On the algebraic structure of factorized S-matrices

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by

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A thesis presented for the degree
of Doctor of Philosophy at the
University of Durham.

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University of Durham

May 1992

Preface

This thesis is based on work done by the author between September 1989 and March 1992 at the University of Durham, supported by a U.K Science and Engineering Research Council studentship. No part of it has been previously submitted for any degree, either in this or any other university.

No claim of originality is made for chapters one and two. Chapters three[1], four[2], five[3] and six[4] and appendix B[5] are based on papers by the author, with the exception of some further review material in sections 4.1, 5.2, 6.1 and 6.3.

I should like to thank all those with whom I have had discussions over the last two years, and in particular my supervisor, Ed Corrigan, for his unfailingly useful advice.

Abstract

This thesis investigates the algebraic structure of certain quantum field theories in one space and one time dimension. These theories are *integrable* - essentially, highly constrained and therefore soluble. Thus, instead of having to use perturbative techniques, it is possible to conjecture their exact S-matrices, which have the property that they are *factorized* into two-particle S-matrices. In particular, there are two types of such theory: in one, scattering is purely elastic, whilst in the other, there is additional structure dictated by the *Yang-Baxter equation.* This thesis explores the algebraic structure of the latter and its links with the former.

We begin, in chapter one, with an informal summary of the development of the subject, followed by a more mathematical exposition in chapter two. Chapter three constructs explicitly some exact factorized S-matrices with Yang-Baxter structure, and comments on their features, both intrinsic and in relation to purely elastic S-matrices. In particular, there is an unexplained close correspondence between the mass spectra and particle fusings in the two types of theory. The next three chapters attempt to shed some light on these features. Chapter four constructs similar S-matrices, but based on quantum-deformed algebras rather than classical algebras. In chapter five we describe the structure of the S-matrices when the particles they describe transform in irreducible representations of classical algebras. This leads us to consider the *Yangian* algebra, the representation theory of which underlies Yang-Baxter dependent S-matrices, and which we therefore review briefly. We begin chapter six by reviewing the work which shows that the Yangian is also the charge algebra of the integrable quantum field theory, and subsequently show that the Yangian is also to a great extent present in the corresponding classical theory.

We conclude with a brief seventh chapter describing the outlook for further research, followed by appendices containing respectively details of the Lagrangians of some integrable quantum field theories, a continuum formulation of the quantum inverse problem, explicit expressions for some of the R-matrices computed in the text, and a summary of known solutions of the Yang-Baxter equation.

Contents

Chapter 1

Introduction

Over the past fifteen years there has been steadily increasing interest in integrable field theories in two dimensions. Much of this can be traced back to the renewal of interest in the mathematics of field theories in the 1970s: firstly, many interesting features of field theories are most easily observed in two dimensions, and secondly, there are interesting direct links between Yang-Mills fields in four dimensions and various integrable field theories in two. (For instance, Toda theories arise as reductions of the self-dual Yang-Mills equations, whilst the dynamics of Yang-Mills loops is that of pointlike excitations of a chiral field.) An excellent survey is given by Polyakov[6]. The other source of this interest is the revival of string theory in the early 1980s, alongside which arose conformal field theories[7]. These are integrable field theories with the property of scale-invariance, which therefore also possess conformal invariance in two dimensions. Although part of their interest derives from the wish to classify string vacua, they also provide the machinery to classify two-dimensional critical phenomena in general, and are thus of great importance to statistical mechanics. In 1989 Zamolodchikov[8] noticed that more general integrable theories could be used to interpolate between conformal field theories with different values of the central charge, providing new stimulus to research.

An equation which is (classically) integrable is essentially one which is highly constrained and therefore soluble; more specifically, it has an infinite number of quantities in involution, and admits soliton solutions. Many such equations are known, and an excellent elementary introduction to the techniques of their solution is given by Drazin and Johnson[9]. We shall, however, be concerned with properties of integrable field theories, which are characterized classically by an infinite number of quantities in involution, and in the quantum case by an infinite number of commuting charges. Examples of such theories include the (affine and non-affine) Toda theories and the principal chiral models, each example of these two types of theory being associated with a Lie (or, in the case of the affine Toda theories, affine) algebra.

A central theme of quantum integrability is the Yang-Baxter equation, and we shall begin with a brief history of its development. Most mathematical detail is left out here, and will be given in chapter two. The Yang-Baxter equation arises in two distinct ways: firstly, in statistical mechanics, as the condition for the transfer matrix to generate commuting charges, and secondly, in quantum field theory, as the condition for factorization of the S-matrix. The former is realized in field theory as the quantum inverse scattering method, which gets its name from one of the best known techniques for solving classically integrable equations. However, the technique actually has its roots in the work of Baxter[10] on exactly soluble models in statistical mechanics. Baxter showed that, when the Boltzmann weights of a statistical mechanical model are chosen in a certain way, the row-to-row transfer matrix $T(u)$ obeys the relation

$$R(u-v)\, T(u) \otimes T(v) = T(v) \otimes T(u)\, R(u-v) \ , \tag{1.1}$$

where R is some matrix function of the spectral parameter u. Writing T as a power series in u and taking the trace of this relation yields infinitely many commuting quantities. For the action of R on tensor products of three transfer matrices to be associative it must obey the Yang-Baxter equation

$$R_{23}(u)R_{13}(u+v)R_{12}(v) = R_{12}(v)R_{13}(u+v)R_{23}(u) \ , \tag{1.2}$$

where the subscripts denote which two of the three spaces the R-matrix acts upon. In particular, Baxter was able to solve the eight-vertex model (and its restrictions the six-vertex and ice models) in this way. In 1979 Faddeev, Sklyanin and Takhtajan[11] applied this result to the sine-Gordon quantum field theory, naming their technique the quantum inverse scattering method (QISM). They did this by discretizing space, so that the Lax pair for the theory gave rise to a transfer matrix satisfying (1.1), and obtained results which agreed with those already known from the sine-Gordon S-matrix. However, their results depended crucially on the fact that they were dealing with two-by-two matrices. The sine-Gordon theory is actually the affine Toda theory (with imaginary coupling constant) associated with $SU(2)$, and the question arises of how to deal with the problem in higher representations of this algebra. The answer was provided by Kulish and Reshetikhin[12], whose technique was to insist on (1.1) at the expense of the usual commutation relations satisfied by the Lie algebra generators in the sine-Gordon Lax pair. They found that requiring that (1.1) be soluble imposes a different set of commutation relations on these generators, which become identical to the usual relations when $\hbar \to 0$. This new algebra became known as the 'quantum group', and was generalized in 1985 by Jimbo[13] and

Drinfeld[14,15] to other Lie algebras. Solutions to the Yang-Baxter equation associated with various representations of these algebras were soon constructed, most notably by Jimbo for higher representations of $SU(2)$[16] and for vector representations of the other classical groups[17]. Baxter's original R-matrix for the six-vertex model corresponded to the solution for the fundamental representation of $SU(2)$. The classification of quantum groups and the structure of their representations has been the subject of much attention from pure mathematicians recently[18,19].

At the same time as he introduced quantum groups[14], Drinfeld also introduced another algebra (associated indirectly with the quantum inverse scattering method) which he named the Yangian, and which may be thought of as the $\hbar \to 0$ limit of the quantum group. The R-matrices associated with this algebra are group-invariant rather than quantum group-invariant, and will be the subject of chapters three and five. It is only now that Yangians are beginning to arouse interest comparable to that attracted by quantum groups, but the importance of the Yangian both for the mathematics of the Yang-Baxter equation and for the physics of integrable field theories will be emphasized throughout this work.

The second way in which the Yang-Baxter equation arises is as the condition for factorization of the S-matrix. The presence of infinitely many conserved charges strongly constrains the S-matrix of a quantum integrable field theory, requiring it to conserve the set of asymptotic momenta of the particles, and leading[20,21] to the factorization of a multi-particle S-matrix element into two-particle elements. In a way which will be made precise in the next chapter, the condition for this factorization to be consistent is equivalent to the Yang-Baxter equation (1.2): it is simply that the three-particle element should be independent of the ordering of its two-particle-element factors. Thus the solutions of the Yang-Baxter equation and the algebraic structures underlying them are fundamental in the study of factorized S-matrices.

It is important at this stage to draw a distinction between the two types of theory which have factorized S-matrices. In the first, the particles appear in multiplets of equal mass. Thus the two-particle S-matrix may exchange particles within these multiplets (since the set of asymptotic momenta will still be conserved) and the Yang-Baxter equation provides a strong constraint on the S-matrix. Examples of such theories are given by the principal chiral, Gross-Neveu and chiral Gross-Neveu models, in which global group invariance leads to equal-mass multiplets in representations of Lie algebras. The S-matrices, also group-invariant, are thus given by solutions of the Yang-Baxter equation associated with Yangians, and the particle multiplets correspond to fundamental representations of the Yangian (which may, however, be reducible as representations of the group). The pioneering work is by the Zamolodchikovs[20], who determine, amongst others, the S-matrix of particles in vector representations of $O(N)$; further S-matrices were computed by Ogievetsky, Reshetikhin and Wiegmann[22,23]. If the Yangian underlies group-invariant R-matrices, and the conserved charges underlie factorized S-matrices, we might expect the charge algebra of theories with these S-matrices to be the Yangian. This is indeed the case, and in 1978 Lüscher derived the action of the charges on asymptotic states, which is now recognizable as that of the Yangian. Appreciation of the algebraic structure underlying the S-matrices has developed slowly and in parallel to the construction of S-matrices and, although the Yangian structure has now been fully elucidated by Bernard[24], much remains unclear.

In the second type of theory, there is no mass-degeneracy*, so that no exchange of quantum numbers within equal-mass multiplets is possible; the S-matrix is then a pure phase, and the Yang-Baxter equation is trivial. Examples of such theories, known as 'Purely Elastic Scattering Theories' (PESTs), include the affine Toda field theories (with real coupling) and Zamolodchikov's integrable deformations of conformal field theories[8]. However, it may still be possible to determine the S-matrix by other means, such as the 'bootstrap procedure' in which intermediate states of S-matrix elements at appropriate poles are regarded as physical states of the theory. In fact, unitarity, analyticity, crossing symmetry and the bootstrap are still enough to determine a complete set of exact S-matrices. Even without a Lagrangian field theory, simply starting with a small set of particles and their S-matrices and implementing the bootstrap leads to a rich structure. The bootstrap only closes on a larger spectrum of particles (*i.e.* all appropriate S-matrix poles correspond to particles already in the spectrum) in certain special cases, the classification of which is

*Actually, there may be some degeneracies but other conserved quantities serve to identify the particles uniquely.

deeply linked with root systems of Lie algebras[25,26].

Throughout the study of two-dimensional integrable quantum field theories one is struck by the universality of the algebraic structures encountered. When the Yang-Baxter equation appears one generally† expects the underlying algebra to be a quantum group or Yangian, and such structures may arise from superficially disparate Lagrangians: for example, Yangian symmetry occurs in both the principal chiral field and in the multicomponent fermionic models obtained as generalizations of the chiral Gross-Neveu model, where the essential requirement is only that the classical theory have a curvature-free conserved current. Furthermore, it seems that the structure of the S-matrices is the same in the quantum-group-invariant case as in the Yangian (group-invariant) case (for general values of the quantum group parameter). To put this another way, any question about general representations of quantum groups and their associated R-matrices can be answered solely by reference to representations of Yangians and theirs.

However, an essential point to be made in this work is that this universality extends much further, into the purely elastic scattering theories. The same bootstrap procedure used there can be applied to the S-matrices with Yang-Baxter structure, and is equivalent to the 'fusion procedure' introduced by Kulish, Sklyanin and Reshetikhin for generating solutions of the Yang-Baxter equation. Where they have been calculated, these exact S-matrices with Yang-Baxter structure have the same mass spectra and fusings‡ as in the PESTs, the point being that the Yang-Baxter structure seems not to be fundamental: the bootstrap alone determines the physics. The difficulty of applying the fusion procedure means that very few solutions of the Yang-Baxter equation have been explicitly calculated in this way, but it seems likely that the underlying fusing structures are the same. The essential step will have to be to relate the description of the fusing structure given by Dorey[25] to the tensor product structure of fundamental representations of the Yangian; encouragingly, progress is being made on the latter by Chari and Pressley[29,30,31].

†Actually, solutions of the YBE also exist[10,27,28] corresponding to spectral parameters on tori and higher genus surfaces; their role in quantum field theory is not yet clear.

‡At least for simply-laced algebras; as we shall see later, the non-simply-laced cases are rather more subtle.

This thesis is laid out as follows:

In chapter two, we describe how the integrability of a quantum field theory leads, through the existence of an infinity of conservation laws, to factorization of its S-matrices. The factorization condition is the Yang-Baxter equation (YBE) and so, when the additional requirements of unitarity, analyticity and crossing-symmetry have been imposed, solutions of the YBE ('R-matrices') give factorized S-matrices. In order to understand factorized S-matrices, we therefore need an understanding of the construction, classification and underlying structure of solutions of the YBE, which we review.

The underlying algebras were originally discovered through the quantum inverse scattering method, via a process which discretized space and altered the Lax formalism correspondingly. Work in progress suggests however that these algebras may be obtainable whilst remaining on the continuum formulation of the theories. However, since a discussion of the quantum inverse scattering method is peripheral to our theme, and since there are deep difficulties with this work, its discussion is relegated to an appendix.

We may also impose the 'bootstrap' principle on factorized S-matrices: that, at simple poles, the intermediate states of the S-matrix should be physical states of the theory. The bootstrap procedure for S-matrices is equivalent to the fusion procedure for R-matrices, which we describe. The bootstrap procedure can be seen at work more simply in the purely elastic scattering theories, and we give a brief description of these theories and the structure of their S-matrices.

It is possible to investigate factorized S-matrices (both with and without Yang-Baxter structure) without reference to any underlying quantum field theory and, since the universality of such S-matrices is a theme of this text, details of the appropriate Lagrangian field theories are left to an appendix.

In chapter three, we describe an algorithm for the fusion procedure for R-matrices for the classical groups and implement it to construct new group-invariant solutions of the YBE associated with $SO(N)$, and hence new factorized S-matrices. The S-matrices we calculate display the same fusing structure (in the simply-laced cases) as those of the purely elastic scattering theories, in contradiction with some previous expectations[23], which held that the fusing structure would be that of the Clebsch-Gordan decomposition of the tensor products of the fundamental representations. Because they act in reducible representations of the group, their structure is more complicated than that of S-matrices in irreducible representations. Although there are simpler ways of writing them, these do not appear to offer any additional insight.

Three avenues for exploration present themselves. Firstly, we construct R-matrices which are quantum group invariant rather than group invariant. This is done in chapter four, and gives the same physics (for general values of the quantum group parameter§) as in the group invariant case. For this reason we henceforth concentrate on group invariant R-matrices.

The second avenue is to examine the structure of group invariant R-matrices in irreducible representations. In general, these do not coincide with the R-matrices from which factorized S-matrices are constructed, but they are much more tractable, and a thorough understanding of them is certainly a prerequisite for solution of the more general case. The techniques of chapter three enable construction of some examples of such R-matrices, and this leads us to a conjecture for their general form. The proof of this conjecture is the subject of chapter five, and introduces a new method based on graphs of tensor products of representations.

It is also in this chapter that the importance of the Yangian emerges, since to classify representations of the Yangian is effectively to classify group invariant R-matrices. In the last section of chapter five, we review what is known about the representation theory of the Yangian. Firstly, there is Drinfeld's work which classifies the representations of the Yangian which are irreducible as representations of the underlying Lie algebra. These coincide with the representations in which R-matrices were found in the first part of the chapter, although Drinfeld does not give the structure of the R-matrices. Secondly, there is an important recent paper by Chari and Pressley[31], contemporary with that upon which

§*i.e.* when the quantum group parameter q is not a root of unity chapter three is based, which constructs the R-matrix in the irreducible representation of the Yangian containing the adjoint representation of the group. Their results for $SO(N)$ match precisely those of chapter three. We finish the chapter with a brief summary of the available methods for the construction of R-matrices and, in an appendix, list all the R-matrices known at present.

The third avenue is to investigate the algebra of conserved charges in the models corresponding to the factorized S-matrices we have constructed. In chapter six, we review the work which shows that the underlying algebra is indeed the Yangian, and show that significant parts of the Yangian are also present in the corresponding classical theories, re-emphasizing the question of the relationship between classical and quantum integrability. We also describe Belavin's implementation of the conserved charge bootstrap in theories with Yangian symmetry; his work, on the a_n case, extends to all other particle multiplets in irreducible representations of the algebra.

To summarize, the overall goal is a complete theory of the structure of factorized S-matrices. In order to achieve this, a common framework is needed for the bootstrap procedure in theories both with and without Yang-Baxter structure, which in turn requires a generalized approach to the fusion procedure and the representation theory of the Yangian. If these aims could be achieved alongside a full exposition of Yangian and quantum group symmetries in physics, we should be approaching a unified algebraic description of quantum integrability in two dimensions. The outlook for this research is discussed in a brief seventh chapter.

Chapter 2

Factorized S-matrices and the Yang-Baxter equation

2.1 Factorized S-matrices

Certain quantum field theories in 1+1 dimensions possess a property which allows their S-matrices to be determined up to an overall factor. This property is known as *factorization*[20,32], which means that a multiparticle S-matrix element, involving the interaction of N particles, can be rewritten as a product of $\frac{N(N-1)}{2}$ two-particle elements. This corresponds to the idea that the physical scattering process is a product of pairwise collisions, with the interacting particles behaving effectively as free particles between these collisions. The proof of this follows from two selection rules:

1. The number of particles of the same mass remains unchanged after the interaction (so that there is no particle production).

2. The final set of two-momenta is the same as the initial one.

These rules arise from the presence, mentioned before, of an infinite number of conserved charges Q_i. If we have*

$$Q_i|p_1^{(a_1)}p_2^{(a_2)}...p_k^{(a_k)};in/out\rangle = \left(\omega_i^{(a_1)}(p_1) + ... + \omega_i^{(a_k)}(p_k)\right)|p_1^{(a_1)}p_2^{(a_2)}...p_k^{(a_k)};in/out\rangle \quad (2.1)$$

(where the (a) label the particle types) then conservation of Q_i implies

$$\sum_{j\in in}\omega_i^{(a_j)}(p_j) = \sum_{j\in out}\omega_i^{(a_j)}(p_j)\ ;$$

requiring this to be true for all i fixes the sets $\{(a_j)\}$ and $\{p_j\}$ and thus implies (1) and (2).

When the particles are sufficiently far from each other, the intermediate states should obey (1) and (2) as well. By performing translations on the asymptotic coordinates of the individual particles, one can arrange arbitrarily large space-time separations between the regions where pairwise collisions occur. In space-time domains sufficiently far from

*Notice that the charges in (2.1) are implicitly *local* rather than *non-local* since they have trivial coproduct (*qv*).

the pairwise collisions, the wave function is approximately that of N free particles. Determining the interaction now reduces to extrapolating this wave function from one such domain to another, which can always be done through regions in which no more than two of the particles are close together. This argument is a synopsis of that given by the Zamolodchikovs[20,32]; a more detailed and rigorous treatment is given by Iagolnitzer[21].

Despite this factorization, the scattering process may still be quite complicated since, if there are mass degeneracies, it can redistribute the momenta among different particles with the same mass. The requirement that it do so consistently leads to the factorization or Yang-Baxter equation. Before we discuss this equation, let us review the other conditions we must impose on the two-particle S-matrix.

Suppose the particles belong to multiplets U, V within which they have equal mass. Let the difference in rapidity† of the scattering particles be θ. The S-matrix then depends only on θ, which lies in the strip $0 \leq \mathrm{Im}\,\theta \leq \pi$, corresponding to the physical sheet of the Mandelstam variable $s = (\mathbf{p}_1 + \mathbf{p}_2)^2$. (In 1+1 dimensions, t and u are not independent of s.) Then $S_{UV}(\theta)$ must satisfy[33]:

i) *Unitarity:* $S_{UV}(\theta)S_{VU}(-\theta) = 1$.

ii) *Crossing symmetry:* $S_{UV}(\theta) = (1 \otimes \mathbf{C}_V)S_{\bar{V}U}(i\pi - \theta)(\mathbf{C}_V \otimes 1)$, where $\mathbf{C}_V$ is the charge conjugation operator on the particle V, $\mathbf{C}_V : V \to \bar{V}$. When V is self-conjugate ($V = \bar{V}$), this reduces to

$$S_{UV}(\theta) = S_{VU}(i\pi - \theta) \ .$$

iii) *Analyticity:* S is a meromorphic function of θ in the physical strip. Its only poles correspond to the formation of bound states in the direct (when the pole has negative residue) or crossed (positive residue) channels.

iv) For *factorization* to be consistent, $S(\theta)$ must satisfy the Yang-Baxter equation.

†The rapidity θ_i of the ith particle is defined by $\mathbf{p}_i = (m\cosh\theta_i, m\sinh\theta_i)$.

Note that (i)-(iv) do not determine the S-matrix completely: we are free to introduce an overall unitary, crossing-symmetric scalar factor, known as a CDD factor[34]. This must be fixed by knowledge of the bound-state/pole structure of S.

2.2 The Yang-Baxter equation

The condition for factorization of the S-matrix to be consistent is that the multiparticle S-matrix element should be independent of the order of the two-particle interactions into which we factorize it. We must require that the two possible orderings for factorization of the three-particle S-matrix are equivalent. Schematically, the condition is shown in figure (2.1), where we imagine time on the vertical and space on the horizontal axis. In symbols,

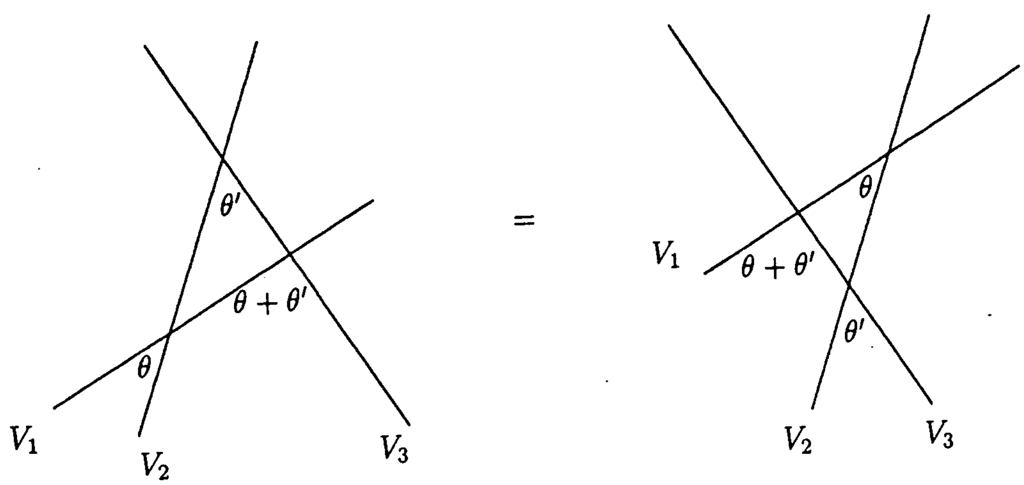


Figure 2.1: The Yang-Baxter equation

this is

$$S_{V_2V_3}(\theta')S_{V_1V_3}(\theta+\theta')S_{V_1V_2}(\theta) = S_{V_1V_2}(\theta)S_{V_1V_3}(\theta+\theta')S_{V_2V_3}(\theta') \ .$$

This equation is the famous Yang-Baxter equation (YBE)[28,35]. For the moment, we wish to discuss the properties of solutions of the Yang-Baxter equation *per se*, without being concerned with the other properties (i)-(iii) of factorized S-matrices. For this reason we shall henceforth call solutions of the YBE 'R-matrices' and use the YBE in the form

$$\begin{aligned}\left(\check{R}_{V_2V_3}(u)\otimes 1\right)\left(1\otimes \check{R}_{V_1V_3}(u+v)\right)\left(\check{R}_{V_1V_2}(v)\otimes 1\right) = \\ \left(1\otimes \check{R}_{V_1V_2}(v)\right)\left(\check{R}_{V_1V_3}(u+v)\otimes 1\right)\left(1\otimes \check{R}_{V_2V_3}(u)\right) \ , \qquad (2.2)\end{aligned}$$

where both sides map $V_1 \otimes V_2 \otimes V_3$ to $V_3 \otimes V_2 \otimes V_1$, and we are now thinking of the V_i as vector spaces‡. Note that R-matrices are unique only up to an overall scale and a scale in u; it is these freedoms to rescale which we shall use to convert R-matrices into S-matrices. As with the S-matrix $S_{V_1V_2}(\theta)$, the R-matrix $\check{R}_{V_1V_2}(u)$ maps $V_1 \otimes V_2$ into $V_2 \otimes V_1$. For this reason, it is sometimes more convenient to use the YBE in the form

$$R_{V_2V_3}(u)R_{V_1V_3}(u+v)R_{V_1V_2}(v) = R_{V_1V_2}(v)R_{V_1V_3}(u+v)R_{V_2V_3}(u) \ , \tag{2.3}$$

where now $\check{R}_{V_iV_j}(u) = \mathbf{P}R_{V_iV_j}(u)$ and $\mathbf{P}(v_1 \otimes v_2) = v_2 \otimes v_1$ for $v_i \in V_i$. This has the advantage that now $R_{V_1V_2}$ maps $V_1 \otimes V_2$ to itself, and thus (2.3) is valued in $End(V_1 \otimes V_2 \otimes V_3)$. Throughout this work we shall use the 'check' symbol to distinguish $\check{R}$ from R.

The YBE is a matrix equation, and one can easily attempt to construct matrix solutions of it. A good example would be the first one found[36,37],

$$R(u) = \begin{pmatrix} 1+u & & & \\ & u & 1 & \\ & 1 & u & \\ & & & 1+u \end{pmatrix} \ , \tag{2.4}$$

in which $V_1 = V_2 = V_3 = \mathbf{C} \times \mathbf{C}$. As more solutions appeared, two important facts emerged about them:

- they are always rational, trigonometric (exponential) or elliptic functions of u
- their classification is deeply bound up with the properties of Lie algebras.

We can begin to explain these facts by looking at the 'classical' limit of the YBE. An R-matrix is said to be quasi-classical if it contains a parameter $\hbar$ such that as $\hbar \to 0$ it takes the form

$$R(u,\hbar) = (scalar).(I + \hbar r(u) + O(\hbar^2)) \ .$$

The terms of order $\hbar^2$ in the YBE then give us

$$[r_{12}(u), r_{13}(u+v)] + [r_{12}(u), r_{23}(v)] + [r_{13}(u+v), r_{23}(v)] = 0 \ , \tag{2.5}$$

‡The fact that S-matrices act on asymptotic states which are particle multiplets naturally leads us to define the R-matrix in terms of its action on a vector space. This is not strictly necessary, however; we really only need an associative algebra B, with R valued in $B^{\otimes 2}$ and the equation valued in $B^{\otimes 3}$.

the classical YBE. It can easily be shown that

$$r(u) = \frac{1}{u} I_a \otimes I_a \tag{2.6}$$

is a solution, where the I_a are generators of a (semi-simple) Lie algebra $\mathcal{A}$. (Summation over repeated indices is always implied.) It has been proved[38] that solutions of the classical YBE are rational, trigonometric or elliptic functions of u; that (2.6) is unique, that trigonometric solutions exist for each $\mathcal{A}$, and that elliptic solutions exist only for $\mathcal{A} = a_n$.

2.3 Quasitriangular Hopf algebras

Soon after this, the full YBE was given an algebraic setting[15], that of quasitriangular Hopf algebras. These algebras originated in the quantum inverse scattering method (QISM), in which the presence of the R-matrix leads to a non-trivial bialgebra structure inconsistent with Lie algebra commutation relations, imposing instead a new set of relations valued in the enveloping algebra. Work in progress[5], based on a method introduced by Bhattacharya and Ghosh[39,40] suggests, interestingly, that such algebras may be derivable from a continuum formulation of the QISM, rather than the usual lattice formulation. However, a discussion of the QISM in general and this work in particular is not necessary to a description of quasitriangular Hopf algebras, and would be peripheral to a work on factorized S-matrices, and is therefore relegated to an appendix.

A quasitriangular Hopf algebra over a field k is an algebra $\mathcal{H}$ (with identity $1_{\mathcal{H}}$ and product $\cdot : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$) with the following additional structures:

- a coproduct $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$
- a counit $\epsilon : \mathcal{H} \to k$
- an antipode $s : \mathcal{H} \to \mathcal{H}$
- a universal R-matrix $\mathcal{R} \in \mathcal{H} \otimes \mathcal{H}$

satisfying, for all $a, b \in \mathcal{H}$,

$$(\Delta \otimes 1_{\mathcal{H}})\Delta = (1_{\mathcal{H}} \otimes \Delta)\Delta$$

$$\Delta(a)(\cdot \otimes \cdot)\Delta(b) = \Delta(a \cdot b) \qquad \Delta(1_{\mathcal{H}}) = 1_{\mathcal{H}} \otimes 1_{\mathcal{H}}$$

$$\epsilon(a)\epsilon(b) = \epsilon(a \cdot b) \qquad \epsilon(1_{\mathcal{H}}) = 1$$

$$s(a) \cdot s(b) = s(b \cdot a)$$

$$(\epsilon \otimes 1_{\mathcal{H}})\Delta(a) = a = (1_{\mathcal{H}} \otimes \epsilon)\Delta(a)$$

$$\cdot(s \otimes 1_{\mathcal{H}})\Delta = \cdot(1_{\mathcal{H}} \otimes s)\Delta = \eta \circ \epsilon \quad (\text{where } \eta : k \to \mathcal{H} \text{ is defined by } \eta(\lambda) = \lambda 1_{\mathcal{H}}\ )$$

and

$$(\Delta \otimes 1_{\mathcal{H}})\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23} \qquad (1_{\mathcal{H}} \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12} \tag{2.7}$$

$$\sigma \circ \Delta(a) = \mathcal{R}\Delta(a)\mathcal{R}^{-1} \tag{2.8}$$

(where $\sigma(x \otimes y) = y \otimes x$, so that in a particular representation $\sigma \circ \Delta(a) = \mathbf{P}\Delta(a)\mathbf{P}$). Physicists should think of Δ as a rule for addition of quantum numbers: since it is a homomorphism, it gives a representation of the action of $\mathcal{H}$ on $\mathcal{H} \otimes \mathcal{H}$; it tells us how the algebra acts on products of asymptotically independent states. For instance, consider the coproduct for a Lie algebra $\mathcal{A}$,

$$\Delta(a) = 1_{\mathcal{A}} \otimes a + a \otimes 1_{\mathcal{A}} \qquad \forall a \in \mathcal{A}\ .$$

In the case of angular momentum $SU(2)$, say, this simply states that J_+, J_- and J_3 act additively on products of states, and can be used to calculate Clebsch-Gordan coefficients through the commutativity of the diagram

$$\begin{array}{ccc} V_1 \otimes V_2 & \xrightarrow{C^{V_3}_{V_1 V_2}} & V_3 \\ \rho_1 \otimes \rho_2(\Delta(a)) \downarrow & & \downarrow \rho_3(a) \\ V_1 \otimes V_2 & \xrightarrow{C^{V_3}_{V_1 V_2}} & V_3 \end{array}$$

It is perhaps because this coproduct is trivial that it is rarely made explicit in physics textbooks. However, whilst the coproduct just given is cocommutative (that is, $\sigma \circ \Delta = \Delta$), this need not necessarily be so. It can be seen from (2.8) that $\mathcal{R}$ gives a measure of the non-cocommutativity of the Hopf algebra, with $\mathcal{R} = 1 \otimes 1$ corresponding to the cocommutative case. (Henceforth we shall leave out the subscript on the identity element.) The Yang-Baxter equation begins to emerge if we calculate $(1 \otimes \Delta)\mathcal{R}$ in two different ways using (2.7) and (2.8), obtaining

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12} \ . \tag{2.9}$$

If we have a matrix representation of this, then we can write it in the form (2.2) instead of (2.3) and obtain

$$(1 \otimes \check{R})(\check{R} \otimes 1)(1 \otimes \check{R}) = (\check{R} \otimes 1)(1 \otimes \check{R})(\check{R} \otimes 1) \ , \tag{2.10}$$

the braid group relation, shown in figure (2.2). (Note that we must have a representation of $\mathcal{R}$ in order to do this since, in $\check{R} = \mathbf{P}R$, $\mathbf{P}$ is defined in terms of its action on vectors.)

The spectral parameter enters the equation if $\mathcal{H}$ has an automorphism T_u $(u \in k)$ such

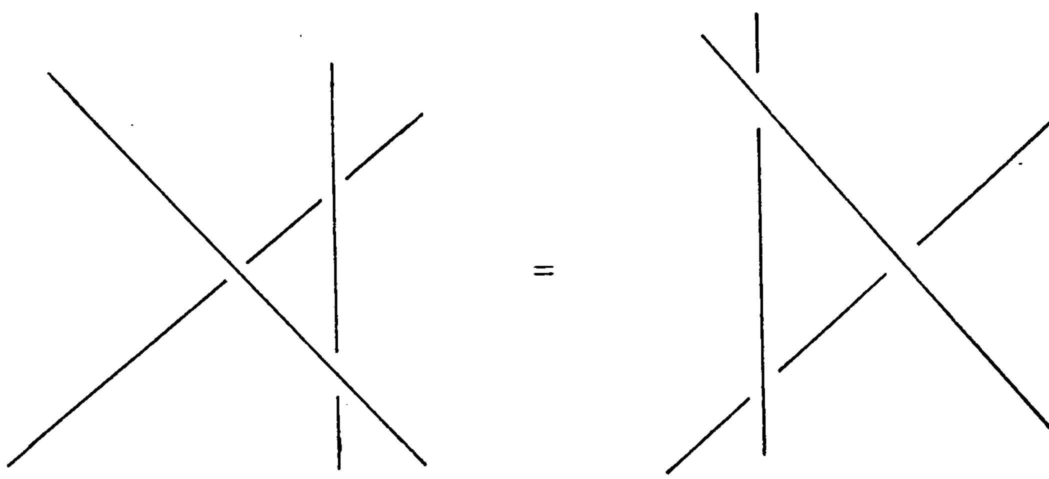


Figure 2.2: The braid group relation

that $T_u T_v = T_{u+v}$, $T_0 = 1$ and $(1 \otimes T_u)\mathcal{R} = (T_{-u} \otimes 1)\mathcal{R} = \mathcal{R}(u)$; then, (2.9) becomes

$$\mathcal{R}_{12}(u)\mathcal{R}_{13}(u+v)\mathcal{R}_{23}(v) = \mathcal{R}_{23}(v)\mathcal{R}_{13}(u+v)\mathcal{R}_{12}(u) \ ,$$

and matrix representations of this give the YBE (2.3). Hence, if we can construct algebras with these properties, their representations will give rise to matrix solutions of the YBE.

There are two types of quasitriangular Hopf algebra in which we are interested: the Yangian, introduced by Drinfeld[14] and related to rational R-matrices, and the quantized universal enveloping algebra or 'quantum group', introduced for $SU(2)$ by Kulish and Reshetikhin[12] and for the general case by Drinfeld[15,14] and Jimbo[16], and related to trigonometric R-matrices.

2.4 Yangians

First, consider a semi-simple Lie algebra $\mathcal{A}$ with generators I_a. In addition to the commutation relations

$$[I_a, I_b] = f^{abc} I_c \ , \tag{2.11}$$

we have

$$\Delta(I_a) = 1 \otimes I_a + I_a \otimes 1 \ , \tag{2.12}$$

$$\epsilon(I_a) = 0 \quad \text{and} \quad s(I_a) = -I_a \ .$$

The Yangian $Y(\mathcal{A})$ is obtained by adding additional generators J_a in the adjoint representation of $\mathcal{A}$,

$$[I_a, J_b] = f^{abc} J_c \tag{2.13}$$

with

$$\Delta(J_a) = 1 \otimes J_a + J_a \otimes 1 + \frac{1}{2} f^{abc} I_c \otimes I_b \ , \tag{2.14}$$

$$\epsilon(J_a) = 0 \quad \text{and} \quad s(J_a) = -J_a + \frac{1}{2} f^{abc} I_c I_b \ , \tag{2.15}$$

and taking the enveloping algebra of the $\{I, J\}$. There are additional constraints on $[J_a, J_b]$ which arise from the requirement that Δ be a homomorphism:

$$[J_a, [J_b, I_c]] - [I_a, [J_b, J_c]] = a_{abcdeg} \{I_d, I_e, I_g\} \ , \tag{2.16}$$

where

$$a_{abcdeg} = \frac{1}{24} f^{adi} f^{bej} f^{cgk} f^{ijk} \quad , \quad \{x_1, x_2, x_3\} = \sum_{i \neq j \neq k} x_i x_j x_k \ ,$$

and

$$[[J_a, J_b], [I_l, J_m]] + [[J_l, J_m], [I_a, J_b]] = \left(a_{abcdeg} f^{lmc} + a_{lmcdeg} f^{abc}\right) \{I_d, I_e, J_g\} \ . \tag{2.17}$$

Since these relations are valued in the enveloping algebra of $\{I, J\}$, they are known as the Yangian Serre relations. For $\mathcal{A} = a_1$, (2.16) is redundant, while for $\mathcal{A} \neq a_1$, (2.17) is redundant. Since it has not appeared elsewhere, we give a brief sketch[41] of the proof of (2.16) here.

First, let u_{ab} be such that $u_{ab} = -u_{ba}$ and

$$u_{ab}[I_a, I_b] = 0 \ . \tag{2.18}$$

Now compute

$$u_{ab}\left(\Delta\left([J_a, J_b]\right) - 1 \otimes [J_a, J_b] - [J_a, J_b] \otimes 1\right) \ .$$

The parts of this expression involving J disappear because of (2.18), whilst the remainder is

$$\frac{1}{2} u_{ab} f^{ade} f^{bgh} f^{dgk} \left(I_k \otimes I_e I_h + I_e I_h \otimes I_k\right) \ . \tag{2.19}$$

Because of (2.18), or $f^{abc}u^{ab} = 0$, we may write u_{ab} as§

$$u_{ab} = v_{dea} f^{deb} - v_{deb} f^{dea} \ ;$$

requiring Δ to be a homomorphism for all v, and using the Jacobi identity twice, we obtain

$$\frac{1}{2} f^{g[ab}[J_{c]}, J_g] = a_{abcdeg}\{I_d, I_e, I_g\} \ , \tag{2.20}$$

or (2.16) (where [ ] denotes the *sum* of terms antisymmetric on the enclosed indices).

$Y(\mathcal{A})$ should be thought of as actually being generated by a whole series of generators in adjoint representations of $\mathcal{A}$ at grades $0, 1, 2, ..$, with the I and J being simply the first two sets, at grades 0 and 1 respectively. The condition (2.16) should then be seen as a constraint on the construction of higher grade generators from products of Js.

The automorphism T_u, with $u \in \mathbf{C}$, is given by

$$T_u : J_a \mapsto J_a + u I_a \quad \text{and} \quad T_u : I_a \mapsto I_a \ . \tag{2.21}$$

§This statement is equivalent to the fact that the second homology group of $\mathcal{A}$, $H_2(\mathcal{A})$, is trivial.

Drinfeld showed that there then exists a formal R-matrix, $\mathcal{R}(u)$, with $(T_w \otimes T_v)\mathcal{R}(u) = \mathcal{R}(u+v-w)$, satisfying

$$(1 \otimes T_u)\sigma \circ \Delta(x) = \mathcal{R}(u)^{-1}(1 \otimes T_u)\Delta(x)\mathcal{R}(u) \qquad (x \in Y(\mathcal{A})) \ ; \tag{2.22}$$

this $\mathcal{R}$ satisfies the Yang-Baxter equation, (2.3), and

$$\mathcal{R}_{21}(u)\mathcal{R}_{12}(-u) = 1 \ .$$

Because T_u is rational, representations of $Y(\mathcal{A})$ give rise to rational matrix solutions of the YBE; because of (2.8) applied to (2.12), these R-matrices are group-invariant. The existence and form of such R-matrices will be the subject of chapter five.

2.5 Quantum Groups

The quantized universal enveloping algebra $U_q\mathcal{A}$, or quantum group, is defined as follows. Take a Lie algebra $\mathcal{A}$ in a Chevalley basis, and to every root assign three generators E_i^+, E_i^- and H_i. Then $U_q\mathcal{A}$ is given by

$$\begin{aligned} [H_i, H_j] &= 0 \\ [H_j, E_i^\pm] &= \pm 2\, \alpha_i.\alpha_j E_i^\pm \\ [E_i^+, E_j^-] &= \delta_{ij} \tfrac{q^{H_i}-q^{-H_i}}{q-q^{-1}} \quad (q \in \mathbf{C}) \\ [E_i^\pm, E_j^\pm] &= 0 \quad \text{if} \quad \alpha_i.\alpha_j = 0 \ , \end{aligned} \tag{2.23}$$

with

$$\begin{aligned} \Delta(H_i) &= 1 \otimes H_i + H_i \otimes 1 \\ \Delta(E_i^\pm) &= E_i^\pm \otimes q^{H_i/2} + q^{-H_i/2} \otimes E_i^\pm \\ \epsilon(E_i^\pm) &= \epsilon(H_i) = 0 \\ s(E_i^\pm) &= -q^{-\rho} E_i^\pm q^{-\rho} \quad \text{where} \quad \rho = \tfrac{1}{2}\textstyle\sum_{\alpha>0} H_\alpha \\ s(H_i) &= -H_i \end{aligned} \tag{2.24}$$

It can be seen immediately that

- the Cartan subalgebra is unchanged from that of the Lie algebra
- the algebra is non-cocommutative
- the $q \to 1$ limit gives the Lie algebra.

Note from (2.23) that $[E_i^+, E_j^-]$ is valued in the enveloping algebra rather than the algebra. We also need the q-analogue of the Serre relations,

$$\sum_{\nu=0}^{1-a_{ij}} (-1)^\nu \begin{bmatrix} 1-a_{ij} \\ \nu \end{bmatrix}_{q^i} \left(E_i^\pm\right)^{1-a_{ij}-\nu} E_j^\pm \left(E_i^\pm\right)^\nu q_i^{-\nu(1-a_{ij}-\nu)/2} = 0 \ , \tag{2.25}$$

where $q_i = q^{\alpha_i^2}$, $i \neq j$, $[A]_q = \frac{q^A - q^{-A}}{q-q^{-1}}$ and

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{[n]_q!}{[m]_q![n-m]_q!} \quad \text{where} \quad [n]_q! = [n]_q[n-1]_q...1 \ .$$

Once again, this algebra is a quasitriangular Hopf algebra with a universal R-matrix satisfying (2.7, 2.8). However, this algebra has no automorphism T_u, and so gives rise only to constant solutions of the YBE (2.9). However, if instead of $\mathcal{A}$ we use an affine Lie algebra, then associated with the distinguished dot on the Dynkin diagram we have an automorphism

$$T_u : E_0^+ \mapsto e^u E_0^+ \quad \text{and} \quad E_0^- \mapsto e^{-u} E_0^- \quad (u \in \mathbf{C}) \ . \tag{2.26}$$

Representations of quantized universal enveloping algebras of affine algebras thus give rise to trigonometric (exponential) solutions of the YBE.

The rational ($q \to 1$) limit of a $U_q\mathcal{A}^{(1)}$-invariant trigonometric R-matrix is the corresponding $Y(\mathcal{A})$-invariant R-matrix. Thus we might also expect $Y(\mathcal{A})$ to appear in the $q \to 1$ limit of the quantum group, and this is indeed the case[42]: taking $U_q\mathcal{A}$ and setting $q = 1 + \epsilon$, where ϵ is a small real parameter, with

$$H_i = H_i^{(0)} \ , \quad E_i^\pm = E_i^{(0)\pm} + i\epsilon E_i^{(1)\pm} \quad \text{and} \quad H_i^{(1)} = [E_i^{(0)+}, E_i^{(0)-}]\delta_{ij} \ ,$$

we obtain $Y(\mathcal{A})$, where generators with the superfix (0) correspond to the I_a and those with superfix (1) to the J_a; higher grade generators are then given at higher powers of ϵ. However, it is not clear to us how the rational limit of $\mathrm{U}_q\mathcal{A}^{(1)}$ corresponds to the Yangian: in other words, how the rational limit of the automorphism (2.26) corresponds to (2.21). This does not seem to be described anywhere in the literature.

It is known[18,19] that the representation theory of quantum groups for q not a root of unity is the same as that of Lie algebras. (If q is a root of unity things become very much more interesting, but such cases lie outside the scope of our discussion.) Thus solutions of (2.9) (and hence solutions of (2.10), braid group generators) exist[43] associated with all representations of Lie algebras. However, representations of the quantized affine algebras and of the Yangian, and hence solutions of the Yang-Baxter equation, are much rarer. A full list of the R-matrices so far constructed, and of the other representations in which they should exist, is given in an appendix.

There are restrictions on both the reducible and irreducible representations of Lie algebras in which we can construct R-matrices, and we shall be discussing these in chapter five. For the moment we make two points. Firstly, this problem can be rephrased as the Baxterization question of Jones[44]: When is it possible to extend a solution of the braid group relation to a solution of the Yang-Baxter equation? Secondly, there is a procedure for constructing R-matrices, discussed below and in chapter three, called the 'fusion procedure', and it turns out that this procedure allows construction of R-matrices in precisely those representations arrived at by other means in chapter five.

2.6 The fusion procedure and the bootstrap equations

The condition (2.8) can be rewritten as

$$[\check{R}(u), \Delta(a)] = 0 \quad \forall a \in \mathcal{H} \ ,$$

so that the R-matrix acting in irreducible representations V, V' of $\mathcal{H}$ may be written, by Schur's lemma, in the form

$$\check{R}_{VV'}(u) = \sum_{W \subset V \otimes V'} \tau_W(u) P_W \ , \tag{2.27}$$

where W are the irreducible components of $V \otimes V'$. (Note that this only applies to decompositions without multiplicities since, otherwise, R acts on the isomorphic components $v^1, .., v^r$ as a matrix $M_{\alpha\beta}(u)$ (where $\alpha, \beta = 1, .., r$), and M will not, in general, be diagonalizable.)

Now suppose that at some value u_0, all but one of the τ_W vanish, so that

$$\check{R}_{VV'}(u_0) = \tau_W(u_0) P_W \tag{2.28}$$

for some specific W. We can use this fact to construct new R-matrices. First, it may easily be verified that

$$\check{R}_{(V\otimes V')U}(u) = \left(\check{R}_{VU}(u+\beta) \otimes 1\right)\left(1 \otimes \check{R}_{V'U}(u+\alpha)\right) \tag{2.29}$$

$$\text{and} \quad \check{R}_{U(V\otimes V')}(u) = \left(1 \otimes \check{R}_{UV'}(u-\alpha)\right)\left(\check{R}_{UV}(u-\beta) \otimes 1\right) \tag{2.30}$$

define new R-matrices on $(V \otimes V') \otimes U$ and $U \otimes (V \otimes V')$. Now consider what happens if we choose $\beta - \alpha = u_0$. The Yang-Baxter equation then becomes

$$\begin{aligned} &\left(\check{R}_{V'U}(u+\alpha) \otimes 1\right)\left(1 \otimes \check{R}_{VU}(u+\alpha+u_0)\right)(P_W \otimes 1) = \\ &\qquad (1 \otimes P_W)\left(\check{R}_{VU}(u+\alpha+u_0) \otimes 1\right)\left(1 \otimes \check{R}_{V'U}(u+\alpha)\right) \quad , \end{aligned} \tag{2.31}$$

which acts on $V \otimes V' \otimes U$. The significance of (2.31) is that P_W passes through the R-matrices, so that we can now consistently restrict (2.29) to act in the space $W \otimes U$. We do this by defining

$$\check{R}_{WU}(u) = (1 \otimes P_W)\left(\check{R}_{VU}(u+\alpha+u_0) \otimes 1\right)\left(1 \otimes \check{R}_{V'U}(u+\alpha)\right) \tag{2.32}$$

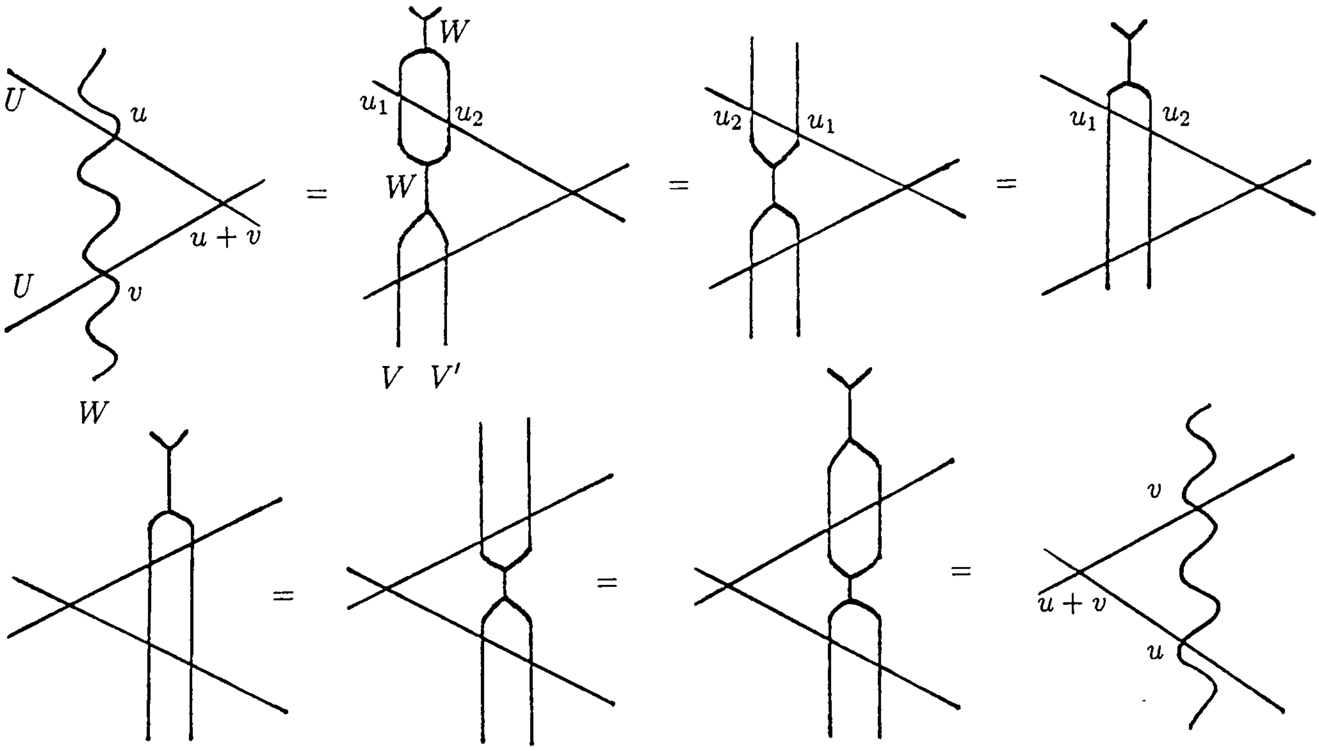
$$\text{and} \quad \check{R}_{UW}(u) = (P_W \otimes 1)\left(1 \otimes \check{R}_{UV'}(u-\alpha)\right)\left(\check{R}_{UV}(u-\alpha-u_0) \otimes 1\right) \ , \tag{2.33}$$

which can be shown to solve the YBE on $U \otimes W \otimes U$ using

$$\check{R}_{WU}(u)\left((1 - P_W) \otimes 1\right) = 0 \ .$$

The proof is given schematically in figure (2.3), in which straight lines correspond to U, V or V' as indicated, wavy lines to W, crossings of lines to R-matrices and fusions of lines to projectors.

Figure 2.3: Schematic proof that the fused R-matrix solves the YBE



where $u_1 = u + \alpha + u_0, \;\; u_2 = u + \alpha$

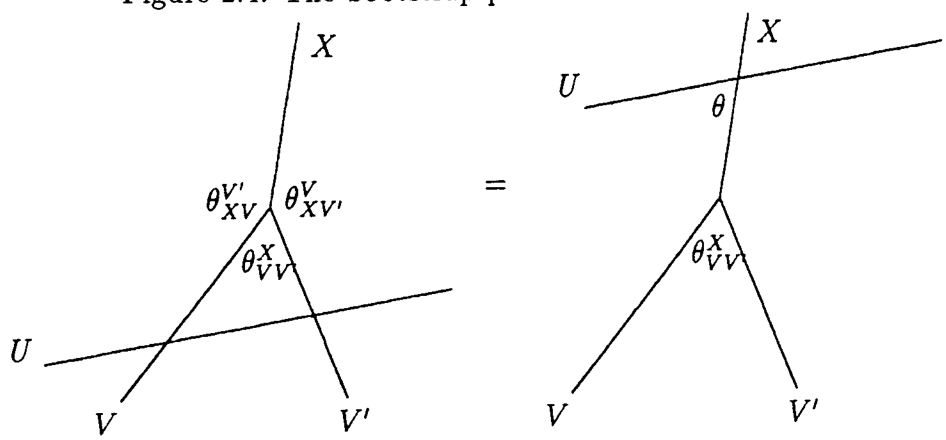
We see that (2.32) and (2.33) are now defined to act not on $V \otimes V' \otimes U$ and $U \otimes V \otimes V'$, but on $W \otimes U$ and $U \otimes W$, giving us new R-matrices. This process is called the fusion procedure[45]: we have taken known solutions of the YBE acting in irreducible representations of $\mathcal{H}$ and constructed from them another solution. The crucial point was the ability to choose u_0 to restrict $\check{R}$ to a particular W. It will be possible to do so only for some W, so that the fusion procedure can be used to construct R-matrices only in certain representations (*cf* the comment at the end of the last section). The procedure can, however, be generalized to construct R-matrices in reducible representations, as we shall see below and in chapter three. In fact, if R can be restricted for some value of u to any subset of the components of $V \otimes V'$, then an R-matrix may be constructed in the corresponding (reducible) representation.

Now compare this with the bootstrap procedure for S-matrices[46]. Suppose that, at some value of the rapidity difference, the two-particle S-matrix has a simple pole. The bootstrap principle says that we should consider the residue at this pole as a physical particle state, in the direct channel if the residue is positive and in the crossed (t-) channel if it is negative. Thus, suppose $S_{VV'}(\theta)$ has a pole at $i\theta_0$, and that

$$S_{VV'}(\theta) \sim \frac{1}{\theta - i\theta_0}\left(c_1 P_{W_1} + .. + c_r P_{W_r}\right)$$

for some numbers c_i. Then we should regard $X = W_1 \oplus ... \oplus W_r$ as a physical multiplet of equal-mass particles, and we say that there is a fusing $VV' \to X$. Now suppose we wish to calculate the S-matrix element $S_{UX}(\theta)$. For consistency of the bootstrap principle, this should be related to S_{UV} and $S_{UV'}$ as shown in figure (2.4), which is equivalent to

Figure 2.4: The bootstrap procedure for S-matrices



$$S_{UX}(\theta) = \left(1 \otimes S_{UV'}(\theta + i\bar{\theta}^{V}_{XV'})\right)\left(S_{UV}(\theta - i\bar{\theta}^{V'}_{XV}) \otimes 1\right)\Big|_X \quad , \tag{2.34}$$

where $\bar{\theta} = \pi - \theta$. This is effectively the same as (2.33): note that $\bar{\theta}^{V}_{XV'} + \bar{\theta}^{V'}_{XV} = \theta_0 \equiv \theta^{X}_{VV'}$. The value of θ_0 determines the mass of X via

$$m_V^2 + m_{V'}^2 + 2m_V m_{V'}\cos(\theta^{X}_{VV'}) = m_X^2 \quad . \tag{2.35}$$

Unlike (2.32, 2.33), (2.34) contains no free parameter α; it has been fixed by the requirement that, if a fusing $VV' \to X$ is allowed, then so are $V'\bar{X} \to \bar{V}$ and $V\bar{X} \to \bar{V}'$, so that applying (2.35) fixes α. We can describe all three fusings as being the result of a three-point coupling $< VV'X >$. In the R-matrix case, α may be fixed by the requirement of $\mathcal{H}$-invariance of $\check{R}_{UX}$, or equivalently by the requirement $\check{R}_{UX}(u)\check{R}_{XU}(-u) = 1$.

This is only the first step in the bootstrap program: we must now examine S_{UX} for simple poles¶ and once again treat its residues as physical particles. We thus continue adding new particles to the spectrum until eventually, we hope, all the simple poles in all the new S-matrices we have calculated correspond to particles already in the spectrum. The bootstrap is then said to have closed. The question of when and whether the bootstrap closes, given an initial set of particles and their S-matrices, is an unanswered one, but it is approached most clearly in those theories where the S-matrix is diagonal and the YBE trivial, since the S-matrix then consists solely of a phase. Such theories are known as purely elastic scattering theories or PESTs.

2.7 The structure of purely elastic scattering

When there is no mass degeneracy, no exchange of 'internal' quantum numbers, *i.e.* change of particle type within equal-mass multiplets, is possible‖. In the theories under consideration, the non-zero spin of the particles also forbids reflection. The only possible scattering process is then transmission, and the S-matrix is (for real θ) a pure phase. However, it is far from being trivial. Unitarity, analyticity, crossing symmetry and the bootstrap principle constrain the S-matrix severely. Such theories arise in two ways: as deformations of coset conformal field theories[48,49,50] (CFTs), in which the S-matrices depend only on θ, and as the affine Toda field theories[47,51](ATFTs), which depend in addition on a coupling constant β, taken to be real. That the two are deeply related was first realized by Eguchi and Yang[52] and by Hollowood and Mansfield[53].

As well as constraining the S-matrix, the bootstrap principle also imposes conditions on the conserved charges. The first of these is (2.35), or momentum conservation. In general, suppose there are further conserved charges Q_s of spins s (*i.e.* which take values $q_s^X e^{s\theta}$ on

¶Investigations of ATFTs[47] indicate that, in addition to the simple poles, physical states may also occur at higher order poles, which should therefore also be included in the bootstrap. As we shall see in the next chapter, this is also true of S-matrices with Yang-Baxter structure. The question of whether or not a given higher-order pole may be expected to correspond to a physical state has not yet been fully answered.

‖In fact, in such theories there are some mass degeneracies, but other conserved quantities serve to distinguish the particles uniquely.

a particle of type X and rapidity θ). The bootstrap principle, figure (2.4), may be imposed on the charges in exactly the same way as it was imposed on the S-matrices, and gives

$$q_s^X = q_s^V e^{-is\bar{\theta}_{XV}^{V'}} + q_s^{V'} e^{is\bar{\theta}_{XV'}^{V}} \ . \tag{2.36}$$

It can easily be imagined that it will be very difficult to find sets of conserved charges and S-matrices satisfying the bootstrap[54]. Those which have been found have a beautiful description[25,26] in terms of root systems of Lie algebras, a short account of which we give here.

There is a (β-independent) solution associated with each simply-laced Lie algebra (associated with the Toda theory corresponding to the untwisted affine algebra). We can assign each particle of the theory to one of the r dots on the Dynkin diagram of the algebra. The spins s of the conserved charges are equal to the exponents of the algebra modulo the Coxeter number h; for each s, the q_s^X form an eigenvector of the Cartan matrix[55,56]. In particular, for $s = 1$ they form the Perron-Frobenius eigenvector, whose entries are all positive and give the masses of the particles. The three point couplings are then given[25] by

$$< XYZ > \neq 0 \quad \Leftrightarrow \quad \exists \text{ roots } \ \alpha \in R_X, \beta \in R_Y, \gamma \in R_Z \ \text{ with } \ \alpha + \beta + \gamma = 0 \ , \tag{2.37}$$

where R_X is the orbit of (either plus or minus) the simple root associated with X under the action of a particular Coxeter element of the Weyl group. We have rather simplified this, but the essential point is that $\alpha + \beta + \gamma = 0$ is an equation in r dimensions; the equations (2.36) are then given by projecting down onto particular planes in root space. The full complexity of the bootstrap equations for the charges has thus been incorporated into a statement about the r dimensional root space of a Lie algebra.

For non-simply-laced algebras the situation is less clear. For the untwisted algebras, solutions to the charge bootstrap may be obtained as above, but the construction and interpretation of S-matrices which correctly describe the quantum ATFT is still in progress[57]. In fact the solutions for each of the non-simply-laced algebras may be obtained by truncating the spectrum for a simply-laced algebra using an automorphism of the Dynkin diagram. It is also possible in this way to obtain solutions associated with twisted affine algebras (and with the corresponding Toda theory) by using the additional (extended) diagram symmetries of the affine simply-laced algebras: such solutions correspond to a subset of the masses (and their fusings) of the parent theory, but once again a consistent description of the quantum ATFT is lacking. For full details see Corrigan *et al.* [47]

We do not give here the full description of the S-matrices[47,25], noting instead that all β-independent S-matrices which solve the bootstrap relations are given in terms of the 'building blocks'

$$\{x\} = (x-1)(x+1) \ , \tag{2.38}$$

where

$$(x) \equiv \frac{\sinh(\frac{\theta}{2} + \frac{i\pi x}{2h})}{\sinh(\frac{\theta}{2} - \frac{i\pi x}{2h})} \ . \tag{2.39}$$

The S-matrices built from (2.38) are believed to describe certain integrable deformations of CFTs[49]. For the affine Toda theories, however, β-dependence must be introduced into the S-matrices. For the simply-laced theories this may be done by replacing this building block with

$$\{x\} = \frac{(x-1)(x+1)}{(x-1+B)(x+1-B)} \ ,$$

where B is a universal function of a coupling constant β, conjectured (initially for the a_n case[58]) to be

$$B(\beta) = \frac{1}{2\pi}\frac{\beta^2}{1+\frac{\beta^2}{4\pi}} \ .$$

The full S-matrix is obtained by raising the blocks to powers of inner products of weights associated with the particles, and can be represented in terms of vertex operators[59]. This then gives the same spectrum of masses and fusings as in the S-matrices constructed from (2.38), and agrees with the perturbation theory[47,51].

For the non-simply-laced affine Toda theories, the masses renormalize differently at one-loop order in the perturbation theory, so that the spectrum is not the same as in the coupling-constant-independent case. It turns out that S-matrices can however be constructed: one needs to replace h in the expressions above by a new non-integer 'renormalized Coxeter number' $H(\beta)$[57]. However, the structure of the bootstrap is then very unclear, since some S-matrix poles are shifted in such a way that they no longer correspond to particles in the canonical spectrum.

Throughout this work we wish to emphasize the apparent universality of the structure of the bootstrap in integrable theories. It is for this reason that we feel it inappropriate to give details of the underlying field theories in this section: for the YBE-dependent S-matrices, for example, the principal chiral field ($G \times G$-invariant non-linear σ-model), Gross-Neveu and generalized chiral Gross-Neveu models, and for the diagonal S-matrices, the deformed CFTs and real-coupling ATFTs. Details of the relevant Lagrangians are given in an appendix.

The point which we shall be making in the next chapter is that the mass spectra and fusing rules, and certain parts of the S-matrix, seem to be the same in the full YBE-dependent S-matrices as in the PESTs. It is much harder to solve the former; it involves a detailed analysis of the classification of solutions of the YBE. However, our results will suggest that the PESTs and their algebraic description, and the solutions of the YBE and theirs, are deeply linked. Furthermore, it seems likely that the bootstrap extends into classical physics: recent work[60] using Hirota's method to obtain soliton solutions of imaginary-coupling a_n ATFTs finds a mass spectrum and fusing structure for these solitons identical to that of the bootstrap.

A few bibliographical notes are in order. Factorized S-matrices were first described in detail by the Zamolodchikovs[20,32], and useful background on the structure of S-matrices in four dimensions can be found in the book by Eden, Landshoff, Olive and Polkinghorne[33]. The Yang-Baxter equation became known as such because of the pioneering works on factorized S-matrices by Yang[37] and on integrable lattice models by Baxter[10]. A comprehensive review of the YBE is given by Jimbo[35], and some useful algebraic background for the non-specialist is given by Majid[61].

The fusion procedure was introduced by Karowski[46] for (YBE dependent) S-matrices and by Kulish, Sklyanin and Reshetikhin[45] for R-matrices, the latter being the first example of its explicit use. The interest in PESTs arose from Zamolodchikov's work[48,49] on deformations of CFTs, whilst the same structures were seen in affine Toda field theories by Braden, Corrigan, Dorey and Sasaki[47,62] and by Christe and Mussardo[51,63]. Recognition of the algebraic structure of PESTs came from Dorey[25] and others[55,56,64,65,66].

Many of the original articles are contained in the collection[28] 'Yang-Baxter Equation in Integrable Systems', whilst useful introductory and review articles can be found[67] in 'Braid Group, Knot Theory and Statistical Mechanics'.

Chapter 3

SO(N)-invariant factorized S-matrices

Consider the fusing rule (2.37) for the PESTs. This is in fact very similar to, but not the same as[68,66], the Clebsch-Gordan (CG) decomposition of the tensor products of fundamental representations of Lie algebras. Specifically, it is found that when there is a fusion $XY \to Z$ in the Toda theory, then $\rho_Z \subset \rho_X \otimes \rho_Y$. (Here, ρ_X is the fundamental representation associated to the same spot on the Dynkin diagram as X.) However, the reverse implication is not always true. The first example of this is in the $d_n^{(1)}$ PESTs, where there is a self-coupling of the second particle (associated with the rank two antisymmetric representation) only for $n = 4$, even though $\rho_2 \subset \rho_2 \otimes \rho_2$ for all n. Such behaviour does not occur in the a_n theories, where the Clebsch-Gordan decomposition is followed by the PEST fusings.

Now, it has been stated[23] that the fusings of Yang-Baxter dependent factorized S-matrices follow the CG decomposition precisely. To see how this would fit in with the fusion procedure, we first determine the representation in which particle multiplet X transforms. The particle multiplet X *is not necessarily the fundamental representation itself*: it may actually be a reducible representation containing the fundamental representation ρ_X as an irreducible component. Then, whenever the S-matrix of two such particles X and Y contains a third particle multiplet $Z \subset X \otimes Y$, it would be expected that there might be a pole in the S-matrix whose residue would be the particle multiplet Z. In the language of the fusion procedure in section (2.3), we may be able to find a u_0 to restrict the R-matrix to Z.

By calculating the S-matrix element for the second particle interacting with itself, we will show that this is not the case: that there is no $2\,2 \to 2$ fusing for $n \neq 4$, even though $\rho_2 \subset \rho_2 \otimes \rho_2$. The 'hole' in the CG decomposition for PEST fusings is also found in YBE dependent S-matrices.

We shall be working with rational R-matrices. Applying (2.22) with $x = I_a$, we see that these are group invariant, so that the decomposition (2.27) applies with the W as irreducible representations of Lie groups. Our basic building block will be the R-matrix in the vector representation V of $SO(N)$ found by the Zamolodchikovs[20],

$$\left(\check{R}_{11}(u)\right)^{ac}_{bd} = \frac{1}{2-u}\left(2\delta_{ab}\delta_{cd} - u\delta_{ad}\delta_{cb} + \frac{2u}{N-2-u}\delta_{bd}\delta_{ac}\right) \quad , \tag{3.1}$$

where $\check{R}_{11}$ acts on $V \otimes V$ by

$$\left[\check{R}_{11}(u)(v \otimes w)\right]^{ac} = \left(\check{R}_{11}(u)\right)^{ac}_{bd} v_b w_d \ .$$

This can be re-expressed in form (2.27) as

$$\begin{aligned}\check{R}_{\square\square}(u) &= \mathbf{P}\left\{P_{\boxtimes\boxtimes} + \left(\frac{u+2}{u-2}\right)P_{\substack{\square\\\square}} + \left(\frac{u+2}{u-2}\right)\left(\frac{u+N-2}{u-N+2}\right)P_0\right\} \\ &= \mathbf{P}(P_{\boxtimes\boxtimes} + [2]\,P_{\substack{\square\\\square}} + [2]\,[N-2]\,P_0)\end{aligned} \tag{3.2}$$

where $\mathbf{P} : u_a v_c \mapsto v_a u_c$ is the transposition operator of factors in a tensor product, $P_{\boxtimes\boxtimes}$ and $P_{\substack{\square\\\square}}$ are the second-rank symmetric traceless and antisymmetric tensors, P_0 is the trace operator, and

$$[a] \equiv \frac{u+a}{u-a} \ . \tag{3.3}$$

We have also introduced extended Young tableaux notation for $SO(N)$, in which a trace may be removed from symmetric indices, indicated by cross-hatching. Note that

$$\check{R}_{\square\square}(u)\check{R}_{\square\square}(-u) = 1 \quad ,$$

and that $\check{R}_{\square\square}(0) = 1$ and $\check{R}_{\square\square}(\infty) = \mathbf{P}$. Also $\mathbf{P}P_{\boxtimes\boxtimes} = P_{\boxtimes\boxtimes}$, $\mathbf{P}P_{\substack{\square\\\square}} = -P_{\substack{\square\\\square}}$, and $\mathbf{P}P_0 = P_0$. (Note also that $\check{R}(-2) = \mathbf{P}P_{\boxtimes\boxtimes}$, so that we could expect to be able to calculate $\check{R}_{\square\boxtimes\boxtimes}$ and $\check{R}_{\square\square\square}$ using the fusion procedure. We shall do this in the next chapter, but for the moment we are interested only in S-matrices, and $\boxtimes\boxtimes$ does not correspond to a particle multiplet, for reasons which we shall explain shortly.)

We can turn $\check{R}_{\square\square}$ into an S-matrix which is unitary, crossing-symmetric and has no poles in the physical strip by defining

$$S_{11}^{min}(\theta) = h(u)\check{R}_{\square\square}(u)$$

(where the min means 'minimal', referring to the lack of poles). The function $h(u)$ must be such that $h(u)h(-u) = 1$ and, setting $u = \frac{C\theta}{i\pi}$ (where $C = N - 2$, the crossing point), must satisfy

$$h(C-u) = h(u)\frac{u(2-C+u)}{(2-u)(C-u)} \ ,$$

which follows from requiring crossing-symmetry in the form

$$h(u)R^{ac}_{bd}(u) = h(C-u)R^{cd}_{ab}(C-u) \ .$$

Finally we need $h(u)$ to have no poles in the physical strip $0 \leq \mathrm{Re}\, u \leq C$, and zeroes at $u = 2$ and $u = C$ to cancel the simple poles in $R_{\square\square}$. Such a function can be found and is given[20,23] in terms of Euler's gamma function by

$$h(u) = \frac{\Gamma(\frac{1}{2}+\frac{u}{2C})\Gamma(\frac{u+2}{2C})\Gamma(\frac{1}{2}+\frac{2-u}{2C})\Gamma(\frac{-u}{2C})}{\Gamma(\frac{1}{2}-\frac{u}{2C})\Gamma(\frac{2-u}{2C})\Gamma(\frac{1}{2}+\frac{2+u}{2C})\Gamma(\frac{u}{2C})} \ .$$

S^{min}_{11} is the conjectured exact two-particle S-matrix for the $O(N)$ σ-model, which is expected (from $\frac{1}{N}$-expansion considerations) to have a spectrum consisting solely of an $O(N)$ vector multiplet of equal mass particles. However, it can also be used to construct a conjectured exact S-matrix for the Gross-Neveu model and for the principal chiral model. The former has $O(N)$ invariance, whilst the latter (defined on the group manifold of $SO(N)$) has $SO(N) \times SO(N)$ invariance (for general G, $G \times G$ invariance). Thus their S-matrices must also have this invariance: those of the former will be built from S^{min}_{11}, those of the latter from $S^{min}_{11} \times S^{min}_{11}$.

Unlike the $O(N)$ σ-model, these models *are* expected to have non-trivial bound state structure. Unitarity and crossing symmetry leave so-called CDD[34] ambiguities in the S-matrix, which can be used to provide this structure. The CDD ambiguities are scalar functions $X(\theta)$ such that $X(\theta)X(-\theta) = 1$ and $X(\theta) = X(i\pi - \theta)$ which are fixed by knowledge of the bound states of S. The full S-matrix is then

$$S(\theta) = X(\theta)S^{min}(\theta) \ ;$$

all of the physical pole structure is contained in the functions $X(\theta)$.

To fix $X_{11}(\theta)$, first note that at $\theta = \frac{2i\pi}{N-2}$ $(u = 2)$, S^{min}_{11} can be restricted to act in the representation $\boxminus \oplus 0$. We shall assume that this corresponds to a new multiplet, which we shall label particle two. Note that since θ is a rapidity, this implies that

$$2m_1^2\left(1+\cos\left(\frac{2\pi}{N-2}\right)\right) = m_2^2$$

(*cf* equation (2.35) in chapter two). We wish the CDD factor to express the fact that two 1-particles have a 2-particle as a bound state at $\theta = \frac{2i\pi}{N-2}$. A suitable factor is then

$$X_{11}(\theta) = (2)(N-4)\ , \tag{3.4}$$

where we have introduced the notation of (2.39). The (2) has been introduced to give the 2-particle bound state whilst the $(N-4)$ ensures crossing symmetry and gives the equivalent bound state in the crossed channel. Thus the full S-matrix is

$$S_{11}(\theta) = (2)(N-4)h(u)\,\check{R}_{\square\square}(u)\ .$$

(This should already be suggestive - in fact the CDD factors of the S-matrices of the principal chiral model are precisely the β-independent S-matrices of the PESTs; it was through this observation that the link between the two types of S-matrix first presented itself.)

An important point to note is that, although $\check{R}(-2) = \mathbf{P}P_{\boxtimes}$, this representation cannot correspond to a physical particle under the bootstrap principle, since crossing symmetry has fixed $u = \frac{C\theta}{i\pi}$, so that $u = -2$ lies outside the physical strip.

3.1 S-matrix fusion and Brauer's algebra

We shall now use the fusion procedure to obtain the S-matrices for scattering a 1-particle with a 2-particle, and for scattering 2-particles with themselves. Hence we shall be using $\mathrm{Res}\check{R}_{\square\square}(2) \propto P_{\boxminus} + (\frac{N}{N-4})P_0$ to define $\check{R}_{\square(\boxminus\oplus 0)}$. In analysing the resulting S-matrices, we shall extend the idea used for the 2-particle, and identify the n-particle with a certain reducible representation. Specifically, suppose the S-matrix projects for some θ onto a reducible representation whose components are various irreducible tensors, and that the only rank-n tensor among these components is the nth fundamental representation. Then we identify this space with the n-particle. In this way, we would expect to finish with r particles, where r is the rank of the algebra[22,23].

We first describe the fusion procedure as applied to S^{min}_{11}. Afterwards, we shall explain how the fusion procedure works on the full S-matrix, describing how the structure of the new S^{min} links up with the poles of the new (fused) CDD factors. We define

$$S^{min}_{12}(\theta) = h(u+1)h(u-1)\check{R}_{\square(\boxminus\oplus 0)}(u) \ , \tag{3.5}$$

where

$$\check{R}_{\square(\boxminus\oplus 0)}(u) = \Big((P_{\boxminus} + P_0)\otimes 1\Big)\Big(1\otimes \check{R}_{\square\square}(u+1)\Big)\Big(\check{R}_{\square\square}(u-1)\otimes 1\Big) \tag{3.6}$$

To see that this solves the YBE, we first introduce the convenient notation

$$Q(x) \equiv P_{\boxminus} + \left(\frac{N}{N-4}\right)^{x} P_0$$

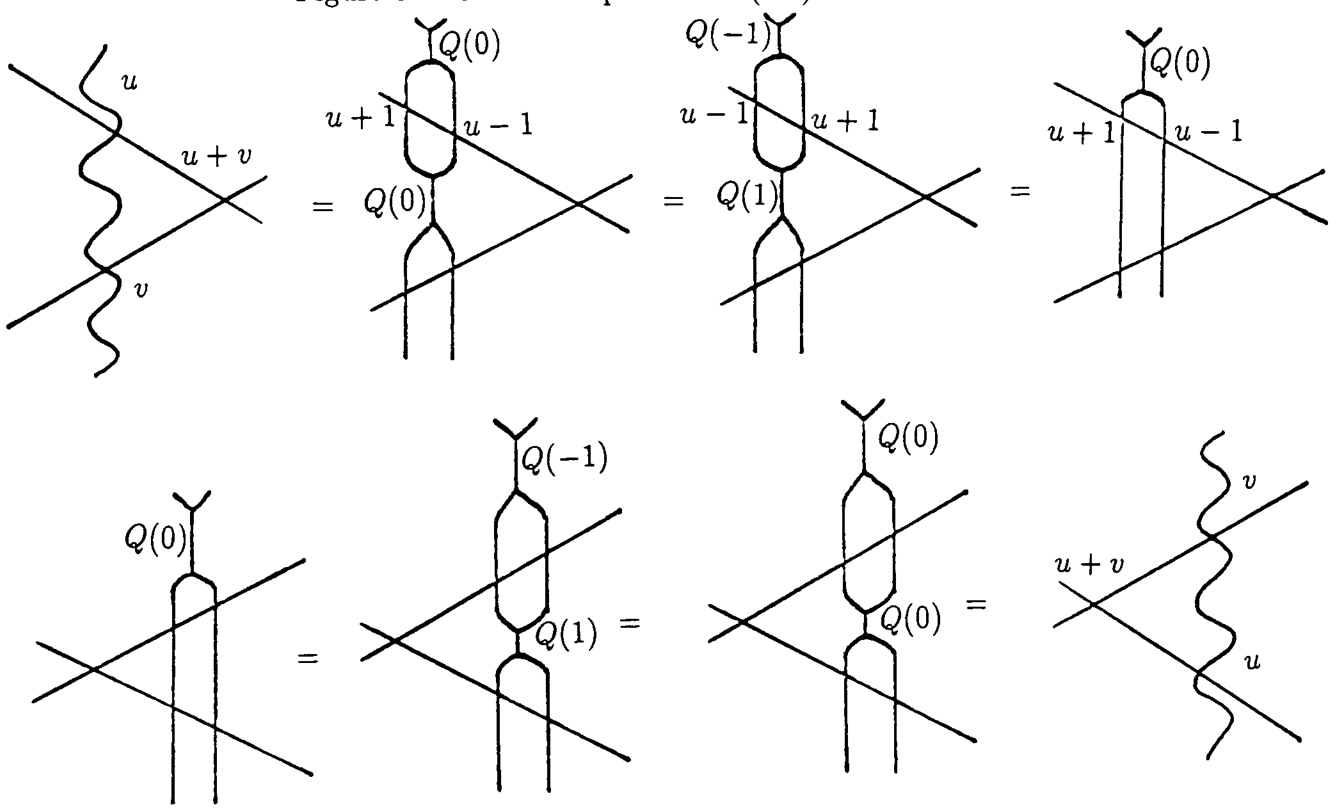
which satisfies $Q(x)Q(y) = Q(x+y)$. Then $\text{Res}\check{R}_{\square\square}(2) \propto P_{\boxminus} + (\frac{N}{N-4})P_0 = Q(1)$, so that

$$\begin{aligned}
\check{R}_{\square(\boxminus\oplus 0)}(u) &= (Q(0)\otimes 1)\Big(1\otimes \check{R}_{\square\square}(u+1)\Big)\Big(\check{R}_{\square\square}(u-1)\otimes 1\Big) \\
&= (Q(-1)\otimes 1)\Big(1\otimes \check{R}_{\square\square}(u-1)\Big)\Big(\check{R}_{\square\square}(u+1)\otimes 1\Big)(1\otimes Q(1)) \\
&= (Q(0)\otimes 1)\Big(1\otimes \check{R}_{\square\square}(u+1)\Big)\Big(\check{R}_{\square\square}(u-1)\otimes 1\Big)(1\otimes Q(0))
\end{aligned}$$

$$\Rightarrow \quad \check{R}_{\square(\boxminus\oplus 0)}(u)\Big(1\otimes\Big(1-(P_{\boxminus}+P_0)\Big)\Big) = 0 \ , \tag{3.7}$$

from which it is easy to show that (3.5), which is unitary and crossing symmetric, also solves the YBE. This is most easily proved schematically, as shown in figure (3.1). In this figure, straight lines correspond to $\square$, wavy lines to $\boxminus \oplus 0$, crossings of lines to R-matrices, and fusions of lines to Q as indicated. The problem now is to find an explicit way to calculate this R-matrix, and to give its spectral decomposition. The expression with which we must deal is (3.1), and we clearly face an unpleasant proliferation of Kronecker deltas and indices. The solution is essentially just a change of notation. This change will allow us to computerize the fusion procedure so that S^{min}_{22} (our next goal) can also be calculated - a feat not possible with calculation by hand.

Figure 3.1: Schematic proof that (3.7) solves the YBE



The basic object $(\check{R}_{11})^{ac}_{bd}$ is composed of three tensors, $\delta_{ab}\delta_{cd}$ (identity), $\delta_{ad}\delta_{cb}$ (permutation) and $\delta_{ac}\delta_{bd}$ (trace). For example, the second rank symmetric traceless tensor of $SO(N)$ is constructed from two vectors u_a, v_c as

$$[P_{\square\square}(u \otimes v)]_{ac} = \frac{1}{2}\left(u_a v_c + u_c v_a\right) - \frac{1}{N}\delta_{ac} u_b v_b = \left[\frac{1}{2}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} + \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) - \frac{1}{N} \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right] u_b v_d \ .$$

In the latter expression we have represented a Kronecker delta on two indices by joining them with a line, introducing a diagrammatic representation of the identity, transposition and trace operators which may be seen simply as a neat way to perform the appropriate index contractions.

When we construct a fused R-matrix using formulae such as (3.6), we must compute products of operators, each of which consists of $\check{R}_{11}$ acting on two of the vectors, whilst the others are left unchanged. For example, the operation $(\check{R}_{11} \otimes 1)^{ace}_{bdf}$, mapping $u_b v_d w_f$ to $u_a v_c w_e$, is composed of the three tensors $\delta_{ab}\delta_{cd}\delta_{ef}$, $\delta_{ad}\delta_{bc}\delta_{ef}$ and $\delta_{ac}\delta_{bd}\delta_{ef}$. Products of these operations are computed by contracting indices appropriately. In general, for a fused

R-matrix defined on $\square^{\otimes n}$, mapping $u^1_{b_1} u^2_{b_2} \ldots u^n_{b_n}$ to $u^1_{a_1} \ldots u^n_{a_n}$, the R-matrix is composed of products of Kronecker deltas $\delta_{a_i b_j}$. Extending the above notation, a convenient way to represent such products is with an $n \times 2$ array of points, joining each point to another in a way corresponding to the contraction of indices. Examples of how products of δs are represented using these symbols are shown in figure (3.2). Multiplying tensors corresponds

Figure 3.2: $\delta_{a_1 b_2} \delta_{a_2 a_3} \delta_{b_1 b_3}$ $\delta_{a_1 b_1} \delta_{a_2 b_3} \delta_{a_3 b_2}$

to concatenating these symbols, and multiplying by the dimension, N, for each closed loop created (since $\delta_{ab}\delta_{ab} = N$). There are clearly $(2n-1)(2n-3)\ldots 5.3 \equiv k$ such symbols; let us call the algebra linearly generated by them $\mathcal{B}_n(N)$. Although it has been introduced in this work as a notational convenience, this algebra is well known[69] to mathematicians*: it is Brauer's algebra, introduced by Brauer[70] in 1937 as the centralizer algebra of the orthogonal and symplectic groups (and subsequently labelled 'somewhat enigmatic' by Weyl). It has been the subject of renewed interest recently because of its relation to the new algebras associated with knot polynomials (which we shall explain in the next chapter). It contains the symmetric group as a subalgebra, $\mathcal{S}_n \subset \mathcal{B}_n(N)$, corresponding to those symbols in which all lines cross from left to right (such as the symbol on the right in figure (3.2)). A useful introduction to diagrammatic techniques in group theory is given by Cvitanovic[71,72].

For $n = 2$ and 3, it is possible to perform calculations in the algebra by hand; one obtains pages of hieroglyphic-like expressions. However, for $n = 4$ one requires a computerizable algorithm, and manipulation for $n = 3$ is also made easier by the use of such an algorithm. The basis of the algorithm is to number the symbols from 1 to k, and then express each term in the fused R-matrix in terms of these basis elements $e_1, e_2, \ldots e_k$.

We must first calculate how the basis elements multiply (*i.e.* how the symbols join). We store the result of joining symbol i to symbol j in the i,jth element of a $k \times k$ array

*I am grateful to H. Morton for pointing this out to me

f, whilst we store the number of loops created in an array l. Then we have

$$e_i e_j = N^{l_{ij}} e_{f_{ij}}.$$

(Note that $f_{ij} \neq f_{ji}$.) This calculation can easily be performed by hand for $n = 3$ ($k = 15$) but must be done by computer for $n = 4$ ($k = 105$). This was done using FORTRAN, after which we used REDUCE to calculate products in the form

$$\left(\sum_{i=1}^{k} s_i(u) e_i\right) \left(\sum_{j=1}^{k} t_j(u) e_j\right) = \sum_{i,j=1}^{k} s_i(u) t_j(u) N^{l_{ij}} e_{f_{ij}}, \tag{3.8}$$

at each stage storing the result in a k-entry column. This puts the fused R-matrices into the form

$$\check{R}(u) = \sum_{i=1}^{k} r_i(u) e_i$$

where each $r_i(u)$ is a rational function of u. We then need to know which combinations of the e_i form orthogonal projectors corresponding to which representations. For example, the rank n totally antisymmetric representation is given by

$$P = \frac{1}{n!} \sum_{i=1}^{n!} (-1)^{\epsilon(\sigma_i)} e_{k(\sigma_i)}$$

where $e_{k(\sigma_i)}$ corresponds to the ith permutation of the n indices, $\delta_{a_1 b_{\sigma_i(1)}} \dots \delta_{a_n b_{\sigma_i(n)}}$, and $\epsilon(\sigma_i)$ is the signature of the permutation. We can check the orthogonality and projection of our candidate projectors by multiplying them together using (3.8). Using REDUCE on SUN systems, computer time for performing operations (3.8) is of the order of seconds for $n = 3$, minutes or hours for $n = 4$, and is prohibitive for $n = 5$. Time to calculate R-matrices grows roughly as nk^2, since such a calculation requires of the order of n iterations of (3.8), and performance of (3.8) grows as k^2.

3.2 Structure of the fused S-matrices

Computing and decomposing (3.5) in this way, we obtain

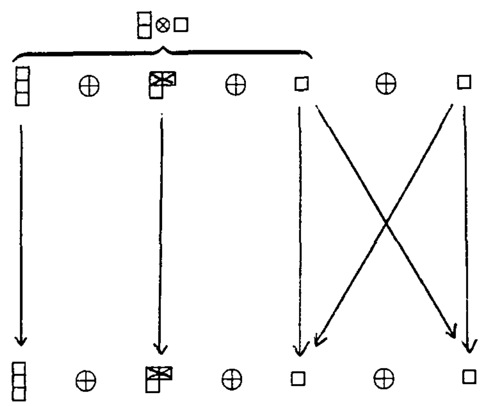
$$S_{12}^{min}(\theta) = \mathbf{P} h(u-1) h(u+1) [1] \left[P^{22}_{\yng(2,1)} + [3] P^{22}_{\yng(1,1,1)} + [3][N-3] (P^{22}_{\yng(1)} + P^{00}_{\yng(1)}) \right.$$

$$
- \frac{2}{(u-3)(u-N+3)}\left((N-1+u)P^{22}_{\square} + (N-1)(u+1)P^{00}_{\square}\right)
+ \; 4\left(P^{20} + \frac{N-4}{N}P^{02}\right)\Big] \qquad (3.9)
$$

In this expression, [a] is defined as in (3.3), the P^{22} terms are the projectors onto irreducible components of $\square \otimes \boxminus$, $P^{00}_{\square}$ is the projector onto $\square \otimes 0$, and P^{02} and P^{20} describe not projectors but intertwiners, $P^{02} : \square \otimes \boxminus \rightarrow \square \otimes 0$ and $P^{20} : \square \otimes 0 \rightarrow \square \otimes \boxminus$, both given the same normalization so that $P^{20}P^{02} = 2N(N-1)P^{22}_{\square}$. In terms of Brauer's algebra, $\mathbf{P}$ and P^{02} are as shown in figure (3.3). Although (3.9) is at first sight somewhat more

Figure 3.3: (a) $\mathbf{P}$ for R_{12} (b) $\mathbf{P}P^{02}$ of S^{min}_{12}

complicated than in the irreducible case, we know from (2.27) that group invariance allows maps between irreducible components only when they are isomorphic. Thus we find that $P^{02}P^{22}_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}} = P^{22}_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}P^{20} = P^{02}P^{22}_{\begin{smallmatrix}\boxtimes&\boxtimes\\\square\end{smallmatrix}} = P^{22}_{\begin{smallmatrix}\boxtimes&\boxtimes\\\square\end{smallmatrix}}P^{20} = 0$, so that we can describe how S^{min}_{12} maps $(\boxminus \oplus 0) \otimes \square$ into itself with the following diagram:



The way we have chosen to write (3.9) is to some extent arbitrary. We have used the [a] notation as far as possible, but the only significant content of the maps between the $\square$ components is where the poles occur (*i.e.* the fact that the numerator of the S-matrix has cancelled the expected pole at $u = N - 1$), and the fact that $S^{min}(\infty) = \mathbf{P}$. However, it will always be possible to rewrite (3.9) in the form

$$S_{12}^{min}(\theta) = \mathbf{P}h(u-1)h(u+1)[1]\left(T_1 + [3]T_2 + [3][N-3]T_3\right) \;, \tag{3.10}$$

where the Ts, although not projectors, are (u-independent) expressions in $\mathcal{B}_3$, combinations of the projectors and intertwiners. (This is because, for each $i = 1, ..., k$, the numerator of the R-matrix is a quadratic polynomial in u the coefficients of which fix $(T_1)_i, (T_2)_i$ and $(T_3)_i$.) An analysis of the algebra of the Ts gives no insight into their meaning, so that writing (3.9) in form (3.10) does not appear likely to be productive.

We note that at $u = 3$, $P_{\begin{smallmatrix}\Box\Box\\\Box\end{smallmatrix}}$ disappears, and S^{min} acts on $\boxminus \oplus \Box \oplus \Box$. In accordance with our earlier comments, we identify this space with the third particle, and so expect a $12 \to 3$ fusing to be reflected in a pole at $u = 3$ in the CDD factor. For $N = 6$, the values 3 and $N - 3$ coincide, and so two zeroes of $h(u-1)h(u+1)$ coincide, dominating the simple pole in $\check{R}_{\Box(\boxminus\oplus 0)}$ in the $12 \to 3$ channel, and allowing only the $12 \to 1$ fusing. Without the $12 \to 3$ channel, the fusion procedure must terminate at this level.

Now we compute S_{22}^{min}. Using (3.7) we find that a unitary, crossing symmetric S-matrix satisfying (2.2) is

$$S_{22}^{min}(\theta) = h\,(u-1)\left(h(u)^2\right)h\,(u+1)\,\check{R}_{(\boxminus\oplus 0)(\boxminus\oplus 0)}(u)$$

where

$$\begin{aligned}
\check{R}_{(\boxminus\oplus 0)(\boxminus\oplus 0)}(u) &= \left(1\otimes(P_{\boxminus}+P_0)\right)\left(\check{R}_{\Box(\boxminus\oplus 0)}(u+1)\otimes 1\right)\left(1\otimes\check{R}_{\Box(\boxminus\oplus 0)}(u-1)\right) \\
&= \left((P_{\boxminus}+P_0)\otimes(P_{\boxminus}+P_0)\right)\left(1\otimes\check{R}_{\Box\Box}(u+2)\otimes 1\right) \\
&\qquad \left(\check{R}_{\Box\Box}(u)\otimes\check{R}_{\Box\Box}(u)\right)\left(1\otimes\check{R}_{\Box\Box}(u-2)\otimes 1\right)
\end{aligned}$$

where the first line acts on $\Box\otimes\Box\otimes(\boxminus\oplus 0)$ and the second on $\Box^{\otimes 4}$.

Upon decomposing this using our algorithm we obtain:

$$
\begin{aligned}
S_{22}^{min}(\theta) \;=\;& \mathbf{P}h\,(u-2)\left(h(u)^2\right)h\,(u+2)\,[2]\Big[P_{\substack{\boxtimes\boxtimes\\ \boxtimes\boxtimes}} + [2]\,[N-4]\,P_{\boxtimes\boxtimes} + [2][4][N-4][N-2]P_0 \\
&+\; [2]P_{\substack{\boxtimes\boxtimes\\ \square\\ \square}} + [2][4][N-4]P_{\substack{\square\\ \square}} + [2][4]P_{\substack{\square\\ \square\\ \square\\ \square}} - \frac{4u(N+u)}{(u-N+4)(u-2)(u-4)}P_{\substack{\square\\ \square}} \\
&-\; \frac{4\left(u^3+2(N-1)u^2+(N^2-4N+4)u-24N\right)}{(u-N+4)(u-N+2)(u-2)(u-4)}P_0 \\
&+\; \frac{2u}{(u-N+4)(u-2)(u-4)}\left({}^{2}_{2}P^{2}_{0} + {}^{2}_{2}P^{0}_{2} + \frac{N-4}{N}\left({}^{0}_{2}P^{2}_{2} + {}^{2}_{0}P^{2}_{2}\right)\right) \\
&+\; \frac{8(N-4)u}{(u-N+4)(u-N+2)(u-2)(u-4)}\left(\frac{N}{N-4}{}^{2}_{2}P^{0}_{0} + \frac{N-4}{N}{}^{0}_{0}P^{2}_{2}\right) \\
&-\; \frac{u\left((u-2)N-u^2-2u+16\right)}{(u-N+4)(u-2)(u-4)}\left({}^{2}_{0}P^{2}_{0} + {}^{0}_{2}P^{0}_{2}\right) - \frac{8(N-4)}{(u-N+4)(u-2)(u-4)}\left({}^{2}_{0}P^{0}_{2} + {}^{0}_{2}P_{0}\right. \\
&+\; \left([2][4][N-2][N-4] - \frac{u(N-1)(8N+4u^2-8u-32)}{(u-N+4)(u-N+2)(u-2)(u-4)}\right){}^{0}_{0}P^{0}_{0}\Big] \qquad (3.1
\end{aligned}
$$

Here $P_{\substack{\boxtimes\boxtimes\\ \boxtimes\boxtimes}}$, ... P_0 describe the projectors in $\substack{\square\\ \square} \otimes \substack{\square\\ \square}$. The ${}^{i}_{k}P^{j}_{l}$ terms, with $ijkl = 0$ or 2, describe intertwining maps ${}^{i}_{k}P^{j}_{l} : W_j \otimes W_l \rightarrow W_i \otimes W_k$ where $W_0 = 0$, $W_2 = \substack{\square\\ \square}$, and have been normalized as follows:

i) The terms with two of $ijkl$ equal to two and two equal to zero are given the same normalization such that

$$
{}^{0}_{2}P^{0}_{2}\,{}^{0}_{2}P^{0}_{2} = {}^{0}_{2}P^{2}_{0}\,{}^{2}_{0}P^{0}_{2} = {}^{0}_{2}P^{0}_{2} \quad , \quad {}^{2}_{2}P^{0}_{0}\,{}^{0}_{0}P^{2}_{2} = (N-1)P_0 \quad ,
$$

and so on.

ii) Those with one of $ijkl$ equal to zero and all the others two are normalized such that

$$
{}^{2}_{2}P^{2}_{0}\,{}^{2}_{0}P^{2}_{2} = 16(N-2)P_{\substack{\square\\ \square}} \quad .
$$

iii) ${}^{0}_{0}P^{0}_{0}$ is normalized such that it is the projector onto the singlet representation $0 \otimes 0$.

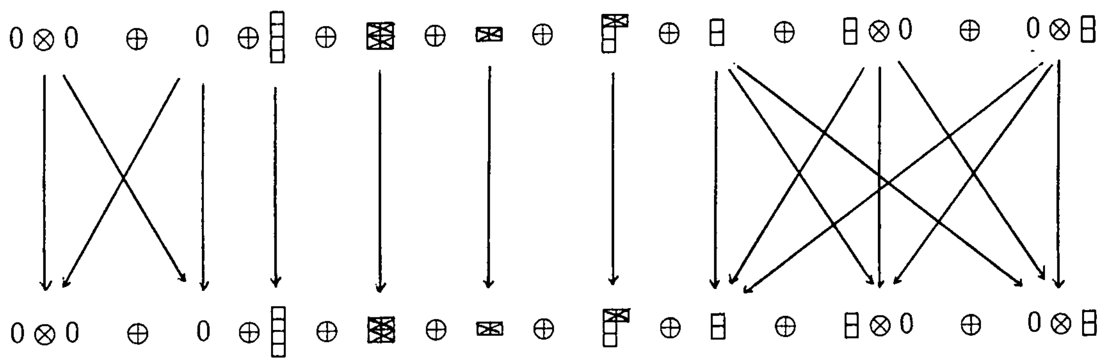
In fact, these normalizations are not central to an understanding of the S-matrix; they are fixed here solely to make our definition exact. The convenience of the values chosen only becomes apparent when doing calculations.

Once again, we may write S_{22}^{min} in a simple form,

$$\begin{aligned} S_{22}^{min}(\theta) &= \mathbf{P}h(u-2)h^2(u)h(u+2)[2]\Big(M_1+[2]M_2 \\ &\quad +[2][4]M_3+[2][N-4]M_4 \\ &\quad [2][4][N-4]M_5+[2][4][N-4][N-2]M_6\Big) \end{aligned} \tag{3.12}$$

Again, the Ms are again not projectors but expressions in $\mathcal{B}_4$ whose meaning is opaque. Here there is actually a degree of freedom in the determination of the Ms, since we are expressing quartic polynomials in terms of $M_1, \dots M_6$. Again, this expression does not seem to carry any insight.

It is found that ${}^2_0P^2_2P_a = 0$ for $a =$ ⊟⊟, $\boxtimes$, $\boxplus$, $\square$ and 0, but that ${}^2_0P^2_2P_{⊟} = {}^2_0P^2_2$ (and $P_{⊟}{}^2_2P^2_0 = {}^2_2P^2_0$ etc.). Hence if we again use a diagram to describe the mappings from $(⊟ \oplus 0)^{\otimes 2}$ to itself :



the diagram has the expected structure: the only intertwiners in the S-matrix are between isomorphic components. As with (3.9), the coefficients of the projectors onto components with multiplicity one in (3.11) are given in the $[a]$ notation, but the only significant content of the coefficients of the projectors and intertwiners onto ⊟ and 0 components is the position of poles and zeros.

We note that, for generic N, at $u = 4$, $\boxtimes$ and $\boxplus$ disappear from the fourth rank terms, and S^{min} acts in ⊟⊟ $\oplus$ ⊟ $\oplus\ 0 \oplus 0$. Identifying this with particle four, we expect a fusing

$22 \to 4$ at $u = 4$ with an appropriate pole in the CDD factor. However, although $P_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$ is present in the S-matrix, there is **no** $22 \to 2$ fusing, because there is no value of u at which it is possible to restrict S to act on $\begin{smallmatrix}\square\\\square\end{smallmatrix} \oplus 0 \oplus 0$. This is the first example of the 'holes' in the Clebsch-Gordan series discussed at the beginning of the chapter.

For $N = 8$, the $22 \to 4$ fusing does not occur. What happens in this case is that two zeroes in $h(u-1)(h(u))^2 h(u+1)$, at 4 and $N-4$, coincide, and dominate the simple poles in the R-matrix. However, there is a non-zero contribution from the double poles in the $P_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$ and P_0 projectors in R, and so we expect a $22 \to 2$ fusing at $u = 4$, **in place** of the $22 \to 4$ fusing. This is the **only** value of N at which the fusing occurs. This may be contrasted with the situation in S_{12}^{min}, where at $N = 6$ the $12 \to 3$ fusing is forbidden, but the dominant $12 \to 1$ fusing is a general feature for all N. We shall have more to say about this in a moment.

At this point we should mention, as an aside, the properties of a_n-invariant R- and S-matrices. There, the fundamental R-matrix in $\square$ of $SU(N)$ is

$$\check{R}_{\square\square}(u) = \mathbf{P}\left(P_{\square\square} + [2]P_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}\right) \ .$$

The fusion procedure can be used to construct R-matrices in both $\square\square$ and $\begin{smallmatrix}\square\\\square\end{smallmatrix}$, and this extends to an ability to construct R-matrices in all representations which have rectangular Young tableaux. In the S-matrix interpretation, particle multiplets are precisely the fundamental representations, and the fusings follow the CG decomposition and correspond to those of the PESTs. In terms of Brauer's algebra, we can restrict to the a_n case by considering only the S_n subalgebra (by formally setting the trace operator equal to zero). Thus in expressions (3.10, 3.12) we are interested only in the third and fourth rank parts respectively, and find that the rank three parts of T_1 and T_2 are $\begin{smallmatrix}\square\square\\\square\end{smallmatrix}$ and $\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}$, and that the rank four parts of N_1 and N_2 are $\begin{smallmatrix}\square\square\\\square\square\end{smallmatrix}$ and $\begin{smallmatrix}\square\square\\\square\\\square\end{smallmatrix}$.

3.3 Conclusions

We have now described all the fusings of the first and second particles in the principal chiral or Gross-Neveu models: we know which fusings occur and, because we know the rapidities at which they occur, we know the corresponding mass ratios. If we analyse the situation in the PESTs, we find that it is exactly as for S_{12} and S_{22}.

In S_{12}, $\square \otimes (\begin{smallmatrix}\square\\\square\end{smallmatrix} \oplus 0)$ contains both $\rho_1 = \square$ and $\rho_3 = \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}$, and the particles associated with these representations both occur as fusings in the theories, *i.e.* S_{12} can be restricted both to $\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix} \oplus \square$ and to $\square$. In S_{22}, however, although $(\begin{smallmatrix}\square\\\square\end{smallmatrix} \oplus 0) \otimes (\begin{smallmatrix}\square\\\square\end{smallmatrix} \oplus 0) \supset \begin{smallmatrix}\square\\\square\\\square\\\square\end{smallmatrix} \oplus \begin{smallmatrix}\square\\\square\end{smallmatrix}$, in general S_{22} may be restricted to $\begin{smallmatrix}\square\\\square\\\square\\\square\end{smallmatrix} \oplus \begin{smallmatrix}\square\\\square\end{smallmatrix} \oplus 0 \oplus 0$ but not to $\begin{smallmatrix}\square\\\square\end{smallmatrix} \oplus 0 \oplus 0$, which is allowed only for d_4. So we have identical situations in the PEST and in our S-matrices: the first 'hole' in the Clebsch-Gordan decomposition occurs in the lack of a $22 \rightarrow 2$ fusing in d_5.

Also, the values at which the fusings occur:

$$\begin{array}{ll} 11 \rightarrow 2 & u = 2 \\ 12 \rightarrow 3 & u = 3 \\ 12 \rightarrow 1 & u = N - 3 \\ 22 \rightarrow 4 & u = 4 \end{array}$$

and, for d_4, $22 \rightarrow 2$ at $u = 4$, are precisely the values of the fusing angles in the $d_n^{(1)}$ $(N = 2n)$ PESTs (and hence the mass ratios are the same). Further, the fusions involving spinor particles, calculated from the spinor R-matrices[23,73], are also the same as for the PESTs.

As mentioned before, the (β-independent) PEST S-matrices are precisely the CDD factors described above for our S-matrices. When we include the CDD factors in our S-matrices and apply the fusion procedure, they must obey

$$X_{UY}(\theta) = X_{UV}(\theta + \gamma) X_{UV'}(\theta + \gamma + \theta_0) \quad ,$$

(where $\theta_0 = \frac{i\pi u_0}{C}$, $\gamma = \frac{i\pi\alpha}{C}$), when $X_{VV'}$ has a simple pole at θ_0. This corresponds to a restriction of $S^{min}_{VV'}$ to the representation associated with particle Y. The constant γ is, as mentioned before, fixed by requiring the group-invariance of $\check{R}_{UX}$. Calculating the CDD factors for the fused S-matrices, we obtain

$$\begin{aligned} X_{12}(\theta) &= (1)(3)(N-5)(N-3) \\ X_{22}(\theta) &= -(2)^2(4)(N-6)(N-4)^2 \ . \end{aligned}$$

Now consider the bootstrap (2.34) for the PEST S-matrices. Initially, we find that the PEST S-matrix for the $11 \to 11$ process is X_{11} of (3.4). In the cases we have studied, γ is fixed to mimic (2.34) for the PESTs precisely, and so the CDD factors obtained are the PEST S-matrices. How this happens is not understood. It is also remarkable that the correspondence between the physical poles in the CDD factors and the restrictions of S^{min} should be preserved by the fusion procedure: the fused CDD factors have physical (negative residue[46]) poles at precisely the values of the S^{min} fusings.

In fact, we are now in a position to write down the full set of S-matrices for d_4, and so for example for the d_4 principal chiral model. Consider the fundamental representations of d_4, which we shall label 1, 2, s (spinor) and s'. We have computed S_{12} and S_{22} from S_{11}[20]. S_{1s} has been calculated by Shankar and Witten[73], and the R-matrix is given (for general N; here, $N=8$) by

$$\check{R}_{1s}(u) = \mathbf{P}(P_T + [N/4]P_{s'}) \quad , \tag{3.13}$$

where P_H, $P_{s'}$ are projectors onto the 'top' and s' representations respectively (for N odd, $s=s'$). For d_4, we can now use the triality property to calculate S_{ss}, $S_{s's'}$ (which are of the same form as S_{11}), S_{1s}, $S_{1s'}$ (same form as S_{1s}) and S_{2s}, $S_{2s'}$ (same form as S_{12}). When this is done, it is found that the complete mass spectrum and set of fusings is the same, and that all the (β-independent) $d_4^{(1)}$ PEST S-matrices[62] are the same as the CDD factors. Note that the pole in $X_{22}(\theta)$ at which $22 \to 2$, $u=4$, is cubic: this is the first example of the need to consider residues of higher-order poles as physical states under the bootstrap principle.

This correspondence extends to all known R-matrices for simply-laced algebras, where the masses and fusings of the $\mathcal{A}$ R-matrices are the same as those of the $\mathcal{A}^{(1)}$ ATFT. For example, the $d_n^{(1)}$ PEST S-matrices match exactly all that is known about the $SO(2n)$ Gross-Neveu model[20,23,73,74,75], whilst the appropriate PEST S-matrices match the known exceptional algebra R-matrices for the basic[22] and adjoint[31] representations.

For non-simply-laced algebras the situation is much less clear. Firstly, the mass spectrum[22] obtained from the R-matrices for $\mathcal{A}$ is that not of the $\mathcal{A}^{(1)}$ ATFT but of the ATFT of one of the twisted algebras. In fact, the full corrsepondence of mass spectra is

$$\begin{array}{rcl}
\text{R}-\text{matrices} & \longleftrightarrow & \text{PEST} \\
a_n & \longleftrightarrow & a_n^{(1)} \\
d_n & \longleftrightarrow & d_n^{(1)} \\
e_{6,7,8} & \longleftrightarrow & e_{6,7,8}^{(1)} \\
b_n & \longleftrightarrow & a_{2n-1}^{(2)} \\
c_n & \longleftrightarrow & d_{n+1}^{(2)} \\
f_4 & \longleftrightarrow & e_6^{(2)} \\
g_2 & \longleftrightarrow & d_4^{(3)} \ ,
\end{array}$$

where the notation is that of Helgason[76,47]. Furthermore, whilst simple poles in the appropriate β-independent PEST S-matrices (and thus the CDD factors of the Yang-Baxter S-matrices) correspond to R-matrix fusings, there are higher-order poles which do not, but which would have corresponded to ATFT fusings had they appeared in the simply-laced ATFT S-matrices. Thus the whole subject - ATFT S-matrices, R-matrices and CDD factors - for non-simply-laced algebras is rather murky. Work to clarify matters is in progress on the c_n case, for which the R-matrices are particularly tractable, as we shall see in chapter five. We believe that the correspondence will turn out to be complete for the simply-laced algebras, and that the non-simply-laced cases will eventually admit of a coherent explanation.

In an attempt to say more about the structure of R-matrices, we shall in later chapters look at rational R-matrices in irreducible representations, the representations of the Yangian (including in detail the work of Chari and Pressley, which overlaps heavily with that of this chapter), and how the Yangian appears in physics. First, however, we shall apply the fusion procedure to the one obvious tractable case: the quantum group analogue of the S-matrices in this section.

Chapter 4

Fused trigonometric R-matrices

We begin by recalling some results for rational R-matrices from the last chapter. What we did was to take the basic rational R-matrix for $SO(N)$,

$$\check{R}_{\square\square}(u) = \mathbf{P}(P_{\boxtimes\boxtimes} + [2]P_{\substack{\square\\\square}} + [2][N-2]P_0) \ ,$$

and apply the fusion procedure to it at $u_0 = 2$ to obtain

$$\check{R}_{\square\,\substack{\square\\\square}\oplus 0}(u) = \mathbf{P}(T_1 + [3]T_2 + [3][N-3]T_3) \tag{4.1}$$

where, because $\substack{\square\\\square} \oplus 0$ is reducible, the T_i were not projectors. Of course, the results of chapter three were also rescaled, and had u rescaled to θ, so that they satisfied the properties (i)-(iii) of S-matrices. We also noted that we could use the fusion procedure at $u_0 = -2$ to obtain $\check{R}_{\square\boxtimes\boxtimes}$. Doing this calculation using our algorithm we obtain

$$\begin{aligned}\check{R}_{\boxtimes\boxtimes\square}(u) &= (1 \otimes P_{\boxtimes\boxtimes})(\check{R}_{\square\square}(u+\alpha-2) \otimes 1)(1 \otimes \check{R}_{\square\square}(u+\alpha)) \\ &= \mathbf{P}\left(P_{\boxtimes\boxtimes\boxtimes} + [3]P_{\substack{\boxtimes\boxtimes\\\square}} + [3][N-1]P_{\square}\right)\end{aligned} \tag{4.2}$$

where (with $\alpha = 1$) $\check{R}_{\boxtimes\boxtimes\square}$ is then group invariant and so has this neat decomposition in terms of projectors. If we leave α general, we obtain

$$R_{\boxtimes\boxtimes\square}(u) = S_1 + [4-\alpha]S_2 + [4-\alpha][N-\alpha]S_3 \ ,$$

where the S_i are no longer projectors (*cf* the R-matrices in reducible representations).

Another option is to use Brauer's algebra to describe not $SO(N)$ but $Sp(N)$. This requires the trace δ_{ac} to be replaced by the $Sp(N)$ antisymmetric 'symplectic trace' f_{ac} (which exists only for N even). In this way we can use the basic R-matrix[77]

$$\check{R}_{\square\square} = \mathbf{P}\left(P_{\substack{\boxtimes\\\boxtimes}} + [-2]P_{\square\square} + [-2][N+2]P_0\right) \tag{4.3}$$

to construct

$$\check{R}_{\substack{\boxtimes\\\boxtimes}\square} = \mathbf{P}\left(P_{\substack{\boxtimes\\\boxtimes\\\boxtimes}} + [-3]P_{\substack{\boxtimes\square\\\boxtimes}} + [-3][N+1]P_{\square}\right) \ , \tag{4.4}$$

where now cross-hatching denotes removal of a symplectic trace from antisymmetric indices.

In this chapter we explore the trigonometric R-matrices corresponding to (4.1, 4.2, 4.4), which are associated with the untwisted quantum affine algebras $SO_q(N)$ and $Sp_q(2n)$.

Remember that we have

$$\left[\check{R}(x), \Delta(a)\right] = 0 \quad \text{for all } a \in \mathrm{U}_q\mathcal{A}^{(1)} \ . \tag{4.5}$$

Suppose we are looking for the R-matrix acting on $V_1^q \otimes V_2^q$, where V_1^q and V_2^q are *irreducible* representations of $\mathrm{U}_q\mathcal{A}$. Recalling (2.27), we see from (4.5) that this may be written

$$\check{R}(x) = \sum_{W^q \subset V_1^q \otimes V_2^q} \tau_{W^q}(x) \check{P}_{W^q} \ , \tag{4.6}$$

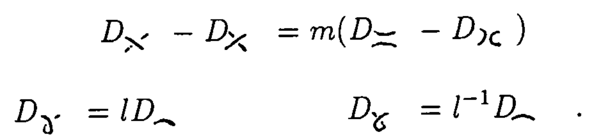
where W^q are irreducible components of $V_1^q \otimes V_2^q$ and the $\check{P} : V_1^q \otimes V_2^q \to V_2^q \otimes V_1^q$ are projectors P composed with the operator $\mathbf{P}$ which transposes V_1^q and V_2^q. Working analogously to the rational case, we shall begin with the R-matrices in □ ⊗ □ constructed* by Jimbo for $SO_q(N)$ and $Sp_q(2n)$, and fuse these to get R-matrices acting in □□ ⊗ □ for $SO_q(N)$ and ⊟ ⊗ □ for $Sp_q(2n)$. These are of the form (4.6), with explicit constructions of the functions τ and the 'twisted' projectors $\check{P}_W$. However, in order to begin, we need to understand the q-analogue of Brauer's algebra.

4.1 The Birman-Wenzl-Murakami algebra

In order to apply the fusion procedure, one needs to know the centralizer algebra of the group or quantum group in question, *i.e.* the algebra which commutes with the group action. Projectors are then constructed as idempotents of the centralizer algebra. For $SU(N)$ the centralizer is the symmetric group, whilst for $SU_q(N)$ it is the Hecke algebra described, for example, by Jimbo[35], which is a q-deformation of the symmetric group. For tensor representations of $SO(N)$ and $Sp(2n)$ it is Brauer's algebra[70], described in the last chapter. The quantum deformation of this turns out to be a specialization of the Birman-Wenzl-Murakami (BWM) algebra, which originally arose as the braid-monoid algebra of the Kauffman polynomial. The Kauffman polynomial for links in its Dubrovnik form[78] is defined as follows:

*Young tableaux are now being used to indicate representations of the *quantum* group.

$$D_{\bigcirc} = \left(1 + \frac{l - l^{-1}}{m}\right)$$



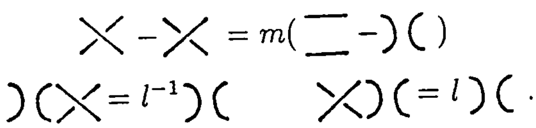
(4.7)

$D(l, m)$ is defined by taking the above relations between polynomials of knots which differ in the way shown at just one crossing point. D is invariant under regular isotopy (Reidemeister moves R2, 'unitarity', and R3, the braid group relation). An excellent discussion of recent developments in knot theory and their relation to theoretical physics is given by Kauffman in his book[79].

Birman and Wenzl[80] and Murakami[81] independently used D to define a new algebra $\mathcal{C}_f(l, m)$. This can be viewed as an algebra of braids on f strands generated by braid and monoid operators acting on the ith and $i + 1$th strands, modulo certain relations. For clarity, we shall use the diagrammatic notation

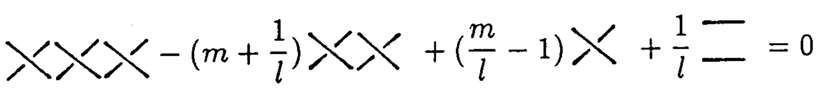
braid ≡  (so that $(\text{braid})^{-1} =$ ) and monoid ≡ 

to give these relations, which fall into two sets: those arising from invariance under regular isotopy, shown in figure (4.1), and those from (4.7),



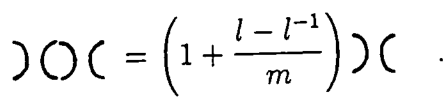
(4.8)

The latter also imply a skein relation for the braid operator:



(4.9)

and an idempotence for the monoid:



The full BWM algebra, linearly generated by all possible products of braids, inverse braids and monoids, can now be expressed diagrammatically using symbols consisting of braids

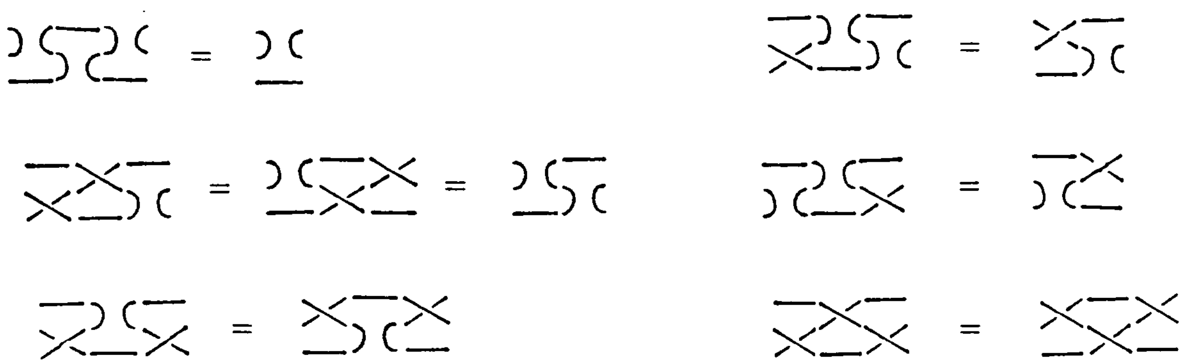


Figure 4.1: BWM algebra relations arising from regular isotopy invariance of D

between two sets of f points, modulo the above relations. (However, not all such braids are in the algebra: for an example see Kauffman[78].) Because of the relations in figure (4.1), multiplication in the algebra corresponds simply to composition of braids. In this work, braids are composed horizontally, as in (4.9) and the last chapter, rather than vertically.

At this point we should note that, because we have defined the algebra using the Dubrovnik form of Kauffman's polynomial rather than the original form[78], our braid operator differs by a factor i from that of Birman and Wenzl and our monoid by a factor -1. We have done this because the classical ($q \to 1$ with m, l as given below) limits of the braid and monoid operators now appear naturally as the transposition and trace operators in the centralizers of the classical groups, and generate Brauer's algebra.

Reshetikhin showed[43] (see also Wenzl[82]) that for particular values $l(q), m(q)$, the BWM algebra becomes the centralizer of the quantum groups $SO_q(N)$ and $Sp_q(2n)$. The strands of the braid then correspond to vector representations V_q of $\mathrm{U}_q\mathcal{A}$, and $\mathcal{C}_f$ is the centralizer of its action on $V_q^{\otimes f}$. Specifically, we have for $SO_q(N)$

$$l(q) = q^{N-1} \quad \text{and} \quad m(q) = q - q^{-1}$$

and for $Sp_q(N)$ (writing $N = 2n$)

$$l(q) = -q^{N+1} \quad \text{and} \quad m(q) = q - q^{-1}\,.$$

Now suppose we need to decompose the tensor product of two vector representations, $\square \otimes \square$, into its irreducible components. This is equivalent to finding the idempotents of $\mathcal{C}_2(l(q), m(q))$. For the $SO_q(N)$ case we find, by direct calculation using the given relations, the following:

$$\begin{aligned}
P_{\square\square} &= \frac{1}{1+q^2}\left[\ = \ +q \times \ - \frac{1+q^{2-N}}{[N-1]+1} \,)(\ \right], \\
P_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} &= \frac{1}{1+\frac{1}{q^2}}\left[\ = \ -\frac{1}{q} \times \ - \frac{1-q^{-N}}{[N-1]+1} \,)(\ \right] \\
\text{and} \quad P_0 &= \frac{1}{[N-1]+1} \,)(\ ,
\end{aligned}$$

where

$$[a] \equiv \frac{q^a - q^{-a}}{q - q^{-1}} ,$$

are orthogonal idempotents of $\mathcal{C}_2(q^{N-1}, q-q^{-1})$ corresponding to the second rank symmetric traceless tensor, second rank antisymmetric tensor, and singlet representations respectively of $SO_q(N)$. These expressions can also be derived from Jimbo's results[35], where matrix expressions for the braid and monoid operators are given, but the BWM algebra structure is not explicit. Note that as $q \to 1$ we recover the appropriate expressions for $SO(N)$.

Similarly, we have for $Sp_q(N)$ the result that

$$\begin{aligned}
P_{\square\square} &= \frac{1}{1+q^2}\left[\ = \ +q \times \ + \frac{1+q^{-2-N}}{[N+1]-1} \,)(\ \right], \\
P_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} &= \frac{1}{1+\frac{1}{q^2}}\left[\ = \ -\frac{1}{q} \times \ + \frac{1-q^{-N}}{[N+1]-1} \,)(\ \right] \\
\text{and} \quad P_0 &= -\frac{1}{[N+1]-1} \,)(
\end{aligned}$$

are the orthogonal idempotents of $\mathcal{C}_2(-q^{N+1}, q - q^{-1})$ corresponding to the appropriate representations of $Sp_q(N)$. Once again, the classical limit gives representations of $Sp(N)$. The monoid now corresponds to the quantized form of the symplectic trace (the rank two antisymmetric invariant of $Sp(N)$, which exists only for N even). The idempotents given for $SO_q(N)$ and $Sp_q(N)$ are found to be eigenvectors of the braid operator with the eigenvalues calculated by Reshetikhin[43]. These eigenvalues may also be obtained by substituting the appropriate values of l and m into the skein relation (4.9).

These results enable us to express the R-matrix for the vector representation of $SO_q(N)$ constructed by Jimbo[35,17,83]

$$\check{R}_{\square\square}(x) = \check{P}_{\boxtimes\boxtimes} - [2]_q \check{P}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} + [2]_q [N-2]_q \check{P}_0 \tag{4.10}$$

(where we have introduced the notation

$$[r]_q \equiv \frac{1-q^r x}{q^r - x} \quad)$$

in terms of braid and monoid operators (and similarly for the $Sp_q(N)$ R-matrix). Note that here we have $\check{P}_{\boxtimes\boxtimes} = P_{\boxtimes\boxtimes}$, $\check{P}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} = -P_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$ and $\check{P}_0 = P_0$, since P acts in the tensor product of $\square$ with itself. The R-matrix is then in a suitable form for application of the fusion procedure. Note also that $\check{R}(0)$ is $\frac{1}{q}$ times the braid operator - an example of the general fact that a trigonometric R-matrix gives a braid group generator (solution of (2.10)) at $x = 0$.

4.2 Fused trigonometric R-matrices

Looking at (4.10) we see that $\check{R}(q^{-2}) = P_{\boxtimes\boxtimes}$, so that the YBE becomes

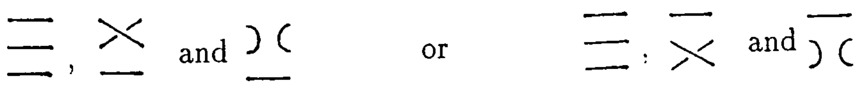
$$(P_{\boxtimes\boxtimes} \otimes 1)\left(1 \otimes \check{R}(xq^{-1})\right)\left(\check{R}(xq) \otimes 1\right) = \left(1 \otimes \check{R}(xq)\right)\left(\check{R}(xq^{-1}) \otimes 1\right)(1 \otimes P_{\boxtimes\boxtimes}) . \tag{4.11}$$

As in the rational case, the fusion procedure gives us that

$$\begin{aligned} &\check{R}_{\square\boxtimes\boxtimes}(x) = (P_{\boxtimes\boxtimes} \otimes 1)\left(1 \otimes \check{R}(xq^{-1})\right)\left(\check{R}(xq) \otimes 1\right) \\ \text{and} \quad &\check{R}_{\boxtimes\boxtimes\square}(x) = (1 \otimes P_{\boxtimes\boxtimes})\left(\check{R}(xq^{-1}) \otimes 1\right)\left(1 \otimes \check{R}(xq)\right) \end{aligned} \tag{4.12}$$

together define a solution of the YBE (2.2) with $V_1 = V_3 = \square, V_2 = \boxtimes\boxtimes$. This is proved by manipulation using (4.11); the proof is precisely analogous to the schematic proof for the rational case in figure (2.3).

The utility of expressing (4.10) in terms of braid and monoid operators becomes apparent when we try to calculate (4.12) explicitly, since each of the factors of (4.12) can be expressed in terms of either



in exactly the same way as the rational case. The algorithm is an extended version of that of the last chapter. In carrying out the calculation, the BWM algebra operations were done by hand, but the addition and factorization of the coefficients was again done using REDUCE, giving us

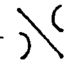
$$\check{R}_{\boxtimes\boxtimes\square}(x) = \check{P}_{\boxtimes\boxtimes\boxtimes} - [3]_q \check{P}_{\boxtimes\boxtimes/\square} + [3]_q [N-1]_q \check{P}_{\square} \ . \tag{4.13}$$

We give expressions for $\check{P}$ in an appendix, using a basis for $\mathcal{C}_3$ whose existence was pointed out by Morton and Traczyk[84] as follows. Recall that Brauer's algebra is generated by the transposition and trace operators. The transposition T satisfies $T = T^{-1}$, and so we can view the algebra as that of braids projected so that over- and under-crossings are not distinguished; the algebra is that of the k symbols described earlier. For the BWM algebra, (4.7) allows the inverse braid operator to be expressed in terms of the braid and monoid operators. We can use this to find k symbols which linearly generate the algebra, *i.e.* to re-express any other symbol in terms of these symbols. On ignoring the distinction between over- and under-crossings, we obtain a natural correspondence between these symbols and the basis of Brauer's algebra. The rational (or 'classical') $q \to 1$ limit of the expressions in the appendix gives the projectors of (4.2), and $\lim_{q\to 1}[a]_q = [a]$, so that the rational limit of (4.13) is (4.2).

The value of $\check{R}$ at $x = 0$ is

$$\check{R}_{\boxtimes\boxtimes\square} \equiv \check{R}_{\boxtimes\boxtimes\square}(0) = \check{P}_{\boxtimes\boxtimes\boxtimes} - q^{-3}\check{P}_{\boxtimes\boxtimes/\square} + q^{-N-2}\check{P}_{\square} \ ,$$

which is $\frac{1}{q^2}$ times the braid group generator found by Reshetikhin[43]. In terms of our basis of k symbols, we have

$$\check{R}_{\boxtimes\boxtimes\square} = \frac{1}{q^2}\left(\frac{1}{1+q^2}\right)\Bigg($$  $$+q$$  $$-\frac{1+q^{2-N}}{[N-1]+1}$$  $$\Bigg)$$

$$= \frac{1}{q^2}$$  $$P_{\boxtimes\boxtimes\otimes\square} \quad \text{on} \quad \square^{\otimes 3} \ .$$

Since $P_{\boxtimes\boxtimes\boxtimes} + P_{\boxtimes\boxtimes/\square} + P_{\square} = 1$, we expect

$$\check{R}_{\boxtimes\boxtimes\square} P_{\boxtimes\boxtimes\boxtimes} = \check{P}_{\boxtimes\boxtimes\boxtimes} \ , \quad \check{R}_{\boxtimes\boxtimes\square} P_{\boxtimes\boxtimes/\square} = -q^{-3}\check{P}_{\boxtimes\boxtimes/\square} \quad \text{and} \quad \check{R}_{\boxtimes\boxtimes\square} P_{\square} = q^{-N-2}\check{P}_{\square} \ .$$

As a check on our results, we should show that the P obtained in this way really are projectors. Owing to the large amount of algebra required, we have done this only for $P_{\Box}$, for which we indeed find that $P_{\Box}^2 = P_{\Box}$.

We can now go back and perform the same calculation for $Sp_q(N)$. We use Jimbo's R-matrix[35,17]

$$\check{R}(x) = P_{\substack{\boxtimes\\\boxtimes}} - [-2]_q P_{\Box\Box} + [-2]_q[N+2]_q P_{\Box}$$

so that $\check{R}(q^2)$ projects onto the antisymmetric second rank tensor. We can then define a new solution of the YBE with

$$\check{R}_{\substack{\boxtimes\\\boxtimes}\Box}(x) \equiv \left(1 \otimes P_{\substack{\boxtimes\\\boxtimes}}\right)\left(\check{R}(xq) \otimes 1\right)\left(1 \otimes \check{R}(xq^{-1})\right)$$

(and $\check{R}_{\Box\substack{\boxtimes\\\boxtimes}}(x)$ defined analogously to (4.12)). Once again, this has a simple decomposition

$$\check{R}_{\substack{\boxtimes\\\boxtimes}\Box}(x) = \check{P}_{\substack{\boxtimes\\\boxtimes\\\boxtimes}} - [-3]_q \check{P}_{\substack{\boxtimes\Box\\\boxtimes}} + [-3]_q[N+1]_q \check{P}_{\Box} \ .$$

In both of the cases we have considered we can set the monoid operator to zero and obtain R-matrices acting in the $\Box\Box \otimes \Box$ and $\substack{\Box\\\Box} \otimes \Box$ representations of $SU_q(N)$. The BWM algebra then reduces to the Hecke algebra described by Jimbo[35], the projectors agree with the known formulae[43,13] for $P_{\Box\Box\Box}$ and $P_{\substack{\Box\\\Box\\\Box}}$ and the classical limit gives the appropriate projectors onto representations of $SU(N)$.

We now turn to the problem of constructing R-matrices in reducible representations. Just as in the rational case, in order to turn $\check{R}_{\Box\Box}(x)$ into an S-matrix, it must be made crossing-symmetric, and this fixes x as a function of rapidity in such a way that the bound state $\substack{\Box\\\Box} \oplus 0$ occurs at a physical value of the rapidity, whereas the bound state $\Box\Box$ occurs at an unphysical value, so that we interpret $\substack{\Box\\\Box} \oplus 0$ as a particle state, but not $\Box\Box$. We use

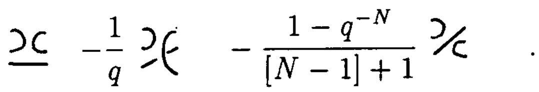
$$\check{R}(q^2) \propto P_{\substack{\Box\\\Box}} + \frac{q^N - 1}{q^{N-2} - q^2} P_0$$

to define

$$\check{R}_{(\substack{\Box\\\Box}\oplus 0)\Box}(x) = \left(1 \otimes (P_{\substack{\Box\\\Box}} + P_0)\right)\left(\check{R}(xq) \otimes 1\right)\left(1 \otimes \check{R}(xq^{-1})\right) \quad (4.14)$$

which, together with $\check{R}_{\square(\boxminus\oplus 0)}$ defined analogously to (4.12), defines a solution of (2.2) with $V_1 = V_3 = \square, V_2 = \boxminus \oplus 0$. Figure (3.1) gives the basic idea of the proof.

As in the rational case, the decomposition includes intertwiners as well as projectors, so that for example the q-analogue of $\mathbf{P}P^{02}$ is



We have not calculated the full decomposition analogous to (3.9), but note instead that the simple form (3.10) of the rational case carries over to the trigonometric case, yielding

$$\check{R}_{(\boxminus\oplus 0)\square}(x) = -[1]_q\left(\check{T}_1 - [3]_q\check{T}_2 + [3]_q[N-3]_q\check{T}_3\right) , \tag{4.15}$$

where the $\check{T}$ are given in the appendix: they have the $\mathbf{P}\,T_i$ of (3.10) as their rational limits, and setting the monoid to zero gives the expected Hecke algebra idempotents. However, as with (3.10), the algebra of the $\check{T}$s gives no additional insights.

Unfortunately the difficulty of calculation makes it impossible to continue and do calculations in $\mathcal{C}_4$. It is clear, though, that the fusion structure of S_{12} mimics the rational case. This is also true of the $U_q a_n$-invariant trigonometric R-matrices[16,85]. So the message seems to be that, for general q, we can learn nothing new from the quantum group case - in other words, any question about trigonometric R-matrices, for general q, can be answered solely by reference to rational R-matrices. Hence it seems best to concentrate on the rational case, where the Lie algebra structure makes calculation easier. An outstanding first question to ask is how the values of the arguments of the $[a]$ (and by extension the $[a]_q$) in both the irreducible (3.2, 3.13, 4.4, 4.2 and so on) and reducible (3.10, 3.12, 4.15) cases are determined, and this is the subject of the next chapter. The irreducible case is dealt with in the first two sections; this work, together with that on the Yangian by Drinfeld and by Chari and Pressley, described in the rest of the chapter, then enables us to say something about the reducible case.

Chapter 5

Rational R-matrices in irreducible representations

In this chapter, we investigate rational R-matrices, discussing both whether they exist in particular representations, and what form they take when they do. In chapter two, we explained how rational R-matrices are related to the Yangian algebra, and we shall be looking at representations of this algebra later in the chapter. For the moment, however, let us adopt the point of view of someone wanting to solve the YBE in form (2.3),

$$R_{12}(u)R_{13}(u+v)R_{23}(v) = R_{23}(v)R_{13}(u+v)R_{12}(u) \qquad (u, v \in \mathbf{C}) \ . \tag{5.1}$$

We already have many examples of solutions to guide us, and we hope to extrapolate from these to a general rule. The group invariance of rational R-matrices tells us that, where $V \otimes V'$ contains no multiplicities,

$$R_{VV'} = \sum_{W \subset V \otimes V'} \tau_W(u) P_W \ . \tag{5.2}$$

In our examples, and others which have been calculated[45,23,22,73,86], the τ_W are built from functions $[a]$, and we wish to understand how. The first step is to notice that in (4.2, 4.1, 4.4), as in all other examples of R-matrices in irreducible representations, we have

$$\frac{\tau_Y(u)}{\tau_X(u)} = [\,\gamma(C_2(X) - C_2(Y))\,] \tag{5.3}$$

for certain X, Y , where C_2 is the quadratic Casimir operator, and γ is a constant of the group. From the examples we have seen, we seem to need an ordering of the representations $W_i \subset V \otimes V'$, setting $X = W_i$ and $Y = W_{i+1}$. Such a proposal was the basis of the conjecture for the general form of trigonometric R-matrices proposed by Ge, Xue and Wu[87].

In fact, the situation is more complicated. To see this, we need another example. Returning to the fusion procedure techniques of chapter three, we see that we can again use $\check{R}_{\square\square}(-2) = \mathbf{P} P_{\boxtimes\boxtimes}$, in the same way that we constructed S_{22}^{min}, to define

$$\begin{aligned}\check{R}_{\boxtimes\boxtimes\,\boxtimes\boxtimes}(u) &= (P_{\boxtimes\boxtimes} \otimes 1)\left(1 \otimes \check{R}_{\boxtimes\boxtimes\square}(u-1)\right)\left(\check{R}_{\boxtimes\boxtimes\square}(u+1) \otimes 1\right) \\ &= (P_{\boxtimes\boxtimes} \otimes P_{\boxtimes\boxtimes})\left(1 \otimes \check{R}_{\square\square}(u-2) \otimes 1\right)\left(\check{R}_{\square\square}(u) \otimes \check{R}_{\square\square}(u)\right)\left(1 \otimes \check{R}_{\square\square}(u+2) \otimes 1\right)\end{aligned}$$

(note that the first line acts on $\boxtimes\boxtimes \otimes \square \otimes \square$, whereas the second line acts on $\square^{\otimes 4}$), which solves the YBE, is group invariant, and has the decomposition

$$\begin{aligned} R_{(2)(2)}(u) &= P_{(4)} + [4]\left(P_{(3,1)} + [N]P_{(2)}\right) \\ &+ [4][2]\left(P_{(2,2)} + [N]P_{(1,1)} + [N][N-2]P_0\right) \end{aligned} \quad (5.4)$$

This agrees with (5.3), but the X, Y to which (5.3) must apply are not the result of a simple ordering of operators. In fact, we can describe them by forming them into a tree: (5.3) applies whenever $X \longrightarrow Y$. So, in this example, the tree is

$$\begin{array}{ccccc} (4) & \longrightarrow^4 & (3,1) & \longrightarrow^2 & (2,2) \\ & & \downarrow_N & & \downarrow_N \\ & & (2) & & (1,1) \\ & & & & \downarrow_{N-2} \\ & & & & 0 \end{array}$$

where we have written the values of the differences of the Casimir operators alongside the connecting arrows. Actually, we could just as well have used

$$\begin{array}{ccccc} (4) & \longrightarrow^4 & (3,1) & \longrightarrow^2 & (2,2) \\ & & \downarrow_N & & \\ & & (2) & \longrightarrow^2 & (1,1) \\ & & & & \downarrow_{N-2} \\ & & & & 0 \end{array}$$

This is the only example known to us of an R-matrix computed explicitly using the fusion procedure where the tree is not a chain, and it shows that something more sophisticated than an ordering of representations is needed.

5.1 Tensor product graphs

The way in which we approach the explanation of this phenomenon is by using a method very similar to that of Kulish, Sklyanin and Reshetikhin's approach to the a_n R-matrices. In their seminal paper[45], they invented the fusion procedure to calculate R-matrices explicitly, but were able to gain insights into their structure using a different method, of which what follows is an extension.

Suppose we seek rational solutions of the Yang-Baxter equation, (5.1), that are both unitary, $R(u)R(-u) = 1$, and have $R(u) \to 1$ as $u \to \infty$. We can then write

$$R(u) = 1 + r(u) + O\left(\frac{1}{u^2}\right) \ , \tag{5.5}$$

and $r(u)$ must now satisfy the classical Yang-Baxter equation,

$$[r_{12}(u), r_{13}(u+v)] + [r_{12}(u), r_{23}(v)] + [r_{13}(u+v), r_{23}(v)] = 0$$

(*cf* equation (2.5)), obtained by substituting (5.5) into (2.2) and examining the leading term, of order $\frac{1}{u^2}$. We examine R-matrices which have as their classical limit the r-matrix described in chapter two,

$$r(u) = \frac{1}{u} I_a \otimes I_a \ ,$$

where I_a are the generators of a semi-simple Lie algebra $\mathcal{A}$.

First, we write

$$R(u) = 1 + r(u) + \frac{T}{u^2} + O\left(\frac{1}{u^3}\right) \ .$$

Next, we use the unitarity condition to find T. Upon examining $R(u)R(-u) = 1$ as $u \to \infty$, we find that

$$T = \frac{1}{2} I_a I_b \otimes I_a I_b \ . \tag{5.6}$$

Our strategy is now to examine the $v \to \infty$ limit of (2.2). Doing this, we obtain

$$R_{12}(u)\left(1 + \frac{1}{u+v} I_a \otimes 1 \otimes I_a + \frac{1}{(u+v)^2} T_{13}\right)\left(1 + \frac{1}{v} 1 \otimes I_b \otimes I_b + \frac{1}{v^2} T_{23}\right) =$$
$$\left(1 + \frac{1}{v} 1 \otimes I_b \otimes I_b + \frac{1}{v^2} T_{23}\right)\left(1 + \frac{1}{u+v} I_a \otimes 1 \otimes I_a + \frac{1}{(u+v)^2} T_{13}\right) R_{12}(u) + O\left(\frac{1}{v^3}\right) \tag{5.7}$$

It should be noted that here we are implicitly assuming the existence of R_{13} and R_{23} for some V_3. This is trivial if $V_1 = V_2 = V_3 = V$ (say), but if $V_1 = V_3 = V$ and $V_2 = V'$ then we are assuming that if $R_{VV'}$ exists, so does R_{VV}.

For R to be a solution of (5.1), it must satisfy (5.7) at each order of $\frac{1}{v}$, so we now expand out the brackets and equate coefficients of $1, \frac{1}{v}$ and $\frac{1}{v^2}$. The results of this can most easily be seen by multiplying through by $v(u+v)$. The left-hand side of (5.7) then becomes

$$R_{12}(u)\Big(v(u+v) + v(1\otimes I_a + I_a\otimes 1)\otimes I_a + u1\otimes I_a\otimes I_a \\ + I_a\otimes I_b\otimes I_aI_b + \frac{u+v}{v}T_{23} + \frac{v}{u+v}T_{13}\Big) \ .$$

We now see that the terms of order v^2 are trivially equal, while those of order v give*

$$[R(u), 1\otimes I_a + I_a\otimes 1] = 0 \ , \tag{5.8}$$

which expresses the invariance of R under the diagonal action of $\mathcal{A}$. This gives us the expected group-invariant form (5.2). As in (2.27), this only applies to decompositions without multiplicities. Our goal is now to solve for the functions τ, which we do by examining the terms of order 1. These are given by

$$R_{12}(u)\,(u1\otimes I_a\otimes I_a + I_a\otimes I_b\otimes I_aI_b + T_{13} + T_{23}) = \\ (u1\otimes I_a\otimes I_a + I_a\otimes I_b\otimes I_bI_a + T_{13} + T_{23})\,R_{12}(u) \ . \tag{5.9}$$

We can simplify this by noticing that

$$T_{13} + T_{23} = \frac{1}{2}\left\{(1\otimes I_a + I_a\otimes 1)(1\otimes I_b + I_b\otimes 1) - I_a\otimes I_b - I_b\otimes I_a\right\}\otimes I_aI_b$$

so that, because of (5.8), we can rewrite (5.9) as

$$R_{12}(u)\left(u1\otimes I_a\otimes I_a + \frac{1}{2}I_a\otimes I_b\otimes [I_a, I_b]\right) = \left(u1\otimes I_a\otimes I_a + \frac{1}{2}I_a\otimes I_b\otimes [I_b, I_a]\right)R_{12}(u) \ .$$

Using

$$[C_2, 1\otimes I_a] \equiv \left[(1\otimes I_d + I_d\otimes 1)^2, 1\otimes I_a\right] = 2f^{abc}I_c\otimes I_b$$

*I am grateful to A. J. Macfarlane for pointing this out to me in the $SU(2)$ case.

(where C_2 is the quadratic Casimir operator $I_a I_a$, here evaluated on the tensor product) we obtain

$$R(u)\left(u1\otimes I_a - \frac{1}{4}[C_2, 1\otimes I_a]\right) = \left(u1\otimes I_a + \frac{1}{4}[C_2, 1\otimes I_a]\right) R(u) \ . \qquad (5.10)$$

This equation is now valued on $V_1 \otimes V_2$ only, and is the final form of the term of order 1. We shall now use (5.10) to find $R_{VV'}$.

In order to obtain a relation between the $\tau_W(u)$, we substitute the form (5.2) for $R_{VV'}(u)$ back into (5.10). Acting on the left with P_Y and on the right with P_X, we obtain

$$\tau_Y(u)P_Y\left(u + \frac{1}{4}\left(C_2(X) - C_2(Y)\right)\right)(1\otimes I_a)\,P_X = \tau_X(u)P_Y\left(u - \frac{1}{4}\left(C_2(X) - C_2(Y)\right)\right)(1\otimes I_a)\,P_X \ . \qquad (5.11)$$

But $1 \otimes I_a$ is an irreducible tensor operator in the adjoint representation, and so we can apply the Wigner-Eckhart theorem to obtain the general form for group-invariant rational R-matrices acting in irreducible representations V, V' of the algebra (where $V \otimes V'$ has no multiplicities)

$$R_{VV'}(u) = \sum_{W\subset V\otimes V'} \tau_W(u) P_W \qquad (5.12)$$

where

$$\frac{\tau_Y(u)}{\tau_X(u)} = \frac{u + \frac{1}{4}\left\{C_2(X) - C_2(Y)\right\}}{u - \frac{1}{4}\left\{C_2(X) - C_2(Y)\right\}} \equiv \left[\frac{1}{4}\left\{C_2(X) - C_2(Y)\right\}\right] \qquad (5.13)$$

for X, Y such that $Y \subset$ *adjoint* $\otimes X$ and $\langle Y||1\otimes I_a||X\rangle$ (the reduced matrix element) is not equal to zero†.

At this stage we should mention the essential difference between our results and those of Kulish *et al.* for the a_n series, which is that their equation is obtained without requiring R to be unitary. This is done by setting V_3 to be the vector representation ($\square$ in the usual Young tableaux notation); $R_{V\square}$ is known for any representation V of a_n, and is linear in $\frac{1}{u}$, so that T vanishes. Their equation for R is equivalent to the YBE; ours only looks at

†In going from (5.11) to (5.13), we have used our freedom to rescale u to set $u \mapsto -u$, so as to be consistent with the R-matrices given earlier. At the expense of this we are able to retain both Drinfeld's notation for the Yangian and a neater form for the R-matrices.

the first two terms in the limit $v \to \infty$. However, because T vanishes, they also have to take into account terms symmetric in $a \leftrightarrow b$ in (5.9), which involve the symmetric third order Casimir operator of a_n, d_{abc}. This adds terms

$$\frac{\langle Y||d_{bca}I_c \otimes I_b||X\rangle}{\langle Y||1 \otimes I_a||X\rangle} \tag{5.14}$$

to both numerator and denominator of (5.13); any representations for which these do not always vanish cannot have unitary R-matrices, since unitarity requires $\tau_X(u)\tau_X(-u) = 1$. The effect is that for those representations of a_n for which unitary R-matrices exist, our equation is the same as theirs, and can be used to obtain those unitary solutions given in their paper. When (5.14) is non-zero for some X, Y, and a unitary solution does not exist, our method has nothing to say about the solution. The other point is that Kulish *et al.* did not continue after reaching their analogue of (5.13), whereas we wish to investigate and classify its solutions.

To deal with (5.13) we first need to know for which X, Y (such that $Y \subset adjoint \otimes X$) the reduced matrix element $\langle Y||1 \otimes I_a||X\rangle$ vanishes. When we are examining $R_{VV}(u)$ (that is to say, R acting in two identical representations $V = V'$) we can split the components of $V \otimes V$ into those appearing symmetrically and those appearing antisymmetrically in the tensor product. Now $\langle Y||1 \otimes I_a + I_a \otimes 1||X\rangle$ vanishes, and so

$$\langle Y||1 \otimes I_a||X\rangle = \frac{1}{2}\langle Y||1 \otimes I_a - I_a \otimes 1||X\rangle \ .$$

Thus for the reduced matrix element to be non-zero, X and Y must have opposite parity[‡]. We now proceed on the assumption that, conversely, when X and Y have opposite parity, the matrix element is non-zero. This is certainly true when $V \otimes V$ only contains two states of weight ω_X, since the highest weight ω_X of X is chosen to be orthogonal to the state of the same weight in Y:

$$\langle \omega_X|1 \otimes I_a + I_a \otimes 1|\omega_Y\rangle = 0 \ .$$

Our system of equations (5.13) then applies to all X, Y of opposite parity such that $Y \subset adjoint \otimes X$.

[‡]Equal to ± 1 and defined by $\eta|X\rangle = \mathbf{P}|X\rangle$ where $|X\rangle$ is given as a vector in $V \otimes V$ by means of the Clebsch-Gordan matrix C^X_{VV}.

For an R-matrix to exist, it is necessary that this system of equations have a solution. In general, however, the system will be overdetermined. We now proceed to investigate the existence and uniqueness of solutions of this system.

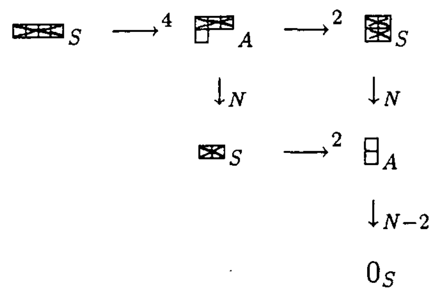
Existence. We check this by forming the representations $W \subset V \otimes V$ into the nodes of a bipartite graph. Starting with the representation of highest weight $\Omega = 2\omega_V$, where ω_V is the highest weight of V, we write $X \longrightarrow Y$ (*i.e.* we draw a directed edge from X to Y) whenever $Y \subset X \otimes$ *adjoint* and X, Y have opposite parity, and label each such edge with the number $C_2(X) - C_2(Y)$. (Note that $X \rightarrow^a Y$ is then equivalent to $Y \rightarrow^{-a} X$.) The set (5.13) is consistent, and thus R is well-defined, if and only if, for every pair of representations $P, Q \subset V \otimes V$, the set of labels on each possible route from P to Q is the same. This is the same as saying that all closed paths on the graph must give $\tau_P/\tau_P = 1$ if the system of equations is to be soluble. A graph for which this is true will be said to be *consistent.*

Uniqueness. The graph described is always connected, since the highest weights of the components of $V \otimes V$ differ by positive roots, and are linked by $1 \otimes I_a - I_a \otimes 1$. Thus (if R exists) any one τ_W is sufficient to determine all of the others. Hence R is defined up to an overall factor, dependent on u. We will choose this factor so that the coefficient of the representation with highest weight Ω is one. Note also that, as a result of (5.13), $\lim_{u\rightarrow\infty} R(u) = 1$.

This reproduces precisely the chains and trees for all the examples of which we know: those given earlier, various others associated with $SO(N)$ and $Sp(2n)$[23,73,86], those in the relevant representations of $SU(N)$[45], and those in the defining representations of all the exceptional groups[22] except e_8. As a brief example we give here the graph corresponding to R_{VV} for V the defining (seven dimensional) representation of g_2, which is one of the R for exceptional groups previously found using the (analytic) Bethe ansatz[22]. Labelling representations in terms of a basis of fundamental weights so that $V = (1, 0)$ the graph is then

$$\begin{array}{ccccc} (2,0)_S & \longrightarrow^1 & (0,1)_A & \longrightarrow^6 & (0,0)_S \\ \downarrow_4 & & & & \\ (1,0)_A & & & & \end{array}$$

This also illustrates the obvious fact that any graph which is a chain or a tree is consistent. The graph for the second rank symmetric traceless representation of $SO(N)$ is



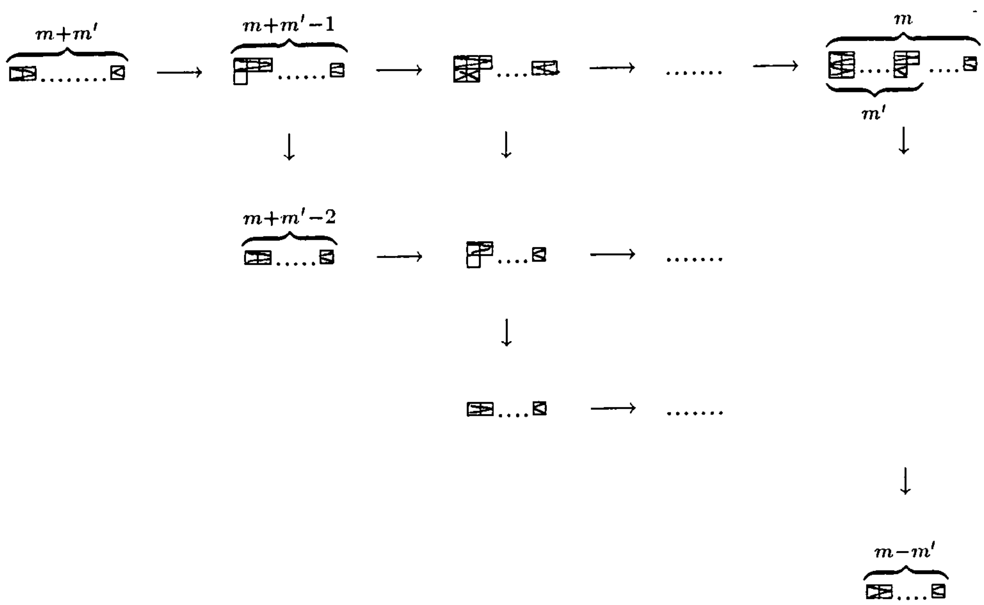
so that the two trees we gave earlier to define the R-matrix are just the two maximal trees of the above consistent graph. Unfortunately, we have not been able to formulate a general method for determining whether a given graph is consistent. This lack of a general method is a severe problem: although we are able to do exhaustive checks, a general method is needed if we are to gain deeper insight.

We have also found some new R-matrices using this method, including those for some representations of a_n whose Young tableaux are rectangular (although we have not shown that the graphs of all such representations are consistent) and, generalizing the example above, for completely symmetric representations of $SO(N)$, which we give in a moment. Furthermore, the graphs for all fundamental representations of c_n are consistent, enabling the construction of the corresponding R-matrices and thus the S-matrices for the particle multiplets in these representations. As mentioned at the end of chapter three, these are currently being investigated. We have tested many other graphs (with the help of the LiE computer algebra package[88]) and found them mostly to be inconsistent. All of our results for individual cases agree with the general characterization found by Drinfeld, which will be explained shortly.

Now consider the cases where $V \neq V'$. We have no general method for determining when $\langle Y||1 \otimes I_a||X\rangle \neq 0$. However, an intriguing fact to emerge from the study of the $V = V'$ graphs is that, in all of the consistent examples, whenever $X \subset adjoint \otimes Y$, X and Y have opposite parity: in other words, it seems that for consistent graphs the parity principle is redundant. We should like to emphasize that this is not true of inconsistent graphs, and that it remains only a conjecture for consistent graphs. If we go ahead and analyse the consistency of $V \neq V'$ graphs on the assumption that, if the graph is going to be consistent, $\langle Y||1 \otimes I_a||X\rangle \neq 0$ whenever $X \subset adjoint \otimes Y$, we obtain matrices $R_{VV'}$, all of which agree with known R-matrices.

As an illustrative example of new R-matrices, we generalize (5.4) by calculating the R-matrices in symmetric, traceless representations of $SO(N)$. These could be used to solve the generalization of the XXX magnet in which the (isotropically coupled) spins take arbitrary directions in N dimensions, as advocated by Reshetikhin[86].

Let m, m' be the representations with highest weight $(m, 0, .., 0)$ and $(m', 0, .., 0)$ (with respect to a basis of fundamental weights), where $m \geq m'$. For these representations, the graph is



(where representations are denoted by the usual Young tableaux, with a trace removed from all symmetric indices). The differences of the Casimirs have not been shown on this diagram. They can be calculated easily using the inverse Cartan matrix, and are found to satisfy the given requirement, *i.e.* that the rectangles in the graph commute. Substituting their values into (5.13) we find that the R-matrix obtained from (5.12, 5.13) is

$$R_{mm'}(u) = \sum_{k=0}^{m'} \sum_{q=0}^{k} \left(\prod_{r=1}^{k} [m + m' + 2 - 2r] \prod_{s=1}^{q} [m + m' + N - 2s - 2] \right) P_{(m+m'-2k,k-q,0,..0)}$$

(where we have rescaled u by a factor of 4 to make the expressions less cumbersome). This agrees with the R-matrices calculated earlier.

If we wish to turn our R-matrices into S-matrices, we find that crossing symmetry fixes the scale of u. It requires that

$$S_{VV}(\theta) = \left(S_{\bar{V}V}(i\pi - \theta)\right)^T \quad ,$$

where θ is the rapidity, and T means transpose and conjugate in the first space, so that if i, j and k, l label states in the incoming and outgoing representations respectively, and $(v_i)^* = v^i$, then $\left(S_{ij}^{kl}\right)^T = S_{ki}^{lj}$. Now $S(0)$ acts as the identity $\delta_i^k \delta_j^l$, and so the crossed version of $S(0)$ is $\delta_l^k \delta_j^i \propto P_0$ (the singlet representation). Hence we need $S(i\pi) \propto P_0$. But where P_0 is present in the decomposition we note that the tensor product of the singlet and adjoint representations is just the adjoint representation, and so in our definition (5.12, 5.13) we must use our freedom to rescale u to put

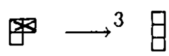
$$\theta = \frac{4\pi i}{c} u \ , \qquad (5.15)$$

where $c = C_2(adjoint)$. Note that c is proportional to the dual Coxeter number of the algebra, where the constant of proportionality is determined by the normalization of the inner product.

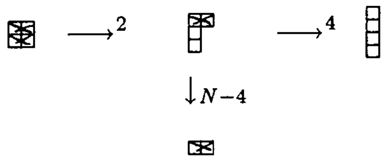
The tensor product graph also has important implications for the fusion procedure. Consider how the fusion procedure works. Given $\check{R}_{VV'}$, if there is a u_0 at which $\check{R}_{VV'}(u_0) \propto P_W$ for some $W \subset V \otimes V'$, then we can calculate $\check{R}_{VW}$, $\check{R}_{V'W}$ and $\check{R}_{WW}$. Using the notions of this section, we can reformulate all the results of the fusion procedure as statements about graphs. For example, if $\check{R}_{VV}(u_0) \propto P_W$, then the statement would be "if there is a unique node W connected to the rest of the graph $V \otimes V$ by just one link with value u_0, then the graphs $W \otimes V$ and $W \otimes W$ are consistent". Our checks also show that the fusion procedure is exhaustive: all consistent graphs can be reached in this way from the graphs of basic representations. Thus we can remove the YBE altogether from a description of the fusion procedure for irreducible representations, which is now purely a statement about representations of Lie algebras.

Other results can also be formulated in terms of graphs: for instance, Drinfeld's result that, for $\mathcal{A} \neq a_n$, there is no R-matrix in the adjoint representation. This says that the graph $adjoint^{\otimes 2}$ is inconsistent - a highly plausible result (since from the definition we can see that this will be the most connected graph), and easy to prove by exhaustion.

We can also make some useful but incomplete remarks about R-matrices in reducible representations on the basis of the results of this chapter so far. Firstly, the foregoing statement about the fusion procedure can be extended to R-matrices in reducible representations: if a part of a graph can be separated from the rest of the graph by removal of an edge, then an R-matrix exists in the corresponding representation. However, the tensor product graph does not completely describe such R-matrices. Consider (3.9) and (3.11). In both cases the coefficients of the representations with multiplicity one are correctly given by the graph:



for the former, and



for the latter. However, in neither case does the graph enable us to say anything about the coefficients of the components of multiplicity greater than one. Further, in general cases we find that we need to know the parity of the components in order to ensure the non-vanishing of the reduced matrix element.

It is thus clear, both from the incompleteness of the graph description and from the lack of a general method for its implementation, that a different approach is needed. Such an approach is provided by considering the representation theory of the Yangian.

5.2 Representations of the Yangian

We shall first describe the work of Drinfeld, who both introduced the Yangian and described many of its representations. Theorem numbers refer to those in his paper[14]. Recall the description of $Y(\mathcal{A})$ in chapter two. Drinfeld sought[14] to construct representations $\tilde{\rho}$ of $Y(\mathcal{A})$ as follows. Starting from a representation ρ of $\mathcal{A}$,

$$\tilde{\rho}(I_\lambda) = \rho(I_\lambda) \ , \tag{5.16}$$

he then needed to define $\tilde{\rho}(J_\lambda)$ in a way consistent with the defining relations of $Y(\mathcal{A})$. One way of doing this is to set

$$\tilde{\rho}(J_\lambda) = 0 \ . \tag{5.17}$$

However, he showed that it is not possible to do this for all irreducible representations. This is because, although $\tilde{\rho}$ is clearly consistent with (2.11 2.13), it is not, in general, consistent with (2.16). Consistency is only possible for representations in which the right hand side of (2.16) vanishes. This is the case for the following representations (theorem seven), although not necessarily only for these representations[41].

Let n_α be the coefficient of the simple root α in the expansion of the highest root $\alpha_{\max}$, and let $k_\alpha = (\alpha_{\max}, \alpha_{\max})/(\alpha, \alpha)$. Let the corresponding fundamental weight be ω_α. The representation of the group with highest weight Ω may then be extended to a representation of $Y(\mathcal{A})$ for

i) $\Omega = \omega_\alpha$ when $n_\alpha = k_\alpha$

and ii) $\Omega = t\omega_\alpha$ when $n_\alpha = 1$ (t a positive integer).

A sketch[41] of the proof of this is as follows. First, we need to know in which representation the right-hand side of (2.16) acts. Since the J_a form an adjoint representation of $\mathcal{A}$, it is clear that the left-hand side of (2.20) is contained in $(adjoint^{\otimes 2})_A$. Further,[§]

$$f^{d[ab}u^{c]d} = 0 \quad \Rightarrow \quad u^{ab} = f^{abc}v^c \ ,$$

so that $W = (adjoint^{\otimes 2})_A - adjoint$. The image of the right-hand side of (2.16) in $End(V)$ is zero if $W \otimes V \not\supset V$ (or $W \not\subset V \otimes V^*$) by the Wigner-Eckhart theorem. Knowing the highest weights of the irreducible components of W, Drinfeld was then able to find the V for which this is true. These were the representations given above, which match precisely the representations with consistent graphs, in the sense that whenever V and V' are in Drinfeld's set, the graphs $V \otimes V$, $V' \otimes V'$ and $V \otimes V'$ are all consistent. The $\mathcal{A}$-representations (ireducible or reducible) which are irreducible as $Y(\mathcal{A})$-representations are of course those in which R-matrices may be constructed, of which a list of those currently known is given in appendix four.

Drinfeld's method does not give the spectral decomposition of the R-matrix. However, the results of section (5.1) can also be derived from the Yangian. We assume $\tilde{\rho}$ in the form (5.16, 5.17) but, instead of investigating the consistency of $\tilde{\rho}$ with the defining relations of $Y(\mathcal{A})$, we consider the implications of $\tilde{\rho}$ for (2.22). Substituting $x = J_a$ and $x = I_a$ respectively into (2.22), we see that $R_{XY}(u) \equiv \rho_X \otimes \rho_Y(\mathcal{R}(u))$ must satisfy

$$R_{XY}(u)\left(u1 \otimes \rho_Y(I_a) - \frac{1}{2}f^{abc}\rho_X(I_c) \otimes \rho_Y(I_b)\right) = \left(u1 \otimes \rho_Y(I_a) + \frac{1}{2}f^{abc}\rho_X(I_c) \otimes \rho_Y(I_b)\right) R_{XY}(u \tag{5.18}$$

and

$$[R_{XY}(u), 1 \otimes \rho_Y(I_a) + \rho_X(I_a) \otimes 1] = 0 \ , \tag{5.19}$$

where 1 is the appropriate representation of the identity. Equation (5.19) is just (5.8), whilst equation (5.18) (theorem four) coincides with (5.10). The general a_n case is dealt with in theorem nine, and reproduces the results of Kulish *et al.* The rest of the analysis of the graph approach now follows through. Notice that since (5.19) and (5.18) are the equations we obtained originally by looking at the terms of orders $\frac{1}{v}$ and $\frac{1}{v^2}$ in the YBE, Drinfeld's results imply that there is no need to go to higher orders to compute the R-

[§]This statement is equivalent to the result that the second cohomology group $H^2(\mathcal{A})$ of a Lie algebra is trivial, *cf* footnote on p22.

matrix, *i.e.* that the complete structure of $Y(\mathcal{A})$ is determined by the generators at grades 0 and 1, as pointed out in chapter two.

The methods we have used can also be applied to trigonometric R-matrices - recall that the R-matrices of chapter four had essentially the same structure as the rational R-matrices. In fact, between submission of the paper on which this chapter is based[3] and its acceptance for publication, a paper appeared[89] by Zhang, Gould and Bracken in which the same ideas of tensor product graphs were introduced in this context. Their method was analogous to that above rather than to that of section (5.1), in that they derived their results from the disappearance of the commutator of the R-matrix with the coproduct of the elements $X_0^{\pm}$ of the quantized affine algebra (which was also the method used by Jimbo to construct (4.10)). The only graphs they investigated in detail were chains, and all of their results agree with ours. The equation (2.8) has also been solved for the exceptional quantum group cases by Sergeev[90], although he did not introduce the concept of the graph. There seems currently to be no approach to trigonometric R-matrices analogous to Drinfeld's approach to the Yangian.

In addition to analysing those representations of $Y(\mathcal{A})$ which are irreducible as representations of $\mathcal{A}$, Drinfeld also showed (theorem eight) that there is always an irreducible representation of $Y(\mathcal{A})$ in the $\mathcal{A}$-representation $X = adjoint \oplus singlet = \mathcal{A} \oplus \mathbf{C}$. (Note that there can be no representation of $Y(\mathcal{A})$ in the adjoint representation alone, since $W \subset adjoint^2$.) The action of $Y(\mathcal{A})$ in this representation is given by

$$\begin{aligned} \rho(I_a)x = [I_a, x] \qquad & \rho(J_a)x = < x, I_a > \\ \rho(I_a)\lambda = 0 \qquad & \rho(J_a)\lambda = d\lambda I_a \end{aligned} \tag{5.20}$$

where $(x, \lambda) \in \mathcal{A} \oplus \mathbf{C}$, $<,>$ is an inner product on $\mathcal{A}$, and $d \in \mathbf{C}$ is a number dependent on $\mathcal{A}$ and on the choice of inner product. Once again, Drinfeld did not construct the R-matrix acting in this representation. This has recently been done by Chari and Pressley[31]. Note in particular that, for $\mathcal{A} = SO(N)$, the adjoint representation is the second-rank antisymmetric tensor representation, so that, in the notation of chapter three, $X = \boxminus \oplus 0$. Thus the R-matrices they construct for the b_n and d_n algebras are precisely (3.11).

Chari and Pressley's work on the Yangian initially concentrated[29,30] on $\mathcal{A} = a_1$, but has more recently dealt with the general case. In their paper[31] they give two main results. The second is the computation of R_{XX}, which we shall describe shortly. The first takes the form of a

Singularity theorem. Let V be an irreducible¶ finite-dimensional representation of $Y(\mathcal{A})$ with highest weight vector v^+, and let $\mathcal{B}$ be a diagram subalgebra‖ of $\mathcal{A}$, where neither $\mathcal{A}$ nor $\mathcal{B}$ is of type e_6 or $a_l\,(l > 1)$. Assume further that $V_{\mathcal{B}}$, the $\mathcal{B}$-subrepresentation of V generated by v^+, is non-trivial. Then $R_{VV}(u)$ has a singularity (*i.e.* is not invertible) at $u = \pm\frac{1}{4}c(\mathcal{B})$, where c is the value of the quadratic Casimir operator in the adjoint representation.

In proving this, the first fact of which to take note is that R_{VV} must have singularities at $\pm\frac{1}{4}c(\mathcal{A})$. For irreducible V this may easily be seen from the tensor product graph: $V \otimes V \supset \textit{adjoint} \oplus \mathbf{C}$ and these two components have opposite parity, so that they are linked by an edge of the graph with label $\frac{1}{4}c(\mathcal{A})$. Chari and Pressley, however, use a much more sophisticated method. Given any representation V of $Y(\mathcal{A})$, the automorphism (2.21) generates a one-parameter family of representations $V(u)$, $u \in \mathbf{C}$. The R-matrix then gives the intertwiner**

$$\check{R} : V \otimes V(u) \longrightarrow V(u) \otimes V \ ,$$

so that the task is to show that $V \otimes V(\pm\frac{1}{4}c)$ and $V(\pm\frac{1}{4}c) \otimes V$ are not isomorphic representations of $Y(\mathcal{A})$. The proof of this requires a number of detailed results about the various duals of V, together with an understanding of Drinfeld polynomials[91] (whose coefficients give the highest weights of representations of $Y(\mathcal{A})$), and is beyond the scope of this work.

In attempting to show that R has singularities at $\pm\frac{1}{4}c(\mathcal{B})$, the problem is encountered that, although $Y(\mathcal{B})$ is a subalgebra of $Y(\mathcal{A})$, it is not a Hopf subalgebra (*i.e.* $\Delta(Y(\mathcal{B})) \not\subset Y(\mathcal{B}) \otimes Y(\mathcal{B})$). Thus on analysing $V \otimes V(u)$ we may either take the tensor product and

¶It need not, however, be irreducible as a representation of $\mathcal{A}$.

‖A diagram or regular subalgebra is generated by the operators associated with a subdiagram of the Dynkin diagram of $\mathcal{A}$.

**Note that in the Yangian representation theory the fusion procedure for R-matrices then describes the decomposition of $V \otimes V(u)$ into irreducible $Y(\mathcal{A})$-representations, analogous to the decomposition of $V \otimes V$ into irreducible $\mathcal{A}$-representations.

then restrict to $Y(\mathcal{B})$ or restrict to $Y(\mathcal{B})$ and then take the tensor product; the results will not necessarily coincide. However, Chari and Pressley prove that the latter procedure gives $V_{\mathcal{B}} \otimes V_{\mathcal{B}}(u)$ as a $Y(\mathcal{B})$-subrepresentation of $V \otimes V(u)$, and subsequently that $R_{V_{\mathcal{B}}V_{\mathcal{B}}}$ carries $V_{\mathcal{B}} \otimes V_{\mathcal{B}}(u)$ into $V_{\mathcal{B}}(u) \otimes V_{\mathcal{B}}$, and that any singularity of $R_{V_{\mathcal{B}}V_{\mathcal{B}}}$ is also a singularity of R_{VV}, giving the required result. In addition to the locations of the singularities u_0, Chari and Pressley also give, for the R-matrices which they calculate, the highest weights of the representations to which the R-matrix is then restricted, $V \otimes V(u_o)$.

At first, one might hope that this theorem would give all the values of u at which we can apply the fusion procedure, and thus describe the bootstrap structure in the representation theory of $Y(\mathcal{A})$. However, there are certain problems. Looking now only at simply-laced cases, we find that singularities in R neither imply nor are implied by particle fusings in the PEST. The forward implication fails because we also need to know the representation associated with the singularity: only those singularities corresponding to fundamental representations[††] of $Y(\mathcal{A})$ give particle fusings. Thus in (3.11) we saw that the singularity at $u = 4$ gave a fusing whilst those at $u = 2$ and $u = N - 4$ in general did not. Although Chari and Pressley compute the representations for the singularities of R_{XX}, the computation for general R_{VV} requires some knowledge of $V \otimes V$ and its tensor product graph.

The reverse implication also fails: we find singularities in R-matrices which are not diagram singularities but which, when these correspond to particle fusings, *are* present in the PEST bootstrap structure. A comparison of the fusing structure of PESTs with the predicted diagram singularities of R-matrices not yet explicitly computed reveals many additional fusings in the PESTs, which may arise in the R-matrix as singularities in coefficients of representations of multiplicity one (which are of the form $[a]$, and are calculable by means of the tensor product graph), but which might also arise in coefficients of representations of multiplicity greater than one (which, as may be seen from (3.11), are more complicated). Thus it seems that to examine the bootstrap procedure at work in the representation theory of the Yangian the singularity theorem and the tensor product graph

[††]The ith fundamental representation of $Y(\mathcal{A})$ may have more than one irreducible component as a representation of $\mathcal{A}$, but its highest component as an $\mathcal{A}$-representation is the ith fundamental representation of $\mathcal{A}$.

are together not enough, and we need some explicit knowledge of the R-matrices.

The other limitation of the singularity theorem is that it only gives the singularities of R_{VV}, not those of $R_{V_1V_2}$ when $V_1 \neq V_2$, and so cannot say anything about the fusing of different particles in the bootstrap. For example, the singularities of (3.9) cannot be obtained in this way.

The procedure by which Chari and Pressley construct $R_{XX}(u)$ for $X = adjoint \oplus \mathbf{C}$ is as follows. First, the coefficients of the components of $X \otimes X$ (as $\mathcal{A}$-representations) with multiplicity one are calculated, which is done predominantly by looking at the diagram singularities and their associated representations. As we discussed above, not all coefficients can be deduced in this way, and in these cases Chari and Pressley use a more sophisticated approach. Recall, however, that all such coefficients may instead be deduced from the tensor product graph.

For the three components of $X \otimes X$ which are isomorphic to $\mathcal{A}$ ($adjoint \otimes \mathbf{C}$, $\mathbf{C} \otimes adjoint$ and $adjoint \subset adjoint^{\otimes 2}$), they use a basis

$$1 \otimes I_a + I_a \otimes 1 \ , \qquad 1 \otimes I_a - I_a \otimes 1 \quad \text{and} \quad \frac{1}{2} f^{abc} I^c \otimes I^b \ .$$

The action of the generators of $Y(\mathcal{A})$ in X is given by (5.20), allowing us to compute the action of $\Delta(I_a)$ and $\Delta(J_a)$ on elements of the $Y(\mathcal{A})$-representation $X(u) \otimes X$. In fact, a Chevalley basis for the algebra is used, in which

$$\check{R}(u)\left(E_\beta^+ \otimes E_\beta^+\right) = E_\beta^+ \otimes E_\beta^+ \tag{5.21}$$

and

$$\check{R}(u)\left(E_\beta^+ \otimes E_{\beta-\alpha_i}^+ - E_{\beta-\alpha_i}^+ \otimes E_\beta^+\right) = -[2]\left(E_\beta^+ \otimes E_{\beta-\alpha_i}^+ - E_{\beta-\alpha_i}^+ \otimes E_\beta^+\right) \ , \tag{5.22}$$

where β is the highest root and α_i the ith simple root. The $-[2]$ corresponds to the a_1 diagram singularity defined by the ith node of the Dynkin diagram.

If we now act‡‡ with $\Delta(J(E_\beta^-))$ on (5.21) and with $\Delta(J(E_{\beta-\alpha_i}^-))$ and $\Delta(J(H_\beta))$ on (5.22), we obtain three equations which we may solve exactly to give the action of $R_{XX}(u)$ on the

‡‡We cannot use the usual I_a, J_a notation in this basis, and so use instead $J(I_a) \equiv J_a$.

adjoint components as a 3×3 matrix. Similar methods give the action on the two singlet components $\mathbf{C} \otimes \mathbf{C}$ and $\mathbf{C} \subset adjoint^{\otimes 2}$.

A final point made by Chari and Pressley determines when a fusing to X may occur in R_{XX}. They prove that such a fusing may occur only if $u_0 = \frac{1}{6}c(\mathcal{A})$ - a result immediate from (2.35, 5.15), in that a pair of V particles can only have a fusing $VV \to \bar{V}$ at relative rapidity $\frac{2i\pi}{3}$.

To summarize: we believe that the exact fusing structure of the bootstrap occurs in the representation theory of the Yangian, since it does so in all known examples of R-matrices. To examine this conjecture we need to know, for any three fundamental representations of the Yangian V_1, V_2 and V_3, whether $R_{V_1V_2}$ has a singularity corresponding to V_3, or equivalently whether there exists some u_0 at which $R_{V_1V_2}(u)$ is restricted to V_3. Various methods give results for some cases:

- The tensor product graph[3] gives all the fusings when V_1 and V_2 are irreducible as representations of $\mathcal{A}$, and $V_1 \otimes V_2$ does not contain multiplicities (and thus all information associated with the bootstrap for a_n and c_n), and gives some of the structure of other R-matrices.
- The singularity theorem[31] gives some information about the possible fusings of $R_{V_1V_1}$, and aids in its computation.
- The construction for all $\mathcal{A}$ of R_{XX} gives all the information about how one particular particle in each theory (except c_n) fuses with itself.
- The fusion procedure[1,45,23] enables the calculation of R-matrices $R_{V_1V_2}$ for certain representations of classical $\mathcal{A}$.

A full summary of the current state of affairs is given in appendix four.

However, whilst such methods have enabled the construction of a number of R-matrices for particular representations or algebras, they are not together enough to achieve a general description, although the presence of the bootstrap structure suggests that one should exist. Such a description, in a form which brings insight into the connection with PESTs, is still a long way off. As Chari and Pressley note[30], the computation of general R-matrices "...remains a difficult open problem."

It is clear that the Yangian is an important algebra for the mathematics of the YBE, where it dictates not only the structure of rational R-matrices but also, for general q, that of trigonometric R-matrices. To quote Cherednik[92] "I think that [Yangians] ..should be more important for mathematics and physics than the q-analogues of universal enveloping algebras now in common use". What has not yet been pointed out is the extent to which the Yangian is important in understanding the *physics* of integrable field theories. This has recently become apparent through Bernard's identification of the Yangian as a physical charge algebra, and will be fully described in the next chapter.

Chapter 6

The Yangian as a charge algebra

If the algebra underlying the S-matrix of an integrable quantum field theory is the Yangian, it might be expected that the charge algebra of the theory would be the Yangian. This chapter begins with the work which shows that this is indeed the case.

Recall the structure of the Yangian. There were commutation relations (2.11, 2.13), the Serre relation (2.16), the coproduct (2.12, 2.14), the automorphism (2.21), and the R-matrix. We shall show that this is precisely the algebra of conserved charges in the field theories in which we are interested: the commutation relations give the commutators of the first two conserved charges, the coproduct gives the action of the charges on products of asymptotically separate states, the T_u automorphism is the Lorentz boost (so that u is proportional to the rapidity) and of course the R-matrix corresponds to the S-matrix.

In fact, some of these results have been around for a long time: as long ago as 1978, Lüscher[93], investigating the conserved charges of the $O(N)$ σ-model, found much of what is actually the $SO(N)$ Yangian. Many other papers[94,95,96,97,98,99] also contained parts of it. Lüscher showed that his equations led to factorization of the S-matrix, a result which would later be paralleled by Drinfeld's proof that the Yangian is a quasitriangular Hopf algebra. However, even after Drinfeld's 1985 paper, the Yangian structure of the conserved charges remained unnoticed, and it was only fully investigated in 1990 by Bernard[24].

Before sketching Bernard's results, we first describe the (classical) definition of the charges. Our starting point is a field theory in 1+1 dimensions with a current $j_\mu(x,t)$ which is conserved,

$$\partial^\mu j_\mu(x,t) = 0 \ , \tag{6.1}$$

Lie algebra valued,

$$j_\mu(x,t) = I_a \, j_\mu^a(x,t)$$

(where I^a generate $\mathcal{A}$), and curl free,

$$\partial_\mu j_\nu - \partial_\nu j_\mu + [j_\mu, j_\nu] = 0 \ . \tag{6.2}$$

The prime example of such theories is the principal chiral field, where the currents $j_\mu^L = g^{-1}\partial_\mu g$ and $j_\mu^R = \partial_\mu g \, g^{-1}$ are conserved (because of the equations of motion) and curl free (because they are pure gauges).

Brézin, Itzykson, Zinn-Justin and Zuber[100] showed that in this case there are actually infinitely many conserved currents. Inductively, suppose we have conserved currents $j_\mu^{(r)}$ for $r = 1, ..., n$, so that $j_\mu^{(r)} = \epsilon_{\mu\nu}\partial^\nu \chi^{(r)}$ for some scalar functions $\chi^{(r)}$, and that for $r \leq n$

$$j_\mu^{(r)} = D_\mu \chi^{(r-1)} \quad \text{where} \quad D_\mu \equiv \partial_\mu + j_\mu \ . \tag{6.3}$$

Then, defining $j_\mu^{(n+1)} = D_\mu \chi^{(n)}$, we have

$$\begin{aligned} \partial^\mu j_\mu^{(n+1)} &= \partial^\mu D_\mu \chi^{(n)} \\ &= D_\mu \partial^\mu \chi^{(n)} \\ &= D_\mu \epsilon^{\mu\nu} j_\nu^{(n)} \\ &= \epsilon^{\mu\nu} D_\mu D_\nu \chi^{(n-1)} \\ &= 0 \end{aligned}$$

since $[D_0, D_1] = 0$ by (6.2). Then we can write $j_\mu^{(n+1)} = \epsilon_{\mu\nu}\partial^\nu \chi^{(n+1)}$. The induction is completed by putting $j_\mu^{(0)} = j_\mu$ and setting $\chi^{(-1)} = 1$.

The second current is

$$j_\mu^{(1)} = D_\mu \chi^{(0)} = \epsilon_\mu{}^\nu j_\nu + j_\mu \chi^{(0)} \ , \quad \text{where} \quad \chi^{(0)} = \int_{(-\infty,t)}^{(x,t)} \epsilon_\mu{}^\nu j_\nu \, dy^\mu \ .$$

Note that $\chi^{(0)}$ is only weakly dependent on the contour, because of conservation of j_μ, and that because of the integral needed to calculate $\chi^{(0)}$, $j^{(1)}$ is *non-local*. In addition, because of the term $j_\mu \chi^{(0)}$, this current is not necessarily valued in the Lie algebra. We shall therefore use instead the Lie algebra valued current

$$j_\mu^{(1)} = \epsilon_\mu{}^\nu j_\nu + \frac{1}{2}\left[j_\mu, \chi^{(0)}\right] \ ,$$

which is also conserved and curl free. We shall therefore be examining the algebra of

$$Q^{(0)\,a} = \int_{-\infty}^{\infty} j_0^a dx \tag{6.4}$$

and

$$Q^{(1)\,a} = \int_{-\infty}^{\infty} j_1^a dx + \frac{1}{2} f^{abc} \int_{-\infty}^{\infty} j_0^b(x) \int_{-\infty}^{x} j_0^c(y)\, dy\, dx \ , \tag{6.5}$$

where we have set

$$\epsilon_{01} = 1 \text{ and } \eta_{11} = -\eta_{00} = 1 \ .$$

In these expressions, the value of t is the same throughout. The $Q^{(i)}$ are conserved because the $j_\mu^{(i)}$ are divergence-free (assuming $j_\mu^{(i)} \to 0$ at spatial infinity). We shall describe the algebraic structures associated with these classical charges shortly, but first we give a brief description of the quantum theory.

6.1 The Yangian as a charge algebra in Quantum Field Theory

The zero curvature equation (6.2) does not hold in quantum field theory; we cannot define $j_\mu^b(x)j_\nu^c(x)$. Instead we must find the short distance operator product expansion (OPE) of the currents. In order to do this, Bernard[24] used an extension of a theorem due to Lüscher[93], which limited the form such an OPE could take on the basis of locality and conservation of j_μ, PT conservation, and Poincaré-covariance. Bernard further assumed that the OPEs close on the currents and their derivatives, and that they have a smooth ultraviolet limit (and hence are chirally split, since the ultraviolet limit is a conformal field theory)*. Note that these assumptions replace entirely the zero-curvature condition: thus the currents, although conserved, *cannot* be used to define an infinite number of quantum conserved charges using the BIZZ procedure as described.

Bernard obtained the equal-time OPE (in light-cone coordinates)

$$\begin{aligned} f^{abc} j_\pm^b(x) j_\pm^c(0) &= \frac{\lambda}{x^\pm} j_\pm^a(0) + \mathcal{O}(\log x) \qquad (6.6) \\ \frac{1}{2} f^{abc}\left(j_+^b(x)j_-^c(0) - j_-^b(x)j_+^c(0)\right) &= -\frac{\lambda}{4}\log(M^2x^+x^-)\left(\partial_+ j_-^a(0) - \partial_- j_+^a(0)\right) + \mathcal{O}(x\log x) \ , \end{aligned}$$

where M is a mass scale and λ is a constant which will be fixed by the charge algebra. One can then define the second conserved current via a point-splitting regularization, as

*Such conditions are true of the principal chiral field and the various Gross-Neveu models, but not of the $O(N)$ σ-models, which undergo a phase-transition in the ultraviolet limit. Lüscher's work dealt specifically with this latter case.

the limit as $\delta \to 0$ of

$$j_\mu^{(1)\,a}(x,t;\delta) = Z(\delta)\epsilon_\mu{}^\nu j_\nu(x,t) + \frac{1}{2} f^{abc} j_\mu^b(x,t)\chi^{(0)\,c}(x-\delta,t) \; ;$$

choosing $Z(\delta) = \frac{\lambda}{2}\log(M\delta) + \mathcal{O}(|\delta|^{1-0})$ then yields a conserved current, so that we can define the first two charges to be

$$Q^{(0)\,a} = \int_{-\infty}^{\infty} j_0^a dx \quad \text{and} \quad Q^{(1)\,a} = \int_{-\infty}^{\infty} j_0^{(1)\,a} dx \; .$$

These charges have commutators

$$\left[Q^{(0)\,a}, Q^{(0)\,b}\right] = i\hbar\, f^{abc} Q^{(0)\,c} \tag{6.7}$$

$$\left[Q^{(0)\,a}, Q^{(1)\,b}\right] = i\hbar\, f^{abc} Q^{(1)\,c} \; . \tag{6.8}$$

The first of these is simple, whilst the second requires application of the Jacobi identity. Note that (6.7) fixes the scale of $Q^{(0)}$ and so fixes $\lambda = \frac{c\hbar}{2\pi}$, where $c = C_2(adjoint)$ as in (5.15).

The coproduct of $Q^{(0)}$ is determined by its locality: it acts additively on asymptotic states, so that

$$\Delta(Q^{(0)\,a}) = 1 \otimes Q^{(0)\,a} + Q^{(0)\,a} \otimes 1 \; . \tag{6.9}$$

The coproduct of $Q^{(1)}$ is more complicated. In general, the coproduct is defined by letting the charge act on products of operators at equal times. Bernard found that

$$\begin{aligned} Q^{(1)\,a}\left(\phi_1(y_1)\phi_2(y_2)\right) &= Q^{(1)\,a}\left(\phi_1(y_1)\right)\phi_2(y_2) + \phi_1(y_1)Q^{(1)\,a}\left(\phi_2(y_2)\right) \\ &- \frac{1}{2} f^{abc} Q^{(0)\,b}((\phi_1(y_1))\; Q^{(0)\,c}((\phi_2(y_2)) \end{aligned}$$

or

$$\Delta(Q^{(1)\,a}) = 1 \otimes Q^{(1)\,a} + Q^{(1)\,a} \otimes 1 + \frac{1}{2} f^{abc} Q^{(0)\,c} \otimes Q^{(0)\,b} \; . \tag{6.10}$$

A schematic proof is shown in figure (6.1). The first integral in $Q^{(1)}$ is shown using the solid line contour, whilst the wavy line contour indicates the integral required to calculate $\chi^{(0)}$. First the solid contour and then the wavy contour are decomposed to obtain the integrals on the right and thus (6.10).

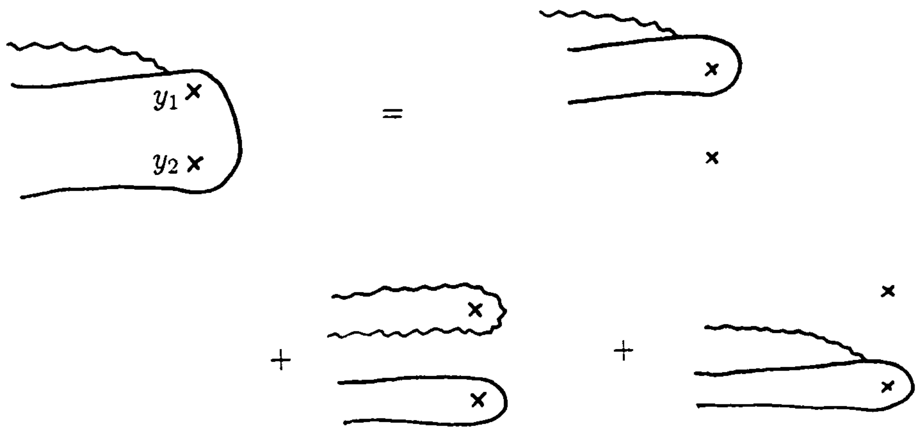


Figure 6.1: Comultiplication of $Q^{(1)}$

The Yangian Serre relation is much more difficult to work out and has not been calculated[101]. The Lorentz boost is obtained using contour techniques once more. The Lorentz boost generator is defined to be

$$M \equiv M_{01} = \int_{-\infty}^{\infty} x\, T_{00}(x,t) - t\, T_{01}(x,t)\; dx \tag{6.11}$$

where $T_{\mu\nu}(x,t)$ is the energy-momentum tensor, but in fact we do not need to know $T_{\mu\nu}$ explicitly in order to calculate the action of M on the charges: because of locality of $Q^{(0)}$,

$$\left[M, Q^{(0)\,a}\right] = 0 \ ,$$

whilst to obtain the action of M on $Q^{(1)}$ a Lorentz boost of $2\pi i$ is used, corresponding to a rotation of 2π in the Euclidean plane, to give $e^{\frac{2\pi M}{\hbar}} Q^{(1)\,a} e^{\frac{-2\pi M}{\hbar}} = Q^{(1)\,a} - \frac{i\hbar}{2} c\, Q^{(0)\,a}$, and thence

$$\begin{aligned} \left[M, Q^{(1)}\right] &= \frac{\hbar}{4\pi} f^{abc} Q^{(0)\,b} Q^{(0)\,c} \\ &= \frac{\hbar^2 c}{4i\pi} Q^{(0)\,a} \ , \end{aligned} \tag{6.12}$$

the method of calculation being as indicated in figure (6.2). This is precisely the normalization required by (5.15).

A Lorentz boost T_θ of rapidity θ then gives

$$Q^{(0)} \mapsto Q^{(0)} \qquad \text{and} \qquad Q^{(1)} \mapsto Q^{(1)} - \frac{\hbar c}{4\pi}\, \theta\, Q^{(0)} \ .$$

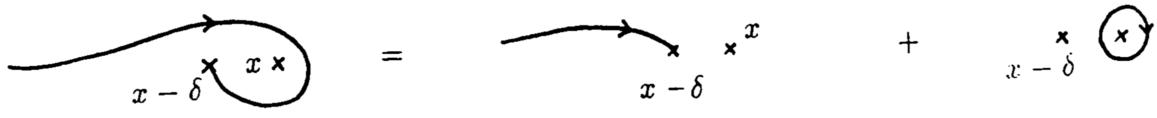


Figure 6.2: Lorentz boost of $2\pi i$ on $Q^{(1)}$

(Note that for $i = 0, 1$ we have

$$T_\theta(Q^{(i)}) \equiv e^{\frac{M\theta}{i\hbar}} Q^{(i)} e^{\frac{-M\theta}{i\hbar}} \quad \text{since} \quad [M,[M,Q^{(1)}]] = 0\ .)$$

This provides the automorphism (2.21) and thus the quasitriangular structure of the Yangian through

$$S(\theta_1-\theta_2)\left(T_{\theta_1}\otimes T_{\theta_2}\Delta(Q^{(i)})\right) = \left(T_{\theta_2}\otimes T_{\theta_1}\Delta(Q^{(i)})\right)S(\theta_1-\theta_2)\ , \tag{6.13}$$

which expresses the conservation of the charge $Q^{(i)}$ during the scattering process $|\theta_1\rangle_{in}\otimes|\theta_2\rangle_{in}\ \rightarrow\ |\theta_2\rangle_{out}\otimes|\theta_1\rangle_{out}$. Note, however, that this equation (along with unitarity and crossing-symmetry) only determines the S-matrix up to an overall CDD factor; it *does not* determine the S-matrix completely. This CDD factor is essential because, as we discussed in chapter three, it determines the pole structure. We have to *postulate* one CDD factor, and then use the bootstrap hypothesis to determine the rest. Thus (6.13) still does not allow us to deduce an exact S-matrix, and we must fall back on the evidence of the $\frac{1}{N}$-expansion or other methods to support our conjecture.

There are a few final points. Firstly, as was the case in chapters two and five, $Q^{(0)}$ and $Q^{(1)}$ alone are enough to determine the S-matrix up to these CDD factors. The first proof of this came from Lüscher, who proved[93] from (6.13) for $i = 0$ and 1 that the $O(N)$ σ-model S-matrix factorizes. Secondly, the action of M on asymptotic states is as $i\hbar\frac{\partial}{\partial\theta}$, so that on asymptotic states

$$Q^{(0)a}|\theta\rangle = i\hbar I_a|\theta\rangle \quad\Rightarrow\quad Q^{(1)a}|\theta\rangle = \frac{\hbar^2 c}{4\pi i}\theta I_a|\theta\rangle\ . \tag{6.14}$$

This is precisely the $Y(\mathcal{A})$-representation $V(u)$ associated with the representation V (5.16, 5.17) defined by Drinfeld.

6.2 The classical origins of Yangian symmetry

The contents of the previous section raise the question: to what extent is $Y(\mathcal{A})$ already present in the classical theory? In this section we shall attempt to answer this by working with the classical charges (6.4, 6.5). Firstly, we shall see that the Poisson brackets of the charges in the corresponding classical theories are (2.11, 2.13), and derive the Serre relation (2.16), not yet computed in either the quantum or classical theories. Secondly, we derive the coproduct (2.12, 2.14); this then leads to the identification of the algebra as a Poisson-Hopf algebra. The Yangian has trivial co-unit, but the antipode map (2.15) appears in both the classical and the quantum theories as the $\mathcal{PT}$ (parity- and time-reversal) transformation. The difference between the classical and quantum cases is that in the former the Lorentz boost acts trivially on the charges, so that the quasitriangular (R-matrix) structure apparently does not appear classically. Similar results have been derived independently for $Y(a_1)$ by Babelon and Bernard[102] in the more general context of dressing symmetries.

The charge algebra

To compute the Poisson brackets of the charges we need the canonical Poisson brackets of the current j^a_μ, which are model-dependent. In the Gross-Neveu and generalized chiral Gross-Neveu models we have[94]

$$\left\{j^a_\mu(x,t), j^b_\nu(y,t)\right\} = f^{abc} j^c_\sigma(x,t)\delta(x-y) \ , \quad \text{where } \sigma = |\mu - \nu| \ . \tag{6.15}$$

Defining $Q^{(0)\,a}$ and $Q^{(1)\,a}$ as in (6.4, 6.5), we obtain[94,98]

$$\begin{aligned} \left\{Q^{(0)\,a}, Q^{(0)\,b}\right\} &= f^{abc} Q^{(0)\,c} \\ \left\{Q^{(0)\,a}, Q^{(1)\,b}\right\} &= f^{abc} Q^{(1)\,c} \ . \end{aligned} \tag{6.16}$$

Having calculated (2.11, 2.13) we must now check (2.16). Attempts to calculate $\{Q^{(1)\,a}, Q^{(1)\,b}\}$

have not proved illuminating in the past[94,98], since it has no simple interpretation in terms of the charges. In fact,

$$\begin{aligned}\left\{Q^{(1)a}, Q^{(1)b}\right\} &= f^{abc}Q^{(2)c} + \frac{1}{4}f^{acd}f^{bef}f^{deg}\left\{Q^{(0)c}Q^{(0)g}Q^{(0)f}\right. \\ &- 2Q^{(0)c}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx_1\, dx_2\, j_0^g(x_1)j_0^f(x_2)\Theta(x_1-x_2) \\ &- \left. 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx_1\, dx_2\, dx_3\, j_0^g(x_1)j_0^c(x_2)j_0^f(x_3)\Theta(x_1-x_2)\Theta(x_1-x_3)\right\} \\ &+ \frac{1}{4}f^{fde}\left(f^{agd}f^{ecb} - f^{bgd}f^{eca}\right) \\ &\qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} j_0^c(x_1)j_0^g(x_2)j_0^f(x_3)\Theta(x_1-x_2)\Theta(x_2-x_3)\ ,\end{aligned}$$

where Θ is the step function

$$\Theta(x) = \begin{cases} 1 & x>0 \\ \frac{1}{2} & x=0 \\ 0 & x<0 \end{cases} .$$

If we now multiply by the structure constants and symmetrize appropriately, the variables of integration are symmetrized in such a way that this expression *does* close on the charges, and we get (2.16):

$$\frac{1}{2}f^{d[ab}\left\{Q^{(1)c]}, Q^{(1)d}\right\} = \frac{1}{24}f^{aip}f^{bjq}f^{ckr}f^{ijk}\left(Q^{(0)p}, Q^{(0)q}, Q^{(0)r}\right) .$$

In the principal chiral model we have, instead of (6.15),

$$\begin{aligned}\left\{j_0^a(x,t), j_0^b(y,t)\right\} &= f^{abc}j_0^c(x,t)\delta(x-y) \\ \left\{j_0^a(x,t), j_1^b(y,t)\right\} &= f^{abc}j_1^c(x,t)\delta(x-y) - \delta^{ab}\frac{\partial}{\partial x}\delta(x-y) \\ \left\{j_1^a(x,t), j_1^b(y,t)\right\} &= 0\ ,\end{aligned} \tag{6.17}$$

leading to the same algebra as above except for an ambiguity proportional to $f^{abc}Q^{(0)c}$ in $\{Q^{(0)a}, Q^{(1)b}\}$. This is caused by the derivative of the delta function in the second expression: integrating this term twice gives a step function, so that the Poisson bracket depends on the order in which the limits of integration tend to infinity. The Poisson bracket is well-defined only if this order is fixed[99]. Note, however, that adding a term $Kf^{abc}I^c$ to the right-hand side of the second equation in (2.13) does not affect the structure of the

Yangian: the coproduct, co-unit and antipode still satisfy Hopf algebra axioms. Note also that Bernard's OPE (6.6), since it antisymmetrizes on $\{a, b\}$, disguises the possibility of such Schwinger terms, which would affect (6.8). This point is commented on in Bernard's lecture notes[103]. A number of other authors have commented upon the problems due to the presence of such terms in Poisson brackets†, particularly in the definition of the transfer matrix. Various resolutions have been proposed[104,105] - including[106] the insertion by hand of the Poisson brackets (6.15).

The coproduct

This is a concept without intrinsic meaning in the classical theory, since the charges are c-number functionals of the currents rather than operators acting on asymptotic states and products thereof. We shall therefore *define* a coproduct as follows.

Let us split space into the two regions $(-\infty, 0)$ and $(0, \infty)$, and define $Q_+^{(i)}$ and $Q_-^{(i)}$ to be the charges obtained by taking all the variables of integration in $Q^{(i)}$ to be positive or negative respectively. (Note that the use of zero as the splitting point is quite arbitrary: any other point would have done as well.) We then have

$$\begin{aligned} Q^{(0)a} &= \int_{-\infty}^{\infty} j_0^a(x)\,dx \\ &= \int_{-\infty}^{0} j_0^a(x)\,dx + \int_0^{\infty} j_0^a(x)\,dx \\ &= Q_-^{(0)a} + Q_+^{(0)a} \end{aligned} \tag{6.18}$$

and

$$\begin{aligned} Q^{(1)a} &= \int_{-\infty}^{\infty} j_1^a(x)\,dx + \frac{1}{2} f^{abc} \int_{-\infty}^{\infty} j_0^b(x) \int_{-\infty}^{x} j_0^c(y)\,dx\,dy \\ &= \int_{-\infty}^{0} j_1^a(x)\,dx + \int_0^{\infty} j_1^a(x)\,dx + \frac{1}{2} f^{abc} \Big\{ \int_0^{\infty} j_0^b(x) \int_0^{x} j_0^c(y)\,dx\,dy \\ &\quad + \int_0^{\infty} j_0^b(x) \int_{-\infty}^{0} j_0^c(y)\,dx\,dy + \int_{-\infty}^{0} j_0^b(x) \int_{-\infty}^{x} j_0^c(y)\,dx\,dy \Big\} \\ &= Q_-^{(1)a} + Q_+^{(1)a} + \frac{1}{2} f^{abc} Q_-^{(0)c} Q_+^{(0)b} \ . \end{aligned} \tag{6.19}$$

†The technical term for Poisson brackets containing derivatives of delta-functions is 'non-ultralocal'.

Let us now define a map $\Delta : Y_C(\mathcal{A}) \to Y_C(\mathcal{A}) \otimes Y_C(\mathcal{A})$ by setting

$$\Delta(\Phi) = \sum_i \Phi_1^{(i)} \otimes \Phi_2^{(i)}$$

whenever

$$\Phi = \sum_i \Phi_{1-}^{(i)} \, \Phi_{2+}^{(i)} \ .$$

Here Φ, $\Phi_1^{(i)}$ and $\Phi_2^{(i)}$ are functionals of j^μ, and the subscripts + and − again indicate the functionals obtained by taking all the variables of integration to be either positive or negative. Using this definition, (6.18) and (6.19) imply

$$\begin{aligned} \Delta\left(Q^{(0)\,a}\right) &= Q^{(0)\,a} \otimes 1 + 1 \otimes Q^{(0)\,a} \\ \text{and} \quad \Delta\left(Q^{(1)\,a}\right) &= Q^{(1)\,a} \otimes 1 + 1 \otimes Q^{(1)\,a} + \frac{1}{2} f^{abc} Q^{(0)\,c} \otimes Q^{(0)\,b} \ . \end{aligned}$$

Coassociativity of Δ is guaranteed by our definition: if we had split space into three instead of two regions, the result obtained would have been independent of the order of the splitting, so that

$$(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta \ .$$

Now note that, with (6.15) or (6.17),

$$\{\Phi_{1-}\Phi_{2+}\,,\ \Psi_{1-}\Psi_{2+}\} = \Phi_{1-}\Psi_{1-}\{\Phi_{2+}, \Psi_{2+}\} + \{\Phi_{1-}, \Psi_{1-}\}\,\Phi_{2+}\Psi_{2+} \ ,$$

since terms of the form $\{\eta_-, \rho_+\}$ vanish. Thus the Poisson bracket $\{\ ,\ \}$ in $Y_C(\mathcal{A}) \otimes Y_C(\mathcal{A})$ is given by

$$\{\Phi_1 \otimes \Phi_2, \Psi_1 \otimes \Psi_2\}_2 \equiv \Phi_1\Psi_1 \otimes \{\Phi_2, \Psi_2\} + \{\Phi_1, \Psi_1\} \otimes \Phi_2\Psi_2 \ ,$$

and this then gives $Y_C(\mathcal{A})$ (when we include the antipode) the structure of a Poisson-Hopf algebra, in the sense that Δ is compatible with the Poisson brackets,

$$\Delta\left(\{\Phi, \Psi\}\right) = \{\Delta(\Phi), \Delta(\Psi)\}_2$$

(see, for example, Takhtajan[107]).

The coproduct acquires a similar interpretation to that of the quantum theory when, following Lüscher and Pohlmeyer[99], we consider a configuration $q(x)$ which appears (at some time) as two separated, localized lumps

$$\begin{array}{lll} \text{lump } 1 & r(x) & x \leq -A \\ \text{lump } 2 & s(x) & x \geq A \ . \end{array}$$

These are assumed to be separate in the sense that integrals of j_μ for lump 1 over $x \geq -A$, and of lump 2 for $x \leq A$, are assumed to be negligible. We then have

$$Q^{(i)}[r] = Q_-^{(i)} \qquad \text{and} \qquad Q^{(i)}[s] = Q_+^{(i)} \ ,$$

so that the coproduct provides a rule for the addition of charges.

The Lax formalism

In order to put the antipode map in its physical setting, we first describe briefly the Lax formalism for this system[108]. In the classical theory, an alternative to the iterative definition of the charges is to take the Lax pair

$$L_\mu(x,t;\lambda) = \frac{\lambda}{\lambda^2 - 1}\left(\lambda j_\mu(x,t) + \epsilon_\mu{}^\nu j_\nu(x,t)\right) \qquad (\lambda \in \mathbf{C}) \ ,$$

for which the zero-curvature condition

$$[\partial_0 + L_0, \partial_1 + L_1] = 0$$

is equivalent to both (6.1) and (6.2), and use it to define a transfer matrix $T(x,y;\lambda)$ via

$$(\partial_1 + L_1(x;\lambda))\, T(x,y;\lambda) = 0 \ . \tag{6.20}$$

The solution of (6.20) is

$$T(x,y;\lambda) = \mathbf{P}\exp\left(\int_y^x L(\xi;\lambda)\, d\xi\right)$$

where $\mathbf{P}$ denotes equal-time path ordering. This then yields the algebra-valued conserved charges‡ in the form[94]

$$T(\lambda) \equiv T(\infty, -\infty; \lambda) = \exp\left(\sum_{r=0}^{\infty} \lambda^{r+1} Q^{(r)}\right) ,$$

‡The original BIZZ charges, valued instead in the enveloping algebra and obtained by integrating (6.3), are given by instead expanding T as $T(\lambda) = 1 + \sum_{r=0}^{\infty} Q^{(r)}$.

giving the algebra and coproduct of the charges via the Poisson bracket relation[109]

$$\{T(\lambda) \overset{\otimes}{,} T(\mu)\} = [r(\lambda,\mu), T(\lambda)\otimes T(\mu)] \qquad \text{where} \qquad r(\lambda,\mu) = \frac{I_a \otimes I_a}{\lambda^{-1} - \mu^{-1}} \tag{6.21}$$

and

$$\Delta(T(\lambda)) = T(\lambda) \otimes T(\lambda) \ . \tag{6.22}$$

The Yangian coproduct follows from (6.22), but we have not yet been able to obtain the Yangian Serre relation from (6.21). Equation (6.22) was first noted, and used to obtain the coproducts of $Q^{(0)}$ and $Q^{(1)}$ for the O(3) σ-model, by Lüscher and Pohlmeyer[99].

The canonical quantization of the algebra replaces $\{\,,\,\}$ with $\frac{1}{i\hbar}[\,,\,]$, giving the correct expressions (6.7, 6.8) for (6.16). Instead of (6.21), however, the following Yang-Baxter expression for the quantum transfer matrix is proposed[95]:

$$R(\lambda,\mu)\,(T(\lambda)\otimes 1)\,(1\otimes T(\mu)) = (1\otimes T(\mu))\,(T(\lambda)\otimes 1)\,R(\lambda,\mu) \ . \tag{6.23}$$

Writing $R(\lambda,\mu) = 1 + i\hbar r(\lambda,\mu) + \mathcal{O}\left(\hbar^2\right)$ then gives

$$[T(\lambda) \overset{\otimes}{,} T(\mu)]_{QFT} = i\hbar\,[r(\lambda,\mu), T(\lambda)\otimes T(\mu)] + \mathcal{O}\left(\hbar^2\right) \ ,$$

implying (6.21). (Note that the brackets on the left denote quantum commutators of operators.) Since there is currently no iterative procedure for defining the quantum charges, the interpretation of T in terms of charges is still tentative. Classically, (6.23) is an identity, since the T are just c-number functions.

The antipode

In both the classical and the quantum theories the $\mathcal{PT}$ transformation[96]

$$s:\ j_\mu(x,t) \mapsto -j_\mu(-x,-t)$$

gives

$$Q^{(0)\,a} \mapsto -Q^{(0)\,a} \qquad \text{and} \qquad Q^{(1)\,a} \mapsto -Q^{(1)\,a} + \frac{1}{2} f^{abc} Q^{(0)\,b} Q^{(0)\,c} \ ,$$

obtained simply by using $\Theta(x) = 1 - \Theta(-x)$, so that s corresponds to the antipode map (2.15)§ Note that the term quadratic in $Q^{(0)}$ in the second expression above vanishes in the

§Actually, the order of $Q^{(0)\,b}$ and $Q^{(0)\,c}$ has been reversed. With definition (2.15), the usual Hopf algebra axioms are satisfied if we use $\sigma \circ \Delta$ instead of Δ.

classical case. Looking at (6.20) we see also that

$$T(\lambda) \mapsto T^{-1}(\lambda)$$

and

$$T(\lambda) \otimes 1 \mapsto 1 \otimes T^{-1}(\mu) \ , \qquad 1 \otimes T(\mu) \mapsto T^{-1}(\lambda) \otimes 1$$

so that from (6.23) we see that R (and thus, in the classical theory, r) is invariant under s, as expected.

The Lorentz boost

Recall the action of the Lorentz boost (6.11) in the quantum theory. To compute its Poisson brackets with the charges in the classical theory we shall need to know $T_{\mu\nu}$ explicitly. In fact if we define the full Poincaré group P, generated by the momenta

$$P_\mu = \int_{-\infty}^{\infty} T_{0\mu}(x,t)\, dx$$

and the Lorentz boost generator

$$M \equiv M_{01} = \int_{-\infty}^{\infty} x\, T_{00}(x,t) - t\, T_{01}(x,t)\ dx \ ,$$

where $T_{\mu\nu}(x,t)$ for the principal chiral field has components

$$\begin{aligned} T_{00} = T_{11} &= \frac{1}{2}\left(j_0^a j_0^a + j_1^a j_1^a\right) \\ \text{and} \quad T_{01} = T_{10} &= j_0^a j_1^a \ , \end{aligned} \tag{6.24}$$

using (6.17) we then obtain

$$\begin{aligned} \{P_0, P_1\} &= 0 \ , \\ \{M, P_\mu\} &= \eta_{\mu 1} P_0 - \eta_{\mu 0} P_1 \ , \\ \{P_\mu, j_\nu\} &= \partial_\mu j_\nu \ , \\ \text{and} \quad \left\{M, j_\mu^a\right\} &= \frac{1}{2}\epsilon_{\rho\sigma}\left(x^\rho \partial^\sigma - x^\sigma \partial^\rho\right) j_\mu + \epsilon_\mu{}^\sigma j_\sigma \ . \end{aligned}$$

Applying M to the charges then gives

$$\left\{M, Q^{(0)}\right\} = \left\{M, Q^{(1)}\right\} = 0 \ , \tag{6.25}$$

so that the full classical symmetry algebra is the direct product $\mathrm{P} \otimes Y_C(\mathcal{A})$. This is to be compared with the quantum commutator (6.12): (6.25) is the expected classical limit of (6.12) via the usual limiting procedure

$$\{\ ,\ \} = \lim_{\hbar \to 0} \frac{1}{i\hbar}\ [\ ,\ ]\ ,$$

but the $\mathcal{O}(\hbar^2)$ anomaly in (6.12) ensures that the classical theory is fundamentally different from the quantum theory: whereas in the former the symmetry algebra is $\mathrm{P} \otimes Y_C(\mathcal{A})$, in the latter P does not commute with $Y_C(\mathcal{A})$. The practical upshot of this is that whereas in the quantum theory there is a rich scattering structure determined by (6.13), in the classical theory we expect that soliton solutions will have trivial scattering. It is intriguing that there should be a physical interpretation in the classical theory for all of $Y(\mathcal{A})$ except the automorphism (2.21). Clearly this automorphism is not generated classically by the Lorentz boost; we do not know if another generator exists.

6.3 The Yangian conserved charge bootstrap

Recall our first description of the bootstrap principle in chapter two, shown in figure (2.4). This principle - that intermediate states of the S-matrix should be interpreted as physical states of the theory - had implications for the S-matrix, (2.34), and for the conserved charges, (2.36) for the PESTs. In chapter three we implemented the bootstrap principle for the S-matrices. Now, having described the underlying conserved charges, we are in a position to implement the conserved charge bootstrap.

A method for doing this was described in a recent preprint by Belavin[110]. He considered the two (commuting) conserved charges

$$\begin{aligned} &\mathrm{Tr}\left(Q^{(0)}Q^{(0)}\right) = Q^{(0)\,a}Q^{(0)\,a} \\ \text{and} \quad &\mathrm{Tr}\left(Q^{(0)}Q^{(1)}\right) = Q^{(0)\,a}Q^{(1)\,a} \end{aligned}$$

and applied the bootstrap principle to the process $lm \to n$. The action of the charges $Q^{(0)}$ and $Q^{(1)}$ on the states is given by (6.14) (where we shall set $i\hbar = 1$), and the action on

$|l\rangle \otimes |m\rangle$ by (6.9, 6.10). (Compare this with the charge bootstrap (2.36) in the PESTs: there, the charges were implicitly local.) Putting these together gives

$$\begin{aligned} (I_aI_a \otimes 1 + 2I_a \otimes I_a + 1 \otimes I_aI_a)\,|l\rangle \otimes |m\rangle &= I_aI_a|n\rangle \\ \text{and} \quad (\theta_l I_aI_a \otimes 1 - \theta_m 1 \otimes I_aI_a + (\theta_l - \theta_m)I_a \otimes I_a)\,|l\rangle \otimes |m\rangle &= \theta_n I_aI_a|n\rangle \ . \end{aligned}$$

We can now use the fact that $I_aI_a|p\rangle = C_2(p)|p\rangle$ and set $\theta_n = 0$ to give

$$\theta_l C_2(l) - \theta_m C_2(m) + \frac{1}{2}(\theta_l - \theta_m)(C_2(n) - C_2(l) - C_2(m)) = 0 \ ,$$

and then work out the fusing angle $\theta_{lm}^n = \theta_l + \theta_m$ by implementing the requirement that $\theta_{lm}^n + \theta_{m\bar{n}}^{\bar{l}} + \theta_{\bar{n}l}^{\bar{m}} = 2i\pi$. To make the expression a little easier on the eye we shall write $\tilde{p} \equiv C_2(p)$. Doing so, we obtain

$$\theta_{lm}^n = \frac{2\tilde{n}\,(\tilde{l} + \tilde{m} - \tilde{n})}{2(\tilde{l}\tilde{n} + \tilde{m}\tilde{n} + \tilde{l}\tilde{m}) - (\tilde{l}^2 + \tilde{m}^2 + \tilde{n}^2)} i\pi \ . \tag{6.26}$$

This method can of course only be used to deduce θ_{lm}^n when the existence of a fusing $lm \to n$ is already known: it gives us no immediate information about the existence of fusings - although, as with the PESTs, the requirement that the bootstrap close might be used to give such information. Using this method, Belavin was able to confirm the values of the fusing angles, and thus the mass spectrum, for the a_n theories already predicted by the R-matrices of Kulish, Sklyanin and Reshetikhin.

An essential ingredient of the above analysis is that the states $|l\rangle$, $|m\rangle$ and $|n\rangle$ are in irreducible representations of $\mathcal{A}$; otherwise, I_aI_a does *not* act as a scalar operator on the states. Recall that, amongst others, this is true for all the particles in the c_n theories, and for the vector and spinor particles in the $SO(N)$ theories. If we apply (6.26) to these particles we obtain precisely the angles and thus the masses expected from the R-matrices constructed via the tensor product graph. However, we do not yet know how to extend Belavin's method to particles in reducible representations, where the action of the Yangian is no longer (5.16, 5.17, 6.14). Once again, irreducible representations are tractable; reducible ones are not.

Chapter 7

Outlook

The presence of a bootstrap principle indicates a unified structure in the representation theory of the Yangian which has hitherto not even been partially realized. The many approaches to particular types of R-matrices, particular methods for their construction and particular Yangian representations have not yielded any insight into how this structure might be revealed, but at least now the goal is clear. In its pursuit we might suggest a number of areas to be explored: extensions of the tensor product graph method, more general approaches to the fusion procedure, further investigation of the charge algebra and Lax formalism in the underlying field theories and the extension of existing methods in Yangian representation theory all spring to mind. All of these are viewed within one of two paradigms: either that of factorized S-matrices satisfying a bootstrap principle in a quantum field theory, or that of Yangian representations whose tensor products may be decomposed. Which will prove the more fruitful is unclear, but it certainly seems that the mathematicians' paradigm, compared with the physicists', has not yet received the attention that is its due.

It may well be that the essential ideas will come from other parts of the arena of the Yangian's significance. There is, so far, no direct descriptive link between PESTs and the Yangian: yet we know, firstly, a great deal about the mathematical structure of PESTs such as Toda theories, and, secondly, that much of the structure of the S-matrices is the same in the two types of theory - to such an extent that Belavin was recently led to conjecture that the underlying symmetry of Toda theory might be $Y(\mathcal{A})/\mathcal{A}$. At present, it is difficult to assign meaning to such a statement in a way which sheds light on the Yangian: but again, now that the goal has been made more explicit, we may be led to new ways of looking at PESTs. It may even be that those cases - the non-simply-laced theories - where the connections between PESTs and the Yangian are even less clear, will be the most productive of new ideas, precisely because the discrepancies must be explained. It is for this reason that we are currently investigating the c_n case. Still further links may be thrown up by studying both classical and quantum solitons in Toda theories with imaginary coupling constant.

It should now have been demonstrated that the Yangian has a wide significance for integrable quantum field theories. We may or may not have seen the full extent of the Yangian's influence, but we certainly do not yet have a synthesis. It seems to the author that a coherent overview of the mathematics and physics of the Yangian would bring us much closer to a general understanding of factorized S-matrices, and thence to the eventual goal of a full algebraic understanding of two-dimensional integrable quantum field theories.

Appendix A

Lagrangian field theories

Throughout the text we have emphasized the universal nature of factorized S-matrices: their structure is largely independent of the underlying field theory, to such an extent that we may study the S-matrices without invoking any particular Lagrangian. However, there *are* Lagrangian field theories which are believed to have these S-matrices, and we supply some details of them here.

It should be stressed that the S-matrices are only *proposed* exact S-matrices for these models; in general, we must use standard field theory techniques to confirm them. For example, the affine Toda theory S-matrices must be checked against perturbation theory, and those for the Gross-Neveu and $O(N)$ σ-models against the $\frac{1}{N}$-expansion. Such techniques can never confirm the S-matrices as exact, but can only provide evidence for their correctness.

A.1 Affine Toda Field Theories

These theories and their proposed exact S-matrices are described in detail in the work of Corrigan *et al*[47]. Affine Toda field theories are described by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu \underline{\phi}.\partial^\mu \underline{\phi} + V_0(\underline{\phi}) + V_{Pert}(\underline{\phi})$$

where

$$V_0(\underline{\phi}) = \frac{m^2}{\beta^2}\sum_{a=1}^{r} n_a e^{\beta \underline{\alpha}_a \cdot \underline{\phi}} \quad \text{and} \quad V_{Pert}(\underline{\phi}) = \frac{m^2}{\beta^2} e^{\beta \underline{\alpha}_0 \cdot \underline{\phi}} \ .$$

Here, ϕ_i, $i = 1, ..., r$, are real scalar fields in 1+1 dimensional spacetime, the $\underline{\alpha}_a$, $a = 1, ..., r$, are the simple roots of a finite Lie algebra $\mathcal{A}$ of rank r, and $\underline{\alpha}_0$ is the lowest root

$$\underline{\alpha}_0 = -\sum_{a=1}^{r} n_a \underline{\alpha}_a \ .$$

The Lagrangian

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu \underline{\phi}.\partial^\mu \underline{\phi} + V_0(\underline{\phi})$$

defines a conformally invariant field theory[111]; V_{Pert} is a deformation which removes this conformal invariance but maintains classical integrability. Note that we have chosen the origin for $\underline{\phi}$ such that the minimum of the potential occurs at $\underline{\phi} = 0$. Further, whilst we have here assumed that the $\underline{\alpha}_i$, $i = 1,..,r$, generate a finite algebra, and $\underline{\alpha}_0$ is the root corresponding to the generator which extends this to an (untwisted) affine algebra, this need not be the case: in fact, $\underline{\alpha}_i$, $i = 0,..,r$, may be any root system, and thus includes the twisted algebras. Throughout the text we follow the conventions of Helgason[47,76] in labelling algebras: $a_n = SU(n+1)$, $b_n = SO(2n+1)$, $c_n = Sp(2n)$, $d_n = SO(2n)$; the superscript (1) denotes the untwisted affine algebra, and higher superscripts denote the various twisted algebras. The notation SU, SO and Sp is used to refer to both the group and its algebra.

A perturbative approach now requires us to expand the potential in powers of ϕ. Doing this, we obtain

$$\begin{aligned} V(\underline{\phi}) &= V_0(\underline{\phi}) + V_{Pert}(\underline{\phi}) \\ &= \frac{m^2}{\beta^2}\sum_{a=0}^{r} n_a + \frac{m^2}{2}\sum_{a=0}^{r} n_a(\underline{\alpha}_a.\underline{\phi})(\underline{\alpha}_a.\underline{\phi}) + \frac{m^2\beta}{6}\sum_{a=0}^{r} n_a(\underline{\alpha}_a.\underline{\phi})(\underline{\alpha}_a.\underline{\phi})(\underline{\alpha}_a.\underline{\phi}) + \ldots. \end{aligned}$$

which we may rewrite in terms of the components of ϕ as

$$V(\underline{\phi}) = \left(\frac{m^2}{2}\sum_{a=0}^{r} n_a(\underline{\alpha}_a)_i(\underline{\alpha}_a)_j\right)\phi_i\phi_j + \left(\frac{m^2\beta}{6}\sum_{a=0}^{r} n_a(\underline{\alpha}_a)_i(\underline{\alpha}_a)_j(\underline{\alpha}_a)_k\right)\phi_i\phi_j\phi_k + \ldots.$$

We can now obtain the (unrenormalized) masses of the r particles by diagonalizing the mass matrix and, having done this, obtain the (bare) three-point couplings by writing the ϕ^3 term in a basis of mass eigenstates. Upon taking the canonical commutation relations, the perturbation theory proceeds via the usual diagrammatic techniques.

A.2 The Principal Chiral Field

This is defined by the Lagrangian

$$\mathcal{L} = \frac{1}{2\lambda}\mathrm{Tr}\,(j_\mu j^\mu) \quad ,$$

where $j_\mu = g^{-1}\partial_\mu g$ and $g(x,t)$ takes its values in the group manifold of a compact, semi-simple Lie group. It is invariant under the left- and right- global transformations $g \mapsto Gg$ and $g \mapsto g\,G$ and is thus sometimes known as the $G \times G$-invariant non-linear σ-model.

Other σ-models may be obtained by letting g take instead values in some coset space: for example, the $O(N)$ σ-model takes g to be valued in the $(N-1)$-sphere $\frac{SO(N)}{SO(N-1)}$.

The equation of motion is $\partial^\mu j_\mu = 0$, whilst j_μ is curvature-free since it is a pure gauge. The canonical conjugate momenta are given by

$$\Pi = \frac{\partial \mathcal{L}}{\partial(\partial^0 g)} = \partial_0 (g^{-1})^T$$

(where T means 'transposed'). In a fully-indexed notation the canonical Poisson bracket is then

$$\{g_{ij}(x,t), \Pi_{kl}(y,t)\} = \delta(x-y)\delta_{ik}\delta_{jl} \ .$$

It is now simple to compute the Poisson brackets (6.17) of j_μ and the energy momentum tensor (6.24) used in chapter six. For details of the quantum theory and the $\frac{1}{N}$-expansion see Zamolodchikov[20] and references therein.

A.3 Fermionic models

The Gross-Neveu model[112]

$$\mathcal{L} = i\bar{\psi}_i \gamma_\mu \partial^\mu \psi_i + g\left(\sum_{i=1}^{N} \bar{\psi}_i \psi_i\right)^2 \ ,$$

where ψ_i are N Majorana fermions and γ_μ are the Dirac matrices. This model has a global $O(N)$ invariance, and the canonical Poisson bracket

$$\left\{\psi_i(x,t), \psi_j^\dagger(y,t)\right\} = \delta_{ij}\delta(x-y)$$

lead to the Poisson bracket (6.15) for the current

$$j^\mu_{ij} = -8ig\left(\bar{\psi}_i \gamma^\mu \psi_j\right) \ ,$$

which is conserved and curvature-free. In the quantum theory, the $\frac{1}{N}$-expansion and semi-classical approaches agree with the bound-state structure predicted by the factorized S-matrices[20,113].

This model may be generalized[94] to the chiral multifermionic model

$$\mathcal{L} = i\bar{\psi}\gamma_\mu\partial^\mu\psi + g\left(\bar{\psi}\gamma_\mu I_a\psi\right)\left(\bar{\psi}\gamma^\mu I_a\psi\right) \quad ,$$

which has global $G \times G$-invariance, where the I_a are the generators of the Lie algebra of G and ψ is a two-dimensional fermion field in a particular representation. The indices ψ_i and $(I_a)_{ij}$ are left implicit. If we set $G = SU(N)$ and put ψ in the N-dimensional representation, we recover the chiral Gross-Neveu model[112].

The current

$$j_{a\,\mu} = -4ig\left(\bar{\psi}\gamma_\mu I_a\psi\right)$$

is conserved and curvature-free, and has Poisson bracket (6.15). These theories have not been investigated in the same detail as the Gross-Neveu model, but should have the structure described in chapter six; the chiral Gross-Neveu model certainly has this structure[95], and its $\frac{1}{N}$-expansion agrees with the proposed S-matrix[114].

Appendix B

A continuum approach to quantum inverse scattering

Recall from the introduction that one origin of the Yang-Baxter equation is the 'quantum inverse scattering' relation (1.1) or, in slightly different notation,

$$R(\lambda,\lambda')\,T_1(\lambda)T_2(\lambda') = T_2(\lambda')T_1(\lambda)\,R(\lambda,\lambda')\ , \tag{B.1}$$

where

$$T_1(\lambda) = T(\lambda)\otimes 1 \qquad \text{and} \qquad T_2(\lambda') = 1\otimes T(\lambda')\ ,$$

This in turn has its origins in the classical inverse scattering method for integrable equations. Suppose we are considering a 1+1 dimensional integrable field theory with a Lax pair *i.e.* a pair of operators $L_x(x,t;\lambda)$, $L_t(x,t;\lambda)$ dependent on some spectral parameter λ in such a way that

$$[\partial_x + L_x, \partial_t + L_t] = 0 \tag{B.2}$$

for all λ is implied by the field equations. We can then obtain scattering data from the transfer matrix defined by

$$(\partial_x + L_x)\,T(x,y;\lambda) = 0\ , \tag{B.3}$$

with

$$T(\lambda) = T(\infty,-\infty;\lambda)\ .$$

If there is a classical r-matrix $r(\lambda,\lambda')$ such that

$$\{L(\lambda) \overset{\otimes}{,} L(\lambda')\} = [r(\lambda,\lambda'), L(\lambda)\otimes 1 + 1\otimes L(\lambda')]\ ,$$

which implies[109]

$$\{T(\lambda) \overset{\otimes}{,} T(\lambda')\} = [r(\lambda,\lambda'), T(\lambda)\otimes T(\lambda')]\ , \tag{B.4}$$

(where r must satisfy (2.5) in order for $\{\,,\}$ to satisfy the Jacobi identity) we can then obtain the infinite number of charges in involution characteristic of the system's integrability by expanding in powers of λ the trace of (B.4), since the trace of the right-hand side vanishes.

In the quantum case the situation is not so clear; the condition (B.2) involves defining products of operators at the same point, which cannot be done directly in the quantum theory. The solution used by Faddeev, Sklyanin and Takhtajan[11] for the sine-Gordon model was to discretize space, making it into a one-dimensional lattice or chain. The Lax operator L_x is altered accordingly, and the transfer matrix, instead of being the path-ordered integral which we obtain as the solution to (B.3), becomes the product of the Lax operators $L_n \equiv L_{x_n}$ defined at successive sites on the lattice. This transfer matrix has the same form as that found by Baxter in his study of the six- and eight-vertex models[28,10], and therefore satisfies the relation (B.1) discovered by Baxter. This relation gives the expected infinite number of commuting charges (the analogues of the classical charges in involution) by expanding the trace of (B.1) in powers of λ, and has the correct classical limit (B.4), given by $R(\lambda, \lambda') = 1 + \hbar r(\lambda, \lambda')$.

However, Faddeev *et al.* were only able to obtain a transfer matrix which satisfied (B.1) because they were dealing with 2×2 matrices. The sine-Gordon theory is the $a_1^{(1)}$ affine Toda theory, and so the question arises "What happens in higher representations of a_1?" This was answered by Kulish and Reshetikhin[12], who took as their method the requirement that the transfer matrix be constructed in such a way as to admit a solution $R(\lambda, \lambda')$ of (B.1). In practical terms this meant that, instead of writing the lattice Lax operator L_n in terms of the usual generators of $a_1^{(1)}$, they used instead operators the algebra of which was left undefined, and then used the requirement that (B.1) have a solution to fix this algebra. The algebra turned out to be precisely what later became known as $SU_q(2)$, which reduces to the classical $SU(2)$ in the fundamental representation.

This method was adopted by Jimbo[17] for the generalized Toda system: working in 1+0 dimensions (which does not affect the nature of the problem), he put the Toda Lax operators on a lattice, replaced the algebra generators by generators whose algebra was not postulated, and used the requirement that (B.1) have a solution* to determine this algebra, which became known as the quantized universal enveloping algebra or quantum group. In fact, equating the various functions of λ in (B.1) gives precisely (2.8) for the various generators, and in fact this is the origin of (2.24).

In this appendix we shall derive the quantum group and Yangian coproducts whilst remaining on the continuum, working respectively from the Lax operators (in 1+1 dimensions) of the affine Toda field theories and of the curvature-free conserved currents. There

*Equivalent, in the language of chapter two, to the requirement that Δ be a homomorphism.

are deep fundamental problems with this, foremost among which is that we are attempting to use classical Lax pairs to describe quantum field theories. For instance, we know from chapter six that the non-local charge $Q^{(1)}$ has to be defined differently in the quantum theory, using a point-splitting regularization. The classical Lax pair then apparently fails to generate the quantum charges correctly. Further, the zero-curvature condition (6.2) does not hold in the quantum theory; (B.2) must instead describe its replacement in the quantum theory, the OPE (6.6). In the affine Toda theories, on the other hand, the difficulty is precisely that found by the originators of quantum groups: although we begin with the Lax operators given in terms of the classical algebra generators, in order to be able to construct an R-matrix the algebra must be put in by hand.

To summarize: our method is only justified if we can show that the Lax operators we use correctly describe the field equations and conserved charges of the quantum field theory. Since we cannot, we simply present it as an interesting phenomenon, and subject for further thought, that it *is* possible to obtain the R-matrix defining relations of Yangians and quantum groups whilst remaining in the continuum.

The method which follows is essentially that of Bhattacharya and Ghosh[39,40], who solved the sine-Gordon model in this way. Their approach is to ours as Faddeev, Sklyanin and Takhtajan's method[11] was to Jimbo's[17], in that they avoid quantum group structure by dealing only with two-by-two matrices.

We begin by taking the spatial component $L(x;\lambda) \equiv -L_x(x;\lambda)$ of a Lax pair,

$$(\partial_x - L(x;\lambda))\,T(x,y;\lambda) = 0 \;,$$

so that

$$T(x,y;\lambda) = \mathbf{P}\exp\left(\int_y^x L(\xi;\lambda)\,d\xi\right)$$

(where $\mathbf{P}$ denotes path ordering on a path at a fixed time, and t is a suppressed label), and define the R-matrix through

$$R(\lambda,\lambda')\,T_1(x,y;\lambda)\,T_2(x,y;\lambda') \;=\; T_2(x,y;\lambda')\,T_1(x,y;\lambda)\,R(\lambda,\lambda') \tag{B.5}$$

where

$$T_1(x,y;\lambda) = T(x,y;\lambda)\otimes 1 \qquad \text{and} \qquad T_2(x,y;\lambda') = 1\otimes T(x,y;\lambda') \ ;$$

this reduces to (B.1) as $x\to\infty$, $y\to-\infty$.

To solve (B.5) we define a 'two-sided' derivative,

$$\Gamma(x;\lambda,\lambda') = \lim_{\Delta\to 0}\Gamma(x;\Delta;\lambda,\lambda') \ ,$$

where

$$\Gamma(x;\Delta;\lambda,\lambda') = \frac{f(x+\Delta)-f(x-\Delta)}{2\Delta} \ .$$

Choosing $f(x) = \{T(x,y;\lambda)\otimes 1\}\,\{1\otimes T(x,y;\lambda')\}$ and setting $y = x-\Delta$, we obtain

$$\Gamma_{12}(x;\Delta;\lambda,\lambda') = \frac{1}{2\Delta}\left[\left\{\mathbf{P}\exp\left(\int_{x-\Delta}^{x+\Delta} L(\xi;\lambda)\,d\xi\right)\otimes 1\right\}\left\{1\otimes\mathbf{P}\exp\left(\int_{x-\Delta}^{x+\Delta} L(\xi;\lambda)\,d\xi\right)\right\} - 1\otimes 1\right. \tag{B.6}$$

We define Γ_{21} similarly, using $f = T_2T_1$ instead of T_1T_2. The object now is to take an explicit Lax operator and solve

$$R(\lambda,\lambda')\Gamma_{12}(x;\lambda,\lambda') = \Gamma_{21}(x;\lambda,\lambda')R(\lambda,\lambda') \ ; \tag{B.7}$$

since R is independent of x, (B.7) then implies (B.5).

A typical term of (B.6) will be an n-fold product of Lax operators integrated over a region whose size is of order Δ^n, divided by 2Δ. Classically, this vanishes for all $n>1$ as $\Delta\to 0$. In the quantum theory, however, singularities in the operator product expansion of L with itself give non-vanishing contributions at each order in Δ. For example, one of the first terms is

$$\frac{1}{2\Delta}\int_{x-\Delta}^{x+\Delta} L(x_1;\lambda)\,dx_1\int_{x-\Delta}^{x_1} L(x_2;\lambda)\,dx_2\otimes 1 \ .$$

Writing the operator product as the sum of a singular part and a normal-ordered part, the contributing term as $\Delta\to 0$ becomes, if L is linear in the fields,

$$\frac{1}{2\Delta}\int_{x-\Delta}^{x+\Delta}\int_{x-\Delta}^{x_1}\left[L^{(+)}(x_1;\lambda), L^{(-)}(x_2;\lambda)\right]\,dx_1\,dx_2\otimes 1 \ ,$$

where $(+)$ and $(-)$ refer to annihilating and creating parts of L respectively, and the brackets denote equal time commutators. (Quantization will be performed on an equal time slice.) This term is of order $\hbar$, and evaluation of all the terms in (B.6) gives a power series in $\hbar$. Examining the coefficients of the various fields or currents in (B.7) then gives a set of relations which R must satisfy. If L is not linear in the fields, as is the case in section (B.1), the re-ordering is more complicated.

B.1 Affine Toda Theory and quantized affine algebras

Following Jimbo[17], we use a Chevalley basis for the Lie algebra $\mathcal{A}$:

$$\begin{aligned} [H_a, H_b] &= 0 \\ \left[H_a, E_b^{\pm}\right] &= (\underline{\alpha}_a . \underline{\alpha}_b)\, E_b^{\pm} \equiv C_{ab} E_b^{\pm} \qquad \text{(no summation)} \\ \left[E_a^{+}, E_b^{-}\right] &= H_a \delta_{ab} \end{aligned} \tag{B.8}$$

where $\underline{\alpha}_a$, $a = 1, \ldots, r$ (where r is the rank of $\mathcal{A}$) are the simple roots. This has the implied normalization

$$\mathrm{Tr}\,(E_a E_{-b}) = \delta_{ab} \qquad \text{and} \qquad \mathrm{Tr}\,(H_a H_b) = C_{ab} \ .$$

(Notice that with this normalization the diagonal of the Cartan matrix C_{ab} is not 2 but $\underline{\alpha}.\underline{\alpha}$.)

In this basis, the affine Toda field equations are

$$\partial^{\mu}\partial_{\mu}\phi_a = -\frac{m^2}{\beta}\sum_b \left(e^{\beta C_{ab}\phi_b} - n_a e^{\beta \underline{\alpha}_0 . \underline{\alpha}_b \phi_b}\right) \ .$$

Here the r real scalar fields $\phi_a(x,t)$ $(a = 1, \ldots, r)$ interact exponentially through the terms on the right; $\underline{\alpha}_0$ is the lowest root, $\underline{\alpha}_0 = -\sum_a n_a \underline{\alpha}_a$, and m and β are constants. This basis is related to that used in appendix A by

$$\underline{\phi} = \sum_b \underline{\alpha}_a \phi_b \qquad \text{or} \qquad \phi_a = C^{-1}_{ab}\, \underline{\alpha}_b . \underline{\phi} \ .$$

The field equations may be written, in Minkowski spacetime, as the zero curvature condition

$$[\partial_x + L_x, \partial_t + L_t] = 0$$

for a Lax pair[115,116]

$$\begin{aligned} L_x(x,t) &= \frac{\beta}{2}\sum_a H_a \partial_t \phi_a + \sum_{a,b} \frac{m}{2}\left[e^{\frac{\beta}{2}C_{ab}\phi_b}\left(E_a^+ + E_a^-\right) + e^{\frac{\beta}{2}\underline{\alpha}_0\cdot\underline{\alpha}_b\phi_b}\left(\lambda E_0^+ + \frac{1}{\lambda}E_0^-\right)\right] \\ L_t(x,t) &= \frac{\beta}{2}\sum_a H_a \partial_x \phi_a + \sum_{a,b} \frac{m}{2}\left[e^{\frac{\beta}{2}C_{ab}\phi_b}\left(E_a^+ - E_a^-\right) + e^{\frac{\beta}{2}\underline{\alpha}_0\cdot\underline{\alpha}_b\phi_b}\left(\lambda E_0^+ - \frac{1}{\lambda}E_0^-\right)\right] \end{aligned}$$

where E_a, H_a are as given and E_0 is the step operator associated with the lowest root, $\left[E_0^+, E_0^-\right] = H_0 \equiv -\sum_a n_a H_a$. Note that, although spectral parameters $\lambda_a \in C$, $a = 0, 1, .., r$ may be introduced via

$$E_a^+ \mapsto \lambda_a E_a^+ \qquad \text{and} \qquad E_a^- \mapsto \frac{1}{\lambda_a} E_a^-$$

without affecting the commutation relations (B.8), all but $\lambda = \lambda_0$ may be removed[116] by transformations of the form $L \mapsto gLg^{-1}$.

With the canonical equal time commutation relations

$$[\,\partial_t \phi_a(x,t), \phi_b(y,t)\,] = i\hbar\delta(x-y)\left(C^{-1}\right)_{ab}$$

and hence

$$\left[\,\partial_t \phi_a^{(+)}(x,t), \phi_b^{(-)}(y,t)\right] = \left[\,\partial_t \phi_a^{(-)}(x,t), \phi_b^{(+)}(y,t)\right] = \frac{1}{2} i\hbar\delta(x-y)\left(C^{-1}\right)_{ab}$$

we can now compute $\Gamma_{12}(x;\lambda,\lambda')$. The exponential terms in the Lax pair are assumed to be normal ordered, so that in computing the singular parts of products of Lax operators we encounter terms like

$$\left[\partial_t \phi_c^{(+)}(x), e^{\frac{\beta}{2}C_{ab}\phi_b^{(-)}(y)}\right] e^{\frac{\beta}{2}C_{ab}\phi_b^{(+)}(y)} = i\hbar\delta(x-y)\delta_{ac}\frac{\beta}{2} : e^{\frac{\beta}{2}C_{ab}\phi_b(y)} : \;.$$

Despite the proliferation of contributing terms at higher orders in $\hbar$, we find that they have simple general expressions. For instance, the term of order $\hbar^n$ involving E_a^+ in the second space is

$$\frac{\epsilon^n}{(n+1)!}\sum_{r=0}^{n}\binom{n+1}{r} H_a^r \otimes E_{a,n-r}^+$$

where

$$E^{\pm}_{a,r} = \sum_{s=0}^{r}(-1)^s H_a^{r-s} E_a^{\pm} H_a^s \qquad \text{and} \qquad \epsilon = \frac{i\hbar\beta}{4} \ .$$

Summing these terms gives

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{\epsilon^n}{(n+1)!} \sum_{r=0}^{n} \binom{n+1}{r} H_a^r \otimes E^+_{a,n-r} &= \sum_{r=0}^{\infty} \frac{\epsilon^r H_a^r}{r!} \otimes \sum_{s=0}^{\infty} \frac{\epsilon^s E^+_{a,s}}{(s+1)!} \\ &= q^{H_a/2} \otimes \widehat{E}_a^+ \ , \end{aligned}$$

where

$$\widehat{E}_a^{\pm} = \sum_{s=0}^{\infty} \frac{\epsilon^s E^{\pm}_{a,s}}{(s+1)!} \qquad \text{and} \qquad q = e^{2\epsilon} \ .$$

In fact, our rather unpleasant-looking series expansion for $\widehat{E}^{\pm}$ can be summed, using (B.8) to commute H_a with $E_a^{\pm}$, to give

$$\widehat{E}^{\pm} = e^{\pm\frac{\epsilon\alpha^2}{2}} \frac{\sinh(\epsilon K^{\mp})}{\epsilon K^{\mp}} E^{\pm}$$

where $K^{\pm} = H \pm \frac{\alpha^2}{2}$; the label a is implicit throughout. Thus, as they stand, the $\widehat{E}^{\pm}$ have commutation relation

$$\left[\widehat{E}^+, \widehat{E}^-\right] = \left(\frac{\sinh(\epsilon K^-)}{\epsilon K^-}\right)^2 E^+E^- - \left(\frac{\sinh(\epsilon K^+)}{\epsilon K^+}\right)^2 E^-E^+ \ , \tag{B.9}$$

which is obtained most neatly by using the relations

$$f(K^{\mp})E^{\pm} = E^{\pm} f(K^{\pm}) \ ,$$

true for any function f.

If we now compute similarly the other terms contributing to Γ, we obtain the full, exact expression

$$\begin{aligned} \Gamma_{12}(x;\lambda,\lambda') &= \frac{\beta}{2}\sum_a \partial_t\phi_a (1\otimes H_a + H_a\otimes 1) \\ &+ \sum_{a,b}\frac{m}{2}\left[e^{\frac{\beta}{2}C_{ab}\phi_b}\left\{q^{H_a/2}\otimes\widehat{E}_a^+ + \widehat{E}_a^+\otimes q^{-H_a/2} + q^{-H_a/2}\otimes\widehat{E}_a^- + \widehat{E}_a^-\otimes q^{H_a/2}\right\}\right. \\ &+ e^{\frac{\beta}{2}\underline{\alpha}_0\cdot\underline{\alpha}_b\phi_b}\left\{\lambda' q^{H_0/2}\otimes\widehat{E}_0^+ + \frac{1}{\lambda'}q^{-H_0/2}\otimes\widehat{E}_0^- + \lambda\widehat{E}_0^+\otimes q^{-H_0/2} + \frac{1}{\lambda}\widehat{E}_0^-\otimes q^{H_0/2}\right\} \end{aligned}$$

Computing Γ_{21} and examining the coefficients of the various ϕ-dependent terms in Γ we see that (B.7) is satisfied if (writing $x = \lambda/\lambda'$)

$$[R(x), 1 \otimes H_a + H_a \otimes 1] = 0$$

$$R(x)\left(q^{\pm H_a/2} \otimes \widehat{E}_a^{\pm} + \widehat{E}_a^{\pm} \otimes q^{\mp H_a/2}\right) = \left(q^{\mp H_a/2} \otimes \widehat{E}_a^{\pm} + \widehat{E}_a^{\pm} \otimes q^{\pm H_a/2}\right) R(x)$$

$$R(x)\left(q^{\pm H_0/2} \otimes \widehat{E}_0^{\pm} + x^{\pm 1}\widehat{E}_0^{\pm} \otimes q^{\mp H_0/2}\right) = \left(q^{\mp H_0/2} \otimes \widehat{E}_0^{\pm} + x^{\pm 1}\widehat{E}_0^{\pm} \otimes q^{\pm H_0/2}\right) R(x) \ .$$

These are the quantized affine algebra coproduct relations of Jimbo[17] and Drinfeld[15,14]; the classical limit coincides with the r-matrix relations found by Olive and Turok[116]. In order for them to be soluble for R, however, we must abandon (B.9) and replace it with the quantum group commutation relation (2.23). It is simple to check that this was always going to be necessary: no $\widehat{E}^{\pm}$ defined by

$$\widehat{E}^{\pm} = f^{\pm}(H)E^{\pm}$$

can give the relation (2.23), whatever the choices of $f^{\pm}(H)$.

B.2 Curvature free conserved currents and Yangians

Recall that classically, for theories with curvature-free conserved currents the zero-curvature condition for the Lax pair

$$L_\mu(x,t) = \frac{\lambda}{\lambda^2 - 1}\left(\lambda j_\mu + \epsilon_\mu{}^\nu j_\nu\right) \qquad (\lambda \in \mathbf{C}) \ , \tag{B.10}$$

is equivalent to both conservation and zero curvature of j_μ. Now let us take the commutation relations of the components of j to be those implied by Bernard's OPE (6.6),

$$\begin{aligned}
\left[j_t^a(x,t), j_t^b(y,t)\right] &= i\hbar f^{abc} j_t(x,t)\delta(x-y) \\
\left[j_x^a(x,t), j_t^b(y,t)\right] &= i\hbar f^{abc} j_x(x,t)\delta(x-y) - k\,\delta^{ab}\partial_x\delta(x-y) \\
\left[j_x^a(x,t), j_x^b(y,t)\right] &= i\hbar f^{abc} j_t(x,t)\delta(x-y) \ .
\end{aligned}$$

With these relations the quantum theory remains integrable, as we saw in chapter six; however, the status of the Lax pair is unclear.

Although the second commutation relation above contains, in general, a Schwinger term, this does not affect $\Gamma(x;\lambda,\lambda')$: all contributions to $\Gamma(x;\Delta;\lambda,\lambda')$ from the Schwinger term disappear in the limit $\Delta\to 0$. The first few terms of $\Gamma_{12}(x;\lambda,\lambda)$ obtained from (B.10) are then

$$\left(\frac{1}{\lambda^2-1}\right)\left(\frac{1}{\lambda'^2-1}\right)\Big[\Big\{\lambda^2(\lambda'^2-1)1\otimes T_a+\lambda'^2(\lambda^2-1)T_a\otimes 1-\lambda\lambda'(\lambda+\lambda')\epsilon f^{abc}T_b\otimes T_c\Big\}\,j^a_x(x,t)$$
$$-\Big\{\lambda(\lambda'^2-1)1\otimes T_a+\lambda'(\lambda^2-1)T_a\otimes 1-\lambda\lambda'(\lambda\lambda'+1)\epsilon f^{abc}T_b\otimes T_c\Big\}\,j^a_t(x,t)\Big]$$

where $\epsilon=\frac{i\hbar}{2}$. The coefficients of j^a_x and j^a_t within the square brackets can be written, respectively, as

$$\lambda\lambda'(\lambda\lambda'+1)(1\otimes T_a+T_a\otimes 1)-\lambda\lambda'(\lambda+\lambda')\left(\frac{1}{\lambda}T_a\otimes 1+\frac{1}{\lambda'}1\otimes T_a+\epsilon f^{abc}T_b\otimes T_c\right)$$

and

$$-\lambda\lambda'(\lambda+\lambda')(1\otimes T_a+T_a\otimes 1)+\lambda\lambda'(\lambda\lambda'+1)\left(\frac{1}{\lambda}T_a\otimes 1+\frac{1}{\lambda'}1\otimes T_a+\epsilon f^{abc}T_b\otimes T_c\right)\ .$$

Thus, writing $u_1=\frac{1}{\lambda}$ and $u_2=\frac{1}{\lambda'}$, the R-matrix must satisfy

$$[R(u_1-u_2),1\otimes T_a+T_a\otimes 1]=0$$

and
$$R(u_1-u_2)\left(u_1T_a\otimes 1+u_2 1\otimes T_a+\epsilon f^{abc}T_b\otimes T_c\right)=\left(u_1T_a\otimes 1+u_2 1\otimes T_a-\epsilon f^{abc}T_b\otimes T_c\right)R(u_1-u_2)\ , \tag{B.11}$$

which are precisely the coproduct relations of the Yangian in its finite dimensional representations $\rho(I_a)=\rho(T_a)$, $\rho(J_a)=u\rho(T_a)$, where ρ is a representation of $\mathcal{A}$.

The higher terms in Γ, in addition to forming a power series in ϵ, also form a power series in C_A, the value of the quadratic Casimir operator in the adjoint representation. For example, the term of order ϵ of the form $\int L\int L\otimes 1$ has a factor $f^{abc}T_bT_c=-C_AT_a$. Thus we must check that (B.11) implies (B.7) at each order in C_A. The coefficient of C_A^0 has been given above, and we have also checked that the result holds at C_A^1 (although we do not give the calculation here). This verifies our result up to $\mathcal{O}(\epsilon^2)$. We do not have a general construction of the coefficient of C_A^n.

Appendix C

Structure of trigonometric R-matrices

Twisted projectors required for trigonometric $\check{R}_{\boxtimes\boxtimes\square}(x)$, (4.13):

$$\check{P}_{\boxtimes\boxtimes\square} = \frac{1}{(q^4+q^2+1)(q^2+1)}\Big[e_1 + q^2\,e_2 + q^2\,e_3 + q\,e_4 + q^3\,e_5$$
$$+q\,e_6 - \frac{q^2-1}{q^{N+2}-1}\Big(q^2\,e_7 + q^4\,e_8 + e_9 + q\,e_{10} + q^3\,e_{11}$$
$$+q\,e_{12} - q^2(q^2-2)\,e_{13} - q^4\,e_{14} - q^3\,e_{15}\Big)\Big]$$

$$\check{P}_{\boxtimes\boxtimes\atop\square} = \frac{1}{(q^4+q^2+1)(q^2+1)}\Big[q^3\,e_1 + q^5\,e_2 - q(q^2+1)\,e_3 + q^4\,e_4 - q^2(q^2+1)\,e_5$$
$$+q^4\,e_6 - \frac{(q^2-1)}{q^{2N}-q^2}\Big\{(q^N-q^2-1)\Big(q^3\,e_7 + q^5\,e_8 + q^2\,e_{10} + q^4\,e_{11}\Big)$$
$$+(q^N+q^4)\Big(q\,e_9 + q^2\,e_{12}\Big) - q(q^{N+4} - 2q^{N+2} - q^6 + q^2 - 1)\,e_{13}$$
$$-q(q^{N+2} + q^N + q^4 + 1)\,e_{14} + q(q^{N+2}+1)\,e_{15}\Big\}\Big]$$

$$\check{P}_{\square} = \frac{q^N(q^2-1)}{(q^{N+2}-1)(q^{2N}-q^2)(q^2+1)}\Big[q^2(q^N-1)\,e_7 + q^4(q^N-1)\,e_8$$
$$-q^N(q^2-1)\,e_9 + q(q^N-1)\,e_{10} + q^3(q^N-1)\,e_{11} - q^{N+1}(q^2-1)\,e_{12}$$
$$-(q^2-1)(q^{N+2}-q^2+1)\,e_{13} + \frac{(q^2-1)^2}{q^N-1}\,e_{14} - q(q^2-1)\,e_{15}\Big]$$

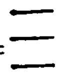
and expressions $\check{T}$ for $\check{R}_{(\boxminus\oplus 0)\square}(x)$, (4.15):

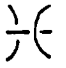
$$\check{T}_1 = \frac{1}{(q^4+q^2+1)(q^2+1)}\Big[q^3\,e_1 + q\,e_2 - q^3(q^2+1)\,e_3 - q^2\,e_4 + q^2(q^2+1)\,e_5$$
$$-q^2\,e_6 + \frac{q^4-1}{q^N-1}\left(-q\,e_7 - q^{-1}\,e_8 + q^2\,e_{10} + e_{11} + \frac{1}{q^2+1}(q\,e_9 - e_{12})\right)$$
$$-\frac{q^2-1}{(q^N-1)^2}\Big(q^{-1}(q^{N+4} - q^{N+2} - q^N + q^6 - q^4 + 1)\,e_{13}$$
$$+q(q^{N+2} + q^N - q^4 - 1)\,e_{14} + (q^N - q^4)\,e_{15}\Big)\Big]$$

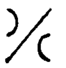
$$\check{T}_2 = \frac{1}{(q^4+q^2+1)(q^2+1)}\Big[q^6\,e_1 + q^4\,e_2 + q^4\,e_3 - q^5\,e_4 - q^3\,e_5$$
$$-q^5\,e_6 + \frac{(q^2-1)}{q^{2N}-q^6}\Big\{(q^{N+2} + q^N - q^4)\Big(-q^4\,e_7 - q^2\,e_8 + q^5\,e_{10} + q^3\,e_{11}\Big)$$
$$-\frac{1}{q^N-1}\Big(q^2(q^{2N}(q^4-q^2-1) - (q^N-q^4)(q^2-1) + q^6)\,e_{13}$$
$$-q^2(q^{2N} + q^{N+6} - q^{N+4} - q^4)\,e_{14} + q^3(q^N-q^4)(q^N+1)\,e_{15}\Big) + (q^N+q^6)\Big(q^4\,e_9 - q^3\,e_{12}$$

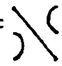
$$\begin{aligned}\check{T}_3 \;=\; & \frac{q^N(q^2-1)}{(q^{2N}-q^6)(q^N-1)(q^2+1)}\Big[(q^N-q^4)\left(q\,e_7+q^{-1}\,e_8-q^2\,e_{10}-e_{11}\right) \\ & +q^N(q^2-1)\,(q\,e_9-e_{12})+\frac{1}{q^N-1}\Big\{q^{-1}(q^N-q^4)(q^N+q^2-1)(q^2-1)\,e_{13} \\ & +q^3(q^2-1)^2\,e_{14}-(q^N-q^4)(q^2-1)\,e_{15}\Big\}\Big]\end{aligned}$$

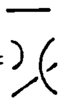
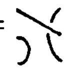
where these are given in terms of the following basis:

$e_1 =$  $e_6 =$  $e_{11} =$ 

$e_2 =$  $e_7 =$  $e_{12} =$ 

$e_3 =$  $e_8 =$  $e_{13} =$ 

$e_4 =$  $e_9 =$  $e_{14} =$ 

$e_5 =$  $e_{10} =$  $e_{15} =$ 

Appendix D

Solutions of the Yang-Baxter equation

This appendix brings together all the rational (Yangian invariant) and trigonometric (quantum group invariant) solutions of the Yang-Baxter equation of which I am aware. It does not include solutions defined in terms of spectral parameters on higher genus Riemann surfaces[117], or those involving more than one additional parameter (such as elliptic solutions[10]); neither does it include the 'exotic' solutions found, for example, by Ge and his collaborators[87,118,119].

For each algebra we begin by giving the Dynkin diagram of $\mathcal{A}$. We then give the decompositions of the fundamental representations v_i of $Y(\mathcal{A})^*$ in terms of the representations of $\mathcal{A}$[22,23,31], the latter being denoted either by Young tableaux or by Dynkin labelling; we also use V_i to denote the ith fundamental representation of $\mathcal{A}$ and 0 to denote the singlet. This is followed by a list of the representations $V \otimes W$ of $\mathcal{A}$ corresponding to which a rational (associated with $Y(\mathcal{A})$) or a trigonometric (associated with $U_q\mathcal{A}^{(1)}$) R-matrix has been constructed, together with a reference to the original paper in which this was done. As far as I am aware, there is no comparable list in print. An early synthesis was provided by Kulish and Sklyanin[120], whilst a good general introduction to the YBE is given by Jimbo[35], who also edited a collection[28] which includes many of the original papers.

1. a_1

The fundamental (two-dimensional) representation of a_1 is also that of $Y(a_1)$. Rational[30,45,121] and trigonometric[10,16,122] R-matrices exist for all representations.

2. a_n

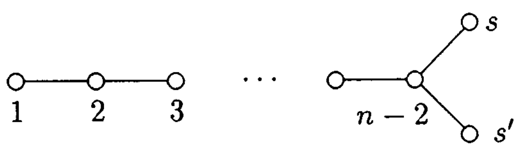
$v_p = V_p, \quad p = 1, \ldots, n.$

*Insofar as they are known. They are obtained by examining the application of the fusion procedure to known R-matrices, *i.e.* how the tensor products of representations of $Y(\mathcal{A})$ decompose.

Rational : $v_1 \otimes v_1$[37]; $X \otimes Y$, where X and Y are any fully symmetric or antisymmetric (fundamental) representations, and $v_1 \otimes \Lambda$ where $\Lambda = (m_1, m_2, .., m_n)$, $m_1 > m_2 > .. > m_n$[45]. We have also constructed (using the tensor product graph) R for $X \otimes Y$, where X and Y are any representations with rectangular Young tableaux (unpublished).

Trigonometric : $v_1 \otimes v_1$[17,123]; $X \otimes Y$, where X is *any* representation and Y is fully symmetric or antisymmetric[13] (see also Hollowood[85]).

3. b_n and d_n



$$v_{s(')} = V_{s(')}, \quad v_1 = V_1, \quad v_p = \bigoplus_{r=0}^{[p/2]} V_{p-2r}, \quad p = 1, .., n-1 \ (b_n) \text{ or } n-2 \ (d_n)$$

Rational : $v_1 \otimes v_1$[20]; $s^{(')} \otimes v_1$, $s^{(')} \otimes s^{(')}$, $s^{(')} \otimes s$[73]; $(m0...0) \otimes (n0...0)$[3]; $s^{(')} \otimes (n0...0)$[86]; $v_1 \otimes v_2$[1]; $v_2 \otimes v_2$[1,31].

Trigonometric : $v_1 \otimes v_1$[17,123]; $s^{(')} \otimes s^{(')}$, $s^{(')} \otimes s$[89,124]; $v_1 \otimes \boxplus$, $v_1 \otimes v_2$[2].

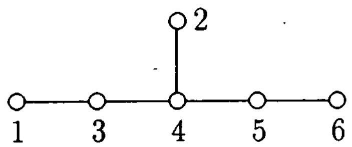
4. c_n

$v_p = V_p$ for all $p = 1, ..., n$

Rational : $v_1 \otimes v_1$[77]; $v_1 \otimes v_p$[23]; $(\square\!\square \oplus 0) \otimes (\square\!\square \oplus 0)$[31]. We have also constructed R-matrices in $v_m \otimes v_n$ using the tensor product graph (unpublished).

Trigonometric : $v_1 \otimes v_1$[17,123]; $v_1 \otimes v_2$[2].

5. e_6

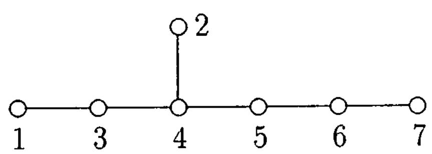


$v_1 = V_1, \quad v_2 = V_2 \oplus 0, \quad v_6 = V_6, \quad v_3 = V_3 \oplus V_6, \quad v_5 = V_5 \oplus V_1, \quad v_4 = V_4 \oplus (100001) \oplus 2\,V_2 \oplus 0.$

Rational : $v_1 \otimes v_1,\ v_6 \otimes v_6,\ v_1 \otimes v_6$[22]; $v_2 \otimes v_2$[31].

Trigonometric : $v_1 \otimes v_1,\ v_6 \otimes v_6$[89,90,125,126]; $v_1 \otimes v_6$[90].

6. e_7

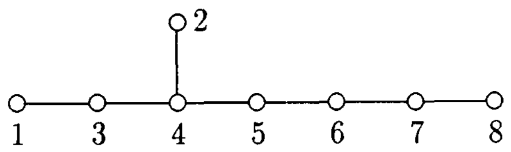


$v_1 = V_1 \oplus 0, \quad v_2 = V_2 \oplus V_7, \quad v_3 = V_3 \oplus V_6 \oplus 2\,V_1 \oplus 0, \quad v_6 = V_6 \oplus V_1 \oplus 0, \quad v_7 = V_7.$

Rational : $v_7 \otimes v_7$[22]; $v_1 \otimes v_1$[31].

Trigonometric : $v_7 \otimes v_7$[89,90,125,126].

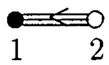
7. e_8



$v_1 = V_1 \oplus V_8 \oplus 0, \quad v_7 = V_7 \oplus V_1 \oplus 2\,V_8 \oplus 0, \quad v_8 = V_8 \oplus 0.$

Rational : $v_8 \otimes v_8$[31].

8. g_2

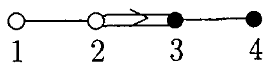


$v_1 = V_1, \quad v_2 = V_2 \oplus 0.$

Rational : $v_1 \otimes v_1$[22,127]; $v_2 \otimes v_2$[31].

Trigonometric : $v_1 \otimes v_1$[128].

9. f_4



$v_1 = V_1 \oplus 0, \quad v_2 = V_2 \oplus (0002) \oplus 2V_1 \oplus 0, \quad v_3 = V_3 \oplus V_1, \quad v_4 = V_4.$

Rational : $v_4 \otimes v_4$[22]; $v_1 \otimes v_1$[31].

Trigonometric : $v_4 \otimes v_4$[129,126]

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'I look at it like this,' he said. 'Before I did this, I was like everyone else. You know what I mean? I was confused and uncertain about all the little details of life. But now,' he brightened up, 'while I'm still confused and uncertain it's on a much higher plane, d'you see, and at least I know I'm bewildered about the really fundamental and important facts of the universe.'

Treatle nodded. 'I hadn't looked at it like that,' he said, 'but you're absolutely right. We've really pushed back the boundaries of ignorance. There's so much about the universe we don't know.'

They both savoured the strange warm glow of being much more ignorant than ordinary people, who were ignorant of only ordinary things.

Terry Pratchett, *Equal Rites*