(W^{*\mu}W^\nu - W^{*\nu}W^\mu)(W^*_\mu W_\nu - W^*_\nu W_\mu)}{(1+|W|^2)^4}. \tag{4.33}$$

The fact that the other Skyrme terms are equivalent to (4.33) follows from the fact that they lead to (4.33) plus a term proportional to the expression

$$\begin{aligned} \eta = 2[&(Z^{\dagger\mu}Z^\nu)(Z^\dagger Z_\mu)(Z^\dagger Z_\nu) - (Z^{\dagger\mu}Z_\mu)(Z^\dagger Z^\nu)(Z^\dagger Z_\nu)] \\ &- (Z^{\dagger\mu}Z_\mu)^2 + (Z^{\dagger\mu}Z^\nu)(Z^\dagger_\mu Z_\nu) \end{aligned} \tag{4.34}$$

which, for $SU(2)$, is zero. Of course (4.28) and (4.29) had to be equivalent since they were already equal at the group manifold level.

4.4 Larger $SU(n)$ Groups, CP^n Spaces And $G_{nm}(\mathbb{C})$Manifolds

Next we consider larger groups. We take, for instance $SU(3)$ and look at (4.28) and (4.29). This time these two expressions are not equal. To see this let us consider a particular element of $SU(3)$, which is a product of two elements from two different subgroups $SU(2)$. Thus we take, say

$$\begin{aligned} U = U_1 U_2 &= \begin{pmatrix} a & 0 & 0 \\ o & a^* & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & ib^* \\ 0 & ib & 0 \end{pmatrix} \\ &= \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & ib^* \\ o & ia^*b & 0 \end{pmatrix} \end{aligned} \tag{4.35}$$

where a and b are complex functions of unit modulus, which we can thus set $a = exp(i\alpha), b = exp(i\beta)$. Next we calculate (4.28) and (4.29) for these terms. We find that (4.28) vanishes while (4.29) is proportional to

$$(\alpha_\mu \beta_\nu - \alpha_\nu \beta_\mu)(\alpha^\mu \beta^\nu - \alpha^\nu \beta^\mu). \tag{4.36}$$

This expression is clearly nonzero. Although this result has been obtained for a very special element of $SU(3)$, it is clear that had we chosen more general elements of $SU(3)$ or larger groups, we would have found similar differences between (4.28) and (4.29). Thus for $SU(n)$ with $n > 2$, we have two candidates for a Skyrme term, namely (4.28) and (4.29).

A similar situation exists when we consider the theory on larger coset CP^n spaces. There is a $U(1)$ principle bundle structure $S^{(2n+1)}(U(1), CP^n)$ over general CP^n spaces[49] but the total manifold $S^{(2n+1)}$ is no longer a group if $n > 1$. Thus we no longer have (4.28) and (4.29) as the starting point for the construction of an additional term to be added to $\widetilde{L}$. Nevertheless we have again two possible Skyrme terms which this time we can take to be given by (4.30) and (4.31). Although they are the same in the $CP^1 = S^2$ case, they are different already for the CP^2 model. To see this we parametrise $Z \in CP^2$ by

$$Z = \frac{(1, W_1, W_2)}{\sqrt{1 + |W_1|^2 + |W|^2}}. \tag{4.37}$$

Then (4.30) is given by

$$F_{\mu\nu}F^{\mu\nu} = \frac{(W^*_{1\mu}W_{1\nu} + W^*_{2\mu}W_{2\mu} + R^*_\mu R_\nu - (\mu \leftrightarrow \nu))^2}{(1+|W_1|^2+|W_2|^2)^4}, \tag{4.38}$$

where $R^\mu = W_1^\mu W_2 - W_1 W_2^\mu$ and where the square implies the summation over μ and ν with the Lorentzian metric.

Next we have to calculate (4.31) However, instead of calculating it explicitly it is convenient to rewrite $-F_{\mu\nu}F^{\mu\nu}$ as

$$-F_{\mu\nu}F^{\mu\nu} = -(A_{\mu\nu} - A_{\nu\mu})(A^{\mu\nu} - A^{\nu\mu}) \tag{4.39}$$

where $A_{\mu\nu} = D_\mu Z^\dagger D_\nu Z$. Then (4.31) is given by

$$4A^\mu_\mu A^\nu_\nu - 2A_{\mu\nu}A^{\mu\nu} - 2A_{\mu\nu}A^{\nu\mu} \tag{4.40}$$

and so we see that the difference of (4.38) and (4.40) is given by $4(A^\mu_\mu A^\nu_\nu - 4A_{\mu\nu}A^{\nu\mu})$. Since in the CP^2 case $A_{\mu\nu}$ is given by

$$A_{\mu\nu} = \frac{(W^*_{1\mu}W_{1\nu} + W^*_{2\mu}W_{2\nu} + R^*_\mu R_\nu)}{(1+|W_1|^2+|W_2|^2)^2}, \tag{4.41}$$

we find that the required difference is given by

$$4\frac{|W_{1\mu}W_{2\nu} - W_{2\mu}W_{1\nu}|^2 + |W_{1\mu}R_\nu - R_\mu W_{1\nu}|^2 + |W_{2\mu}R_\nu - R_\mu W_{2\nu}|^2}{(1+|W_1|^2+|W_2|^2)4}, \tag{4.42}$$

where the summation over μ and ν is again understood with the Lorentzian metric. The derived expression clearly does not vanish in general (of course, it is zero if $W_2 = 0$, which correspond to the previous CP^1 case). Similar results can be obtained for other CP^n models.

In addition we will try to extend our handling of the CP^n spaces to the Grassmannian manifolds $G_{nm}(\mathbb{C})$. Since the basic Lagrangian for the Grassmannian models possess a $U(m)$ local symmetry, we can think of the Lagrangian as giving a constant function on the orbits of the $U(m)$ group realised on the $G_{nm}(\mathbb{C})$ spaces. Therefore the picture looks locally as a product of two spaces $G_{nm}(\mathbb{C}) \times U(m)$.

To construct both the basic Lagrangian and the additional terms in $G_{nm}(\mathbb{C})$ spaces we will consider the line bundle E, whose cross section is the $n \times m$ field matrix of the Grassmannian base space. Furthermore, its structure group is the unitary group of order M, *i.e.*$(U(m))$. To show that $U(m)$ is the structure group, consider two overlapping patches say (U_i, U_j) such that $U_i \cap U_j \neq \phi$, and consider the local trivialisation maps ψ_i:

$$\psi_i : E \rightarrow \pi^{-1}(U_i).$$

Then the transition function t_{ij} at the point $Z \in G_{nm}(\mathbb{C})$ which takes the form

$$t_{ij} = \psi_j^{-1} \circ \psi_i \tag{4.43}$$

is an element of $U(m)$, that is $t_{ij} \in U(m)$.

To complete the description of E, we also need to specify how covariant variations of Z are performed. For this purpose we introduce the one form connection, used to parallel transport structures in E, by the unique splitting of the fibre space into two orthogonal subspaces, one of which is (called) the horizontal subspace and the other the vertical one.

So let $\Gamma[c]$ be the transport of Z with respect to the curve c. Then $\Gamma[c]$ belongs to $U(m)$ and it takes, infinitesimally, the form[23]

$$\Gamma[c] = P(e^{\int dZA}), \tag{4.44}$$

where P in the above expression stands for the path ordering and A is the one form gauge potential, which in our case of the $m \times m$ matrix, can be written as

$$A = Z^{+}dZ. \tag{4.45}$$

From this connection we determine the covariant derivative on E which is given by

$$DZ = dZ - AZ \qquad\qquad DZ^{+} = dZ^{+} + Z^{+}A. \tag{4.46}$$

It is easy to show that DZ transforms under $U(m)$ as required.

Another important object, which transforms covariantly under $U(m)$, is the curvature of the bundle E, which is obtained form the one-form connection by the covariant differentiation

$$\begin{aligned} F &= DA = DZ^{+} \wedge DZ \\ &= dZ^{+} - A \wedge A. \end{aligned} \tag{4.47}$$

Following the same steps as in the CP^n models, let our fields be mappings from the space-time manifold to the target manifold E:

$$\phi : M \rightarrow E, \tag{4.48}$$

where M is chosen to be a Lorentzian manifold with dimension $d+1$ and parametrised by the coordinates $x^{\mu} = (x^{\circ}, ..., x^{d})$. Consequently, forms defined on E induce forms on M by the pullback mapping ϕ^*. In this description the basic Lagrangian in $G_{nm(\mathbb{C})}$ models is the bilinear form on the space of one forms, namely

$$L_{\circ} = tr\phi^*(DZ)^{\dagger} \wedge *\phi^*(DZ) \tag{4.49}$$

or written with respect to a coordinates system on E,

$$L_{\circ} = tr(D_{\mu}Z)^{\dagger}(D^{\mu}Z). \tag{4.50}$$

Another equivalent approach in constructing the basic Lagrangian and possibly additional terms on $G_{nm}(\mathbb{C})$ spaces is to construct expressions on the group manifold $U(n)$ and then restrict the group elements to the proper coset by the appropriate projection. Obviously the most difficult part of the second approach is to find the projection from the group manifold to the Grassmannian space. In fact Uhlenbeck[46] showed that in this case the projection operator has the form

$$U = i(I\!I - 2I\!P), \tag{4.51}$$

with the condition that $I\!P$ is a matrix of rank M; in other words let $(Z_1, Z_2, ..., Z_m)$ be a set of orthonormal vectors then $I\!P$ is given by

$$I\!P = Z_1^{\dagger}Z_1 + Z_2^{\dagger}Z_2 + ... + Z_m^{\dagger}Z_m. \tag{4.52}$$

In any model building process one needs covariant objects on the group manifold $U(n)$ and the left invariant forms are the best candidates to play this role for us. Thus in the other description the basic Lagrangian has the form

$$\begin{aligned} L_\circ &= tr\phi^*(\theta) \wedge *\phi^*(\theta); \qquad \phi^*(\theta) = \theta_\mu = U^{-1}\partial_\mu U \\ &= \partial_\mu I\!P \partial^\mu I\!P, \end{aligned} \tag{4.53}$$

provided that U satisfies (4.54).

The additional terms on $G_{nm}(\mathbb{C})$ spaces are of two different types; namely, those which are topological and others which are not. For the topological ones we are interested in constructing an analogue of the WZW term of $SU(2)$ or CP^1 on $G_{nm}(\mathbb{C})$ spaces. In $(2+0)$ dimensions a reasonable candidate for the WZW term is

$$\begin{aligned} L_{WZW} &= \phi(\theta) \wedge \phi(\theta) \wedge \phi(\theta) \\ &= \epsilon^{\mu\nu\rho} tr((1 - I\!P) I\!P_\mu (1 - I\!P) I\!P_\nu (1 - I\!P) I\!P_\rho). \end{aligned} \tag{4.54}$$

However, due to the trace properties, the above term is symmetric in $\{\mu\nu\rho\}$ and on the other hand, it is proportional to the alternating tensor. Therefore the WZW term identically vanishes on the Grassmannian coset.

For Skyrme terms one can define four possible terms. The first one takes the form

$$L_1 = tr\phi^*(F) \wedge *\phi^*(F), \tag{4.55}$$

or, in terms of a coordinates system on E,

$$L_1 = trF_{\mu\nu}F^{\mu\nu} \tag{4.56}$$

where $F_{\mu\nu}$ is given by

$$\begin{aligned} F_{\mu\nu} &= (D_\mu Z)^+(D_\nu Z) - (D_\nu Z)^+(D_\mu Z) \\ &= \partial_\mu Z^+ \partial_\nu Z - \partial_\nu Z^+ \partial_\mu Z + [A_\mu, A_\nu]. \end{aligned} \tag{4.57}$$

The second candidate, just as in CP^1 case, is written as

$$L_2 = \epsilon^{\mu\nu\gamma} tr(D_\mu Z^+ D^\rho Z + c.c)\epsilon_{\rho\sigma\gamma} tr(D_\nu Z^+ D^\sigma Z + c.c), \tag{4.58}$$

whereas the third term is given by

$$L_3 = tr[\phi^*(\theta^+ \wedge \theta) \wedge *\phi^*(\theta^+ \wedge \theta)], \tag{4.59}$$

which in our case reduces to

$$L_3 = tr[\mathbb{P}_\mu \mathbb{P}_\nu \mathbb{P}^\mu \mathbb{P}^\nu - \mathbb{P}_\nu \mathbb{P}_\mu \mathbb{P}^\mu \mathbb{P}^\nu]. \tag{4.60}$$

The fourth term is simply given as

$$L_4 = -[tr(\theta_\mu^+ \theta^\mu)]^2 + tr[\theta_\mu^+ \theta_\nu] tr[\theta^{\mu +} \theta^\nu], \tag{4.61}$$

which upon projection onto the Grassmannian manifold takes the form

$$L_4 = -tr(\mathbb{P}_\mu \mathbb{P}^\mu)^2 + tr(\mathbb{P}_\mu \mathbb{P}_\nu) tr(\mathbb{P}^\mu \mathbb{P}^\nu). \tag{4.62}$$

Are any of these Skyrme terms equivalent? To check this we resort to a parametrisation of the Grassmannian spaces due to Macfarlane[50] . He wrote the field matrix as

$$Z = \begin{pmatrix} P \\ Q \end{pmatrix}, \tag{4.63}$$

where P and Q are $(n-m, m)$ and (m, m) matrices, respectively, and then exploited the $U(m)$ local symmetry to rewrite it as

$$Z = \begin{pmatrix} KL \\ L \end{pmatrix} \tag{4.64}$$

where L is a hermitian matrix.

After a straightforward but tedious calculation one can show that the four terms take the following forms respectively

$$
\begin{aligned}
L_1 &= tr(L\partial_\mu K^+[1 - KL^2K^+ + (KL^2K^+)]\partial_\nu KL - (\mu \leftrightarrow \nu)), \\
L_2 &= 2\tau(tr\partial_\mu K^+\partial^\mu K)^2 - \tau^2 tr(\partial_\mu K^+\partial^\nu K + \partial_\nu K^+\partial^\mu K)tr(\mu \leftrightarrow \nu), \\
L_3 &= tr(\partial_\mu(KL^2K^+)\partial_\nu(KL^2K^+) + \partial_\mu(KL^2)\partial_\nu(L^2K^+))^2 \\
&\quad tr\{(\partial_\mu(KL^2K^+)\partial_\nu(KL^2) + \partial_\mu(KL^2)\partial_\nu(L^2))(\mu \leftrightarrow \nu)^+\} \\
&\quad tr\{(\partial_\mu L^2K\partial_\nu KL^2K^+ + \partial_\mu L^2\partial_\nu L^2K^+)(\mu \leftrightarrow \nu)^+\} \\
&\quad + tr(\partial_\mu LrK^+\partial_\nu KL^2 + \partial_\mu L^2\partial_\nu L^2)^2 \\
&\quad - tr(\partial_\mu KL^2K^+\partial_\nu KL^2K^+ + \partial_\mu KL^2\partial_\nu L^2K^+)(\mu \leftrightarrow \nu)^2 \\
&\quad - tr(\partial_\mu L^2K^+\partial_\nu KL^2 + \partial_\mu L^2\partial_\nu L^2)(\mu \leftrightarrow \nu) \\
&\quad - tr|\partial_\mu KL^2K^+\partial_\nu KL^2 + \partial_\mu KL^2\partial_\nu L^2|^2 \\
&\quad - tr|\partial_\mu L^2K^+\partial_\nu KL^2K^+ + \partial_\mu L^2\partial_\nu L^2K^+|^2, \\
L_4 &= tr\{(\partial_\mu KL^2K^+\partial_\nu KL^2K^+ + \partial_\mu KL^2\partial_\nu L^2K^+) \\
&\quad tr(\partial_\mu L^2K^+\partial_\nu KL^2 + \partial_\mu L^2\partial_\nu L^2)(\mu \leftrightarrow \nu)\} \\
&\quad - tr(\partial_\mu KL^2K^+\partial^\mu KL^2K^+ + \partial_\mu KL^2\partial^\mu L^2K^+ + \partial_\mu L^2K^+\partial^\mu KL^2\partial_\nu L^2\partial^\nu L^2)^2.
\end{aligned}
\tag{4.65}
$$

Finally, we should add a few comments about the generalisation of the terms (4.24) and (4.25). Of these, the first contributes to the equations of motion only in $(1+1)$ and $(2+0)$ dimensions. In these cases its contribution to the action comes from the fact that (4.26) can be locally written as $d^{-1}tr(\theta \wedge \theta \wedge \theta)$ since $tr(\theta \wedge \theta \wedge \theta)$ is closed but not exact. The presence of the WZW term in the action implies that the original $SU(2)$ current algebra associated with (4.16) is replaced by the Kac-Moody algebra; also quantum considerations require the quantisation of the coefficient of WZW term[51] . In (2+1) dimensions, we can consider the three-form (4.24) as an additional contribution to the action. This term is locally a total divergence and so it contributes only to the quantum properties of the extended structures that the theory may posses. This pattern also holds in higher dimensions if we consider larger $SU(n)$ groups. For instance, in the general even dimensional case, the WZW term determines the "Schwinger terms" in the algebra[52] .

Let us now look at (4.25). As already mentioned by Wilczek and Zee[47] , (4.25) is also locally a total divergence. Nevertheless, and although they look different, the local expressions (4.24) and (4.25) are in fact equal. This can be seen by setting the two components z_1 and z_2 of Z as

$$z_1 = a_1 + ia_2 \qquad z_2 = a_3 + ia_4 \tag{4.66}$$

where $\sum_i a_i^2 = 1$, and then observing that in these variables (4.25) becomes

$$h = -\frac{1}{3}\epsilon^{abcd}\epsilon^{\mu\nu\lambda}a_a\partial_\mu a_b\partial_\nu a_c\partial_\lambda a_d. \tag{4.67}$$

But (4.67) is the expression for the topological charge density of the $SU(2)$ model in three dimensions, which is, in fact, given by the form (4.24). In fact both (4.67) and (4.24) come from the invariant volume form on S^3, which is given by the well known expression

$$\frac{da_1 \wedge da_2 \wedge da_3}{\sqrt{1-a^2}} \tag{4.68}$$

and which is equivalent to $tr(\theta \wedge \theta \wedge \theta)$.

The expression (4.25) is not locally a total divergence when we consider it for CP^n with $n > 1$, *i.e.* when Z has more than two components. Of course, introduced this way, the term looses its topological character and so it should not be called the "Hopf term", a name which is reserved for (4.25). However, from a pure field theoretic point of view, we could include such a term in the total action for CP^n, $n > 1$. We could calculate it explicitly, *e.g.* for the CP^2 case using the parametrisation (4.37). In this case one finds

$$\widetilde{h} = \frac{\epsilon_{\alpha\beta\gamma}}{(1+|W_1|^2+|W_2|^2)^2}[\partial_\alpha W_1^*\partial_\beta W_1(W_2^*\partial_\gamma W_2 - W_2\partial_\gamma W_2^*) + (W_1 \leftrightarrow W_2)]. \tag{4.69}$$

All in all, we have studied various σ models which one can use for seeking relativistic field equations with soliton-like extended structures in (2+1) dimensions. Such models can be thought of as lower dimensional analogues of the (3+1) dimensional Skyrme model. Some of the discussion was not restricted to (2+1) dimensions. In particular, we have found that when we consider $SU(3)$ or larger spaces we have two possible Skyrme terms which can be used to stabilise the solitons; for $SU(2)$ both of these terms coincide and give the conventional Skyrme term. Of course, for extended sructures, and solitons in particular, we should consider their stability. It may be that in practice, for stability reasons, we may be able to restrict our attention in (3+1) dimensions. However, when we consider more general numerical simulations there seems to be very little to choose between these two types of terms and we suggest that both are included.

When we restrict ourselves to (2+1) dimensions, and consider CP^n models, the situation is similar and, again, we have two terms involving four derivatives and in addition we have also one term involving three derivatives. In the CP^1 case the two four derivatives terms are the same and the three derivative term becomes topological (it becomes the Hopf term). For higher n the two terms are different and the possible additional three derivative term is no longer topological. In the next chapter we will discuss some implications of the addition of the Skyrme terms to the basic CP^n models and will discuss static solutions (Skyrmions) of these models.

V. SIGMA MODELS WITH SOLITONS IN (2+1) DIMENSIONS

5.1 Introduction

It is generally believed that most of the properties displayed by solitons in their scattering are associated with the integrability of the solitonic models, *i.e.* with the existence of an infinite number of conservation laws which restrict the form of the scattering and also partially determine some of its properties. Most integrable models correspond to field theories which describe the time evolution of one dimensional systems; very often the underlying field theory is nonrelativistic and the solitonic properties of the solution of this theory are achieved by a subtle interplay between the dispersive and the nonlinear terms. In many physical applications however, we are interested in models in higher dimensions, particularly those which are Lorentz invariant. But already in (2+1) dimensions only very special models are integrable *e.g.* the modified chiral model[53] or the Davey-Stewartson equation[54] and the Kadomtsev-Petviashvili equation[55]. Moreover none of these models is relativistically invariant. So, as we have argued in chapter one, if we want to consider Lorentz invariant models with solitonic-like behaviour in (2+1) dimensions or even higher, we have to go beyond integrable models and look at some quasi-integrable models in (2+1) dimensions.

At the same time it is not clear from the physical point of view whether we would prefer to restrict ourselves to purely integrable models, at least as far as the radiation effects are concerned. In fact, most physical processes, such as *e.g.* the proton-proton scattering, do indeed show some radiation effects but, as it turns out, these effects are often rather small. Thus, in the proton-proton scattering case, the elastic cross section dominates the inelastic one, especially for not too high energies. So may be the idea of going beyond the integrable models is physically more sound than it may seem at first sight-as long as all radiation effects observed in the quasi-integrable models are not too large.

In chapter 1 we have argued that the simplest of such quasi-integrable models is the modified CP^1 model whose Lagrangian is given by equation (1.12). However, having presented such a model, a question then arise as to the uniqueness of this model and its solutions. What are the properties of various terms in the model? What is their role? How much freedom does the model possess? We have already looked at these questions in the previous chapter where we studied the uniqueness of the Skyrme term. In this chapter we will concentrate on the potential term.

5.2 Skyrme Model

To perform our analysis of the modified CP^1 model, let us start by observing that the additional potential term (a term with no derivatives), whatever it is, must break the global $O(3)$ invariance. Thus let us take it to be a function of only one component of $\vec{\phi}$, say, ϕ^3, (1.8).e. we take it as

$$L_V = f(\phi^3). \tag{5.1}$$

It is clear that the model based on the Lagrangian consisting of the sum of L_o (basic Lagrangian) , L_S (Skyrme term) and L_V (potential term) is still Lorentz invariant and for positive values of θ_1 and reasonable choices of $f(\phi^3)$, its Hamiltonian is positive definite. Moreover, despite the appearance to the contrary, the Lagrangian does not contain time derivatives higher than two and so its equation of motion takes the conventional form.

How unique are the two new terms? This problem was considered in chapter four where it was shown that for a model with a $O(3)$ symmetry the L_S term is unique; the other term is very nonunique and we will discuss here some of the choices one can make.

What is the role of all our additional terms whatever the choice of L_V? To study this point let us consider an arbitrary field configuration $\vec{\phi} = \vec{\phi}_1 = \vec{\phi}(t, x, y)$ and compare it with $\vec{\phi}_2 = \vec{\phi}(\mu t, \mu x, \mu y)$ (and use the corresponding expressions for $\vec{\phi}$). In particular, let us look at $E_0 = \int L_o\, dxdy$, $E_S = \int L_S\, dxdy$ and $E_V = \int L_V\, dxdy$ and compare their values for both field configurations. Clearly, for $\vec{\phi}_2$ we can change the variables of integration ($t \to \mu t$, $x \to \mu x$, $y \to \mu y$) and so find that

$$\begin{aligned} E_0(\vec{\phi}_2) &= E_0(\vec{\phi}_1) \\ E_S(\vec{\phi}_2) &= \mu^2 E_S(\vec{\phi}_1) \\ E_V(\vec{\phi}_2) &= \mu^{-2} E_V(\vec{\phi}_1). \end{aligned}$$

We see that as the scaling properties of the last two terms are opposite, the combined effect of the inclusion of both of them is to introduce a scale and to stabilise the solitons. Moreover, the value of μ for which $E(\vec{\phi}_2) = E_0(\vec{\phi}_2) + E_S(\vec{\phi}_2) + E_V(\vec{\phi}_2)$ is minimal is given by $\mu = (E_V(\vec{\phi}_1)/E_S(\vec{\phi}_1))^{1/4}$. So, for that value of μ, $E_S(\vec{\phi}) = E_V(\vec{\phi})$ and $E(\vec{\phi}) =$

$E_0(\vec{\phi}_1) + 2 * (E_V(\vec{\phi}_1) * E_S(\vec{\phi}_1))^{1/2}$. We have thus proved that for every solution $\vec{\phi}$ of the Skyrme model the energies of the additional terms are the same ($E_S(\vec{\phi}) = E_V(\vec{\phi})$). This was, in fact, observed in the model introduced in ref.[17] , where L_V was taken to be given by

$$L_V = -\frac{1}{4}\theta_2(1+\phi^3)^4, \tag{5.2}$$

where θ_2 is a new parameter of the model. The equation of motion of this model is then given by

$$\begin{aligned} &\partial_\mu\partial^\mu\phi^i - (\vec{\phi}\,.\,\partial_\mu\partial^\mu\vec{\phi})\phi^i - 2\theta_1\big[\partial_\mu\partial^\mu\phi^i(\partial_\nu\vec{\phi}\,.\,\partial^\nu\vec{\phi}) + \partial_\nu\phi^i(\partial_\mu\partial^\nu\vec{\phi}\,.\,\partial^\mu\vec{\phi}) \\ &\quad -\partial_\nu\partial_\mu\phi^i(\partial^\nu\vec{\phi}\,.\,\partial^\mu\vec{\phi}) - \partial_\mu\phi^i(\partial^\nu\partial_\nu\vec{\phi}\,.\,\partial^\mu\vec{\phi}) + (\partial_\mu\vec{\phi}\,.\,\partial^\mu\vec{\phi})(\partial_\nu\vec{\phi}\,.\,\partial^\nu\vec{\phi})\phi^i \\ &\quad - (\partial_\nu\vec{\phi}\,.\,\partial_\mu\vec{\phi})(\partial^\nu\vec{\phi}\,.\,\partial^\mu\vec{\phi})\phi^i\big] + 2\theta_2(1+\phi^3)^3\,(\delta_{i3} - \phi^i\phi^3) = 0, \end{aligned} \tag{5.3}$$

and, as shown in ref.[17] , the model possesses a static one soliton solution. This solution has a simple form in the W formulation. In fact, it is given by

$$W = \lambda(x_+ - a), \tag{5.4}$$

where λ takes the value

$$\lambda = \lambda_0 = \sqrt[4]{\frac{\theta_2}{2\theta_1}}. \tag{5.5}$$

Observe that this is a particular case of the one instanton solution of the S^2 σ-model (3.59), but with the fixed "size" (determined by λ_0). In what follows we shall refer to this solution as a skyrmion. The total energy of the field configuration (5.4) is given by

$$E(\lambda) = 2\pi\big(1 + \frac{4}{3}\theta_1\lambda^2 + \frac{2}{3}\frac{\theta_2}{\lambda^2}\big). \tag{5.6}$$

The solution (5.4) was then used in many further investigations[17][18] which have shown, among other things, that

(1) the skyrmion is stable,

(2) there exist small repulsive forces between two skyrmions,

(3) the forces between a skyrmion and a anti-skyrmion are attractive,

(4) if two skyrmions are sent towards each other at a zero impact parameter and at sufficiently high velocity, then they come out of the interaction region at 90° to the direction of the original motion.

(5) the mechanism of this 90° scattering proceeds through the process of the formation of a ring (when two skyrmions are on top of each other).

Even though most of these results were determined in numerical simulations, some of them may in fact be established by analytical considerations. In particular, in the next section we discuss the stability properties of the solutions of our modified models; most of that analysis is actually applicable to the whole class of potential-like terms (*i.e.* going beyond (5.2).)

5.3 Modified Skyrme Model

To proceed with our construction of more generalised models we will use the formulation in terms of the W field (instead of $\vec{\phi}$ field), since this makes the description of results simpler. Let us consider more general potential terms, namely corresponding to L_V given by

$$L_V = -4\,\theta_2 \frac{|V(W)|^2}{(1\,+\,|W|^2)^4}, \tag{5.7}$$

where V is any function of W. As is easy to check, (5.2) corresponds to $V = 1$.

Let us also adopt the convention that W_x denotes $\frac{\partial W}{\partial x}$ and that W_i, $i = 1,2$ stand for W_x and W_y respectively. And let W_+ and W_- denote $\frac{\partial W}{\partial x_+}$ and $\frac{\partial W}{\partial x_-}$ respectively. The equation of motion for the static solutions of our family of models is then given by

$$\begin{aligned} W_{+-} - \frac{2W^*W_+W_-}{(1\,+\,|W|^2)} &+ \frac{4\theta_1}{(1\,+\,|W|^2)^2}\left[2W^*_{+-}W_+W_- - W^*_{++}W_+^2 - W^*_{--}W_-^2 + \right.\\ &\left. + \; W_{++}W^*_+W_- \,+\, W_{--}W^*_-W_+ \,-\, W_{+-}(|W_+|^2 + |W_-|^2)\right] \\ &+ \frac{8\theta_1 W}{(1\,+\,|W|^2)^3}(|W_+|^2 - |W_-|^2)^2 \\ &+ \frac{\theta_2}{(1\,+\,|W|^2)^3}\left((1\,+\,|W|^2)\frac{\partial V}{\partial W}^* V - 4\,W\,|V|^2\right) = 0. \end{aligned} \tag{5.8}$$

We will shortly restrict our attention to some particular choices for V but first, let us describe some properties of the models which hold for any choice of V. Thus we observe that the energy density for the static solutions can be expressed in terms of W as follows

$$E = \frac{|W_i|^2}{(1+|W|^2)^2} + 2\theta_1 [\frac{(iW_x^* W_y - iW_y^* W_x)^2}{(1+|W|^2)^4} + 2\frac{\theta_2}{\theta_1}\frac{|V(W)|^2}{(1+|W|^2)^4}]. \tag{5.9}$$

Moreover, following Tchrakian[56] we can define

$$Q = \int \frac{i\,\epsilon_{ij}\, W_i^*\, W_j}{(1+|W|^2)^2}\, \tilde{U}\, dxdy, \tag{5.10}$$

which is a topological-like charge for most suitable real functions $\tilde{U}$ dependent on W and its complex conjugate (for more details see the appendix). In fact, (5.10) is equivalent to the topological charge of the unmodified CP^1 when $\tilde{U} = 1$. But as

$$\begin{aligned} &\int |\frac{W_i + i\,\epsilon_{ij}\, W_j}{(1+|W|^2)^2}|^2\, dxdy \geq 0, \\ \text{and} \quad &\int |\frac{i\,\epsilon_{ij}\, W_i^*\, W_j}{(1+|W|^2)^2} + U|^2\, dxdy, \geq 0 \end{aligned} \tag{5.11}$$

we see that (for U real)

$$\begin{aligned} E_0 &= \int \frac{|W_i|^2}{(1+|W|^2)^2}\, dxdy \geq \int \frac{i\epsilon_{ij} W_i^*\, W_j}{(1+|W|^2)^2}\, dxdy, \\ E_1 &= \int U^2 + \frac{|i\,\epsilon_{ij} W_i^*\, W_j|^2}{(1+|W|^2)^4}\, dxdy \geq \int \frac{2\, i\, U\, \epsilon_{ij} W_i^*\, W_j}{(1+|W|^2)^2}\, dxdy. \end{aligned} \tag{5.12}$$

Next we observe that the right hand sides of the above inequalities are two particular cases of (5.10) corresponding to $\tilde{U} = 1$ and $\tilde{U} = 2U$. Taking $U = 2\frac{\theta_2}{\theta_1}\frac{|V(W)|^2}{(1+|W|^2)^4}$, and as $E = E_0 + 2\theta_1 E_1$ we see that the topological nature of Q implies that all solutions which satisfy

$$\begin{aligned} W_i + i\,\epsilon_{ij} W_j &= 0, \\ U + \frac{i\,\epsilon_{ij} W_i^*\, W_j}{(1+|W|^2)^2} &= 0 \end{aligned} \tag{5.13}$$

are stable static solutions of the general model. In terms of the complex variables x_+ and x_-, these equations can be rewritten as

$$\begin{aligned} &W_+ = 0, \\ &\frac{\theta_2}{2\theta_1}|V|^2 = |W_+|^4 \end{aligned} \tag{5.14}$$

(as a matter of fact, every solution of the pair of equations (5.14) is also a solution of (5.8).) As the one skyrmion solution (5.4) is also a solution of (5.14) for $V = 1$ when (5.5) is satisfied, we have shown that this solution is stable.

It is interesting to determine the zero modes of our solutions. This is quite tedious for a general configuration, but can be done quite easily for (5.4).

To determine the fluctuations around the solution (5.4) of the model with L_V as given by (5.2) we look at the terms that are of the the second order in δW in the expansion of the total action S (ignoring the time derivatives), around the solution (5.4) *i.e.* $W_0 = \lambda(x_+ - a)$. For simplicity, we shall take $a = 0$ (exploiting the translational invariance). The expansion of S, to the second order in δW, is

$$\begin{aligned} S(W) =& S(W_0) + \frac{1}{2}\int dx_+ dx_- \left\{ 2\delta W^*(x_-) \frac{\delta^2 S}{\delta W^*(x_-)\delta W(x_+)}\bigg|_{W=W_0} \delta W(x_+) \right. \\ & + \delta W(x_-) \frac{\delta^2 S}{\delta W(x_-)\delta W(x_+)}\bigg|_{W=W_0} \delta W(x_+) \\ & \left. + \delta W^*(x_-) \frac{\delta^2 S}{\delta W^*(x_-)\delta W^*(x_+)}\bigg|_{W=W_0} \delta W^*(x_+) \right\} \\ & + O(\delta W^3) \equiv S(W_0) + S^{(2)} + O(\delta W^3) \end{aligned} \tag{5.15}$$

where the first order terms cancel because W_0 is a solution of the Euler-Lagrange equations for S. A short calculation shows that in this case,

$$S^{(2)} = 4\int dx_+ dx_- \left\{ \left| \frac{\partial_- \delta W}{1 + |W_0|^2} \right|^2 + \theta_1 \left| \frac{2\lambda^* \partial_+ \delta W}{(1 + |W_0|^2)^2} + \frac{2\lambda \partial_- \delta W^*}{(1 + |W_0|^2)^2} \right|^2 \right\} \geq 0\,, \tag{5.16}$$

where, as before, $\partial_- \equiv \frac{\partial}{\partial x_-}$, $\partial_+ \equiv \frac{\partial}{\partial x_+}$. Equation (5.16) implies that W_0 is a minimum of the action S.

Next we look at the the zero modes of the solution, *i.e.* the values of δW for which $S^{(2)} = 0$. In our case the zero modes are the solutions of the system of equations

$$\begin{aligned} &\partial_- \delta W = 0, \\ &\lambda^* \partial_+ \delta W + \lambda \partial_- \delta W^* = 0\,, \end{aligned} \tag{5.17}$$

the general solution of which is $\delta W = Ax_+ + B$, with $A^* = -\frac{\lambda^*}{\lambda} A$.

Clearly, all these modes correspond to either a translation ($x_+ \mapsto x_+ + b$) or a rotation ($x_+ \mapsto x_+ e^{i\phi}$). In fact, if $W_0 = \lambda x_+$ is a solution, then $e^{i\phi}\lambda x_+ + \lambda b$ is also a solution. Then, $\delta W = e^{i\phi}\lambda x_+ + \lambda b - \lambda x_+$ must be a zero mode. But δW can be written, to first order in ϕ and b, as

$$\delta W = i\phi\lambda x_+ + \lambda b = Ax_+ + B\ , \tag{5.18}$$

with $A = i\phi\lambda$, $B = \lambda b$. Note, however, that

$$A^* = -i\phi\lambda^* = \frac{\lambda^*}{\lambda} i\phi\lambda = -\frac{\lambda^*}{\lambda} A\ , \tag{5.19}$$

thus we see that the only zero modes relative to the static solution (5.4) are those of translations and rotations.

5.4 A Skyrme Model With Static Pairs

The choice of V was somewhat arbitrary so we should check what happens when we take other expressions for L_V which correspond to only small modifications of (5.2). Thus let us take, first,

$$V = |W + \lambda a^2|^2, \tag{5.20}$$

where λ and a are the new parameters of the model. Clearly, the appearance of a introduces a new scale into the model and, as is easy to check,

$$W = \lambda\left(x_+^2 - a^2\right), \tag{5.21}$$

satisfies (5.14) if

$$\theta_1 = \frac{\theta_2}{32\lambda^2}$$

and corresponds to a stable solution of (5.8). The parameter a of the model sets the scale of the relative distance between the two skyrmions in the solution (5.21). Is this new solution stable and how general is it? What are its properties?

Like (5.2) the solution of the previous model, (5.21) also satisfies (5.13) and so by the same argument as before it is stable. Again, its zero modes can be computed explicitly. To do this, like in section 3 we consider the fluctuations around the static solution W_0

given by (5.21). Then, the second order term $S^{(2)}$ is given by

$$S^{(2)} = \int dx_+ dx_- \Bigg\{ 8 \left| \frac{\partial_- \delta W}{1+|W_0|^2} \right|^2 + 64\theta_1 \left| \frac{\lambda^* x_- \partial_+ \delta W + \lambda x_+ \partial_- \delta W^* - \lambda^* \frac{x_-}{x_+} \delta W - \lambda \frac{x_+}{x_-} \delta W^*}{(1+|W_0|^2)^2} \right|^2 \Bigg\} \geq 0\,, \tag{5.22}$$

and so we see that the zero modes of (5.21) are the solutions of

$$\partial_- \delta W = 0, \qquad \lambda^* x_- \partial_+ \delta W + \lambda x_+ \partial_- \delta W^* - \lambda^* \frac{x_-}{x_+} \delta W - \lambda \frac{x_+}{x_-} \delta W^* = 0\,. \tag{5.23}$$

Clearly, the general solution of this system of equations is given by δW of the form: $\delta W = A x_+^2 + B x_+$, with the condition $A^* = -\frac{\lambda^*}{\lambda} A$. But, as in the previous case, the new action is invariant under translations and rotations. Thus,

$$\delta W = \lambda(e^{2i\phi}(x_+ + b)^2 - a^2) - \lambda(x_+^2 - a^2) \approx 2i\phi\lambda x_+^2 + 2b\lambda x_+ \tag{5.24}$$

with ϕ and b small, must be a zero mode. But this is precisely of the form $\delta W = A x_+^2 + B x_+$ with $A = 2i\phi\lambda$, $B = 2\lambda b$ and fulfills the condition $A^* = -\frac{\lambda^*}{\lambda} A$. Thus we see that there are no zero modes other than the global translations and rotations.

What are the properties of our new solutions? Some preliminary results on the scattering properties of the field configurations like (5.21) have already been obtained[57] In particular, they show that when the two skyrmions are displaced a little from their positions of equilibrium $\pm a$ they oscillate around these positions and if the displacement is quite large they may even scatter at 90° during these oscillations. If the simulations are performed with absorbing boundary conditions then the skyrmions, while oscillating, gradually settle at their positions of equilibrium.

What would happen if one started the simulations with an initial field configuration corresponding to one skyrmion *i.e.* given by (5.4)? This, as is easy to check, is not a static solution, so some evolution is to be expected. The surprising result is that the preliminary results showed very little evolution; *i.e.* the fields behave as if (5.4) were a static solution. The only resolution of this paradox is that there exists a solution which is very close to (5.4). However, it is difficult to find its analytical form. All our attempts at finding it have failed. Of course due to the existence of a topological charge we can easily write down field configurations which lie in the $Q = 1$ sector of the theory. Then choosing the configuration for which the action is minimal would provide us with a solution of the equation of motion. However, there is no guarantee that such a procedure will give a well behaved solution; namely that it will lead to a solution which is not singular and reasonably localised (as seen in our simulations). For a start let us choose the configuration

$$W = \alpha x_+ + \beta x_- + \gamma. \tag{5.25}$$

What is the topological charge? Note that if the fields of CP^n models are given by $Z = \vec{f}/|f|$, then the topological charge takes the form

$$Q = \int d^2x \frac{|\partial_+ f|^2 - |\partial_- f|^2}{[|f|^2]^2}. \tag{5.26}$$

But for the CP^1 model in the sterographic parametrisation, $\vec{f}$ takes the form $\vec{f} = (1, W)$, and so the topological charge is given by

$$Q = \int d^2x \frac{|\partial_+ W|^2 - |\partial_- W|^2}{[1 + |W|^2]^2}. \tag{5.27}$$

Inserting the configuration (5.25) into the expression of the topological charge gives us

$$Q = \int d^2x \frac{|\alpha|^2 - |\beta|^2}{[1 + |W|^2]^2}. \tag{5.28}$$

To perform the integration, first make the substitution $x \to x + \frac{\gamma}{\alpha+\beta}$. Then the integral becomes

$$Q = \frac{|\alpha|^2 - |\beta|^2}{\pi} \int_0^{2\pi} d\theta \int_0^{\infty} r dr \frac{1}{[1 + |(\alpha+\beta) r cos\theta + i(\alpha - \beta) r sin\theta|^2]^2}. \tag{5.29}$$

Next we integrate over the radial variable which yields

$$Q = \frac{|\alpha|^2 - |\beta|^2}{\pi} J, \tag{5.30}$$

where

$$J = \int_0^{2\pi} d\theta \frac{1}{|a cos\theta + b sin\theta|^2}. \tag{5.31}$$

To evaluate the integral J, we made the substitution $t = tan\theta$, calculate the resultant integral by the method of residues and find

$$J = \frac{4\pi}{||\alpha|^2 - |\beta|^2|}. \tag{5.32}$$

Therefore the topological charge of our configuration is

$$Q = sign(|\alpha|^2 - |\beta|^2) \tag{5.33}$$

where *sign* in the above equation is the sign function defined as

$$sign(x) = \frac{x}{|x|}. \tag{5.34}$$

Note that if the parameter α is larger than β then the configuration (5.25) is close to an instanton configuration, whereas if the converse is true then the same configuration is closer to an anti-instanton configuration. So may be this configuration will be the required solution for some choice of α and β. Unfortunately this is not the case.

Next we examined another configuration, namely

$$W = \frac{\alpha x_+ + \beta x_- + \gamma}{k x_+ + m x_- + l}. \tag{5.35}$$

The topological charge associated with this configuration is given by

$$Q = \frac{1}{\pi} \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \frac{N}{D^2} \tag{5.36}$$

with

$$\begin{aligned} N = {} & x([\alpha m - \beta k][(\alpha^* + \beta^*)l^* - \gamma^*(k^* + m^*)] + c.c) \\ & + iy([\alpha^* m^* - \beta^* k^*][(\alpha - \beta)l - \gamma(k - m)] - c.c) \\ & + (|\alpha l - \gamma k|^2 - |\beta l - \gamma m|^2), \end{aligned} \tag{5.37}$$

and

$$D = (1 + |W|^2). \tag{5.38}$$

In evaluating this integral we decided to take a different strategy, namely first integrating over x using the residue techniques and then integrating the remainder (y integration) by ordinary methods. In doing so we encountered the following two integrals, the first of which is

$$I_1 = \int_{-\infty}^{+\infty} dx \frac{\alpha' x + \beta'}{[ax^2 + bx + c]^2}. \tag{5.39}$$

Evaluating this integral by the residue method we found that I_1 is

$$I_1 = \frac{2\pi a}{|\triangle|^{3/2}}(-\frac{\alpha'}{a}b + 2\beta') \tag{5.40}$$

where $\triangle = b^2 - 4ac$ (discriminant). The second integral is

$$\begin{aligned} I_2 &= \int_{-\infty}^{+\infty} dy \frac{\alpha' y + \beta'}{[ay^2 + by + c]^{3/2}} \\ &= \frac{2(\beta' - \frac{\alpha' b}{2a})}{\sqrt{a}(c - \frac{b^2}{4a})} \end{aligned} \tag{5.41}$$

Puting everything together we found that the topological charge of the configuration (5.5) is given by equation (5.41) provided that

$$\begin{aligned} \alpha' &= (2i(\alpha^* m^* - \beta^* k^*)[(|k+m|^2 + |\alpha+\beta|^2)((\alpha-\beta)l - \gamma(k-m)) \\ &\quad - ((\alpha-\beta)l - (k+m)\gamma)((m^*k - k^*m) + (\alpha^*\beta - \beta^*\alpha))] - c.c), \\ \beta' &= (|k+m|^2 + |\alpha+\beta|^2)(|\alpha l - \gamma k|^2 - |\beta l - \gamma m|^2) \\ &\quad - 4(Re(\alpha^* m^* - \beta^* k^*)[(\alpha+\beta)l - (k+m)\gamma]Re(k+m)L^* + (\alpha+\beta)\gamma^*), \\ a &= 4[|k+m|^2 + |\alpha+\beta|^2 + ((mk^* - km^*) + (\alpha^*\beta - \alpha\beta^*))^2] \\ b &= 2i[(-(k+m)^2(k^* - m^*)l^* + c.c) + |k+m|^2((k-m)l^* - l(k^* - m^*)) \\ &\quad + (-(k+m)(\alpha+\beta)(l^*(\alpha^* - \beta^*) + \gamma^*(k^* - m^*)) + c.c) \\ &\quad + ((k+m)(\alpha^* + \beta^*)(l^*(\alpha-\beta) - \gamma(k^* - m^*)) - c.c) \\ &\quad + ((\alpha^* + \beta^*)^2\gamma(\alpha-\beta) + c.c) + |\alpha+\beta|^2(\gamma^*(\alpha-\beta) - c.c)] \\ &\quad - 4i(|k+m|^2 + |\alpha+\beta|^2)(l^*(k-m) + \gamma^*(\alpha-\beta) - c.c), \\ c &= 4(|k+m|^2 + |\alpha+\beta|^2)(|l|^2 + |\gamma|^2) + (((k+m)l^* + c.c) + ((\alpha+\beta)\gamma^* + c.c))^2. \end{aligned} \tag{5.42}$$

It is easy to check this expression by considering the two limits when our configuration is either an instanton or an anti-instanton. Indeed the limits are just as expected *i.e.* ± 1 respectively. Once again, it turns out that the configuration (5.38) does not solve the equations of motion, for any choice of its parameters.

All other methods of finding the analytical form of the one Skyrmion configuration has been unsuccessful, so the challenge is still there.

5.5 Models with k Skyrmions and General Comments

One can go even further and seek models which possess solutions with k static skyrmions. A superposition of k skyrmions will in general be described by a configuration corresponding to a polynomial of degree k in x_+ for W. Rather than looking at a general configuration, we will investigate the most symmetric case namely the one for which the k skyrmions form a regular polygon of order k. Our solution will thus be of the type

$$W = \lambda(x_+^k - a^k), \tag{5.43}$$

where a will be the distance from each skyrmion to the origin (the position of the skyrmion are given by the zeros of W). Notice that when $a = 0$ the k skyrmions are on top of each other. What is the potential for which (5.43) is a solution of the model?

In fact it is easy to check that if V is given by

$$V = (W + \lambda a^k)^{2(k-1)/k}, \tag{5.44}$$

corresponding to

$$L_V = -\frac{\theta_2}{4}[(\phi_1 + \lambda a^k(1 + \phi_3))^2 + \phi_2^2)(1 + \phi_3)^2]^{2(k-1)/k}, \tag{5.45}$$

then (5.43) is a solution of (5.14), if

$$\theta_1 = \frac{|\lambda|^{-4/k}\theta_2}{2k^4}. \tag{5.46}$$

Moreover this solution is stable as can be shown by looking at the fluctuation around it. It is important to note that L_V is given by a non-integer power of W except for k equal to 1 or 2.

We have presented a class of models which possess solitons-like static solutions. All these models are based on the S^2 σ-model. Their Lagrangian consist of the Lagrangian of the S^2 model supplemented by additional Skyrme-like and potential terms. The two additional terms, taken together, stabilise the solitons. As the Skyrme term is unique we see that we have some freedom in the choice of the potential term. The simplest choice of the potential term gives a model which possesses a one-skyrmion static solution; the other choices have static solutions with other skyrmion numbers. Thus in these other models there are rather complicated forces in multi-skyrmion channels; some attractive and some repulsive and such that for some special configurations of skyrmions all forces cancel allowing us to have multi-skyrmion configurations as their static solutions.

We have looked at some properties of these models; in particular we have analysed the stabilities of their multi-skyrmion solutions. We have found them all to be stable; hence these solutions can be the starting points of the investigations of their scattering properties. A numerical investigation of some of these properties is currently being carried out[57] .

VI. SOLITON SCATTERING IN CP^2 MODEL

6.1 Introduction

In this chapter we will consider the solitonic properties of the static solutions of CP^2 models in $(2+1)$ dimensions. Along the way we will investigate the impact of adding extra terms to the basic CP^2 Lagrangian, particularly two sorts of terms, the first of which is a generalisation of the Hopf term to the CP^2 space, and the second corresponds to generalisations of the Skyrme term.

In chapters one and five we have argued that both quasi-integrable and integrable models share the same property of having extended structures (solitons) in (2+1) dimensions. However, for quasi-integrable models, unlike integrable ones, their solitons are not very much constrained in their scatterings by the existence of an infinite number of conservation laws. Thus to have a further understanding of the scattering properties of solitons in quasi-integrable models Zakrzewski and others[18] have investigated this aspect in the CP^1 model.

So what are the scattering properties of the extended structures as they come close together in, say, head-on collisions? Simulations have shown that in all cases the scattering proceeds through the same intermediate stages: first the extended structures come close to each other, then they form a ring and finally they emerge out of the ring at 90° to the original direction of motion.

The formation of the intermediate stage in the form of a ring is one of the properties of the CP^1 model, and prevents us from following too closely the trajectories of the solitons; clearly when the solitons are close together they overlap and due to their indistinguishability their trajectories lose their meaning. Comparing the expressions for the positions of the solitons when they are far apart, and considering their speed, it was found that, in analogy with a similar result in (1+1) dimensions, the solitons are shifted along their trajectories. However, due to the ring structure of the intermediate state the qualitative assessment of the shift is somewhat difficult to perform. What happens when the two solitons are on top of each other, say, at the origin? As is easy to see, the energy density of such configuration is in the shape of a ring centered at the origin. Thus it would seem natural to assume that the two solitons come on top of each other before they scatter at 90°.

To study the importance of the ring formation and/or the shrinking of solitons one ought to go beyond the simplest S^2 model. The effects of shrinking were taken care of by adding the "Skyrme-like" and potential terms; here we concentrate on the ring formation and the phase shift along the trajectory.

To go beyond the formation of a ring, when the solitons are on top of each other, one has to consider a model with a larger target manifold space; the simplest such model is the CP^2 model.

6.2 CP^2 Model

As we showed in chapter three and four, the CP^2 model is based on the Lagrangian

$$L_0 = (D^\mu z)^\dagger D_\mu z, \tag{6.1}$$

where the basic field vector z has three components $z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$, which are constrained to satisfy $z_i^\dagger z_i = 1$, and where the covariant derivative $D_\mu Z$ is defined as in chapter three.

The Lagrangian (6.1) is invariant under $U(1)$ local gauge transformations $z \rightarrow z\, e^{i\phi}$ which allows us to set

$$z = \frac{(1, W_1, W_2)}{\sqrt{1 + |W_1|^2 + |W_2|^2}} \tag{6.2}$$

and so consider the complex W_1 and W_2 fields as the independent fields of the theory. (the CP^1 case corresponds to setting, say, $W_2 = 0$). If we rewrite our Lagrangian in terms of W_1 and W_2 fields as defined above it takes the form

$$L = \frac{\partial_\mu W_1 \partial^\mu W_1^\star + \partial_\mu W_2 \partial^\mu W_2^\star + (W_1 \partial_\mu W_2 - W_2 \partial_\mu W_1)(W_1 \partial^\mu W_2 - W_2 \partial^\mu W_1)^\star}{(1 + |W_1|^2 + |W_2|^2)^2}. \tag{6.3}$$

But from our past experience with the CP^1 model it is more convenient, from the point view of simulations and the accumulated numerical errors, to work with a different parametrisation, in which the fields at spatial infinity go to finite values. In the case of the CP^1 model, this corresponds to using $O(3)$ variables ϕ_i $i = 1, 2, 3$. Inspired by the

CP^1 model, let us try the following parametrisation and check whether it leads to the equations of motion which are free of any singularities:

$$Z = e^{i\chi} \begin{pmatrix} \phi_1 \\ i(\phi_2 + \phi_3) \\ i(\phi_4 + \phi_5) \end{pmatrix}, \tag{6.4}$$

where the fields $\phi_i' s$ parametrise locally the sphere S^4 *i.e.*

$$\vec{\phi} \cdot \vec{\phi} = 1. \tag{6.5}$$

In chapter 3 we showed, at some length, that choosing a particular parametrisation is equivalent to choosing a metric for the background space in which we embed the sphere S^4 locally. But with respect to this parametrisation the CP^2 Lagrangian density takes the new form

$$L = \partial_\mu \vec{\phi} \cdot \partial^\mu \vec{\phi} - (\phi_2 \phi_{3\mu} - \phi_3 \phi_{2\mu} + \phi_4 \phi_{5\mu} - \phi_5 \phi_{4\mu})^2, \tag{6.6}$$

which shows that the metric of the background space is not Euclidean. Therefore one may wonder whether we can repeat our construction in the CP^1 case, that is finding a nonlinear transformation to linearise the metric on the background space. In the new metric the Lagrangian would take the simple form; something like

$$L = \partial_\mu \vec{\Psi} \cdot \partial^\mu \vec{\Psi}, \qquad \vec{\Psi} \cdot \vec{\Psi} = 1, \tag{6.7}$$

with some "small" corrections. Unfortunately all our attempts to find such transformation have failed. However, (6.6) is not convenient to use as the equations of motion obtained from it are not free of singularities; in fact they have a singularity at the origin. So we have to use the W_i formulation.

The equations of motion of the W_i fields are given by

$$\begin{aligned} \partial_t^2 W_1 \; &= \; \frac{2W_1^\star \left((\partial_t W_1)^2 - (\partial_x W_1)^2 - (\partial_y W_1)^2 \right)}{1 + |W_1|^2 + |W_2|^2} \\ +2W_2^\star &\left(\frac{(\partial_t W_1)(\partial_t W_2) - (\partial_x W_1)(\partial_x W_2) - (\partial_y W_1)(\partial_y W_2)}{1 + |W_1|^2 + |W_2|^2} \right) + \partial_x^2 W_1 + \partial_y^2 W_1 , \end{aligned} \tag{6.8}$$

and a similar equation for W_2, obtained from (6.8) by the interchange $(1 \leftrightarrow 2)$. If we set $W_2 = 0$ in (6.3) and in (6.8) we reduce the problem to the $CP^1 = S^2$ case studied before;

however, now we are interested in the CP^2 case and so we consider W_1 and W_2 as two independent fields. What are the static solutions of the equations of motion for W_1 and W_2?

First of all we observe that any finite action solution of the CP^2 model in two Euclidean dimensions is a static solution of (6.8). Moreover, the topological charge of their instanton solutions corresponds to the number of solitons in our case. Thus a static one soliton configuration can be chosen to be given by

$$W_1 = \mu z - b_1 \qquad W_2 = \mu z - b_2. \tag{6.9}$$

Clearly (6.9) solves (6.8). Moreover, although (6.9) is not the most general field configuration describing one soliton, it is sufficient for our purposes. It describes a soliton which is located at $\frac{b_1+b_2}{\mu}$ and whose size is proportional to $|b_1 - b_2|^2 + 2$.

To see this substitute (6.9) in the static part of L_o, giving

$$L_o = \frac{2|\mu|^2(2 + |b_1 - b_2|^2)}{[1 + 2|\mu|^2|z - \frac{b_1+b_2}{2\mu}|^2 + \frac{|b_1-b_2|^2}{2}]^2}. \tag{6.10}$$

By generalising (6.9), it is easy to see that $W_1 = \lambda z^2$, $W_2 = \mu z$, where $z = x + iy$, is also a static solution of the equations of motion and describes two solitons on top of each other (and located at $z = 0$). For a general choice of the parameters λ and μ the energy density of the configuration has a ring-like structure (like in the CP^1 case); to see this one has to work out the expression of energy density for this configuration which, as it turns out, has the form

$$E = 2\frac{|\mu|^2 + 4|\lambda|^2r^2 + |\lambda|^2|\mu|^2r^4}{(1 + |\mu|^2r^2 + |\lambda|^2r^4)^2}. \tag{6.11}$$

It is very clear that this expression is radially symmetric and it has a maximum which is not at the origin; thus this expression describes a ring shape. However, when the parameters μ and λ satisfy $\mu^2 = 2\lambda$ the energy density takes the shape of a single peak (*i.e.* the ring becomes a peak) due to the fact the energy density in this case has the new form

$$E = 2\frac{|\mu|}{(|\mu|r^2 + 1)^2}, \tag{6.12}$$

an expression which is radially symmetric and has a maximum at the origin, hence it describes a single peak. We can displace the solitons initially by choosing $W_1 = \lambda(z^2 - a)$

for some reasonable value of a, and then taking W_2 as above with $\mu^2 = 2\lambda$, set the two solitons moving towards each other by taking as initial conditions $\frac{dW_1}{dt} = aV$, $\frac{dW_2}{dt} = 0$. With such an initial value problem the solitons are set to expand as well, so when they, emerge out of the interaction region they do not shrink too fast.

To perform any numerical simulation we have to choose the appropriate boundary conditions on our fields. As is clear from the expression of the $W_i^{'s}$ given above, the $W_i^{'s}$ are the largest at the boundary of our grid and there they vary a lot. Hence we cannot impose any "fixed" boundary conditions; instead, following the ideas developed in[18] , we have chosen to update the fields on the boundary using a linear or quadratic extrapolation from the values of the fields inside the lattice grid. Such an extrapolation produces an exact result for the initial values for the fields corresponding to one or two solitons and it allows for their change of size. The extrapolation is not good when waves of the radiation generated in the scattering reach the boundary. At this stage, gradually, our boundary conditions introduce small distortions which, after a while, lead to numerical instabilities and the results of the simulations cannot be trusted. However, all of these instabilities manifest themselves as extra peaks in the energy density (at the boundaries), and the total energy is no longer conserved. Luckily, they take some time to arise, due to their localisation, and they do not perturb significantly the solitons or their motion. They do prevent us, however, from carrying our simulations with the W_i formalism for more than only limited periods of time. Luckily, these periods are long enough to see what is going on.

We have performed many simulations corresponding to different values of the initial velocity V. All our simulations were performed at various workstations in Durham and at Los Alamos working in double precision and using a fourth order Rung-Kutta method of simulating the time evolution. Almost all of the simulations were performed on a fixed 201×201 lattice, with lattice spacing $\delta x = \delta y = 0.03$, the time step being 0.01. Some simulations were rechecked on larger grids (301×301 or even 401×401) or on different machines including the Los Alamos connection machine in which case the grid involved 512×512 lattice points.

All our simulations showed a 90° scattering. Moreover, they also showed a shift along the trajectory as seen initially in the CP^1 case. In fig.1 we display typical trajectories of our solitons and in fig.2a, 2b and 2c we show the time dependence of the distance between the solitons for simulations started with three different values of V. We clearly see a shift

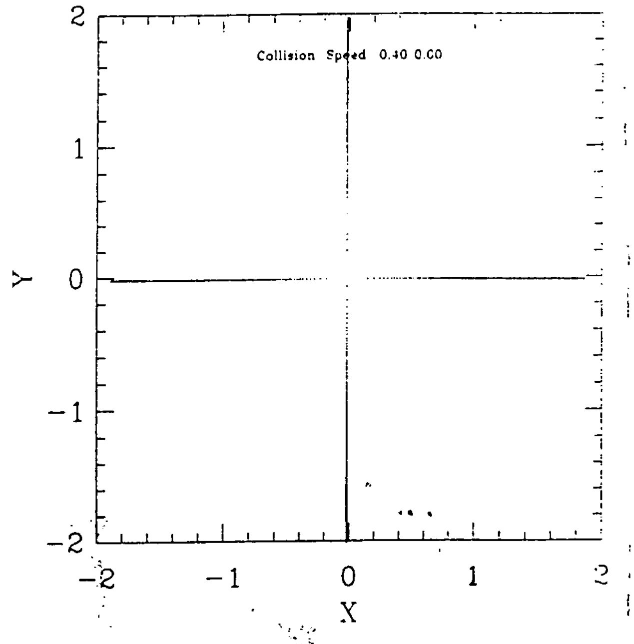

*Fig.*1 Trajectories of the two solitons with $\kappa = 0$.

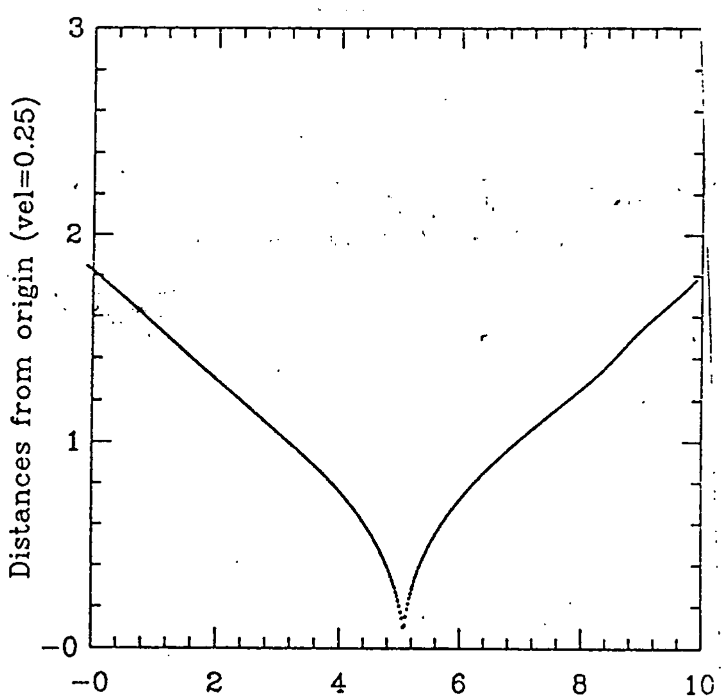

*Fig.*2*a* The time dependence of the distance between the two solitons for $V = 0.25$.

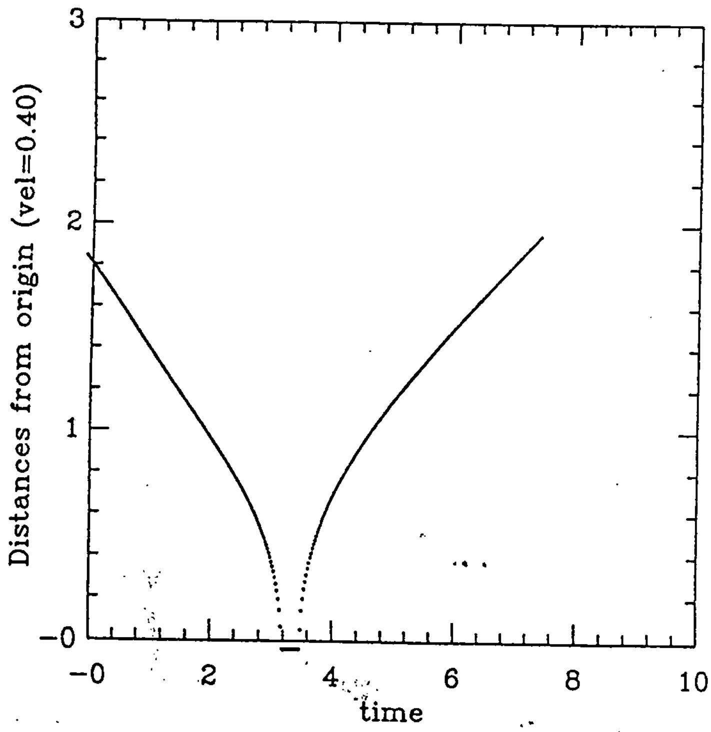

Fig.2b The time dependence of the distancens between the two solitons for simulations started with $V = 0.40$.

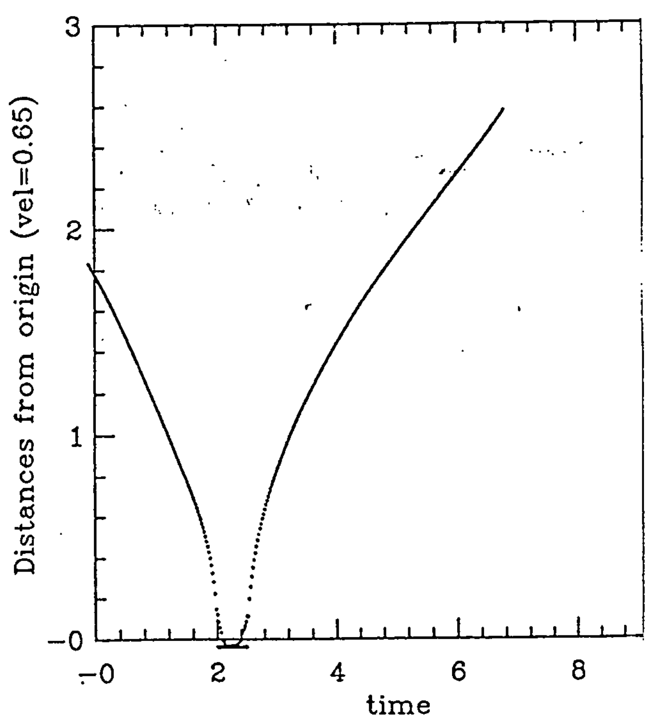

Fig.2c The time dependence of the distance between the two solitons for simulations started with $V = 0.65$.

along the trajectory which is similar to the one observed in the CP^1 case except that this time the interpretation is easier (our picture suggests that as the solitons are close together they speed up and then come on top of each other where they spend some time after which they separate and gradually, as they leave the interaction region, they regain their initial speed). Clearly this is only a qualitative picture of their interaction; when they are close together they loose their identity, and like in the (1+1) dimensional case, it makes little sense of talking about their trajectories. Moreover, as is easy to check, the shift along the trajectories does not depend on V (and in the case of the simulations shown in fig.2 its value is $\delta = 1$, if we assume that the solitons go through the origin).

In addition we would like to add that the Durham soliton group[58] has also looked at some field configurations corresponding to one soliton and one anti-soliton. As the forces between them are basically attractive, placed some distance apart, solitons and anti-solitons move towards each other and then annihilate into pure radiation. The angular dependence of the outgoing radiation is not uniform; most of it is, again, sent out at 90° to the direction of their final approach (just before the annihilation).

Hence, we see that apart from the fact that the energy density of two solitons on top of each other can have any shape between a single peak and a ring, the scattering properties of the solitons in the CP^2 model are very similar to those of the CP^1 model. However, the CP^2 model, involving two complex fields, has more degrees of freedom and so it allows the addition of extra terms to the Lagrangian density. these terms in the CP^1 case either vanish identically or are given by total derivatives. An example of such a term is the generalisation of the Hopf term. This term, in the CP^1 case is purely topological and, as such, is locally a total divergence (and so does not contribute to the equations of motion); in the CP^2 case it ceases to be topological and so can affect the dynamics. We will study the effects of this term in the next section. On top of that "Skyrme-like" terms in CP^2 spaces are considered and a brief study of their effects is given.

6.3 Generalised Hopf term and its effects

To introduce the generalised Hopf term we have to go back a little and look at the Lagrangian (6.1). How unique is this expression? Are there any terms we could add to (6.1)? Clearly, there are many but if we restrict ourselves to terms involving not more than three derivatives we are only left with

$$L_{Hopf} = (D_\mu z)^\dagger (D_\nu z)(z^\dagger \partial_\alpha z)\epsilon^{\mu\nu\alpha}, \qquad (6.13)$$

or the terms derived from it, in a particular representation like (6.2), with the additional factors $1+|W_1|^2+|W_2|^2$ in the denominator. It is easy to check that L_{Hopf} is locally a total divergence if the z field has only two components and so describes a CP^1 field. In this case, the explicit substitution of (6.2) with $W_2 = 0$ into L_{Hopf} gives a vanishing contribution. However, for CP^2 field L_{Hopf} does not vanish. So what are the equations of motion of the model with the Lagrangian given by $L = L_0 + \kappa L_{Hopf}$? A little calculation shows that this equation is given by

$$(1 - zz^\dagger)D_\mu D^\mu z - 2\kappa\epsilon^{\mu\nu\alpha}D_\alpha z((D_\mu z)^\dagger D_\nu z) = 0, \tag{6.14}$$

where, as before $z^\dagger z = 1$. Inserting the specific form of z given by (6.2) gives us the two equations for W_1 and W_2; they are given by

$$\begin{aligned}
\frac{\partial^2 W_1}{\partial t^2} &= \frac{\partial^2 W_1}{\partial x^2} + \frac{\partial^2 W_1}{\partial y^2} + \frac{2}{[1+|W_1|^2+|W_2|^2]} \\
&\{W_1^\star((\frac{\partial W_1}{\partial t})^2 - (\frac{\partial W_1}{\partial x})^2 - (\frac{\partial W_1}{\partial y}^2)) + W_2^\star(\frac{\partial W_1}{\partial t}\frac{\partial W_2}{\partial t} - \frac{\partial W_1}{\partial x}\frac{\partial W_2}{\partial x} - \frac{\partial W_1}{\partial y}\frac{\partial W_2}{\partial y})\} \\
&-\kappa[\frac{1}{[1+|W_1|^2+|W_2|^2]}]^2\{((1+|W_1|^2)\frac{\partial W_2^\star}{\partial x} - W_1W_2^\star\frac{\partial W_1^\star}{\partial x})(\frac{\partial W_1}{\partial y}\frac{\partial W_2}{\partial t} - \frac{\partial W_2}{\partial y}\frac{\partial W_1}{\partial t}) \\
&+ ((1+|W_1|^2)\frac{\partial W_2^\star}{\partial y} - W_1W_2^\star\frac{\partial W_1^\star}{\partial y})(\frac{\partial W_1}{\partial t}\frac{\partial W_2}{\partial x} - \frac{\partial W_2}{\partial t}\frac{\partial W_1}{\partial x}) \\
&+ ((1+|W_1|^2)\frac{\partial W_2^\star}{\partial t} - W_1W_2^\star\frac{\partial W_1^\star}{\partial t})(\frac{\partial W_1}{\partial x}\frac{\partial W_2}{\partial y} - \frac{\partial W_2}{\partial x}\frac{\partial W_1}{\partial y})\}
\end{aligned} \tag{6.15}$$

and a similar equation for W_2.

These expressions are rather complicated but their form is completely straightforward. What is the role of the new term and its contribution to the equations of motion?

Looking at the effects of the additional term we observe that due to the $\epsilon^{\mu\nu\rho}$ symbol and the three derivatives in (6.14) these effects vanish for static fields. Hence the additional term in (6.14) or (6.15) resemble a little the familiar Lorentz force of classical dynamics.

To test this we have decided to analyse the generalised Hopf term further and so we have looked at the energy momentum tensor $T_{\mu\nu}$. Then a tedious but straightforward calculation shows that the energy momentum tensor does not receive any contribution from L_{Hopf}. It is a well known fact that the energy momentum tensor is the result of varying the Lagrangian with respect to the metric but as in our case the Hopf term is independent of the metric by construction then one should expect the contribution of the

Hopf term to be zero. Thus the additional term plays the role of the "internal magnetic field" and the additional contributions to the equations of motion resemble a little the "internal Lorentz force".

We have analysed the role of the additional terms by performing some numerical simulations carried out with various values of κ. First of all we have looked at the effects of the new term on the behaviour of a single soliton *i.e.* on the behaviour of the field configuration described by (6.9) of the previous section. In the previous section we considered the parameters μ and b_i constant, corresponding thus to a static field configuration, and so the energy density (2.10) corresponds to the potential energy density. If we assume a specific time dependence of μ and b_i we can determine the total energy density of the initial one soliton configuration. For all reasonable assumptions as to this time dependence the shapes of the total and potential energy densities are very similar (and the kinetic energy is quite small).

We performed several simulations starting with different initial assumptions for b_1, b_2 and μ. We have found that the new term only alters the rate of shrinking or expanding of the soliton (*i.e.* $b_1 - b_2$). And it always acts in the direction of reducing the effect; thus it reduces the shrinking if the soliton is shrinking or reduces the expansion if the soliton is expanding. In fig.3 we show the plots of the maximum of the energy density of the three simulations in which the initial soliton is shrinking. (As the total energy is conserved, the height of the energy peak is inversely proportional to the square of the size of the soliton). In fig.4 we show similar plots for the expanding solitons.

Incidentally, it is easy to check that when we restrict ourselves to the one soliton configurations mentioned above and consider only time dependent b_1 and b_2 then the contribution of the new term to the evolution of b_1 and b_2 treated as collective coordinate vanishes. Hence our results are also an indirect test of the validity of the collective coordinates description of the evolution. We see that although this approximation is quite good for small values of κ its validity decreases as κ increases. Clearly the new term affects the field configuration in a rather complicated way (and different parts of it differently) and as we will argue below some of its effects can be approximated by a rotation.

To get a better understanding we can go beyond the collective coordinate approximation and substitute the field configurations (6.9) into the full equations (6.15). However, it is easy to check that the fields (6.9) do not solve (6.15) for any choice of $b_i(t)$.

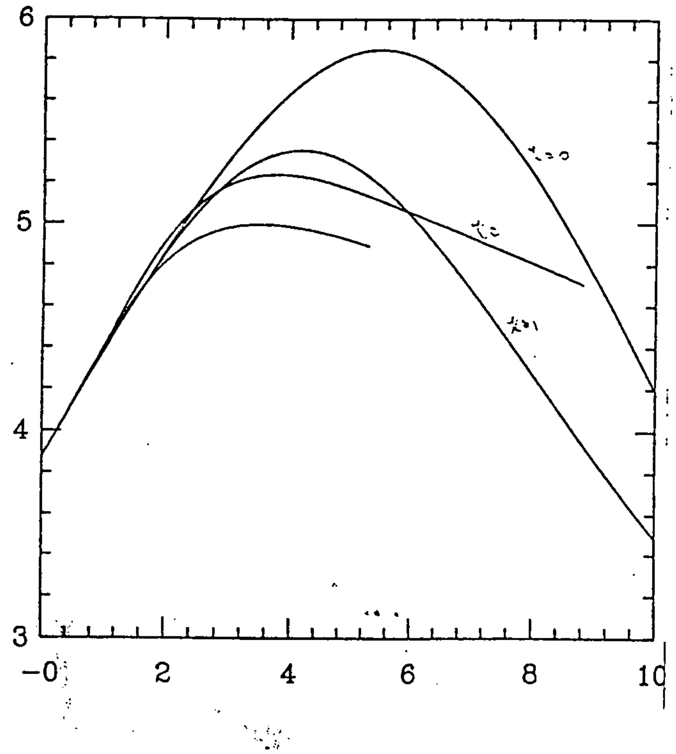

Fig.3 Plots of the maximum energy density of the three simulations in which the initial soiton is shrinking.

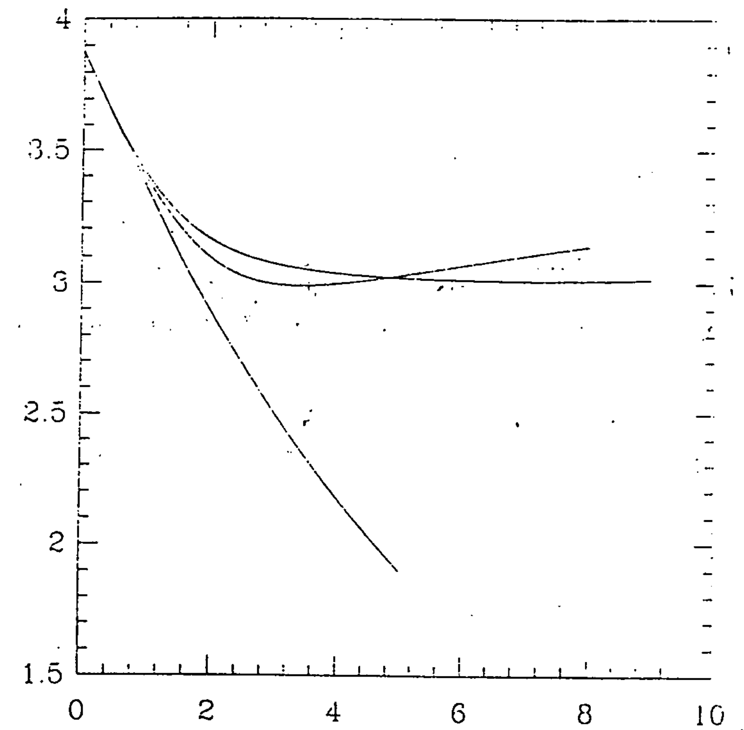

Fig.4 Plots of the maximum energy density of the three simulations in which the initial soliton is expanding.

Thus, clearly, even if initially $W_i(x,y,t)$ are given by (6.9) their evolution takes them away from this form. However, disregarding this fact, we find that the substitution of (6.9) to (6.15) gives us

$$-\ddot{R} = \frac{(\dot{R}^2 - \dot{S}^2)}{2}\frac{(2\mu^* x_- - R^*)}{[1+|W_1|^2+|W_2|^2]} - \frac{(\dot{R}\dot{S}S^* + \frac{\dot{S}R^{*2}}{2} + \frac{\dot{R}^2 R^*}{2})}{[1+|W_1|^2+|W_2|^2]} + \frac{2i\kappa|\mu|^2\dot{S}S^*(2\mu x_+ - R)}{[1+|W_1|^2+|W_2|^2]^2} \tag{6.16}$$

and

$$\ddot{S} = \frac{(\dot{R}^2 - \dot{S}^2)S^*}{2[1+|W_1|^2+|W_2|^2]} + \frac{2\mu^* x_- \dot{S}\dot{R}}{[1+|W_1|^2+|W_2|^2]} - \frac{(\dot{S}\dot{R}R^* + \frac{\dot{S}^2 S^*}{2} + \frac{\dot{R}S^*}{2})}{[1+|W_1|^2+|W_2|^2]} - \frac{2i\mu^*\kappa\dot{S}(2+|S|^2)}{[1+|W_1|^2+|W_2|^2]^2}, \tag{6.17}$$

where we have introduced the soliton position $R = b_1 + b_2$ and the soliton size $S = b_1 - b_2$.

As we have said before these equations cannot be satisfied at all values of x and y. However, we expect the field configurations around the maximum of the energy density to be the most important. Hence we propose a new approximation based on replacing the x and y dependence in (6.16) and (6.17) by their values at the point of the maximum energy density. It is difficult to assess the validity of this new approximation; the obtained results are its best test.

In our studies of the effects of the modified Hopf term on the shrinking or expansion of a single soliton we used as our initial conditions $R(0) = 0$, $\frac{dR}{dt}(0) = 0, S(0) = A$, $\frac{dS}{dt}(0) = B$, where A and B were taken to be real and the sign of $\frac{B}{A}$ determines whether the soliton was initially expanding or shrinking. Keeping these initial values and observing that the maximum of the energy density corresponds to $x = y = 0$ we find that (6.16) tells us that R remains zero at all times in agreement with the results of the full simulations. The equation for S (6.17) reduces to

$$\ddot{S} = \frac{\dot{S}^2 S^* + 4i\kappa\mu^*\dot{S}}{[1+\frac{|S|^2}{2}]}. \tag{6.18}$$

This equation is clearly nonlinear but it is easy to see that if $\kappa = 0$ (i.e. there is no modified Hopf term), S remains real at all times (as A and B are real). However, the appearance of i in front of κ in (6.18) gradually introduces a phase to S. Thus, $\dot{S}$ becomes complex and its time dependent phase corresponds to a rotation of the soliton. Of course this rotation will not be uniform (different parts of the soliton can rotate unequally) and only the full simulations can reveal what is going on-our approximations will only measure the gross effects of this rotation.

In addition the rotation of the soliton will affect its size; to see this we have to solve (6.18) numerically or introduce a further approximation. Thus, in particular, we can put

$$S(t) = f(t)e^{i\phi(t)} \tag{6.19}$$

and then perform the Taylor series expansion around the initial value $t = 0$. We find

$$f(t) = A + Bt + \frac{B^2 A}{2+A^2}t^2 + \frac{B^2(2+A^2) - 32\kappa^2\mu^{*2}B}{3(2+A^2)^2}t^3 + O(t^4) \tag{6.20}$$

and

$$\phi(t) = \frac{4\kappa\mu^* B}{A(2+A^2)}t^2 + O(t^3) \tag{6.21}$$

We note that as the contribution κ to $f(t)$ is negative the nonvanishing κ tends to reduce the shrinking or expanding of the soliton in agreement with what we have observed in our full simulations.

To go beyond the Taylor series expansion we must resort to some numerical work. In fig.5 we present the curves of $f(t)$ obtained for three values of κ and in fig.6 the corresponding curves for $\phi(t)$ for four values of κ (including $\kappa = 0$ when $\phi = 0$). We see that an effect of the nonvanishing value of κ is to slow down the original expansion or shrinking of the soliton, reverse it and, for large enough κ to replace it by a periodic variation of the soliton size $f = |S|$.

To compare with the results of our simulations we have to translate the information about $f(t)$ into the behaviour of the maximum of the energy density (2.7). Hence we do not need $f(t)$ but instead an expression proportional to $G(t) = \frac{1}{2+f^2}$. In fig.7 we present the plots of $G(t)$ corresponding to the plots of $f(t)$. To compare with our numerical results we have to insert a further overall factor. In fig.8a we present the time dependence of the maximum of the energy density found in our full simulation for $\kappa = 0$ and $\kappa = 1$ and in fig.8b the corresponding plots based on our approximation. We notice a very good agreement despite the crudeness of our approximation. Hence the additional term acts a little like an internal rotation and slows down all the shrinking or expanding of the solitons.

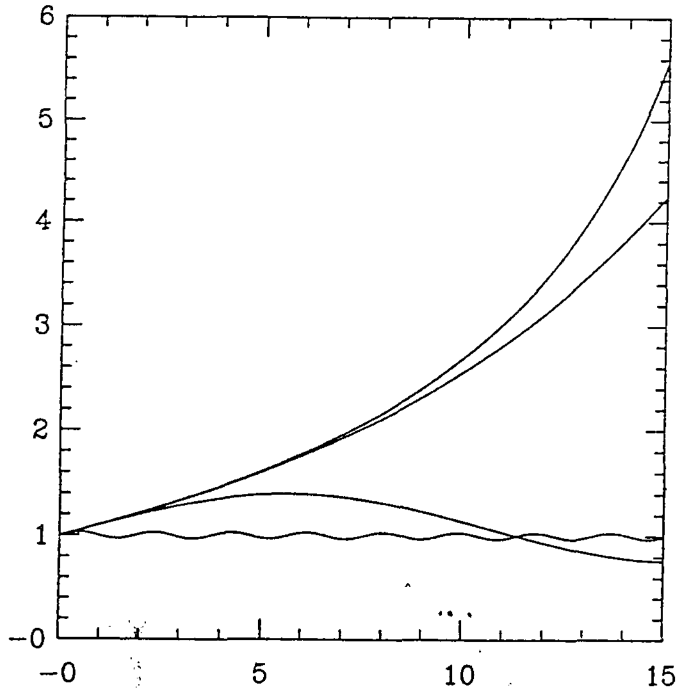

Fig.5 Curves of $f(t)$ obtained for three values of κ.

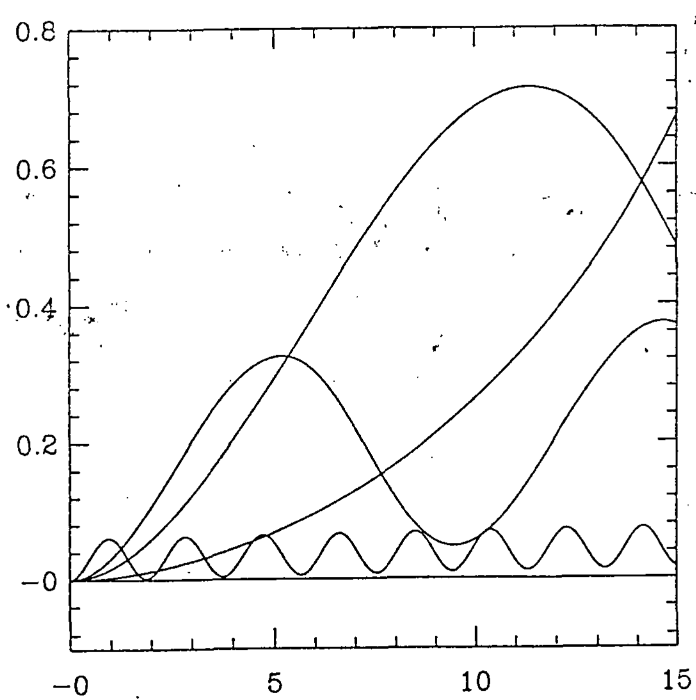

Fig.6 Curves of $\phi(t)$ obtained for three values of κ.

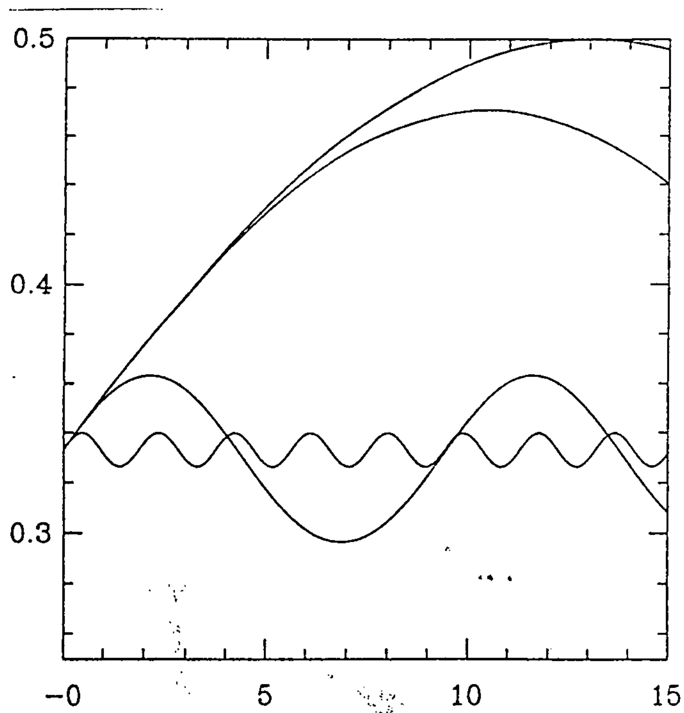

*Fig.*7 Plots of $G(t)$ corresponding to the plots of $f(t)$.

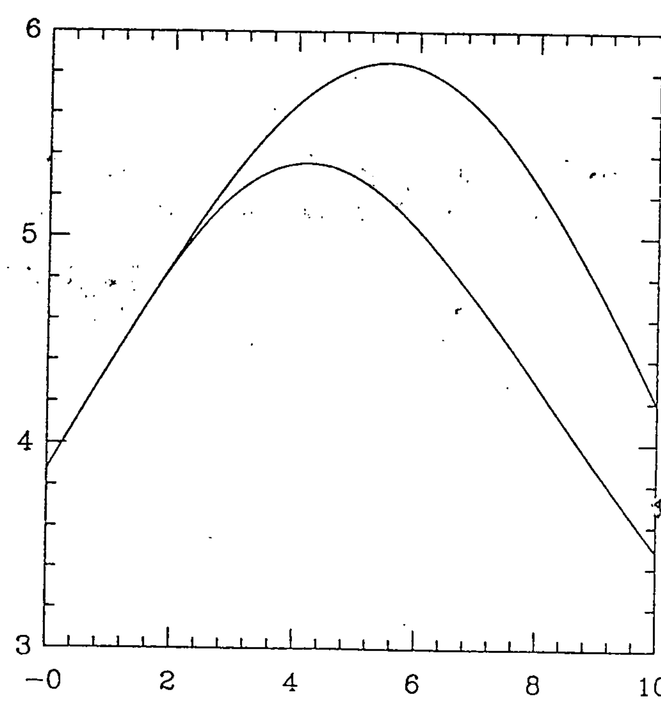

*Fig.*8*a* The time dependence of the maximum of the energy density found in the simulations for $\kappa = 0, 1$.

Given this rotation-like action of the additional term what would be the effects of the new term on the scattering properties of solitons? To answer this question we have performed many simulations for the initial configuration of two solitons for different values of κ. In all simulations the solitons were given an initial velocity towards each other (so that for $\kappa = 0$ their scattering would have resulted in a head-on collision, which as we mentioned in the previous section leads to the 90° scattering). The results were very much as expected. The trajectories of the solitons were deflected, with the degree of the deflection increasing with κ. In fig.9 we present some trajectories corresponding to different values of κ. We see that the effects of the extra term resemble a little the effects of a rotation. If we look in detail at the pictures of the energy densities at various times during the scattering we find various irregularities, thus the rotation is non-uniform (see fig.10). In consequence, sooner or later the variation of the fields gets so large that the simulations develop numerical errors and the results of the simulations cannot be trusted. However, this takes a while so that the gross features of our results are still reliable. To go further we would need another formulation of the model (which would avoid using W_1 and W_2 fields, replacing them by constrained fields, which cannot get too large). In the CP^1 case such a formulation is given by the real S^2 variables. The price paid by this choice is the reduction of the speed of the simulations and larger memory requirements, the gain is a better control of the numerical errors. Our simulations in the CP^1 case have shown that although the simulations involving the complex W field have larger errors these errors are relatively insignificant when the fields do not vary too much and until some divergencies arise at the boundaries; unfortunately we do not have a similar convenient formulation in the CP^2 case (apart from, of course, the formulation in terms of the z_i fields mentioned earlier) which is not too demanding on the computer memory.

6.4 Skyrme model on CP^2 spaces

In chapter 4 we showed that in the CP^2 spaces one has the liberty to define two inequivalent Skyrme terms given by (4.32) and (4.33). In this section we will consider answers to the following two questions: What is the nature of the difference between the two CP^2 Skyrme terms? And, what is the influence of either of these two terms on the fixing of the size of the extended structures (skyrmions).

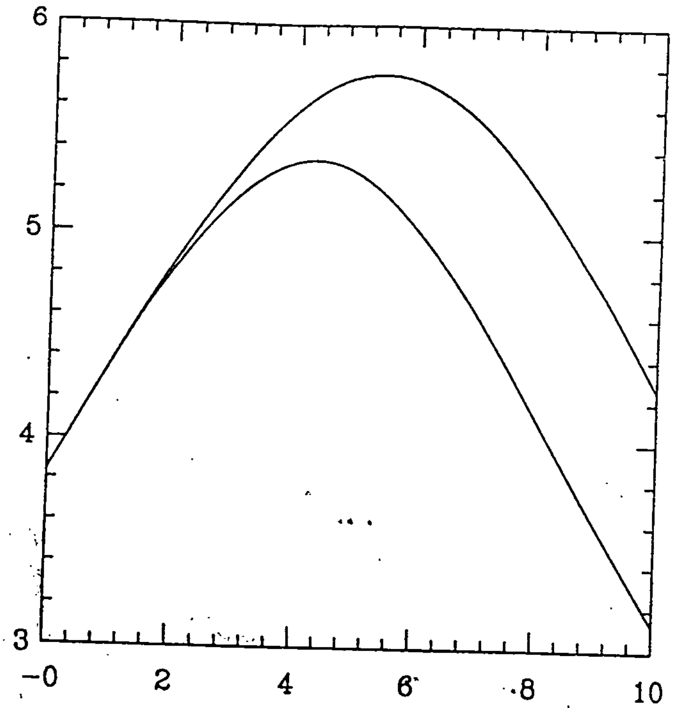

Fig.8b The time dependence of the maximum of the energy density based on the new approximation.

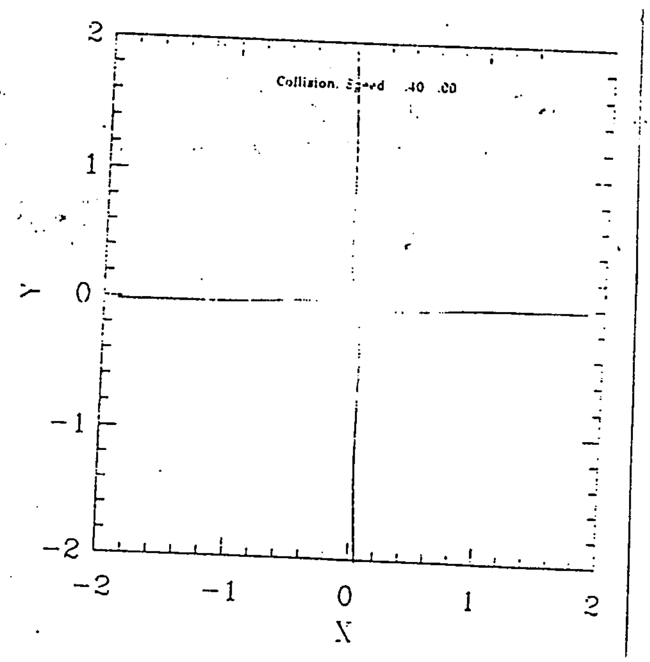

Fig.9a The trajectories corresponding to $\kappa = 0.025$.

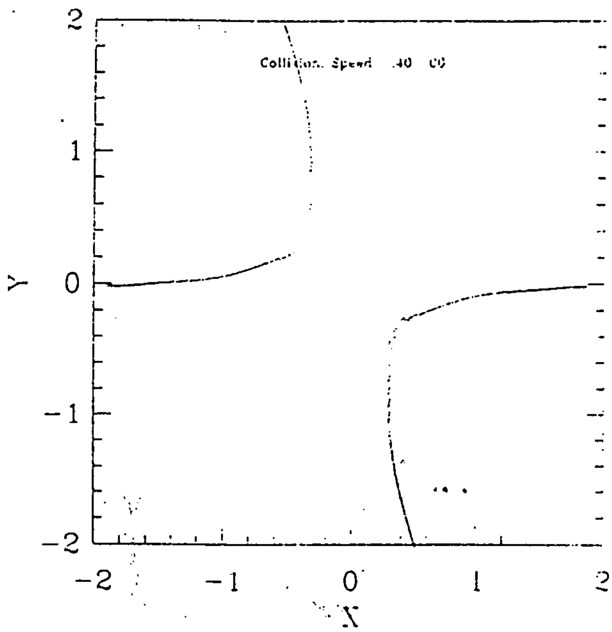

Fig.9b The trajectories corresponding to $\kappa = 0.2$.

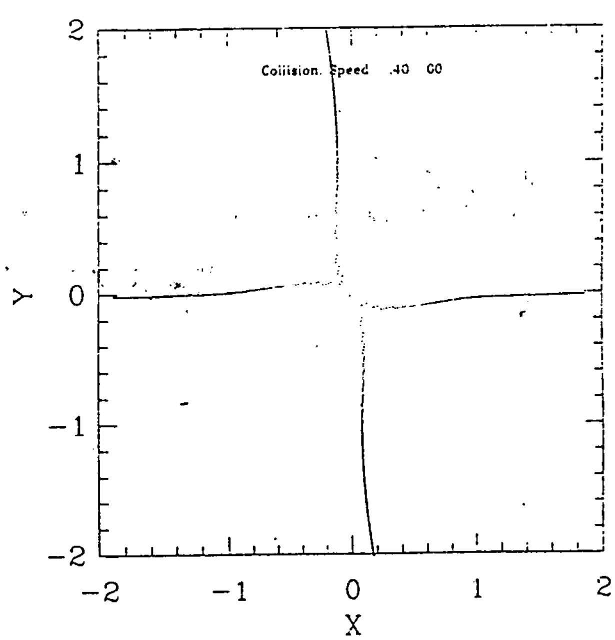

Fig.9c The trajectories corresponding to $\kappa = 0.35$.

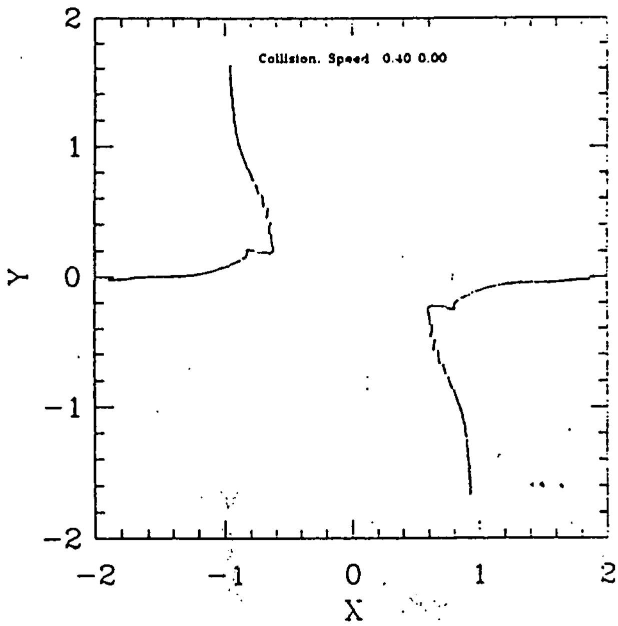

Fig.9*d* The trajectories corresponding to $\kappa = 3$.

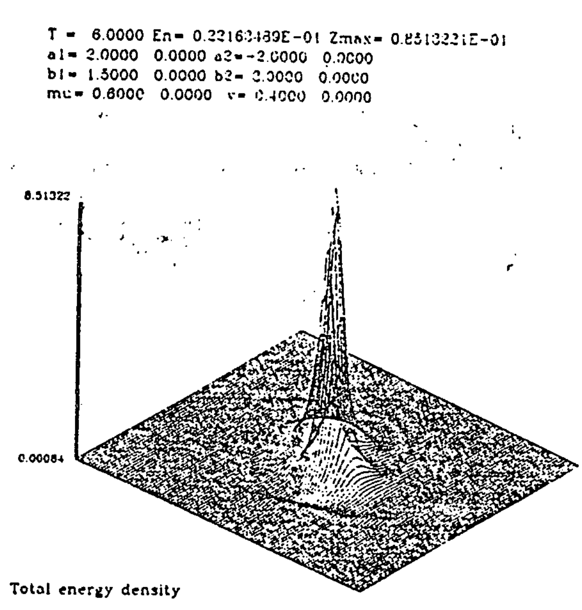

Fig.10 The total energy density seen in a typical simulation.

As to the difference between the two terms one can show, after lengthy and tedious calculations, that the difference (Δ_{Sky}) takes the form

$$\begin{aligned}\Delta_{Sky} = \frac{1}{[1+|W_1|^2+|W_2|^2]^4}\{&|W_{1t}|^2[(1+|W_1|^2)|W_{2+}|^2+|W_2|^2|W_{1+}|^2] \\ &+|W_{2t}|^2[(1+|W_2|^2)|W_{1+}|^2+|W_2|^2|W_{2+}|^2] \\ &-(W_{1t}W_{2t}^*[W_{1-}^*W_{2+}+(|W_1|^2+|W_2|^2)W_{-1}W_{2-}^*]+c.c)\},\end{aligned} \tag{6.22}$$

where this expression has been evaluated only for the holomorphic maps. Of course, it is even more complicated for more general fields. Observe that (6.22) vanishes identically in (2+0) dimensions *i.e.* for static configurations. This observation suggests either that the difference between the two terms is topological, or that there exist two different evolution patterns in the configuration space with the same initial conditions. Each pattern provides a different evolution trajectory, corresponding to the Skyrme term used in the model.

To investigate the first possibility one can either try to rewrite the difference as a total divergence, or equally well, work out its contribution to the equations of motion and check whether this contribution vanishes. In doing so we have found that the contribution of the difference term to the equations of motion does not vanish. Therefore the difference term is not topological.

So if we want to have a further understanding of the impact of the difference term on the evolution of a system of skyrmions, we have to perform numerical simulations or employ some reliable approximation scheme. In fact an approximation method like the collective coordinate approach can be used to investigate this problem or, perhaps, only the numerical simulations can answer all these questions.

In the remainder of this chapter we will compute the equations of motion of the CP^2 model modified by the addition of the Skyrme term given in chapter 4 by (4.30). To begin with we write down the form of the modified Lagrangian

$$\begin{aligned}L_{mod} = L_{CP^2} - \frac{\theta_1}{[1+|W_1|^2+|W_2|^2]}[&(W_{1\mu}^*W_{1\nu}-c.c)^2 \\ &+2(W_{1\mu}^*W_{1\nu}-c.c)(W_2^{\mu*}W_2^{\nu}-c.c)+2(W_{1\mu}^*W_{1\nu}-c.c)(R^{\mu*}R^{\nu}-c.c) \\ &+2(W_{2\mu}^*W_{2\nu}-c.c)^2+2(W_{2\mu}^*W_{2\nu}-c.c)(R^{\mu*}R^{\nu}-c.c) \\ &+(R^{\mu*}R^{\nu}-c.c)^2]\end{aligned} \tag{6.23}$$

where R_μ is given by (4.38). Then applying the variational principle to the modified

Lagrangian results in the following equation

$$
\begin{aligned}
&(1+|W_2|^2)W_{1\sigma}^{\sigma} - W_1W_2^*W_{2\sigma}^{\sigma} - \frac{2}{[1+|W_1|^2+|W_2|^2]}\{\\
&\quad(1+|W_2|^2)W_1^*W_{1\sigma}W_1^{\sigma} + (1-|W_1|^2+|W_2|^2)W_2^*W_{1\sigma}W_2^{\sigma} - 2W_1^2W_2^*W_2^{\sigma} - W_2^{2*}W_1W_{2\sigma}W_2^{\sigma}\}\\
&\quad - 4\frac{\theta_1}{[1+|W_1|^2+|W_2|^2]}\{W_{1\sigma}^{\sigma*}(W_{1\mu}W_1^{\mu} + W_2^*W_{1\nu}R^{\nu}) - W_{1\sigma}^{\sigma}(W_{1\mu}W_1^{\mu*} + W_2^*W_{1\mu}^*R^{\mu})\\
&\quad + W_{1\mu\sigma}(2W_1^{\sigma*}W_1^{\mu} - W_1^{\mu*}W_1^{\sigma} + W_2^*W_1^{\sigma*}R^{\mu}) - W_1\mu\sigma^*(W_1^{\mu}W_1^{\sigma} + W_2^*W_1^{\sigma}R^{\mu})\\
&\quad + W_{1\mu}(R_{\sigma}^{\mu}R^{\sigma*} + R^{\mu}R_{\sigma}^{\sigma*} - R^{\sigma}R_{\sigma}^{\mu*} - R^{\mu*}R_{\sigma}^{\sigma}) - W_{2\sigma}^{\sigma}(W_{1\mu}W_2^{\mu*} + W_2^*W_{2\mu}^*R^{\mu})\\
&\quad + W_{2\sigma}^{\sigma*}(W_{1\mu}W_2^{\mu} + W_2^*W_{2\mu}R^{\mu}) + W_{2\sigma\mu}(W_2^{\sigma*}W_1^{\mu} + W_2^{\sigma*}R^{\mu}W_2^* - W_2^*W_2^{\sigma}R^{\mu})\\
&\quad - W_{2\mu\sigma}^*W_2^{\sigma}W_1^{\mu} + W_{2\sigma}^*(W_1^{\sigma*}W_{1\nu}R^{\nu} + W_{2\mu}R^{\mu}W_2^{\sigma*} - W_{2\mu}^*R^{\mu}W_2^{\sigma} + R_{\nu}R^{\nu}R^{\sigma*} - R^{\sigma}R_{\mu}^*R^{\mu}\\
&\quad - W_1^{\sigma}W_{1\mu}^*R^{\mu}) + W_2^*(R_{\nu}R_{\sigma}^{\nu}R^{\sigma*} + R_{\nu}R^{\nu}R_{\sigma}^{\sigma*})\\
&\quad - \frac{1}{2}[((W_{1\mu}^*W_{1\nu} - c.c) + (W_{2\mu}^*W_{2\nu} - c.c) + (R_{\mu}^*R_{\nu} - c.c))(W_2^{\nu*}R^{\mu} - W_2^{\mu*}R^{\nu})]\}\\
&\quad + \frac{4\theta_1}{[1+|W_1|^2+|W_2|^2]^3}\{4W_1^{\sigma*}W_{1\mu}W_1^{\mu}(W_1W_{1\sigma}^* + W_2W_{2\sigma}^* + W_{2\sigma}W_2^*)\\
&\quad - 4W_1^{\sigma}W_{1\mu}^*W_1^{\mu}(W_1W_{1\sigma}^* + W_2W_{2\sigma}^* + W_2^*W_{2\sigma} + 4W_2^{\sigma*}W_{1\mu}W_2^{\mu}(W_1W_{1\sigma}^* + W_{2\sigma}W_2^* + W_2W_{2\sigma}^*)\\
&\quad - 4W_2^{\sigma}W_{1\mu}W_2^{\mu*}(W_1W_{1\sigma}^* + W_2^*W_{2\sigma} + W_2^*W_{2\sigma}) + 4W_{1\mu}R^{\mu}R^{\sigma*}(W_1W_{1\sigma}^* + W_2^*W_{2\sigma} + W_2W_{2\sigma}^*)\\
&\quad - 4W_{1\mu}R^{\mu*}R^{\sigma}(W_1W_{1\sigma}^* + W_2^*W_{2\sigma} + W_2W_{2\sigma}^*) + 4W_2^*W_1^{\sigma*}W_{1\nu}R^{\nu}\\
&\quad (W_1W_{1\sigma}^* + W_2^*W_{2\sigma} + W_2W_{2\sigma}^*)\\
&\quad - 4W_2^*W_1^{\sigma}W_{1\nu}^*R^{\nu}(W_1W_{1\sigma}^* + W_2^*W_{2\sigma} + W_2W_{2\sigma}^*) + 8W_2R_{\nu}R^{\nu}R^{\sigma*}\\
&\quad (W_1W_{1\sigma} + W_2^*W_{2\sigma} + W_2W_{2\sigma}^*)\\
&\quad - 4W_2^*R^{\sigma*}R_{\mu}^*R^{\mu}(2W_1^*W_{1\sigma} + W_1W_{1\sigma}^* + W_2^*W_{2\sigma} + W_2W_{2\sigma}^*)\\
&\quad - 4W_2^{\sigma}W_{2\mu}R^{\mu}W_2^*(W_1W_{1\sigma}^* + W_2W_{2\sigma}^* + W_2^*W_{2\sigma})\\
&\quad 4W_2^{\sigma*}W_{2\mu}R^{\mu}W_1^*W_2^*W_{1\sigma} + 4W_1^*W_2^*R_{\nu}R^{\nu}R^{\sigma*}W_{1\sigma} - W_1[(W_{1\mu}^*W_{1\nu} - c.c)^2\\
&\quad + 2(W_{1\mu}^*W_{1\nu} - c.c)(W_2^{\mu*}W_2^{\nu} - c.c) + 2(W_{1\mu}^*W_{1\nu} - c.c)(R^{\mu*}R^{\nu} - c.c)\\
&\quad + (W_{2\mu}^*W_{2\nu} - c.c)^2 + 2(W_{2\mu}^*W_{2\nu} - c.c)(R^{\mu*}R^{\nu} - c.c) + (R_{\mu}^*R_{\nu} - R_{\mu}R_{\nu}^*)^2]\} = 0
\end{aligned}
\tag{6.24}
$$

and a similar equation for W_2 obtained by the interchange $(1 \leftrightarrow 2)$. As a check, we can take the CP^1 limit (*i.e.* set $W_2 = 0$) and compare our equation with the one in ref.[17] ; the two expressions agree.

To construct classical static solutions for the modified model, we exploit the freedom of adding a potential term. After a few pages of algebra one can show that the configuration

$$W_1 = \lambda_1 x_+ \qquad W_2 = \lambda_2 x_+ \tag{6.25}$$

is a static solution with

$$\sqrt{\lambda_1^2+\lambda_2^2} = (\frac{\theta_1}{2\theta_2})^{1/4}, \tag{6.26}$$

provided that we add to the modified Lagrangian an extra potential term which has the form

$$L_{pot} = -\frac{4\theta_2}{[1+|W_1|^2+|W_2|^2]^4}. \tag{6.27}$$

What is the meaning of this result? In fact, what we have done so far is just the embedding of the modified CP^1 Skyrmion into the larger CP^2 target manifold. To see this choose W_1, W_2 as the basis of the solution space. Then any vector belonging to the solution space is given by

$$U = \alpha W_1 + \beta W_2, \tag{6.28}$$

but one can easily show that this configuration is equivalent to the expression

$$\begin{pmatrix} 1 \\ 0 \\ \nu U \end{pmatrix}, \tag{6.29}$$

which is a solution of the modified CP^2 model provided ν is given as $\sqrt{\lambda_1^2+\lambda_2^2}$ and it satisfies (6.26). However the configuration (6.29) is just a CP^1 Skyrmion. Hence by making a global rotation inside the subspace spanned by the W_1 and W_2 one can rewrite our solution as a CP^1 embedding.

6.5 Some further remarks and conclusions

We have studied the scattering properties of soliton like structures in a CP^2 model in (2+1) dimensions. We have found that most of their properties resemble those seen in the CP^1 case. In head-on collisions the solitons scatter at 90° to the direction of their initial motion. The scattering tends to destabilise the solitons; in most scattering processes they shrink. In the CP^1 case they can be stabilised by the addition of extra terms to the Lagrangian; however, these extra terms involve higher derivatives and require more computing power and in the CP^1 case most of their effects can be reduced to being responsible for the required stability of solitons. In the CP^2 case the stabilising terms are non-unique and it would be interesting to see what the effects of their different choices are; however, due to the large computing power required we have not performed these tests yet. Also, all our simulations have been performed in the W formulation and so we can trust our results only until solitons have not shrunk too much. The results of the CP^1 model support this expectation.

Since the CP^2 model has field configurations more general than the CP^1 model, we have looked at the mechanism of scattering. We have found that, indeed, the solitons come on top of each other before they scatter at 90°. During this they experience a shift in their trajectories; our results have suggested that the shift along the trajectory is approximately independent of the velocity (if relativistic effects are properly taken into account).

We have also looked at the effects of a particular additional term in the Lagrangian that can be added to the CP^2 model Lagrangian. This term vanishes for static configurations and is identically zero in the CP^1 case. It does not contribute to the energy momentum tensor but it does alter the equations of motion of the CP^2 model. We have found that its effect resembles a little the effect of rotation. This rotation can stabilise a single soliton but it alters the trajectories of solitons in motion.

All in all, the CP^2 model possesses a rich spectrum of extended structures which to a large extend behave like solitons. The scattering properties of these structures are rich and highly nontrivial. The results obtained so far are only a foretaste of what can be expected in physically more relevant (3+1) models.

VII. CONCLUSIONS

At first sight it appears that solitons owe their existence to the integrability of the models in which they arise. For example, in the literature, there are quite a number of integrable models in (1+1) dimensions which admit solitonic solutions. Moreover, in (2+0) dimensions a few σ-models, which are integrable, have been constructed (CP^n, $G_{nm}(\mathbb{C})$ and $U(m)$). However, introducing time to these models destroys their integrability. At this stage the following question may arise: is it unlikely for solitons to appear in these models in (2+1) dimensions?

In this thesis we have answered this question by showing that in (2+1) dimensions the CP^1 model, if modified by the addition of two stabilising terms, to cure the size instability of its lumps, admits solitonic solutions. Furthermore, the scattering properties of these solutions are very much like the scattering properties of the solitons of the integrable models in (1+1) dimensions.

Then we have argued that the modified CP^1 model is unique up to a potential term. However, this uniqueness does not generalise to the other CP^n models (*i.e.* $n > 1$).

We have also studied the scattering properties of the static CP^2 lumps in (2+1) dimensions, and like the CP^1 case, they undergo a shift in their trajectories and in their head-on collisions they show 90° scattering. However, in the CP^2 case, to treat the instability of the CP^2 lumps, one has the liberty of either repeating the handling of the CP^1 model *i.e.* adding the Skyrme terms and a potential term to the basic Lagrangian, but unlike the CP^1 case, the Skyrme term is not unique, or to consider other terms, containing fewer derivatives than Skyrme terms, *e.g.* the generalised Hopf term which is not topological in the CP^2 case, though it is independent of the metric. We have shown that this term has a subtle rotational effect on the CP^2 lumps, in the sense that it rotates different parts of the lump unequally. Thus it always acts in the direction of reducing the effects.

One interesting thing left for future investigation is to understand the nature of the difference between the two Skyrme terms in the CP^2 case. For this purpose one has to employ some reliable approximation. One poewrful approximation, frequently used in the study of the lumps interactions, which can be considered in our case is the collective coordinates approximation, or perhaps resorting to numerical simulations could give an answer to this question.

Another interesting thing to do is to consider the supersymmetric version of the modified CP^1 model, and look at the various topics considered in this thesis again for the supersymmetrized model.

In fact, one may wonder whether the addition of extra terms to the basic Lagrangian of a given model is the only way of stabilising the size of its lumps. Probably an alternative way could be by considering models on target manifolds whose topologies do not allow for conformal invariance of the underlying model. Of course, an answer to the feasibility of this alternate resides in giving an example of such a quasi-integrable model, if it exists.

One would have to note that in our study of CP^n models, only the scattering properties of instantonic solutions have received much attention. Therefore, it would be interesting to investigate the scattering properties of non-instantonic solutions. However, one would a priory expect their scattering properties to be more complicated than the instantonic ones due to the fact that the non-instantonic solutions are not stable for topological reasons.

Finally it would be interesting to study the consequences of building CP^n models on quantum planes (so far all the CP^n models have been constructed on either bosonic spaces, or in their supersymmetric version, on superspaces) where the commutativity properties of the coordinates parametrising these planes depends on the deformation parameter.

APPENDIX

To establish the topological nature of the quantity Q introduced in chapter 3 (5.10) we observe that if we consider $(W_x^\dagger W_y - W_y^\dagger W_x)P$, where P is any function of W and $W^\dagger$, we can rewrite it as

$$\begin{aligned}(W_x^\dagger W_y - W_y^\dagger W_x)P &= -\frac{1}{2}(W_x^\dagger W_y - W_y^\dagger W_x)(W^\dagger P_{W^\dagger} + W P_W) \\ +\frac{1}{2}\frac{\partial}{\partial x}((W^\dagger W_y - W_y^\dagger W)P) &- \frac{1}{2}\frac{\partial}{\partial y}((W^\dagger W_x - W_x^\dagger W)P),\end{aligned} \tag{A1}$$

where P_W denotes $\frac{\partial P}{\partial W}$ etc. Thus we see that $(W_x^\dagger W_y - W_y^\dagger W_x)(P + \frac{1}{2}(P_{W^\dagger} + WP_W))$ is given by

$$q = \frac{1}{2}\frac{\partial}{\partial x}((W^\dagger W_y - W_y^\dagger W)P) - \frac{1}{2}\frac{\partial}{\partial y}((W^\dagger W_x - W_x^\dagger W)P),$$

and so is really topological in nature. To see this last point we integrate q over the whole space and use the divergence theorem in 2 dimensions to obtain

$$\begin{aligned}\int dxdy\,[\frac{\partial}{\partial x}((W^\dagger W_y - W_y^\dagger W)P) - \frac{\partial}{\partial y}((W^\dagger W_x - W_x^\dagger W)P)] \\ = \oint 2(W^\dagger W_z P - W(W_z)^\dagger P)dz,\end{aligned} \tag{A2}$$

where we have introduced a complex variable $z = x + iy$ and used the fact that $W = W(z)$ only (*i.e.* is not a function of $z^\dagger$). The line integral in (A2) is along a large circle at infinity and around any singularities of the integrand. Thus, if there are no singularities in the finite plane the integral of q is determined only by the behaviour of the fields (W and $W^\dagger$) at infinity. In this sense it is topological.

To apply this observation to our problem we notice that all we have to do is to relate $f = \frac{\tilde{U}}{(1+WW^\dagger)^2}$ in (5.10) to $(P + \frac{1}{2}(P_{W^\dagger} + WP_W))$, *i.e.* find P such that $f = P + \frac{1}{2}(P_{W^\dagger} + WP_W)$. We see that the problem has been reduced to having to solve one simple equation (linear, first order partial differential equation for real P). It is clear that this equation always has a solution. The expression for it depends, of course, on the specific form of $\tilde{U}$. In particular when $\tilde{U} = 1$ it is easy to check that P is given by $P = \frac{1}{(1+WW^\dagger)}$, while if $\tilde{U} = \frac{1}{(1+WW^\dagger)^4}$ (our original model) P is proportional to $\frac{1}{WW^\dagger}\frac{(1+WW^\dagger)^5-1}{(1+WW^\dagger)^5}$. For a more general choice of $V(W, W^\dagger)$ the calculation of P is more involved. However, if we look at the case of $V = WW^\dagger$ (a special case of the choice made in section 4) then P is proportional to

$$\frac{1}{WW^\dagger}\left(-\frac{1}{3}\frac{1}{(1+WW^\dagger)^3}+\frac{1}{2}\frac{1}{(1+WW^\dagger)^4}-\frac{1}{5}\frac{1}{(1+WW^\dagger)^5}+\frac{1}{30}\right). \tag{A3}$$

The expression for P in the case of $V(W,W^\dagger) = |W + \lambda a^2|^2$ is much more involved but its explicit form can be found with relative ease.

In each of the cases, for z on the large circle $z = re^{i\phi}$, where $W \sim z^\alpha \sim r^\alpha e^{i\alpha\phi}$, the asymptotic behaviour of P is given by $P \sim \frac{const}{WW^\dagger}$ and so it corresponds to $P \sim const \times r^{-2\alpha}$. On the other hand the integrand of the line integral in (A2) is proportional to $r^{2\alpha} d\phi$ and so we see that, in the limit of the large circle the powers of r cancel, the integration over ϕ gives 2π and we obtain a finite nonvanishing number that that is proportional to the overall constant in the asymptotic form of P. Thus we see that Q is topological and we know how to find its value. The topological nature of Q holds for most choices of V-as long its form does not introduce any singularities or spoil the asymptotic behaviour.

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