Geometry of arithmetic surfaces

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Geometry of Arithmetic Surfaces

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Abstract

In this thesis my emphasis is on the resolution of the singularities of fibre products of Arithmetic Surfaces.

In chapter one as an introduction to my thesis some elementary concepts related to regular and singular points are reviewed and the concept of tangent cone is defined for schemes over a discrete valuation ring. The concept of arithmetic surfaces is introduced briefly in the end of this chapter.

In chapter 2 my new procedures namely the procedure of Mojgan[1] and the procedure of Mahtab[2] and a new operator called Moje are introduced. Also the concept of tangent space is defined for schemes over a discrete valuation ring.

In chapter 3 the singularities of schemes which are the fibre products of some surfaces with ordinary double points are resolved. It is done in two different methods. The results from both methods are consistent.

In chapter 4, I have tried to resolve the singularities of a special class of arithmetic three-folds, namely those which are the fibre product of two arithmetic surfaces, which were very helpful to achieve my final results about the resolution of singularities of fibre products of the minimal regular models of Tate.

Chapter 5 includes my final results which are about the resolution of singularities of the fibre product of two minimal regular models of Tate.

[1]In the Persian Language Moje means eyelash and Mojgan is its plural.

[2]In the Persian Language Mahtab means moonlight.

Preface

This thesis is the result of my research work in the Department of Mathematical Sciences at the University of Durham, between October 1993 and July 1996 under the supervision of Professor A.J. Scholl. No part of it has been previously submitted for any degree and no part is published anywhere. It is believed that the material presented in this thesis is entirely original.

I thank Prof. Scholl for his careful supervision and his encouragement. In addition I would like to thank all of the lecturers whose courses I attended. In particular I should thank Prof. Scholl and Dr Oxbury for their courses in my subject under the titles of schemes and algebraic geometry respectively, which were excellent.

I respect the previous heads of our Department Dr Armstrong, Prof. Squires (whose recent death is regretted by us all), Prof. Scholl and the present head of the Department Prof. E.F. Corrigan and wish continuing success for the Department in future.

I would deeply like to thank the Iranian Ministry of Culture and Higher Education and also the Isfahan University of Technology for their financial support during this period of my studies in England.

My special thanks go to my children for being so patient, tolerant and understanding. This thesis is dedicated to them.

I will never forget the memory of this period of my studies, especially for having very kind fellow-students.

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Contents

Chapter 1

Introduction

The basis of this thesis is the resolution of singularities of the fibre product of Arithmetic Surfaces. The blowing-up of a point of a variety or a subvariety of a variety is discussed before, see for example [11], [4], [9], [10] and [20]. We can find about the blowing-up of a noetherian scheme with respect to a coherent sheaf of ideals in [11]. Some discussions about the blowing-up are purely algebraic, see for example [14], [28] and [11]. My research work started with proposition 2.0.2 and remark 2.1.2 in Scholl's paper (see [21]), and lemma 5.5 in Deligne's paper (see [7]) and I tried to find a resolution for the singularities of a fibre product of the arithmetic surfaces with ordinary double points and could find the answer when our surfaces are over $\operatorname{Spec} k[t]$, where k is an algebraically closed field. Later on, I realized that the results carry through without change if k is replaced by an arbitrary regular ring R. Then there was an attempt for the substitution of $R[t]$ by a discrete valuation ring R with an algebraically closed residue field $k = R/(\pi)$ and it was possible. These results are collected in chapter 3. The rest of my thesis is mainly about the resolution of singularities of the fibre product of arithmetic surfaces of genus one.

Before introducing more details about the research work, let's have a quick review on the history of the resolution of singularities, the question which has always been interesting to mathematicians.

The resolution of curves was known earlier. Walker proved it for surfaces over $\mathbb{C}$ in 1935. The first algebraic proof for resolution of surfaces and then the embedded resolution for surfaces and resolution for three-folds over a field k (Char. k=0) was given by Zariski, 1939 and 1944 respectively, see [28] for the first one.

Hironaka proved the resolution and embedded resolution in all dimensions in characteristic 0, see [12].

In 1966 Abhyankar proved resolution of three-folds in characteristic $p \geq 7$.

The resolution of singularities of arithmetic schemes (*eg* of schemes of finite type over $\mathbb{Z}$ or a dvr) is in general unknown. But it is known for arithmetic surfaces (see [23]). For arithmetic surfaces of genus one there is a nice algorithm to obtain the minimal regular model (due to Tate), see [25] for details.

The aim of this thesis is to investigate the case of those Arithmetic 3-folds which are fibre products of two arithmetic surfaces. The simplest case is when the singularities in the fibres of the surfaces are ordinary double points, so locally for the etale topology isomorphic to Spec $R[x,y]/(xy-\pi)$. In this case we can find an explicit desingularisation for the fibre products of an arbitrary number of surfaces. The existence of such a desingularisation (at least in the geometric case) has been known for a long time. In 1969 Deligne gave a sheaf of ideals which under blowing-up, would resolve such a singularity. In chapter 3, Deligne's method is reproduced with details. We also give another method to desingularise a product of double points, following Scholl's remark (see [21]) and prove that these two methods give the same answer.

In the last section of chapter 3 we imitate the methods of Deligne and Scholl in the arithmetic case, and prove that they give a (common) desingularisation over a dvr.

The final results in this thesis are in chapters 4 and 5, where we show that (under certain hypothesis) the fibre product of two arithmetic surfaces of genus one has a regular model, by taking the product of two Tate models and applying birational transformations.

1.1 Definitions + Conventions

Recall that a noetherian local ring A of dimension d and with maximal ideal $\mathfrak{m}$ and residue field $k = A/\mathbf{m}$ is regular if $\dim_k \mathfrak{m}/\mathfrak{m}^2 = d$ (see [3]), or equivalently the maximal ideal of A is generated by d elements, which also means that the associated graded ring $G_m = \bigoplus_{n\geq 0}\mathfrak{m}^n/\mathfrak{m}^{n+1}$ is a polynomial ring in d variables over the field $A/\mathbf{m}$, see [27].

Also recall that for a commutative ring A with identity and a prime ideal P of A, the localization of A at P is called the local ring of Spec A at P and the field A_P/PA_P is said to be the residue class field of Spec A at P and is denoted by $k(P)$ which is actually the field of fractions of A/P, see [13] and [11].

When A is a commutative ring with identity and has the spectrum (Spec A, $\mathcal{O}$), the stalk $\mathcal{O}_P$ of the sheaf $\mathcal{O}$ is isomorphic to the local ring A_P , see [11]. A scheme X is regular if all of its local rings are regular local rings. If for $P \in X$, $\mathcal{O}_P$ is not regular we say that P is a singular point of X.

Recall that for a scheme X the Zariski tangent space to X at $P \in X$ is the dual of the $k(P)$-vector space $\mathbf{m}_P/\mathbf{m}_P^2$, *ie*, $\mathrm{Hom}_{k(P)}(\mathbf{m}_P/\mathbf{m}_P^2, k(P))$ which is described completely in [17]. If k is an algebraically closed field and X is a scheme of finite type over k and P is a closed point of X and $U \subset X$ is an affine open neighbourhood of P, a closed immersion $i{:}U \longrightarrow \mathbf{A}_k^n$ makes U isomorphic with a subscheme Spec $(k[x_1, ..., x_n]/A)$ of $\mathbf{A}_k^n$. Then by using suitable translation we can assume that $x_1, ..., x_n \in I(P)$ or equivalently $i(P) = P_0$ the origin in $\mathbf{A}_k^n$. For each $f \in k[x_1, ..., x_n]$ we use f^* to denote the leading form of f, *ie*, if $f = \Sigma_{i=r}^n f_i$, where f_i is homogeneous of degree i (and $f_r \neq 0$), then $f^* = f_r$ and we use A^* for the ideal of $k[x_1, ..., x_n]$ generated by all polynomials f^* (for $f \in A$) and call Spec $(k[x_1, ..., x_n]/A^*)$ the tangent cone of X at P. It is the same as Spec $(Gr(\mathcal{O}_P))$, see [17], page 216. Recall that a point $P \in X$ is a regular point if the tangent cone and the Zariski tangent space at P are the same.

For example if k is an algebraically closed field of Char. 0 and X =Spec $k[x, y]/(y^2 - x^2(x + 1))$, then at P_0=$(0, 0)$ the tangent cone consists of the line pair Spec $k[x, y]/(y^2 - x^2)$ and the tangent space is $\mathbf{A}_k^2$. So P_0 is a singular point.

Remark 1.1 In the above discussion we cannot substitute k by an arbitrary ring, but if R is a discrete valuation ring with algebraically closed residue field $k = R/(\pi)$ we can define the tangent cone at $P \in X$ in a similar way.

Here a closed point on the special fibre $\mathbb{A}_k^n$ of $\mathbb{A}_R^n$ say $P = (a_1, ..., a_n)$ corresponds to the maximal ideal $\mathcal{M}_P = (x_1 - \alpha_1, ..., x_n - \alpha_n, \pi)$ of $R[x_1, x_2, ..., x_n]$ where $\alpha_i \in R$ and $\bar{\alpha}_i = a_i$, *ie*, a_i is the reduction of $\alpha_i \pmod{\pi}$.

Now let X be a scheme of finite type over R and P be a closed point of X and $U \subset X$ be an affine open neighbourhood of P and $i : U \longrightarrow \mathbf{A}_R^n$ be a closed immersion making U isomorphic with the subscheme $V = \mathrm{Spec}\,(R[x_1, ..., x_n]/A)$ of $\mathbf{A}_R^n$ and use suitable translation such that $i(P) = P_0$, where P_0 is the origin of the special fibre of $\mathbf{A}_R^n$, *ie*, $\mathbf{A}_k^n$.

Let f be a non-zero polynomial in A. Since $i(P) = P_0 \in V$, we have $f \equiv 0 (\mathrm{mod}\, \mathbf{M})$ (*ie*, $P_0 \in \mathbf{A}_k^n$ lies on $\bar{f} = 0$), where $\mathbf{M} = (\pi, x_1, ..., x_n)$. So there exists $r \geq 0$ such that $f \in \mathbf{M}^r$ and $f \notin \mathbf{M}^{r+1}$. Suppose that $f = (\Sigma_{i=1}^t g_i) + h$ where g_i is a monomial in $R[x_1, ..., x_n]$ and $g_i \in \mathbf{M}^r \setminus \mathbf{M}^{r+1}$ and $h \in \mathbf{M}^{r+1}$.

If $g_i = u_i \pi^{\alpha_{i_0}} x_1^{\alpha_{i_1}} ... x_n^{\alpha_{i_n}}$ (where $u_i \in R^*$), then $\alpha_{i_0} + \alpha_{i_1} + ... + \alpha_{i_n} = r$ and u_i is well-defined $(\mathrm{mod}\, \pi)$ as g_i is well-defined $(\mathrm{mod}\, \mathbf{M}^{r+1})$. Now replace π by an indeterminate say x_0. Let $g_i^* = \bar{u}_i x_0^{\alpha_0} x_1^{\alpha_1} ... x_n^{\alpha_n} \in k[x_0, ..., x_n]$ and $f^* = \Sigma_{i=1}^t g_i^*$. Then f^* is a homogeneous polynomial of degree r in $k[x_0, x_1, ..., x_n]$ which is determined uniquely by f. Let A^* be the ideal of $k[x_0, x_1, ..., x_n]$ generated by all $f^{*\prime}s$ (for $f \in A$). Now we define the tangent cone at $P \in X$ as follows:

Definition 1.2 With the above notation

$$\text{Tangent cone of } X \text{ at } P = \mathrm{Spec}\, k[x_0, x_1, ..., x_n]/A^*$$

Remark 1.3 In definition (2.5) of chapter 2 we introduce the concept of tangent space and then we give an example about tangent cone and tangent space.

Convention 1.4 In this thesis valuation ring means a discrete valuation ring with uniformisor π and algebraically closed residue field $k = R/(\pi)$.

Remark 1.5 In [23] the concept of arithmetic surfaces is defined over a Dedekind domain, but since singularity is a local property and the localization of a Dedekind domain at any of its non-zero primes is a discrete valuation ring, we study our arithmetic surfaces over a discrete valuation ring.

Convention 1.6 Let R be a discrete valuation ring. By an arithmetic surface we mean a regular scheme V purely of dimension 2, which is flat over R.

Remark 1.7 Let $X \longrightarrow \operatorname{Spec} R$ be a separated integral scheme which is flat and of finite type. By a desingularisation of X we mean a birational, proper morphism $X' \longrightarrow X$ such that X' is flat over $\operatorname{Spec} R$ and is a regular scheme.

Remark 1.8 For more details and several examples of arithmetic surfaces see pp 311-318 in [23]. The blowing-up of an arithmetic surface over a discrete valuation ring is discussed in remark 7.7, pp 345-347 in [23].

The theorem related to the resolution of singularities of arithmetic surfaces and also existence of minimal proper regular models for arithmetic surfaces is discussed in theorem 4.5, page 317 in [23]. For nice examples over a discrete valuation ring see Tate's algorithm in [25].

All schemes which I have used in this thesis as components in the fibre products, are examples for arithmetic surfaces.

Chapter 2

Procedures

2.1 Introduction

To determine the singular points of a geometric scheme, one uses the Jacobian criterion. In the arithmetic case there is no Jacobian criterion, but we find an analogous procedure to determine the singular points, which we call the procedure of Mojgan (§2.2 and §2.3). For resolution of singularities we mainly use a sequence of blowings-up. To do these with a tidy method, we employ a procedure which we call Mahtab (§2.4).

2.2 The Procedure of Mojgan

As it was mentioned in chapter one, the tangent cones and tangent spaces are helpful for the determination of singular points. Here we try for another method. We start with another interpretation of Jacobian which would be generalized later on.

Let k be an algebraically closed field and $X = \operatorname{Spec} k[x_1, ..., x_n]/(f_1(x_1, ..., x_n), ..., f_m(x_1, ..., x_n))$ and $P = (a_1, ..., a_n) \in X$. Then define $\mathrm{Moje}(P, f_i)$ as follows:

Moje(P, f_i)=Linear form of $f_i(T_1 + a_1, ..., T_n + a_n)$,

say

Moje$(P, f_i) = C_{i1}T_1 + C_{i2}T_2 + ... + C_{in}T_n$ and let $C = (C_{ij})$ where $1 \leq i \leq m$ and $1 \leq j \leq n$, and P corresponds to the maximal ideal

$$\mathcal{M}_P = (x_1 - a_1, x_2 - a_2, ..., x_n - a_n)/(f_1, ..., f_m) \tag{2.1}$$

of $\Gamma(X, \mathcal{O}_X)$ and

$$\mathcal{M}_P/\mathcal{M}_P^2 = (T_1, ..., T_n)/((T_1, ..., T_n)^2 + \text{Span}(\text{Moje}(P, f_i))_{1 \leq i \leq m}). \tag{2.2}$$

Let $r = \text{rank}(C_{ij})$. Then $\dim \mathcal{M}_P/\mathcal{M}_P^2 = n - r$. If $n-r=\dim X$, then by Jacobian criterion P is a regular point, otherwise it is singular, see [11].

Remark 2.1 For the calculation of singular points of schemes over a field of characteristic zero, or determination of singular points of the special fibre (when our schemes are over a dvr) and also for solving the system of equations, I have used WMAPLE53.

Example 2.2 Let $X = \text{Spec}\, k[x_1, y_1, x_2, y_2, x_3, y_3]/(x_1y_1 - x_2y_2, x_1y_1 - x_3y_3)$. Then

$$f_1(x_1, y_1, x_2, y_2, x_3, y_3) = x_1y_1 - x_2y_2$$

and

$$f_2(x_1, y_1, x_2, y_2, x_3, y_3) = x_1y_1 - x_3y_3.$$

If $P = (a_1, b_1, a_2, b_2, a_3, b_3) \in X$, then it corresponds to

$$\begin{aligned} \mathcal{M}_P &= (x_1 - a_1, y_1 - b_1, ..., x_3 - a_3, y_3 - b_3)/(f_1, f_2) \\ &= (T_1, S_1, ..., T_3, S_3)/(f_1(T_1 + a_1, S_1 + b_1, ..., T_3 + a_3, S_3 + b_3), \\ &\quad f_2(T_1 + a_1, S_1 + b_1, ..., T_3 + a_3, S_3 + b_3)) \end{aligned} \tag{2.3}$$

but

$$\begin{aligned} &f_1(T_1 + a_1, S_1 + b_1, ..., T_3 + a_3, S_3 + b_3) = \\ &T_1S_1 - T_2S_2 + (a_1S_1 + b_1T_1 - a_2S_2 - b_2T_2), \end{aligned} \tag{2.4}$$

ie,

$$\mathrm{Moje}(P, f_1) = b_1T_1 + a_1S_1 - b_2T_2 - a_2S_2. \tag{2.5}$$

Similarly we can check that

$$\mathrm{Moje}(P, f_2) = b_1T_1 + a_1S_1 - b_3T_3 - a_3S_3, \tag{2.6}$$

so

$$C(P) = \begin{pmatrix} b_1 & a_1 & -b_2 & -a_2 & 0 & 0 \\ b_1 & a_1 & 0 & 0 & -b_3 & -a_3 \end{pmatrix}. \tag{2.7}$$

If we equate the minor determinants to zero and use them as the equations of a system, we see that P is singular if at least two pairs (a_i, b_i) are zero $(1 \le i \le 3)$.

Example 2.3 Let

$$X = \mathrm{Spec}\, k[x_1, y_1, ..., x_n, y_n]/(x_1y_1 - x_iy_i)_{2\le i\le n}, \tag{2.8}$$

and $P = (a_1, b_1, ..., a_n, b_n) \in X$. Consider $f_i = x_1y_1 - x_iy_i$ where $2 \le i \le n$. Then

$$\mathrm{Moje}(P, f_i) = b_1T_1 + a_1S_1 - b_iT_i - a_iS_i, \tag{2.9}$$

ie,

$$C(P) = \begin{pmatrix} b_1 & a_1 & -b_2 & -a_2 & 0 & 0 & \dots & \dots & 0 & 0 \\ b_1 & a_1 & 0 & 0 & -b_3 & -a_3 & 0 & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ b_1 & a_1 & 0 & 0 & \dots & \dots & 0 & 0 & -b_n & -a_n \end{pmatrix}. \tag{2.10}$$

This matrix shows us that if at least two pairs (a_i, b_i) are zero where $2 \le i \le n$, then P is a singular point.

2.3 Generalization

Let R be a discrete valuation ring with algebraically closed residue field $k = \frac{R}{(\pi)}$ and

$$X = \operatorname{Spec} R[x_1, ..., x_n]/(f_1(x_1, ..., x_n), ..., f_m(x_1, ..., x_n)). \tag{2.11}$$

Then the closed points of X are on the special fibre of X, *ie*, on the scheme

$$X_\pi = \operatorname{Spec} k[x_1, ..., x_n]/(\bar{f}_1(x_1, ..., x_n), ..., \bar{f}_m(x_1, ..., x_n)), \tag{2.12}$$

where $\bar{f}_i$ denotes the reduction of $f_i \,(\operatorname{mod} \pi)$. Let $P = (a_1, ..., a_n) \in X_\pi$ be a closed point of X and choose $\alpha_1, ..., \alpha_n \in R$ such that $\bar{\alpha}_i \equiv a_i (\operatorname{mod} \pi)$. Then as a point of X, P corresponds to the maximal ideal of $\Gamma(X, \mathcal{O}_X)$ generated by $x_1 - \alpha_1, ..., x_n - \alpha_n$ and π, *ie*,

$$\begin{aligned} \mathcal{M}_P &= (x_1 - \alpha_1, x_2 - \alpha_2, ..., x_n - \alpha_n, \pi)/ \\ &\qquad (f_1(x_1, ..., x_n), ..., f_m(x_1, ..., x_n)). \end{aligned} \tag{2.13}$$

By using the translation $T_i = x_i - \alpha_i$, we get

$$\begin{aligned} \mathcal{M}_P &= (T_1, T_2, ..., T_n, \pi)/ \\ &\qquad (f_1(T_1 + \alpha_1, ..., T_n + \alpha_n), ..., f_m(T_1 + \alpha_1, ..., T_n + \alpha_n)). \end{aligned} \tag{2.14}$$

So

$$\begin{aligned} \mathcal{M}_P/\mathcal{M}_P^2 &= (T_1, T_2, ..., T_n, \pi)/ \\ &\qquad ((T_1, T_2, ..., \pi)^2 + (T_1, T_2, ..., T_n, \pi) \cap \\ &\qquad (f_1(T_1 + \alpha_1, ..., T_n + \alpha_n), ..., f_m(T_1 + \alpha_1, ..., T_n + \alpha_n))), \end{aligned} \tag{2.15}$$

ie,

$$\begin{aligned} \mathcal{M}_P/\mathcal{M}_P^2 &= (T_1, T_2, ..., T_n, \pi)/ \\ &\qquad ((T_1, T_2, ..., T_n, \pi)^2 + \{(\operatorname{Moje}(P, f_i)\}_{1 \le i \le m})), \end{aligned} \tag{2.16}$$

where $\operatorname{Moje}(P, f_i)$ is the linear part in $T_1, ..., T_n$ and π of $f_i(T_1 + \alpha_1, ..., T_n + \alpha_n)$, say

$$\operatorname{Moje}(P, f_i) = d_{i0}\pi + d_{i1}T_1 + d_{i2}T_2 + ... + d_{in}T_n, \tag{2.17}$$

where $1 \le i \le m$ and $d_{ij} \in R$, *ie*, $\text{Moje}(P, f_i) = f_i(T_1 + \alpha_1, ..., T_n + \alpha_n) \mod (T_1, ..., T_n, \pi)^2$.

Now consider the matrix

$$D(P) = \begin{pmatrix} \bar{d}_{10} & \bar{d}_{11} & \bar{d}_{12} & \dots & \dots & \dots & \bar{d}_{1n} \\ \bar{d}_{20} & \bar{d}_{21} & \bar{d}_{22} & \dots & \dots & \dots & \bar{d}_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \bar{d}_{m0} & \bar{d}_{m1} & \bar{d}_{m2} & \dots & \dots & \dots & \bar{d}_{mn} \end{pmatrix}, \tag{2.18}$$

where $\bar{d}_{ij}$ is the reduction of $d_{ij} \pmod{\pi}$. If $r = rank\, D(P)$, then

$$\dim \mathcal{M}_P/\mathcal{M}_P^2 = n + 1 - r. \tag{2.19}$$

Thus we obtain:

Proposition 2.4 (Procedure of Mojgan)

P is a regular point of X if $n + 1 - r = \dim X$, otherwise it is singular.

Definition 2.5 By using the same notation as above we consider

$$\text{moje}(P, f_i) = \bar{d}_{i0}T_0 + \bar{d}_{i1}T_1 + ... + \bar{d}_{in}T_n,$$

corresponding to $\text{Moje}(P, f_i)$ (T_0 corresponds to π), and define

Tangent space of X at P=$\text{Spec}\, k[T_0, T_1, T_2, ..., T_n]/(\{\text{moje}(P, f_i)\}_{1 \le i \le m})$

Example 2.6 Let $X = \text{Spec}\, R[x_1, y_1, x_2, y_2]/(x_1y_1 - y_2^2 + x_2^3, x_1y_1 - \pi)$. Consider $f_1 = x_1y_1 - y_2^2 + x_2^3$, $f_2 = x_1y_1 - \pi$. Then $f_1^* = x_1y_1 - y_2^2$ and $f_2^* = -\pi := -x_0$. Hence

Tangent cone of X at P_0=$\text{Spec}\, k[x_0, x_1, y_1, x_2, y_2]/(x_1y_1 - y_2^2, x_0)$=
$\text{Spec}\, k[x_1, y_1, x_2, y_2]/(x_1y_1 - y_2^2)$. But
$\text{moje}(P_0, f_1) = 0$ and $\text{moje}(P_0, f_2) = -T_0$, *ie*,
Tangent space of X at P_0=$\text{Spec}\, k[T_0, T_1, S_1, T_2, S_2]/(0, T_0) =$
$\text{Spec}\, k[T_1, S_1, T_2, S_2] = \mathbb{A}_k^4$, so P_0 is a singular point.

Lemma 2.7 Let Y be a regular scheme. Then $\mathbb{A}^n_Y = \mathbb{A}^n \times_{\text{Spec}\,\mathbf{Z}} Y$ is regular.

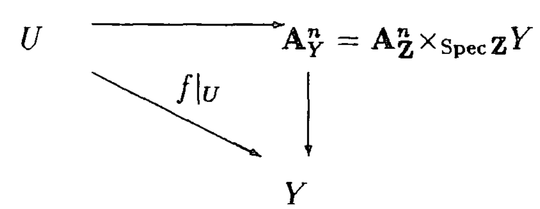
Proof : Let $Y = \text{Spec}\,R$, where R is a regular ring. Then $\mathbb{A}^n_Y = \mathbb{A}^n \times_{\text{Spec}\,\mathbf{Z}} \text{Spec}\,R = \text{Spec}\,(\mathbb{Z}[x_1, ..., x_n] \otimes_{\mathbf{Z}} R) = \text{Spec}\,R[x_1, ..., x_n]$ which is regular (see theorem 40, page 126 in [15]).

If Y is non-affine, then we consider Y as a union of affine schemes and use the above result for the product of $\mathbb{A}^n$ with the affine pieces and then glue them together. □

Theorem 2.8 Let $f : X \longrightarrow Y$ be etale. If Y is regular, then X is regular.
Proof: See page 27, prop. 3.17-(c) in [16]. □

Theorem 2.9 Let Y be a regular scheme and X be smooth over Y. Then X is a regular scheme.

Proof : Let $f : X \longrightarrow Y$ be the smooth morphism. Then by definition (see definition (1.1), page 128 in [1]), each $x \in X$ has an open neighbourhood U such that $U \longrightarrow \mathbf{A}^n_Y$ is etale and $\mathbf{A}^n_Y \longrightarrow Y$ is the projection on the second factor and the following diagram is commutative:



Since Y is regular by lemma (2.7), A^n_Y is regular. By using theorem (2.8) we conclude that U is regular. □

Remark 2.10 Note that the other definition of smooth morphism as it is stated in page 304 of [17] or page 305 in [23] is equivalent with definition (1.1), page 128 in [1] (for the proof see prop. 3.24, page 31 in [16]).

Corollary 2.11 Let X and Y be schemes over S. If Y is regular and X is smooth over S, then $X \times_S Y$ is a regular scheme.

Proof: $X \longrightarrow S$ is smooth. After base extension we get $X \times_S Y \longrightarrow Y$. Now Y is regular and $X \times_S Y \longrightarrow Y$ is smooth (see [1], page 129, prop. (1.7)-(*iii*)). By using theorem (2.9) we conclude that $X \times_S Y$ is regular. □

Here we will have an application of the procedure of Mojgan which is very useful in coming chapters.

Lemma 2.12 Let R be a discrete valuation ring and $f(x_1, x_2, ..., x_n)$ be a non-constant monomial in $R[x_1, x_2, ..., x_n]$ such that f is not divisible by π. Then $X = \text{Spec}\, R[x_1, x_2, ..., x_n]/(f(x_1, x_2, ..., x_n) - \pi)$ is a regular scheme.

Proof: If f does not contain $x_1, x_2, ..., x_l$ (for $l < n$), then we get

$$X = \mathbf{A}^l_R \times_{\text{Spec}\, R} \text{Spec}\, R[x_{l+1}, x_{l+2}, ..., x_n]/((f(x_{l+1}, x_{l+2}, ..., x_n) - \pi). \tag{2.20}$$

So without loss of generality we can assume that f contains all x_i's, say $f(x_1, x_2, ..., x_n) = x_1^{m_1} x_2^{m_2} ... x_n^{m_n}$. Now let $g(x_1, x_2, ..., x_n) = f(x_1, x_2, ..., x_n) - \pi$, *ie*, $g(x_1, x_2, ..., x_n) = x_1^{m_1} x_2^{m_2} ... x_n^{m_n} - \pi$. Then the special fibre of X is

$$\begin{aligned} X_\pi &= \text{Spec}\, k[x_1, x_2, ..., x_n]/(\bar{g}(x_1, x_2, ..., x_n)) = \\ &\quad \text{Spec}\, k[x_1, x_2, ..., x_n]/(x_1^{m_1} x_2^{m_2} ... x_n^{m_n}). \end{aligned} \tag{2.21}$$

We can find the singular points of X_π by using the Jacobian criterion. Let $Q = (a_1, a_2, ..., a_n) \in X_\pi$. Then as a point of X, Q corresponds to the ideal of $\Gamma(X, \mathcal{O}_X)$ generated by $x_1 - \alpha_1, x_2 - \alpha_2, ..., x_n - \alpha_n$ and π, *ie*, Q corresponds to

$$\mathcal{M}_Q = (x_1 - \alpha_1, x_2 - \alpha_2, ..., x_n - \alpha_n, \pi)/(g(x_1, x_2, ..., x_n)), \tag{2.22}$$

where $\alpha_i \in R$ and $\bar{\alpha}_i \equiv a_i (\text{mod} \pi)$. Since $Q \in X_\pi$, we have $\prod_{i=1}^{n} a_i = 0$, which means that at least one a_i is zero. Without loss of generality assume $a_1 = 0$, *ie*, $\pi | \alpha_1$, say $\alpha_1 = u_1 \pi$. According to the values of m_1 we consider two different cases as follows:

Case one $\qquad m_1 = 1$

We get $g(x_1, x_2, ..., x_n) = x_1 x_2^{m_2} ... x_n^{m_n} - \pi$. Hence

$$\begin{aligned}
&g(T_1+\alpha_1, T_2+\alpha_2, ..., T_n+\alpha_n) = \\
&(T_1+u_1\pi)(T_2+\alpha_2)^{m_2}...(T_n+\alpha_n)^{m_n} - \pi = \\
&(u_1\pi\alpha_2^{m_2}...\alpha_n^{m_n} - \pi) + \alpha_2^{m_2}...\alpha_n^{m_n}T_1 + \\
&m_2(u_1\pi)\alpha_2^{m_2-1}\alpha_3^{m_3}...\alpha_n^{m_n}T_2 + m_3(u_1\pi)\alpha_2^{m_2}\alpha_3^{m_3-1}\alpha_4^{m_4}...\alpha_n^{m_n}T_3 + \\
&... + m_n(u_1\pi)\alpha_2^{m_2}...\alpha_{(n-1)}^{m_{(n-1)}}\alpha_n^{m_n-1}T_n + ... \qquad (2.23)
\end{aligned}$$

ie,

$$\begin{aligned}
&\text{Moje}(Q,g) = (u_1\alpha_2^{m_2}...\alpha_n^{m_n} - 1)\pi + \alpha_2^{m_2}...\alpha_n^{m_n}T_1 + \\
&m_2(u_1\pi)\alpha_2^{m_2-1}\alpha_3^{m_3}...\alpha_n^{m_n}T_2 + m_3(u_1\pi)\alpha_2^{m_2}\alpha_3^{m_3-1}\alpha_4^{m_4}...\alpha_n^{m_n}T_3 + \\
&... + m_n(u_1\pi)\alpha_2^{m_2}...\alpha_{(n-1)}^{m_{(n-1)}}\alpha_n^{m_n-1}T_n \,(\text{mod}\,(T_1, ..., T_n, \pi)^2). \qquad (2.24)
\end{aligned}$$

So

$$D(Q) \;=\; (\,\bar{u_1}a_2^{m_2}...a_n^{m_n} - 1 \quad a_2^{m_2}...a_n^{m_n} \quad 0 \quad 0 \quad ... \quad 0 \quad 0\,) \qquad (2.25)$$

This means that Q is singular if $\bar{u_1}a_2^{m_2}...a_n^{m_n} - 1 = a_2^{m_2}...a_n^{m_n} = 0$ which is impossible, so Q is a regular point.

Case two $\qquad m_1 > 1$

In this case we get

$$\begin{aligned}
&g(T_1+\alpha_1, T_2+\alpha_2, ..., T_n+\alpha_n) = \\
&(T_1+u_1\pi)^{m_1}(T_2+\alpha_2)^{m_2}...(T_n+\alpha_n)^{m_n} - \pi = \\
&({u_1}^{m_1}\pi^{m_1}\alpha_2^{m_2}...\alpha_n^{m_n} - \pi) + m_1(u_1\pi)^{m_1-1}\alpha_2^{m_2}...\alpha_n^{m_n}T_1 + \\
&m_2(u_1\pi)^{m_1}\alpha_2^{m_2-1}\alpha_3^{m_3}...\alpha_n^{m_n}T_2 + m_3(u_1\pi)^{m_1}\alpha_2^{m_2}\alpha_3^{m_3-1}\alpha_4^{m_4}...\alpha_n^{m_n}T_3 + \\
&... + m_n(u_1\pi)^{m_1}\alpha_2^{m_2}...\alpha_{n-1}^{m_{n-1}}\alpha_n^{m_n-1}T_n + ..., \qquad (2.26)
\end{aligned}$$

ie,

$$\begin{aligned}
&\text{Moje}(Q,g) = (\pi^{m_1-1}{u_1}^{m_1}\alpha_2^{m_2}...\alpha_n^{m_n} - 1)\pi + m_1(u_1\pi)^{m_1-1}\alpha_2^{m_2}...\alpha_n^{m_n}T_1 + \\
&m_2(u_1\pi)^{m_1}\alpha_2^{m_2-1}\alpha_3^{m_3}...\alpha_n^{m_n}T_2 + m_3(u_1\pi)^{m_1}\alpha_2^{m_2}\alpha_3^{m_3-1}\alpha_4^{m_4}...\alpha_n^{m_n}T_3 + \\
&... + m_n(u_1\pi)^{m_1}\alpha_2^{m_2}...\alpha_{n-1}^{m_{n-1}}\alpha_n^{m_n-1}T_n \,(\text{mod}\,(T_1, ..., T_n, \pi)^2). \qquad (2.27)
\end{aligned}$$

Hence

$$D(Q) = (\,-1 \quad 0 \quad 0 \quad \dots \quad \dots \quad \dots \quad 0 \quad 0\,), \tag{2.28}$$

which has rank 1. So $\dim \mathcal{M}_Q/\mathcal{M}_Q^2 = (n+1) - 1 = n = \dim X$, *ie*, Q is a regular point. □

Corollary 2.13 The scheme $X = \operatorname{Spec} R[x_1, x_2, ..., x_{n-1}, x_n, y_n]/((x_1 x_2 ... x_{n-1})^2 x_n y_n - \pi)$ is a regular scheme.

Now we generalise lemma (2.12) as follows:

Theorem 2.14 Let $g_1(x_1, ..., x_n), ..., g_m(x_1, ..., x_n) \in R[x_1, ..., x_n]$ such that $f(x_1, ..., x_2) = g_1(x_1, ..., x_n)...g_m(x_1, ..., x_n)$ is not divisible by π and $h(x_1, ..., x_n) = f(x_1, ..., x_x) - \pi$. If for each i, the polynomial g_i is a linear form or the k-variety $W_i = \operatorname{Spec} k[x_1, ..., x_n]/(\bar{g}_i(x_1, ..., x_n))$ is non-singular, then

$$Y = \operatorname{Spec} R[x_1, ..., x_n]/(g_1(x_1, ..., x_n)...g_m(x_1, ..., x_n) - \pi) \tag{2.29}$$

is a regular scheme.

Proof: Let $P = (a_1, ..., a_n) \in Y_\pi$ and $\alpha_1, ..., \alpha_n \in R$ such that $\alpha_i \equiv a_i (\operatorname{mod} \pi)$, (for $i = 1, ..., n$). Surely $f(\alpha_1, ..., \alpha_n) \equiv 0 (\operatorname{mod} \pi)$ and $\frac{\partial f}{\partial x_i}(\alpha_1, ..., \alpha_n) \equiv 0 (\operatorname{mod} \pi)$ if $P \in Y_\pi^{\mathrm{Sing}}$.

Now let $P \in Y_\pi^{\mathrm{Sing}}$ and $T_i = x_i - \alpha_i$. Then $\pi | \frac{\partial f}{\partial x_i}(\alpha_1, ..., \alpha_n)$ and so

$$\frac{\partial f}{\partial x_i}(\alpha_1, ..., \alpha_n) T_i \equiv 0 (\operatorname{mod} (T_1, ..., T_n, \pi)^2) \tag{2.30}$$

But

$$f(T_1 + \alpha_1, ..., T_n + \alpha_n) = f(\alpha_1, ..., \alpha_n) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\alpha_1, ..., \alpha_n) T_i + \dots . \tag{2.31}$$

Hence

$$\begin{aligned} \operatorname{Moje}(P, h) &= \{\{f(\alpha_1, ..., \alpha_n) - \pi\} + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\alpha_1, ..., \alpha_n) T_i\} (\operatorname{mod} (T_1, ..., T_n, \pi)^2) \\ &\equiv \{f(\alpha_1, ..., \alpha_n) - \pi\} (\operatorname{mod} \pi^2). \end{aligned} \tag{2.32}$$

So $P \in Y^{\text{Sing}}$ if $f(\alpha_1, ..., \alpha_n) - \pi \equiv 0 (\text{mod}\, \pi^2)$ which means that $\pi | f(\alpha_1, ..., \alpha_n)$ and $\pi^2 \not| f(\alpha_1, ..., \alpha_n)$. Without loss of generality assume that $\pi | g_1(\alpha_1, ..., \alpha_n)$ and $\pi \not| g_2(\alpha_1, ..., \alpha_n) ... g_m(\alpha_1, ..., \alpha_n)$. Note that

$$\frac{\partial f}{\partial x_i} = \frac{\partial g_1}{\partial x_i} g_2 g_3 ... g_m + g_1 \{ (\frac{\partial g_2}{\partial x_i}) g_3 ... g_m + ... + g_2 ... g_{m-1} (\frac{\partial g_m}{\partial x_i}) \}. \tag{2.33}$$

So

$$\frac{\partial f}{\partial x_i}(\alpha_1, ..., \alpha_n) = \frac{\partial g_1}{\partial x_i}(\alpha_1, ..., \alpha_n) g_2(\alpha_1, ..., \alpha_n) ... g_m(\alpha_1, ..., \alpha_n) \, (\text{mod}\, \pi). \tag{2.34}$$

If $P \in Y^{\text{Sing}}$, we get $\frac{\partial f}{\partial x_i}(\alpha_1, ..., \alpha_n) \equiv 0 (\text{mod}\, \pi)$ which implies that $\frac{\partial g_1}{\partial x_i}(\alpha_1, ..., \alpha_n) \equiv 0 \, (\text{mod}\, \pi)$ (recall that $\pi \not| g_2(\alpha_1, ..., \alpha_n) ... g_m(\alpha_1, ..., \alpha_n))$ and $g_1(\alpha_1, ..., \alpha_n) \equiv 0 \, (\text{mod}\, \pi)$, hence $P \in {W_1}^{\text{Sing}}$. □

Example 2.15 Let

$$Y = \text{Spec}\, R[x_1, y_1, x_2, y_2] / (x_1^2 (y_2^2 + x_2^3 + 1)(x_1^2 + y_2) - \pi). \tag{2.35}$$

By using $g_1(x_1, y_1, x_2, y_2) = g_2(x_1, y_1, x_2, y_2) = x_1$ and $g_3(x_1, y_1, x_2, y_2) = y_2^2 + x_2^3 + 1$ and $g_4(x_1, y_1, x_2, y_2) = x_1^2 + y_2$ and applying theorem (2.14), we conclude that Y is a regular scheme.

2.4 The Procedure of Mahtab

Let X be a noetherian scheme and $\mathcal{J}$ be a coherent sheaf of ideals on X and $\mathcal{S} = \bigoplus_{d \geq 0} \mathcal{J}^d$ (where $\mathcal{J}^d$ is the dth power of the ideal $\mathcal{J}$ with $\mathcal{J}^0 = \mathcal{O}_X$). Then $\mathcal{S}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules which has a structure of a sheaf of $\mathcal{O}_X$-algebras. The scheme $\widetilde{X} = \mathbf{Proj} \mathcal{S}$ is called the blowing-up of X with respect to $\mathcal{J}$. If Y is a closed subscheme of X corresponding to $\mathcal{J}$, then $\widetilde{X}$ is called the blowing-up of X along Y or with centre Y. For more details see [11].

As a trivial example we can consider the case $X = \mathbf{A}_k^n$ and blow-up $P_0 = (0, 0, ..., 0, 0) \in X$. Let $A = \Gamma(X, \mathcal{O}_X) = k[x_1, x_2, ..., x_n]$. Then P_0 corresponds to the ideal $I = (x_1, x_2, ..., x_n)$ of A and $\widetilde{X}$ is isomorphic with a closed subscheme of $\mathrm{Proj}A[\mu_1, \mu_2, ..., \mu_n] = \mathbb{P}_A^{n-1} = \mathrm{Spec}\, A \times_{\mathrm{Spec}\, k} \mathbb{P}_k^{n-1}$ defined by $\mathrm{Ker}\,\alpha = (\{x_i\mu_j - x_j\mu_i\})_{1\leq i,j\leq n}$, where α is the following epimorphism:

$$\alpha : A[\mu_1, \mu_2, ..., \mu_n] \longrightarrow S = \bigoplus_{d\geq 0} I^d$$

$$\mu_i \longmapsto x_i.$$

When we are involving with a scheme X over a field of characteristic zero and blow-up a suitable subscheme Y of X, by Hironaka's proof there exists a regular scheme $\widetilde{X}$ birational to X such that $\widetilde{X} - E$ is isomorphic to $X - Y$, where E is the exceptional divisor.

Remark 2.16 After blowing-up a scheme X we get some open pieces for the covering of $\widetilde{X}$ and then glue them together. For more details about the gluing of schemes see [11], [13], [20] vol II, and [8]. If we continue using the process of blowings-up, we use $\widetilde{X_1}$, $\widetilde{X_2}$,...,$\widetilde{X_n}$, for the result of gluing of the open pieces after first,second,...,nth blowings-up.

Remark 2.17 Let R be a ring and M be an R- module. A sequence $x_1, x_2, ..., x_r$ of elements of R is called a regular sequence for M if x_1 is not a zero divisor in M, and for all $i = 2, ..., r$, x_i is not a zero divisor in $M/(x_1, ..., x_{i-1})M$. For more details see [11], page 184.

Remark 2.18 Recall that a scheme X is normal if all of its local rings are integrally closed domains. Let X be an integral scheme and for each open affine subset $U = \mathrm{Spec}\, A$ of X, let $\tilde{A}$ be the integral closure of A in its quotient field and let $\tilde{U} = \mathrm{Spec}\, \tilde{A}$. By gluing the schemes $\tilde{U}$ we obtain a normal integral scheme $\widetilde{X}$ called the normalization of X, see [11] and [13].

If a scheme is not normal after blowing-up the singular part would be worse than the singular part of the original scheme. So before each blowing-up, if the scheme is not normal we find its normalization and then blow-up this normalized scheme. In some cases just after normalization we get a regular scheme. For instance the normalization of the singular scheme $X = \mathrm{Spec}\, k[x, y]/(y^2 - x^3)$ is isomorphic with A_k^1. Of course I don't use it in my examples except in example (2.21).

In coming chapters we will use several times the procedure of Mahtab . Here we introduce this procedure.

Let $X = \text{Spec}\, R[x_1, x_2, ..., x_n]/(f_1, ..., f_m)$ such that X is flat over Spec R, and Y be a subscheme of X and $g_1, g_2, ..., g_r$ be a regular sequence generating the ideal $I = I(Y)$ of Y in $A = \Gamma(X, \mathcal{O}_X)$. Define

$$\alpha : A[\mu_1, ..., \mu_r] \longrightarrow S = \bigoplus_{d \geq 0} I^d$$
$$\mu_i \longmapsto g_i.$$

Then α is an epimorphism and

$$\text{Ker}\alpha = (\{\mu_i g_j - \mu_j g_i\})_{1 \leq i,j \leq r}$$

and $A[\mu_1, \mu_2, ..., \mu_r]/\text{Ker}\alpha \cong S$, which induces an isomorphism

$$\phi : \widetilde{X} = \text{Proj}S \longrightarrow \text{Proj}(A[\mu_1, \mu_2, ..., \mu_r]/\text{Ker}\,\alpha).$$

The right hand side scheme is a closed subscheme of

$$\text{Proj}A[\mu_1, \mu_2, ..., \mu_r] = \mathbf{P}_A^{r-1} = \text{Spec}\, A \times_{\text{Spec}\, R} \mathbf{P}_R^{r-1}. \tag{2.36}$$

The standard open covering of $\mathbf{P}_R^{r-1}$, *ie*,

$$\begin{aligned} U_1^0 &= \text{Spec}\, R[\frac{\mu_2}{\mu_1}, \frac{\mu_3}{\mu_1}, ..., \frac{\mu_r}{\mu_1}], \\ . &= ... \\ . &= ... \\ . &= ... \\ U_r^0 &= \text{Spec}\, R[\frac{\mu_1}{\mu_r}, \frac{\mu_2}{\mu_r}, ..., \frac{\mu_{r-1}}{\mu_r}], \end{aligned} \tag{2.37}$$

induces an open covering for $\widetilde{X}$, say $\widetilde{X} = V_1^0 \cup ... \cup V_r^0$ where

$$\begin{aligned} V_1^0 = \; & \text{Spec}\,(A \otimes_R \Gamma(U_1^0, \mathcal{O})\,(\text{mod}\, Ker\alpha)) = \\ & \text{Spec}\, R[x_1, ..., x_n, \frac{\mu_2}{\mu_1}, ..., \frac{\mu_r}{\mu_1}]/(f_1(x_1, ..., x_n), ..., f_m(x_1, ..., x_n), \\ & \frac{\mu_2}{\mu_1} - \frac{g_2}{g_1}, \frac{\mu_3}{\mu_1} - \frac{g_3}{g_1}, ..., \frac{\mu_r}{\mu_1} - \frac{g_r}{g_1}, ...) = \\ & \text{Spec}\, R[x_1, x_2, ..., x_n, T_{12}, T_{13}, ..., T_{1r}]/ \\ & (f_1(x_1, ..., x_n), ..., f_m(x_1, ...x_n), \{T_{1j}g_1 - g_j\}_{2 \leq j \leq r}). \end{aligned} \tag{2.38}$$

Similarly for $2 \leq i \leq r$ we get

$$\begin{aligned} V_i^0 &= \text{Spec}\,(A \otimes_R \Gamma(U_i^0, \mathcal{O})\,(\text{mod Ker}\alpha)) = \\ &\text{Spec}\,R[x_1, ..., x_n, \frac{\mu_1}{\mu_i}, \frac{\mu_2}{\mu_i}, ..., \frac{\mu_r}{\mu_i}]/(f_1(x_1, ..., x_n), ..., f_m(x_1, ..., x_n), \\ &\frac{\mu_1}{\mu_i} - \frac{g_1}{g_i}, \frac{\mu_2}{\mu_i} - \frac{g_2}{g_i}, ..., \frac{\mu_r}{\mu_i} - \frac{g_r}{g_i}, ...) = \\ &\text{Spec}\,R[x_1, ..., x_n, T_{i1}, T_{i2}, ..., T_{ii-1}, T_{ii+1}, ..., T_{ir}]/ \\ &(f_1(x_1, ..., x_n), ..., f_m(x_1, ..., x_n), \{T_{ij}g_i - g_j\}_{1 \leq j \leq r}, \ j \neq i\ ). \end{aligned} \tag{2.39}$$

If all the pieces V_i^0 are regular, their gluing gives us the regular scheme $\widetilde{X}$. If any V_i^0 is singular after normalization we blow-up its singular part or a subscheme of its singular part and continue this process until getting all pieces regular and then glue them through their overlap.

We call the above procedure, the procedure of Mahtab.

Example 2.19 Let k be a field and $X = \text{Spec}\,k[x_1, y_1, x_2, y_2]/(x_1y_1 - x_2y_2)$. By using the Jacobian criterion we can check that $P_0 = (0, 0, 0, 0)$ is the only singular point of X and as a point of X it corresponds to the maximal ideal I of $A = \Gamma(X, \mathcal{O}_X)$ generated by x_1, y_1, x_2 and y_2. Now consider the ring homomorphism

$$\begin{aligned} \alpha : A[\mu_1, \eta_1, \mu_2, \eta_2] &\longrightarrow S = \bigoplus\nolimits_{d \geq 0} I^d \\ \mu_i &\longrightarrow x_i \\ \eta_i &\longrightarrow y_i. \end{aligned} \tag{2.40}$$

Then α is an epimorphism and $\text{Ker}\alpha$ is the ideal of $A[\mu_1, \eta_1, \mu_2, \eta_2]$ generated by

$$\{x_i\mu_j - x_j\mu_i, x_i\eta_j - y_j\mu_i, y_i\eta_j - y_j\eta_i\}_{1 \leq i,j \leq 2} \tag{2.41}$$

and we have

$$A[\mu_1, \eta_1, \mu_2, \eta_2]/\text{Ker}\alpha \cong S, \tag{2.42}$$

which induces an isomorphism

$$\widetilde{X} = \text{Proj}S \longrightarrow \text{Proj}A[\mu_1, \eta_1, \mu_2, \eta_2]/\text{Ker}\alpha. \tag{2.43}$$

The right hand side scheme is a closed subscheme of

$$\begin{aligned} \mathrm{Proj}A[\mu_1, \eta_1, \mu_2, \eta_2] &\cong \mathrm{Spec}\, A \times_{\mathrm{Spec}\, k} \mathrm{Proj} k[\mu_1, \eta_1, \mu_2, \eta_2] \\ &= \mathrm{Spec}\, A \times_{\mathrm{Spec}\, k} \mathbb{P}^3_k. \end{aligned} \tag{2.44}$$

So we have the following diagram for blowing-up:

$$\begin{array}{ccc} \widetilde{X} & \longrightarrow & \mathrm{Spec}\, A \times_{\mathrm{Spec}\, k} \mathbb{P}^3_k \\ & \overset{\phi}{\searrow} & \downarrow \\ & & X = \mathrm{Spec}\, A \end{array}$$

The standard open covering of $\mathbb{P}^3_k = \mathrm{Proj} k[\mu_1, \eta_1, \mu_2, \eta_2]$, *ie,*

$$U_1^0 = \mathrm{Spec}\, k[\frac{\eta_1}{\mu_1}, \frac{\mu_2}{\mu_1}, \frac{\eta_2}{\mu_1}] \tag{2.45}$$

$$U_1^1 = \mathrm{Spec}\, k[\frac{\mu_1}{\eta_1}, \frac{\mu_2}{\eta_1}, \frac{\eta_2}{\eta_1}] \tag{2.46}$$

$$U_2^0 = \mathrm{Spec}\, k[\frac{\mu_1}{\mu_2}, \frac{\eta_1}{\mu_2}, \frac{\eta_2}{\mu_2}] \tag{2.47}$$

$$U_2^1 = \mathrm{Spec}\, k[\frac{\mu_1}{\eta_2}, \frac{\eta_1}{\eta_2}, \frac{\mu_2}{\eta_2}], \tag{2.48}$$

induces an open covering for $\widetilde{X}$, say W_1^0,W_1^1,W_2^0 and W_2^1 where

$$\mu_1 \neq 0, \qquad W_1^0 = \mathrm{Spec}\, k[x_1, \frac{x_2}{x_1}, \frac{y_2}{x_1}] \tag{2.49}$$

$$\eta_1 \neq 0, \qquad W_1^1 = \mathrm{Spec}\, k[y_1, \frac{x_2}{y_1}, \frac{y_2}{y_1}] \tag{2.50}$$

$$\mu_2 \neq 0, \qquad W_2^0 = \mathrm{Spec}\, k[\frac{x_1}{x_2}, \frac{y_1}{x_2}, x_2] \tag{2.51}$$

$$\eta_2 \neq 0, \qquad W_2^1 = \mathrm{Spec}\, k[\frac{x_1}{y_2}, \frac{y_1}{y_2}, y_2]. \tag{2.52}$$

The schemes W_1^0, W_1^1, W_2^0 and W_2^1 are all regular schemes and are pieces for the covering of $\widetilde{X}$. □

Example 2.20 Let k be a field and

$$X = \operatorname{Spec} k[x_1, y_1, x_2, y_2]/(y_1^2 - y_2^2 - y_1x_1^2 + y_2x_2^2). \tag{2.53}$$

By using the Jacobian criterion we can check that $P_0 = (0,0,0,0)$ is the only singular point of X which corresponds to the ideal of $A = \Gamma(X, \mathcal{O}_X)$ generated by x_1, y_1, x_2 and y_2, *ie*,

$$I = \mathcal{M}_{P_0} = (x_1, y_1, x_2, y_2)/(y_1^2 - y_2^2 - y_1x_1^2 + y_2x_2^2). \tag{2.54}$$

Now we blow-up P_0. Consider the epimorphism

$$\alpha : A[\mu_1, \eta_1, \mu_2, \eta_2] \longrightarrow S = \bigoplus_{d\geq 0} I^d$$

$$\mu_i \longmapsto x_i$$

$$\eta_i \longmapsto y_i.$$

If we use the the procedure of Mahtab, we get four pieces for the covering of $\widetilde{X}$, say V_1^0, V_1^1, V_2^0 and V_2^1 as follows:

Chart 1

$$\begin{aligned} V_1^0 &= \operatorname{Spec} k[x_1, \frac{y_1}{x_1}, \frac{x_2}{x_1}, \frac{y_2}{x_1}]/((\frac{y_1}{x_1})^2 - (\frac{y_2}{x_1})^2 - x_1(\frac{y_1}{x_1}) + x_1(\frac{y_2}{x_1})(\frac{x_2}{x_1})^2) = \\ & \operatorname{Spec} k[X_1, Y_1, X_2, Y_2]/(Y_1^2 - Y_2^2 - X_1Y_1 + X_1Y_2X_2^2). \end{aligned} \tag{2.55}$$

Let $f(X_1, Y_1, X_2, Y_2) = Y_1^2 - Y_2^2 - X_1Y_1 + X_1Y_2X_2^2$. If $Q = (a_1, b_1, a_2, b_2) \in V_1^0$, then

$$\operatorname{Moje}(Q, f) = (a_2^2b_2 - b_1)T_1 + (2b_1 - a_1)S_1 + 2a_1a_2b_2T_2 + (a_1a_2^2 - 2b_2)S_2, \tag{2.56}$$

ie,

$$C(Q) = (\, a_2^2b_2 - b_1 \quad 2b_1 - a_1 \quad 2a_1a_2b_2 \quad a_1a_2^2 - 2b_2 \,). \tag{2.57}$$

We can check that $C(Q)$ has rank zero if and only if $a_1 = b_1 = b_2 = 0$. Hence

$$(V_1^0)^{\text{Sing}} = \{(a_1, b_1, a_2, b_2) \in (V_1^0)_\pi | a_1 = b_1 = b_2 = 0\} := S. \tag{2.58}$$

Now we blow-up S. Let $A{=}\Gamma(V_1^0, \mathcal{O}_{\widetilde{X_1}}|_{V_1^0})$ and I be the ideal of A generated by x_1, y_1, y_2 and define

$$\alpha : A[\mu_1, \eta_1, \eta_2] \longrightarrow S = \bigoplus_{d\geq 0} I^d$$

$$\eta_i \longmapsto y_i$$

$$\mu_1 \longmapsto x_1.$$

If we use the rest of details about the the procedure of Mahtab we get three pieces for the covering of $\widetilde{V_1^0}$, say V_{11}^{00}, V_{11}^{01} and V_{12}^{01} as follows:

Chart 1.1

$$\begin{aligned} V_{11}^{00} &= \operatorname{Spec} k[x_1, \frac{y_1}{x_1}, x_2, \frac{y_2}{x_1}]/((\frac{y_1}{x_1})^2 - (\frac{y_2}{x_1})^2 - (\frac{y_1}{x_1}) + x_2^2(\frac{y_2}{x_1})) = \\ & \operatorname{Spec} k[X_1, Y_1, X_2, Y_2]/(Y_1^2 - Y_2^2 - Y_1 - X_2^2 Y_2). \end{aligned} \quad (2.59)$$

Let $Q = (a_1, b_1, a_2, b_2) \in V_{11}^{00}$ and $f(X_1, Y_1, X_2, Y_2) = Y_1^2 - Y_2^2 - Y_1 - X_2^2 Y_2$. Then

$$\operatorname{Moje}(Q, f) = (2b_1 - 1)S_1 - 2a_2b_2T_2 - (a_2^2 + 2b_2)S_2, \quad (2.60)$$

so

$$C(Q) = (\,0 \quad 2b_1 - 1 \quad -2a_2b_2 \quad -(a_2^2 + 2b_2)\,). \quad (2.61)$$

Hence $C(Q)$ has rank zero if $b_1 = \frac{1}{2}, a_2 = b_2 = 0$ which is impossible. This means that Q is a regular point and consequently V_{11}^{00} is a regular scheme.

Chart 1.2

$$V_{11}^{01} = \operatorname{Spec} k[X_1, Y_1, X_2, Y_2]/(1 - Y_2^2 - X_1 + X_1Y_2X_2^2), \quad (2.62)$$

which is a regular scheme (look at $C(Q)$).

Chart 1.4

$$V_{12}^{01} = \operatorname{Spec} k[X_1, Y_1, X_2, Y_2]/(Y_1^2 - 1 - X_1Y_1 + X_1X_2^2) \quad (2.63)$$

which is a regular scheme (look at $C(Q)$). The gluing of V_{11}^{00}, V_{11}^{01} and V_{12}^{01} gives us the regular scheme $\widetilde{V_1^0}$.

Chart 2

$$\begin{aligned} V_1^1 &= \operatorname{Spec} k[\frac{x_1}{y_1}, y_1, \frac{x_2}{y_1}, \frac{y_2}{y_1}]/(1-(\frac{y_2}{y_1})^2 - y_1(\frac{x_1}{y_1})^2 + y_1(\frac{y_2}{y_1})(\frac{x_2}{y_1})^2) = \\ & \operatorname{Spec} k[X_1, Y_1, X_2, Y_2]/(1 - Y_2^2 - Y_1X_1^2 + Y_1Y_2X_2^2). \end{aligned} \tag{2.64}$$

Let $f(X_1, Y_1, X_2, Y_2) = 1 - Y_2^2 - Y_1X_1^2 + Y_1Y_2X_2^2$ and $Q = (a_1, b_1, a_2, b_2) \in V_1^0$. Then

$$\text{Moje}(Q, f) = -2a_1b_1T_1 + (b_2a_2^2 - a_1^2)S_1 + 2b_1a_2b_2T_2 + (b_1a_2^2 - 2b_2)S_2, \tag{2.65}$$

so

$$C(Q) = (\, -2a_1b_1 \quad b_2a_2^2 - a_1^2 \quad 2b_1a_2b_2 \quad b_1a_2^2 - 2b_2 \,), \tag{2.66}$$

which has rank zero if $-2a_1b_1 = b_2a_2^2 - a_1^2 = 2b_1a_2b_2 = b_1a_2^2 - 2b_2 = 0$. This system has the solution $a_1 = b_1 = b_2 = 0$ or $a_1 = a_2 = b_2 = 0$. Considering $f(a_1, b_1, a_2, b_2) = 0$, we find out that none of these solutions are acceptable. Hence Q is a regular point and consequently V_1^1 is a regular scheme.

Chart 3

$$\begin{aligned} V_2^0 &= \operatorname{Spec} k[\frac{x_1}{x_2}, \frac{y_1}{x_2}, x_2, \frac{y_2}{x_2}]/((\frac{y_1}{x_2})^2 - (\frac{y_2}{x_2})^2 - x_2(\frac{y_1}{x_2})(\frac{x_1}{x_2})^2 + x_2(\frac{y_2}{x_2})) = \\ & \operatorname{Spec} k[X_1, Y_1, X_2, Y_2]/(Y_1^2 - Y_2^2 - X_2Y_1X_1^2 + X_2Y_2). \end{aligned} \tag{2.67}$$

Let $f(X_1, Y_1, X_2, Y_2) = Y_1^2 - Y_2^2 - X_2Y_1X_1^2 + X_2Y_2$ and $Q = (a_1, b_1, a_2, b_2) \in V_2^0$. Then

$$\text{Moje}(Q, f) = (-2a_1a_2b_1)T_1 + (2b_1 - a_1^2a_2)S_1 + (b_2 - a_1^2b_1)T_2 + (a_2 - 2b_2)S_2, \tag{2.68}$$

ie,

$$C(Q) = (\, -2a_1a_2b_1 \quad 2b_1 - a_1^2a_2 \quad b_2 - a_1^2b_1 \quad a_2 - 2b_2 \,). \tag{2.69}$$

So Q is singular if $-2a_1a_2b_1 = 2b_1 - a_1^2a_2 = b_2 - a_1^2b_1 = a_2 - 2b_2 = 0$. This system has the solution $b_1 = a_2 = b_2 = 0$ (note that $a_1 = b_1 = a_2 = b_2 = 0$ is also a solution for the system, but actually it is a special case of $b_1 = a_2 = b_2 = 0$), *ie*,

$$(V_2^0)^{\text{Sing}} = \{(a_1, b_1, a_2, b_2) \in V_1^0 \mid b_1 = a_2 = b_2 = 0\}. \tag{2.70}$$

Now we blow-up $(V_2^0)^{\text{Sing}}$. By using the procedure of Mahtab we get three pieces for the covering of $\widetilde{V_2^0}$, say V_{21}^{01},V_{22}^{00} and V_{22}^{01} as follows:

Chart 3.2

$$V_{21}^{01} = \operatorname{Spec} k[X_1, Y_1, X_2, Y_2]/(1 - Y_2^2 - X_2X_1^2 + X_2Y_2), \tag{2.71}$$

which is a regular scheme (just check C(Q)).

Chart 3.3

$$V_{22}^{00} = \operatorname{Spec} k[X_1, Y_1, X_2, Y_2]/(Y_1^2 - Y_2^2 - Y_1X_1^2 + Y_2). \tag{2.72}$$

Let $f(X_1, Y_1, X_2, Y_2) = Y_1^2 - Y_2^2 - Y_1X_1^2 + Y_2$ and $Q = (a_1, b_1, a_2, b_2) \in V_{22}^{00}$. Then

$$\text{Moje}(Q, f) = -2a_1b_1T_1 + (2b_1 - a_1^2)S_1 + (1 - 2b_2)S_2, \tag{2.73}$$

so

$$C(Q) = (\,-2a_1b_1 \quad 2b_1 - a_1^2 \quad 0 \quad 1 - 2b_2\,). \tag{2.74}$$

So Q is singular if $-2a_1b_1 = 2b_1 - a_1^2 = 1 - 2b_2 = 0$. But this system has the solution $a_1 = b_1 = b_2 - \frac{1}{2} = 0$ which is not acceptable. Hence Q is a regular point and as a result V_{22}^{00} is a regular scheme.

Chart 3.4

$$V_{22}^{01} = \operatorname{Spec} k[X_1, Y_1, X_2, Y_2]/(Y_1^2 - 1 - X_2Y_1X_1^2 + X_2), \tag{2.75}$$

which is a regular scheme (just check C(Q)). The gluing of V_{21}^{01}, V_{22}^{00} and V_{22}^{01} gives us the regular scheme $\widetilde{V_2^0}$.

Chart 4

$$V_2^1 = \operatorname{Spec} k[X_1, Y_1, X_2, Y_2]/(Y_1^2 - 1 - Y_2Y_1X_1^2 + Y_2X_2^2), \tag{2.76}$$

which is a regular scheme (just check $C(Q)$). The gluing of $\widetilde{V_1^0}, V_1^1, \widetilde{V_2^0}$ and V_2^1 gives us the regular scheme $\widetilde{X}$.

Example 2.21 Let R be a discrete valuation ring and $U = \operatorname{Spec} R[x,y]/(x^2 + \pi^2 y^3 - \pi^4)$. Since $(\frac{x}{\pi})^2 + y^3 - \pi^2 = 0$, $(\frac{x}{\pi})$ is integral over $\Gamma(U, \mathcal{O}_U)$. We get $V = U^{\text{nor}} = \operatorname{Spec} R[\frac{x}{\pi}, y]/((\frac{x}{\pi})^2 + y^3 - \pi^2) = \operatorname{Spec} R[X,Y]/(X^2 + Y^3 - \pi^2)$. By using the Jacobian criterion we can check that

$$V_\pi^{\text{Sing}} = \{(a,b) \in V_\pi | a = b = 0\}. \tag{2.77}$$

Let $f(X,Y) = X^2 + Y^3 - \pi^2$ and $P_0 = (0,0) \in V_\pi$. Then for u and v in R we get

$$f(T + u\pi, S + v\pi) = (T + u\pi)^2 + (S + v\pi)^3 - \pi^2 \tag{2.78}$$

So $\operatorname{Moje}(P_0, f) = 0$. Hence

$$D(P_0) = (0 \quad 0 \quad 0), \tag{2.79}$$

which has rank zero, *ie*, $P_0 = (0,0) \in V_\pi$ is the only singular point of V.

Let $A = \Gamma(V, \mathcal{O}_V)$. Then as a point of V, P_0 corresponds to the maximal ideal $I = (x, y, \pi)$ of A. Now we blow-up P_0. Consider the ring homomorphism

$$\phi : A[\mu, \eta, \theta] \longrightarrow S = \bigoplus\nolimits_{d \geq 0} I^d$$

$$\mu \longmapsto x$$

$$\eta \longmapsto y$$

$$\theta \longmapsto \pi.$$

By using the rest details of the procedure of Mahtab, we get three pieces for the covering of $\widetilde{V}$ as follows:

Chart 1

$$\begin{aligned} V_1^0 &= \operatorname{Spec} R[x, \frac{y}{x}, \frac{\pi}{x}]/(1 + x(\frac{y}{x})^3 - (\frac{\pi}{x})^2) = \\ & \operatorname{Spec} R[X,Y,Z]/(1 + XY^3 - Z^2, XZ - \pi). \end{aligned} \tag{2.80}$$

So

$$(V_1^0)_\pi = \operatorname{Spec} k[X,Y,Z]/(1 + XY^3 - Z^2, XZ). \tag{2.81}$$

By using the Jacobian criterion, we can check that

$$(V_1^0)_\pi^{\text{Sing}} = \phi. \tag{2.82}$$

which means that $(V_1^0)^{\text{Sing}} = \phi$, *ie*, V_1^0 is a regular scheme.

Chart 2

$$\begin{aligned} V_1^1 &= \operatorname{Spec} R[\frac{x}{y}, y, \frac{\pi}{y}]/((\frac{x}{y})^2 + y - (\frac{\pi}{y})^2) \\ & \operatorname{Spec} R[X,Y,Z]/(X^2 + Y - Z^2, YZ - \pi) \\ & \operatorname{Spec} R[X,Z]/(Z^3 - X^2 Z - \pi), \end{aligned} \tag{2.83}$$

so

$$(V_1^1)_\pi^{\text{Sing}} = \{(a,c) \in (V_1^1)_\pi | x = z = 0\}. \tag{2.84}$$

Let $g(X,Z) = Z^3 - X^2 Z - \pi$. Then

$$g(T + u\pi, W + w\pi) = (W + w\pi)^3 - (T + u\pi)^2 (W + w\pi) - \pi. \tag{2.85}$$

So $\text{Moje}(Q, g) = -\pi$, *ie*,

$$D(Q) = (\,-1 \quad 0 \quad 0\,), \tag{2.86}$$

which has rank one. Hence Q is a regular point of V_1^1 and consequently V_1^1 is a regular scheme.

Chart 3

$$\begin{aligned} V_\theta^\theta &= \operatorname{Spec} R[\frac{x}{\pi}, \frac{y}{\pi}]/((\frac{x}{\pi})^2 + \pi(\frac{y}{\pi})^3 - 1) = \\ & \operatorname{Spec} R[X,Y]/(X^2 + \pi Y^3 - 1), \end{aligned} \tag{2.87}$$

and we get

$$(V_\theta^\theta)_\pi = \operatorname{Spec} k[X,Y]/(X^2-1). \tag{2.88}$$

By using the Jacobian criterion we can check that $(V_\theta^\theta)_\pi^{\text{Sing}} = \phi$, so $(V_\theta^\theta)^{\text{Sing}} = \phi$, *ie*, V_θ^θ is a regular scheme. The gluing of V_1^0, V_1^1 and V_θ^θ gives us the regular scheme $\widetilde{V}$.

Remark 2.22 In the rest of this thesis each time that we are involving with a scheme of the form $X = \operatorname{Spec} R[x_1,y_1,x_2,y_2]/(F(x_1,y_1,x_2,y_2), G(x_1,y_1,x_2,y_2))$, we use f_1 and f_2 rather than F and G respectively. If we have an additional indeterminate Z and

$$\begin{aligned} X = {} & \operatorname{Spec} R[x_1,y_1,x_2,y_2,Z]/ \\ & (F(x_1,y_1,x_2,y_2,Z), G(x_1,y_1,x_2,y_2,Z), H(x_1,y_1,x_2,y_2,Z)), \end{aligned} \tag{2.89}$$

we use f_1, f_2 and f_3 instead of F, G and H respectively.

For the calculation of Moje at a point $P = (a_1,b_1,a_2,b_2,c) \in X_\pi$, we use α_1, β_1, α_2, β_2, $\gamma \in R$ such that $\alpha_i \equiv a_i (\operatorname{mod} \pi)$ and $\beta_i \equiv b_i (\operatorname{mod} \pi)$ and $\gamma \equiv c (\operatorname{mod} \pi)$. If $a_i = 0$ or $b_i = 0$ or $c = 0$ we use $\alpha_i = u_i\pi$ or $\beta_i = v_i\pi$ or $\gamma = w\pi$. For new indeterminates after translation we use T_i, S_i and W corresponding to x_i, y_i and Z respectively.

The lower and upper index show us the indeterminate involving with the blowing-up. To be more precise, we use X_1^0 to show that the involving indeterminate in the blowing-up is x_1, X_1^1 when it is y_1, X_2^0 when it is x_2, X_2^1 when it is y_2 and finally X_θ^θ when the involving indeterminate is Z.

Chapter 3

Product of ordinary double points

3.1 Introduction

In this chapter R always denotes an arbitrary regular ring (except in §3.5 where R denotes a dvr). We introduce $\mathbf{Y}^{12\ldots n}$ in the second section and try to resolve its singularities later. We find this desingularisation in two different methods. The results from the both methods are the same. In the last section of this chapter, we introduce $X^{12\ldots n}$ and consider our schemes over Spec$R$ where R is a discrete valuation ring with algebraically closed residue field $k = \frac{R}{\pi}$.

3.2 Proposed schemes

Consider the surface

$$Y^i = \text{Spec } R[t][x_i, y_i]/(x_i y_i - t) \cong \mathbf{A}^2_k, \quad 1 \leq i \leq n, \tag{3.1}$$

over Spec $R[t]$. Then Y^i over Spec $R[t]$ is smooth everywhere except at $x_1 = y_1 = 0$, where Y^i has an ordinary double point on the fibre corresponding to $t = 0$, but Y^i is a regular scheme. The fibre product of

$$Y^1 = \text{Spec } R[t][x_1, y_1]/(x_1y_1 - t) \tag{3.2}$$

and

$$Y^2 = \text{Spec } R[t][x_2, y_2]/(x_2y_2 - t) \tag{3.3}$$

is

$$\begin{aligned} Y^{12} &= \text{Spec}\, R[t][x_1, y_1, x_2, y_2]/(x_1y_1 - x_2y_2, x_1y_1 - t) \\ &= \text{Spec}\, R[x_1, y_1, x_2, y_2]/B_2 \end{aligned} \tag{3.4}$$

where B_2 is the ideal $(x_1y_1 - x_2y_2)$ of $A_2 = R[x_1, y_1, x_2, y_2]$. Inductively for each $n \in \mathbb{N}$ we get

$$\begin{aligned} Y^{12\ldots n} &= \text{Spec}\, R[x_1, y_1, \cdots, x_n, y_n]/(x_1y_1 - x_2y_2, x_1y_1 - x_3y_3, \cdots, x_1y_1 - x_ny_n) \\ &= \text{Spec}\, R[x_1, y_1, \cdots, x_n, y_n]/(x_1y_1 - x_iy_i)_{2 \le i \le n} \\ &:= \text{Spec}\, R[x_1, y_1, \cdots, x_n, y_n]/B_n, \end{aligned} \tag{3.5}$$

where

$$B_n = (x_1y_1 - x_iy_i)_{2 \le i \le n.} \tag{3.6}$$

In fact in the affine space $\mathbf{A}_k^{2n}$, $Y^{12\ldots n}$ is determined by the equations

$$x_1y_1 = x_2y_2 = \ldots = x_ny_n. \tag{3.7}$$

Remark 3.1 In the other sections of this chapter we will try to find singular points and also a desingularisation of $Y^{12\ldots n}$ in two different methods. These desingularisations would be the results of some successive blowings-up.

3.3 First Method

In this section we will find the singular points of the scheme $Y^{12\ldots n}$ and by successive blowings-up, we will try to resolve its singularities. This is the resolution described (without proof) in Scholl's paper (see [21]).

Convention 3.2 In the rest of this section we consider $Y^{12\dots n}$ as

$$Y^{12\dots n} = \text{Spec } R[x_1, y_1, \dots, x_n, y_n]/B_n \tag{3.8}$$

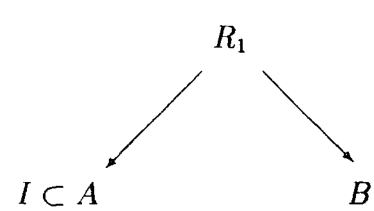
where

$$B_n = (x_1y_1 - x_2y_2, x_1y_1 - x_3y_3, \dots, x_1y_1 - x_ny_n). \tag{3.9}$$

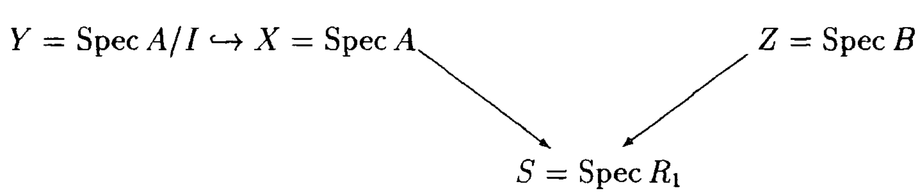
We also use A_n for the polynomial ring $k[x_1, y_1, \dots, x_n, y_n]$ and $A_{12\dots n}$ for the quotient ring $A_n/B_n = \Gamma(Y^{12\dots n}, \mathcal{O}_{Y^{12\dots n}})$.

Lemma 3.3 Let X and Z be schemes over S such that Z is flat over S, and Y is a closed subscheme of X. Then the blowing-up of $X \times_S Z$ at $Y \times_S Z$ is isomorphic to $\widetilde{X} \times_S Z$, where $\widetilde{X}$ is the blowing-up of X at Y.

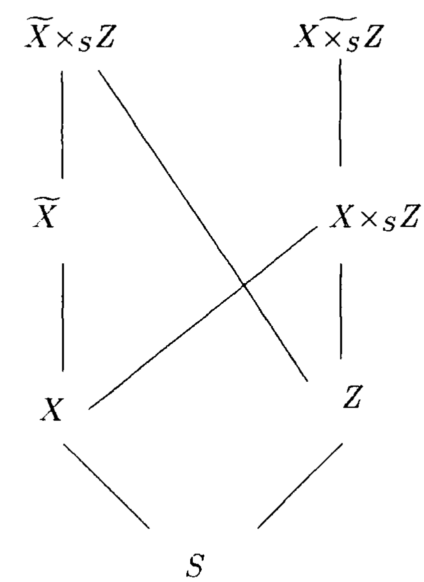
Proof : We prove it in the affine case. Let $X = \text{Spec } A$, $Y = \text{Spec } A/I$, $Z = \text{Spec } B$ and $S = \text{Spec } R_1$, *ie*,



which induces



The following diagram shows us the scheme of the proof.



Since B is flat over R_1, $I \hookrightarrow A$ implies that $I\otimes_{R_1}B \hookrightarrow A\otimes_{R_1}B$ is an injection. Let J be the image of $I\otimes_{R_1}B$ in $A\otimes_{R_1}B$. Then $J \cong I\otimes_{R_1}B$. Now notice that $\widetilde{X}\times_S Z$=$(\mathbf{Proj}\bigoplus_{d\geq 0}I^d)\times_S\text{Spec }B$ and $\widetilde{X\times_S Z}$= $\mathbf{Proj}\bigoplus_{d\geq 0}J^d = \mathbf{Proj}\bigoplus_{d\geq 0}(I\otimes_{R_1}B)^d$ $\cong \mathbf{Proj}((\bigoplus_{d\geq 0}I^d)\otimes_{R_1}B) \cong (\mathbf{Proj}\bigoplus_{d\geq 0}I^d)\times_{\text{Spec }R_1}\text{Spec }B$ $=\widetilde{X}\times_{\text{Spec }R_1}Z$.

If X and Z are not affine, then we consider them as the union of affine schemes and use the above result for the affine pieces and then glue them. □

Example 3.4 Consider the scheme Y^{123} = Spec $R[x_1, y_1, ..., x_3, y_3]/(f_1, f_2)$ where

$$f_1(x_1, y_1, ..., x_3, y_3) = x_1y_1 - x_2y_2 \tag{3.10}$$

and

$$f_2(x_1, y_1, ..., x_3, y_3) = x_1y_1 - x_3y_3. \tag{3.11}$$

By using the Jacobian criterion we can check that the singular points of Y^{123} are points of the form $(a_1, b_1, a_2, b_2, a_3, b_3)$ such that at least two pairs (a_i, b_i) are zero (see example(2.2)). In particular $P_0 = (0,0,0,0,0,0)$ is a singular point. Now we blow-up P_0. The following diagram is related to this blowing-up:

$$\begin{array}{ccc} \widetilde{Y_1^{123}} & \longrightarrow & \operatorname{Spec} A_{123} \times_{\operatorname{Spec} R} \mathbb{P}^5_R \\ & \searrow^{\phi_{123}} & \downarrow \\ & & Y^{123} = \operatorname{Spec} A_{123} \end{array}$$

By using the procedure of Mahtab we can check that the standard open covering for $\mathbb{P}^5_R = \operatorname{Proj} R[\mu_1, \eta_1, ..., \mu_3, \eta_3]$, *ie*,

$$\begin{aligned} U_1^0 &= \operatorname{Spec} R[\frac{\eta_1}{\mu_1}, \frac{\mu_2}{\mu_1}, ..., \frac{\mu_3}{\mu_1}, \frac{\eta_3}{\mu_1}] \\ . &= \ldots \\ . &= \ldots \\ . &= \ldots \\ U_3^1 &= \operatorname{Spec} R[\frac{\mu_1}{\eta_3}, \frac{\eta_1}{\eta_3}, ..., \frac{\eta_2}{\eta_3}, \frac{\mu_3}{\eta_3}], \end{aligned} \tag{3.12}$$

induces an open covering for $\widetilde{Y_1^{123}}$, say W_1^0, W_1^1,...,W_3^0,W_3^1, where

$$\begin{aligned} \mu_1 &\neq 0, \\ W_1^0 &= \mathbb{A}^1_R \times_{\operatorname{Spec} R} \operatorname{Spec} R[x_2, y_2, x_3, y_3]/(x_2 y_2 - x_3 y_3) \\ &= \mathbb{A}^1_R \times_{\operatorname{Spec} R} Y^{23}, \end{aligned} \tag{3.13}$$

and Y^{23} has $P_0 = (0,0,0,0)$ as its only singular point. Hence $(W_1^0)^{\text{Sing}} = \mathbb{A}^1_R \times_{\operatorname{Spec} R} \{P_0\}$ in $\mathbb{A}^1_R \times_{\operatorname{Spec} R} Y^{23}$. According to lemma (3.3), the blowing-up of W_1^0 along $(W_1^0)^{\text{Sing}}$ is isomorphic to

$$\begin{aligned} &\mathbb{A}^1_R \times_{\operatorname{Spec} R} \widetilde{Y^{23}} \cong \mathbb{A}^1_R \times_{\operatorname{Spec} R} (W_2^0 \cup W_2^1 \cup W_3^0 \cup W_3^1) \cong \\ &(\mathbb{A}^1_R \times_{\operatorname{Spec} R} W_2^0) \cup (\mathbb{A}^1_R \times_{\operatorname{Spec} R} W_2^1) \cup (\mathbb{A}^1_R \times_{\operatorname{Spec} R} W_3^0) \cup (\mathbb{A}^1_R \times_{\operatorname{Spec} R} W_3^1) \cong \\ &\operatorname{Spec} R[x_1] \times_{\operatorname{Spec} R} \operatorname{Spec} R[x_2, \frac{x_3}{x_2}, \frac{y_3}{x_2}] \cup ... \cup \operatorname{Spec} R[x_1] \times_{\operatorname{Spec} R} \operatorname{Spec} R[\frac{x_2}{y_3}, \frac{y_2}{y_3}, y_3] \\ &\cong \operatorname{Spec} R[x_1, \frac{x_2}{x_1}, \frac{x_3}{x_2} \frac{y_3}{x_2}] \cup ... \cup \operatorname{Spec} R[x_1, \frac{x_2}{y_3}, \frac{y_2}{y_3}, \frac{y_3}{x_1}] = \\ &W_{12}^{00} \cup W_{12}^{01} \cup W_{13}^{00} \cup W_{13}^{01}. \end{aligned} \tag{3.14}$$

Using the same method for the resolution of singularities of $W_1^1, W_2^0, ..., W_3^1$ we get

$$\widetilde{Y_2^{123}} = W_{12}^{00} \cup W_{12}^{01} \cup W_{13}^{00} \cup W_{13}^{01} ... \cup W_{31}^{10} \cup W_{31}^{11} \cup W_{32}^{10} \cup W_{32}^{11}, \tag{3.15}$$

where $W_{ij}^{\alpha\beta}$'s are 24 pieces for the covering of $\widetilde{Y_2^{123}}$ and

$$\begin{aligned} W_{12}^{00} &= \operatorname{Spec} R[x_1, \frac{x_2}{x_1}, \frac{x_3}{x_2}, \frac{y_3}{x_2}] \\ W_{12}^{01} &= \operatorname{Spec} R[x_1, \frac{y_2}{x_1}, \frac{x_3}{y_2}, \frac{y_3}{y_2}] \\ . &= ... \\ . &= ... \\ . &= ... \\ W_{32}^{11} &= \operatorname{Spec} R[\frac{x_1}{y_2}, \frac{y_1}{y_2}, \frac{y_2}{y_3}, y_3], \end{aligned} \tag{3.16}$$

are all regular schemes. So after two successive blowings-up we get the regular scheme $\widetilde{Y_2^{123}}$.

Remark 3.5 By using the Jacobian criterion we can check that points of the form $P = (a_1, b_1, ..., a_n, b_n) \in Y^{12...n}$ such that at least two pairs (a_i, b_i) are zero, are singular (see example (2.3)).

Theorem 3.6 After $n-1$ successive blowings-up of $Y^{12...n}$ we get a regular scheme $\widetilde{Y_{n-1}^{12...n}}$ as the gluing of $2^{n-1}(n!)$ open pieces of its covering.

Proof : We use induction on n. For $n = 2$ and 3 we already have checked (examples 2.19 and 3.4). Suppose it is true for $n < m$ (with $m \geq 4$) and recall that $P_0 = (0, 0, ..., 0) \in Y^{12...m}$ is a singular point of $Y^{12...m}$ which corresponds to the ideal $I_{12...m}$ of $\Gamma(Y^{12...m}, \mathcal{O}_{Y^{12...m}})$ generated by $x_1, y_1, ..., x_m, y_m$, *ie*, $\mathfrak{m}_{P_0}$ is the maximal ideal of A_m generated by $x_1, y_1, ..., x_m, y_m$. We blow-up the origin in $Y^{12...m}$ to obtain a scheme $\widetilde{Y^{12...m}}$. By the procedure of Mahtab, it has an open covering $W_1^0, W_1^1, ..., W_m^0, W_m^1$ where

$$\mu_1 \neq 0, \;\; W_1^0 =$$
$$\operatorname{Spec}(R[\frac{\eta_1}{\mu_1}, \frac{\mu_2}{\mu_1}, ..., \frac{\mu_m}{\mu_1}, \frac{\eta_m}{\mu_1}] \bigotimes\nolimits_R R[x_1, y_1, ..., x_m, y_m]/B_m) \text{ (mod Relations)}$$

$$\begin{aligned}
&= \operatorname{Spec} R[\frac{\eta_1}{\mu_1}, \frac{\mu_2}{\mu_1}, \dots, \frac{\mu_m}{\mu_1}, \frac{\eta_m}{\mu_1}, x_1, y_1, \dots, x_m, y_m]/ \\
&\quad (x_1y_1 - x_2y_2, x_1y_1 - x_3y_3, \dots, x_1y_1 - x_my_m, \text{ Relations}) \\
&= \operatorname{Spec} R[\frac{y_1}{x_1}, \frac{x_2}{x_1}, \dots, \frac{x_m}{x_1}, \frac{y_m}{x_1}, x_1, y_1, \dots, x_m, y_m]/ \\
&\quad\quad (x_1y_1 - x_2y_2, x_1y_1 - x_3y_3, \dots, x_1y_1 - x_my_m) \\
&= \operatorname{Spec} R[x_1, \frac{y_1}{x_1}, \frac{x_2}{x_1}, \frac{y_2}{x_1}, \dots, \frac{x_m}{x_1}, \frac{y_m}{x_1}]/ \\
&\quad (\frac{y_1}{x_1} - (\frac{x_2}{x_1})(\frac{y_2}{x_1}), \frac{y_1}{x_1} - (\frac{x_3}{x_1})(\frac{y_3}{x_1}), \dots, \frac{y_1}{x_1} - (\frac{x_m}{x_1})(\frac{y_m}{x_1})) \\
&= \operatorname{Spec} R[x_1, \frac{y_1}{x_1}, \frac{x_2}{x_1}, \frac{y_2}{x_1}, \dots, \frac{x_m}{x_1}, \frac{y_m}{x_1}]/ \\
&\quad ((\frac{x_2}{x_1})(\frac{y_2}{x_1}) - (\frac{x_3}{x_1})(\frac{y_3}{x_1}), (\frac{x_2}{x_1})(\frac{y_2}{x_1}) - (\frac{x_4}{x_1})(\frac{y_4}{x_1}), \dots, (\frac{x_2}{x_1})(\frac{y_2}{x_1}) - (\frac{x_m}{x_1})(\frac{y_m}{x_1})) \\
&= \mathbf{A}^1_R \times_{\operatorname{Spec} R} \operatorname{Spec} R[X_1, Y_1, \dots, X_m, Y_m]/(X_2Y_2 - X_3Y_3, \dots, X_2Y_2 - X_mY_m) \\
&= \mathbf{A}^1_R \times_{\operatorname{Spec} R} Y^{2\dots m}. \qquad (3.17)
\end{aligned}$$

Note that $(W_1^0)^{\text{Sing}} = \mathbf{A}^1_R \times_{specR} (Y^{23\dots m})^{\text{Sing}}$ and $P_0 = (0, 0, \dots, 0, 0) \in Y^{23\dots m}$ is a singular point. Let $S = \mathbf{A}^1_R \times_{\operatorname{Spec} R} \{P_0\} \subset (W_1^0)^{\text{Sing}}$. If we find the blow-up of W_1^0 at S, by lemma (3.3), the blowing-up of W_1^0 is isomorphic to $\mathbf{A}^1_R \times_{\operatorname{Spec} R} \widetilde{Y^{23\dots m}}$. By using the same calculations as above we can check that

$$\begin{aligned}
W_1^0 &= \mathbf{A}^1_R \times_{\operatorname{Spec} R} Y^{23\dots m} \\
W_1^1 &= \mathbf{A}^1_R \times_{\operatorname{Spec} R} Y^{23\dots m} \\
. &= \dots \\
. &= \dots \\
. &= \dots \\
W_m^0 &= \mathbf{A}^1_R \times_{\operatorname{Spec} R} Y^{12\dots(m-1)} \\
W_m^1 &= \mathbf{A}^1_R \times_{\operatorname{Spec} R} Y^{12\dots(m-1)}. \qquad (3.18)
\end{aligned}$$

If we use the assumption of the induction for $Y^{23\dots m}, \dots, Y^{12\dots(m-1)}$, after $m-2$ successive blowings-up we can resolve the singularities of each piece (Note that all of these pieces are isomorphic) and for each of them we find $2^{m-2}(m-1)!$ open pieces for the covering, so totally we get $2m(2^{m-2}(m-1)!) = 2^{m-1}(m!)$ open pieces for the covering of $\widetilde{Y^{12\dots m}_{m-1}}$ and their gluing through their overlap gives us $\widetilde{Y^{12\dots m}_{m-1}}$. □

Remark 3.7 As we have seen in theorem (3.6), we can use successive blowings-up for desingularisation of $Y^{12\ldots m}$ and get $2^{m-1}(m!)$ open pieces for the covering. To get an explicit formula for each piece we index the open pieces as $W^{\alpha_1\alpha_2\ldots\alpha_{m-1}}_{n_1n_2\ldots n_{m-1}}$ where each $\alpha_i \in \{0,1\}$ and $n_i \in \{1,2,\ldots,m\} ;= I_m$ and $n_i \neq n_j$. Let $\hat{n}$ be the remaining element of I_m and $Z_l^t = x_l$ if $t = 0$, y_l if $t = 1$. Then

$$W^{\alpha_1\alpha_2\ldots\alpha_{m-1}}_{n_1n_2\ldots n_{m-1}} = \text{Spec}\, R[Z^{\alpha_1}_{n_1}, \frac{Z^{\alpha_2}_{n_2}}{Z^{\alpha_1}_{n_1}}, \frac{Z^{\alpha_3}_{n_3}}{Z^{\alpha_2}_{n_2}}, \ldots, \frac{Z^{\alpha_{m-1}}_{n_{m-1}}}{Z^{\alpha_{m-2}}_{n_{m-2}}}, \frac{x_{\hat{n}}}{Z^{\alpha_{m-1}}_{n_{m-1}}}, \frac{y_{\hat{n}}}{Z^{\alpha_{m-1}}_{n_{m-1}}}]. \tag{3.19}$$

We find it convenient to express it by using the following matrix

$$W^{\alpha_1\alpha_2\ldots\alpha_{m-1}}_{n_1n_2\ldots n_{m-1}} \longleftrightarrow \left\{ \begin{array}{cccccc} \alpha_1 & \alpha_2 & \ldots & \alpha_{m-1} & 0 & 1 \\ n_1 & n_2 & \ldots & n_{m-1} & \hat{n} & \hat{n} \end{array} \right\} \tag{3.20}$$

For instance if $\alpha_1 = \alpha_2 = \ldots = \alpha_{m-1} = 0$ and $n_t = t$ for $1 \leq t \leq m-1$, then $\hat{n} = m$ and we get

$$W^{00\ldots00}_{12\ldots(m-2)(m-1)} \longleftrightarrow \left\{ \begin{array}{cccccc} 0 & 0 & \ldots & 0 & 0 & 1 \\ 1 & 2 & \ldots & (m-1) & m & m \end{array} \right\} \tag{3.21}$$

and

$$W^{00\ldots00}_{12\ldots(m-2)(m-1)} = \text{Spec}\, R[x_1, \frac{x_2}{x_1}, \frac{x_3}{x_2}, \ldots, \frac{x_{m-1}}{x_{m-2}}, \frac{x_m}{x_{m-1}}, \frac{y_m}{x_{m-1}}]. \tag{3.22}$$

Example 3.8 Let $n = 8$. Then

$$W^{0111001}_{4537218} \longleftrightarrow \left\{ \begin{array}{ccccccccc} 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 4 & 5 & 3 & 7 & 2 & 1 & 8 & 6 & 6 \end{array} \right\}, \tag{3.23}$$

hence

$$W^{0111001}_{4537218} = \text{Spec}\, R[x_4, \frac{y_5}{x_4}, \frac{y_3}{y_5}, \frac{y_7}{y_3}, \frac{x_2}{y_7}, \frac{x_1}{x_2}, \frac{y_8}{x_1}, \frac{x_6}{y_8}, \frac{y_6}{y_8}], \tag{3.24}$$

which is one of $5160960 (= 2^7(8!))$ pieces which appears in the collection of the open pieces for the covering of 7th blowing-up of $Y^{12\ldots8}$, *ie,* $\widetilde{Y_7^{12\ldots8}}$.

3.4 Second Method

In this section we will try to blow-up a specific ideal and show that just one blowing-up of this ideal gives us the same result as we got in theorem (3.6). For case $n = 2$ this method gives us the same result as we had in example (2.19). This ideal was given by Deligne in [7].

Convention 3.9 Let J_n be the ideal of $\Gamma(Y^{12...n}, \mathcal{O}_{Y^{12...n}})$ generated by the elements of the form $Z^0_{\sigma(1)}Z^1_{\sigma(2)}...Z^{n-1}_{\sigma(n)}$ where $Z^{t-1}_{\sigma(t)}$ means $Z_{\sigma(t)}$ to power $t-1$ and $Z_{\sigma(t)}{=}x_{\sigma(t)}$ or $y_{\sigma(t)}$, $t = 1, 2, ..., n$ and $\sigma \in \mathbf{S_n}$. Since $Z^0_{\sigma(1)}{=}1$ this term is not effective and so J_n is generated by $2^{n-1}(n!)$ elements of this form. There are a lot of relations between the generators of J_n which we will use later.

Theorem 3.10 Let $Y^{12...n}{=}\text{Spec}\, R[x_1, y_1, ..., x_n, y_n]/B_n$. If we blow-up the ideal J_n, then we get the regular scheme $\widetilde{Y^{12...n}}$.

Proof: Let $N = 2^{n-1}(n!)$, $\sigma \in \mathbf{S_n}$ and $i_1, i_2, ..., i_{n-1} \in \{0, 1\}$. We consider N variables $\mu_\sigma^{i_1,i_2,...,i_{n-1}}$(corresponding to the generators of J_n say $A_\sigma^{i_1,i_2,...,i_{n-1}}$'s) and define the ring homomorphism

$$\alpha_n : A_{12...n}[\mu_{\sigma_1}^{00...00}, ..., \mu_{\sigma_{n!}}^{11...11}] \longrightarrow S^{12...n} = \bigoplus\nolimits_{d\geq 0} J_n^d \tag{3.25}$$

$$\mu_\sigma^{i_1,i_2,...,i_{n-1}} \longmapsto A_\sigma^{i_1,i_2,...,i_{n-1}}.$$

Then α_n is an epimorphism and we have

$$S^{12...n} = \bigoplus\nolimits_{d\geq 0} J_n^d \cong A_{12...n}[\mu_{\sigma_1}^{00...00}, ..., \mu_{\sigma_{n!}}^{11...11}]/\text{Ker}\alpha_n, \tag{3.26}$$

where

$\text{Ker}\alpha_n = (\mu_{\sigma_1}^{00...00}A_{\sigma_2}^{00...00} - A_{\sigma_1}^{00...00}\mu_{\sigma_2}^{00...00}, ..., \mu_{\sigma_{(n!)-1}}^{11...11}A_{\sigma_{n!}}^{11...11} - A_{\sigma_{(n!)-1}}^{11...11}\mu_{\sigma_{n!}}^{11...11})$,
and α_n induces an isomorphism

$$\widetilde{Y^{12...n}} = \text{Proj}S^{12...n} \cong \text{Proj}(A_{12...n}[\mu_{\sigma_1}^{00...00}, ..., \mu_{\sigma_{n!}}^{11...11}]/\text{Ker}\alpha_n), \tag{3.27}$$

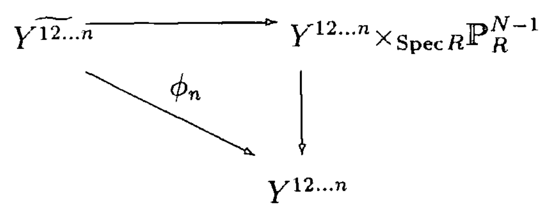
which is a closed subscheme of

$$\mathbf{P}^{N-1}_{A_{12...n}} = \text{Proj}A_{12...n}[\mu_{\sigma_1}^{00...00}, ..., \mu_{\sigma_{n!}}^{11...11}] = \text{Spec}\, A_{12...n}\times_{\text{Spec}\, R}\mathbf{P}_R^{N-1}. \tag{3.28}$$

The standard open covering of $\mathbb{P}_k^{N-1}$, *ie,*

$$
\begin{aligned}
\mu_{\sigma_1}^{00...00} \neq 0, \quad U_{\sigma_1}^{00...00} &= \operatorname{Spec} R[\frac{\mu_{\sigma_2}^{00...00}}{\mu_{\sigma_1}^{00...00}}, ..., \frac{\mu_{\sigma_{n!}}^{11...11}}{\mu_{\sigma_1}^{00...00}}], \\
. &= ... \\
. &= ... \\
. &= ... \\
\mu_{\sigma_{n!}}^{11...11} \neq 0, \ , U_{\sigma_{n!}}^{11...11} &= \operatorname{Spec} R[\frac{\mu_{\sigma_1}^{00...00}}{\mu_{\sigma_{n!}}^{00...00}}, ..., \frac{\mu_{\sigma_{n!-1}}^{11...11}}{\mu_{\sigma_{n!}}^{11...11}}],
\end{aligned}
\tag{3.29}
$$

induces an open covering for $\widetilde{Y^{12...n}}$, say



$$
\widetilde{Y^{12...n}} = V_{\sigma_1}^{00...00} \cup ... \cup V_{\sigma_{n!}}^{11...11},
\tag{3.30}
$$

where

$$
\begin{aligned}
V_{\sigma_1}^{00...00} &= \operatorname{Spec} R[x_1, y_1, ..., x_n, y_n, \frac{\mu_{\sigma_2}^{00...00}}{\mu_{\sigma_1}^{00...00}}, ..., \frac{\mu_{\sigma_{n!}}^{11...11}}{\mu_{\sigma_1}^{00...00}}]/ \\
&\qquad (x_1y_1 - x_2y_2, ..., x_1y_1 - x_ny_n, \frac{1}{\mu_{\sigma_1}^{00...00}}(\text{Relations})) \\
. &= ... \\
. &= ... \\
. &= ... \\
V_{\sigma_{n!}}^{11...11} &= \operatorname{Spec} R[x_1, y_1, ..., x_n, y_n, \frac{\mu_{\sigma_1}^{00...00}}{\mu_{\sigma_{n!}}^{11...11}}, ..., \frac{\mu_{\sigma_{n!-1}}^{11...11}}{\mu_{\sigma_{n!}}^{11...11}}]/ \\
&\qquad (x_1y_1 - x_2y_2, ..., x_1y_1 - x_ny_n, \frac{1}{\mu_{\sigma_{n!}}^{11...11}}(\text{Relations})),
\end{aligned}
\tag{3.31}
$$

By using the relations in our calculations as it is stated in Deligne's paper (see lemma (5.5) in [7]), we get

$$V_{\sigma_1}^{00\ldots00} = \text{Spec } R[\frac{x_1}{x_2}, \frac{x_2}{x_3}, \ldots, \frac{x_{n-1}}{x_n}, x_n, \frac{y_1}{x_2}]. \tag{3.32}$$

The regularity of $V_{\sigma_1}^{00\ldots00}$ is also stated in lemma 5.5 in [7]. We can do the same calculations for the other pieces of the covering of $\widetilde{Y^{12\ldots n}}$. □

Remark 3.11 To get an explicit formula for pieces of the covering of $\widetilde{Y^{12\ldots n}}$ stated in theorem (3.10), we index the open pieces as $V_\sigma^{\beta_1\beta_2\ldots\beta_{n-1}}$ where each $\beta_i \in \{0, 1\}$ and $\sigma \in \mathbf{S_n}$. Then

$$V_\sigma^{\beta_1\beta_2\ldots\beta_{n-1}} = \text{Spec } R[\frac{T_{\sigma(1)}^{\beta_1}}{T_{\sigma(2)}^{\beta_1}}, \frac{T_{\sigma(2)}^{\beta_2{}^\sharp}}{T_{\sigma(3)}^{\beta_2}}, \frac{T_{\sigma(3)}^{\beta_3{}^\sharp}}{T_{\sigma(4)}^{\beta_3}}, \ldots, \frac{T_{\sigma(n-1)}^{\beta_{n-1}{}^\sharp}}{T_{\sigma(n)}^{\beta_{n-1}}}, T_{\sigma(n)}^{\beta_{n-1}}, \frac{T_{\sigma(1)}^{\beta_1^*}}{T_{\sigma(2)}^{\beta_1}}], \tag{3.33}$$

where

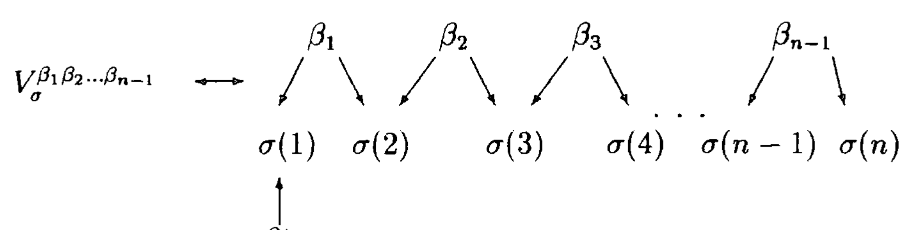
$$\beta_1^* = 1 - \beta_1\,, \tag{3.34}$$

$$\beta_j{}^\sharp = \beta_{j-1} \tag{3.35}$$

and

$$T_{\sigma(j)}^{\beta_j} = \begin{cases} x_{\sigma(j)} & if \quad \beta_j = 0 \\ y_{\sigma(j)} & if \quad \beta_j = 1. \end{cases} \tag{3.36}$$

We can use a diagram to get $V_\sigma^{\beta_1\beta_2\ldots\beta_n}$ as follows:



Which means that we start from β_1 and consider $T_{\sigma(1)}^{\beta_1}$ corresponding to β_1 and $\sigma(1)$ and divide it by $T_{\sigma(2)}^{\beta_1}$ corresponding to the arrow between β_1 and $\sigma(2)$. Then we start with β_2 and divide $T_{\sigma(2)}^{\beta_2{}^\sharp}$ by $T_{\sigma(3)}^{\beta_2}$ and continue this process up-to division of $T_{\sigma(n-1)}^{\beta_{n-1}^\sharp}$ by $T_{\sigma(n)}^{\beta_{n-1}}$. Later on, we write the term $T_{\sigma(n)}^{\beta_{n-1}}$ alone and finally we find the division of $T_{\sigma(1)}^{\beta_1^*}$ by $T_{\sigma(2)}^{\beta_1}$.

For instance if $\beta_1 = \beta_2 = \ldots = \beta_{n-1} = 0$ and

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \ldots & n \\ 1 & 2 & 3 & \ldots & n \end{pmatrix}, \tag{3.37}$$

then

$$\begin{aligned} V_\sigma^{00\ldots00} &= \operatorname{Spec} R[\frac{x_1}{x_2}, \frac{x_2}{x_3}, \frac{x_3}{x_4}, \ldots, \frac{x_{n-1}}{x_n}, x_n, \frac{y_1}{x_2}] = \\ & \operatorname{Spec} R[x_n, \frac{x_{n-1}}{x_n}, \frac{x_{n-2}}{x_{n-1}}, \ldots, \frac{x_3}{x_4}, \frac{x_2}{x_3}, \frac{x_1}{x_2}, \frac{y_1}{x_2}]. \end{aligned} \tag{3.38}$$

Lemma 3.12 By using the same notation as it was used in theorem (3.6) and theorem (3.10), we get

$$W_{n_1 n_2 \ldots n_{m-1}}^{\alpha_1 \alpha_2 \ldots \alpha_{m-1}} = V_\sigma^{\alpha_{m-1} \ldots \alpha_2 \alpha_1}, \tag{3.39}$$

where

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \ldots & m \\ \hat{n} & n_{m-1} & n_{m-2} & \ldots & n_1 \end{pmatrix}. \tag{3.40}$$

Proof: Let $\sigma \in \mathbb{S}_m$, say

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \ldots & m \\ \sigma(1) & n_{m-1} & n_{m-2} & \ldots & n_1 \end{pmatrix} \tag{3.41}$$

and $\beta_i = \alpha_{m-i}$. Then

$$\begin{aligned} & V_\sigma^{\alpha_{m-1}\alpha_{m-2}\ldots\alpha_2\alpha_1} = V_\sigma^{\beta_1\beta_2\ldots\beta_{m-1}} = \\ & \operatorname{Spec} R[\frac{T_{\sigma(1)}^{\beta_1}}{T_{\sigma(2)}^{\beta_1}}, \frac{T_{\sigma(2)}^{\beta_1}}{T_{\sigma(3)}^{\beta_2}}, \frac{T_{\sigma(3)}^{\beta_2}}{T_{\sigma(4)}^{\beta_3}}, \ldots, \frac{T_{\sigma(m-2)}^{\beta_{m-3}}}{T_{\sigma(m-1)}^{\beta_{m-2}}}, \frac{T_{\sigma(m-1)}^{\beta_{m-2}}}{T_{\sigma(m)}^{\beta_{m-1}}}, T_{\sigma(m)}^{\beta_{m-1}}, \frac{T_{\sigma(1)}^{\beta_1 *}}{T_{\sigma(2)}^{\beta_1}}] = \\ & \operatorname{Spec} R[T_{\sigma(m)}^{\beta_{m-1}}, \frac{T_{\sigma(m-1)}^{\beta_{m-2}}}{T_{\sigma(m)}^{\beta_{m-1}}}, \frac{T_{\sigma(m-2)}^{\beta_{m-3}}}{T_{\sigma(m-1)}^{\beta_{m-2}}}, \ldots, \frac{T_{\sigma(3)}^{\beta_2}}{T_{\sigma(4)}^{\beta_3}}, \frac{T_{\sigma(2)}^{\beta_1}}{T_{\sigma(3)}^{\beta_2}}, \frac{T_{\sigma(1)}^{\beta_1}}{T_{\sigma(2)}^{\beta_1}}, \frac{T_{\sigma(1)}^{\beta_1 *}}{T_{\sigma(2)}^{\beta_1}}] = \\ & \operatorname{Spec} R[Z_{n_1}^{\alpha_1}, \frac{Z_{n_2}^{\alpha_2}}{Z_{n_1}^{\alpha_1}}, \frac{Z_{n_3}^{\alpha_3}}{Z_{n_2}^{\alpha_2}}, \ldots, \frac{Z_{n_{m-1}}^{\alpha_{m-1}}}{Z_{n_{m-2}}^{\alpha_{m-2}}}, \frac{x_{\hat{n}}}{Z_{n_{m-1}}^{\alpha_{m-1}}}, \frac{y_{\hat{n}}}{Z_{n_{m-1}}^{\alpha_{m-1}}}] = \\ & W_{n_1 n_2 \ldots n_{m-1}}^{\alpha_1, \alpha_2 \ldots \alpha_{m-1}}, \end{aligned} \tag{3.42}$$

where $\hat{n} = \sigma(1)$. □

Example 3.13 Compare $W = W_{(13)2578(11)(10)6143(12)}^{100100010111}$ and $V = V_\sigma^{111010001001}$, where

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\ 9 & 12 & 3 & 4 & 1 & 6 & 10 & 11 & 8 & 7 & 5 & 2 & 13 \end{pmatrix}. \tag{3.43}$$

Solution

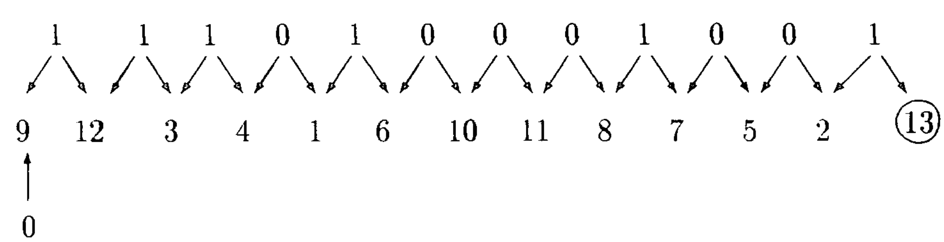
We have

$$W \leftrightarrow \begin{Bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 13 & 2 & 5 & 7 & 8 & 11 & 10 & 6 & 1 & 4 & 3 & 12 & 9 & 9 \end{Bmatrix}. \tag{3.44}$$

Hence

$$W = \operatorname{Spec} R[y_{13}, \frac{x_2}{y_{13}}, \frac{x_5}{x_2}, \frac{y_7}{x_5}, \frac{x_8}{y_7}, \frac{x_{11}}{x_8}, \frac{x_{10}}{x_{11}}, \frac{y_6}{x_{10}}, \frac{x_1}{y_6}, \frac{y_4}{x_1}, \frac{y_3}{y_4}, \frac{y_{12}}{y_3}, \frac{x_9}{y_{12}}, \frac{y_9}{y_{12}}]$$

and



so we get

$$V = \operatorname{Spec} R[\frac{y_9}{y_{12}}, \frac{y_{12}}{y_3}, \frac{y_3}{y_4}, \frac{y_4}{x_1}, \frac{x_1}{y_6}, \frac{y_6}{x_{10}}, \frac{x_{10}}{x_{11}}, \frac{x_{11}}{x_8}, \frac{x_8}{y_7}, \frac{y_7}{x_5}, \frac{x_5}{x_2}, \frac{x_2}{y_{13}}, y_{13}, \frac{x_9}{y_{12}}], \tag{3.45}$$

ie,

$$W = V.$$

3.5 Over a dvr

Convention 3.14 In this section we will introduce $X^{12\ldots n}$ and try to resolve its singularities later. R always denotes a discrete valuation ring with field of fractions K, prime ideal (π) and algebraically closed residue field $k = \frac{R}{(\pi)}$.

Now consider the arithmetic surface $X^i = \operatorname{Spec} R[x_i, y_i]/(x_i y_i - \pi)$, $1 \le i \le n$, over $\operatorname{Spec} R$. Then X^i over $\operatorname{Spec} R$ is smooth everywhere except at $x_i = y_i = 0$ where the special fibre has an ordinary double point, but X^i is a regular scheme. The fibre product of $X^1 = \operatorname{Spec} R[x_1, y_1]/(x_1 y_1 - \pi)$ and $X^2 = \operatorname{Spec} R[x_2, y_2]/(x_2 y_2 - \pi)$ is $X^{12} = \operatorname{Spec} R[x_1, y_1, x_2, y_2]/(x_1 y_1 - x_2 y_2, x_1 y_1 - \pi) := \operatorname{Spec} R[x_1, y_1, x_2, y_2]/B'_2$ where B'_2 is the ideal $(x_1 y_1 - x_2 y_2, x_1 y_1 - \pi)$ of $A'_2 = R[x_1, y_1, x_2, y_2]$.

Inductively for each $n \in \mathbb{N}$, $n > 2$ we get $X^{12\ldots n} = \operatorname{Spec} R[x_1, y_1, \ldots, x_n, y_n]/ (x_1 y_1 - x_2 y_2, \ldots, x_1 y_1 - x_n y_n, x_1 y_1 - \pi) = \operatorname{Spec} R[x_1, y_1, \ldots, x_n, y_n]/(x_i y_i - \pi)_{1 \le i \le n} := \operatorname{Spec} R[x_1, y_1, \ldots, x_n, y_n]/B'_n$ where $B'_n = (x_1 y_1 - x_2 y_2, \ldots, x_1 y_1 - x_n y_n, x_1 y_1 - \pi)$. In fact in the affine space $\mathbf{A}^{2n}_R$, $X^{12\ldots n}$ is determined by the equations

$$x_1 y_1 = x_2 y_2 = \ldots = x_n y_n = \pi. \tag{3.46}$$

Remark 3.15 In the rest of this section we will try to find the singular points and also a desingularisation of $X^{12\ldots n}$ in two different methods. These desingularisations are the results of some successive blowings-up, analogous to those performed in the geometric case.

Lemma 3.16 The only singular point of the scheme X^{12} is the point $P_0 \in (X^{12})_\pi$ (*ie*, $x_1 = y_1 = x_2 = y_2 = \pi = 0$).

Proof: Recall that $X^{12} = \operatorname{Spec} R[x_1, y_1, x_2, y_2]/(x_1 y_1 - x_2 y_2, x_1 y_1 - \pi)$. Let $A = R[x_1, y_1]/(x_1 y_1 - \pi)$. By using theorem (2.14) (or just by using Moje) we can check that A is a regular ring. Note that

$$X^{12} = \operatorname{Spec} A[x_2, y_2]/(x_1 y_1 - x_2 y_2). \tag{3.47}$$

Let $\phi(x_2, y_2) = x_1 y_1 - x_2 y_2$. Then $\frac{\partial \phi}{\partial x_2} = -y_2$ and $\frac{\partial \phi}{\partial y_2} = -x_2$. The system $\frac{\partial \phi}{\partial x_2} = \frac{\partial \phi}{\partial y_2} = 0$ has the solution $x_2 = y_2 = 0$. So X^{12} is smooth over A everywhere except at those points where $x_2 = y_2 = 0$.

Now let $B = \operatorname{Spec} R[x_2, y_2]/(x_2 y_2 - \pi)$. By using theorem (2.14) (or just by using Moje) we can check that B is a regular ring. Note that $X^{12} = \operatorname{Spec} B[x_1, y_1]/ (x_1 y_1 - x_2 y_2)$. Let $\psi(x_1, y_1) = x_1 y_1 - x_2 y_2$. Then $\frac{\partial \psi}{\partial x_1} = y_1$ and $\frac{\partial \psi}{\partial y_1} = x_1$. The system $\frac{\partial\psi}{\partial x_1} = \frac{\partial\psi}{\partial y_1} = 0$ has the solution $x_1 = y_1 = 0$. So X^{12} is smooth over B everywhere except at those points where $x_1 = y_1 = 0$. Considering the above discussion we can conclude that X^{12} is regular everywhere except possibly at $x_1 = y_1 = x_2 = y_2 = 0$. But $\text{Moje}(P_0, f_1) = 0$, so P_0 is the only singular point of X^{12}, *ie,*

$$(X^{12})^{\text{Sing}} = \{P_0\}. \ \square$$

Lemma 3.17 After one blowing-up of X^{12} at P_0 we get the regular scheme $\widetilde{X^{12}}$.

Proof: If we do exactly the same with whatever we did in example (2.19) for the scheme $X_2 = \text{Spec}\, R[x_1, y_1, x_2, y_2]/(x_1y_1 - x_2y_2) := \text{Spec}\, A^2$ over $\text{Spec}\, R[t]$, with the structure morphism induced by the ring homomorphism

$$R[t] \longrightarrow R[x_1, y_1, x_2, y_2]/(x_1y_1 - x_2y_2) = A^2 \tag{3.48}$$

$$t \longmapsto x_1y_1$$

we get a regular scheme $\widetilde{X_2}$ which is the gluing of the following open pieces of the covering:

$$\begin{aligned} \mu_1 \neq 0, \ W_1^0 &= \text{Spec}\, R[x_1, \frac{x_2}{x_1}, \frac{y_2}{x_1}] \\ \eta_1 \neq 0, \ W_1^1 &= \text{Spec}\, R[y_1, \frac{x_2}{y_1}, \frac{y_2}{y_1}] \\ \mu_2 \neq 0, \ W_2^0 &= \text{Spec}\, R[\frac{x_1}{x_2}, \frac{y_1}{x_2}, x_2] \\ \eta_2 \neq 0, \ W_2^1 &= \text{Spec}\, R[\frac{x_1}{y_2}, \frac{y_1}{y_2}, y_2]. \end{aligned} \tag{3.49}$$

Now consider the ring homomorphism

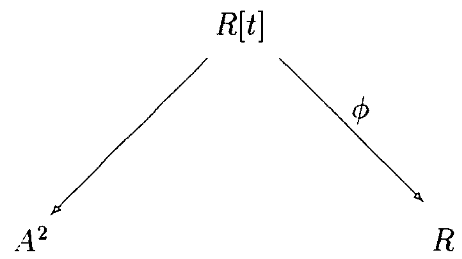
$$\phi : R[t] \longrightarrow R$$

$$t \longmapsto \pi.$$

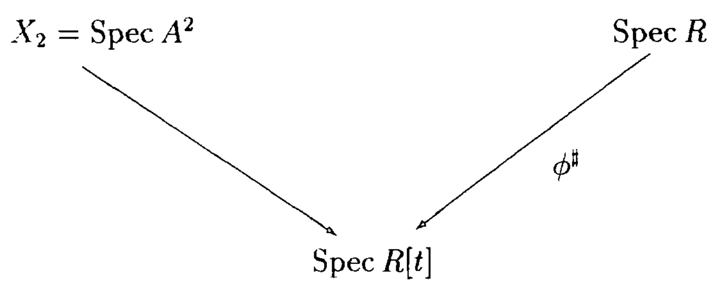
which induces the structure morphism

$$\phi^\sharp : \text{Spec}\, R \longrightarrow \text{Spec}\, R[t]\ .$$

So we have



which induces



and we get $X^{12} = X_2 \times_{\mathrm{Spec}\, R[t], \phi^\sharp} \mathrm{Spec}\, R$.

By using the lemma (3.3), we have $\widetilde{X_1^{12}} = \widetilde{X_2} \times_{\mathrm{Spec}\, R[t], \phi^\sharp} \mathrm{Spec}\, R$.

To find the affine open pieces of the covering of $\widetilde{X_1^{12}}$ we can do as follows:

$$V_1^0 = W_1^0 \times_{\mathrm{Spec}\, R[t], \phi^\sharp} \mathrm{Spec}\, R = \mathrm{Spec}\, R[x_1, \frac{x_2}{x_1}, \frac{y_2}{x_1}] \times_{\mathrm{Spec}\, R[t], \phi^\sharp} \mathrm{Spec}\, R =$$
$$\mathrm{Spec}\, R[x_1, \frac{x_2}{x_1}, \frac{y_2}{x_1}]/(x_2 y_2 - \pi) = \mathrm{Spec}\, R[X_1, X_2, Y_2]/(X_1^2 X_2 Y_2 - \pi),$$

which is a regular scheme (by corollary 2.13).

Using the same method, gives us

$$\begin{aligned} V_1^1 &= \mathrm{Spec}\, R[Y_1, X_2, Y_2]/(Y_1^2 X_2 Y_2 - \pi) \\ V_2^0 &= \mathrm{Spec}\, R[X_1, Y_1, X_2]/(X_2^2 X_1 Y_1 - \pi) \\ V_2^1 &= \mathrm{Spec}\, R[X_1, Y_1, Y_2]/(Y_2^2 X_1 Y_1 - \pi). \end{aligned} \tag{3.50}$$

which are all regular schemes and their gluing through their overlap, gives us the desired regular scheme $\widetilde{X_1^{12}}$. □

Lemma 3.18 Let $P = (a_1, b_1, ..., a_n, b_n) \in (X^{12...n})_\pi$ such that between $a_1, b_1, ..., a_n, b_n$ at least two pairs (a_i, b_i) are zero. Then P is a singular point of $X^{12...n}$.

Proof : Without loss of generality let $a_1 = b_1 = a_j = b_j = 0$ (for a fixed j, $2 \leq j \leq n$).

Now consider

$$X^{12...n} = \mathrm{Spec}\, R[x_1, y_1, ..., x_n, y_n]/(f_1, f_2, ..., f_n), \tag{3.51}$$

where

$$f_i(x_1, y_1, ..., x_n, y_n) = x_1y_1 - x_{i+1}y_{i+1}, \text{ for } i = 1, 2, ..., n-1 \tag{3.52}$$

and $f_n(x_1, y_1, ..., x_n, y_n) = x_1y_1 - \pi$.

Note that for points of the form $Q = (0, 0, a_2, b_2, ..., a_n, b_n) \in (X^{12...n})_\pi^{\mathrm{Sing}}$, $\mathrm{Moje}(Q, f_i) = -\beta_{i+1}T_{i+1} - \alpha_{i+1}S_{i+1}$ (for $1 \leq i \leq n-1$) and $\mathrm{Moje}(Q, f_n) = -\pi$, *ie*,

$$D(Q) = \begin{pmatrix} 0 & 0 & 0 & -b_2 & -a_2 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -b_3 & -a_3 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & -b_n & -a_n \\ -1 & 0 & 0 & \dots & \dots & \dots & \dots & \dots & 0 & 0 \end{pmatrix}. \tag{3.53}$$

Now let $Q = P$, *ie*, $a_j = b_j = 0$. Then $\mathrm{Moje}(Q, f_{j-1}) = 0$. So *rank* $D(P) < n$. Hence P is a singular point. □

Theorem 3.19 After $n-1$ blowings-up of $X^{12...n}$ starting with P_0 (*ie*, $x_1 = y_1 = ... = x_n = y_n = \pi = 0$) we get $2^{n-1}(n!)$ regular pieces such that their gluing through the overlap gives us the regular scheme $\widetilde{X^{12...n}}$.

Proof: If we do exactly the same with whatever that we did in theorem (3.6) for the scheme

$$X_n = \mathrm{Spec}\, R[x_1, y_1, x_2, y_2, \ldots, x_n, y_n]/$$
$$(x_1y_1 - x_2y_2, x_1y_1 - x_3y_3, \ldots, x_1y_1 - x_ny_n) := \mathrm{Spec}\, A^n, \qquad (3.54)$$

over Spec $R[t]$, with the structure morphism induced by the ring homomorphism

$$R[t] \longrightarrow A^n$$
$$t \longmapsto x_1y_1,$$

after $n-1$ successive blowings-up we get $2^{n-1}(n!)$ open pieces for the covering of the regular scheme $\widetilde{X}_n$, say $W^{\alpha_1\alpha_2\ldots\alpha_{n-1}}_{m_1m_2\ldots m_{n-1}}$ where $\alpha_i \in \{0,1\}$ and $m_j \in \{1,2,\ldots,n\}$. Consider $\phi^\sharp$: Spec $R \longrightarrow$ Spec $R[t]$ as it was defined in lemma (3.17). Then we have

$$\begin{array}{ccc} & R[t] & \\ \swarrow & & \searrow^{\phi} \\ A^n & & R \end{array}$$

which induces

$$\begin{array}{ccc} X_n = \mathrm{Spec}\, A^n & & \mathrm{Spec}\, R \\ \searrow & & \swarrow_{\phi^\sharp} \\ & \mathrm{Spec}\, R[t] & \end{array}$$

By using lemma (3.3) we have

$$\widetilde{X^{12\ldots n}_{n-1}} = \widetilde{X_n} \times_{\mathrm{Spec}\, R[t], \phi^\sharp} \mathrm{Spec}\, R. \qquad (3.55)$$

The affine pieces for the covering of $\widetilde{X^{12\ldots n}_{n-1}}$ are of the form

$$V^{\alpha_1\alpha_2\ldots\alpha_{n-1}}_{m_1m_2\ldots m_{n-1}} = W^{\alpha_1\alpha_2\ldots\alpha_{n-1}}_{m_1m_2\ldots m_{n-1}} \times_{\mathrm{Spec}\,R[t],\phi^\sharp} \mathrm{Spec}\,R \tag{3.56}$$

The gluing of these open pieces together through their overlaps, gives us the regular scheme $\widetilde{X^{12\ldots n}_{n-1}}$. □

Example 3.20 As it is shown in example (3.8) for the case $n = 8$ we get

$$W^{0111001}_{4537218} = \mathrm{Spec}\,R[x_4, \frac{y_5}{x_4}, \frac{y_3}{y_5}, \frac{y_7}{y_3}, \frac{x_2}{y_7}, \frac{x_1}{x_2}, \frac{y_8}{x_1}, \frac{x_6}{y_8}, \frac{y_6}{y_8}]. \tag{3.57}$$

So

$$\begin{aligned}
&V^{0111001}_{4537218} = W^{0111001}_{4537218} \times_{\mathrm{Spec}\,R[t],\phi^\sharp} \mathrm{Spec}\,R = \\
&\mathrm{Spec}\,R[x_4, \frac{y_5}{x_4}, \frac{y_3}{y_5}, \frac{y_7}{y_3}, \frac{x_2}{y_7}, \frac{x_1}{x_2}, \frac{y_8}{x_1}, \frac{x_6}{y_8}, \frac{y_6}{y_8}]/(x_6y_6 - \pi) = \\
&\mathrm{Spec}\,R[x_4, \frac{y_5}{x_4}, \frac{y_3}{y_5}, \frac{y_7}{y_3}, \frac{x_2}{y_7}, \frac{x_1}{x_2}, \frac{y_8}{x_1}, \frac{x_6}{y_8}, \frac{y_6}{y_8}]/ \\
&(x_4^2(\frac{y_5}{x_4})^2(\frac{y_3}{y_5})^2(\frac{y_7}{y_3})^2(\frac{x_2}{y_7})^2(\frac{x_1}{x_2})^2(\frac{y_8}{x_1})^2(\frac{x_6}{y_8})(\frac{y_6}{y_8}) - \pi) = \\
&\mathrm{Spec}\,R[X_4, Y_5, Y_3, Y_7, X_2, X_1, Y_8, X_6, Y_6]/ \\
&((X_4Y_5Y_3Y_7X_2X_1Y_8)^2X_6Y_6 - \pi),
\end{aligned} \tag{3.58}$$

which is a regular scheme by corollary (2.13).

Remark 3.21 Now we will try to blow-up a specific ideal and show that just one blowing-up of this ideal gives us the same result as we got in theorem (3.6). For the case $n = 2$ this method gives us the same result with what we had in example (2.19) (just by using R rather than k).

Remark 3.22 We use J_n to show the ideal of $\Gamma(X^{12\ldots n}, \mathcal{O}_{X^{12\ldots n}})$ generated by the elements of the form $Z^0_{\sigma(1)}Z^1_{\sigma(2)}Z^2_{\sigma(3)}\ldots Z^{n-1}_{\sigma(n)}$ with the details stated in convention (3.9).

Theorem 3.23 Let $I_n = (x_1y_1 - x_2y_2, x_1y_1 - x_3y_3, \ldots, x_1y_1 - x_ny_n, x_1y_1 - \pi)$ and $X^{12\ldots n}$=Spec $R[x_1, y_1, x_2, y_2, \ldots, x_n, y_n]/I_n$. If we blow-up the ideal J_n, then we get $2^{n-1}(n!)$ open pieces of regular schemes and their gluing gives us the regular scheme $\widetilde{X^{12\ldots n}}$.

Proof: Let

$$X_n = \operatorname{Spec} R[x_1, y_1, ..., x_n, y_n]/$$
$$(x_1y_1 - x_2y_2, x_1y_1 - x_3y_3, ..., x_1y_1 - x_ny_n) := \operatorname{Spec} A^n. \quad (3.59)$$

If we do exactly the same with what we did in theorem (3.10) for the scheme $X_n = \operatorname{Spec} A^n$ over $\operatorname{Spec} R[t]$, with the structure morphism induced by the ring homomorphism

$$R[t] \longrightarrow A^n$$
$$t \longmapsto x_1y_1$$

and blow-up the ideal $(J_n + B_n)/B_n$ of $R[x_1, y_1, ..., x_n, y_n]/B_n$, we get $2^{n-1}(n!)$ pieces for the covering of $\widetilde{X^{12...n}}$ which are all regular schemes, say $V_{\sigma_l}^{\alpha_1\alpha_2...\alpha_{n-1}}$'s where $\alpha_i \in \{0,1\}$ and $\sigma_l \in \mathbf{S_n}$, as they were calculated in theorem (3.10). Considering ϕ and $\phi^\sharp$ as in lemma (3.17), we have $X^{12...n} = X_n \times_{\operatorname{Spec} R[t], \phi^\sharp} \operatorname{Spec} R$. By using lemma (3.3) we have $\widetilde{X^{12...n}} = \widetilde{X_n} \times_{\operatorname{Spec} R[t], \phi^\sharp} \operatorname{Spec} R$. Note that the open pieces for the covering of $\widetilde{X^{12...n}}$ (which are $2^{n-1}(n!)$ pieces) are of the form $V_{\sigma_l}^{\alpha_1\alpha_2...\alpha_{n-1}} \times_{\operatorname{Spec} R[t], \phi^\sharp} \operatorname{Spec} R$ for $i, j \in \{0,1\}$ and $l \in \{1, 2, ..., n!\}$, $\sigma_l \in \mathbf{S_n}$ and each piece is isomorphic with one $V_{m_1m_2...m_{n-1}}^{\alpha_1\alpha_2...\alpha_{n-1}}$ calculated in theorem (3.19). Recall that $V_{\sigma_l}^{\alpha_1\alpha_2...\alpha_{n-1}}$'s are calculated in theorem (3.10). □

Example 3.24 As we have seen in example (3.13) for the given σ in that example we get

$$V_\sigma^{111010001001} = \operatorname{Spec} R[\frac{y_9}{y_{12}}, \frac{y_{12}}{y_3}, \frac{y_3}{y_4}, ..., \frac{x_2}{y_{13}}, y_{13}, \frac{x_9}{y_{12}}]. \quad (3.60)$$

So we have

$$V' = \operatorname{Spec} R[\frac{y_9}{y_{12}}, \frac{y_{12}}{y_3}, ..., \frac{x_2}{y_{13}}, y_{13}, \frac{x_9}{y_{12}}]/(x_9y_9 - \pi) =$$
$$\operatorname{Spec} R[y_{13}, \frac{x_2}{y_{13}}, \frac{x_5}{x_2}, ..., \frac{y_{12}}{y_3}, \frac{x_9}{y_{12}}, \frac{y_9}{y_{12}}]/$$
$$(y_{13}^2(\frac{x_2}{y_{13}})^2(\frac{x_5}{x_2})^2...(\frac{y_{12}}{y_3})^2(\frac{x_9}{y_{12}})(\frac{y_9}{y_{12}}) - \pi) =$$
$$\operatorname{Spec} R[Y_{13}, X_2, X_5, ..., Y_{12}, X_9, Y_9]/(Y_{13}^2X_2^2X_5^2...Y_{12}^2X_9Y_9 - \pi), \quad (3.61)$$

which is regular by corollary (2.13).

Chapter 4

Desingularisation of a special class of arithmetic three-folds

4.1 Introduction

As before R denotes a discrete valuation ring with field of fractions K, maximal ideal (π) and algebraically closed residue field $k = \frac{R}{(\pi)}$. By an arithmetic surface over $\operatorname{Spec} R$ we mean a regular scheme V purely of dimension 2, which is flat over $\operatorname{Spec} R$. Our aim in this chapter will be to attempt to settle the resolution of singularities of a certain class of arithmetic three-folds, namely those which are the fibre product of two Arithmetic Surfaces. Let V_1 and V_2 be arithmetic surfaces over $\operatorname{Spec} R$, where R is a discrete valuation ring and consider the fibre product $X = V_1 \times_{\operatorname{Spec} R} V_2$. This is an arithmetic three-fold. In general it will not be regular. It may have some singularities at points (y_1, y_2) if $f_1(y_1) = f_2(y_2) = (\pi) := \xi$ (where $f_1 : V_1 \longrightarrow \operatorname{Spec} R$, $f_2 : V_2 \longrightarrow \operatorname{Spec} R$ are the structural morphisms and both y_1 and y_2 are singular points of the special fibres $(V_1)_\pi = f_1^{-1}(\xi)$ and $(V_2)_\pi = f_2^{-1}(\xi)$. In this Chapter we mainly try to resolve the singularities of some three-folds of this sort which are useful in chapter 5. The arithmetic surfaces which we use in this chapter have one of the following forms:

$$\begin{aligned} I_1 &\longleftrightarrow \operatorname{Spec} R[x,y]/(xy-\pi) \\ I_2 &\longleftrightarrow \operatorname{Spec} R[x,y]/(y^2-x^3-\pi) \\ I_3 &\longleftrightarrow \operatorname{Spec} R[x,y]/(y(y-x^2)-\pi) \\ I_4 &\longleftrightarrow \operatorname{Spec} R[x,y]/(xy(x-y)-\pi) \\ I_5 &\longleftrightarrow \operatorname{Spec} R[x,y]/(x^m y^n-\pi) \end{aligned} \tag{4.1}$$

We can summarize our results as follows:

Theorem 4.1 Let R be a dvr and $p = \operatorname{char} k \neq 2,3$ and V_1 and V_2 be arithmetic surfaces of the forms I_i and I_j respectively, where either $1 \leq i \leq j \leq 4$, $(i,j) \neq (2,2)$, $(3,4)$ or $(i,j) = (1,5)$. Then there exists a desingularisation for $X = V_1 \times_{\operatorname{Spec} R} V_2$.

Lemma 4.2 Let $\alpha, \beta \geq 0$ and $g(x_2,y_2) \in R[x_2,y_2]$ such that $\pi \nmid g$ and

$$V = \operatorname{Spec} R[x_1,y_1,x_2,y_2]/(x_1y_1 - g_2(x_2,y_2),\ x_1y_1x_2^\alpha y_2^\beta - \pi).$$

If $A = R[x_2,y_2]/(x_2^\alpha y_2^\beta g(x_2,y_2)-\pi)$ is regular, then

$$V^{\text{Sing}} = \{(0,0,a_2,b_2) \in V_\pi | \frac{\partial \bar{g}}{\partial x_2}(a_2,b_2) = \frac{\partial \bar{g}}{\partial y_2}(a_2,b_2) = 0\},$$

where $\bar{g}$ denotes the reduction of g (modπ).

Proof: We have

$$V = \operatorname{Spec} A[x_1,y_1]/(x_1y_1 - g(x_2,y_2)). \tag{4.2}$$

Let $\phi(x_1,y_1) = x_1y_1 - g(x_2,y_2)$. By using the Jacobian criterion we find out that V is smooth over A everywhere except at those points where $\frac{\partial\phi}{\partial x_1} = \frac{\partial\phi}{\partial y_1} = 0$, *ie*, $x_1 = y_1 = 0$. Considering $\phi(x_1,y_1) = 0$, we get $g(x_2,y_2) = 0$. To have these points on the special fibre, we need $\bar{g}(x_2,y_2) = 0$. So

$$\begin{aligned} V^{\text{Sing}} \subseteq\ & \{P = (a_1,b_1,a_2,b_2) \in V_\pi \mid a_1 = b_1 = \bar{g}(a_2,b_2) = 0\} = \\ & \{P = (0,0,a_2,b_2) \in V_\pi \mid \bar{g}(a_2,b_2) = 0\} := S \subset V_\pi. \end{aligned} \tag{4.3}$$

Let $P = (0, 0, a_2, b_2) \in S$ and $\alpha_2, \beta_2 \in R$ such that $\alpha_2 \equiv a_2(\operatorname{mod}\pi)$ and $\beta_2 \equiv b_2(\operatorname{mod}\pi)$. Considering $f_1(x_1, y_1, x_2, y_2) = x_1y_1 - g(x_2, y_2)$ and $f_2(x_1, y_1, x_2, y_2) = x_1y_1x_2^{\alpha}y_2^{\beta} - \pi$ we get

$$\begin{aligned} &f_1(T_1 + u_1\pi, S_1 + v_1\pi, T_2 + \alpha_2, S_2 + \beta_2) = f_1(u_1\pi, v_1\pi, \alpha_2, \beta_2) + \\ &\frac{\partial f_1}{\partial x_1}(u_1\pi, v_1\pi, \alpha_2, \beta_2)T_1 + \frac{\partial f_1}{\partial y_1}(u_1\pi, v_1\pi, \alpha_2, \beta_2)S_1 + \frac{\partial f_1}{\partial x_2}(u_1\pi, v_1\pi, \alpha_2, \beta_2)T_2 \\ &+\frac{\partial f_1}{\partial y_2}(u_1\pi, v_1\pi, \alpha_2, \beta_2)S_2 + \ldots = (u_1\pi)(v_1\pi) - g(\alpha_2, \beta_2) + v_1\pi T_1 + u_1\pi S_1 - \\ &\frac{\partial g}{\partial x_2}(\alpha_2, \beta_2)T_2 - \frac{\partial g}{\partial y_2}(\alpha_2, \beta_2)S_2 + \ldots \end{aligned} \tag{4.4}$$

So

$$\operatorname{Moje}(P, f_1) = \{-g(\alpha_2, \beta_2) - \frac{\partial g}{\partial x_2}(\alpha_2, \beta_2)T_2 - \frac{\partial g}{\partial y_2}(\alpha_2, \beta_2)S_2\} \,(\operatorname{mod}\,(T_2, S_2, \pi)^2) \tag{4.5}$$

and $\operatorname{Moje}(P, f_2) = -\pi$, *ie*,

$$D(Q) = \begin{pmatrix} -\bar{g}(a_2, b_2) & 0 & 0 & -\frac{\partial \bar{g}}{\partial x_2}(a_2, b_2) & -\frac{\partial \bar{g}}{\partial y_2}(a_2, b_2) \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.6}$$

So Q is singular if $\frac{\partial \bar{g}}{\partial x_2}(a_2, b_2) = \frac{\partial \bar{g}}{\partial y_2}(a_2, b_2) = 0$ (since P is on the special fibre, $\bar{g}(a_2, b_2) = 0$). □

Convention 4.3 For the most cases in this chapter the second row of $D(Q)$ corresponding to the point Q is $(\,-1 \quad 0 \quad 0 \quad 0 \quad 0\,)$. For abbreviation we use $D(Q)_{r_1}$ for the first row of $D(Q)$. In the cases that the second row is different from $(\,-1 \quad 0 \quad 0 \quad 0 \quad 0\,)$ we write down $D(Q)$.

4.2 One component is I_1

In this section our arithmetic three-folds are those which are the fibre products of two arithmetic surfaces such that one of them is of the form I_1.

Lemma 4.4 Let $V_1 = V_2 = \operatorname{Spec} R[x, y]/(xy - \pi)$. Then $X = V_1 \times_{\operatorname{Spec} R} V_2$ is singular and just after one blowing-up we can resolve its singularity.

Proof: See lemma (3.17). □

Lemma 4.5 Let $p = char\, k \neq 2$ and $V_1 = \mathrm{Spec}\, R[x,y]/(xy-\pi)$ and $V_2 = \mathrm{Spec}\, R[x,y]/(y^2-x^3-\pi)$. Then $X = V_1 \times_{\mathrm{Spec} R} V_2$ is singular and after one blowing-up we can resolve its singularity.

Proof: In fact $X = \mathrm{Spec}\, R[x_1,y_1,x_2,y_2]/(x_1y_1 - y_2^2 + x_2^3\,,\, x_1y_1 - \pi)$. By using corollary (2.11) we can show that P_0 is the only singular point of X. Now we blow-up P_0. Using the procedure of Mahtab gives us four open pieces for the covering of $\widetilde{X}$ as follows:

Chart 1

$$V_1^0 = \mathrm{Spec}\, R[X_1,X_2,Y_2]/(X_1^2Y_2^2 - X_1^3X_2^3 - \pi). \tag{4.7}$$

By using the Jacobian criterion we can check that

$$(V_1^0)_\pi^{\mathrm{Sing}} = \{(a_1,a_2,b_2) \in (V_1^0)_\pi | a_1 = 0 \,\mathrm{or}\, a_2 = b_2 = 0\}. \tag{4.8}$$

Let $Q \in (V_1^0)_\pi^{\mathrm{Sing}}$. Then

$$D(Q) = (\,-1 \quad 0 \quad 0 \quad 0 \quad 0\,), \tag{4.9}$$

ie, Q is a regular point. So V_1^0 is a regular scheme.

Chart 2

$$V_1^1 = \mathrm{Spec}\, R[Y_1,X_2,Y_2]/(Y_1^2Y_2^2 - Y_1^3X_2^3 - \pi) \cong V_1^0. \tag{4.10}$$

Chart 3

$$V_2^0 = \mathrm{Spec}\, R[X_1,Y_1,X_2,Y_2]/(X_1Y_1 - Y_2^2 + X_2\,,\, X_2^2X_1Y_1 - \pi). \tag{4.11}$$

Let $A = R[X_2,Y_2]/(X_2^2(Y_2^2 - X_2) - \pi)$ and $g(X_2,Y_2) = Y_2^2 - X_2$. Then A is a regular ring (by theorem (2.14) and $V_2^0 = \mathrm{Spec}\, A[X_1,Y_1]/(X_1Y_1 - g(X_2,Y_2))$. By using lemma (4.2) we get

$$(V_2^0)^{\text{Sing}} = \{P = (0,0,a_2,b_2) \in (V_2^0)_\pi | \frac{\partial \bar{g}}{\partial X_2}(a_2,b_2) = \frac{\partial \bar{g}}{\partial Y_2}(a_2,b_2) = 0\} =$$
$$\{(0,0,a_2,b_2) \in (V_2^0)_\pi | 2b_2 = -1\} = \{P = (0,0,a_2,-\frac{1}{2}) \mid \bar{g}(0,-\frac{1}{2}) = 0\} =$$
$$\{P = (0,0,a_2,-\frac{1}{2}) \mid (\frac{1}{2})^2 = 0\} = \phi. \tag{4.12}$$

So V_2^0 is a regular scheme.

Chart 4

$$V_2^1 = \operatorname{Spec} R[X_1,Y_1,X_2,Y_2]/(X_1Y_1 - 1 + Y_2X_2^3\,,\, Y_2^2X_1Y_1 - \pi). \tag{4.13}$$

Let $A = R[X_2,Y_2]/(Y_2^2(1-Y_2X_2^3) - \pi)$ and $g(X_2,Y_2) = 1 - Y_2X_2^3$. Then A is regular (by theorem (2.14)) and

$$V_2^1 = \operatorname{Spec} A[X_1,Y_1]/(X_1Y_1 - g(X_2,Y_2)).$$

By using lemma (4.2) we get

$$(V_2^1)^{\text{Sing}} = \{P = (0,0,a_2,b_2) \in (V_2^1)_\pi | \frac{\partial \bar{g}}{\partial X_2}(a_2,b_2) = \frac{\partial \bar{g}}{\partial Y_2}(a_2,b_2) = 0\}$$
$$\{P = (0,0,a_2,b_2) \in (V_1^0)_\pi | a_2 = 0\} =$$
$$\{P = (0,0,0,b_2) \in (V_1^0)_\pi | \bar{g}(0,b_2) = 0\} = \phi. \tag{4.14}$$

So V_2^1 is a regular scheme. The gluing of V_1^0, V_1^1, V_2^0 and V_2^1 gives us the regular scheme $\widetilde{X}$ which is the blowing-up of X at P_0. □

Lemma 4.6 Let $p = char\, k \neq 2$ and $V_1 = \operatorname{Spec} R[x,y]/(xy - \pi)$ and $V_2 = \operatorname{Spec} R[x,y]/(y(y-x^2) - \pi)$. Then $X = V_1 \times_{\operatorname{Spec} R} V_2$ is singular and after two blowings-up we can resolve its singularity.

Proof: The fibre product of V_1 and V_2 is

$$X = \operatorname{Spec} R[x_1,y_1,x_2,y_2]/(x_1y_1 - y_2^2 + y_2x_2^2\,,\, x_1y_1 - \pi). \tag{4.15}$$

By using corollary (2.11) we can check that P_0 is the only singular point of X.

Now we blow-up P_0. By using the procedure of Mahtab we get four open pieces for the covering of $\widetilde{X}$ as follows:

Chart 1

$$V_1^0 = \operatorname{Spec} R[X_1, X_2, Y_2]/(X_1^2 Y_2^2 - X_1^3 X_2^2 Y_2 - \pi). \tag{4.16}$$

By using theorem (2.14) we can check that V_1^0 is a regular scheme.

Chart 2

$$V_1^1 = \operatorname{Spec} R[Y_1, X_2, Y_2]/(Y_1^2 Y_2^2 - Y_1^3 Y_2 X_2^2 - \pi) \cong V_1^0. \tag{4.17}$$

Chart 3

$$V_2^0 = \operatorname{Spec} R[X_1, Y_1, X_2, Y_2]/(X_1 Y_1 - Y_2^2 + X_2 Y_2\,,\ X_2^2 X_1 Y_1 - \pi). \tag{4.18}$$

Let $A = R[X_2, Y_2]/(X_2^2 Y_2(Y_2 - X_2) - \pi)$ and $g(X_2, Y_2) = Y_2(Y_2 - X_2)$. Then A is regular (by using theorem (2.14)) and

$$V_2^0 = \operatorname{Spec} A[X_1, Y_1]/(X_1 Y_1 - g(X_2, Y_2)).$$

By using lemma (4.2) we get

$$\begin{aligned}(V_2^0)^{\text{Sing}} &= \{P = (0,0,a_2,b_2) \in (V_2^0)_\pi \mid \frac{\partial \bar{g}}{\partial X_2}(a_2,b_2) = \frac{\partial \bar{g}}{\partial Y_2}(a_2,b_2) = 0\} = \\ &\quad \{P = (0,0,a_2,b_2) \in (V_2^0)_\pi \mid b_2 = a_2 - 2b_2 = 0\} = \{P_0\}. \end{aligned} \tag{4.19}$$

Now we blow-up P_0. By using the procedure of Mahtab we get four open pieces for the covering of $\widetilde{V}_2^0$ as follows:

Chart 3.1

$$V_{21}^{00} = \operatorname{Spec} R[X_1, X_2, Y_2]/(X_1^4 X_2^2 Y_2^2 - X_1^4 X_2^3 Y_2 - \pi). \tag{4.20}$$

By using theorem (2.14) we can check that V_{21}^{00} is a regular scheme.

Chart 3.2

$$V_{21}^{01} = \text{Spec}\, R[Y_1, X_2, Y_2]/(Y_1^4X_2^2Y_2^2 - Y_1^4X_2^3Y_2 - \pi) \cong V_{21}^{00}. \tag{4.21}$$

Chart 3.3

$$V_{22}^{00} = \text{Spec}\, R[X_1, Y_1, X_2, Y_2]/(X_1Y_1 - Y_2^2 + Y_2\,,\ X_2^4X_1Y_1 - \pi). \tag{4.22}$$

Let $A = R[X_2, Y_2]/(X_2^4Y_2(Y_2 - 1) - \pi)$ and $g(X_2, Y_2) = Y_2(Y_2 - 1)$. Then A is regular (by theorem (2.14)) and

$$V_{22}^{00} = \text{Spec}\, A[X_1, Y_1]/(X_1Y_1 - g(X_2, Y_2)).$$

By using lemma (4.2) we get

$$\begin{aligned}(V_{22}^{00})^{\text{Sing}} &= \{P = (0,0,a_2,b_2) \in (V_{22}^{00})_\pi | \frac{\partial \bar{g}}{\partial X_2}(a_2,b_2) = \frac{\partial \bar{g}}{Y_2}(a_2,b_2) = 0\} = \\ &\quad \{P = (0,0,a_2,b_2) \in (V_{22}^{00})_\pi | 2b_2 - 1 = 0\} = \\ &\quad \{P = (0,0,a_2,\frac{1}{2}) | \bar{g}(a_2,\frac{1}{2}) = 0\} = \phi. \end{aligned} \tag{4.23}$$

So V_{22}^{00} is a regular scheme.

Chart 3.4

$$V_{22}^{01} = \text{Spec}\, R[X_1, Y_1, Y_2]/(Y_2^4X_1Y_1(1 + X_1^2Y_1^2 - 2X_1Y_1) - \pi). \tag{4.24}$$

By using theorem (2.14) we can check that V_{22}^{00} is a regular scheme. The gluing of V_{21}^{00}, V_{21}^{01}, V_{22}^{00} and V_{22}^{01} gives us the regular scheme $\widetilde{V_2^0}$.

Chart 4

$$V_2^1 = \text{Spec}\, R[X_1, Y_1, X_2, Y_2]/(X_1Y_1 - 1 + Y_2X_2^2\,,\ Y_2^2X_1Y_1 - \pi). \tag{4.25}$$

Let $A = R[X_2, Y_2]/(Y_2^2(1 - Y_2X_2^2) - \pi)$ and $g(X_2, Y_2) = 1 - Y_2X_2^2$. Then A is regular (by theorem (2.14)) and

$$V_2^1 = \text{Spec}\, A[X_1, Y_1]/(X_1Y_1 - g(X_2Y_2)).$$

By using lemma (4.2) we get

$$\begin{aligned}(V_2^1)^{\text{Sing}} &= \{P = (0,0,a_2,b_2) \in (V_2^1)_\pi | \frac{\partial \bar{g}}{\partial X_2}(a_2,b_2) = \frac{\partial \bar{g}}{\partial Y_2}(a_2,b_2) = 0\} = \\ &\{P = (0,0,a_2,b_2) \in (V_2^1)_\pi | a_2 = 0\} = \phi. \end{aligned} \tag{4.26}$$

So V_2^1 is a regular scheme. The gluing of V_1^0,V_1^1,$\widetilde{V_2^0}$ and V_2^1 gives us the regular scheme $\widetilde{X}$. □

Lemma 4.7 Let $p = char\, k \neq 2$ and $V_1 = \text{Spec}\, R[x,y]/(xy - \pi)$ and $V_2 = \text{Spec}\, R[x,y]/(xy(x-y) - \pi)$. Then $X = V_1 \times_{\text{Spec}R} V_2$ is singular and after three blowings-up we can resolve its singularity.

Proof: In fact

$$X = \text{Spec}\, R[x_1, y_1, x_2, y_2]/(x_1y_1 - x_2^2y_2 + x_2y_2^2\, ,\, x_1y_1 - \pi). \tag{4.27}$$

By using corollary (2.11) we can check that $P_0 = (0,0,0,0) \in X_\pi$ is the only singular point of X. Now we blow-up P_0. By using the procedure of Mahtab we get four pieces for the covering of $\widetilde{X_1}$ as follows:

Chart 1

$$\begin{aligned} V_1^0 &= \text{Spec}\, R[X_1, Y_1, X_2, Y_2]/(Y_1 - X_1X_2^2Y_2 + X_1X_2Y_2^2\, ,\, X_1^2Y_1 - \pi) = \\ &\text{Spec}\, R[X_1, X_2, Y_2]/(X_1^3X_2^2Y_2 - X_1^3X_2Y_2^2 - \pi). \end{aligned} \tag{4.28}$$

By using corollary (2.14) we can check that V_1^0 is a regular scheme.

Chart 2

$$V_1^1 = \text{Spec}\, R[Y_1, X_2, Y_2]/(Y_1^3X_2^2Y_2 - Y_1^3X_2Y_2^2 - \pi) \cong V_1^0. \tag{4.29}$$

Chart 3

$$V_2^0 = \operatorname{Spec} R[X_1, Y_1, X_2, Y_2]/(X_1Y_1 - X_2Y_2 + X_2Y_2^2\,,\, X_2^2X_1Y_1 - \pi). \qquad (4.30)$$

Let $A = R[X_2, Y_2]/(X_2^3Y_2(1 - Y_2) - \pi)$ and $g(X_2, Y_2) = X_2Y_2(1 - Y_2)$. Then A is regular (by theorem (2.14)) and

$$V_2^0 = \operatorname{Spec} A[X_1, Y_1]/(X_1Y_1 - g(X_2, Y_2)).$$

By using lemma (4.2) we get

$$\begin{aligned}(V_2^0)^{\text{Sing}} &= \{P = (0,0,a_2,b_2) \in (V_2^0)_\pi | \frac{\partial \bar{g}}{\partial X_2}(a_2, b_2) = \frac{\partial \bar{g}}{\partial Y_2}(a_2, b_2) = 0\} = \\ &\quad \{P = (0,0,a_2,b_2) \in (V_2^0)_\pi | b_2(1 - b_2) = a_2(1 - 2b_2) = 0\} = \\ &\quad \{(0,0,0,0), (0,0,0,1)\}. \qquad (4.31)\end{aligned}$$

We will try to resolve the singularities of V_2^0 later on.

Chart 4

$$V_2^1 = \operatorname{Spec} R[X_1, Y_1, X_2, Y_2]/(X_1Y_1 - Y_2X_2^2 + X_2Y_2\,,\, Y_2^2X_1Y_1 - \pi) \cong V_2^0. \quad (4.32)$$

Recall that

$$V_2^0 = \operatorname{Spec} R[X_1, Y_1, X_2, Y_2]/(X_1Y_1 - X_2Y_2 + X_2Y_2^2\,,\, X_2^2X_1Y_1 - \pi) \qquad (4.33)$$

is singular and $P_0 = (0,0,0,0)$ is a singular point of V_2^0. Now we blow-up P_0. By using the procedure of Mahtab we get four pieces for the covering of $\widetilde{V_1^0}$ as follows:

Chart 3.1

$$\begin{aligned}&V_{21}^{00} = \operatorname{Spec} R[X_1, Y_1, X_2, Y_2]/(Y_1 - X_2Y_2 + X_1X_2Y_2^2\,,\, X_1^4X_2^2Y_1 - \pi) = \\ &\operatorname{Spec} R[X_1, X_2, Y_2]/(X_1^4X_2^3Y_2 - X_1^5X_2^3Y_2^2 - \pi). \qquad (4.34)\end{aligned}$$

By using theorem (2.14) we can check that V_{21}^{00} is a regular scheme.

Chart 3.2

$$V_{21}^{01} = \text{Spec}\, R[X_1, Y_1, X_2, Y_2]/(X_1 - X_2Y_2 + Y_1X_2Y_2^2\,,\ Y_1^4X_2^2X_1 - \pi) =$$
$$\text{Spec}\, R[Y_1, X_2, Y_2]/(Y_1^4X_2^3Y_2 - Y_1^5X_2^3Y_2^2 - \pi) \cong V_{21}^{00}. \tag{4.35}$$

Chart 3.3

$$V_{22}^{00} = \text{Spec}\,[X_1, Y_1, X_2, Y_2]/(X_1Y_1 - Y_2 + X_2Y_2^2, X_2^4X_1Y_1 - \pi). \tag{4.36}$$

Let $A = R[X_2, Y_2]/(X_2^4Y_2(1 - X_2Y_2) - \pi)$ and $g(X_2, Y_2) = Y_2(1 - X_2Y_2)$. Then A is regular (by theorem (2.14)) and

$$V_{22}^{00} = \text{Spec}\, A[X_1, Y_1]/(X_1Y_1 - g(X_2, Y_2)).$$

By using lemma (4.2) we get

$$(V_{22}^{00})^{\text{Sing}} = \{P = (0, 0, a_2, b_2) \in (V_{22}^{00})_\pi | \frac{\partial \bar{g}}{\partial X_2}(a_2, b_2) = \frac{\partial \bar{g}}{\partial Y_2}(a_2, b_2) = 0\} =$$
$$\{P = (0, 0, a_2, b_2) \in (V_{22}^{00})_\pi | b_2 = 0 \text{ and } 2a_2b_2 = 1\} = \phi. \tag{4.37}$$

So V_{22}^{00} is a regular scheme.

Chart 3.4

$$V_{22}^{01} = \text{Spec}\, R[X_1, Y_1, X_2, Y_2]/(X_1Y_1 - X_2 + X_2Y_2\,,\ Y_2^4X_2^2X_1Y_1 - \pi). \tag{4.38}$$

Let $A = R[X_2, Y_2]/(Y_2^4X_2^3(1 - Y_2) - \pi)$ and $g(X_2, Y_2) = X_2(1 - Y_2)$. Then A is regular (by theorem (2.14)) and

$$V_{22}^{01} = \text{Spec}\, A[X_1, Y_1]/(X_1Y_1 - g(X_2, Y_2)).$$

By using lemma (4.2) we get

$$(V_{22}^{01})^{\text{Sing}} = \{P = (0, 0, a_2, b_2) \in (V_{22}^{01})_\pi | \frac{\partial \bar{g}}{\partial X_2}(a_2, b_2) = \frac{\partial \bar{g}}{\partial Y_2}(a_2, b_2) = 0\} =$$
$$\{P = (0, 0, a_2, b_2) \in (V_1^0)_\pi | a_2 = 0 \text{ and } b_2 = 1\} = \{(0, 0, 0, 1)\}. \tag{4.39}$$

Hence the only singular point of V_{22}^{01} is $Q = (0,0,0,1)$. Now we blow-up $Q \in (V_{22}^{01})^{\text{Sing}}$. For convenience first we use the translation $Y_2 = y_2 + 1$ to get the scheme

$$\begin{aligned} &W_{22}^{01} = \\ &\text{Spec}\, R[x_1, y_1, x_2, y_2]/(x_1y_1 - x_2 + x_2(y_2+1), (y_2+1)^4 x_2^2 x_1 y_1 - \pi) = \\ &\text{Spec}\, R[x_1, y_1, x_2, y_2]/(x_1y_1 + x_2y_2\,,\ (y_2+1)^4 x_2^2 x_1 y_1 - \pi). \end{aligned} \tag{4.40}$$

and surely

$$(W_{22}^{01})^{\text{Sing}} \supset \{P_0 \in (W_{22}^{01})_\pi\}. \tag{4.41}$$

Now we blow-up P_0. By using the procedure of Mahtab we get four pieces for the covering of $\widetilde{W_{22}^{01}}$ as follows:

Chart 3.4.1

$$\begin{aligned} &W_{221}^{010} = \\ &\text{Spec}\, R[x_1, \frac{y_1}{x_1}, \frac{x_2}{x_1}, \frac{y_2}{x_1}]/(\frac{y_1}{x_1} + (\frac{x_2}{x_1})(\frac{y_2}{x_1})\,,\ x_1^4(x_1(\frac{y_2}{x_1}) + 1)^4 (\frac{y_1}{x_1})(\frac{x_2}{x_1})^2 - \pi) \\ &= \text{Spec}\, R[X_1, X_2, Y_2]/(-X_1^4(X_1Y_2+1)^4 X_2^3 Y_2 - \pi) \end{aligned} \tag{4.42}$$

which is a regular scheme (use theorem (2.14)).

Chart 3.4.2

$$\begin{aligned} &W_{221}^{011} = \text{Spec}\, R[\frac{x_1}{y_1}, y_1, \frac{x_2}{y_1}, \frac{y_2}{y_1}]/(\frac{x_1}{y_1} + (\frac{x_2}{y_1})(\frac{y_2}{y_1})\,,\ y_1^4(y_1(\frac{y_2}{y_1}) + 1)^4 (\frac{x_2}{y_1})^2 (\frac{x_1}{y_1}) - \pi) \\ &= \text{Spec}\, R[Y_1, X_2, Y_2]/(-Y_1^4(Y_1Y_2+1)^4 X_2^3 Y_2 - \pi) \cong W_{221}^{010}. \end{aligned} \tag{4.43}$$

Chart 3.4.3

$$\begin{aligned} &W_{222}^{010} = \\ &\text{Spec}\, R[\frac{x_1}{x_2}, \frac{y_1}{x_2}, x_2, \frac{y_2}{x_2}]/((\frac{x_1}{x_2})(\frac{y_1}{x_2}) + \frac{y_2}{x_2}\,,\ x_2^4(x_2(\frac{y_2}{x_2}) + 1)^4 (\frac{x_1}{x_2})(\frac{y_1}{x_2}) - \pi) \\ &= \text{Spec}\, R[X_1, Y_1, X_2]/(X_2^4(-X_1Y_1X_2+1)^4 X_1 Y_1 - \pi) \end{aligned} \tag{4.44}$$

which is a regular scheme (use theorem (2.14)).

Chart 3.4.4

$$W_{222}^{011} =$$
$$\text{Spec}\, R[\frac{x_1}{y_2}, \frac{y_1}{y_2}, \frac{x_2}{y_2}, y_2]/((\frac{x_1}{y_2})(\frac{y_1}{y_2}) + \frac{x_2}{y_2}\,,\, y_2^4(y_2+1)^4(\frac{x_2}{y_2})^2(\frac{x_1}{y_2})(\frac{y_1}{y_2}) - \pi)$$
$$= \text{Spec}\, R[X_1, Y_1, Y_2]/(Y_2^4(Y_2+1)^4 X_1^3 Y_1^3 - \pi) \qquad (4.45)$$

which is a regular scheme (use theorem (2.14)). The gluing of W_{221}^{010}, W_{221}^{011}, W_{222}^{010} and W_{222}^{011} gives us the the regular scheme $\widetilde{W_{22}^{01}}$. If we glue all regular schemes which we have had so far, we get the desired regular scheme $\widetilde{X}$. □

4.3 One component is I_2

In this section our arithmetic three-folds are the fibre products of two arithmetic surfaces such that one of them is of the form I_2 and apart from the cases discussed in section two. When V_1 and V_2 are both of the form I_2, the output of the MAPLE in calculations related to the singular points of the special fibre of $X = V_1 \times_{\text{Spec}R} V_2$ shows us that after two blowings-up the resolution of singularities of the charts is much more complicated than other cases. It is part (a) of open problem (4.21).

Lemma 4.8 Let $p = char\, k \neq 2,3$ and $V_1 = \text{Spec}\, R[x,y]/(y^2 - x^3 - \pi)$ and $V_2 = \text{Spec}\, R[x,y]/(y^2 - yx^2 - \pi)$. Then $X = V_1 \times_{\text{Spec}R} V_2$ is singular and after some blowings-up we can resolve its singularity.

Proof: In fact

$$X = \text{Spec}\, R[x_1, y_1, x_2, y_2]/(y_1^2 - x_1^3 - y_2^2 + y_2 x_2^2\,,\, y_1^2 - x_1^3 - \pi). \qquad (4.46)$$

By using corollary (2.11) we can check that P_0 is the only singular point of X. Now we blow-up P_0. By using the procedure of Mahtab we get four pieces for the covering of $\widetilde{X}$ as follows:

Chart 1

$$V_1^0 = \mathrm{Spec}\, R[X_1, Y_1, X_2, Y_2]/(Y_1^2 - X_1 - Y_2^2 + X_1Y_2X_2^2\,,\, X_1^2Y_1^2 - X_1^3 - \pi). \tag{4.47}$$

Let $A = R[X_1, Y_1]/(X_1^2(Y_1^2 - X_1) - \pi)$. By using theorem (2.14) we can check that A is a regular ring. Note that

$$V_1^0 = \mathrm{Spec}\, A[X_2, Y_2]/(Y_1^2 - X_1 - Y_2^2 + X_1Y_2X_2^2). \tag{4.48}$$

Let $\phi(X_2, Y_2) = Y_1^2 - X_1 - Y_2^2 + X_1Y_2X_2^2$. Then $\frac{\partial\phi}{\partial X_2} = 2X_1Y_2X_2$ and $\frac{\partial\phi}{\partial Y_2} = -2Y_2 + X_1X_2^2$. Since $p = char\, k \neq 2$, the system $\frac{\partial\phi}{\partial X_2} = \frac{\partial\phi}{\partial Y_2} = 0$ can be written as $2X_1Y_2X_2 = -2Y_2 + X_1X_2^2 = 0$. So V_1^0 is smooth over A everywhere except at those points where $2X_1Y_2X_2 = -2Y_2 + X_1X_2^2 = 0$. To have the points on the special fibre of V_1^0, we need $X_1^2(Y_1^2 - X_1) = 0$. Considering $\phi(X_2, Y_2) = \frac{\partial\phi}{\partial X_2} = \frac{\partial\phi}{\partial Y_2} = X_1^2(Y_1^2 - X_1) = 0$ we get $(V_1^0)^{\mathrm{Sing}} \subset S \subset (V_1^0)_\pi$ where S contains points of the form $(a_1, b_1, a_2, b_2) \in (V_1^0)_\pi$ satisfying one of the following conditions:

(1) $a_1 = b_1 = b_2 = 0$;

(2) $a_1 - b_1^2 = a_2 = b_2 = 0$.

Let $Q_1 = (0, 0, a_2, 0) \in S$. Then

$$\mathrm{Moje}(Q_1, f_1) = -u_1\pi - T_1 \text{ and } \mathrm{Moje}(Q_1, f_2) = -\pi, \tag{4.49}$$

ie,

$$D(Q_1)_{r_1} = (\,-\bar{u}_1 \quad -1 \quad 0 \quad 0 \quad 0\,). \tag{4.50}$$

So *rank* $D(Q_1) = 2$. Hence Q_1 is a regular point.

Let $Q_2 = (a_1, b_1, 0, 0) \in S$ such that $a_1 - b_1^2 = 0$. Then $\mathrm{Moje}(Q_2, f_1) = -T_1 + 2\beta_1S_1$ and

$$\mathrm{Moje}(Q_2, f_2) = (2\alpha_1\beta_1^2 - 3\alpha_1^2)T_1 + 2\beta_1\alpha_1^2S_1 - \pi, \tag{4.51}$$

ie,

$$D(Q_2) = \begin{pmatrix} 0 & -1 & 2b_1 & 0 & 0 \\ -1 & 2a_1b_1^2 - 3a_1^2 & 2b_1a_1^2 & 0 & 0 \end{pmatrix}. \tag{4.52}$$

So *rank* $D(Q_2) = 2$. Hence $D(Q_2)$ is a regular point. This means that V_1^0 is a regular scheme.

Chart 2

$$V_1^1 = \operatorname{Spec} R[X_1, Y_1, X_2, Y_2]/(1 - Y_1X_1^3 - Y_2^2 + Y_1Y_2X_2^2\,,\, Y_1^2 - Y_1^3X_1^3 - \pi). \tag{4.53}$$

Now let $A = R[X_1, Y_1]/(Y_1^2 - Y_1^3X_1^3 - \pi)$. By using theorem (2.14) we can check that A is a regular ring. Note that

$$V_1^1 = \operatorname{Spec} A[X_2, Y_2]/(1 - Y_1X_1^3 - Y_2^2 + Y_1Y_2X_2^2). \tag{4.54}$$

Let $\phi(X_2, Y_2) = 1 - Y_1X_1^3 - Y_2^2 + Y_1Y_2X_2^2$. Then $\frac{\partial\phi}{\partial X_2} = 2Y_1Y_2X_2$ and $\frac{\partial\phi}{\partial Y_2} = -2Y_2 + Y_1X_2^2$. Since $p = char\, k \neq 2$, the system $\frac{\partial\phi}{\partial X_2} = \frac{\partial\phi}{\partial Y_2} = 0$ can be written as $2Y_1Y_2X_2 = -2Y_2 + Y_1X_2^2 = 0$. So V_1^1 is smooth over A everywhere except at those points where $2Y_1Y_2X_2 = -2Y_2 + Y_1X_2^2 = 0$. To have the points on the special fibre, we need $Y_1^2(1 - Y_1X_1^3) = 0$. Considering $\phi(X_2, Y_2) = \frac{\partial\phi}{\partial X_2} = \frac{\partial\phi}{\partial Y_2} = Y_1^2(1 - Y_1X_1^3) = 0$ we get $(V_1^1)^{\text{Sing}} \subset S \subset (V_1^1)_\pi$ where S contains points of the form $(a_1, b_1, a_2, b_2) \in (V_1^1)_\pi$ satisfying the following condition:

$$a_2 = b_2 = a_1^3b_1 - 1 = 0.$$

Let $Q = (a_1, b_1, 0, 0) \in S$ such that $a_1^3b_1 - 1 = 0$. Then $\operatorname{Moje}(Q, f_1) = -3\alpha_1^2\beta_1T_1 - \alpha_1^3S_1$ and

$$\operatorname{Moje}(Q, f_2) = -3\alpha_1^2\beta_1^3T_1 + \beta_1(2 - 3\alpha_1^3\beta_1)S_1 - \pi, \tag{4.55}$$

ie,

$$D(Q) = \begin{pmatrix} 0 & -3a_1^2b_1 & -a_1^3 & 0 & 0 \\ -1 & -3a_1^2b_1^3 & b_1(2 - 3a_1^3b_1) & 0 & 0 \end{pmatrix}. \tag{4.56}$$

So Q is singular if $-3a_1^2b_1 = -a_1^3 = a_1^3b_1 - 1 = 0$ which is impossible. Hence Q is a regular point and as a result V_1^1 is a regular scheme.

Chart 3

$$V_2^0 = \operatorname{Spec} R[X_1, Y_1, X_2, Y_2]/(Y_1^2 - X_2X_1^3 - Y_2^2 + X_2Y_2\,,\, X_2^2Y_1^2 - X_2^3X_1^3 - \pi). \tag{4.57}$$

From the first equation we get $Y_1^2 - X_2X_1^3 = Y_2^2 - X_2Y_2$. By using it in the other equation we can write V_2^0 as follows:

$$V_2^0 = \operatorname{Spec} R[X_1, Y_1, X_2, Y_2]/(Y_1^2 - X_2X_1^3 - Y_2^2 + X_2Y_2\,,\; X_2^2Y_2^2 - X_2^3Y_2 - \pi). \tag{4.58}$$

Now suppose that $A = R[X_2, Y_2]/(X_2^2Y_2(Y_2 - X_2) - \pi)$. We can check that A is a regular ring (by using theorem (2.14)). Note that

$$V_2^0 = \operatorname{Spec} A[X_1, Y_1]/(Y_1^2 - X_2X_1^3 - Y_2^2 + X_2Y_2). \tag{4.59}$$

Let $\phi(X_1, Y_1) = Y_1^2 - X_2X_1^3 - Y_2^2 + X_2Y_2$. Then $\frac{\partial\phi}{\partial X_1} = -3X_2X_1^2$ and $\frac{\partial\phi}{\partial Y_1} = 2Y_1$. Since $p = char\, k \neq 2, 3$, the system $\frac{\partial\phi}{\partial X_1} = \frac{\partial\phi}{\partial Y_1} = 0$ can be written as $-3X_2X_1^2 = 2Y_1 = 0$. So V_2^0 is smooth over A everywhere except at those points where $-3X_2X_1^2 = 2Y_1 = 0$. To have the points on the special fibre of V_2^0, we need $X_2^2Y_1^2 - X_2^3X_1^3 = 0$. Considering $\phi(X_1, Y_1) = \frac{\partial\phi}{\partial X_1} = \frac{\partial\phi}{\partial Y_1} = X_2^2Y_1^2 - X_2^3X_1^3 = 0$ we get $(V_2^0)^{\text{Sing}} \subset S \subset (V_2^0)_\pi$ where S contains points of the form $(a_1, b_1, a_2, b_2) \in (V_2^0)_\pi$ satisfying one of the following conditions:

(1) $a_1 = b_1 = b_2 = 0$;

(2) $a_1 = b_1 = a_2 - b_2 = 0$;

(3) $b_1 = a_2 = b_2 = 0$.

Let $Q_1 = (0, 0, a_2, 0) \in S$. Then $\text{Moje}(Q_1, f_1) = \alpha_2 v_2\pi + \alpha_2 S_2$ and $\text{Moje}(Q_1, f_2) = -\pi$, *ie*,

$$D(Q_1)_{r_1} = \begin{pmatrix} a_2\bar{v}_2 & 0 & 0 & 0 & a_2 \end{pmatrix}. \tag{4.60}$$

So Q_1 is singular if $a_2 = 0$, *ie*, $Q_1 = P_0$.

Let $Q_2 = (0, 0, a_2, b_2) \in S$ such that $a_2 - b_2 = 0$. Then

$$\text{Moje}(Q_2, f_1) = \beta_2 T_2 + (\alpha_2 - 2\beta_2)S_2 \tag{4.61}$$

and $\text{Moje}(Q_2, f_2) = -\pi$, *ie*,

$$D(Q_2)_{r_1} = \begin{pmatrix} 0 & 0 & 0 & b_2 & (a_2 - 2b_2) \end{pmatrix}. \tag{4.62}$$

So Q_2 is singular if $a_2 = b_2 = 0$ *ie* $Q_2 = P_0$.

Let $Q_3 = (a_1, 0, 0, 0) \in S$. Then

$$\text{Moje}(Q_3, f_1) = -\alpha_1^3 u_2\pi - \alpha_1^3 T_2 \text{ and } \text{Moje}(Q_3, f_2) = -\pi, \tag{4.63}$$

ie,

$$D(Q_3)_{r_1} = (\,-a_1^3\bar{u}_2 \quad 0 \quad 0 \quad -a_1^3 \quad 0\,). \tag{4.64}$$

So Q_3 is singular if $a_1 = 0$, *ie*, $Q_3 = P_0$. Hence

$$(V_2^0)^{\mathrm{Sing}} = \{P_0 \in (V_2^0)_\pi\}.$$

We will try to resolve this singular point later.

Chart 4

$$V_2^1 = \operatorname{Spec} R[X_1, Y_1, X_2, Y_2]/(Y_1^2 - Y_2X_1^3 - 1 + Y_2X_2^2\,,\, Y_2^2Y_1^2 - Y_2^3X_1^3 - \pi). \tag{4.65}$$

From the first equation we get $Y_1^2 - Y_2X_1^3 = 1 - Y_2X_2^2$. By using it in the other equation we get

$$V_2^1 = \operatorname{Spec} R[X_1, Y_1, X_2, Y_2]/(Y_1^2 - Y_2X_1^3 - 1 + Y_2X_2^2\,,\, Y_2^2(1 - Y_2X_2^2) - \pi). \tag{4.66}$$

Now let $A = R[X_2, Y_2]/(Y_2^2(1 - Y_2X_2^2) - \pi)$. Considering theorem (2.14), we can check that A is a regular ring. Notice that

$$V_2^1 = \operatorname{Spec} A[X_1, Y_1]/(Y_1^2 - Y_2X_1^3 - 1 + Y_2X_2^2). \tag{4.67}$$

Let $\phi(X_1, Y_1) = Y_1^2 - Y_2X_1^3 - 1 + Y_2X_2^2$. Then $\frac{\partial\phi}{\partial X_1} = -3Y_2X_1^2$ and $\frac{\partial\phi}{\partial Y_1} = 2Y_1$. Since $p = char\, k \neq 2\,, 3$, the system $\frac{\partial\phi}{\partial X_1} = \frac{\partial\phi}{\partial Y_1} = 0$ can be written as $-3Y_2X_1^2 = 2Y_1 = 0$. So V_2^1 is smooth over A everywhere except at those points where $-3Y_2X_1^2 = 2Y_1 = 0$. To have the points on the special fibre of V_2^1, we need $Y_2^2Y_1^2 - Y_2^3X_1^3 = 0$. Considering $\phi(X_1, Y_1) = \frac{\partial\phi}{\partial X_1} = \frac{\partial\phi}{\partial Y_1} = Y_2^2Y_1^2 - Y_2^3X_1^3 = 0$ we get $(V_2^1)^{\mathrm{Sing}} \subset S \subset (V_2^1)_\pi$ where S contains points of the form $(a_1, b_1, a_2, b_2) \in (V_2^1)_\pi$ satisfying the following condition:

$$a_1 = b_1 = a_2^2b_2 - 1 = 0.$$

Let $Q = (0, 0, a_2, b_2) \in S$ such that $a_2^2b_2 - 1 = 0$. Then $\mathrm{Moje}(Q, f_1) = 2\alpha_2\beta_2T_2 + \alpha_2^2S_2$ and $\mathrm{Moje}(Q, f_2) = -\pi$, *ie*,

$$D(Q)_{r_1} = (\,0 \quad 0 \quad 0 \quad 2a_2b_2 \quad a_2^2\,). \tag{4.68}$$

So Q is singular if $a_2^2 = 2a_2b_2 = a_2^2b_2 - 1 = 0$ which is impossible. So Q is a regular point and as a result V_2^1 is a regular scheme.

Recall that

$$V_2^0 = \operatorname{Spec} R[X_1, Y_1, X_2, Y_2]/(Y_1^2 - X_2X_1^3 - Y_2^2 + X_2Y_2\,,\, X_2^2Y_1^2 - X_2^3X_1^3 - \pi) \tag{4.69}$$

is singular and P_0 is its singular point. Now we blow-up P_0. By using the procedure of Mahtab we get four pieces for the covering of V_2^0 as follows:

Chart 3.1

$$V_{21}^{00} = \operatorname{Spec} R[X_1, Y_1, X_2, Y_2]/(Y_1^2 - X_1^2X_2 - Y_2^2 + X_2Y_2\,,\, X_1^4X_2^2Y_1^2 - X_1^6X_2^3 - \pi). \tag{4.70}$$

From the first equation we get $Y_1^2 - X_1^2X_2 = Y_2^2 - X_2Y_2$. By using it in the other equation we can write V_{21}^{00} as follows:

$$V_{21}^{00} = \operatorname{Spec} R[X_1, Y_1, X_2, Y_2]/(Y_1^2 - X_1^2X_2 - Y_2^2 + X_2Y_2\,,\, X_1^4X_2^2Y_2(Y_2 - X_2) - \pi). \tag{4.71}$$

Suppose that $A = R[X_1, X_2, Y_2]/(X_1^4X_2^2Y_2(Y_2 - X_2) - \pi)$. We can use theorem (2.14) to check that A is a regular ring. Note that

$$V_{21}^{00} = \operatorname{Spec} A[Y_1]/(Y_1^2 - X_1^2X_2 - Y_2^2 + X_2Y_2). \tag{4.72}$$

Let $\phi(Y_1) = Y_1^2 - X_1^2X_2 - Y_2^2 + X_2Y_2$. Then $\frac{\partial\phi}{\partial Y_1} = 2Y_1$. Since $p = char\, k \neq 2$, the equation $\frac{\partial\phi}{\partial Y_1} = 0$ can be written as $2Y_1 = 0$. So V_{21}^{00} is etale over A everywhere except at those points where $Y_1 = 0$. To have the points on the special fibre of V_{21}^{00}, we need $X_1^4X_2^2Y_1^2 - X_1^6X_2^3 = 0$. Considering $\phi(Y_1) = \frac{\partial\phi}{\partial Y_1} = X_1^4X_2^2Y_1^2 - X_1^6X_2^3 = 0$ we get $(V_{21}^{00})^{\text{Sing}} \subset S \subset (V_{21}^{00})_\pi$ where S contains points of the form $(a_1, b_1, a_2, b_2) \in (V_{21}^{00})_\pi$ satisfying one of the following conditions:

(1) $a_1 = b_1 = b_2 = 0;$

(2) $a_1 = b_1 = a_2 - b_2 = 0;$

(3) $b_1 = a_2 = b_2 = 0.$

Let $Q_1 = (0, 0, a_2, 0) \in S$. Then

$$\text{Moje}(Q_1, f_1) = \alpha_2 v_2 \pi + \alpha_2 S_2 \text{ and } \text{Moje}(Q_1, f_2) = -\pi, \tag{4.73}$$

ie,

$$D(Q_1)_{r_1} = (\, a_2\bar{v}_2 \quad 0 \quad 0 \quad 0 \quad a_2\,). \tag{4.74}$$

So Q_1 is singular if $a_2 = 0$, *ie*, $Q_1 = P_0$.

Let $Q_2 = (0, 0, a_2, b_2) \in S$ such that $a_2 - b_2 = 0$. Then

$$\text{Moje}(Q_2, f_1) = \beta_2 T_2 + (\alpha_2 - 2\beta_2) S_2 \text{ and } \text{Moje}(Q_2, f_2) = -\pi, \tag{4.75}$$

ie,

$$D(Q_2)_{r_1} = (\, 0 \quad 0 \quad 0 \quad b_2 \quad a_2 - 2b_2\,). \tag{4.76}$$

So Q_2 is singular if $a_2 = b_2 = 0$, *ie*, $Q_2 = P_0$.

Let $Q_3 = (a_1, 0, 0, 0) \in S$. Then

$$\text{Moje}(Q_3, f_1) = -\alpha_1^2 u_2 \pi - \alpha_1^2 T_2 \text{ and } \text{Moje}(Q_3, f_2) = -\pi, \tag{4.77}$$

ie,

$$D(Q_3)_{r_1} = (\, -a_1^2\bar{u}_2 \quad 0 \quad 0 \quad -a_1^2 \quad 0\,). \tag{4.78}$$

So Q_3 is singular if $a_1 = 0$, *ie*, $Q_3 = P_0$. This means that

$$(V_{21}^{00})^{\text{Sing}} = \{P_0\}.$$

We will try to resolve this singular point later.

Chart 3.2

$$V_{21}^{01} =$$

$$\text{Spec}\, R[X_1, Y_1, X_2, Y_2]/(1 - Y_1^2 X_2 X_1^3 - Y_2^2 + X_2 Y_2\,,\ Y_1^4 X_2^2 - Y_1^6 X_2^3 X_1^3 - \pi). \tag{4.79}$$

Let $A = R[X_1, Y_1, X_2]/(Y_1^4 X_2^2(1 - Y_1^2 X_2 X_1^3) - \pi)$. By using theorem (2.14) we can check that A is a regular ring. Note that

$$V_{21}^{01} = \operatorname{Spec} A[Y_2]/(1 - Y_1^2 X_2 X_1^3 - Y_2^2 + X_2 Y_2). \tag{4.80}$$

Let $\phi(Y_2) = 1 - Y_1^2 X_2 X_1^3 - Y_2^2 + X_2 Y_2$. Then $\frac{\partial \phi}{\partial Y_2} = -2Y_2 + X_2$. Since $p = char\, k \neq 2$, the equation $\frac{\partial \phi}{\partial Y_2} = 0$ can be written as $-2Y_2 + X_2 = 0$. So V_{21}^{01} is etale over A everywhere except at those points where $X_2 - 2Y_2 = 0$. To have the points on the special fibre of V_{21}^{01}, we need $Y_1^4 X_2^2(1 - Y_1^2 X_2 X_1^3) = 0$. Considering $\phi(Y_2) = \frac{\partial \phi}{\partial Y_2} = Y_1^4 X_2^2(1 - Y_1^2 X_2 X_1^3) = 0$, we get $(V_{21}^{01})^{\text{Sing}} \subset S \subset (V_{21}^{01})_\pi$ where S contains points of the form $(a_1, b_1, a_2, b_2) \in (V_{21}^{01})_\pi$ satisfying the following condition:

$$b_1 = a_2 - 2b_2 = 4 + a_2^2 = 0.$$

Let $Q = (a_1, 0, a_2, b_2) \in S$ such that $a_2 - 2b_2 = 4 + a_2^2 = 0$. Then $\text{Moje}(Q, f_1) = \beta_2 T_2 + (\alpha_2 - 2\beta_2) S_2$ and $\text{Moje}(Q, f_2) = -\pi$, *ie*,

$$D(Q)_{r_1} = (\,0 \quad 0 \quad 0 \quad b_2 \quad (a_2 - 2b_2)\,). \tag{4.81}$$

So Q is singular if $b_2 = a_2 - 2b_2 = 4 + a_2^2 = 0$ which is impossible. Hence Q is a regular point and as a result V_{21}^{01} is a regular scheme.

Chart 3.3

$$V_{22}^{00} = \operatorname{Spec} R[X_1, Y_1, X_2, Y_2]/(Y_1^2 - X_2^2 X_1^3 - Y_2^2 + Y_2\,,\; X_2^4 Y_1^2 - X_2^6 X_1^3 - \pi). \tag{4.82}$$

From the first equation we get $Y_1^2 - X_2^2 X_1^3 = Y_2^2 - Y_2$. By using it in the other equation we get

$$V_{22}^{00} = \operatorname{Spec} R[X_1, Y_1, X_2, Y_2]/(Y_1^2 - X_2^2 X_1^3 - Y_2^2 + Y_2\,,\; X_2^4(Y_2^2 - Y_2) - \pi). \tag{4.83}$$

Now let $A = R[X_2, Y_2]/(X_2^4(Y_2^2 - Y_2) - \pi)$. By using theorem (2.14) we can check that A is a regular ring. Notice that

$$V_{22}^{00} = \operatorname{Spec} A[X_1, Y_1]/(Y_1^2 - X_2^2 X_1^3 - Y_2^2 + Y_2). \tag{4.84}$$

Let $\phi(X_1, Y_1) = Y_1^2 - X_2^2X_1^3 - Y_2^2 + Y_2$. Then $\frac{\partial\phi}{\partial X_1} = -3X_2^2X_1^2$ and $\frac{\partial\phi}{\partial Y_1} = 2Y_1$. Since $p = char\, k \neq 2, 3$, the system $\frac{\partial\phi}{\partial X_1} = \frac{\partial\phi}{\partial Y_1} = 0$ can be written as $-3X_2^2X_1^2 = 2Y_1 = 0$. So V_{22}^{00} is smooth over A everywhere except at those points where $-3X_2^2X_1^2 = 2Y_1 = 0$. To have the points on the special fibre of V_{22}^{00}, we need $X_2^4Y_1^2 - X_2^6X_1^3 = 0$. Considering $\phi(X_1, Y_1) = \frac{\partial\phi}{\partial X_1} = \frac{\partial\phi}{\partial Y_1} = X_2^4Y_1^2 - X_2^6X_1^3 = 0$, we get $(V_{22}^{00})^{\text{Sing}} \subset S \subset (V_{22}^{00})_\pi$ where S contains points of the form $(a_1, b_1, a_2, b_2) \in (V_{22}^{00})_\pi$ satisfying one of the the following conditions:

(1) $b_1 = a_2 = b_2 = 0$;

(2) $b_1 = a_2 = b_2 - 1 = 0$;

(3) $a_1 = b_1 = b_2 = 0$;

(4) $a_1 = b_1 = b_2 - 1 = 0$.

Let $Q_1 = (a_1, 0, 0, 0) \in S$. Then

$$\text{Moje}(Q_1, f_1) = S_2 + v_2\pi \quad \text{and} \quad \text{Moje}(Q_1, f_2) = -\pi, \tag{4.85}$$

ie,

$$D(Q_1)_{r_1} = (\,\bar{v}_2 \quad 0 \quad 0 \quad 0 \quad 1\,). \tag{4.86}$$

So $D(Q_1)$ has rank two. Hence Q_1 is a regular point.

Let $Q_2 = (a_1, 0, 0, 1) \in S$. Then $\text{Moje}(Q_2, f_1) = (1 - 2\beta_2)S_2$ where $\beta_2 \equiv 1 (\text{mod}\, \pi)$ and $\text{Moje}(Q_2, f_2) = -\pi$, *ie*,

$$D(Q_2)_{r_1} = (\,0 \quad 0 \quad 0 \quad 0 \quad -1\,). \tag{4.87}$$

So $D(Q_2)$ has rank two. Hence Q_2 is a regular point.

Let $Q_3 = (0, 0, a_2, 0) \in S$. Then

$$\text{Moje}(Q_3, f_1) = v_2\pi + S_2 \quad \text{and} \quad \text{Moje}(Q_3, f_2) = -\pi, \tag{4.88}$$

ie,

$$D(Q_3)_{r_1} = (\,\bar{v}_2 \quad 0 \quad 0 \quad 0 \quad 1\,). \tag{4.89}$$

So $D(Q_3)$ has rank two. Hence Q_3 is a regular point.

Let $Q_4 = (0, 0, a_2, 1) \in S$. Then $\text{Moje}(Q_4, f_1) = (1 - 2\beta_2)S_2$ where $\beta_2 \equiv 1 (\text{mod}\, \pi)$ and $\text{Moje}(Q_4, f_2) = -\pi$, *ie*,

$$D(Q_4)_{r_1} = (\,0 \quad 0 \quad 0 \quad 0 \quad -1\,). \tag{4.90}$$

So $D(Q_4)$ has rank two. Hence Q_4 is a regular point and as a result V_{22}^{00} is a regular scheme.

Chart 3.4

$$V_{22}^{01} = \operatorname{Spec} R[X_1, Y_1, X_2, Y_2]/(Y_1^2 - Y_2^2X_2X_1^3 - 1 + X_2\,,\, Y_2^4X_2^2Y_1^2 - Y_2^6X_2^3X_1^3 - \pi). \tag{4.91}$$

From the first equation we get $Y_1^2 - Y_2^2X_2X_1^3 = 1 - X_2$. By using it in the other equation we get

$$V_{22}^{01} = \operatorname{Spec} R[X_1, Y_1, X_2, Y_2]/(Y_1^2 - Y_2^2X_2X_1^3 - 1 + X_2\,,\, Y_2^4X_2^2(1 - X_2) - \pi). \tag{4.92}$$

Now suppose that $A = R[X_2, Y_2]/(Y_2^4X_2^2(1 - X_2) - \pi)$. We can check that A is a regular ring (by using theorem (2.14)). Note that

$$V_{22}^{01} = \operatorname{Spec} A[X_1, Y_1]/(Y_1^2 - Y_2^2X_2X_1^3 - 1 + X_2). \tag{4.93}$$

Let $\phi(X_1, Y_1) = Y_1^2 - Y_2^2X_2X_1^3 - 1 + X_2$. Then $\frac{\partial\phi}{\partial X_1} = -3Y_2^2X_2X_1^2$ and $\frac{\partial\phi}{\partial Y_1} = 2Y_1$. Since $p = char\,k \neq 2, 3$, the system $\frac{\partial\phi}{\partial X_1} = \frac{\partial\phi}{\partial Y_1} = 0$ can be written as $-3Y_2^2X_2X_1^2 = 2Y_1 = 0$. So V_{22}^{01} is smooth over A everywhere except at those points where $-3Y_2^2X_2X_1^2 = 2Y_1 = 0$. To have the points on the special fibre of V_{22}^{01}, we need $Y_2^4X_2^2Y_1^2 - Y_2^6X_2^3X_1^3 = 0$. Considering $\phi(X_1, Y_1) = \frac{\partial\phi}{\partial X_1} = \frac{\partial\phi}{\partial Y_1} = Y_2^4X_2^2Y_1^2 - Y_2^6X_2^3X_1^3 = 0$, we get $(V_{22}^{01})^{\text{Sing}} \subset S \subset (V_{22}^{01})_\pi$ where S contains points of the form $(a_1, b_1, a_2, b_2) \in (V_{22}^{01})_\pi$ satisfying one of the following conditions:

(1) $b_1 = a_2 - 1 = b_2 = 0$;

(2) $a_1 = b_1 = a_2 - 1 = 0$.

Let $Q_1 = (a_1, 0, 1, 0) \in S$. Then

$$\text{Moje}(Q_1, f_1) = T_2 \text{ and } \text{Moje}(Q_1, f_2) = -\pi, \tag{4.94}$$

ie,

$$D(Q_1)_{r_1} = (0 \quad 0 \quad 0 \quad 1 \quad 0). \tag{4.95}$$

So Q_1 is a regular point.

Let $Q_2 = (0, 0, 1, b_2) \in S$. Then $\text{Moje}(Q_2, f_1) = T_2$ and $\text{Moje}(Q_2, f_2) = -\pi$ where $\alpha_2 \equiv 1 (\text{mod}\, \pi)$, *ie*,

$$D(Q_2)_{r_1} = (\,0 \quad 0 \quad 0 \quad 1 \quad 0\,). \tag{4.96}$$

So $rank\, D(Q_2) = 2$. Hence Q_2 is a regular point and as result V_{22}^{01} is a regular scheme.

Recall that

$$V_{21}^{00} = \text{Spec}\, R[X_1, Y_1, X_2, Y_2]/(Y_1^2 - x_1^2 X_2 - Y_2^2 + X_2 Y_2\,,\, X_1^4 X_2^2 Y_1^2 - X_1^6 X_2^3 - \pi) \tag{4.97}$$

is singular and P_0 is its singular point. Now we blow-up P_0. By using the procedure of Mahtab we get four pieces for the covering of $\widetilde{V_{21}^{00}}$ as follows:

Chart 3.1.1

$$V_{211}^{000} = \text{Spec}\, R[X_1, Y_1, X_2, Y_2]/(Y_1^2 - X_1 X_2 - Y_2^2 + X_2 Y_2\,,\, X_1^8 X_2^2 Y_1^2 - X_1^9 X_2^3 - \pi). \tag{4.98}$$

From the first equation we get $Y_1^2 - X_1 X_2 = Y_2^2 - X_2 Y_2$. By using it in the equation we get

$$V_{211}^{000} = \text{Spec}\, R[X_1, Y_1, X_2, Y_2]/(Y_1^2 - X_1 X_2 - Y_2^2 + X_2 Y_2\,,\, X_1^8 X_2^2 Y_2 (Y_2 - X_2) - \pi). \tag{4.99}$$

Assume that $A = R[X_1, X_2, Y_2]/(X_1^8 X_2^2 Y_2 (Y_2 - X_2) - \pi)$. We can use theorem (2.14) to check that A is a regular ring. Note that

$$V_{211}^{000} = \text{Spec}\, A[Y_1]/(Y_1^2 - X_1 X_2 - Y_2^2 + X_2 Y_2). \tag{4.100}$$

Let $\phi(Y_1) = Y_1^2 - X_1X_2 - Y_2^2 + X_2Y_2$. Then $\frac{\partial\phi}{\partial Y_1} = 2Y_1$. Since $p = char\, k \neq 2$, the equation $\frac{\partial\phi}{\partial Y_1} = 0$ can be written as $2Y_1 = 0$. So V_{211}^{000} is etale over A everywhere except at those points where $Y_1 = 0$. To have the points on the special fibre of V_{211}^{000}, we need $X_1^8X_2^2Y_1^2 - X_1^9X_2^3 = 0$. Considering $\phi(Y_1) = \frac{\partial\phi}{\partial Y_1} = X_1^8X_2^2Y_1^2 - X_1^9X_2^3 = 0$ we get $(V_{211}^{000})^{\text{Sing}} \subset S \subset (V_{211}^{000})_\pi$ where S contains points of the form $(a_1, b_1, a_2, b_2) \in (V_{211}^{000})_\pi$ satisfying one of the following conditions:

(1) $a_1 = b_1 = b_2 = 0$;

(2) $a_1 = b_1 = a_2 - b_2 = 0$;

(3) $b_1 = a_2 = b_2 = 0$.

Let $Q_1 = (0, 0, a_2, 0) \in S$. Then

$$\text{Moje}(Q_1, f_1) = \alpha_2(v_2 - u_1)\pi - \alpha_2T_1 + \alpha_2S_2 \tag{4.101}$$

and $\text{Moje}(Q_1, f_2) = -\pi$, *ie*,

$$D(Q_1)_{r_1} = (\, a_2(\bar{v}_2 - \bar{u}_1) \quad -a_2 \quad 0 \quad 0 \quad a_2\,). \tag{4.102}$$

So Q_1 is singular if $a_2 = 0$, *ie*, $Q_1 = P_0$.

Let $Q_2 = (0, 0, a_2, b_2) \in S$ such that $a_2 - b_2 = 0$. Then

$$\text{Moje}(Q_2, f_1) = -\alpha_2u_1\pi - \alpha_2T_1 + \beta_2T_2 + (\alpha_2 - 2\beta_2)S_2 \tag{4.103}$$

and $\text{Moje}(Q_2, f_2) = -\pi$, *ie*,

$$D(Q_2)_{r_1} = (\, -a_2\bar{u}_1 \quad -a_2 \quad 0 \quad b_2 \quad a_2 - 2b_2\,). \tag{4.104}$$

So Q_2 is singular if $a_2 - 2b_2 = b_2 = 0$, *ie*, $Q_2 = P_0$.

Finally let $Q_3 = (a_1, 0, 0, 0) \in S$. Then

$$\text{Moje}(Q_3, f_1) = -\alpha_1u_2\pi - \alpha_1T_2 \text{ and } \text{Moje}(Q_3, f_2) = -\pi, \tag{4.105}$$

ie,

$$D(Q_3)_{r_1} = (\, -a_1\bar{u}_2 \quad 0 \quad 0 \quad -a_1 \quad 0\,). \tag{4.106}$$

So Q_3 is singular if $a_1 = 0$, *ie*, $Q_3 = P_0$. Hence

$$(V_{211}^{000})_\pi^{\text{Sing}} = \{P_0\}.$$

We will try to resolve this singularity later.

Chart 3.1.2

$$V_{211}^{001} = \text{Spec}\, R[X_1, Y_1, X_2, Y_2]/$$
$$(1 - Y_1X_1^2X_2 - Y_2^2 + X_2Y_2\,,\ Y_1^8X_1^4X_2^2 - Y_1^9X_1^6X_2^3 - \pi). \quad (4.107)$$

Now assume that $A = R[X_1, Y_1, X_2]/(Y_1^8X_1^4X_2^2(1 - Y_1X_1^2X_2) - \pi)$. Considering theorem (2.14) we can check that A is a regular ring. Notice that

$$V_{211}^{001} = \text{Spec}\, A[Y_2]/(1 - Y_1X_1^2X_2 - Y_2^2 + X_2Y_2). \quad (4.108)$$

Let $\phi(Y_2) = 1 - Y_1X_1^2X_2 - Y_2^2 + X_2Y_2$. Then $\frac{\partial\phi}{\partial Y_2} = -2Y_2 + X_2$. Since $p = char\, k \neq 2$, the equation $\frac{\partial\phi}{\partial Y_2} = 0$ can be written as $X_2 - 2Y_2 = 0$. So V_{211}^{001} is etale over A everywhere except at those points where $X_2 - 2Y_2 = 0$. To have the points on the special fibre of V_{211}^{001}, we need $Y_1^8X_1^4X_2^2 - Y_1^9X_1^6X_2^3 = 0$. Considering the system $\phi(Y_2) = \frac{\partial\phi}{\partial Y_2} = Y_1^8X_1^4X_2^2 - Y_1^9X_1^6X_2^3 = 0$, we can check that $(V_{211}^{001})^{\text{Sing}} \subset S \subset (V_{211}^{001})_\pi$ where S contains points of the form $(a_1, b_1, a_2, b_2) \in (V_{211}^{001})_\pi$ satisfying one of the following conditions:

(1) $a_1 = a_2 - 2b_2 = 4 + a_2^2 = 0$;

(2) $b_1 = a_2 - 2b_2 = 4 + a_2^2 = 0$.

Let $Q_1 = (0, b_1, a_2, b_2) \in S$ such that $a_2 = 2b_2$ and $4 + a_2^2 = 0$. Then

$$\text{Moje}(Q_1, f_1) = \beta_2T_2 + (\alpha_2 - 2\beta_2)S_2 \quad (4.109)$$

and $\text{Moje}(Q_1, f_2) = -\pi$, *ie*,

$$D(Q)_{r_1} = (\,0 \quad 0 \quad 0 \quad b_2 \quad a_2 - 2b_2\,). \quad (4.110)$$

So Q_1 is singular if $b_2 = a_2 - 2b_2 = 4 + a_2^2 = 0$ which is impossible. Hence Q_1 is a regular point.

Let $Q_2 = (a_1, 0, a_2, b_2) \in S$ such that $a_2 - 2b_2 = 4 + a_2^2 = 0$. Then

$$\text{Moje}(Q_2, f_1) = -\alpha_1^2\alpha_2v_1\pi - \alpha_1^2\alpha_2S_1 + \beta_2T_2 + (\alpha_2 - 2\beta_2)S_2 \quad (4.111)$$

and $\text{Moje}(Q_2, f_2) = -\pi$, *ie*,

$$D(Q_2)_{r_1} = (\, -a_1^2 a_2 \bar{v}_1 \quad 0 \quad -a_1^2 a_2 \quad b_2 \quad a_2 - 2b_2 \,). \tag{4.112}$$

So Q_2 is singular if $a_1^2 a_2 = b_2 = a_2 - 2b_2 = 4 + a_2^2 = 0$ which is impossible. Hence Q_2 is a regular point and as a result V_{211}^{001} is a regular scheme.

Chart 3.1.3

$$V_{212}^{000} = \text{Spec}\, R[X_1, Y_1, X_2, Y_2]/(Y_1^2 - X_2X_1^2 - Y_2^2 + Y_2\,,\, X_2^8X_1^4Y_1^2 - X_2^9X_1^6 - \pi). \tag{4.113}$$

From the first equation we get $Y_1^2 - X_2X_1^2 = Y_2^2 - Y_2$. By using it in the other equation we can write

$$V_{212}^{000} = \text{Spec}\, R[X_1, Y_1, X_2, Y_2]/(Y_1^2 - X_2X_1^2 - Y_2^2 + Y_2\,,\, X_2^8X_1^4(Y_2^2 - Y_2) - \pi). \tag{4.114}$$

Now suppose that $A = R[X_1, X_2, Y_2]/(X_2^8X_1^4(Y_2^2 - Y_2) - \pi)$. We can check that A is a regular ring (by using theorem (2.14)). We can write

$$V_{212}^{000} = \text{Spec}\, A[Y_1]/(Y_1^2 - X_2X_1^2 - Y_2^2 + Y_2). \tag{4.115}$$

Let $\phi(Y_1) = Y_1^2 - X_2X_1^2 - Y_2^2 + Y_2$. Then $\frac{\partial\phi}{\partial Y_1} = 2Y_1$. Since $p = char\, k \neq 2$, the equation $\frac{\partial\phi}{\partial Y_1} = 0$ can be written as $2Y_1 = 0$. So V_{212}^{000} is etale over A everywhere except at those points where $Y_1 = 0$. To have the points on the special fibre of V_{212}^{000}, we need $X_2^8X_1^4Y_1^2 - X_2^9X_1^6 = 0$. Considering $\phi(Y_1) = \frac{\partial\phi}{\partial Y_1} = X_2^8X_1^4Y_1^2 - X_2^9X_1^6 = 0$, we get $(V_{212}^{000})^{\text{Sing}} \subset S \subset (V_{212}^{000})_\pi$ where S contains points of the form $(a_1, b_1, a_2, b_2) \in (V_{212}^{000})_\pi$ satisfying one of the following conditions:

(1) $b_1 = a_2 = b_2 = 0$;

(2) $b_1 = a_2 = b_2 - 1 = 0$;

(3) $b_1 = a_1 = b_2 = 0$;

(4) $b_1 = a_1 = b_2 - 1 = 0$.

Let $Q_1 = (a_1, 0, 0, 0) \in S$. Then

$$\text{Moje}(Q_1, f_1) = (v_2 - \alpha_1^2 u_2)\pi - \alpha_1^2 T_2 + S_2 \text{ and } \text{Moje}(Q_1, f_2) = -\pi, \tag{4.116}$$

ie,

$$D(Q_1)_{r_1} = (\,(\bar{v}_2 - a_1^2\bar{u}_2) \quad 0 \quad 0 \quad -a_1^2 \quad 1\,). \tag{4.117}$$

So $D(Q_1)$ has rank two. Hence Q_1 is a regular point.

Let $Q_2 = (a_1, 0, 0, 1) \in S$. Then

$$\text{Moje}(Q_2, f_1) = -\alpha_1^2 u_2\pi - \alpha_1^2 T_2 + (1 - 2\beta_2)S_2 \tag{4.118}$$

where $\beta_2 \equiv 1(\text{mod}\,\pi)$ and $\text{Moje}(Q_2, f_2) = -\pi$, *ie*,

$$D(Q_2)_{r_1} = (\,-a_1^2\bar{u}_2 \quad 0 \quad 0 \quad -a_1^2 \quad -1\,). \tag{4.119}$$

So $D(Q_2)$ has rank two. Hence Q_2 is a regular point.

Let $Q_3 = (0, 0, a_2, 0) \in S$. Then

$$\text{Moje}(Q_3, f_1) = v_2\pi + S_2 \;\text{ and }\; \text{Moje}(Q_3, f_2) = -\pi, \tag{4.120}$$

ie,

$$D(Q_3)_{r_1} = (\,\bar{v}_2 \quad 0 \quad 0 \quad 0 \quad 1\,). \tag{4.121}$$

So $D(Q_3)$ has rank two. Hence Q_3 is a regular point.

Let $Q_4 = (0, 0, a_2, 1) \in S$. Then

$$\text{Moje}(Q_4, f_1) = (1 - 2\beta_2)S_2 \;\text{ and }\; \text{Moje}(Q_4, f_2) = -\pi, \tag{4.122}$$

ie,

$$D(Q_4)_{r_1} = (\,0 \quad 0 \quad 0 \quad 0 \quad -1\,). \tag{4.123}$$

So $D(Q_2)$ has rank two. Hence Q_2 is a regular point and as a result V_{212}^{000} is a regular scheme.

Chart 3.1.4

$$V_{212}^{001} = \text{Spec}\, R[X_1, Y_1, X_2, Y_2]/$$
$$(Y_1^2 - Y_2X_1^2X_2 - 1 + X_2\,,\; Y_2^8X_1^4X_2^2Y_1^2 - Y_2^9X_1^6X_2^3 - \pi). \tag{4.124}$$

From the first equation we get $Y_1^2 - Y_2X_1^2X_2 = 1 - X_2$. By using it in the other equation we get

$$V_{212}^{001} = \operatorname{Spec} R[X_1, Y_1, X_2, Y_2]/(Y_1^2 - Y_2X_1^2X_2 - 1 + X_2\,,\; Y_2^8X_1^4X_2^2(1 - X_2) - \pi). \tag{4.125}$$

Let $A = R[X_1, X_2, Y_2]/(Y_2^8X_1^4X_2^2(1 - X_2) - \pi)$. By using theorem (2.14) we can check that A is a regular ring. Note that

$$V_{212}^{001} = \operatorname{Spec} A[Y_1]/(Y_1^2 - Y_2X_1^2X_2 - 1 + X_2). \tag{4.126}$$

Let $\phi(Y_1) = Y_1^2 - Y_2X_1^2X_2 - 1 + X_2$. Then $\frac{\partial\phi}{\partial Y_1} = 2Y_1$. Since $p = char\, k \neq 2$, the system $\frac{\partial\phi}{\partial Y_1} = 0$ can be written as $2Y_1 = 0$. So V_{212}^{001} is etale over A everywhere except at those points where $Y_1 = 0$. To have the points on the special fibre of V_{212}^{001}, we need $Y_2^8X_1^4X_2^2Y_1^2 - Y_2^9X_1^6X_2^3 = 0$. Considering $\phi(Y_1) = \frac{\partial\phi}{\partial Y_1} = Y_2^8X_1^4X_2^2Y_1^2 - Y_2^9X_1^6X_2^3 = 0$, we can check that $(V_{212}^{001})^{\text{Sing}} \subset S \subset (V_{212}^{001})_\pi$ where S contains points of the form $(a_1, b_1, a_2, b_2) \in (V_{212}^{001})_\pi$ satisfying one of the following conditions:

(1) $b_1 = a_2 - 1 = b_2 = 0$;

(2) $a_1 = b_1 = a_2 - 1 = 0$.

Let $Q_1 = (a_1, 0, 1, 0) \in S$. Then

$$\text{Moje}(Q_1, f_1) = -\alpha_1^2\alpha_2 v_2\pi + T_2 - \alpha_1^2\alpha_2 S_2 \tag{4.127}$$

where $\alpha_2 \equiv 1 (\text{mod}\, \pi)$ and $\text{Moje}(Q_1, f_2) = -\pi$, *ie*,

$$D(Q_1)_{r_1} = (\, -a_1^2\bar{v}_2 \quad 0 \quad 0 \quad 1 \quad -a_1^2\,). \tag{4.128}$$

SO $D(Q_1)$ has rank two. Hence Q_1 is a regular point.

Let $Q_2 = (0, 0, 1, b_2) \in S$. Then

$$\text{Moje}(Q_2, f_1) = T_2 \text{ and } \text{Moje}(Q_2, f_2) = -\pi, \tag{4.129}$$

ie,

$$D(Q_2)_{r_1} = (\, 0 \quad 0 \quad 0 \quad 1 \quad 0\,). \tag{4.130}$$

So $D(Q_2)$ has rank two. Hence Q_2 is a regular point and as a result V_{212}^{001} is a regular scheme.

Recall that

$$V_{211}^{000} = \operatorname{Spec} R[X_1, Y_1, X_2, Y_2]/(Y_1^2 - X_1X_2 - Y_2^2 + X_2Y_2\,,\; X_1^8X_2^2Y_1^2 - X_1^9X_2^3 - \pi) \tag{4.131}$$

is singular and P_0 is its singular point. Now we blow-up P_0. By using the procedure of Mahtab we get four pieces for the covering of V_{211}^{000} as follows:

Chart 3.1.1.1

$$V_{2111}^{0000} = \operatorname{Spec} R[X_1, Y_1, X_2, Y_2]/(Y_1^2 - X_2 - Y_2^2 + X_2Y_2\,,\; X_1^{12}X_2^2Y_1^2 - X_1^{12}X_2^3 - \pi). \tag{4.132}$$

Let $A = R[X_1, Y_1, X_2]/(X_1^{12}X_2^2Y_1^2 - X_1^{12}X_2^3 - \pi)$. By using theorem (2.14) we can check that A is a regular ring. Note that

$$V_{2111}^{0000} = \operatorname{Spec} A[Y_2]/(Y_1^2 - X_2 - Y_2^2 + X_2Y_2). \tag{4.133}$$

Let $\phi(Y_2) = Y_1^2 - X_2 - Y_2^2 + X_2Y_2$. Then $\frac{\partial\phi}{\partial Y_2} = -2Y_2 + X_2$. Since $p = char\, k \neq 2$, the equation $\frac{\partial\phi}{\partial Y_2} = 0$ can be written as $X_2 - 2Y_2 = 0$. So V_{2111}^{0000} is etale over A everywhere except at those points where $X_2 - 2Y_2 = 0$. To have the points on the special fibre of V_{2111}^{0000}, we need $X_1^{12}X_2^2(Y_1^2 - X_2) = 0$. Considering $\phi(Y_2) = \frac{\partial\phi}{\partial Y_2} = X_1^{12}X_2^2(Y_1^2 - X_2) = 0$, we can check that $(V_{2111}^{0000})^{\text{Sing}} \subset S \subset (V_{2111}^{0000})_\pi$ where S contains points of the form $(a_1, b_1, a_2, b_2) \in (V_{2111}^{0000})_\pi$ satisfying one of the following conditions:

(1) $a_1 = a_2 - 2b_2 = 4b_1^2 - 4a_2 + a_2^2 = 0;$

(2) $b_1 = a_2 = b_2 = 0.$

Let $Q_1 = (0, b_1, a_2, b_2) \in S$ such that $a_2 - 2b_2 = 4b_1^2 - 4a_2 + a_2^2 = 0$. Then

$$\operatorname{Moje}(Q_1, f_1) = 2\beta_1S_1 + (\beta_2 - 1)T_2 + (\alpha_2 - 2\beta_2)S_2 \tag{4.134}$$

and $\operatorname{Moje}(Q_1, f_2) = -\pi$, *ie*,

$$D(Q_1)_{r_1} = (\,0 \quad 0 \quad 2b_1 \quad b_2 - 1 \quad a_2 - 2b_2\,). \tag{4.135}$$

So Q_1 is singular if $2b_1 = b_2 - 1 = 4b_1^2 - 4a_2 + a_2^2 = 0$ which is impossible. Hence Q_1 is a regular point.

Let $Q_2 = (a_1, 0, 0, 0) \in S$. Then

$$\text{Moje}(Q_2, f_1) = -u_2\pi - T_2 \text{ and } \text{Moje}(Q_2, f_2) = -\pi, \tag{4.136}$$

ie,

$$D(Q_2)_{r_1} = (\, -\bar{u}_2 \quad 0 \quad 0 \quad -1 \quad 0\,). \tag{4.137}$$

So $D(Q_2)$ has rank two. Hence Q_2 is a regular point and as a result V_{2111}^{0000} is a regular scheme.

Chart 3.1.1.2

$$\begin{aligned} V_{2111}^{0001} &= \text{Spec}\, R[X_1, Y_1, X_2, Y_2]/ \\ &(1 - X_1X_2 - Y_2^2 + X_2Y_2, Y_1^{12}X_1^8X_2^2 - Y_1^{12}X_1^9X_2^3 - \pi). \end{aligned} \tag{4.138}$$

Suppose that $A = R[X_1, Y_1, X_2]/(Y_1^{12}X_1^8X_2^2(1 - X_1X_2) - \pi)$. Considering theorem (2.14) we can check that A is a regular ring. Notice that

$$V_{2111}^{0001} = \text{Spec}\, A[Y_2]/(1 - X_1X_2 - Y_2^2 + X_2Y_2). \tag{4.139}$$

Let $\phi(Y_2) = 1 - X_1X_2 - Y_2^2 + X_2Y_2$. Then $\frac{\partial\phi}{\partial Y_2} = -2Y_2 + X_2$. Since $p = char\, k \neq 2$, the equation $\frac{\partial\phi}{\partial Y_2} = 0$ can be written as $X_2 - 2Y_2 = 0$. So V_{2111}^{0001} is etale over A everywhere except at those points where $X_2 - 2Y_2 = 0$. To have the points on the special fibre of V_{2111}^{0001}, we need $Y_1^{12}X_1^8X_2^2(1 - X_1X_2) = 0$. Considering $\phi(Y_2) = \frac{\partial\phi}{\partial Y_2} = Y_1^{12}X_1^8X_2^2(1 - X_1X_2) = 0$, we can check that $(V_{2111}^{0001})^{\text{Sing}} \subset S \subset (V_{2111}^{0001})_\pi$ where S contains points of the form $(a_1, b_1, a_2, b_2) \in (V_{2111}^{0001})_\pi$ satisfying one of the following conditions:

(1) $b_1 = a_2 - 2b_2 = 4a_1a_2 - 4 - a_2^2 = 0$;

(2) $a_1 = a_2 - 2b_2 = a_2^2 + 4 = 0$.

Let $Q_1 = (a_1, 0, a_2, b_2) \in S$ such that $a_2 - 2b_2 = 4a_1a_2 - 4 - a_2^2 = 0$. Then

$$\text{Moje}(Q_1, f_1) = -\alpha_2T_1 + (\beta_2 - \alpha_1)T_2 + (\alpha_2 - 2\beta_2)S_2, \tag{4.140}$$

and $\text{Moje}(Q_1, f_2) = -\pi$, *ie*,

$$D(Q_1)_{r_1} = (\, 0 \quad -a_2 \quad 0 \quad b_2 - a_1 \quad 0\,). \tag{4.141}$$

So Q_1 is singular if $-a_2 = b_2 - a_1 = a_2 - 2b_2 = 4a_1a_2 - 4 - a_2^2 = 0$ which is impossible. Hence Q_1 is a regular point.

Let $Q_2 = (0, b_1, a_2, b_2) \in S$ such that $a_2 - 2b_2 = a_2^2 + 4 = 0$. Then

$$\text{Moje}(Q_2, f_1) = -\alpha_2 u_1 \pi - \alpha_2 T_1 + \beta_2 T_2 + (\alpha_2 - 2\beta_2) S_2 \tag{4.142}$$

and $\text{Moje}(Q_2, f_2) = -\pi$, *ie*,

$$D(Q_2)_{r_1} = (\, -a_2\bar{u}_1 \quad -a_2 \quad 0 \quad b_2 \quad 0\,). \tag{4.143}$$

So Q_2 is singular if $-a_2 = b_2 = a_2 - 2b_2 = a_2^2 + 4 = 0$ which is impossible. Hence Q_2 is a regular point and as a result V_{2111}^{0001} is a regular scheme.

Chart 3.1.1.3

$$\begin{aligned} V_{2112}^{0000} &= \operatorname{Spec} R[X_1, Y_1, X_2, Y_2]/ \\ &(Y_1^2 - X_1 - Y_2^2 + Y_2\,,\ X_2^{12}X_1^8Y_1^2 - X_2^{12}X_1^9 - \pi). \end{aligned} \tag{4.144}$$

Suppose that $A = R[X_1, Y_1, X_2]/(X_2^{12}X_1^8(Y_1^2 - X_1) - \pi)$. We can use theorem (2.14) to prove that A is a regular ring. Notice that

$$V_{2112}^{0000} = \operatorname{Spec} A[Y_2]/(Y_1^2 - X_1 - Y_2^2 + Y_2). \tag{4.145}$$

Let $\phi(Y_2) = Y_1^2 - X_1 - Y_2^2 + Y_2$. Then $\frac{\partial\phi}{\partial Y_2} = -2Y_2 + 1$. Since $p = char\, k \neq 2$, the equation $\frac{\partial\phi}{\partial Y_2} = 0$ can be written as $2Y_2 - 1 = 0$. So V_{2112}^{0000} is etale over A everywhere except at those points where $2Y_2 - 1 = 0$. To have the points on the special fibre of V_{2112}^{0000}, we need $X_2^{12}X_1^8(Y_1^2 - X_1) = 0$. Considering $\phi(Y_2) = \frac{\partial\phi}{\partial Y_2} = X_2^{12}X_1^8(Y_1^2 - X_1) = 0$, we can check that $(V_{2112}^{0000})^{\text{Sing}} \subset S \subset (V_{2112}^{0000})_\pi$ where S contains points of the form $(a_1, b_1, a_2, b_2) \in (V_{2112}^{0000})_\pi$ satisfying one of the following conditions:

(1) $\quad a_2 = b_2 - \frac{1}{2} = a_1 - b_1^2 - \frac{1}{4} = 0$;

(2) $a_1 = b_2 - \frac{1}{2} = 4b_1^2 + 1 = 0.$

Let $Q_1 = (a_1, b_1, 0, b_2) \in S$ such that $b_2 - \frac{1}{2} = a_1 - b_1^2 - \frac{1}{4} = 0$. Then

$$\text{Moje}(Q_1, f_1) = -T_1 + 2\beta_1 S_1 + (1 - 2\beta_2) S_2 \tag{4.146}$$

and $\text{Moje}(Q_1, f_2) = -\pi$, *ie*,

$$D(Q_1)_{r_1} = (\,0 \quad -1 \quad 2b_1 \quad 0 \quad 0\,). \tag{4.147}$$

So *rank* $D(Q_1) = 2$. Hence Q_1 is a regular point.

Let $Q_2 = (0, b_1, a_2, \frac{1}{2}) \in S$ such that $b_2 - \frac{1}{2} = 4b_1^2 + 1 = 0$. Then

$$\text{Moje}(Q_2, f_1) = -u_1\pi - T_1 + 2\beta_1 S_1 + (1 - 2\beta_2) S_2 \tag{4.148}$$

where $\beta_2 \equiv \frac{1}{2}(\text{mod}\,\pi)$ and $\text{Moje}(Q_2, f_2) = -\pi$, *ie*,

$$D(Q_2)_{r_1} = (\,-\bar{u}_1 \quad -1 \quad 2b_1 \quad 0 \quad 0\,). \tag{4.149}$$

So $D(Q_2)$ has rank two. Hence Q_2 is a regular point and as a result V_{2112}^{0000} is a regular scheme.

Chart 3.1.1.4

$$\begin{aligned} V_{2112}^{0001} = &\text{ Spec } R[X_1, Y_1, X_2, Y_2]/ \\ &(Y_1^2 - X_1X_2 - 1 + X_2\,,\ Y_2^{12}X_1^8X_2^2Y_1^2 - Y_2^{12}X_1^9X_2^3 - \pi). \end{aligned} \tag{4.150}$$

From the first equation we get $Y_1^2 - X_1X_2 = 1 - X_2$. By using it in the other equation we get

$$V_{2112}^{0001} = \text{Spec } R[X_1, Y_1, X_2, Y_2]/(Y_1^2 - X_1X_2 - 1 + X_2\,,\ Y_2^{12}X_1^8X_2^2(1 - X_2) - \pi). \tag{4.151}$$

Let $A = R[X_1, X_2, Y_2]/(Y_2^{12}X_1^8X_2^2(1 - X_2) - \pi)$. By using theorem (2.14) we can check that A is a regular ring. Note that

$$V_{2112}^{0001} = \text{Spec } A[Y_1]/(Y_1^2 - X_1X_2 - 1 + X_2). \tag{4.152}$$

Let $\phi(Y_1) = Y_1^2 - X_1X_2 - 1 + X_2$. Then $\frac{\partial\phi}{\partial Y_1} = 2Y_1$. Since $p = char\, k \neq 2$, the equation $\frac{\partial\phi}{\partial Y_1} = 0$ can be written as $2Y_1 = 0$. So V_{2112}^{0001} is etale over A everywhere except at those points where $Y_1 = 0$. To have the points on the special fibre of V_{2112}^{0001}, we need $Y_2^{12}X_1^8X_2^2Y_1^2 - Y_2^{12}X_1^9X_2^3 = 0$. Considering $\phi(Y_1) = \frac{\partial\phi}{\partial Y_1} = Y_2^{12}X_1^8X_2^2Y_1^2 - Y_2^{12}X_1^9X_2^3 = 0$, we can check that $(V_{2112}^{0001})^{\text{Sing}} \subset S \subset (V_{2112}^{0001})_\pi$ where S contains points of the form $(a_1, b_1, a_2, b_2) \in (V_{2112}^{0001})_\pi$ satisfying one of the following conditions:

(1) $\quad b_1 = b_2 = a_1a_2 + 1 - a_2 = 0$;

(2) $\quad a_1 = b_1 = a_2 - 1 = 0$.

Let $Q_1 = (a_1, 0, a_2, 0) \in S$ such that $a_1a_2 + 1 - a_2 = 0$. Then $\text{Moje}(Q_1, f_1) = -\alpha_2T_1 + (1-\alpha_1)T_2$ and $\text{Moje}(Q_1, f_2) = -\pi$, *ie*,

$$D(Q)_{r_1} = (\,0 \quad -a_2 \quad 0 \quad 1-a_1 \quad 0\,). \tag{4.153}$$

So Q_1 is singular if $-a_2 = 1 - a_1 = a_1a_2 + 1 - a_2 = 0$ which is impossible. Hence Q_1 is a regular point.

Let $Q_2 = (0, 0, 1, b_2) \in S$. Then $\text{Moje}(Q_2, f_1) = -\alpha_2u_1\pi - \alpha_2T_1 + T_2$ where $\alpha_2 \equiv 1 (\text{mod}\, \pi)$ and $\text{Moje}(Q_2, f_2) = -\pi$, *ie*,

$$D(Q_2)_{r_1} = (\,-a_2\bar{u}_1 \quad -a_2 \quad 0 \quad 1 \quad 0\,). \tag{4.154}$$

So $rank\ D(Q_2) = 2$. Hence Q_2 is a regular point and as a result V_{2112}^{0001} is a regular scheme. The gluing of the regular schemes which we have had so far, gives us the regular scheme $\widetilde{X}$. □

Lemma 4.9 Let $p = char\, k \neq 2, 3$ and $V_1 = \text{Spec}\, R[x,y]/(y^2 - x^3 - \pi)$ and $V_2 = \text{Spec}\, R[x,y]/(x^2y - xy^2 - \pi)$. Then $X = V_1 \times_{\text{Spec}R} V_2$ is singular and after some blowings-up we get a regular scheme $\widetilde{X}$.

Proof: We have

$$X = \text{Spec}\, R[x_1, y_1, x_2, y_2]/(y_1^2 - x_1^3 - x_2^2y_2 + x_2y_2^2\,,\, y_1^2 - x_1^3 - \pi). \tag{4.155}$$

By using corollary (2.11) we can check that $P_0 \in X_\pi$ is the only singular point of X. Now we blow-up P_0. By using the procedure of Mahtab we get four pieces for the covering of $\widetilde{X}$ as follows:

Chart 1

$$V_1^0 = \text{Spec}\, R[X_1, Y_1, X_2, Y_2]/(Y_1^2 - X_1 - X_1X_2^2Y_2 + X_1X_2Y_2^2\,,\, X_1^2Y_1^2 - X_1^3 - \pi). \tag{4.156}$$

Assume that $A = R[X_1, Y_1]/(X_1^2(Y_1^2 - X_1) - \pi)$. We can check that A is a regular ring (by using theorem (2.14)). Note that

$$V_1^0 = \text{Spec}\, A[X_2, Y_2]/((Y_1^2 - X_1 - X_1X_2^2Y_2 + X_1X_2Y_2^2). \tag{4.157}$$

Let $\phi(X_2, Y_2) = Y_1^2 - X_1 - X_1X_2^2Y_2 + X_1X_2Y_2^2$. Then $\frac{\partial\phi}{\partial X_2} = -2X_1X_2Y_2 + X_1Y_2^2$ and $\frac{\partial\phi}{\partial Y_2} = -X_1X_2^2 + 2X_1X_2Y_2$. Since $p = char\, k \neq 2$, the system $\frac{\partial\phi}{\partial X_2} = \frac{\partial\phi}{\partial Y_2} = 0$ can be written as $X_1Y_2(Y_2 - 2X_2) = X_1X_2(2Y_2 - X_2) = 0$. So V_1^0 is smooth over A everywhere except at the points where $X_1Y_2(Y_2 - 2X_2) = X_1X_2(2Y_2 - X_2) = 0$. To have the points on the special fibre of V_1^0, we need $X_1^2(Y_1^2 - X_1) = 0$. Considering $\phi(X_2, Y_2) = \frac{\partial\phi}{\partial X_2} = \frac{\partial\phi}{\partial Y_2} = X_1^2(Y_1^2 - X_1) = 0$ we get $(V_1^0)^{\text{Sing}} \subset S \subset (V_1^0)_\pi$ where S contains points of the form $(a_1, b_1, a_2, b_2) \in (V_1^0)_\pi$ satisfying one of the following conditions:

(1) $a_1 = b_1 = 0$;

(2) $a_2 = b_2 = a_1 - b_1^2 = 0$.

Let $Q_1 = (0, 0, a_2, b_2) \in S$. Then

$$\text{Moje}(Q_1, f_1) = (\alpha_2\beta_2^2 - \alpha_2^2\beta_2 - 1)u_1\pi + (\alpha_2\beta_2^2 - \alpha_2^2\beta_2 - 1)T_1 \tag{4.158}$$

and $\text{Moje}(Q_1, f_2) = -\pi$, *ie*,

$$D(Q_1)_{r_1} = (\, (a_2b_2^2 - a_2^2b_2 - 1)\bar{u}_1 \quad (a_2b_2^2 - a_2^2b_2 - 1) \quad 0 \quad 0 \quad 0\,). \tag{4.159}$$

So Q_1 is singular if $a_2b_2^2 - a_2^2b_2 - 1 = 0$.

Let $Q_2 = (a_1, b_1, 0, 0) \in S$ such that $a_1 - b_1^2 = 0$. Then $\text{Moje}(Q_2, f_1) = -T_1 + 2\beta_1S_1$ and

$$\text{Moje}(Q_2, f_2) = (2\alpha_1\beta_1^2 - 3\alpha_1^2)T_1 + 2\alpha_1^2\beta_1S_1 - \pi, \tag{4.160}$$

ie,

$$D(Q_2) = \begin{pmatrix} 0 & -1 & 2b_1 & 0 & 0 \\ -1 & 2a_1b_1^2 - 3a_1^2 & 2a_1^2b_1 & 0 & 0 \end{pmatrix}. \tag{4.161}$$

So $D(Q_2)$ has rank two. Hence Q_2 is a regular point. This means that

$$(V_1^0)^{\text{Sing}} = \{(a_1, b_1, a_2, b_2) \in (V_1^0)_\pi \mid a_1 = b_1 = a_2b_2^2 - a_2^2b_2 - 1 = 0\}.$$

Now we blow-up $(V_1^0)^{\text{Sing}}$. By using the procedure of Mahtab we get three pieces for the covering of $\widetilde{V_1^0}$ as follows:

Chart 1.1

$$\begin{aligned}
&V_{11}^{00} = \operatorname{Spec} R[x_1, \frac{y_1}{x_1}, x_2, y_2, \frac{x_2y_2^2 - x_2^2y_2 - 1}{x_1}]/((\frac{y_1}{x_1})^2 + (\frac{x_2y_2^2 - x_2^2y_2 - 1}{x_1}),\\
&x_1^4(\frac{y_1}{x_1})^2 - x_1^3 - \pi) = \operatorname{Spec} R[X_1, Y_1, X_2, Y_2, Z]/(Y_1^2 + Z\,,\, X_1^4Y_1^2 - X_1^3 - \pi\,,\\
&X_1Z - X_2Y_2^2 + X_2^2Y_2 + 1) = \operatorname{Spec} R[X_1, Y_1, X_2, Y_2]/\\
&(-X_1Y_1^2 - X_2Y_2^2 + X_2^2Y_2 + 1\,,\, X_1^4Y_1^2 - X_1^3 - \pi).
\end{aligned} \tag{4.162}$$

Let $A = R[X_1, Y_1]/(X_1^3(X_1Y_1^2 - 1) - \pi)$. By using theorem (2.14) we can check that A is a regular ring. Note that

$$V_{11}^{00} = \operatorname{Spec} A[X_2, Y_2]/(-X_1Y_1^2 - X_2Y_2^2 + X_2^2Y_2 + 1). \tag{4.163}$$

Let $\phi(X_2, Y_2) = -X_1Y_1^2 - X_2Y_2^2 + X_2^2Y_2 + 1$. Then $\frac{\partial\phi}{\partial X_2} = -Y_2^2 + 2X_2Y_2$ and $\frac{\partial\phi}{\partial Y_2} = -2X_2Y_2 + X_2^2$. Since $p = char\, k \neq 2$, the system $\frac{\partial\phi}{\partial X_2} = \frac{\partial\phi}{\partial Y_2} = 0$ can be written as $Y_2(2X_2 - Y_2) = X_2(X_2 - 2Y_2) = 0$. So V_{11}^{00} is smooth over A everywhere except at the points where $Y_2(2X_2 - Y_2) = X_2(X_2 - 2Y_2) = 0$. To have the points on the special fibre of V_{11}^{00}, we need $X_1^3(X_1Y_1^2 - 1) = 0$. Considering $\phi(X_2, Y_2) = \frac{\partial\phi}{\partial X_2} = \frac{\partial\phi}{\partial Y_2} = X_1^3(X_1Y_1^2 - 1) = 0$, we can check that $(V_{11}^{00})^{\text{Sing}} \subset S \subset (V_{11}^{00})_\pi$ where S contains points of the form $(a_1, b_1, a_2, b_2) \in (V_{11}^{00})_\pi$ satisfying the following condition:

$$a_2 = b_2 = a_1b_1^2 - 1 = 0.$$

Let $Q = (a_1, b_1, 0, 0) \in S$ such that $a_1b_1^2 = 0$. Then $\text{Moje}(Q, f_1) = -\beta_1^2T_1 - 2\alpha_1\beta_1S_1$ and

$$\text{Moje}(Q, f_2) = (4\alpha_1^3\beta_1^2 - 3\alpha_1^2)T_1 + 2\alpha_1^4\beta_1S_1 - \pi, \tag{4.164}$$

ie,

$$D(Q) = \begin{pmatrix} 0 & -b_1^2 & -2a_1b_1 & 0 & 0 \\ -1 & (4a_1^3b_1^2 - 3a_1^2) & 2a_1^4b_1 & 0 & 0 \end{pmatrix}. \tag{4.165}$$

So Q is singular if $-b_1^2 = -2a_1b_1 = a_1b_1^2 - 1 = 0$ which is impossible. Hence Q is a regular point and as a result V_{11}^{00} is a regular scheme.

Chart 1.2

$$V_{11}^{01} = \text{Spec } R[\frac{x_1}{y_1}, y_1, x_2, y_2, \frac{x_2y_2^2 - x_2^2y_2 - 1}{y_1}]/(1 + \frac{x_1}{y_1}(\frac{x_2y_2^2 - x_2^2y_2 - 1}{y_1}),$$
$$y_1^4(\frac{x_1}{y_1})^2 - y_1^3(\frac{x_1}{y_1})^3 - \pi) = \text{Spec } R[X_1, Y_1, X_2, Y_2, Z]/(1 + X_1Z,$$
$$Y_1^4X_1^2 - Y_1^3X_1^3 - \pi, Y_1Z - X_2Y_2^2 + X_2^2Y_2 + 1). \tag{4.166}$$

Suppose that $A = R[X_1, Y_1]/(X_1^2Y_1^3(Y_1 - X_1) - \pi)$. By using theorem (2.14) we can check that A is a regular ring. Notice that

$$V_{11}^{01} = \text{Spec } A[Z, X_2, Y_2]/(1 + X_1Z, Y_1Z - X_2Y_2^2 + X_2^2Y_2 + 1) =$$
$$\text{Spec } A[X_1^{-1}][X_2, Y_2]/(Y_1 + X_1X_2Y_2^2 - X_1X_2^2Y_2 - X_1). \tag{4.167}$$

Note that $B = A[X_1^{-1}] = A[Z]/(1 + X_1Z)$ and Spec B is etale over A. So $A[X_1^{-1}]$ is a regular ring.

Let $\phi(X_2, Y_2) = Y_1 + X_1X_2Y_2^2 - X_1X_2^2Y_2 - X_1$. Then $\frac{\partial\phi}{\partial X_2} = X_1Y_2^2 - 2X_1X_2Y_2$ and $\frac{\partial\phi}{\partial Y_2} = 2X_1X_2Y_2 - X_1X_2^2$. Since $p = char\, k \neq 2$, and $X_1 \neq 0$, the system $\frac{\partial\phi}{\partial X_2} = \frac{\partial\phi}{\partial Y_2} = 0$ can be written as $Y_2^2 - 2X_2Y_2 = 2X_2Y_2 - X_2^2 = 0$. So V_{11}^{01} is smooth over $A[X_1^{-1}]$ everywhere except at the points where $Y_2^2 - 2X_2Y_2 = 2X_2Y_2 - X_2^2 = 0$. To have the points on the special fibre of V_{11}^{01}, we need $X_1^2Y_1^3(Y_1 - X_1) = 0$. Considering $\phi(X_2, Y_2) = \frac{\partial\phi}{\partial X_2} = \frac{\partial\phi}{\partial Y_2} = X_1^2Y_1^3(Y_1 - X_1) = X_1Z + 1 = 0$, we can check that $(V_{11}^{01})^{\text{Sing}} \subset S \subset (V_{11}^{01})_\pi$ where S contains points of the form $(a_1, b_1, a_2, b_2, c) \in (V_{11}^{01})_\pi$ satisfying the following condition:

$$a_1 - b_1 = a_2 = b_2 = a_1c + 1 = 0.$$

Let $Q = (a_1, b_1, 0, 0, c) \in S$. Then Moje$(Q, f_1) = \alpha_1W + \gamma T_1$ and

$$\text{Moje}(Q, f_2) = (2\alpha_1\beta_1^4 - 3\alpha_1^2\beta_1^3)T_1 + (4\alpha_1^2\beta_1^3 - 3\alpha_1^3\beta_1^2)S_1 - \pi \tag{4.168}$$

and Moje$(Q, f_3) = \gamma S_1 + \beta_1 W$, *ie*,

$$D(Q) = \begin{pmatrix} 0 & c & 0 & 0 & 0 & a_1 \\ -1 & -a_1^5 & a_1^5 & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & b_1 \end{pmatrix}. \tag{4.169}$$

The determinant consisting of the first three columns of $D(Q)$ is equal to $c^2 = a^{-2} \neq 0$. So $rank D(Q) = 2$. Hence Q is a regular point and as a result V_{11}^{01} is a regular scheme.

Chart 1.5

Let $Z = x_2y_2^2 - x_2^2y_2 - 1$. Then

$$V_{1\theta}^{0\theta} = \operatorname{Spec} R[\frac{x_1}{Z}, \frac{y_1}{Z}, x_2, y_2, Z]/((\frac{y_1}{Z})^2 + \frac{x_1}{Z}, Z^4(\frac{x_1}{Z})^2(\frac{y_1}{Z})^2 - Z^3(\frac{x_1}{Z})^3 - \pi,$$
$$Z - x_2y_2^2 + x_2^2y_2 + 1) = \operatorname{Spec} R[X_1, Y_1, X_2, Y_2, Z]/(Y_1^2 + X_1, Z^4X_1^2Y_1^2 -$$
$$Z^3X_1^3 - \pi, Z - X_2Y_2^2 + X_2^2Y_2 + 1) = \operatorname{Spec} R[Y_1, X_2, Y_2, Z]/$$
$$(Z - X_2Y_2^2 + X_2^2Y_2^2 + 1, Z^4Y_1^6 - Z^3Y_1^6 - \pi) = \operatorname{Spec} R[Y_1, X_2, Y_2]/$$
$$((X_2Y_2^2 - X_2^2Y_2 - 1)^3Y_1^6(X_2Y_2^2 - X_2^2Y_2 - 2) - \pi). \tag{4.170}$$

Since $X_2Y_2^2 - X_2^2Y_2 - 1 = 0$ and $X_2Y_2^2 - X_2^2Y_2 - 2 = 0$ in $\mathbf{A}_k^3$ define non-singular k-varieties (by Jacobian criterion), by using theorem (2.14) we conclude that $V_{1\theta}^{0\theta}$ is a regular scheme.

Chart 2

$$V_1^1 = \operatorname{Spec} R[\frac{x_1}{y_1}, y_1, \frac{x_2}{y_1}, \frac{y_2}{y_1}]/(1 - y_1(\frac{x_1}{y_1})^3 - y_1(\frac{x_2}{y_1})^2(\frac{y_2}{y_1}) + y_1(\frac{x_2}{y_1})(\frac{y_2}{y_1})^2,$$
$$y_1^2 - y_1^3(\frac{x_1}{y_1})^3 - \pi) = \operatorname{Spec} R[X_1, Y_1, X_2, Y_2]/$$
$$(1 - Y_1X_1^3 - Y_1X_2^2Y_2 + Y_1X_2Y_2^2, Y_1^2 - Y_1^3X_1^3 - \pi). \tag{4.171}$$

Now let $A = R[X_1, Y_1]/(Y_1^2(1 - Y_1X_1^3) - \pi)$. We can use theorem (2.14) to show that A is a regular ring. Notice that

$$V_1^1 = \operatorname{Spec} A[X_2, Y_2]/(1 - Y_1X_1^3 - Y_1X_2^2Y_2 + Y_1X_2Y_2^2). \tag{4.172}$$

Let $\phi(X_2, Y_2) = 1 - Y_1X_1^3 - Y_1X_2^2Y_2 + Y_1X_2Y_2^2$. Then $\frac{\partial\phi}{\partial X_2} = -2Y_1X_2Y_2 + Y_1Y_2^2$ and $\frac{\partial\phi}{\partial Y_2} = -Y_1X_2^2 + 2Y_1X_2Y_2$. Since $p = char\, k \neq 2$, we can write the system $\frac{\partial\phi}{\partial X_2} = \frac{\partial\phi}{\partial Y_2} = 0$ as $-2Y_1X_2Y_2 + Y_1Y_2^2 = -Y_1X_2^2 + 2Y_1X_2Y_2 = 0$. So V_1^1 is smooth over A everywhere except at the points where $-2Y_1X_2Y_2 + Y_1Y_2^2 = -Y_1X_2^2 + 2Y_1X_2Y_2 = 0$. To have the points on the special fibre of V_1^1, we need $Y_1^2(1 - Y_1X_1^3) = 0$. Considering $\phi(X_2, Y_2) = \frac{\partial\phi}{\partial X_2} = \frac{\partial\phi}{\partial Y_2} = Y_1^2(1 - Y_1X_1^3) = 0$, we can check that $(V_1^1)^{\text{Sing}} \subset S \subset (V_1^1)_\pi$ where S contains points of the form $(a_1, b_1, a_2, b_2) \in (V_1^1)_\pi$ satisfying the following condition:

$$a_2 = b_2 = a_1^3b_1 - 1 = 0.$$

Let $Q = (a_1, b_1, 0, 0) \in S$ such that $a_1^3b_1 - 1 = 0$. Then $\text{Moje}(Q, f_1) = -3\alpha_1^2\beta_1T_1 - \alpha_1^3S_1$ and

$$\text{Moje}(Q, f_2) = -3\alpha_1^2\beta_1^3T_1 + \beta_1(2 - 3\alpha_1^3\beta_1)S_1 - \pi, \tag{4.173}$$

ie,

$$D(Q) = \begin{pmatrix} 0 & -3a_1^2b_1 & -a_1^3 & 0 & 0 \\ -1 & -3a_1^2b_1^3 & b_1(2 - 3a_1^3b_1) & 0 & 0 \end{pmatrix}. \tag{4.174}$$

So Q is singular if $3a_1^2b_1 = a_1^3 = 6a_1^2b_1^2(a_1^3b_1 - 1) = a_1^3b_1 - 1 = 0$ which is impossible. Hence Q is a regular point and as a result V_1^1 is a regular scheme.

Chart 3

$$\begin{aligned} V_2^0 &= \text{Spec}\, R[\frac{x_1}{x_2}, \frac{y_1}{x_2}, x_2, \frac{y_2}{x_2}]/((\frac{y_1}{x_2})^2 - x_2(\frac{x_1}{x_2})^3 - x_2(\frac{y_2}{x_2}) + x_2(\frac{y_2}{x_2})^2, \\ &\quad x_2^2(\frac{y_1}{x_2})^2 - x_2^3(\frac{x_1}{x_2})^3 - \pi) = \text{Spec}\, R[X_1, Y_1, X_2, Y_2]/ \\ &\quad (Y_1^2 - X_2X_1^3 - X_2Y_2 + X_2Y_2^2, X_2^2Y_1^2 - X_2^3X_1^3 - \pi). \end{aligned} \tag{4.175}$$

From the first equation we get $Y_1^2 - X_2X_1^3 = X_2Y_2(1 - Y_2)$. By using it in the other equation, we can write

$$V_2^0 = \text{Spec}\, R[X_1, Y_1, X_2, Y_2]/(Y_1^2 - X_2X_1^3 - X_2Y_2 + X_2Y_2^2, X_2^3Y_2(1 - Y_2) - \pi). \tag{4.176}$$

Now assume that $A = R[X_2, Y_2]/(X_2^3 Y_2(1 - Y_2) - \pi)$. Considering theorem (2.14) we can check that A is a regular ring. Notice that

$$V_2^0 = \operatorname{Spec} A[X_1, Y_1]/(Y_1^2 - X_2X_1^3 - X_2Y_2 + X_2Y_2^2). \tag{4.177}$$

Let $\phi(X_1, Y_1) = Y_1^2 - X_2X_1^3 - X_2Y_2 + X_2Y_2^2$. Then $\frac{\partial \phi}{\partial X_1} = -3X_2X_1^2$ and $\frac{\partial \phi}{\partial Y_1} = 2Y_1$. Since $p = char\, k \neq 2, 3$, the system $\frac{\partial \phi}{\partial X_1} = \frac{\partial \phi}{\partial Y_1} = 0$ can be written as $-3X_2X_1^2 = 2Y_1 = 0$. So V_2^0 is smooth over A everywhere except at the points where $-3X_2X_1^2 = 2Y_1 = 0$. To have the points on the special fibre of V_2^0, we need $X_2^2Y_1^2 - X_2^3X_1^3 = 0$. Considering $\phi(X_1, Y_1) = \frac{\partial \phi}{\partial X_1} = \frac{\partial \phi}{\partial Y_1} = X_2^2Y_1^2 - X_2^3X_1^3 = 0$, we can check that $(V_2^0)^{\text{Sing}} \subset S \subset (V_2^0)_\pi$ where S contains points of the form $(a_1, b_1, a_2, b_2) \in (V_2^0)_\pi$ satisfying one of the following conditions:

(1) $b_1 = a_2 = 0$;

(2) $a_1 = b_1 = b_2 = 0$;

(3) $a_1 = b_1 = b_2 - 1 = 0$.

Let $Q = (a_1, 0, 0, b_2) \in S$. Then

$$\text{Moje}(Q_1, f_1) = (\beta_2^2 - \beta_2 - \alpha_1^3)u_2\pi + (\beta_2^2 - \beta_2 - \alpha_1^3)T_2 \tag{4.178}$$

and $\text{Moje}(Q_1, f_2) = -\pi$, *ie*,

$$D(Q_1)_{r_1} = (\, (b_2^2 - b_2 - a_1^3)\bar{u}_2 \quad 0 \quad 0 \quad b_2^2 - b_2 - a_1^3 \quad 0\,). \tag{4.179}$$

So Q_1 is singular if $b_2^2 - b_2 - a_1^3 = 0$.

Let $Q_2 = (0, 0, a_2, 0) \in S$. Then

$$\text{Moje}(Q_2, f_1) = -\alpha_2 v_2 \pi - \alpha_2 S_2 \text{ and } \text{Moje}(Q_2, f_2) = -\pi, \tag{4.180}$$

ie,

$$D(Q_2)_{r_1} = (\, -a_2\bar{v}_2 \quad 0 \quad 0 \quad 0 \quad -a_2\,). \tag{4.181}$$

So $D(Q_2)$ is singular if $a_2 = 0$ *ie* $Q_2 = P_0$. Note that P_0 satisfies the conditions $b_1 = a_2 = b_2^2 - b_2 - a_1^3 = 0$.

Finally let $Q_3 = (0, 0, a_2, 1) \in S$. Then

$$\text{Moje}(Q_3, f_1) = \beta_2(\beta_2 - 1)T_2 + \alpha_2(2\beta_2 - 1)S_2 \tag{4.182}$$

and Moje$(Q_3, f_2) = -\pi$, *ie*,

$$D(Q_3)_{r_1} = (\,0 \quad 0 \quad 0 \quad 0 \quad a_2\,). \tag{4.183}$$

So Q_3 is singular if $a_2 = 0$, *ie*, $Q_3 = (0,0,0,1)$. Note that Q_3 also satisfies the conditions $b_1 = a_2 = b_2^2 - b_2 - a_1^3 = 0$. Hence

$$(V_2^0)^{\mathrm{Sing}} = \{(a_1, b_1, a_2, b_2) \in (V_2^0)_\pi \mid b_1 = a_2 = a_1^3 + b_2 - b_2^2 = 0\}. \tag{4.184}$$

Now we blow-up $(V_2^0)^{\mathrm{Sing}}$. By using the procedure of Mahtab we get three pieces for the covering of $\widetilde{V_2^0}$ as follows:

Chart 3.2

$$\begin{aligned} V_{21}^{01} &= \operatorname{Spec} R[x_1, y_1, \frac{x_2}{y_1}, \frac{x_1^3 + y_2 - y_2^2}{y_1}]/(1 - \frac{x_2}{y_1}(\frac{x_1^3 + y_2 - y_2^2}{y_1}), \\ &\quad y_1^4(\frac{x_2}{y_1})^2 - y_1^3(\frac{x_2}{y_1})^3 x_1^3 - \pi) = \operatorname{Spec} R[X_1, Y_1, X_2, Y_2, Z]/(1 - X_2 Z\,, \\ &\quad Y_1^4 X_2^2 - Y_1^3 X_2^3 X_1^3 - \pi\,,\; Y_1 Z - X_1^3 - Y_2 + Y_2^2). \end{aligned} \tag{4.185}$$

Suppose that $A = R[X_1, Y_1, X_2]/(Y_1^3 X_2^2(Y_1 - X_2 X_1^3) - \pi)$. We can check that A is a regular ring (by using theorem (2.14)). We can write

$$\begin{aligned} &V_{21}^{01} = \operatorname{Spec} A[Z, Y_2]/(1 - X_2 Z\,,\; Y_1 Z - X_1^3 - Y_2 + Y_2^2) = \\ &\operatorname{Spec} A[X_2^{-1}][Y_2]/(Y_1 - X_1^3 X_2 - X_2 Y_2 + X_2 Y_2^2). \end{aligned} \tag{4.186}$$

Note that $B = A[X_2^{-1}] = A[Z]/(1 - X_2 Z)]$ and $\operatorname{Spec} B$ is etale over A. So $A[X_2^{-1}]$ is a regular ring.

Now let $\phi(Y_2) = Y_1 - X_1^3 X_2 - X_2 Y_2 + X_2 Y_2^2$. Then $\frac{\partial \phi}{\partial Y_2} = -X_2 + 2X_2 Y_2$. Since $p = char\, k \neq 2$ and $X_2 \neq 0$ we get $2Y_2 - 1 = 0$. So V_{21}^{01} is etale over $A[X_2^{-1}]$ everywhere except at the points where $2Y_2 - 1 = 0$. To have the points on the special fibre of V_{21}^{01}, we need $Y_1^3 X_2^2(Y_1 - X_2 X_1^3) = 0$. Considering $\phi(Y_2) = \frac{\partial \phi}{\partial Y_2} = Y_1^3 X_2^2(Y_1 - X_2 X_1^3) = 1 - X_2 Z = 0$, we can check that $(V_{21}^{01})^{\mathrm{Sing}} \subset S \subset (V_{21}^{01})_\pi$ where S contains points of the form $(a_1, b_1, a_2, b_2, c) \in (V_{21}^{01})_\pi$ satisfying the following condition:

$$b_1 = b_2 - \frac{1}{2} = 1 + 4a_1^3 = ca_2 - 1 = 0.$$

Let $Q = (a_1, 0, a_2, \frac{1}{2}, c) \in S$. Then $\text{Moje}(Q, f_1) = -\gamma T_2 - \alpha_2 W$ and $\text{Moje}(Q, f_2) = -\pi$ and

$$\text{Moje}(Q, f_3) = \gamma v_1 \pi - 3\alpha_1^2 T_1 + \gamma S_1 + (-1 + 2\beta_2) S_2 \tag{4.187}$$

where $\beta_2 \equiv \frac{1}{2} (\text{mod}\, \pi)$, *ie*,

$$D(Q) = \begin{pmatrix} 0 & 0 & 0 & -c & 0 & -a_2 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ c\bar{v}_1 & -3a_1^2 & c & 0 & 0 & 0 \end{pmatrix}. \tag{4.188}$$

The 3×3 matrix consisting of the first, second and fourth columns of $D(Q)$ has determinant $-3a_1^2 c \neq 0$ (note that $ca_2 - 1 = 0$ and $1 + 4a_1^3 = 0$). So $D(Q)$ has rank three. Hence Q is a regular point and as a result V_{21}^{01} is a regular scheme.

Chart 3.3

$$V_{22}^{00} = \text{Spec}\, R[x_1, \frac{y_1}{x_2}, x_2, y_2, \frac{x_1^3 + y_2 - y_2^2}{x_2}]/((\frac{y_1}{x_2})^2 - (\frac{x_1^3 + y_2 - y_2^2}{x_2}),$$
$$x_2^4 (\frac{y_1}{x_2})^2 - x_2^3 x_1^3 - \pi) = \text{Spec}\, R[X_1, Y_1, X_2, Y_2, Z]/(Y_1^2 - Z,$$
$$X_2^4 Y_1^2 - X_2^3 X_1^3 - \pi,\ X_2 Z - X_1^3 - Y_2 + Y_2^2) = \text{Spec}\, R[X_1, Y_1, X_2, Y_2]/$$
$$(X_2 Y_1^2 - X_1^3 - Y_2 + Y_2^2,\ X_2^4 Y_1^2 - X_2^3 X_1^3 - \pi). \tag{4.189}$$

From the first equation we get $X_2 Y_1^2 - X_1^3 = Y_2 - Y_2^2$. By using it in the other equation we get

$$V_{22}^{00} = \text{Spec}\, R[X_1, Y_1, X_2, Y_2]/(X_2 Y_1^2 - X_1^3 - Y_2 + Y_2^2,\ X_2^3 Y_2 (1 - Y_2) - \pi). \tag{4.190}$$

Assume that $A = R[X_2, Y_2]/(X_2^3 Y_2 (1 - Y_2) - \pi)$. By using theorem (2.14) we can check that A is a regular ring. Note that

$$V_{22}^{00} = \text{Spec}\, A[X_1, Y_1]/(X_2 Y_1^2 - X_1^3 - Y_2 + Y_2^2). \tag{4.191}$$

Let $\phi(X_1, Y_1) = X_2 Y_1^2 - X_1^3 - Y_2 + Y_2^2$. Then $\frac{\partial \phi}{\partial X_1} = -3X_1^2$ and $\frac{\partial \phi}{\partial Y_1} = 2X_2 Y_1$. Since $p = char\, k \neq 2, 3$, the system $\frac{\partial \phi}{\partial X_1} = \frac{\partial \phi}{\partial Y_1} = 0$ can be written as $-3X_1^2 = 2X_2 Y_1 = 0$. So V_{22}^{00} is smooth over A everywhere except at the points where

$-3X_1^2 = 2X_2Y_1 = 0$. To have the points on the special fibre of V_{22}^{00}, we need $X_2^4Y_1^2 - X_2^3X_1^3 = 0$. Considering $\phi(X_1, Y_1) = \frac{\partial\phi}{\partial X_1} = \frac{\partial\phi}{\partial Y_1} = X_2^4Y_1^2 - X_2^3X_1^3 = 0$, we can check that $(V_{22}^{00})^{\mathrm{Sing}} \subset S \subset (V_{22}^{00})_\pi$ where S contains points of the form $(a_1, b_1, a_2, b_2) \in (V_{22}^{00})_\pi$ satisfying one of the following conditions:

(1) $a_1 = a_2 = b_2 = 0$;

(2) $a_1 = b_1 = b_2 = 0$;

(3) $a_1 = b_1 = b_2 - 1 = 0$;

(4) $a_1 = a_2 = b_2 - 1 = 0$.

Let $Q_1 = (0, b_1, 0, 0) \in S$. Then

$$\mathrm{Moje}(Q_1, f_1) = (\beta_1^2 u_2 - v_2)\pi + \beta_1^2 T_2 - S_2 \text{ and } \mathrm{Moje}(Q_1, f_2) = -\pi, \tag{4.192}$$

ie,

$$D(Q_1)_{r_1} = (\,(b_1^2\bar{u}_2 - \bar{v}_2) \quad 0 \quad 0 \quad b_1^2 \quad -1\,). \tag{4.193}$$

So $D(Q_1)$ has rank two. Hence Q_1 is a regular point.

Let $Q_2 = (0, 0, a_2, 0) \in S$. Then

$$\mathrm{Moje}(Q_2, f_1) = -v_2\pi - S_2 \text{ and } \mathrm{Moje}(Q_2, f_2) = -\pi, \tag{4.194}$$

ie,

$$D(Q_2)_{r_1} = (\,-\bar{v}_2 \quad 0 \quad 0 \quad 0 \quad -1\,). \tag{4.195}$$

So $D(Q_2)$ has rank two. Hence Q_2 is a regular point.

Let $Q_3 = (0, 0, a_2, 1) \in S$. Then $\mathrm{Moje}(Q_3, f_1) = (2\beta_2 - 1)S_2$ where $\beta_2 \equiv 1 \pmod{\pi}$ and $\mathrm{Moje}(Q_3, f_2) = -\pi$, *ie*,

$$D(Q_3) = (\,0 \quad 0 \quad 0 \quad 0 \quad 1\,) \tag{4.196}$$

which means that $D(Q_3)$ has rank two. Hence Q_3 is a regular point.

Let $Q_4 = (0, b_1, 0, 1) \in S$. Then $\mathrm{Moje}(Q_4, f_1) = \beta_1^2 u_2\pi + \beta_1^2 T_2 + (2\beta_2 - 1)S_2$ where $\beta_2 \equiv 1 (\mathrm{mod}\, \pi)$ and $\mathrm{Moje}(Q_4, f_2) = -\pi$, *ie*,

$$D(Q_4)_{r_1} = (\,b_1^2\bar{u}_2 \quad 0 \quad 0 \quad b_1^2 \quad 1\,). \tag{4.197}$$

So $D(Q_4)$ has rank two. Hence Q_4 is a regular point and as a result V_{22}^{00} is a regular scheme.

Chart 3.5

Let $Z = x_1^3 + y_2 - y_2^2$. Then

$$V_{2\theta}^{0\theta} = \text{Spec}\, R[x_1, \frac{y_1}{Z}, \frac{x_2}{Z}, y_2, Z]/((\frac{y_1}{Z})^2 - \frac{x_2}{Z}, Z^4(\frac{y_1}{Z})^2(\frac{x_2}{Z})^2 - Z^3(\frac{x_2}{Z})^3 x_1^3 - \pi, \\ Z - x_1^3 - y_2 + y_2^2) = \text{Spec}\, R[X_1, Y_1, X_2, Y_2, Z]/(Y_1^2 - X_2, Z - X_1^3 - Y_2 + Y_2^2, \\ Z^4 Y_1^2 X_2^2 - Z^3 X_2^3 X_1^3 - \pi) = \text{Spec}\, R[X_1, Y_1, Y_2, Z]/(Z - X_1^3 - Y_2 + Y_2^2, Y_1^6 Z^4 \\ - X_1^3 Y_1^6 Z^3 - \pi) = \text{Spec}\, R[X_1, Y_1, Y_2]/((Y_2 - Y_2^2 + X_1^3)^3 Y_1^6 (Y_2 - Y_2^2) - \pi) \tag{4.198}$$

Since $Y_2 - Y_2^2 + X_1^3 = 0$ and $Y_2 - Y_2^2 = 0$ in $\mathbb{A}_k^3$ define non-singular k-varieties, by using the theorem (2.14) we conclude that $V_{2\theta}^{0\theta}$ is a regular scheme.

Chart 4

$$V_2^1 = \text{Spec}\, R[\frac{x_1}{y_2}, \frac{y_1}{y_2}, \frac{x_2}{y_2}, y_2]/((\frac{y_1}{y_2})^2 - y_2(\frac{x_1}{y_2})^3 - y_2(\frac{x_2}{y_2})^2 + y_2(\frac{x_2}{y_2}), \\ y_2^2(\frac{y_1}{y_2})^2 - y_2^3(\frac{x_1}{y_2})^3 - \pi) = \text{Spec}\, R[X_1, Y_1, X_2, Y_2]/ \\ (Y_1^2 - Y_2 X_1^3 - Y_2 X_2^2 + Y_2 X_2, Y_2^2 Y_1^2 - Y_2^3 X_1^3 - \pi). \tag{4.199}$$

Recall that

$$V_2^0 = \text{Spec}\, R[X_1, Y_1, X_2, Y_2]/(Y_1^2 - X_2 X_1^3 - X_2 Y_2 + X_2 Y_2^2, X_2^2 Y_1^2 - X_2^3 X_1^3 - \pi). \tag{4.200}$$

The ring homomorphism

$$\alpha : \Gamma(V_2^0, \mathcal{O}_{V_2^0}) \longrightarrow \Gamma(V_2^1, \mathcal{O}_{V_2^1})$$

$$X_1 \longmapsto -X_1$$

$$Y_1 \longmapsto Y_1$$

$$X_2 \longmapsto -Y_2$$

$$Y_2 \longmapsto X_2$$

is an isomorphism and induces $V_2^1 \cong V_2^0$. Hence $\widetilde{V_2^1} \cong \widetilde{V_2^0}$. The gluing of the regular schemes which we have had so far, gives us the regular scheme $\widetilde{X}$. □

4.4 One component is I_3

In this section our arithmetic three-folds are the products of two arithmetic surfaces such that one of them is of the form I_3 and apart from the cases discussed before.

Lemma 4.10 Let $p = char\, k \neq 2\,,\, 3$ and $V_1 = V_2 = \text{Spec}\, R[x,y]/(y^2 - yx^2 - \pi)$. Then $X = V_1 \times_{\text{Spec} R} V_2$ is singular and after some blowings-up we get a regular scheme $\widetilde{X}$.

Proof: We have

$$X = \text{Spec}\, R[x_1, y_1, x_2, y_2]/(y_1^2 - y_2^2 - y_1 x_1^2 + y_2 x_2^2\,,\, y_1^2 - y_1 x_1^2 - \pi). \tag{4.201}$$

By using corollary (2.11) we can check that $P_0 = (0,0,0,0) \in X_\pi$ is the only singular point of X. We blow-up P_0. By using the procedure of Mahtab we get four pieces for the covering of $\widetilde{X_1}$ as follows:

Chart 1

$$V_1^0 = \text{Spec}\, R[X_1, Y_1, X_2, Y_2]/(Y_1^2 - Y_2^2 - X_1 Y_1 + X_1 Y_2 X_2^2\,,\, X_1^2 Y_1^2 - X_1^3 Y_1 - \pi). \tag{4.202}$$

Suppose that $A = R[X_1, Y_1]/(X_1^2 Y_1 (Y_1 - X_1) - \pi)$. We can use theorem (2.14) to show that A is a regular ring. Notice that

$$V_1^0 = \text{Spec}\, R[X_2, Y_2]/(Y_1^2 - Y_2^2 - X_1 Y_1 + X_1 Y_2 X_2^2). \tag{4.203}$$

Let $\phi(X_2, Y_2) = Y_1^2 - Y_2^2 - X_1 Y_1 + X_1 Y_2 X_2^2$. Then $\frac{\partial \phi}{\partial X_2} = 2X_1 Y_2 X_2$ and $\frac{\partial \phi}{\partial Y_2} = -2Y_2 + X_1 X_2^2$. Since $p = char\, k \neq 2$, the system $\frac{\partial \phi}{\partial X_2} = \frac{\partial \phi}{\partial Y_2} = 0$ can be written as $2X_1 Y_2 X_2 = -2Y_2 + X_1 X_2^2 = 0$. So V_1^0 is smooth over A everywhere except at the points where $2X_1 Y_2 X_2 = -2Y_2 + X_1 X_2^2 = 0$. To have the points on the special fibre of V_1^0, we need $X_1^2 Y_1 (Y_1 - X_1) = 0$. Considering $\phi(X_2, Y_2) = \frac{\partial \phi}{\partial X_2} = \frac{\partial \phi}{\partial Y_2} = X_1^2 Y_1 (Y_1 - X_1) = 0$, we get $(V_1^0)^{\text{Sing}} \subset S \subset (V_1^0)_\pi$, where S contains points of the form $(a_1, b_1, a_2, b_2) \in (V_1^0)_\pi$ satisfying one of the following conditions:

(1) $\quad a_1 = b_1 = b_2 = 0;$

(2) $b_1 = a_2 = b_2 = 0;$

(3) $a_1 - b_1 = a_2 = b_2 = 0.$

Let $Q_1 = (0, 0, a_2, 0) \in S$. Then $\text{Moje}(Q_1, f_1) = 0$ and $\text{Moje}(Q_1, f_2) = -\pi$. So $D(Q_1)$ has rank one hence Q_1 is a singular point.

Let $Q_2 = (a_1, 0, 0, 0) \in S$. Then $\text{Moje}(Q_2, f_1) = -\alpha_1 v_1 \pi - \alpha_1 S_1$ and

$$\text{Moje}(Q_2, f_2) = -(\alpha_1^3 v_1 + 1)\pi - \alpha_1^3 S_1, \tag{4.204}$$

ie,

$$D(Q_2) = \begin{pmatrix} -a_1\bar{v}_1 & 0 & -a_1 & 0 & 0 \\ -(a_1^3\bar{v}_1 + 1) & 0 & -a_1^3 & 0 & 0 \end{pmatrix}. \tag{4.205}$$

So Q_2 is singular if $(a_1\bar{v}_1)(a_1^3) - (a_1^3\bar{v}_1 + 1)a_1 = 0$ *ie* $a_1 = 0$ which means that $Q_2 = P_0$.

Let $Q_3 = (a_1, b_1, 0, 0) \in S$ such that $a_1 - b_1 = 0$. Then $\text{Moje}(Q_3, f_1) = -\beta_1 T_1 + (2\beta_1 - \alpha_1)S_1$ and

$$\text{Moje}(Q_3, f_2) = -\pi + \alpha_1(2\beta_1^2 - 3\alpha_1\beta_1)T_1 + \alpha_1^2(2\beta_1 - \alpha_1)S_1, \tag{4.206}$$

ie,

$$D(Q_3) = \begin{pmatrix} 0 & -b_1 & 2b_1 - a_1 & 0 & 0 \\ -1 & a_1(2b_1^2 - 3a_1b_1) & a_1^2(2b_1 - a_1) & 0 & 0 \end{pmatrix}. \tag{4.207}$$

Since $a_1 = b_1$ we get

$$D(Q_3) = \begin{pmatrix} 0 & -a_1 & a_1 & 0 & 0 \\ -1 & -a_1^3 & a_1^3 & 0 & 0 \end{pmatrix}. \tag{4.208}$$

So Q_3 is singular if $a_1 = 0$, *ie*, $Q_3 = P_0$. Hence

$$(V_1^0)^{\text{Sing}} = \{(a_1, b_1, a_2, b_2) \in (V_1^0)_\pi \mid a_1 = b_1 = b_2 = 0\}.$$

Now we blow-up $(V_1^0)^{\text{Sing}}$. By using the procedure of Mahtab we get three pieces for the covering of $\widetilde{V_1^0}$ as follows:

Chart 1.1

$$V_{11}^{00} = \text{Spec}\, R[X_1, Y_1, X_2, Y_2]/(Y_1^2 - Y_2^2 - Y_1 + Y_2X_2^2,\ X_1^4Y_1^2 - X_1^4Y_1 - \pi). \tag{4.209}$$

Now let $A = R[X_1, Y_1]/(X_1^4 Y_1(Y_1 - 1) - \pi)$. By using theorem (2.14) we can check that A is a regular ring. We can write

$$V_{11}^{00} = \operatorname{Spec} A[X_2, Y_2]/(Y_1^2 - Y_2^2 - Y_1 + Y_2 X_2^2). \tag{4.210}$$

Let $\phi(X_2, Y_2) = Y_1^2 - Y_2^2 - Y_1 + Y_2 X_2^2$. Then $\frac{\partial \phi}{\partial X_2} = 2X_2 Y_2$ and $\frac{\partial \phi}{\partial Y_2} = -2Y_2 + X_2^2$. Since $p = char\, k \neq 2$, the system $\frac{\partial \phi}{\partial X_2} = \frac{\partial \phi}{\partial Y_2} = 0$ can be written as $2X_2 Y_2 = -2Y_2 + X_2^2 = 0$. So V_{11}^{00} is smooth everywhere except at those points where $2X_2 Y_2 = -2Y_2 + X_2^2 = 0$. To have the points on the special fibre of V_{11}^{00}, we need $X_1^4 Y_1(Y_1 - 1) = 0$. Considering $\phi(X_2, Y_2) = \frac{\partial \phi}{\partial X_2} = \frac{\partial \phi}{\partial Y_2} = X_1^4 Y_1(Y_1 - 1) = 0$, we get $(V_{11}^{00})^{\text{Sing}} \subset S \subset (V_{11}^{00})_\pi$, where S contains points of the form $(a_1, b_1, a_2, b_2) \in (V_{11}^{00})_\pi$ satisfying one of the following conditions:

(1) $b_1 = a_2 = b_2 = 0$;

(2) $b_1 - 1 = a_2 = b_2 = 0$.

Let $Q_1 = (a_1, 0, 0, 0) \in S$. Then $\text{Moje}(Q_1, f_1) = -v_1\pi - S_1$ and

$$\text{Moje}(Q_1, f_2) = -(\alpha_1^4 v_1 + 1)\pi - \alpha_1^4 S_1, \tag{4.211}$$

ie,

$$D(Q_1) = \begin{pmatrix} -\bar{v}_1 & 0 & -1 & 0 & 0 \\ -(a_1^4 \bar{v}_1 + 1) & 0 & -a_1^4 & 0 & 0 \end{pmatrix}. \tag{4.212}$$

Since $a_1^4 \bar{v}_1 - (a_1^4 \bar{v}_1 + 1) = -1 \neq 0$, $D(Q_1)$ has rank two. Hence Q_1 is a regular point.

Let $Q_2 = (a_1, 1, 0, 0) \in S$. Then $\text{Moje}(Q_2, f_1) = (2\beta_1 - 1)S_1$ and

$$\text{Moje}(Q_2, f_2) = 4\alpha_1^3 \beta_1(\beta_1 - 1)T_1 + \alpha_1^4(2\beta_1 - 1)S_1 - \pi \tag{4.213}$$

where $\beta_1 \equiv 1 (\operatorname{mod} \pi)$, *ie*,

$$D(Q_2) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & a_1^4 & 0 & 0 \end{pmatrix} \tag{4.214}$$

which has rank two. Hence Q_2 is a regular point and as a result V_{11}^{00} is a regular scheme.

Chart 1.2

$$V_{11}^{01} = \operatorname{Spec} R[X_1, Y_1, X_2, Y_2]/(1 - Y_2^2 - X_1 + X_1Y_2X_2^2\,,\, Y_1^4X_1^2 - Y_1^4X_1^3 - \pi). \tag{4.215}$$

Now suppose that $A = R[X_1, Y_1]/(Y_1^4X_1^2(1 - X_1) - \pi)$. By using theorem (2.14), we can check that A is a regular ring. Notice that

$$V_{11}^{01} = \operatorname{Spec} A[X_2, Y_2]/(1 - Y_2^2 - X_1 + X_1Y_2X_2^2). \tag{4.216}$$

Let $\phi(X_2, Y_2) = 1 - Y_2^2 - X_1 + X_1Y_2X_2^2$. Then $\frac{\partial\phi}{\partial X_2} = 2X_1X_2Y_2$ and $\frac{\partial\phi}{\partial Y_2} = -2Y_2 + X_1X_2^2$. Since $p = char\, k \neq 2$, the system $\frac{\partial\phi}{\partial X_2} = \frac{\partial\phi}{\partial Y_2} = 0$ can be written as $2X_1X_2Y_2 = -2Y_2 + X_1X_2^2 = 0$. So V_{11}^{01} is smooth over A everywhere except at the points where $2X_1X_2Y_2 = -2Y_2 + X_1X_2^2 = 0$. To have the points on the special fibre of V_{11}^{01}, we need $Y_1^4X_1^2(1 - X_1) = 0$. Considering $\phi(X_2, Y_2) = \frac{\partial\phi}{\partial X_2} = \frac{\partial\phi}{\partial Y_2} = Y_1^4X_1^2(1 - X_1) = 0$, we get $(V_{11}^{01})^{\text{Sing}} \subset S \subset (V_{11}^{01})_\pi$, where S contains points of the form $(a_1, b_1, a_2, b_2) \in (V_{11}^{01})_\pi$ satisfying the following condition:

$$a_1 - 1 = a_2 = b_2 = 0$$

Let $Q = (1, b_1, 0, 0) \in S$. Then $\text{Moje}(Q, f_1) = -T_1$ and

$$\text{Moje}(Q, f_2) = -\pi + \alpha_1\beta_1^4(2 - 3\alpha_1)T_1 + 4\alpha_1^2\beta_1^3(1 - \alpha_1)S_1 \tag{4.217}$$

where $\alpha_1 \equiv 1(\text{mod}\,\pi)$, *ie*,

$$D(Q) = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ -1 & -b_1^4 & 0 & 0 & 0 \end{pmatrix}. \tag{4.218}$$

So $D(Q)$ has rank two. Hence Q is a regular point and as a result V_{11}^{01} is a regular scheme.

Chart 1.4

$$V_{12}^{01} = \operatorname{Spec} R[X_1, Y_1, X_2, Y_2]/(Y_1^2 - 1 - X_1Y_1 + X_1X_2^2\,,\, Y_2^4X_1^2Y_1^2 - Y_2^4X_1^3Y_1 - \pi). \tag{4.219}$$

Now suppose that $A = R[X_1, Y_1, Y_2]/(Y_2^4X_1^2Y_1(Y_1 - X_1) - \pi)$. Considering theorem (2.14), we can check that A is a regular ring. Note that

$$V_{12}^{01} = \operatorname{Spec} A[X_2]/(Y_1^2 - 1 - X_1Y_1 + X_1X_2^2). \tag{4.220}$$

Let $\phi(X_2) = Y_1^2 - 1 - X_1Y_1 + X_1X_2^2$. Then $\frac{\partial\phi}{\partial X_2} = 2X_1X_2$. Since $p = char k \neq 2$, the equation $\frac{\partial\phi}{\partial X_2} = 0$ can be written as $2X_1X_2 = 0$. So V_{12}^{01} is etale over A everywhere except at the points where $2X_1X_2 = 0$. To have the points on the special fibre of V_{12}^{01}, we need $Y_2^4X_1^2Y_1(Y_1 - X_1) = 0$. Considering $\phi(X_2) = \frac{\partial\phi}{\partial X_2} = Y_2^4X_1^2Y_1(Y_1 - X_1) = 0$, we get $(V_{12}^{01})^{\text{Sing}} \subset S \subset (V_{12}^{01})_\pi$, where S contains points of the form $(a_1, b_1, a_2, b_2) \in (V_{12}^{01})_\pi$ satisfying one of the following conditions:

(1) $a_1 = b_1 - 1 = 0$;

(2) $a_1 = b_1 + 1 = 0$;

(3) $a_2 = b_2 = a_1b_1 - b_1^2 + 1 = 0$.

Let $Q_1 = (0, 1, a_2, b_2) \in S$. Then

$$\text{Moje}(Q_1, f_1) = (\alpha_2^2 - \beta_1)u_1\pi + (\alpha_2^2 - \beta_1)T_1 + 2\beta_1S_1 \tag{4.221}$$

where $\beta_1 \equiv 1 (\text{mod}\,\pi)$ and $\text{Moje}(Q_1, f_2) = -\pi$, *ie*,

$$D(Q_1)_{r_1} = (\,(a_2^2 - 1)\bar{u}_1 \quad a_2^2 - 1 \quad 2 \quad 0 \quad 0\,). \tag{4.222}$$

So $D(Q_1)$ has rank two. Hence Q_1 is a regular point.

Let $Q_2 = (0, -1, a_2, b_2) \in S$. Then

$$\text{Moje}(Q_2, f_1) = (\alpha_2^2 - \beta_1)u_1\pi + (\alpha_2^2 - \beta_1)T_1 + 2\beta_1S_1 \tag{4.223}$$

where $\beta_1 \equiv -1 (\text{mod}\,\pi)$ and $\text{Moje}(Q_2, f_2) = -\pi$, *ie*,

$$D(Q_2)_{r_1} = (\,(a_2^2 + 1)\bar{u}_1 \quad a^2 + 1 \quad -2 \quad 0 \quad 0\,). \tag{4.224}$$

So $rank\, D(Q_2) = 2$. Hence Q_2 is a regular point.

Let $Q_3 = (a_1, b_1, 0, 0) \in S$ such that $a_1b_1 - b_1^2 + 1 = 0$. Then

$$\text{Moje}(Q_3, f_1) = -\beta_1T_1 + (2\beta_1 - \alpha_1)S_1 \text{ and } \text{Moje}(Q_3, f_2) = -\pi, \tag{4.225}$$

ie,

$$D(Q_3)_{r_1} = (\,0 \quad -b_1 \quad (2b_1 - a_1) \quad 0 \quad 0\,). \tag{4.226}$$

So Q_3 is singular if $a_1 = b_1 = a_1b_1 - b_1^2 + 1 = 0$, which is impossible. Hence Q_3 is a regular point and as a result V_{12}^{01} is a regular scheme.

Chart 2

$$V_1^1 = \operatorname{Spec} R[X_1, Y_1, X_2, Y_2]/(1 - Y_2^2 - Y_1X_1^2 + Y_1Y_2X_2^2\,,\; Y_1^2 - Y_1^3X_1^2 - \pi). \tag{4.227}$$

Let $A = R[X_1, Y_1]/(Y_1^2(1 - Y_1X_1^2) - \pi)$. We can check that A is a regular ring (by using theorem (2.14)). Notice that

$$V_1^1 = \operatorname{Spec} A[X_2, Y_2]/(1 - Y_2^2 - Y_1X_1^2 + Y_1Y_2X_2^2). \tag{4.228}$$

Now suppose that $\phi(X_2, Y_2) = 1 - Y_2^2 - Y_1X_1^2 + Y_1Y_2X_2^2$. Then $\frac{\partial\phi}{\partial X_2} = 2Y_1Y_2X_2$ and $\frac{\partial\phi}{\partial Y_2} = -2Y_2 + Y_1X_2^2$. Since $p = char\, k \neq 2$, the system $\frac{\partial\phi}{\partial X_2} = \frac{\partial\phi}{\partial Y_2} = 0$ can be written as $2Y_1Y_2X_2 = -2Y_2 + Y_1X_2^2 = 0$. So V_1^1 is smooth over A everywhere except at the points where $2Y_1Y_2X_2 = -2Y_2 + Y_1X_2^2 = 0$. To have the points on the special fibre of V_1^1, we need $Y_1^2(1 - Y_1X_1^2) = 0$. Considering $\phi(X_2, Y_2) = \frac{\partial\phi}{\partial X_2} = \frac{\partial\phi}{\partial Y_2} = Y_1^2(1 - Y_1X_1^2) = 0$, we get $(V_1^1)^{\mathrm{Sing}} \subset S \subset (V_1^1)_\pi$, where S contains points of the form $(a_1, b_1, a_2, b_2) \in (V_1^1)_\pi$ satisfying the following condition:

$$a_2 = b_1a_1^2 - 1 = b_2 = 0$$

Let $Q = (a_1, b_1, 0, 0) \in S$ such that $b_1a_1^2 - 1 = 0$. Then $\mathrm{Moje}(Q, f_1) = -2\alpha_1\beta_1T_1 - \alpha_1^2S_1$ and

$$\mathrm{Moje}(Q, f_2) = -2\alpha_1\beta_1^3T_1 + \beta_1(2 - 3\alpha_1^2\beta_1)S_1 - \pi, \tag{4.229}$$

ie,

$$D(Q) = \begin{pmatrix} 0 & -2a_1b_1 & -a_1^2 & 0 & 0 \\ -1 & -2a_1b_1^3 & b_1(2 - 3a_1^2b_1) & 0 & 0 \end{pmatrix}. \tag{4.230}$$

So Q is singular if $-2a_1b_1 = -a_1^2 = b_1a_1^2 - 1 = 0$ which is impossible. So Q is a regular point and as a result V_1^1 is a regular scheme.

Chart 3

$$V_2^0 = \operatorname{Spec} R[X_1, Y_1, X_2, Y_2]/(Y_1^2 - Y_2^2 - X_2Y_1X_1^2 + X_2Y_2\,,\; X_2^2Y_1^2 - X_2^3Y_1X_1^2 - \pi). \tag{4.231}$$

Recall that

$$\begin{aligned}
&V_1^0 = \operatorname{Spec} R[X_1, Y_1, X_2, Y_2]/ \\
&(Y_1^2 - Y_2^2 - X_1Y_1 + X_1Y_2X_2^2\,,\; X_1^2Y_1^2 - X_1^3Y_1 - \pi) = \\
&\operatorname{Spec} R[X_1, Y_1, X_2, Y_2]/(Y_1^2 - Y_2^2 - X_1Y_1 + X_1Y_2X_2^2\,,\; X_1^2Y_2^2 - X_1^3Y_2X_2^2 - \pi),
\end{aligned} \tag{4.232}$$

(we used $Y_1^2 - X_1Y_1 = Y_2^2 - X_1Y_2X_2^2$). We can write

$$V_2^0 = \operatorname{Spec} R[X_1, Y_1, X_2, Y_2]/(Y_2^2 - Y_1^2 - X_2Y_2 + X_2Y_1X_1^2\,,\; X_2^2Y_1^2 - X_2^3Y_1X_1^2 - \pi). \tag{4.233}$$

The ring homomrphism

$$\begin{aligned}
\alpha \,:\, \Gamma(V_1^0, \mathcal{O}_{V_1^0}) &\longrightarrow \Gamma(V_2^0, \mathcal{O}_{V_2^0}) \\
X_1 &\longmapsto X_2 \\
Y_1 &\longmapsto Y_2 \\
X_2 &\longmapsto X_1 \\
Y_2 &\longmapsto Y_1
\end{aligned}$$

is an isomorphism and induces $V_2^0 \cong V_1^0$.

Chart 4

$$\begin{aligned}
V_2^1 \;=\; & \operatorname{Spec} R[X_1, Y_1, X_2, Y_2]/ \\
& (Y_1^2 - 1 - Y_2Y_1X_1^2 + Y_2X_2^2\,,\; Y_2^2Y_1^2 - Y_2^3Y_1X_1^2 - \pi).
\end{aligned} \tag{4.234}$$

Recall that

$$\begin{aligned}
&V_1^1 = \operatorname{Spec} R[X_1, Y_1, X_2, Y_2]/(1 - Y_2^2 - Y_1X_1^2 + Y_1Y_2X_2^2\,,\; Y_1^2 - Y_1^3X_1^2 - \pi) = \\
&\operatorname{Spec} R[X_1, Y_1, X_2, Y_2]/(1 - Y_2^2 - Y_1X_1^2 + Y_1Y_2X_2^2\,,\; Y_1^2Y_2^2 - Y_1^3Y_2X_2^2 - \pi),
\end{aligned} \tag{4.235}$$

(we used $1 - Y_1X_1^2 = Y_2^2 - Y_1Y_2X_2^2$). Note that V_2^1 can be written as

$$V_2^1 = \operatorname{Spec} R[X_1, Y_1, X_2, Y_2]/(1 - Y_1^2 - Y_2X_2^2 + Y_2Y_1X_1^2\,,\, Y_2^2Y_1^2 - Y_2^3Y_1X_1^2 - \pi). \tag{4.236}$$

The ring homomorphism

$$\alpha : \Gamma(V_1^1, \mathcal{O}_{V_1^1}) \longrightarrow \Gamma(V_2^1, \mathcal{O}_{V_2^1})$$

$$X_1 \longmapsto X_2$$

$$Y_1 \longmapsto Y_2$$

$$X_2 \longmapsto X_1$$

$$Y_2 \longmapsto Y_1$$

is an isomorphism and induces $V_2^1 \cong V_1^1$. The gluing of the regular schemes which we have had so far gives us the regular scheme $\widetilde{X}$. □

4.5 One component is I_4

In this section our arithmetic three-folds are the products of two arithmetic surfaces such that one of them is of the form I_4 and apart from the cases discussed before.

Lemma 4.11 Let $p = \operatorname{char} k \neq 3$ and $V_1 = V_2 = \operatorname{Spec} R[x, y]/(x^2y - xy^2 - \pi)$. Then $X = V_1 \times_{\operatorname{Spec} R} V_2$ is singular and after one blowing-up we get a regular scheme $\widetilde{X}$.

Proof: By using corollary (2.11) we can check that P_0 is the only singular point of

$$X = \operatorname{Spec} R[x_1, y_1, x_2, y_2]/(x_1^2y_1 - x_1y_1^2 - x_2^2y_2 + x_2y_2^2\,,\, x_1^2y_1 - x_1y_1^2 - \pi). \tag{4.237}$$

Now we blow-up P_0. By using the procedure of Mahtab we get four pieces for the covering of $\widetilde{X}$ as follows:

Chart 1

$$V_1^0 = \operatorname{Spec} R[X_1, Y_1, X_2, Y_2]/ \\ (Y_1 - Y_1^2 - X_2^2Y_2 + X_2Y_2^2\,,\, X_1^3Y_1 - X_1^3Y_1^2 - \pi). \tag{4.238}$$

Now let $A = R[X_1, Y_1]/(X_1^3Y_1 - X_1^3Y_1^2 - \pi)$. Considering theorem (2.14), we can check that A is a regular ring. We can write

$$V_1^0 = \operatorname{Spec} A[X_2, Y_2]/(Y_1 - Y_1^2 - X_2^2Y_2 + X_2Y_2^2). \tag{4.239}$$

Let $\phi(X_2, Y_2) = Y_1 - Y_1^2 - X_2^2Y_2 + X_2Y_2^2$. Then

$$\frac{\partial\phi}{\partial X_2} = -2X_2Y_2 + Y_2^2 \text{ and } \frac{\partial\phi}{\partial Y_2} = -X_2^2 + 2X_2Y_2. \tag{4.240}$$

Since $p = chark \neq 3$, the system $\frac{\partial\phi}{\partial X_2} = \frac{\partial\phi}{\partial Y_2} = 0$ can be written as $-2X_2Y_2 + Y_2^2 = -X_2^2 + 2X_2Y_2 = 0$. So V_1^0 is smooth over A everywhere except at the points where $-2X_2Y_2 + Y_2^2 = -X_2^2 + 2X_2Y_2 = 0$. To have these points on the special fibre of V_1^0, we need $X_1^3(1 - Y_1)Y_1 = 0$. Considering $\phi(X_2, Y_2) = \frac{\partial\phi}{\partial X_2} = \frac{\partial\phi}{\partial Y_2} = X_1^3(1 - Y_1)Y_1 = 0$, we get $(V_1^0)^{\text{Sing}} \subset S \subset (V_1^0)_\pi$, where S contains points of the form $(a_1, b_1, a_2, b_2) \in (V_1^0)_\pi$ satisfying one of the following conditions:

(1) $\quad b_1 = a_2 = b_2 = 0;$

(2) $\quad b_1 - 1 = a_2 = b_2 = 0.$

Let $Q_1 = (a_1, 0, 0, 0) \in S$. Then

$$\text{Moje}(Q_1, f_1) = v_1\pi + S_1 \text{ and } \text{Moje}(Q_1, f_2) = \alpha_1^3v_1\pi + \alpha_1^3S_1 - \pi, \tag{4.241}$$

ie,

$$D(Q_1) = \begin{pmatrix} \bar{v}_1 & 0 & 1 & 0 & 0 \\ (a_1^3\bar{v}_1 - 1) & 0 & a_1^3 & 0 & 0 \end{pmatrix}, \tag{4.242}$$

which has rank 2. So Q_1 is a regular point.

Let $Q_2 = (a_1, 1, 0, 0) \in S$. Then $\text{Moje}(Q_2, f_1) = (1 - 2\beta_1)S_1$ where $\beta_1 \equiv 1(\text{mod}\pi)$ and

$$\text{Moje}(Q_2, f_2) = 3\alpha_1^2\beta_1(1 - \beta_1)T_1 + \alpha_1^3(1 - 2\beta_1)S_1 - \pi, \tag{4.243}$$

ie,

$$D(Q_2) = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & -\alpha_1^3 & 0 & 0 \end{pmatrix}, \tag{4.244}$$

which has rank two. So Q_2 is a regular point. This means that V_1^0 is a regular scheme.

Chart 2

$$\begin{aligned} V_1^1 &= \operatorname{Spec} R[X_1, Y_1, X_2, Y_2]/(X_1^2 - X_1 - X_2^2Y_2 + X_2Y_2^2, Y_1^3X_1^2 - Y_1^3X_1 - \pi) \\ &= \operatorname{Spec} R[X_1, Y_1, X_2, Y_2]/(X_1 - X_1^2 - Y_2^2X_2 + Y_2X_2^2\,,\ -Y_1^3X_1 + Y_1^3X_1^2 - \pi). \end{aligned} \tag{4.245}$$

The homomorphism

$$\alpha : \Gamma(V_1^0, \mathcal{O}_{V_1^0}) \longrightarrow \Gamma(V_1^1, \mathcal{O}_{V_1^1})$$

$$X_1 \longmapsto -Y_1$$

$$Y_1 \longmapsto X_1$$

$$X_2 \longmapsto Y_2$$

$$Y_2 \longmapsto X_2$$

is an isomorphism of rings which induces

$$V_1^1 \cong V_1^0$$

Chart 3

$$\begin{aligned} V_2^0 = {} & \operatorname{Spec} R[X_1, Y_1, X_2, Y_2]/ \\ & (X_1^2Y_1 - X_1Y_1^2 - Y_2 + Y_2^2\,,\ X_2^3X_1^2Y_1 - X_2^3X_1Y_1^2 - \pi). \end{aligned} \tag{4.246}$$

In $\Gamma(V_1^0, \mathcal{O}_{V_1^0})$ from the first equation we get $Y_1 - Y_1^2 = X_2^2Y_2 - X_2Y_2^2$. So in the other equation of V_2^0 we can use

$$X_1^3(Y_1 - Y_1^2) = X_1^3(X_2^2Y_2 - X_2Y_2^2) = X_1^3X_2^2Y_2 - X_1^3X_2Y_2^2,$$

ie,

$$V_1^0 = \operatorname{Spec} R[X_1, Y_1, X_2, Y_2]/(X_2^2Y_2 - X_2Y_2^2 - Y_1 + Y_1^2\,,\ X_1^3X_2^2Y_2 - X_1^3X_2Y_2^2 - \pi). \tag{4.247}$$

The homomorphism

$$\alpha : \Gamma(V_1^0, \mathcal{O}_{V_1^0}) \longrightarrow \Gamma(V_2^0, \mathcal{O}_{V_2^0})$$

$$X_1 \longmapsto X_2$$

$$Y_1 \longmapsto Y_2$$

$$X_2 \longmapsto X_1$$

$$Y_2 \longmapsto Y_1$$

is an isomorphism of rings and induces

$$V_2^0 \cong V_1^0.$$

Chart 4

$$V_2^1 = \operatorname{Spec} R[X_1, Y_1, X_2, Y_2]/(X_1^2Y_1 - X_1Y_1^2 - X_2^2 + X_2\,,\ Y_2^3X_1^2Y_1 - Y_2^3X_1Y_1^2 - \pi). \tag{4.248}$$

Considering V_1^0 in the form which was used in chart 3, we can write

$$\begin{aligned} V_1^0 = {} & \operatorname{Spec} R[X_1, Y_1, X_2, Y_2]/ \\ & (Y_2^2X_2 - Y_2X_2^2 - Y_1^2 + Y_1\,,\ -X_1^3X_2Y_2^2 + X_1^3X_2^2Y_2 - \pi). \end{aligned} \tag{4.249}$$

The homomorphism

$$\alpha : \Gamma(V_1^0, \mathcal{O}_{V_1^0}) \longrightarrow \Gamma(V_2^1, \mathcal{O}_{V_2^1})$$

$$X_1 \longmapsto -Y_2$$

$$Y_1 \longmapsto X_2$$

$$X_2 \longmapsto Y_1$$

$$Y_2 \longmapsto X_1$$

is an isomorphism and induces

$$V_2^1 \cong V_1^0.$$

The gluing of V_1^0,V_1^1, V_2^0 and V_2^1 gives us the regular scheme $\widetilde{X}$. □

4.6 One component is I_5

In this section we will discuss about some arithmetic three-folds and then about those arithmetic three-folds which are the products of two arithmetic surfaces such that one of them is of the form I_5.

Convention 4.12 For non-negative integers a, b, m and n, let

$$V_{a,b,m,n} = \text{Spec}\, R[x_1, y_1, x_2, y_2]/(x_1y_1 - x_2^a y_2^b, x_2^m y_2^n - \pi). \tag{4.250}$$

Then the fibre product of two arithmetic surfaces of the forms I_1 and I_5 is $V_{m,n,m,n}$.

Remark 4.13 We will try to find a desingularisation for $V_{a,b,m,n}$ and finally we consider the case $a = m$ and $b = n$.

Lemma 4.14 The three-fold $V_{1,1,m,n}$ is singular and just after one blowing-up we can resolve its singularity.

Proof : Let $Y = V_{1,1,m,n} = \text{Spec}\, R[x_1, y_1, x_2, y_2]/(x_1y_1 - x_2y_2, x_2^m y_2^n - \pi)$. By using the procedure of Mojgan we can check that $P_0 \in Y^{\text{Sing}}$. Now we blow-up P_0. By using the procedure of Mahtab we get four pieces for the covering of $\widetilde{Y}$ as follows:

Chart 1

$$\begin{aligned} V_1^0 &= \text{Spec}\, R[x_1, \frac{y_1}{x_1}, \frac{x_2}{x_1}, \frac{y_2}{x_1}]/(\frac{y_1}{x_1} - (\frac{x_2}{x_1})(\frac{y_2}{x_1}), x_1^{m+n}(\frac{x_2}{x_1})^m(\frac{y_2}{x_1})^n - \pi) = \\ &\quad \text{Spec}\, R[X_1, X_2, Y_2]/(X_1^{m+n} X_2^m Y_2^n - \pi) \end{aligned} \tag{4.251}$$

which is a regular scheme (by lemma (2.12)). By using similar calculation we can check that V_1^1, V_2^0 and V_2^1 are regular schemes. The gluing of V_1^0, V_1^1, V_2^0 and V_2^1 gives us the regular scheme $\widetilde{Y}$. □

Lemma 4.15 There exists a desingularisation for $V_{a,0,m,n}$.

Proof: We discuss it in three different possible cases as follows:

Case 1 If $a = 0$, then

$$V_{0,0,m,n} = \operatorname{Spec} R[x_1, y_1, x_2, y_2]/(x_1y_1 - 1, x_2^m y_2^n - \pi). \tag{4.252}$$

Let $A = R[x_2, y_2]/(x_2^m y_2^n - \pi)$. Then A is a regular ring. We can write

$$V_{0,0,m,n} = \operatorname{Spec} A[x_1, y_1]/(x_1y_1 - 1). \tag{4.253}$$

By using the Jacobian criterion we can check that $V_{0,0,m,n}$ is smooth over A and so it is regular.

Case 2 If $a = 1$, then

$$\begin{aligned} V_{1,0,m,n} &= \operatorname{Spec} R[x_1, y_1, x_2, y_2]/(x_1y_1 - x_2, x_2^m y_2^n - \pi) = \\ &\operatorname{Spec} R[x_1, y_1, y_2]/(x_2^m y_2^n - \pi) \end{aligned} \tag{4.254}$$

which is a regular scheme (by lemma (2.12)).

Case 3 If $a > 1$, then

$$Y := V_{a,0,m,n} = \operatorname{Spec} R[x_1, y_1, x_2, y_2]/(x_1y_1 - x_2^a, x_2^m y_2^n - \pi). \tag{4.255}$$

By using the procedure of Mojgan we can check that

$$Y^{\text{Sing}} \supset \{(a_1, b_1, a_2, b_2) \in Y_\pi \mid a_1 = b_1 = a_2 = 0\} := S \tag{4.256}$$

Now we find the blowing-up of Y with centre S. By using the procedure of Mahtab we get three pieces for the covering of $\widetilde{Y}$ as follows:

Chart 1

$$\begin{aligned} V_1^0 &= \operatorname{Spec} R[x_1, \frac{y_1}{x_1}, \frac{x_2}{x_1}, y_2]/(\frac{y_1}{x_1} - x_1^{a-2}(\frac{x_2}{x_1})^a, x_1^m(\frac{x_2}{x_1})^m y_2^n - \pi) = \\ & \operatorname{Spec} R[X_1, Y_1, X_2, Y_2]/(Y_1 - X_1^{a-2}X_2^a, X_1^m X_2^m Y_2^n - \pi) = \\ & \operatorname{Spec} R[X_1, X_2, Y_2]/(X_1^m X_2^m Y_2^n - \pi) \end{aligned} \tag{4.257}$$

which is a regular scheme.

Chart 2

$$\begin{aligned} V_1^1 &= \operatorname{Spec} R[\frac{x_1}{y_1}, y_1, \frac{x_2}{y_1}, y_2]/(\frac{x_1}{y_1} - y_1^{a-2}(\frac{x_1}{y_1}), y_1^m(\frac{x_2}{y_1})^m y_2^n - \pi) = \\ & \operatorname{Spec} R[Y_1, X_2, Y_2]/(Y_1^m X_2^m Y_2^n - \pi) \cong V_1^0. \end{aligned} \tag{4.258}$$

Chart 3

$$\begin{aligned} & V_2^0 = \operatorname{Spec} R[\frac{x_1}{x_2}, \frac{y_1}{x_2}, x_2, y_2]/((\frac{x_1}{x_2})(\frac{y_1}{x_2}) - x_2^{a-2}, x_2^m y_2^n - \pi) = \\ & \operatorname{Spec} R[X_1, Y_1, X_2, Y_2]/(X_1 Y_1 - X_2^{a-2}, X_2^m Y_2^n - \pi) = V_{a-2,0,m,n}. \end{aligned} \tag{4.259}$$

If $a = 2$, then

$$\begin{aligned} & V_{a-2,0,m,n} = V_{0,0,m,n} = \\ & \operatorname{Spec} R[x_1, y_1, x_1, y_2]/(x_1 y_1 - 1, x_2^m y_2^n - \pi) \end{aligned} \tag{4.260}$$

which is regular (we discussed it in the case one).

If $a = 3$ then

$$V_{a-2,0,m,n} = V_{1,0,m,n} = \operatorname{Spec} R[x_1, y_1, y_2]/(x_1^m y_1^m y_2^n - \pi) \tag{4.261}$$

which is a regular scheme.

If $a > 3$ then we continue the process of blowings-up of $V_{a,0,m,n}$. Each blowing-up reduces a by two. So finally we get $V_{0,0,m,n}$ or $V_{1,0,m,n}$ which were discussed in cases 1 and 2. Hence there exists a desingularisation for $V_{a,0,m,n}$. □

Remark 4.16 For the three-fold $V_{0,b,m,n}$ we can change the variables x_2 and y_2 with each other and use lemma (4.15) for $V_{b,0,n,m}$.

Lemma 4.17 There exists a desingularisation for

$$V_{a,1,m,n} = \operatorname{Spec} R[x_1, y_1, x_2, y_2]/(x_1y_1 - x_2^a y_2, x_2^m y_2^n - \pi).$$

Proof: We consider the following cases:

Case 1 If $a = 0$, then

$$\begin{aligned} V_{0,1,m,n} &= \operatorname{Spec} R[x_1, y_1, x_2, y_2]/(x_1y_1 - y_2, x_2^m y_2^n) = \\ &\operatorname{Spec} R[x_1, y_1, x_2]/(x_2^m x_1^n y_1^n - \pi) \end{aligned} \tag{4.262}$$

which is regular.

Case 2 If $a = 1$, then

$$V_{1,1,m,n} = \operatorname{Spec} R[x_1, y_1, x_2, y_2]/(x_1y_1 - x_2y_2, x_2^m y_2^n - \pi) \tag{4.263}$$

which was discussed in lemma (4.14).

Case 3 If $a > 1$, then we can check that

$$V_{a,1,m,n}^{\text{Sing}} \supset \{(a_1, b_1, a_2, b_2) \in (V_{a,1,m,n})_\pi \mid a_1 = b_1 = a_2 = 0\} := S. \tag{4.264}$$

Now we blow-up S. By using the procedure of Mahtab we get three pieces for the covering of $\widetilde{V_{a,1,m,n}}$ as follows:

Chart 1

$$\begin{aligned} V_1^0 &= \operatorname{Spec} R[x_1, \frac{y_1}{x_1}, \frac{x_2}{x_1}, y_2]/(\frac{y_1}{x_1} - x_1^{a-2}(\frac{x_2}{x_1})^{a_2} y_2, x_1^m(\frac{x_2}{x_1})^m y_2^n - \pi) = \\ &\operatorname{Spec} R[X_1, X_2, Y_2]/(X_1^m X_2^m Y_2^n - \pi) \end{aligned} \tag{4.265}$$

which is a regular scheme.

Chart 2

$$V_1^1 \cong V_1^0. \tag{4.266}$$

Chart 3

$$V_2^0 = \operatorname{Spec} R[\frac{x_1}{x_2}, \frac{y_1}{x_2}, x_2, y_2]/((\frac{x_1}{x_2})(\frac{y_1}{x_2}) - x_2^{a-2}y_2, x_2^m y_2^n - \pi) =$$
$$\operatorname{Spec} R[X_1, Y_1, X_2, Y_2]/(X_1Y_1 - X_2^{a-2}Y_2, X_2^m Y_2^n - \pi) = V_{a-2,1,m,n}. \tag{4.267}$$

For $a = 2$, $V_{a-2,1,m,n} = V_{0,1,m,n}$ which is regular (see case one). For $a = 3$, $V_{a-2,1,m,n} = V_{1,1,m,n}$ which was discussed in lemma (4.14). If $a > 3$ the process of blowings-up decreases a by two after each blowing-up. So after some blowings-up and gluing the regular pieces of the covering we get a regular scheme $\widetilde{V_{a,1,m,n}}$. □

Remark 4.18 For $V_{1,b,m,n}$ we change x_1 and y_1 with each other and use lemma (4.17) for $V_{b,1,m,n}$.

Theorem 4.19 There exists a desingularisation for

$$V_{a,b,m,n} = \operatorname{Spec} R[x_1, y_1, x_2, y_2]/(x_1y_1 - x_2^a y_2^b, x_2^m y_2^n - \pi).$$

Proof: Without loss of generality let $b \geq a$. If $a \geq 2$ then by using the procedure of Mojgan we can check that

$$V_{a,b,m,n}^{\text{Sing}} \supset \{(a_1, b_1, a_2, b_2) \in (V_{a,b,m,n})_\pi \mid a_1 = b_1 = a_2b_2 = 0\} := S. \tag{4.268}$$

Now we find the blowing-up of $V_{a,b,m,n}$ with centre S. By using the procedure of Mahtab we get three pieces for the covering of $\widetilde{V_{a,b,m,n}}$ as follows:

Chart 1

$$V_1^0 = \operatorname{Spec} R[x_1, \frac{y_1}{x_1}, x_2, y_2, \frac{x_2y_2}{x_1}]/(\frac{y_1}{x_1} - x_1^{a-2}(\frac{x_2y_2}{x_1})^a y_2^{b-a}, x_2^m y_2^n - \pi) =$$
$$\operatorname{Spec} R[X_1, Y_1, X_2, Y_2, Z]/(Y_1 - X_1^{a-2}Z^aY_2^{b-a}, X_1Z - X_2Y_2, X_2^mY_2^n - \pi) =$$
$$\operatorname{Spec} R[X_1, X_2, Y_2, Z]/(X_1Z - X_2Y_2, X_2^mY_2^n - \pi) \cong V_{1,1,m,n}. \tag{4.269}$$

which was discussed in lemma (4.14).

Chart 2

$$V_1^1 = \text{Spec}\, R[\frac{x_1}{y_1}, y_1, x_2, y_2, \frac{x_2y_2}{y_1}]/(\frac{x_1}{y_1} - y_1^{a-2}(\frac{x_2y_2}{y_1})^a y_2^{b-a}\,,\, x_2^m y_2^n - \pi) =$$

$$\text{Spec}\, R[X_1, Y_1, X_2, Y_2, Z]/(X_1 - Y_1^{a-2}Z^aY_2^{b-a}\,,\, Y_1Z - X_2Y_2\,,\, X_2^mY_2^n - \pi) =$$

$$\text{Spec}\, R[Y_1, X_2, Y_2, Z]/(ZY_1 - X_2Y_2\,,\, X_2^mY_2^n - \pi) \cong V_{1,1,m,n}. \qquad (4.270)$$

Chart 5

Let $Z = x_2y_2$. Then

$$V_\theta^\theta = \text{Spec}\, R[\frac{x_1}{Z}, \frac{y_1}{Z}, x_2, y_2, Z]/$$

$$((\frac{x_1}{Z})(\frac{y_1}{Z}) - Z^{a-2}(\frac{x_2y_2}{Z})^a y_2^{b-a}\,,\, Z - x_2y_2\,,\, x_2^m y_2^n - \pi) =$$

$$\text{Spec}\, R[X_1, Y_1, X_2, Y_2, Z]/(X_1Y_1 - Z^{a-2}Y_2^{b-a}\,,\, Z - X_2Y_2\,,\, X_2^mY_2^n - \pi) =$$

$$\text{Spec}\, R[X_1, Y_1, X_2, Y_2]/(X_1Y_1 - X_2^{a-2}Y_2^{a-2}Y_2^{b-a}\,,\, X_2^mY_2^n - \pi) =$$

$$\text{Spec}\, R[X_1, Y_1, X_2, Y_2]/(X_1Y_1 - X_2^{a-2}Y_2^{b-2}\,,\, X_2^mY_2^n - \pi)$$

$$\cong V_{a-2,b-2,m,n}. \qquad (4.271)$$

By continuing the process of blowings-up with centre S after each blowing-up a and b reduce by two. We continue the process of blowings-up until getting $V_{\alpha,\beta,m,n}$ such that $\beta = 0$ or 1. So it is enough to discuss about $V_{\alpha,0,m,n}$ and $V_{\alpha,1,m,n}$. But the first one is discussed in lemma (4.15) and the second one is discussed in lemma (4.17). □

Corollary 4.20 Let $V_1 = \text{Spec}\, R[x_1, y_1]/(x_1y_1 - \pi)$ and $V_2 = \text{Spec}\, R[x_2, y_2]/(x_2^m y_2^n - \pi)$. Then $X = V_1 \times_{\text{Spec}R} V_2$ is singular and after some blowings-up we can resolve its singularities.

Proof: In theorem (4.19) put $a = m$ and $b = n$. □

Open problem 4.21 There exists a desingularisation for $X = V_1 \times_{\text{Spec}R} V_2$ where V_1 is of the form I_i and V_2 is of the form I_j in the following cases:

(a) $i = j = 2$;

(b) $i = 3$ and $j = 4$;

(c) $i = 5$ and $j = 2, 3, 4, 5$.

Chapter 5

Desingularisation of the fibre product of minimal regular models of Tate

5.1 Introduction

Any non-singular cubic with a K-rational point can be transformed into Weierstrass normal form. The affine equation of a Weierstrass normal form is

$$E = \operatorname{Spec} K[x,y]/(y^2 + a_1xy + a_3y - x^3 - a_2x^2 - a_4x - a_6). \qquad (5.1)$$

For more details about $b_2, b_4, b_8, c_4, c_6, \Delta$ and j see page 36 in [25]. Recall that for a UFD, R with $K = Q(R)$, $p \in R$ irreducible and $k(p) = Q(R/pR)$, the function

$$r_p : \mathbb{P}_n(R) \longrightarrow \mathbb{P}_n(k(p))$$

$$[y_0, y_1, ..., y_n] \longmapsto [\tilde{y_0}, \tilde{y_1}, ..., \tilde{y_n}]$$

(where $y_i \in R$ with no common factor), is called reduced map (mod$p$)-function. The representation $[\tilde{y_0}, \tilde{y_1}, ..., \tilde{y_n}]$ for $[y_0, y_1, ..., y_n]$ is called reduced. Note that the reduction $\tilde{E}$ of E is

$$\tilde{E} = \operatorname{Spec} k(p)[x,y]/(y^2 + \tilde{a}_1xy + \tilde{a}_3y - x^3 - \tilde{a}_2x^2 - \tilde{a}_4x - \tilde{a}_6). \tag{5.2}$$

If R is a discrete valuation ring with $Q(R) = K$, then $k(\xi)=Q(R/(\pi))=k$, and we write

$$\tilde{E} = \operatorname{Spec} k[x,y]/(y^2 + \tilde{a}_1xy + \tilde{a}_3y - x^3 - \tilde{a}_2x^2 - \tilde{a}_4x - \tilde{a}_6). \tag{5.3}$$

If for some choice of Weierstrass model (5.1) with $a_i \in R$, $\tilde{E}$ is non-singular, it is said that E over k has good reduction, otherwise E has bad reduction.

5.2 Neron model and Tate's algorithm

Let R be a Dedekind domain with $K = Q(R)$ and E/K be an elliptic curve. A Neron model for E/K is a smooth group scheme $\mathcal{E}/R$ whose generic fibre is E/K and satisfies the following universal property:

If $\mathcal{H}/K$ is a smooth R-scheme (*ie*, $\mathcal{H}$ is smooth over R) with generic fibre X/K and $\phi_K : X/K \longrightarrow E/K$ is a rational map defined over K, then there exists a unique R-morphism $\phi_R : \mathcal{H}/R \longrightarrow \mathcal{E}/R$ extending ϕ_K.

The algorithm of Tate computes the reduction type of an elliptic curve given by Weierstrass equation. He discusses about $\tilde{\mathcal{C}}/k$ which is the special fibre of $\mathcal{C}$ ($\mathcal{C}$ is a minimal proper regular model of E over R), *ie*, $\tilde{\mathcal{C}} = \mathcal{C} \times_{\operatorname{Spec} R} \operatorname{Spec} k$ and would be one of the types $I_0, I_n, II, III, IV, I_0^*, I_n^*, IV^*, III^*$ and II^*, see page 46 in [25] or page 365 in [23].

In this algorithm Tate starts with the given Weierstrass equation and uses a sequence of blowings-up to produce a minimal regular model for E.

Convention 5.1 We classify the minimal regular models of Tate as follows:

J_1 : $\tilde{E}$ is of the form I_n $(n > 0)$;

J_2 : $\tilde{E}$ is of the form II;

J_3 : $\tilde{E}$ is of the form III;

J_4 : $\tilde{E}$ is of the form IV;

J_5 : $\tilde{E}$ has one of the forms I_0^*, I_n^* for $n > 0$, IV^*, III^* and II^*.

Remark 5.2 Let X and Y be noetherian schemes (over R) and $f : X \longrightarrow Y$ be a morphism of finite type, $x \in X$, $y \in Y$ and $y = f(x)$, such that $k(x) \cong k(y)$. By theorem 3, page 249 in [17], $\hat{O}_{X,x} \cong \hat{O}_{Y,y}$ if and only if f is etale in a neighbourhood of x. If X and Y are regular schemes, to check that $\hat{O}_{X,x} \cong \hat{O}_{Y,y}$, it is enough to show it for tangent spaces, *ie*, to show that

$$\mathrm{m}_x/\mathrm{m}_x^2 \cong \mathrm{m}_y/\mathrm{m}_y^2.$$

For more details see pages 249-254 in [17] and page 116 in [1].

Definition 5.3 Let X and Y be schemes over R, $x \in X$ and $y \in Y$. We say (X, x) is an etale neighbourhood of (Y, y) if there exists a morphism $f : X \longrightarrow Y$ such that $f(x) = y$ and f is etale at x.

The smallest equivalence relation on pairs (Y, y) such that if (X, x) is an etale neighbourhood of (Y, y) then $(X, x) \sim (Y, y)$, is called etale equivalence or sometimes is called local isomorphism for etale topology.

Remark 5.4 With the notation as in definition (5.3) let $(X, x) \sim (Y, y)$. Then

X is regular at x if and only if Y is regular at y.

In fact by using remark (5.2) we get $\hat{O}_{X,x} \cong \hat{O}_{Y,y}$. But a noetherian local ring is regular if and only if its completion is regular (see [3], page 124, prop. 11.24).

Remark 5.5 In this section we want to show that

(a) Every minimal regular model of Tate at each singular point of its special fibre is etale equivalent to one V_j, $j = 1, 2, ..., 5$.

(b) The same chain of blowings-up used for resolution of singularities of the fibre products of V_j's (used in chapter 4) does the same on the fibre products of the Tate's minimal regular models.

Theorem 5.6 Let R be a dvr and p=char $k \neq 2, 3, d$ (where $d = (m, n)$ for the case i=5) and W_i be a minimal regular model of Tate of the form J_i for (E/K) and $Q \in (W_i)_\pi$ be a singular point of $(W_i)_\pi$. Then (W_i, Q) is etale equivalent to (V_i, P), for a singular point $P \in (V_i)_\pi$ (where V_i is an affine scheme of the form I_i).

Proof: Recall that $k = \frac{R}{(\pi)}$ is algebraically closed. We prove the theorem in five possible cases as follows:

CASE ONE ($i = 1$)

We can choose an affine neighbourhood $Z_1 = \operatorname{Spec} A_1$ of the singular point $Q \in (W_1)_\pi$ such that A_1 is a finitely generated R-algebra and $Z_1 \cap (W_1)_\pi$ is the divisor generated by $\tilde{u}\tilde{v} = 0$, where u , $v \in A_1$. This means that $\pi | uv$, *ie* $uv = \pi\epsilon$ for a unit element $\epsilon \in A_1$ and we get $u(v\epsilon^{-1}) = \pi$.

Now let $f : Z_1 \longrightarrow V_1 = \operatorname{Spec} R[x,y]/(xy - \pi)$ be the morphism induced by the ring homomorphism

$$f^\sharp : R[x,y]/(xy - \pi) \longrightarrow A_1$$

$$x \longmapsto u$$

$$y \longmapsto \epsilon^{-1}v.$$

Then f is of finite type and $f(Q) = P_0$. We can check that $k(Q) = k(P_0)$. Notice that Z_1 and V_1 are two-dimensional regular schemes. So $T_{Z_1,Q} \cong T_{V_1,P_0}$ (we can also use definition (2.5) to check that tangent spaces are isomorphic), which means that $\mathbf{m}_{P_0}/\mathbf{m}_{P_0}^2 \cong \mathbf{m}_Q/\mathbf{m}_Q^2$. This implies that $\hat{\mathcal{O}}_{V_1,P_0} \cong \hat{\mathcal{O}}_{Z_1,Q}$, which in turn implies that f is etale at Q (see corollary (4.5), page 116 in [1]). So it is etale in a neighbourhood of Q (see prop. (4.6), page 116 in [1]). Now we get the following diagram:

$$Z_1 = \operatorname{Spec} B_1$$

$$\swarrow \qquad \searrow$$

$$V_1 \qquad\qquad W_1$$

where $Z_1 \longrightarrow W_1$ is the inclusion. Note that if k is not algebraically closed, we can not assume that the components of the special fibres and singular points are defined over k. They are defined over a finite separable extension k'/k and there exists a discrete valuation ring R', etale over R with residue field k'. But by making the base change $R \longrightarrow R'$ we are reduced to the case $k = k'$.

CASE TWO ($i = 2$)

In this case the special fibre has a cusp at $Q \in (W_2)_\pi$. As it is shown in Tate's algorithm (see [25] or [23]), the minimal regular model W_2 has the following affine equation

$$v^2 - u^3 - \pi(a_{2,1}u^2 - a_{1,1}uv + a_{4,1}u - a_{3,1}v + a_{6,1}) = 0. \tag{5.4}$$

Let $w = a_{2,1}u^2 - a_{1,1}uv + a_{4,1}u - a_{3,1}v + a_{6,1}$ and $A = R[u,v]/(v^2 - u^3 - \pi w)$.

Since $\pi^2 \nmid a_6$ we get $\pi \nmid a_{6,1}$, w is invertible in an open set $U_2 = \operatorname{Spec} A[w^{-1}]$ containing the singular point Q. Let $B = A[w^{-1}, \alpha]/(\alpha^6 - w)$ and $Z_2 = \operatorname{Spec} B$. Then $Z_2 \longrightarrow \operatorname{Spec} A[w^{-1}] = U_2$ is etale (by Jacobian criterion). So locally for the etale topology $(U_2, Q) \cong (\operatorname{Spec} B, Q')$ where Q' is the singular point of $(Z_2)_\pi$. Recall that $B = R[u, v, w^{-1}, w^{\frac{1}{6}}]/(v^2 - u^3 - \pi w)$ and let $C = R[x,y]/(y^2 - x^3 - \pi)$. Then the morphism $f : Z_2 = \operatorname{Spec} B \longrightarrow \operatorname{Spec} C = V_2$ induced by the ring homomorphism

$$f^\sharp : R[x,y]/(y^2 - x^3 - \pi) \longrightarrow R[u, v, w^{-1}, w^{\frac{1}{6}}]/(v^2 - u^3 - \pi)$$

$$x \longmapsto w^{\frac{-1}{3}} u$$

$$y \longmapsto w^{\frac{-1}{2}} v$$

is etale at Q' (just compare the tangent spaces), so singularities are locally isomorphic for the etale topology, *ie*, $(\operatorname{Spec} B, Q') \sim (\operatorname{Spec} C, P_0)$. Now we have

$$V_2 = \operatorname{Spec} C \longleftarrow Z_2 = \operatorname{Spec} B \longrightarrow U_2$$

where both morphisms are etale. So we get $(V_2, P_0) \sim (Z_2, Q')$ and $(Z_2, Q') \sim (U_2, Q)$ which implies that $(V_2, P_0) \sim (U_2, Q)$, *ie*, (W_2, Q) is etale equivalent to (V_2, P_0).

CASE THREE ($i = 3$)

We can choose an affine neighborhood $U_3 = \operatorname{Spec} A_3$ (where A_3 is a finitely generated R-algebra) of Q such that $U_3 \cap (W_3)_\pi$ is the divisor generated by

$y'(y'-x'^2)$, where x' and y' are reductions $(\bmod \pi)$ of some u and v, so $v(v-u^2) \equiv 0(\bmod \pi)$, *ie*, $v(v-u^2) = \pi\epsilon$, $\epsilon \in A_3{}^*$ and we get

$$\pi = \frac{1}{\epsilon}v(v-u^2) = \frac{v}{\epsilon^{\frac{1}{2}}}(\frac{v}{\epsilon^{\frac{1}{2}}} - (\frac{u}{\epsilon^{\frac{1}{4}}})^2).$$

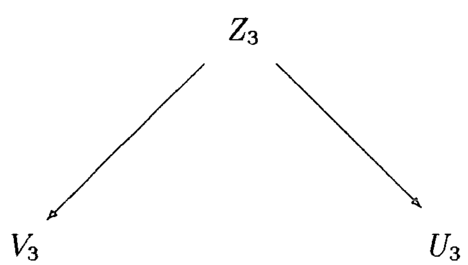
Now let $Z_3 = \operatorname{Spec} A_3[\epsilon^{\frac{1}{4}}]$ and $f : Z_3 \longrightarrow V_3 = \operatorname{Spec} R[x,y]/(y(y-x^2)-\pi)$ be the morphism induced by the ring homomorphism

$$f^\sharp : R[x,y]/(y(y-x^2)-\pi) \longrightarrow A_3[\epsilon^{\frac{1}{4}}] \tag{5.5}$$

$$x \longmapsto \frac{u}{\epsilon^{\frac{1}{4}}}$$

$$y \longmapsto \frac{v}{\epsilon^{\frac{1}{2}}}.$$

Then f is of finite type and $f(Q') = P_0$ where Q' is the singular point of $(Z_3)_\pi$. Now we can check that $k(Q') = k(P_0)$ and by doing the same conclusion as it was done in case one, $T_{Z_3,Q'} \cong T_{V_3,P_0}$. Since Z_3 and V_3 are both regular, we get $\widehat{\mathcal{O}}_{V_3,P_0} \cong \widehat{\mathcal{O}}_{Z_3,Q'}$. This implies that f is etale at Q' and so it is etale in a neighbourhood of Q'. We can check that $Z_3 = \operatorname{Spec} A_3[\epsilon^{\frac{1}{4}}] \longrightarrow \operatorname{Spec} A_3 = U_3$ is etale (by Jacobian criterion). So we get



where both morphisms are etale. Now we get $(V_3, P_0) \sim (Z_3, Q')$ and $(Z_3, Q') \sim (U_3, Q)$, which implies that $(V_3, P_0) \sim (U_3, Q)$, *ie*, (W_3, Q) is etale equivalent to (V_3, P_0).

CASE FOUR $(i = 4)$

Here the special fibre consists of three rational curves passing through a point $Q \in (W_4)_\pi$. We can choose an affine scheme $U_4 = \operatorname{Spec} A_4$ (where A_4 is a finitely generated R-algebra) such that $(U_4) \cap (W_4)_\pi$ is the divisor generated by $x'y'(x'-y')$, where x' and y' are reductions $(\bmod \pi)$ of some u and v. So

$uv(u-v) \equiv 0(\mathrm{mod}\pi)$, *ie*, $uv(u-v) = \pi\epsilon$, $\epsilon \in A_4{}^*$ and we get $\pi = \frac{1}{\epsilon}uv(u-v) = (\frac{u}{\epsilon^{\frac{1}{3}}})(\frac{v}{\epsilon^{\frac{1}{3}}})((\frac{u}{\epsilon^{\frac{1}{3}}}) - (\frac{v}{\epsilon^{\frac{1}{3}}}))$. Now let $A_4' = A_4[\epsilon^{\frac{1}{3}}]$, $Z_4 = \mathrm{Spec}\, A_4'$ and

$$f : Z_4 \longrightarrow V_4 = \mathrm{Spec}\, R[x,y]/(xy(x-y) - \pi)$$

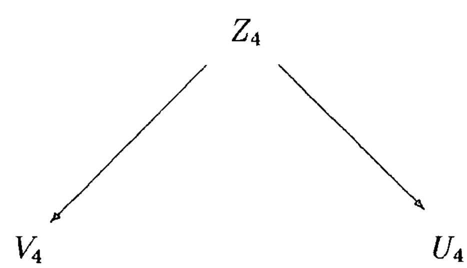
be the morphism induced by the ring homomorphism

$$f^\sharp : R[x,y]/(xy(x-y) - \pi) \longrightarrow A_4[\epsilon^{\frac{1}{3}}]$$

$$x \longmapsto \frac{u}{\epsilon^{\frac{1}{3}}}$$

$$y \longmapsto \frac{v}{\epsilon^{\frac{1}{3}}}.$$

Then f is of finite type and $f(Q') = P_0$, where Q' is the singular point of $(Z_4)_\pi$. As we showed in case 3, we can check that f is etale in a neighbourhood of Q'. We can also check that $Z_4 = \mathrm{Spec}\, A_4[\epsilon^{\frac{1}{3}}] \longrightarrow \mathrm{Spec}\, A_4 = U_4$ is etale (by Jacobian criterion). So we get



where both morphisms are etale. By using the same discussion as we did in case 3, we can show that (W_4, Q) is etale equivalent to (V_4, P_0).

CASE FIVE $(i = 5)$

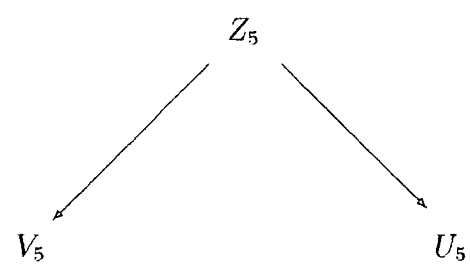
Here the special fibre is the union of some rational curves with multiplicities which is a normal crossing as a divisor. Let $Q \in (W_5)_\pi$ be the intersection of two of these rational curves. We can choose an affine scheme $U_5 = \mathrm{Spec}\, A_5$ (where A_5 is a finitely generated R-algebra) of Q such that $U_5 \cap (W_5)_\pi$ is the divisor generated by $x'^m y'^n$, where x' and y' are reductions $(\mathrm{mod}\pi)$ of some u and v. So $u^m v^n \cong 0(\mathrm{mod}\pi)$, *ie*, $u^m v^n = \pi\epsilon$, $\epsilon \in A_5{}^*$. Let d=gcd$(m,n) = am + bn$ (which is a unit, by our assumption). Then we get $u^m v^n = (\epsilon^d)^{\frac{1}{d}}\pi = \epsilon\pi$ which implies that

$$(\frac{u}{\epsilon^{\frac{a}{d}}})^m (\frac{v}{\epsilon^{\frac{b}{d}}})^n = \pi.$$

Let $A'_5 = A_5[\epsilon^{\frac{1}{d}}]$, $Z_5 = \operatorname{Spec} A'_5$ and $f : Z_5 \longrightarrow V_5 = \operatorname{Spec} R[x,y]/(x^m y^n - \pi)$ be the morphism induced by the ring homomorphism

$$f^\sharp : R[x,y]/(x^m y^n - \pi) \longrightarrow A_5[\epsilon^{\frac{1}{d}}]$$
$$x \longmapsto \frac{u}{\epsilon^{\frac{a}{d}}}$$
$$y \longmapsto \frac{v}{\epsilon^{\frac{b}{d}}}.$$

we can do the same conclusion as we did in cases 3 and 4, to show that f is etale in a neighbourhood of Q' (Q' is the corresponding singular point of $(Z_5)_\pi$). We can also check that $Z_5 = \operatorname{Spec} A_5[\epsilon^{\frac{1}{d}}] \longrightarrow \operatorname{Spec} A_5 = U_5$ is is etale (by Jacobian criterion). So we get



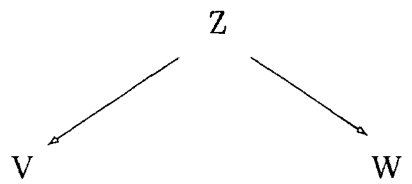
where both morphisms are etale. As we discussed in cases 3 and 4, this diagram shows that (W_5, Q) is etale equivalent to (V_5, P_0). □

Remark 5.7 The above discussion tells us that each minimal regular model of Tate at each singular point of its special fibre is etale equivalent to one V_j, $j = 1, 2, ..., 5$. In the following theorem W_α shows a minimal regular model of Tate of the form J_α, V_α is an arithmetic surface of the form I_α and Z_α is as it was used in theorem (5.6). We also use $W = W_\alpha \times_{\operatorname{Spec} R} W_\beta$, $V = V_\alpha \times_{\operatorname{Spec} R} V_\beta$ and $Z = Z_\alpha \times_{\operatorname{Spec} R} Z_\beta$, for $\alpha, \beta \in \{1, 2, ..., 5\}$.

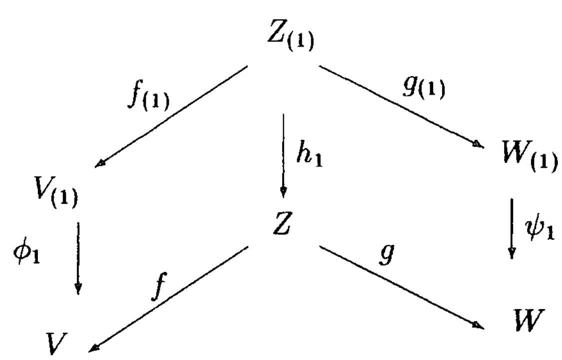
Recall that so far $\widetilde{X_n}$ was used for the n-th blowing-up of X with the centre X^{Sing} or a subscheme of X^{Sing}. Here we use $X_{(n)} = \widetilde{X_n}, n \geq 0$ (where $X_{(0)} = X$) and $\phi_n : X_{(n)} \longrightarrow X_{(n-1)}$ to show that X_n is the blowing-up of $X_{(n-1)}$ with the centre $X^{\text{Sing}}_{(n-1)}$.

Theorem 5.8 Let $p = \operatorname{char} k \neq 2, 3, d$ (d the same as in theorem (5.6)). If we are not in the cases $W_2 \times_{\operatorname{Spec} R} W_2$, $W_3 \times_{\operatorname{Spec} R} W_4$ and $W_5 \times_{\operatorname{Spec} R} W_j$ (for $j = 2, 3, 4, 5$), then there exists a desingularisation for $W = W_\alpha \times_{\operatorname{Spec} R} W_\beta$, $(1 \leq \alpha\ ,\ \beta \leq 5)$.

Proof: Let $W = W_{(0)} = W_\alpha \times_{\mathrm{Spec} R} W_\beta$ and consider $\psi_n : W_{(n)} \longrightarrow W_{(n-1)}$ as the blowing-up of $W_{(n-1)}$ with centre $W_{(n-1)}^{\mathrm{Sing}}$. Since the product of etale morphisms is etale, by using the results of the theorem (5.6), for each $Q \in W^{\mathrm{Sing}}$ we get the following diagram:

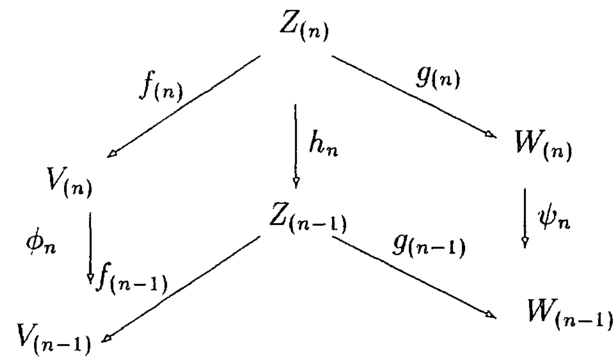


where $f : Z \longrightarrow V$ and $g : Z \longrightarrow W$ are etale and $Q \in g(Z)$. We define $Z_{(n)}$, h_n and also $V_{(n)}$, ϕ_n in the similar way that we defined $W_{(n)}$ and ψ_n. Inductively we find $f_{(n)}$ and $g_{(n)}$ by base change and surely they are etale (note that the blow-ups commute with flat basechange, so the squares are cartesian and we can deduce that the top arrows are etale from the knowledge that the bottom ones are). For $n = 1$ we get the following diagram:



Since $f_{(1)}$ and $g_{(1)}$ are etale, we get $f_{(1)}^{-1}(V_{(1)}^{\mathrm{Sing}}) = Z_{(1)}^{\mathrm{Sing}} = g_{(1)}^{-1}(W_{(1)}^{\mathrm{Sing}})$. If $V_{(1)}$ is regular we get $V_{(1)}^{\mathrm{Sing}} = \phi$, so $g_{(1)}^{-1}(W_{(1)}^{\mathrm{Sing}}) = \phi$, which means that $W_{(1)}^{\mathrm{Sing}} = \phi$, *ie*, $W_{(1)}$ is regular at points P such that $\psi_1(P) \in g(Z)$.

Now we do the same for other Z's so that $g(Z)$ covers W. If in all of such cases $W_{(1)}^{\mathrm{Sing}} = \phi$, we are done and the gluing of these regular schemes is the answer. Otherwise we continue the process of blowings-up and inductively we get the following diagram:



Suppose that $V_{(n-1)}$ is not regular. Since products and base extensions preserve the etale property of morphisms, so $f_{(n)}$ and $g_{(n)}$ are etale and the above diagram is commutative. Hence

$$f_{(n)}^{-1}(V_{(n)}^{\text{Sing}}) = Z_{(n)}^{\text{Sing}} = g_{(n)}^{-1}(W_{(n)}^{\text{Sing}})$$

By doing similar discussion with what we did about $W_{(1)}$ we can conclude that if $V_{(n)}$ is regular, then $W_{(n)}$ is regular at points P such that $\psi_1 \circ \psi_2 \circ \ldots \circ \psi_{n-1} \circ \psi_n(P) \in g(Z)$.

If $V_{(n)}$ is not regular we continue the process of blowings-up. Recall that in chapter 4 we have proved the existence of a desingularisation for V (in the involving cases). So there exists $k_0 \in \mathbb{N}$ such that for $n \geq k_0$, $V_{(n)}^{\text{Sing}} = \phi$. By using the same discussion as above we conclude that for $n \geq k_0$, $W_{(n)}$ is regular at any point whose image in W lies in $g(Z)$. Now we do it for other Z's so that $g(Z)$ covers W. Hence for sufficiently large n, $W_{(n)}$ is regular. This means that there exists a desingularisation for W. □

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