Gauge fields and quantum theory

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by

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To the memory of Euan Squires

Abstract

This thesis investigates the problems within quantum mechanics for the Bohm model caused by Lorentz invariance and the existence of photons. A model describing the electromagnetic interactions of fermions is produced which does not use photons and avoids these problems. It is then shown how these techniques can be extended to linearised gravitational interactions. Finally semi-classical gravity and the possibility of gravitationally induced collapse are considered.

In the first part of the thesis two modifications to the Bohm model are proposed. One takes account of Lorentz invariance, and the other is capable of describing photons. The main part of the thesis is devoted to describing interactions in a way which does not need extra gauge particles, and so is in the same spirit as the Bohm model.

Electromagnetic interactions are formed using a 4-potential operator which is calculated directly, without imposing commutation relations on the 4-potential. This leads to an expression for the 4-potential in terms of the Dirac field, and results in there being no photon states. There are various ways of constructing the theory and the scattering matrix of standard QED is compared to the scattering matrix of the version which appears to be most similar. Considering only the matrix elements between fermion states, they are found to be in agreement at the order e^2, but disagree at the order e^4. It follows that this model, which otherwise appears to be a self consistent theory of QED, cannot agree with experiment.

The same techniques can be used to quantise General Relativity when it is linearised about the Minkowski metric. The metric operator is calculated in terms of the Dirac field. The interaction is similar to that of electrodynamics, being of order 4 in the Dirac field. Finally issues relating to gravitational collapse are discussed.

Preface

The work presented in this thesis was carried out between October 1992 and September 1996 in the Department of Mathematical Sciences at the University of Durham, under the supervision of Professor Euan Squires. Tragically on 6$^{\text{th}}$ June 1996 Professor Squires died, and my supervision was completed by Dr. Lucien Hardy and Professor Ed Corrigan.

The material in this thesis has not been submitted for any degree in this or in any other university.

No claim of originality is made for the review in chapter 2 for the description of particles using wave equations, except for the derivation of the single time equation of Bohm and Hiley from the many-time system. The material in chapter 3 is based on work by Euan Squires and the author and appears in [1] and [2]. Some of the material of chapter 4 appears in [11] and [12]. The detailed treatment of the non-relativistic potential and of relativistic interactions however is believed to be original, and was completed by the author. With the exception of the review of QED and the introduction to the interaction representation, chapters 5, 6, 7 and 8 are original. Chapter 9 is based upon work by Euan Squires and the author, and is also believed to be original.

Firstly I would like to acknowledge the gratitude I have to my original supervisor, Euan Squires, for his help and support during my 3 years at Durham. It was with a deep sense of shock and sadness that I learned of his death in June 1996. I would like thank David Fairlie, Ed Corrigan, and Lucien Hardy for the concern they showed over the future of my PhD. I would also like to thank Lucien Hardy not only for completing my supervision, but for always being available for conversations during my time in Durham, and Ed Corrigan for his help with my supervision just before I submitted.

I have very much enjoyed my time in Durham. I would like to show my appreciation to my friends, my family and Susannah, without whose love and support I would not have been able to survive. Finally I would like to thank EPSRC for funding my PhD.

Statement of Copyright

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Chapter 1

Introduction

1.1 Quantum Mechanics and its Early Successes

Around the beginning of the century it was concluded that classical mechanics was not adequate for describing all physical phenomena. For instance classical electrodynamics predicts all atoms are unstable, and this is clearly contradicted by observation. This led to the birth of quantum mechanics. It was a radical theory and its concepts were very different.

A quantum mechanical system is described using observables which are linear operators, and a state vector $|\Phi>$. Observables include position $\hat{x}$ and momentum $\hat{p}$. If a system is such that an observable O has the definite value o' then

$$O|\Phi> = o'|\Phi>. \tag{1.1}$$

For instance we can use the state $|x'>$ to describe a particle that is at a position x' and this state obeys

$$\hat{x}|x'> = x'|x'>. \tag{1.2}$$

This notation allows states which do not correspond to a definite position such as

$$|x'> + |x''>, \tag{1.3}$$

where $x' \neq x''$. Similarly particles do not have to have a definite momentum. The position and momentum operators in a one dimensional system $\hat{x}$ and $\hat{p}$ obey the commutation relations

$$[\hat{x}, \hat{p}] = i\hbar. \tag{1.4}$$

This means that a particle cannot have a definite position and a definite momentum simultaneously.

The motion of particles can be described in two different ways. In the Schrödinger picture the dynamical variable is a state vector $|\Phi, t\rangle$ and the operators representing position and momentum remain constant. The state $|\Phi, t\rangle$ obeys the Schrödinger equation,

$$i\hbar\frac{\partial}{\partial t}|\Phi, t\rangle = H|\Phi, t\rangle, \tag{1.5}$$

where H is the Hamiltonian operator which represents the energy of the system. In the Heisenberg picture the dynamical variables are the observables, such as position and momentum, and the states remain constant. Each observable O obeys the Heisenberg equation

$$i\hbar\frac{\partial}{\partial t}O = [O, H]. \tag{1.6}$$

Amongst the first successes of quantum mechanics was the description of the energy levels of the hydrogen atom. The hydrogen atom can be modelled using the Schrödinger equation and predicts the energy levels with reasonable accuracy.

1.2 The Predictions and Limitations of the Dirac Equation

At a similar time the theory of special relativity was being developed by Einstein and this describes large fast moving particles very accurately. The principles of relativity can be extended to quantum mechanics and a single relativistic particle can be described by either the Klein-Gordon equation or the Dirac equation. The predictions of the Dirac equation for the energy levels of the hydrogen atom are much more accurate than that of the Schrödinger or the Klein-Gordon equations. They nevertheless do not completely agree with experiment.

The Dirac equation can be written in the form

$$i\hbar\frac{\partial}{\partial t}\Psi_{(\alpha)}(\mathbf{x},t) = H_{(\alpha\beta)}\Psi_{(\beta)}(\mathbf{x},t), \tag{1.7}$$

where $\mathbf{x} \in \mathbf{R}^3$. The Dirac particle Hamiltonian $H$is a 4×4 matrix operator, with $H_{(\alpha\beta)}$ its matrix elements. The object $\Psi_\alpha(\mathbf{x},t)$ is a 4 component spinor and Dirac particles, that is spin $\frac{1}{2}$ particles, come in 4 varieties. Each particle has a positive or negative energy, and an up or down spin. Dirac concluded that the observed vacuum state was a sea of negative energy particles and that the observed particles were positive energy particles and anti-particles, which are the lack of a negative energy particle in the sea. The existence of anti-particles and of particles with different spins has been experimentally verified.

The great success of the Dirac equation did come at a price. Its interpretation is even less easy that of the Schrödinger equation. It is also far from obvious how to describe interacting spin $\frac{1}{2}$ particles. When the hydrogen atom is modelled with the Dirac equation, the model is really one of a single particle, whose mass is the reduced mass of the electron and proton, in a background electromagnetic field which is the field produced classically by a stationary charged particle. In reality a hydrogen atom contains two charged particles, one electron and one proton, undergoing electromagnetic interactions. These two systems are not exactly equivalent and this is one reason why the energy levels calculated using the Dirac equation are not those experimentally observed. It is also less than obvious how to introduce full electromagnetic interactions into quantum mechanics.

1.3 Quantum Electrodynamics

In order to handle genuine interactions Quantum Electrodynamics (QED) was devised. QED is a field theory and is based on different premises to normal quantum mechanics. Field theory takes quantum mechanics one step further and performs what is often called second quantisation. To see the differences we can briefly consider a scalar field theory. This is described with the field $\phi(\mathbf{x},t)$, which is initially treated as a wave-function and obeys the Klein-Gordon equation. The field $\phi(\mathbf{x},t)$ is a dynamical variable and has a conjugate momentum, which is denoted by $\pi(\mathbf{x},t)$. An analogy is drawn between position and momentum, and $\phi(\mathbf{x},t)$ and $\pi(\mathbf{x},t)$, and the commutation relations

$$[\phi(\mathbf{x},t),\pi(\mathbf{x}',t)] = i\hbar\delta^3(\mathbf{x}-\mathbf{x}'), \qquad [\phi(\mathbf{x},t),\phi(\mathbf{x}',t)] = 0, \tag{1.8}$$

are imposed.

QED contains two fields, a Dirac field $\psi(\mathbf{x},t)$ which is a spinor and a photon field $A^\mu(\mathbf{x},t)$. It views electrodynamics as the interactions between Dirac particles and photons. This is very unlike what is imagined classically, where photons do not exist. A brief introduction to QED can be found in section 6.2.

QED gives many predictions and these agree remarkably accurately with experiment, one of which is the corrections to the energy levels of the hydrogen atom, known as the Lamb shift. From scattering theory it can be deduced that the electromagnetic field produced by a bound state is not that calculated from the obvious classical analogy, but contains small corrections. The energy levels calculated with the single particle Dirac equation using the corrected field as a background field are found to be much more accurate.

1.4 A Brief Overview of the Thesis

This thesis investigates the problems within quantum mechanics caused by Lorentz invariance, the Bohm model and the existence of photons (gravitons). Chapter 2 is a review of the evolution equations of quantum mechanics. Chapter 3 proposes two modifications of the Bohm model, one which takes account of Lorentz invariance, and another which is compatible with the existence of photons. The main part of the thesis is concerned with a description of the interactions of Dirac particles which do not need gauge particles.

A method of quantising the electromagnetic field is developed, using only standard quantum mechanics, and which contains no photons. There are different ways of formulating the theory. The scattering matrix is calculated for one particular formulation, which appears to be the most similar to QED, and it is compared to the scattering matrix of QED. Considering only the elements of the scattering matrices between fermion states, they are found to be in agreement up to the order of e^2 but do not agree at the order of e^4.

It is then shown how the techniques used to describe electromagnetic interactions can be successfully adapted to linear gravity. This produces a theory whose interaction Hamiltonian is of order 4 in the Dirac field, and which is of a similar form to the previous theory of electromagnetism.

Finally issues relating to quantum state reduction are discussed.

1.5 The Wave Function Description of Quantum Mechanics

Chapter 2 considers the wave function description of quantum mechanics and particles in background fields. Many Dirac particles can be described by spinors where each particle has its own spinor index, so for instance n Dirac particles can be described by $\Psi_{(\alpha_1 \ldots \alpha_n)}$. How to write down the equation(s) of motion is not obvious. Bell [3] suggested that to allow for the time translation produced by Lorentz invariance, a two particle wave-function Φ should have two times and obey two equations of the form

$$i\hbar \frac{\partial}{\partial t_1}\Phi = H_1\Phi, \qquad i\hbar \frac{\partial}{\partial t_2}\Phi = H_2\Phi. \tag{1.9}$$

This thesis starts by describing a system of Dirac particles by giving each particle its own space-time variable and its own equation. If there are n particles then the equation for particle k is

$$i\hbar \frac{\partial}{\partial t_k}\Psi_{n(\alpha_1 \ldots \alpha_{k-1}\alpha_k\alpha_{k+1}\ldots\alpha_n)}(x_1, \ldots, x_n) = H_{k(\alpha_k\beta)}\Psi_{n(\alpha_1 \ldots \alpha_{k-1}\beta\alpha_{k+1}\ldots\alpha_n)}(x_1, \ldots, x_n). \tag{1.10}$$

$H_{k(\alpha\beta)}$ are the matrix elements of the Dirac Hamiltonian for particle k which can contain background electromagnetic and/or gravitational fields. In any frame the space-time variable x_k contains its own time t_k meaning that Ψ_n has many times. It will be shown that for such an n particle system we can deduce the single time equation

$$i\hbar\frac{\partial}{\partial t}\Psi_{n(\alpha_1\ldots\alpha_n)}(\mathbf{x}_1,\ldots,\mathbf{x}_n,t) = H_{1(\alpha_1\beta)}\Psi_{n(\beta\ldots\alpha_n)}(\mathbf{x}_1,\ldots,\mathbf{x}_n,t)$$
$$+\ldots+H_{n(\alpha_n\beta)}\Psi_{n(\alpha_1\ldots\beta)}(\mathbf{x}_1,\ldots,\mathbf{x}_n,t). \quad (1.11)$$

This equation has been proposed by Bohm and Hiley. Here $\mathbf{x}$ denotes a spatial position and $x = (\mathbf{x}, t)$ a position in space-time. When we attempt to describe interacting systems using the original set of many Dirac equations as in Eq.(1.10), we find that the equations are not compatible. Eq.(1.11) however has no such compatibility problems. It is thus postulated that Dirac particles in a background electromagnetic and/or gravitational field obey Eq.(1.11).

1.6 The Bohm Model

Our experience of the world is that objects have a precise position. Indeed any measurement will give a result that is definite (even if it is not accurate). In Quantum mechanics everything seems 'fuzzy'. Particles occupy a distribution of positions, rather than a single value. How is it possible to interpret quantum mechanics in a way which is consistent with experience?

One of the most successful explanations is that of Bohm [4]. It gives each particle a trajectory in a way which preserves the statistical predictions of quantum mechanics. Can the Bohm model be made compatible with theories such as QED?

Since the work of Bell [5], it has been known that any hidden-variable model of quantum theory must be non-local. That it must also violate Lorentz-invariance was shown by Hardy [6] (see also [7]). The Bohm model can be modified so that it takes account of Lorentz invariance [8]. Clearly this modified model, which we call the Retarded Bohm Model, will violate quantum theory. It is not clear, however, whether it violates the results of any actual experiments.

The first part of chapter 3 shows how the Retarded Bohm Model evades the Hardy proof of the lack of Lorentz-invariance of hidden-variable models. Another possible test of the retarded model is also revealed.

QED contains photons as well as fermions and it is possible to describe photons using the Bohm model. The second part of Chapter 3 proposes a version of the Bohm model which is suitable when there are particles present which do not have trajectories, such as photons.

1.7 Electromagnetic Interactions

The remainder of the thesis discusses a method of producing both non-relativistic and relativistic interactions. This is done in a way which does not use extra particles and is consistent with the spirit of the Bohm model in that it only has particles where the Bohm model has trajectories. Chapter 4 describes non-relativistic scalar particles. An operator field $\phi(\mathrm{x})$ is introduced which can be compared to the operator field in scalar field theory, and which obeys the commutation relations

$$[\phi(\mathrm{x}), \phi^{\dagger}(\mathrm{x}')] = \delta^3(\mathrm{x} - \mathrm{x}'), \qquad [\phi(\mathrm{x}), \phi(\mathrm{x}')] = 0. \tag{1.12}$$

$\phi^{\dagger}(\mathrm{x})$ creates a particle at a position $\mathbf{x}$ in space. The motivation behind the commutation relations is that they lead to standard normalisations of bosons (see section 4.2). No analogy is drawn between the $\phi(\mathrm{x})$ field and a wave-function, it is merely a set of operators that annihilate particles. The $\phi(\mathrm{x})$ operators are used to construct other operators and from them the equation of motion for the state $|\Phi, t\rangle$ is built. Field theory in a similar context is discussed by Schweber [11] and Lawrie [12].

The advantage of this notation is it allows the simple and rigorous construction of physical fields and interactions. The theory of quantum electrostatics is considered in detail and it is shown that the theory can be constructed so that self interactions are either included or excluded. It is then shown how relativistic interactions can be formed.

Chapter 5 shows how this notation can be extended to Dirac particles. There are four types of Dirac particle and so we use 4 corresponding operator fields $\psi_\alpha(\mathbf{x})$ which obey the anti-commutation relations

$$\{\psi^\dagger_\alpha(\mathbf{x}), \psi_\beta(\mathbf{x}')\} = \delta_{\alpha\beta}\delta^3(\mathbf{x}-\mathbf{x}'), \qquad \{\psi_\alpha(\mathbf{x}), \psi_\beta(\mathbf{x}')\} = 0. \tag{1.13}$$

It is shown that the n-particle components of the equation

$$i\hbar\frac{\partial}{\partial t}|\Psi, t\rangle = \int d^3\mathbf{x}\, \psi^\dagger_\alpha(\mathbf{x}) H_{(\alpha\beta)}\psi_\beta(\mathbf{x})|\Psi, t\rangle, \tag{1.14}$$

are Eq.(1.11), where $|\Psi, t\rangle$ is expanded in terms of the wave-functions Ψ_n. The H in Eq.(1.14) is the Hamiltonian of each single particle and is dependent on $\mathbf{x}$ and its derivatives. The particles in Eq.(1.14) are identical. The H_k in Eq.(1.11) is the Hamiltonian of particle (k), and is formed from H by replacing $\mathbf{x}$ with $\mathbf{x}_k$. This implies that Eq.(1.14) can be used to describe the evolution of the state vector $|\Psi, t\rangle$ through time.

At the moment, the Hamiltonian H contains only background fields. To produce electromagnetic interactions we must replace the background electromagnetic field in Eq.(1.14) with the correct electromagnetic field operator. It can be argued that the differences between the description of Dirac particles here and in standard field theory are only philosophical, and that mathematically the theories are equivalent. The description of electromagnetic interactions however is genuinely different. Classically the electromagnetic field can be expressed by the electromagnetic field tensor $F^{\mu\nu}$ which obeys

$$\partial_\mu F^{\mu\nu} = j^\nu, \tag{1.15}$$

where j^ν is the 4-current vector. QED describes interactions by imposing commutation relations on the electromagnetic field operator. It regards photons as real particles. The system described here forms electromagnetic interactions by imposing an operator version of Eq.(1.15). This equation can be solved and we gain an expression for the electromagnetic field operator in terms of the $\psi_\alpha(\mathbf{x})$ fields. The electromagnetic field operator is analogous to the classical electromagnetic field and photons do not have an independent existence.

Many experiments have been performed which handle individual photons and there is much evidence claimed for their existence. Unless these photons existed at the beginning of time they always come from a source, and their characteristics are determined by this source. An atom can be made to jump to a higher energy level by a photon of the correct frequency, so it is said. However this photon must have originated somewhere and the jump can be considered merely as an interaction between the source and the atom, which will proceed if the energy lost by the source is the same as the energy gained by the atom. Experiments have also been performed using superpositions of photon states. This can be considered as merely a superposition of the states of the source.

The success of such a theory can only be determined by its experimental predictions, and how closely it agrees with standard QED. There is a choice of possible 4-potential operators, and the one which appears to give the best results is chosen. In chapter 6 the scattering matrix is calculated for this 4-potential, and it is compared to the scattering matrix of standard QED. The matrix elements between fermion states are found to be in agreement for the terms of the order of e^2 but do not agree at the order of e^4.

1.8 Quantum Gravity

The failure of these techniques to produce an accurate theory of electromagnetism does not automatically mean that they will fail with gravity. It is well known that there are many difficulties in constructing a theory of quantum gravity in the conventional way, so this alternative method is worth trying. A theory of quantum linear gravity can be constructed and it may be possible to generalise this to non-linear gravity.

Dirac particles in a background gravitational field can be described using an equation of the form of Eq.(1.11). As before this is equivalent to the state vector equation Eq.(1.14). To form gravitational interactions we simply replace the background metric in Eq.(1.14) with a metric operator. In linear gravity the metric operator obeys an operator version of the linearised Einstein's equation, an equation which can be solved. The un-linearised Einstein's equation in general is not solvable and it is uncertain as to whether these techniques can be extended to a full theory of quantum gravity. Chapter 7 discusses the metric operator. Chapter 8 is a detailed discussion of Quantum Linear Gravity. The interaction Hamiltonian is of order 4 in the Dirac field and it is shown how to calculate a scattering matrix for this theory.

Penrose [13] argues that any theory of quantum gravity must necessarily lead to quantum state reduction. His arguments are based upon the assumption that part of the state represents the gravitational field (i.e. the assumption that gravitons exist). This assumption is violated by the theory developed in chapters 7 and 8 and indeed they do not involve quantum state reduction.

Nevertheless his arguments should not be dismissed. There is no evidence either way as to whether gravitons actually do exist. Penrose argues that the collapse time of a superposition of states is related to its gravitational self energy. The theory of semi-classical gravity is investigated and it is shown that this energy appears within it. However the mechanism for collapse is unclear. In semi-classical gravity this self energy acts as an attractive force between superposed states, and there is no such collapse in semi classical gravity.

Chapter 2

The Wave Function Description

2.1 Introduction

This chapter concentrates on the wave function description of quantum mechanics and the motion of particles in background fields. A brief review of non-relativistic quantum mechanics and the Schrödinger equation is given before relativistic wave equations are considered. A single relativistic scalar particle can be described by the Klein-Gordon equation and a single particle of spin-$\frac{1}{2}$ by the Dirac equation. Electrons are spin-$\frac{1}{2}$ and the Dirac equation has given successful experimental predictions. Many body relativistic systems which contain interactions are much more difficult to describe and there are a number of possible approaches. The method chosen here involves sacrificing manifest Lorentz invariance, and this leads to the many body Dirac equation proposed by Bohm and Hiley [14].

The motion of Dirac particles in background electromagnetic and gravitational fields is reviewed. We are particularly interested in linear gravity. The metric caused by a space-time containing a single fixed particle is found within the context of linear gravity. The motion of Dirac particles in this space-time is discussed and it is shown how it is in agreement with Newtonian gravity.

2.2 Non-relativistic Particles

Classically a single non-relativistic particle of momentum $\mathbf{p}$ and mass m moving in a force field with potential $V(\mathbf{x})$ obeys the energy equation

$$E = \frac{\mathbf{p}^2}{2m} + V(\mathbf{x}), \tag{2.1}$$

where $\mathbf{x}, \mathbf{p} \in \mathbf{R}^3$. The energy E of the system is a constant.

Quantum systems are described by a state vector $|\Phi, t>$. The state $|\mathbf{x}>$ represents a particle in an eigen-state of position with eigenvalue $\mathbf{x}$, which is normalised according to

$$< \mathbf{x}|\mathbf{y} >= \delta^3(\mathbf{x}-\mathbf{y}). \tag{2.2}$$

A general state of a single particle can be expanded as

$$|\Phi, t> = \int d^3\mathbf{x}\, \Phi(\mathbf{x}, t)|\mathbf{x}>, \tag{2.3}$$

where using Eq.(2.2) the wave-function is,

$$\Phi(\mathbf{x}, t) =< \mathbf{x}|\Phi, t > . \tag{2.4}$$

Note that we differentiate between the concepts of 'wave-function' and 'state'. An analogy can be drawn between quantum mechanics and vectors on $\mathbf{R}^2$. We can expand a general point as

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2, \tag{2.5}$$

where $\mathbf{e}_1$ and $\mathbf{e}_2$ are the basis vectors of $\mathbf{R}^2$ and a_1 and a_2 are numbers. The a_i are the co-ordinates of the vector but not the vector itself, and are analagous to the wave-function. Eq.(2.5) itself is a vector and analogous to the state.

We see that the quantum state is completely determined by its expansion in the position basis. We postulate that the state which represents a particle of momentum $\mathbf{p}$ is

$$|\mathbf{p}> = (2\pi\hbar)^{-\frac{3}{2}} \int d^3\mathbf{x} \exp\frac{i\mathbf{x}.\mathbf{p}}{\hbar}|\mathbf{x}>. \tag{2.6}$$

Since

$$-i\hbar\nabla \exp\frac{i\mathbf{x}.\mathbf{p}}{\hbar} = \mathbf{p}\exp\frac{i\mathbf{x}.\mathbf{p}}{\hbar}, \tag{2.7}$$

the momentum operator, that is the operator $\hat{\mathbf{p}}$ which obeys

$$\hat{\mathbf{p}}|\mathbf{p}'> = \mathbf{p}'|\mathbf{p}'>, \tag{2.8}$$

is

$$\hat{\mathbf{p}} = \int d^3\mathbf{x}|\mathbf{x}>(-i\hbar\nabla) < \mathbf{x}|. \tag{2.9}$$

We also postulate that the operator $i\hbar\frac{\partial}{\partial t}$ represents the total energy of the system.

The equation of motion of a single non-relativistic particle is formed from Eq.(2.1). We perform the substitution

$$E \rightarrow i\hbar\frac{\partial}{\partial t}, \qquad \mathbf{p} \rightarrow -i\hbar\nabla, \tag{2.10}$$

and $\Phi(\mathbf{x}, t)$ obeys

$$i\hbar\frac{\partial}{\partial t}\Phi(\mathbf{x}, t) = -\frac{\hbar^2\nabla^2}{2m}\Phi(\mathbf{x}, t) + V(\mathbf{x})\Phi(\mathbf{x}, t). \tag{2.11}$$

This can be easily extended to larger numbers of particles. Classically a system of n non-interacting particles moving in an external force with a potential $V(\mathbf{x})$ obeys

$$E = \frac{\mathbf{p}_1^2}{2m} + V(\mathbf{x}_1) + \frac{\mathbf{p}_2^2}{2m} + V(\mathbf{x}_2) + \ldots + \frac{\mathbf{p}_n^2}{2m} + V(\mathbf{x}_n). \tag{2.12}$$

Each particle (j) is at the spatial position $\mathbf{x}_j$, has a momentum of $\mathbf{p}_j$ and has a mass m.

The state $|\mathbf{x}_1; \mathbf{x}_2; \ldots; \mathbf{x}_n>$ represents n identifiable particles, each of which is an eigenstate of position with the eigenvalue of particle (j) being $\mathbf{x}_j$. This state is normalised according to

$$< \mathbf{x}_1; \mathbf{x}_2; \ldots; \mathbf{x}_n|\mathbf{y}_1; \mathbf{y}_2; \ldots; \mathbf{y}_n >= \delta^3(\mathbf{x}_1 - \mathbf{y}_1)\delta^3(\mathbf{x}_2 - \mathbf{y}_2)\ldots\delta^3(\mathbf{x}_n - \mathbf{y}_n). \tag{2.13}$$

A general state of a n particles can be expanded as

$$|\Phi, t> = \int d^3\mathbf{x}_1 \ldots d^3\mathbf{x}_n\, \Phi(\mathbf{x}_1, \ldots, \mathbf{x}_n, t)|\mathbf{x}_1; \ldots; \mathbf{x}_n>, \tag{2.14}$$

where using Eq.(2.13) the wave-function is,

$$\Phi(\mathbf{x}_1, \ldots, \mathbf{x}_n, t) = < \mathbf{x}_1; \ldots; \mathbf{x}_n | \Phi, t > . \qquad (2.15)$$

Again the state is entirely determined by the its expansion in the position basis. The state which represents n particles with momenta $\mathbf{p}_1, \ldots, \mathbf{p}_n$ is

$$|\mathbf{p}_1; \ldots; \mathbf{p}_n> = (2\pi\hbar)^{-\frac{3n}{2}} \int d^3\mathbf{x}_1 \ldots d^3\mathbf{x}_n \, \exp i \left(\frac{\mathbf{x}_1.\mathbf{p}_1 + \ldots + \mathbf{x}_n.\mathbf{p}_n}{\hbar} \right) |\mathbf{x}_1; \ldots; \mathbf{x}_n>. \qquad (2.16)$$

We see that

$$-i\hbar\nabla_j \exp i \left(\frac{\mathbf{x}_1.\mathbf{p}_1 + \ldots + \mathbf{x}_n.\mathbf{p}_n}{\hbar} \right) = \mathbf{p}_j \exp i \frac{\mathbf{x}_1.\mathbf{p}_1 + \ldots + \mathbf{x}_n.\mathbf{p}_n}{\hbar}, \qquad (2.17)$$

and the momentum operator for particle (j) is thus

$$\mathbf{p}_j = \int d^3\mathbf{x}_1 \ldots d^3\mathbf{x}_n |\mathbf{x}_1; \ldots; \mathbf{x}_n> (-i\hbar\nabla_j) < \mathbf{x}_1; \ldots; \mathbf{x}_n|. \qquad (2.18)$$

Here ∇_j refers to differentiation with respect to $\mathbf{x}_j$.

The equation of motion for a system of n non-relativistic particles in an external force, each of mass m, is formed from Eq.(2.12) by performing the substitution

$$E \to i\hbar\frac{\partial}{\partial t}, \qquad \mathbf{p}_j \to -i\hbar\nabla_j. \qquad (2.19)$$

The n particle wave-function $\Phi(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n, t)$ obeys

$$i\hbar\frac{\partial}{\partial t}\Phi = \left[-\hbar^2\frac{\nabla_1^2}{2m} + V(\mathbf{x}_1) - \hbar^2\frac{\nabla_2^2}{2m} + V(\mathbf{x}_2) + \ldots - \hbar^2\frac{\nabla_n^2}{2m} + V(\mathbf{x}_n) \right] \Phi. \qquad (2.20)$$

Consider the case where the external force is caused by a particle of charge e fixed at the origin, and where each of the particles described by Eq.(2.20) also has charge e. The potential obeys

$$\nabla^2 V(\mathbf{x}) = -e\rho(\mathbf{x}), \qquad (2.21)$$

where $\rho(\mathbf{x})$ is the charge density field. Here and throughout the thesis we use Heaviside's units. The particle fixed at the spatial origin has a charge density of $\rho(\mathbf{x}) = e\delta^3(\mathbf{x})$. We see that the solution of Eq.(2.21) which vanishes at infinity is

$$V(\mathbf{x}) = \frac{e^2}{4\pi|\mathbf{x}|}, \qquad (2.22)$$

and Eq.(2.20) can be written explicity as

$$i\hbar\frac{\partial}{\partial t}\Phi = \left[-\hbar^2\frac{\nabla_1^2}{2m} + \frac{e^2}{4\pi|\mathbf{x}_1|} - \hbar^2\frac{\nabla_2^2}{2m} + \frac{e^2}{4\pi|\mathbf{x}_2|} + \ldots - \hbar^2\frac{\nabla_n^2}{2m} + \frac{e^2}{4\pi|\mathbf{x}_n|}\right]\Phi. \quad (2.23)$$

We can also produce equations of motion for interacting particles in a similar manner. We discuss interactions later in section 4.3.

2.3 Relativistic Particles

Classically a free single relativistic particle of mass m obeys

$$p^2 = m^2, \quad (2.24)$$

where $p \in \mathbf{R}^4$ and we are using the Minkowski metric with a $(+1,-1,-1,-1)$ signature. This system can be quantised in a similar manner to the previous section. We replace p_μ with the operator $-i\hbar\partial_\mu$ and obtain the Klein-Gordon equation

$$-\hbar^2\partial^\mu\partial_\mu\Phi(x) = m^2\Phi(x), \quad (2.25)$$

where $x \in \mathbf{R}^4$. To describe a single particle of charge e in an electromagnetic field given by the 4-potential $A^\mu(x)$ we perform the substitution $-i\hbar\partial_\mu \rightarrow -i\hbar\partial_\mu + eA_\mu$ and its wave function $\Phi(x)$ obeys

$$(-i\hbar\partial^\mu + eA^\mu(x))^2\Phi(x) = m^2\Phi(x). \quad (2.26)$$

The question of how to describe two relativistic particles is less easy to answer. In a background electromagnetic field $A^\mu(x)$ we would expect to use

$$(i\hbar\partial_1^\mu - eA^\mu(x_1))^2\Phi(x_1,x_2) = m^2\Phi(x_1,x_2),$$

$$(i\hbar\partial_2^\mu - eA^\mu(x_2))^2\Phi(x_1,x_2) = m^2\Phi(x_1,x_2), \quad (2.27)$$

where $\partial_j^\mu = \frac{\partial}{\partial x_{j\mu}}$. There are two major problems here. The first is that since there are two space-time variables x_1 and x_2, in any frame there are two times t_1 and t_2. It is not at all obvious how to interpret this situation. The second problem arises when we try to include interactions. The electromagnetic field is now dependent on the position of both particles, $A^\mu(\mathbf{x}_1, \mathbf{x}_2)$. The second problem is that in general the equations of the interacting system are not compatible with each other. Much has been written on this subject and how to alter these equations so that they are compatible. See for instance Komar [15][16], Horwitz and Rohrlich [17], Crater and Van Alstine [18], Sazdjian [19] [20], and Horwitz and Rotbart [21]. In this thesis a different route shall be followed.

The method can best be explained in terms of spin-$\frac{1}{2}$ particles. A single free Dirac particle can be described by a spinor $\Psi(\mathbf{x}, t)$ which obeys

$$i\hbar\frac{\partial}{\partial t}\Psi = H\Psi. \tag{2.28}$$

The Hamiltonian is

$$H = \gamma^0 \left(-i\hbar\gamma^j\partial_j + m\right). \tag{2.29}$$

j sums from 1 to 3, and the γ^μ are 4×4 matrices obeying the anti-commutation relations

$$\{\gamma^\mu, \gamma^\nu\} = 2r^{\mu\nu}. \tag{2.30}$$

In writing the Dirac equation in this manner a particular frame has been selected. Within this frame we use $\mathbf{x} \in \mathbf{R}^3$ to denote spatial co-ordinates, t is the time and $x = (\mathbf{x}, t)$ is a point of space-time. To emphasise that H is a matrix operator and that Ψ is a spinor we write Eq.(2.28) in component form,

$$i\hbar\frac{\partial}{\partial t}\Psi_{(\alpha)} = H_{(\alpha\beta)}\Psi_{(\beta)}. \tag{2.31}$$

This equation has two types of solution. One is

$$\Psi(\mathbf{x}, t) = \exp\frac{-it\sqrt{\mathbf{p}^2 + m^2} + i\mathbf{x}.\mathbf{p}}{\hbar}u_\tau(\mathbf{p}), \qquad \tau = 1, 2, \tag{2.32}$$

which represents a positive energy particle of spatial momentum $\mathbf{p}$. $u_1(\mathbf{p})$ and $u_2(\mathbf{p})$ are the usual 4 component spinors for positive energy particles. The exact expression is dependent upon the basis being used. The second type is

$$\Psi(\mathbf{x}, t) = \exp\frac{it\sqrt{\mathbf{p}^2 + m^2} - i\mathbf{x}.\mathbf{p}}{\hbar}v_\tau(\mathbf{p}), \qquad \tau = 3, 4, \tag{2.33}$$

which represents a negative energy particle of spatial momentum $\mathbf{p}$. $v_1(\mathbf{p})$ and $v_2(\mathbf{p})$ are the usual 4-component spinors for negative energy particles. In order to express these solutions in one equation we define the object κ_τ by

$$\kappa_\tau = \begin{cases} 1 & \tau = 1, 2, \\ -1 & \tau = 3, 4. \end{cases} \tag{2.34}$$

The solutions Eqs.(2.32,2.33) can now be written as the single equation

$$\Psi(\mathbf{x}, t) = \exp \frac{\kappa_\tau \left(-it\sqrt{\mathbf{p}^2 + m^2} + i\mathbf{x}.\mathbf{p}\right)}{\hbar} u_\tau(\mathbf{p}). \tag{2.35}$$

Here we use the notation $u_3(\mathbf{p}) = v_1(\mathbf{p})$ and $u_4(\mathbf{p}) = v_2(\mathbf{p})$, and this is continued throughout the thesis. We see that Eq.(2.28) is an energy equation,

$$\begin{aligned} & i\hbar \frac{\partial}{\partial t} \exp \frac{\kappa_\tau \left(-it\sqrt{\mathbf{p}^2 + m^2} + i\mathbf{x}.\mathbf{p}\right)}{\hbar} u_{\tau(\alpha)}(\mathbf{p}) \\ &= H_{(\alpha\beta)} \exp \frac{\kappa_\tau \left(-it\sqrt{\mathbf{p}^2 + m^2} + i\mathbf{x}.\mathbf{p}\right)}{\hbar} u_{\tau(\beta)}(\mathbf{p}) \\ &= \kappa_\tau \sqrt{\mathbf{p}^2 + m^2} \exp \frac{\kappa_\tau \left(-i\sqrt{\mathbf{p}^2 + m^2} + i\mathbf{x}.\mathbf{p}\right)}{\hbar} u_{\tau(\alpha)}(\mathbf{p}), \end{aligned} \tag{2.36}$$

where $u_{\tau(\alpha)}(\mathbf{p})$ denotes the α component of $u_\tau(\mathbf{p})$.

Several free Dirac particles can be described by multiple Dirac equations. The wave-function of n free particles is now a multiple spinor $\Psi_{n(\alpha_1\alpha_2\ldots\alpha_n)}(x_1, x_2, \ldots, x_n)$. Each particle has its own space-time position, its own spinor index and its own equation. This means that

$$i\hbar \frac{\partial}{\partial t_k} \Psi_{n(\alpha_1\ldots\alpha_{k-1}\alpha_k\alpha_{k+1}\ldots\alpha_n)} = H_{k(\alpha_k\beta)} \Psi_{n(\alpha_1\ldots\alpha_{k-1}\beta\alpha_{k+1}\ldots n)}, \tag{2.37}$$

for each k where $1 \le k \le n$. H_k is the Hamiltonian in Eq.(2.29) with each of the variables x substituted by x_k, so that H_k is the Hamiltonian for particle (k).

To avoid confusion over indices we consider the two particle case, where there are two equations

$$i\hbar \frac{\partial}{\partial t_1} \Psi_{2(\alpha\beta)} = H_{1(\alpha\tau)} \Psi_{2(\tau\beta)}, \qquad i\hbar \frac{\partial}{\partial t_2} \Psi_{2(\alpha\beta)} = H_{2(\beta\tau)} \Psi_{2(\alpha\tau)}. \tag{2.38}$$

Adding these equations together gives

$$i\hbar\left(\frac{\partial}{\partial t_1}+\frac{\partial}{\partial t_2}\right)\Psi_{2(\alpha\beta)}=H_{1(\alpha\tau)}\Psi_{2(\tau\beta)}+H_{2(\beta\tau)}\Psi_{2(\alpha\tau)}. \tag{2.39}$$

Consider (t_1, t_2) space and the line parameterised by t, given by $t = t_1 = t_2$. On this line

$$\frac{\partial}{\partial t}=\frac{\partial}{\partial t_1}+\frac{\partial}{\partial t_2}, \tag{2.40}$$

and so on the surface of configuration space-time in which $t_1 = t_2$, we can replace Eq.(2.39) by

$$i\hbar\frac{\partial}{\partial t}\Psi_{2(\alpha\beta)}=H_{1(\alpha,\tau)}\Psi_{2(\tau\beta)}+H_{2(\beta\tau)}\Psi_{2(\alpha\tau)}. \tag{2.41}$$

In replacing two equations by a single equation we have clearly lost some information. Eq.(2.41) only applies to the surface $t_1 = t_2$. Eqs.(2.38) applies over all t_1 and t_2.

This result extends to the many particle case. If we add each of the Eqs.(2.37), then we obtain

$$i\hbar\left(\frac{\partial}{\partial t_1}+\ldots+\frac{\partial}{\partial t_n}\right)\Psi_{(\alpha_1,\ldots,\alpha_n)}=\sum_{k=1}^{n}H_{k(\alpha_k\beta)}\Psi_{n(\alpha_1\ldots\alpha_{k-1}\beta\alpha_{k+1}\ldots\alpha_n)}. \tag{2.42}$$

On the path in $(t_1, \ldots, t_n)$ space, $t = t_1 = \ldots = t_n$, which is parameterised by t we have

$$\frac{\partial}{\partial t}=\frac{\partial}{\partial t_1}+\ldots+\frac{\partial}{\partial t_n}, \tag{2.43}$$

and on this path

$$i\hbar\frac{\partial}{\partial t}\Psi_{(\alpha_1\ldots\alpha_n)}=\sum_{k=1}^{n}H_{k(\alpha_k\beta)}\Psi_{(\alpha_1\ldots\alpha_{k-1}\beta\alpha_{k+1}\ldots\alpha_n)}. \tag{2.44}$$

This equation was proposed by Bohm and Hiley [14] to describe multiple Dirac particles. A solution to this equation, without using components, is the multiple spinor

$$\Psi_n(\mathbf{x}_1,\ldots,\mathbf{x}_n,t)=\exp\frac{\kappa_{\tau_1}\left(-it\sqrt{\mathbf{p}_1^2+m^2}+i\mathbf{x}_1.\mathbf{p}_1\right)}{\hbar}$$

$$\times\ldots\times\exp\frac{\kappa_{\tau_n}\left(-it\sqrt{\mathbf{p}_n^2+m^2}+i\mathbf{x}_n.\mathbf{p}_n\right)}{\hbar}u_{\tau_1}(\mathbf{p}_2)\ldots u_{\tau_n}(\mathbf{p}_n), \tag{2.45}$$

where κ_τ is given by Eq.(2.34).

In order to produce Eq.(2.44) a frame has to be selected. Eq.(2.44) only describes a small part of the configuration space of Eqs.(2.37,2.42) and is not manifestly Lorentz invariant. Eqs.(2.37,2.42) are manifestly Lorentz invariant, since the original Dirac equation is Lorentz invariant. A Lorentz transformation of Eq.(2.42) will cause the surface $t = t_1 = \ldots = t_n$ to move.

To describe a single Dirac particle with an electric charge e in an electromagnetic field given by the 4-potential $A^\mu(x)$ we again perform the substitution $-i\hbar\partial_\mu \to -i\hbar\partial_\mu + eA_\mu$. The evolution of a single Dirac particle in an electromagnetic field A^μ is given by

$$i\hbar\frac{\partial}{\partial t}\Psi_1 = H\Psi_1, \tag{2.46}$$

where

$$H = eA^0 + \gamma^0\left(\gamma^j\left(-i\hbar\partial_j + eA_j\right) + m\right). \tag{2.47}$$

The previous discussion holds in a background electromagnetic field, where we replace the free particle Hamiltonian with the new Hamiltonian. We could choose to describe many particles in a background electromagnetic field with Eq.(2.37). This is equivalent to Eq.(2.44) along the surface $t = t_1 = \ldots = t_n$.

Background fields are useful for constructing theories. However we have no reason to believe that they actually occur in reality. As far as is known all physical fields are caused by particles. Consider a system of two particles which are moving in the electromagnetic field caused by each other. We might expect we could describe this system by using an electromagnetic field which is not a function of a single position, but which is dependent on the positions of both particles, $A^\mu(\mathbf{x}_1, \mathbf{x}_2, t)$. Using $A^\mu(\mathbf{x}_1, \mathbf{x}_2, t)$ leads to interactions, though at the moment we have no way of calculating this field. The question that we wish to ask at the moment is not how to calculate this field, but given a 4-potential $A^\mu(\mathbf{x}_1, \mathbf{x}_2, t)$, which equation of motion should be used?

We could choose to describe two Dirac particles with Eqs.(2.38). If the particles are interacting then we cannot assume that $[H_1, H_2] = 0$. If $[H_1, H_2] \neq 0$ then

$$\frac{\partial^2}{\partial t_1 \partial t_2}\Psi_2 \neq \frac{\partial^2}{\partial t_2 \partial t_1}\Psi_2, \tag{2.48}$$

and Eqs.(2.38) are incompatible. Eq.(2.41) has no such compatability problems. It should be remembered that Eqs.(2.38) are manifestly Lorentz invariant whereas Eq.(2.41) is not. This does not mean that we should completely discount multi-time quantum mechanics, and the possibility of manifest Lorentz invariance. Eq.(2.39) has no compatability problems, and is manifestly Lorentz invariant. However for the purposes of this thesis it is postulated that Dirac particles can be described by the single time equation

$$i\hbar\frac{\partial}{\partial t}\Psi_{(\alpha_1 \ldots \alpha_n)} = \sum_{k=1}^{n} H_{k(\alpha_k \beta)}\Psi_{(\alpha_1 \ldots \alpha_{k-1}\beta\alpha_{k+1}\ldots\alpha_n)}, \tag{2.49}$$

where H_k is the Hamiltonian for particle k.

To gain an intuitive insight as the motion of Dirac particles in a background electromagnetic field we take the slow speed approximation. We write

$$\Psi_1 = \begin{pmatrix} X \\ Y \end{pmatrix}, \tag{2.50}$$

where X and Y are two-component spinors and use the explicit representation of the γ^μ matrices given by

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}. \tag{2.51}$$

I is the unit 2×2 unit matrix and σ^j are the Pauli matrices. In the slow speed approximation it can be shown that

$$i\hbar\frac{\partial X}{\partial t} = HX, \tag{2.52}$$

where

$$H = \frac{-1}{2m}(\hbar\nabla - ie\mathbf{A})^2 + \frac{\hbar e}{2m}\sigma.\mathbf{B} + (A^0 + m), \tag{2.53}$$

$A^\mu = (A^0, \mathbf{A})$ and $\mathbf{B}$ is the magnetic field. This is the Pauli equation. A derivation can be found in Itzykson and Zuber [22]. The first term on the right hand side is a kinetic energy term. The second term is a spin term. The third term is the potential energy, which is the sum of the mass energy and the electrostatic potential A^0.

Eq.(2.52) is

$$i\hbar\frac{\partial}{\partial t}\Psi_{1(j)} = H_{(jl)}\Psi_{1(l)}, \qquad j = 1,2, \tag{2.54}$$

where l sums over 1 and 2, and is an equation only involving the first two components of the spinor. This equation generalises to many particles,

$$i\hbar\frac{\partial}{\partial t}\Psi_{1(j_1\ldots j_n)} = H_{1(j_1 l)}\Psi_{1(l\ldots j_n)} + \ldots + H_{n(j_n l)}\Psi_{1(j_1\ldots l)}, \tag{2.55}$$

where each of the j_k are 1 or 2 and l sums over 1 and 2.

2.4 A Background Gravitational field

The Dirac equation for a single particle in a background curved space-time can be written as

$$\left(i\hbar\gamma^\mu(\mathbf{x},t)\mathcal{D}_\mu - m\right)\Psi = 0. \tag{2.56}$$

where again j sums from 1 to 3 but now the $\gamma^\mu(\mathbf{x},t)$ are 4×4 matrix fields obeying the anti-commutation relations

$$\{\gamma^\mu(\mathbf{x},t), \gamma^\nu(\mathbf{x},t)\} = 2g^{\mu\nu}(\mathbf{x},t). \tag{2.57}$$

Spinors have their own special transformation properties, and $\mathcal{D}_\mu$ is the spinor co-variant derivative corresponding to ∂_μ. This is discussed in Weinberg [23]. In order to find the covariant derivative at a point X of space-time a locally inertial set of co-ordinates ξ^μ_X are used. The matrix

$$V^\alpha{}_\mu(X) \equiv \left(\frac{\partial\xi^\alpha_X(x)}{\partial x^\mu}\right)_{x=X} \tag{2.58}$$

transforms vectors from the basis of the x^μ co-ordinates, to the basis of the ξ^μ co-ordinates. Weinberg shows that the co-variant derivative which corresponds to the derivative $V_\alpha{}^\mu\frac{\partial}{\partial x^\mu}$ can be written in the form

$$V_\alpha{}^\mu\left[\frac{\partial}{\partial x^\mu} + \Gamma_\mu(\mathbf{x},t)\right], \tag{2.59}$$

where the Γ_μ are each 4×4 matrices. The exact form of Γ_μ can be found in Weinberg. We thus deduce that the covariant derivative which corresponds to the derivative $\frac{\partial}{\partial x^\mu}$ is

$$\mathcal{D}_\mu \equiv \frac{\partial}{\partial x^\mu} + \Gamma_\mu(\mathbf{x}, t). \tag{2.60}$$

It should be noted that this is not the notation that is adopted in Weinberg.

In order to form an equation of motion for an arbitrary number of particles in a background gravitational field we must write Eq.(2.56) in the form

$$i\hbar\frac{\partial}{\partial t}\Psi = H\Psi, \tag{2.61}$$

where H a Hamiltonian operator. If we write

$$H = \left(\gamma^0(\mathbf{x}, t)\right)^{-1}\left(-i\hbar\gamma^j(\mathbf{x}, t)\mathcal{D}_j + m\right) - i\hbar\Gamma_0, \tag{2.62}$$

then Eq.(2.61) is equivalent to Eq.(2.56). Using the results of the previous section we can describe many Dirac particles with the analogue of Eq.(2.49). Each particle has a Hamiltonian of the form of Eq.(2.62) and H_k is produced by substituting $\mathbf{x}$ with $\mathbf{x}_k$.

We now consider the gravitational field produced by a fixed particle at the origin. This thesis is mainly concerned with weak gravitational fields and linear gravity, and in order to help future results we consider only gravity linearised about the Minkowski metric.

We assume that space-time is topologically equivalent to Minkowski space. On each point of $\mathbf{R}^4$ is placed a metric

$$g_{\mu\nu}(\mathbf{x}, t) = r_{\mu\nu} + \varepsilon_{\mu\nu}(\mathbf{x}, t), \tag{2.63}$$

where it is assumed $\varepsilon_{\mu\nu}(\mathbf{x}, t)$ is small. Linear gravity carries out calculations to lowest order in $\varepsilon_{\mu\nu}(\mathbf{x}, t)$. For instance to this order the inverse metric is

$$g^{\mu\nu}(\mathbf{x}, t) = r^{\mu\nu} - \varepsilon^{\mu\nu}(\mathbf{x}, t). \tag{2.64}$$

Also, if we write

$$\bar{\varepsilon}_{\mu\nu}(\mathbf{x}, t) = \varepsilon_{\mu\nu}(\mathbf{x}, t) - \frac{1}{2}r_{\mu\nu}\varepsilon^\alpha_\alpha(\mathbf{x}, t), \tag{2.65}$$

then Einstein's tensor is found to be

$$G_{\mu\nu}(\mathbf{x},t) = -\frac{1}{2}\partial^\tau\partial_\tau\bar{\varepsilon}_{\mu\nu}(\mathbf{x},t) + \partial^\tau\partial_\nu\bar{\varepsilon}_{\mu\tau}(\mathbf{x},t) + \partial^\tau\partial_\mu\bar{\varepsilon}_{\nu\tau}(\mathbf{x},t) - \frac{1}{2}r_{\mu\nu}\partial^\tau\partial^\sigma\bar{\varepsilon}_{\tau\sigma}(\mathbf{x},t). \tag{2.66}$$

Any space-time manifold M can be described by a large number of different metric fields. If $g_{\mu\nu}(\mathbf{x},t)$ and $g'_{\mu\nu}(\mathbf{x},t)$ are two such metric fields then there is a gauge transformation which transforms $g_{\mu\nu}(\mathbf{x},t)$ into $g'_{\mu\nu}(\mathbf{x},t)$,

$$g_{\mu\nu}(\mathbf{x},t) \rightarrow g'_{\mu\nu}(\mathbf{x},t). \tag{2.67}$$

In linearised gravity we limit the metric fields to those which are perturbed about $r_{\mu\nu}$ by $\varepsilon_{\mu\nu}(\mathbf{x},t)$ where $\varepsilon_{\mu\nu}(\mathbf{x},t)$ is small. This removes much of the gauge freedom. However linearised gravity still has the gauge freedom generated by a vector field $\xi_\alpha(\mathbf{x},t)$,

$$\varepsilon_{\mu\nu}(\mathbf{x},t) \rightarrow \varepsilon_{\mu\nu}(\mathbf{x},t) + \partial_\mu\xi_\nu(\mathbf{x},t) + \partial_\nu\xi_\mu(\mathbf{x},t). \tag{2.68}$$

This is closely analagous to the gauge freedom of electromagnetism, $A_\mu \rightarrow A_\mu + \partial_\mu\chi$. It is possible to make a gauge transformation to obtain

$$\partial^\mu\bar{\varepsilon}_{\mu\nu}(\mathbf{x},t) = 0, \tag{2.69}$$

which is the analogue of the Lorentz gauge condition. In this gauge, the expression for the Einstein tensor simplifies to become

$$G_{\mu\nu}(\mathbf{x},t) = -\frac{1}{2}\partial^\mu\partial_\mu\bar{\varepsilon}_{\mu\nu}(\mathbf{x},t), \tag{2.70}$$

and the linearised Einstein equation is

$$\partial_\tau\partial^\tau\bar{\varepsilon}_{\mu\nu}(\mathbf{x},t) = -16\pi G T_{\mu\nu}(\mathbf{x},t). \tag{2.71}$$

A more detailed derivation is contained in Wald [24].

In Minkowski space the stress-energy tensor of a particle of mass m stationary at the spatial origin is

$$T_{\mu\nu}(\mathbf{x}) = \begin{cases} m\delta^3(\mathbf{x}) & \mu = \nu = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2.72}$$

We extend the overline notation to general tensors,

$$\overline{A}_{\mu\nu} = A_{\mu\nu} - \frac{1}{2}r_{\mu\nu}A^{\tau}_{\tau}, \tag{2.73}$$

and so

$$\overline{T}_{\mu\nu} = \frac{1}{2}m\delta^3(\mathbf{x})\delta_{\mu\nu}. \tag{2.74}$$

Now

$$\overline{r}_{\mu\nu} = -r_{\mu\nu}, \tag{2.75}$$

and so

$$\overline{\overline{\varepsilon}}_{\mu\nu}(\mathbf{x},t) = \overline{\varepsilon}_{\mu\nu}(\mathbf{x},t) + \frac{1}{2}r_{\mu\nu}\varepsilon^{\tau}_{\tau}(\mathbf{x},t) = \varepsilon_{\mu\nu}(\mathbf{x},t). \tag{2.76}$$

Hence

$$\partial^2\varepsilon_{\mu\nu}(\mathbf{x},t) = -16\pi G\overline{T}_{\mu\nu}(\mathbf{x},t), \tag{2.77}$$

and in the leading order of $\varepsilon_{\mu\nu}(\mathbf{x},t)$,

$$\partial^2\varepsilon_{\mu\nu}(\mathbf{x},t) = -8\pi mG\delta^3(\mathbf{x})\delta_{\mu\nu}. \tag{2.78}$$

The solution which vanishes asymptotically is

$$\varepsilon_{\mu\nu}(\mathbf{x},t) = -\frac{\delta_{\mu\nu}2mG}{|\mathbf{x}|}, \tag{2.79}$$

where $|\mathbf{x}|$ denotes the Cartesian length of the spatial co-ordinates $\mathbf{x}$,

$$|\mathbf{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2}, \tag{2.80}$$

and

$$ds^2 = \left(1 - \frac{2mG}{|\mathbf{x}|}\right)dt^2 - \left(1 + \frac{2mG}{|\mathbf{x}|}\right)dx^j dx^j. \tag{2.81}$$

If we transfer to (r, θ, ϕ) co-ordinates given by

$$x^1 = r\sin\theta\sin\phi, \quad x^2 = r\sin\theta\cos\phi, \quad x^3 = r\cos\theta, \tag{2.82}$$

the metric Eq.(2.81) can be written as

$$ds^2 = \left(1 - \frac{2mG}{r}\right)dt^2 - \left(1 + \frac{2mG}{r}\right)\left(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\right) \tag{2.83}$$

$$= \left(1 - \frac{2mG}{r}\right) dt^2 - \left(1 - \frac{2mG}{r}\right)^{-1} \left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right) + O\left(\frac{1}{r^2}\right). \quad (2.84)$$

According to full general relativity, the metric caused by a particle of mass m stationary at the spatial origin is the Schwarzschild metric,

$$ds^2 = \left(1 - \frac{2mG}{r}\right) dt^2 - \left(1 - \frac{2mG}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2. \quad (2.85)$$

The difference between Eq.(2.83) and the Schwarzschild metric is caused by the linearisation.

We now find a linear approximation to the Hamiltonian Eq.(2.62) in terms of $\varepsilon^{\mu\nu}(\mathbf{x}, t)$. We are most interested in the principles underlying a theory of gravity, and not of producing exact results which take account of all the subtle effects of spin. It is first shown that for a first approximation the Γ_μ part of the spinor covariant derivatives has a very small contribution and for a can be safely ignored. Consider a spinor $\Psi(\mathbf{x}, t)$ describing a single particle moving with a velocity of magnitude v. We then have

$$i\hbar\mathcal{D}_\mu\Psi = i\hbar\left(\partial_\mu + \Gamma_\mu\right)\Psi = i\hbar\partial_\mu\Psi + O(\hbar)\Psi \approx i\hbar\partial_\mu\Psi. \quad (2.86)$$

The derivative part $i\hbar\partial_\mu\Psi$ dominates the expression because $i\hbar\partial_\mu$ is the momentum operator. For $\mu = 1, 2, 3$ it is of the order of $mv\Psi$ and for $\mu = 0$ it is of the order $m\sqrt{1 + v^2}\,\Psi$. Each of the Γ_μ are small because it is assumed the perturbation $\varepsilon_{\mu\nu}(\mathbf{x}, t)$ about the Minkowski metric is small. We can thus approximate the Hamiltonian Eq.(2.62) by

$$H = \left(\gamma^0(\mathbf{x}, t)\right)^{-1} \left(-i\hbar\gamma^j(\mathbf{x}, t)\partial_j + m\right). \quad (2.87)$$

The next task is to find an expression for the matrix fields $\gamma^\mu(\mathbf{x}, t)$. Let $\gamma^{\mu\prime}$ be 4×4 matrices which obey

$$\{\gamma^{\mu\prime}, \gamma^{\nu\prime}\} = 2\tau^{\mu\nu}. \quad (2.88)$$

We perturb the $\gamma^\mu(\mathbf{x}, t)$ about $\gamma^{\mu\prime}$ and put

$$\gamma^\mu(\mathbf{x}, t) = \gamma^{\mu\prime} + \gamma^{\nu\prime}\alpha^\mu_\nu(\mathbf{x}, t), \quad (2.89)$$

where it is assumed $\alpha^{\mu}_{\nu}(\mathbf{x},t)$ is small. Up to linear order in $\alpha^{\mu}_{\nu}(\mathbf{x},t)$

$$2g^{\mu\nu}(\mathbf{x},t) = \{\gamma^{\mu}(\mathbf{x},t),\gamma^{\nu}(\mathbf{x},t)\} = \{\gamma^{\mu\prime},\gamma^{\nu\prime}\} + 2\alpha^{\mu}_{\tau}(\mathbf{x},t)\{\gamma^{\tau\prime},\gamma^{\nu\prime}\} = 2r^{\mu\nu} + 4\alpha^{\mu\nu}(\mathbf{x},t), \tag{2.90}$$

and so

$$\alpha^{\mu\nu}(\mathbf{x},t) = -\frac{1}{2}\varepsilon^{\mu\nu}(\mathbf{x},t). \tag{2.91}$$

We also have

$$\left(\gamma^{0}(\mathbf{x},t)\right)^{-1} = \gamma^{0\prime} + \frac{1}{2}\varepsilon^{0}_{\nu}(\mathbf{x},t)\gamma^{\nu\prime}, \tag{2.92}$$

up to linear order in $\varepsilon^{\mu\nu}(\mathbf{x},t)$ and so the approximate Hamiltonian Eq.(2.87) becomes

$$H = \left(\gamma^{0\prime} + \frac{1}{2}\varepsilon^{0}_{\tau}(\mathbf{x},t)\gamma^{\tau\prime}\right)\left(-i\hbar\left(\gamma^{j\prime} - \frac{1}{2}\varepsilon^{j}_{\tau}(\mathbf{x},t)\gamma^{\tau\prime}\right)\partial_j + m\right). \tag{2.93}$$

We are interested in the motion of a particle in the background gravitational field produced by a particle fixed at the origin. The Hamiltonian of such a particle in the linear approximation is provided by substituting the metric Eq.(2.81) into the Hamiltonian Eq.(2.93) and this gives

$$H = \gamma^{0\prime}\left(1 - \frac{mG}{|\mathbf{x}|}\right)\left(-i\hbar\left(1 - \frac{mG}{|\mathbf{x}|}\right)\gamma^{j\prime}\partial_j + m\right). \tag{2.94}$$

In order to gain insight into this motion we take the slow speed approximation, using the same techniques that led to the Pauli equation. We write

$$\Psi_1 = \begin{pmatrix} X \\ Y \end{pmatrix}, \tag{2.95}$$

where X and Y are two-component spinors and use the explicit representation

$$\gamma^{0\prime} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \gamma^{j\prime} = \begin{pmatrix} 0 & \sigma^{j} \\ -\sigma^{j} & 0 \end{pmatrix}, \tag{2.96}$$

where again σ^j are the Pauli matrices. The Dirac equation can be written as

$$i\hbar\frac{\partial X}{\partial t} = -i\hbar\left(1+V\right)^2\sigma_j\partial_j Y + m\left(1+V\right)X, \tag{2.97}$$

$$i\hbar\frac{\partial Y}{\partial t} = -i\hbar\left(1+V\right)^2\sigma_j\partial_j X - m\left(1+V\right)Y, \tag{2.98}$$

where $V = -\frac{mG}{|\mathbf{x}|}$. For low velocities and positive energies $Y \ll X$, $i\hbar\frac{\partial Y}{\partial t}$ is small and

$$Y \approx -\frac{i\hbar(1+V)\sigma_j\partial_j X}{m}. \tag{2.99}$$

Putting Eq.(2.99) into Eq.(2.97) we find

$$i\hbar\frac{\partial X}{\partial t} = -\frac{\hbar^2\sigma_j\sigma_k(1+V)\partial_j(1+V)^2\partial_k X}{m} + m(1+V)X. \tag{2.100}$$

If we let

$$S = -\frac{\hbar^2\sigma_j\sigma_k(1+V)\left[\partial_j(1+V)^2\right]\partial_k}{m}, \tag{2.101}$$

then

$$i\hbar\frac{\partial X}{\partial t} = -\frac{\hbar^2\sigma_j\sigma_k(1+V)^3\partial_j\partial_k X}{m} + m(1+V)X + SX. \tag{2.102}$$

SX contains terms dependent on spin, is analogous to the spin terms in the Pauli equation, and is of the order $O(v\hbar)$. Now

$$\sigma_j\sigma_k = \frac{1}{2}\{\sigma_j,\sigma_k\} + \frac{1}{2}[\sigma_j,\sigma_k], \tag{2.103}$$

and since $\partial_j\partial_k$ is symmetric in j and k

$$\sigma_j\sigma_k\partial_j\partial_k = \frac{1}{2}\{\sigma_j,\sigma_k\}\partial_j\partial_k = \frac{1}{2}\delta_{jk}\partial_j\partial_k = \frac{1}{2}\partial_j\partial_j. \tag{2.104}$$

Hence

$$i\hbar\frac{\partial X}{\partial t} = -\frac{\hbar^2\left(1-\frac{mG}{|\mathbf{x}|}\right)^3\partial_j\partial_j}{2m}X + \left(m - \frac{m^2G}{|\mathbf{x}|}\right)X + SX. \tag{2.105}$$

The first term is the kinetic energy term. The factor of $\left(1-\frac{mG}{\mathbf{x}}\right)^3$ is due to the lengthening of the spatial part of the metric and the shortening of the time-like part. The second term is the mass energy m and the gravitational potential energy $-\frac{m^2G}{|\mathbf{x}|}$. The final term is of magnitude $O(\hbar)$ and is dependent on spin. If the original Hamiltonian Eq.(2.62) had been used then there would have been further spin terms.

Chapter 3

Two Modified Bohm Models

3.1 Introduction

If we want to use the Bohm model in conjunction with theories such as QED then there are major problems. The first is that QED is, or at least appears to be, Lorentz invariant. Yet it has been shown by Hardy [6] (see also [7]) that any hidden variable interpretation of quantum mechanics must violate Lorentz invariance. Secondly it is impossible to describe bosons, and in particular photons, using the Bohm model.

This chapter discusses two modifications to the Bohm model. The first is the Retarded Bohm Model [8] which is an attempt to make a model which is local and Lorentz-invariant. This model must disagree with quantum mechanics, but this does not necessarily imply that it disagrees with experiment. It is shown how this model evades the Hardy proof of lack of Lorentz invariance of hidden variable models, and another experimental test is discussed. The second modification gives a model which can be used when particles which do not have trajectories, such as photons, are present.

3.2 The Standard Bohm Model

Unlike classical mechanics, the interpretation of quantum mechanics is far from easy. If a wave-function $\Phi(\mathbf{x}, \mathbf{y}, t)$ describes the motion of two particles then what are these particles doing? If we look at them then what do we see? The wave-function gives a probability distribution $|\Phi(\mathbf{x}, \mathbf{y}, t)|^2$ of the likelihood of our observing particle (1) at the position $\mathbf{x}$ and particle (2) position $\mathbf{y}$, but does not tell us exactly where they will be. This leaves room for a variety of interpretations.

One of the more successful is that of Bohm. Imagine there are is a fluid flowing on configuration space $(\mathbf{x}, \mathbf{y})$. This fluid flows so that the density at any point $(\mathbf{x}, \mathbf{y})$ is

$$\rho(\mathbf{x}, \mathbf{y}, t) = |\Phi(\mathbf{x}, \mathbf{y}, t)|^2. \tag{3.1}$$

At the point $(\mathbf{x}, \mathbf{y})$ the velocity of the fluid is given by

$$\dot{\mathbf{x}} = \mathbf{Re}\left(\frac{-i\hbar\nabla_x\Phi(\mathbf{x}, \mathbf{y})}{m\Phi(\mathbf{x}, \mathbf{y})}\right), \qquad \dot{\mathbf{y}} = \mathbf{Re}\left(\frac{-i\hbar\nabla_y\Phi(\mathbf{x}, \mathbf{y})}{m\Phi(\mathbf{x}, \mathbf{y})}\right). \tag{3.2}$$

Returning back to the original system of 2 particles, we can use Eq.(3.2) as guidance equations, which determine trajectories $\mathbf{x}(t)$, $\mathbf{y}(t)$ of the two particles. The Bohm model states that all observations are determined by observations of position, and the observed positions are given by $\mathbf{x}(t)$ and $\mathbf{y}(t)$. This model gives results which statistically agree with those of quantum theory.

This naturally generalises to any number of particles and can also be generalised to Dirac particles. Indeed it is tempting to imagine the whole observed universe to be determined in this way.

3.3 The Bohm Model and Lorentz Invariance

3.3.1 The Mach-Zehnder interferometer

The important features of the Mach-Zehnder interferometer are shown in fig. 3.1. A particle enters at point A and encounters a beam splitter, which separates the

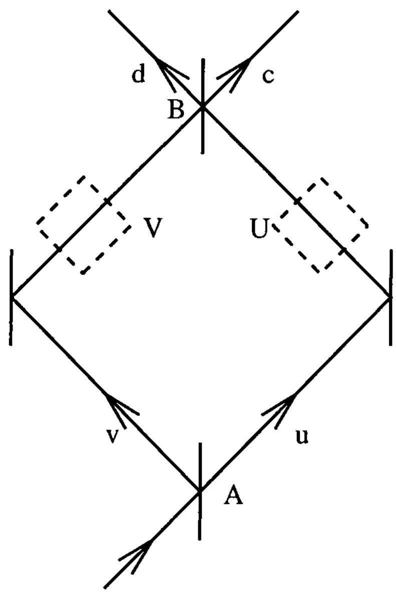

Figure 3.1: A Mach-Zehnder type interferometer

wave-packet into two parts of equal magnitude:

$$|\Psi> \rightarrow \frac{1}{\sqrt{2}}(|u_0> + |v_0>). \tag{3.3}$$

The wave-function then evolves with time according to

$$|\Psi_t> = \frac{1}{\sqrt{2}}(|u_t> + |v_t>), \tag{3.4}$$

until either it is observed by the detectors, U or V, or the wave-packets reach region B. Here there is a second beam splitter. This again splits the incident beam into two parts. In fact it can be arranged that:

$$|u_t> \rightarrow \frac{1}{\sqrt{2}}(|c_t> + |d_t>) \tag{3.5}$$

and

$$|v_t> \rightarrow \frac{1}{\sqrt{2}}(|c_t> - |d_t>), \tag{3.6}$$

where path $|c>$ will cause the C detector to fire and path $|d>$ will cause the D detector to fire.

Hence, if detectors U and V are missing, we have for $t > T$, where T is the time when the wave packets reach the beam splitter:

$$|\Psi_t> = |c_t>. \tag{3.7}$$

On the other hand, if the U(V) detector is present, it will either register the particle, in which case

$$|\Psi_t> = 0, \tag{3.8}$$

or it will fail to register, leading to

$$|\Psi_t> = \frac{1}{\sqrt{2}}(|c_t> \pm |d_t>). \tag{3.9}$$

It is possible to describe the interferometer in one space and one time dimension. Essentially this means taking time as the vertical axis in fig.3.1 and distance as the horizontal. The beam-splitter at $x = 0$ can be modelled by a potential $V(x) = \frac{\hbar^2}{m}\Omega\delta(x)$. A wave-packet travels towards the potential, and interacts with it at $t = 0$. This produces two wave-packets leaving A in opposite directions. Mirrors are placed at $kx = \pm 2\pi n$, where $n \in \mathbf{Z}$, and completely reflect the two parts of the wave-function back towards $x = 0$.

The solution of the scattering problem for plane waves is

$$\Psi(x) = \begin{cases} e^{ikx} + Re^{-ikx} & x < 0 \\ Te^{ikx} & x > 0, \end{cases} \tag{3.10}$$

where

$$R = -\frac{\Omega^2 + ik\Omega}{k^2 + \Omega^2} \quad \text{and} \quad T = \frac{k^2 - ik\Omega}{k^2 + \Omega^2}. \tag{3.11}$$

A wave-packet whose momenta peaks sharply at $k = \Omega$ will be half transmitted and half reflected as required. For $k = \Omega$ the above solution becomes

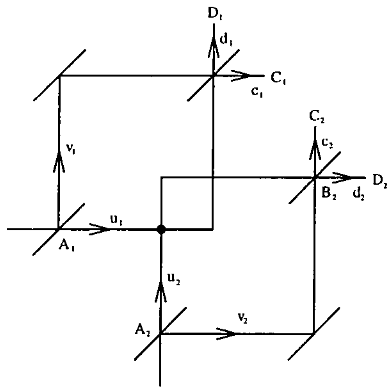

Figure 3.2: The Hardy Experiment with two overlapping interferometers

$$\Psi(x) = \begin{cases} e^{ikx} - \frac{1+i}{2}e^{-ikx} & x < 0 \\ \frac{1-i}{2}e^{ikx} & x > 0. \end{cases} \tag{3.12}$$

Here Schrödinger's equation is invariant under $x \rightarrow -x$ and $\Psi \rightarrow \Psi^*$. Hence

$$\Psi(x) = \begin{cases} \frac{1+i}{2}e^{ikx} & x < 0 \\ e^{ikx} - \frac{1-i}{2}e^{-ikx} & x > 0 \end{cases} \tag{3.13}$$

is also a solution.

The reflection of $-\frac{1+i}{2}e^{-ikx}$ in the left mirror leads to $\frac{1+i}{2}e^{ikx}$ and the reflection of $\frac{1-i}{2}e^{ikx}$ in the right to $-\frac{1-i}{2}e^{-ikx}$. Hence, when the two wave-packets meet again at $x = 0$, they will combine constructively for $x > 0$ and destructively for $x < 0$. Thus a single wave-packet will leave $x = 0$ heading in the direction $x \rightarrow +\infty$. In other words, the reflected direction corresponds to the d-path, and with the single interferometer no particles will be observed on it.

3.3.2 The Hardy Experiment

Here we have two overlapping interferometers, one for electrons and one for positrons, labelled (1) and (2) respectively. This means the state will exist in 3-dimensional configuration space, with $\Psi = \Psi(x_1, x_2, t)$. The interferometers are made to overlap as in fig.3.2 so that, if the electron and positron are in the states $|u_1\rangle$ and $|u_2\rangle$, they will annihilate each other with probability 1. The quantum evolution then depends on the sign of $(T_1 - T_2)$ where T_1, T_2 are the times when the particles reach the respective interference regions B_1, B_2. We have, after the annihilation region has been passed,

$$\begin{aligned} \Psi &\rightarrow u_1v_2 + v_1u_2 + v_1v_2 && t < T_1 \text{ and } t < T_2 && \text{(a)} \\ &\rightarrow \begin{cases} u_1c_2 - u_1d_2 + 2v_1c_2 & T_2 < t < T_1 \qquad \text{(b)} \\ c_1u_2 - d_1u_2 + 2c_1v_2 & T_1 < t < T_2 \qquad \text{(c)} \end{cases} \\ &\rightarrow 3c_1c_2 - c_1d_2 - c_2d_1 - d_1d_2 && t > T_1 \text{ and } t > T_2. && \text{(d)} \end{aligned} \tag{3.14}$$

If the regions where the two $|u_i>$ and $|v_i>$ meet, B_1 and B_2, are arranged to have a space-like separation, then the description of the experiment depends on the choice of a particular time coordinate, i.e., on a particular foliation of space-time by a series of space-like hypersurfaces, each labelled by a time coordinate. Clearly it is possible for $(T_1 - T_2)$ to have either sign, according to the choice of foliation, and hence for either (b) or (c) of Eq.(3.14) to be appropriate.

To see how this leads to the Hardy contradiction with Lorentz-invariance we note first that, according to Eq.(3.14(d)), there is a probability of $\frac{1}{12}$ that both particles will end up at the dark detectors, i.e., along the paths d_1, d_2. The standard Bohm model has trajectories given by

$$\dot{x}_i(x_1, x_2, \mathbf{S}) = \mathbf{Re}\frac{\mathbf{p}_i\Psi(x_1, x_2, \mathbf{S})}{m\Psi(x_1, x_2, \mathbf{S})}, \tag{3.15}$$

where $\mathbf{S}$ is the hypersurface $t = constant$, t being the time variable in some Lorentz frame. The model is designed to give exactly the statistical predictions of orthodox quantum theory and so must contain trajectories going along these paths. Let us consider such trajectories from the point of view of a Lorentz frame in which $T_1 < T_2$. Then, from Eq.(3.14(c)), it is clear that a path along d_1 requires that particle 2 is on the u_2 path. Hence, in this frame we have a unique description of the event: particle 2 went along u_2 and particle 1 went along v_1. It could not have gone along u_1, otherwise it would have been annihilated.

However, it is clear that from a different Lorentz frame in which $T_2 < T_1$, we have exactly the opposite description: particle 1 went along u_1 and particle 2 along

v_2. In the Bohm model the paths actually exist, so only one of the descriptions can correspond to what actually happened. Hence there is a preferred frame of reference, in clear violation of Lorentz-invariance.

It should be noted that this proof has been challenged by Berndl and Goldstein [9]. They question some of Hardy's assumptions, and state that multi-time theories can serve as a counter example.

3.3.3 The Retarded Model

In [8] Squires introduces the Retarded Bohm model. This is a version of the Bohm model where, instead of each particle reacting according to the spatial positions of the other particles on the surfaces $t = const$, each particle reacts according to where the other particles are on its backward light-cone.

The retarded Bohm model is governed by a similar expression to that in Eq.(3.15) except that we replace the space-like surfaces $\mathbf{S}$, by the backward light-cone from the i^{th} particle. This introduces the assumption that such a generalised wave-function exists. In a relativistic universe its existence might be expected, but cannot be known.

Since B_1 and B_2 have a space-like separation each particle will react as though it reached the area of interference first. Thus, using Eq.(3.14(c)), when B_1 is reached by particle 1 it will go along c_1 or d_1 on the basis of whether particle 2 was along u_2 or v_2. In particular, it can go along d_1 only if particle 2 is on the path u_2. (Note that the time when the wave-packets reach the annihilation region is inside the backward light-cones from the times when the particles reach the respective regions B). Likewise, using now Eq.(3.14(b)), particle 2 can go along d_2 only if particle 1 is on path u_1. Hence, for both particles to end up in the dark detectors, D_1 and D_2, they have to have followed the u_1 and u_2 paths, respectively. Since this is not possible, for they would then have annihilated, we conclude that in the retarded model there are no such events.

It follows that the Hardy proof of a lack of Lorentz-invariance cannot be used.

Of course the reason why we have been able to evade the Hardy Theorem is that we have violated one of the assumptions, namely that the hidden variable model should in all cases give the results of orthodox quantum theory. The retarded model fails in this respect, so it can in principle be distinguished from orthodox quantum theory by experiment. One possible such experiment is discussed in the next section.

3.3.4 Experimental Tests

The previous section suggests a very clear experimental test of the retarded Bohm model (see [8] [10] for other tests). If the Hardy Experiment could actually be performed then any event in which D_1 and D_2 record particles would immediately rule out the retarded model.

Unfortunately such experiments are not 100% efficient and it is necessary to see what happens if the annihilation mechanism is not perfect. Perhaps surprisingly it turns out that an imperfect annihilation mechanism (which of course reduces the d_1, d_2 probability in the quantum theory case) increases it from zero in the retarded model. To see this suppose the probability of a $|u_1u_2>$ state not being annihilated is α^2. Then, according to orthodox quantum mechanics, we have, after the annihilation region has been passed,

$$\begin{aligned}
\Psi \;&\rightarrow\; \alpha u_1u_2 + u_1v_2 + v_1u_2 + v_1v_2 \qquad\qquad t < T_1 \text{ and } t < T_2 \qquad \text{(a)} \\
&\rightarrow \left\{ \begin{array}{lll} (1+\alpha)u_1c_2 - (1-\alpha)u_1d_2 + 2v_1c_2 & T_2 < t < T_1 & \text{(b)} \\ (1+\alpha)c_1u_2 - (1-\alpha)d_1u_2 + 2c_1v_2 & T_1 < t < T_2. & \text{(c)} \end{array} \right.
\end{aligned} \qquad (3.16)$$

These equations replace the previous Eqs.(3.14(a,b,c)). The important point to note now is that in the retarded Bohm model the path taken by particle 2 at the second beam splitter is determined by Eq.(3.16(b)). Thus it will go along d_2 only if particle 1 is on u_1 and then with probability

$$\frac{(1-\alpha)^2}{(1+\alpha)^2+(1-\alpha)^2} = \frac{(1-\alpha)^2}{2(1+\alpha^2)}\,. \qquad (3.17)$$

A similar argument holds for particle 1. Thus the probability of obtaining d_1 and d_2 in the retarded model is given by

$$\begin{aligned} P_R(d_1 d_2) &= \frac{(1-\alpha)^4}{4(1+\alpha^2)^2} P(u_1 u_2) \\ &= \frac{\alpha^2(1-\alpha)^4}{4(3+\alpha^2)(1+\alpha^2)^2}, \end{aligned} \tag{3.18}$$

where the $P(u_1 u_2)$ is the probability of $u_1 u_2$ paths calculated from Eq.(3.16(a)).

We want to compare this with the result of orthodox quantum theory. From the result

$$\Psi \to (3+\alpha)c_1c_2 - (1-\alpha)c_1d_2 - (1-\alpha)d_1c_2 - (1-\alpha)d_1d_2 \quad t > T_1 \text{ and } t > T_2, \tag{3.19}$$

we see that the probability of obtaining d_1 and d_2 in standard quantum theory is given by

$$P_S(d_1 d_2) = \frac{(1-\alpha)^2}{4(3+\alpha^2)}. \tag{3.20}$$

It is clear that $P_S(d_1 d_2) \gg P_R(d_1 d_2)$. In fact:

$$\frac{P_R}{P_S} = \frac{2\alpha^3(\alpha-1)^3}{(1+\alpha^2)(\alpha^2+2\alpha-1)} \tag{3.21}$$

which reaches a maximum value of $(12+8\sqrt{2})^{-1} \approx 0.0429$ at $\alpha = \sqrt{2}-1 \approx 0.414$. This maximum value of α corresponds to the case where it is hardest to distinguish between the retarded and standard models. Hence, so long as the efficiency α is known, the two results can easily be distinguished.

3.4 The Bohm Model and Fermion-Boson Correlations

In most versions of the Bohm hidden-variable model, there are trajectories for fermions, the particles of matter, but not for photons (see for example refs [25], [26], [27]). Bohm himself has devised an ontological model for photons, but this regards their wave function as a wave functional $\psi(\ldots\phi(\mathbf{x},t)\ldots)$ over photon fields

[28]. The principle reason there are no trajectories for photons is that there is no natural definition of a positive probability current for relativistic bosons. In many situations the apparent existence of photon trajectories, e.g. the detection of a photon at only one point as in the experiments of Aspect and Grangier [29], follows simply from the existence of fermion trajectories. We give an example at the end of this section. However the usual guidance equation for the fermion trajectories cannot in general be adequate if there are no bosons trajectories.

To see this we consider a correlated system of a fermion, with position variable $\mathbf{x}$, and a boson, with position variable $\mathbf{y}$. Then the Bohm model guidance equation for the fermion position is

$$\dot{\mathbf{x}} = \mathbf{Re}\left(\frac{\mathbf{p}_x \Psi(\mathbf{x},\mathbf{y})}{m\Psi(\mathbf{x},\mathbf{y})}\right), \tag{3.22}$$

where Ψ is the wave-function of the correlated system, and $\mathbf{p}_x$ denotes the momentum operator of the fermion. Clearly, in such an expression there is no unique fermion trajectory unless we have a value of $\mathbf{y}$.

In order to obtain a suitable replacement for this equation, i.e. one that does have a well-defined fermion trajectory, we recall that it can be derived by demanding that a distribution of positions corresponding to $|\Psi|^2$ at some initial time t, retains this property at future times, with Ψ varying according to the Schrödinger equation. If there are no boson trajectories, then we require this same result to hold when the boson positions are 'not observed', i.e. when we average over boson positions. Then the fermion distribution is given by

$$\rho(\mathbf{x}) = \int d^3\mathbf{y}|\Psi(\mathbf{x},\mathbf{y})|^2. \tag{3.23}$$

In order to preserve this distribution for all times we require the fermions to have a velocity given by

$$\nabla_x.(\rho\dot{\mathbf{x}}) = \frac{\partial \rho}{\partial t}. \tag{3.24}$$

Using Eq.(3.23) and the Schrödinger equation for Ψ we readily find

$$\frac{\partial \rho}{\partial t} = \frac{i\hbar}{2m}\nabla_x.\int d^3\mathbf{y}(\Psi^*\nabla_x\Psi - \Psi\nabla_x\Psi^*). \tag{3.25}$$

Comparing the last two equations we have the solution

$$\dot{\mathbf{x}} = \mathbf{Re}\left(\frac{\int d^3\mathbf{y}\Psi^*\mathbf{p}_x\Psi}{m\int d^3\mathbf{y}\Psi^*\Psi}\right). \tag{3.26}$$

The solution of Eq.(3.24) is clearly not unique, any more than in the standard Bohm model; we have here taken the simplest solution analagous to Eq.(3.22).

It is proposed that Eq.(3.26) should be used instead of Eq.(3.22) in cases where there are particles which do not have trajectories. Essentially the same equation was considered in [25] as an alternative to the Bohm formula (Eq.(3.22)), with the apparent advantage that it did not involve a quantum potential in the configuration space. However, as pointed out by Holland, [25] such a procedure is not in general correct because it does not take into account the observable effects of one trajectory on another. It is important therefore to emphasise that the suggestion made here is different: we propose that Eq.(3.26) should be used only when there is no trajectory for the y-particle. Hence we are not proposing an alternative to the usual Bohm model, but an extension of the model where the usual form does not apply. In general of course we can let the vector $\mathbf{x}$ represent the position of all the particles with trajectories, and similarly let $\mathbf{y}$ represent the remaining positions.

Since we are considering a model in which all observations are measurements of fermion positions, the model is guaranteed to give correct answers, in exactly the same way as the standard Bohm model. Nevertheless, it is instructive to see how it works in practice, and to this end we consider the simple measurement situation situation used by Squires [8]. We work in one space-time dimension and suppose that a photon is emitted from the origin in two wave-packets, $\Phi_L(y)$ travelling to the left, and $\Phi_R(x)$, travelling to the right. We consider the simplest case, where these have equal amplitudes. For detectors we use free particles, initially in stationary Gaussian wave-packets centred at $\pm l$, respectively. Their wave-functions are given by

$$\Psi_{L,R} = \left(\frac{a}{\pi}\right)^{\frac{1}{4}} \exp\left[-\frac{1}{2}a(x_{L,R} \pm l)^2\right]. \tag{3.27}$$

We suppose that the photon interacts with the L, R detector by giving it a momentum $\mp p$, with $p > 0$. Hence, after the interaction, the complete state of the system has the form

$$\Psi_C = 2^{-\frac{1}{2}}[\Phi_L(y)\Psi_L^{-p}(x_L)\Psi_R(x_R) + \Phi_R(y)\Psi_L(x_L)\Psi_R^p(x_R)], \tag{3.28}$$

where the $\Psi^{\pm p}$ represent the moving wave-packets, e.g.

$$\Psi_L^{-p}(x_L) = \left(\frac{a}{\pi}\right) \exp\left[i\left(x_L p + \frac{p^2 t}{2m}\right) - \frac{1}{2}a\left(x_L + l + \frac{pt}{m}\right)^2\right]. \tag{3.29}$$

For simplicity we have here, as in [27], ignored the quantum spreading of the wave-function.

When we use Eq.(3.26) to calculate the trajectories we obtain

$$m\dot{x}_L = \mathbf{Re}\left(\frac{|\Psi_R|^2\Psi_L^{-p*}p_L^{op}\Psi_L^{-p} + |\Psi_R^p|^2\Psi_L^* p_L^{op}\Psi_L}{|\Psi_R|^2|\Psi_L^{-p}|^2 + |\Psi_R^p|^2|\Psi_L|^2}\right), \tag{3.30}$$

and

$$m\dot{x}_R = \mathbf{Re}\left(\frac{|\Psi_L|^2\Psi_R^{p*}p_R^{op}\Psi_R^p + |\Psi_L^{-p}|^2\Psi_R^* p_R^{op}\Psi_R}{|\Psi_R|^2|\Psi_L^{-p}|^2 + |\Psi_R^p|^2|\Psi_L|^2}\right), \tag{3.31}$$

where $p_{L,R}^{op}$ are the momentum operators for the particles in the L, R detectors respectively. Note that the cross terms, which appear when the wave-function of Eq.(3.28) is squared, vanish when the y-integral is performed because we can assume there is no overlap between the left and right moving photon wave-packets. Using Eq.(3.29), we find

$$m\dot{x}_L = \frac{-p\exp\left[-a(x_R - l)^2 - a\left(x_L + l + \frac{pt}{m}\right)^2\right]}{\exp\left[-a(x_R - l)^2 - a\left(x_L + l + \frac{pt}{m}\right)^2\right] + \exp\left[-a(x_L + l)^2 - a\left(x_R - l - \frac{pt}{m}\right)^2\right]}, \tag{3.32}$$

and

$$m\dot{x}_R = \frac{p\exp\left[-a(x_L + l)^2 - a\left(x_R - l - \frac{pt}{m}\right)^2\right]}{\exp\left[-a(x_R - l)^2 - a\left(x_L + l + \frac{pt}{m}\right)^2\right] + \exp\left[-a(x_L + l)^2 - a\left(x_R - l - \frac{pt}{m}\right)^2\right]}. \tag{3.33}$$

We can simplify these equations if we choose units in which $a = 1$ and $\frac{p}{m} = 1$, and introduce u and v defined by

$$u = x_R - l, \tag{3.34}$$

and

$$v = -(x_L + l). \tag{3.35}$$

Then we find

$$\dot{u} = \frac{1}{1+\exp[2t(v-u)]}, \tag{3.36}$$

and

$$\dot{v} = \frac{1}{1+\exp[-2t(v-u)]}. \tag{3.37}$$

Adding these equations immediately gives $\dot{u}+\dot{v}=1$, or

$$u+v = t + u_0 + v_0, \tag{3.38}$$

where u_0 and v_0 give the initial positions of the detector particles and we have assumed the instantaneous interaction happens at time zero. From Eq.(3.38) we see that at least one of u or v must go to infinity as t goes to infinity. Hence, at least one of the detectors will 'see' the photon. To rule out the possibility that both might, we divide Eq.(3.37) by Eq.(3.36) to obtain

$$\frac{\partial v}{\partial u} = \exp[2t(v-u)], \tag{3.39}$$

or by eliminating t using Eq.(3.38),

$$\frac{\partial v}{\partial u} = \exp\left(2\left[v-\frac{1}{2}(v_0-u_0)\right]^2 - 2\left[u-\frac{1}{2}(v_0+u_0)\right]^2\right). \tag{3.40}$$

This separates to give

$$\int_{-(v_0-u_0)/2}^{(v_0-u_0)/2} ds\ \exp(-2s^2) = \int_{u-(v_0-u_0)/2}^{v-(v_0-u_0)/2} ds\ \exp(-2s^2), \tag{3.41}$$

which clearly shows that, except in the special case where $u_0=v_0$, v and u cannot both become large. Explicity, if $v_0>u_0$ then the right hand side is positive, so it is v that becomes large, and conversely if $u_0>v_0$. These results are identical to those obtained in the model used in [8] where the photon was assumed to be absorbed in the interaction. They give again the expected agreement with quantum theory. In particular, although there is no photon position, a single photon will be recorded at one place, the actual place being determined by the hidden variables (positions) of the detector particles.

Chapter 4

Non-Relativistic Bosons

4.1 Introduction

In chapter 3 we saw how the Bohm model naturally treats fermions and bosons in a different way. The fermions, the particles of matter, are like classical particles in that they follow trajectories. On the other hand there are no trajectories for bosons, which are the particles of the gauge fields.

In the remainder of the thesis we shall see whether we can remove the idea of bosons entirely from the formulation of quantum theory. This chapter serves as an introduction, and discusses non-relativistic particles. Subsequent chapters discuss relativistic particles.

A powerful notation is introduced which allows the discussion of gauge fields, and through them non-relativistic and relativistic interactions, without the need for extra gauge particles. This notation used is that of field theory, but this is within a framework which is no more than standard quantum mechanics. Field theory in a similar context is discussed by Schweber [11] and Lawrie [12].

4.2 Operators and States

First we consider non-relativistic scalar bosons. Let $\phi(\mathbf{x})$ be an operator field which obeys the commutation relations

$$[\phi(\mathbf{x}), \phi^\dagger(\mathbf{x}')] = \delta^3(\mathbf{x} - \mathbf{x}'), \qquad [\phi(\mathbf{x}), \phi(\mathbf{x}')] = 0, \tag{4.1}$$

that is for each $\mathbf{x}$, $\phi(\mathbf{x})$ is an independent annihilation operator. $\phi^\dagger(\mathbf{x})$ and $\phi(\mathbf{x})$ create and annihilate particles. Let the vacuum state $|0>$ be a state that obeys

$$\phi(\mathbf{x})|0> = 0, \tag{4.2}$$

for every $\mathbf{x}$, and which is normalised according to

$$< 0|0 >= 1. \tag{4.3}$$

It is not assumed that $|0>$ is unique, and there may be another vacuum state $|0'>$ obeying the same conditions. Define $|\mathbf{x}>$ by

$$|\mathbf{x}> = \phi^\dagger(\mathbf{x})|0>. \tag{4.4}$$

This is a single particle in an eigenstate of position. Two particle boson states are formed in the same way

$$|\mathbf{x}_1, \mathbf{x}_2> = |\mathbf{x}_2, \mathbf{x}_1> = \phi^\dagger(\mathbf{x}_1)\phi^\dagger(\mathbf{x}_2)|0>, \tag{4.5}$$

and similarly

$$< \mathbf{x}_1, \mathbf{x}_2| =< \mathbf{x}_2, \mathbf{x}_1| =< 0|\phi(\mathbf{x}_1)\phi(\mathbf{x}_2). \tag{4.6}$$

We see that the two particles are not identifiable and that the particles are bosons. The commutation relations Eqs.(4.1) and the normalisation Eq.(4.3) give

$$< \mathbf{x}_1|\mathbf{x}_2 >= \delta^3(\mathbf{x}_1 - \mathbf{x}_2), \tag{4.7}$$

and

$$< \mathbf{x}_1, \mathbf{x}_2|\mathbf{x}_3, \mathbf{x}_4 >= \delta^3(\mathbf{x}_1 - \mathbf{x}_3)\delta^3(\mathbf{x}_2 - \mathbf{x}_4) + \delta^3(\mathbf{x}_1 - \mathbf{x}_4)\delta^3(\mathbf{x}_2 - \mathbf{x}_3). \tag{4.8}$$

This equation should be compared to Eq.(2.13), the equation giving the normalisation of identifiable particles.

Now let

$$a(\mathbf{p}) = (2\pi\hbar)^{-\frac{3}{2}} \int d^3\mathbf{x} \exp \frac{-i\mathbf{p}.\mathbf{x}}{\hbar} \phi(\mathbf{x}). \tag{4.9}$$

We can then deduce

$$[a(\mathbf{p}), a^\dagger(\mathbf{p}')] = \delta^3(\mathbf{p} - \mathbf{p}'), \qquad [a(\mathbf{p}), a(\mathbf{p}')] = 0. \tag{4.10}$$

$a(\mathbf{p})$ is the annihilation operator for a particle of momentum $\mathbf{p}$. We define $|\mathbf{p}> = a^\dagger(\mathbf{p})|0>$ and $|\mathbf{p}_1, \mathbf{p}_2> = a^\dagger(\mathbf{p}_1)a^\dagger(\mathbf{p}_2)|0>$. In particular, acting with the adjoint of Eq.(4.9) on the vacuum state gives

$$|\mathbf{p}> = (2\pi\hbar)^{-\frac{3}{2}} \int d^3\mathbf{p} \exp \frac{i\mathbf{p}.\mathbf{x}}{\hbar} |\mathbf{x}>, \tag{4.11}$$

which is Eq.(2.6), that is the relationship between position and momentum is the same here as in the standard description of quantum mechanics.

From the commutation relations Eq.(4.1) we find

$$[\phi^\dagger(\mathbf{x}')\phi(\mathbf{x}'), \phi^\dagger(\mathbf{x})] = \delta^3(\mathbf{x} - \mathbf{x}')\phi^\dagger(\mathbf{x}), \tag{4.12}$$

and so

$$\phi^\dagger(\mathbf{x})\phi(\mathbf{x})|\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n> \\ = \Big(\delta^3(\mathbf{x}_1 - \mathbf{x}) + \delta^3(\mathbf{x}_2 - \mathbf{x}) + \ldots + \delta^3(\mathbf{x}_n - \mathbf{x})\Big)|\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n>. \tag{4.13}$$

We thus see that

$$\int d^3\mathbf{x} f(\mathbf{x})\phi^\dagger(\mathbf{x})\phi(\mathbf{x})|\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n> \\ = (f(\mathbf{x}_1) + f(\mathbf{x}_2) + \ldots + f(\mathbf{x}_n))|\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n>, \tag{4.14}$$

for any function $f(\mathbf{x})$. In particular we can define a position operator

$$\mathbf{x}^{op} \equiv \int d^3\mathbf{x}\, \phi^\dagger(\mathbf{x})\mathbf{x}\phi(\mathbf{x}), \tag{4.15}$$

and this obeys the eigenvalue equation

$$\mathbf{x}^{op}|\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n> = (\mathbf{x}_1+\ldots+\mathbf{x}_n)|\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n>. \tag{4.16}$$

Now let

$$\left(\mathbf{x}^2\right)^{op} \equiv \int d^3\mathbf{x}\,\phi^\dagger(\mathbf{x})\mathbf{x}^2\phi(\mathbf{x}). \tag{4.17}$$

Then

$$\left(\mathbf{x}^2\right)^{op}|\mathbf{x}_1,\ldots,\mathbf{x}_n> = \left(\mathbf{x}_1^2+\ldots+\mathbf{x}_n^2\right)|\mathbf{x}_1,\ldots,\mathbf{x}_n>. \tag{4.18}$$

The position operator squared obeys

$$(\mathbf{x}^{op})^2|\mathbf{x}_1,\ldots,\mathbf{x}_n> = (\mathbf{x}_1+\ldots+\mathbf{x}_n)^2|\mathbf{x}_1,\ldots,\mathbf{x}_n>, \tag{4.19}$$

from Eq.(4.16), and we note that $(\mathbf{x}^2)^{op} \neq (\mathbf{x}^{op})^2$.

This can be generalised to operators which are not straight forward functions of position. The single particle momentum operator is $-i\hbar\nabla$ in the position basis. We now define a momentum operator for an arbitrary number of particles,

$$\mathbf{p}^{op} \equiv -i\hbar\int d^3\mathbf{x}\,\phi^\dagger(\mathbf{x})\nabla\phi(\mathbf{x}). \tag{4.20}$$

This obeys the eigen-value equation

$$\mathbf{p}^{op}|\mathbf{p}_1,\ldots,\mathbf{p}_n> = (\mathbf{p}_1+\ldots+\mathbf{p}_n)|\mathbf{p}_1,\ldots,\mathbf{p}_n>. \tag{4.21}$$

We also define the momentum-squared operator as

$$\left(\mathbf{p}^2\right)^{op} \equiv -\hbar^2\int d^3\mathbf{x}\,\phi^\dagger(\mathbf{x})\nabla^2\phi(\mathbf{x}), \tag{4.22}$$

which obeys the eigen-value equation

$$\left(\mathbf{p}^2\right)^{op}|\mathbf{p}_1,\ldots,\mathbf{p}_n> = \left(\mathbf{p}_1^2+\ldots+\mathbf{p}_n^2\right)|\mathbf{p}_1,\ldots,\mathbf{p}_n>. \tag{4.23}$$

If we assume that the vacuum state $|0>$ is unique then a general state of scalar bosons can be expanded as

$$|\Phi,t> = \Phi_0(t)|0>$$

$$+\int d^3\mathbf{x}_1\,\Phi_1(\mathbf{x}_1,t)\phi^\dagger(\mathbf{x}_1)|0> + \frac{1}{2}\int d^3\mathbf{x}_1 d^3\mathbf{x}_2\,\Phi_2(\mathbf{x}_1,\mathbf{x}_2,t)\phi^\dagger(\mathbf{x}_1)\phi^\dagger(\mathbf{x}_2)|0> + \ldots$$
$$+\frac{1}{n!}\int d^3\mathbf{x}_1\ldots d^3\mathbf{x}_n\,\Phi_n(\mathbf{x}_1,\ldots,\mathbf{x}_n,t)\phi^\dagger(\mathbf{x}_1)\ldots\phi^\dagger(\mathbf{x}_n)|0> + \ldots \qquad (4.24)$$

where each Φ_j is completely symmetric in each of the spatial variables. The state has wave functions for each number of particles. For instance the n particle wave-function is the $\phi^\dagger(\mathbf{x}_1)\phi^\dagger(\mathbf{x}_2)\ldots\phi^\dagger(\mathbf{x}_n)|0>$ component of $|\Psi,t>$, that is

$$< \mathbf{x}_1,\ldots,\mathbf{x}_n|\Phi,t> = \Phi_n(\mathbf{x}_1,\ldots,\mathbf{x}_n,t). \qquad (4.25)$$

The Hamiltonian of a single non-relativistic particle in an external potential $V(\mathbf{x})$ is

$$H = -\frac{\hbar^2\nabla^2}{2m} + V(\mathbf{x}). \qquad (4.26)$$

We now write the equation of motion for non-relativistic bosons in an external potential $V(\mathbf{x})$ as

$$i\hbar\frac{\partial}{\partial t}|\Phi,t> = \int d^3\mathbf{x}\,\phi^\dagger(\mathbf{x})H\phi(\mathbf{x})|\Phi,t>. \qquad (4.27)$$

Taking the $|\mathbf{x}_1,\ldots,\mathbf{x}_n>$ component we obtain

$$i\hbar\frac{\partial}{\partial t}\Phi_n = [H_1+\ldots+H_n]\,\Phi_n, \qquad (4.28)$$

where H_k is H with each $\mathbf{x}$ replaced by $\mathbf{x}_k$, that is H_k is the Hamiltonian of particle k. This is the equation of motion for n non-relativistic particles given in chapter 2.

4.3 Quantum Electrostatics

This section examines quantum electrostatics in detail. The method we shall employ to produce the electrostatic interaction can be generalised to electromagnetism and quantum linear gravity.

Consider a non-relativistic one particle wave function $\Phi(\mathbf{x},t)$. This has charge density $\rho(\mathbf{x},t) = e\Phi^*(\mathbf{x},t)\Phi(\mathbf{x},t)$, where e is the charge of the particle. We postulate that the charge density operator is the equivalent operator formed by replacing $\Phi(\mathbf{x},t)$ by $\phi(\mathbf{x})$, that is $e\phi^\dagger(\mathbf{x})\phi(\mathbf{x})$. This operator obeys the eigen-value equation

$$e\phi^\dagger(\mathbf{x})\phi(\mathbf{x})|\mathbf{x}_1,\ldots,\mathbf{x}_n> = e\left(\delta^3(\mathbf{x}_1-\mathbf{x})+\ldots+\delta^3(\mathbf{x}_n-\mathbf{x})\right)|\mathbf{x}_1,\ldots,\mathbf{x}_n>. \qquad (4.29)$$

We see that the eigen-value is the value of the charge density at the point $\mathbf{x}$ caused by point particles at the positions $\mathbf{x}_1$, ..., $\mathbf{x}_n$, each with charge e. The potential energy operator obeys

$$\nabla^2 V(\mathbf{x}) = -e^2\phi^\dagger(\mathbf{x})\phi(\mathbf{x}). \tag{4.30}$$

The solution of this equation which vanishes at infinity is

$$V(\mathbf{x}) = \frac{e^2}{4\pi}\int d^3\mathbf{x}'\,\frac{\phi^\dagger(\mathbf{x}')\phi(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}, \tag{4.31}$$

representing the case where there is no external field. This operator obeys the eigen-value equation

$$V(\mathbf{x})|\mathbf{x}_1,\ldots,\mathbf{x}_n\rangle = \left(\frac{e^2}{4\pi|\mathbf{x}-\mathbf{x}_1|}+\ldots+\frac{e^2}{4\pi|\mathbf{x}-\mathbf{x}_n|}\right)|\mathbf{x}_1,\ldots,\mathbf{x}_n\rangle. \tag{4.32}$$

The eigen-value is the value of the electrostatic potential energy at $\mathbf{x}$ caused by particles at $\mathbf{x}_1$, $\ldots\mathbf{x}_n$.

$V(\mathbf{x})$ is the potential energy at a particular point (that is at $\mathbf{x}$). What about the total potential energy of the system? Classically the energy of an electromagnetic field $\mathbf{E}(\mathbf{x}) = -\nabla\tilde{V}(\mathbf{x})$ is given by

$$\mathcal{E} = \frac{1}{2}\int d^3\mathbf{x}\,\mathbf{E}(\mathbf{x}).\mathbf{E}(\mathbf{x}), \tag{4.33}$$

(in Heaviside's units) and integrating by parts gives

$$\mathcal{E} = \frac{1}{2}\int d^3\mathbf{x}\,\tilde{V}(\mathbf{x})\rho(\mathbf{x}). \tag{4.34}$$

The $\tilde{V}(\mathbf{x})$ used here is the total electrostatic potential of the field independent of the charge of any particle moving in that field. It corresponds to $V(\mathbf{x})/e$ where $V(\mathbf{x})$ is the total potential energy of a particle of charge e in the field. This result extends to the quantum system. Remembering that the quantum charge density operator is $e\phi^\dagger(\mathbf{x})\phi(\mathbf{x})$ we define the interaction Hamiltonian,

$$H_{Iex} = \frac{1}{2}\int d^3\mathbf{x}\,\phi^\dagger(\mathbf{x})V(\mathbf{x})\phi(\mathbf{x}) = \frac{e^2}{8\pi}\int d^3\mathbf{x}d^3\mathbf{x}'\,\frac{\phi^\dagger(\mathbf{x})\phi^\dagger(\mathbf{x})\phi(\mathbf{x}')\phi(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}. \tag{4.35}$$

We thus write the equation of motion for particles interacting through electrostatics by

$$i\hbar\frac{\partial}{\partial t}|\Phi,t\rangle = [H_0 + H_{Iex}]|\Phi,t\rangle, \tag{4.36}$$

where H_0 is the total free particle Hamiltonian,

$$H_0 = -\frac{\hbar^2}{2m}\int d^3\mathbf{x}\,\phi^\dagger(\mathbf{x})\nabla^2\phi(\mathbf{x}), \tag{4.37}$$

and H_{Iex} is the total interaction Hamiltonian given by Eq.(4.35). Remembering that particles are bosons and normalised according to Eq.(4.8), we find for example

$$H_{Iex}|\mathbf{x}_3,\mathbf{x}_4> = \frac{e^2}{8\pi}\int d^3\mathbf{x}_1 d^3\mathbf{x}_2 \frac{|\mathbf{x}_1,\mathbf{x}_2> < \mathbf{x}_1,\mathbf{x}_2|}{|\mathbf{x}_1-\mathbf{x}_2|}|\mathbf{x}_3,\mathbf{x}_4> = \frac{e^2}{4\pi|\mathbf{x}_3-\mathbf{x}_4|}|\mathbf{x}_3,\mathbf{x}_4>. \tag{4.38}$$

Taking the $|\mathbf{x}_1,\mathbf{x}_2>$ component of Eq.(4.36) we obtain the two particle equation,

$$i\hbar\frac{\partial}{\partial t}\Phi_2 = \left[-\frac{\hbar^2\nabla_1^2}{2m} - \frac{\hbar^2\nabla_2^2}{2m} + \frac{e^2}{4\pi|\mathbf{x}_1-\mathbf{x}_2|}\right]\Phi_2, \tag{4.39}$$

where Φ_2 is given by Eq.(4.25).

The equation *ex*cludes self interactions. Generalising Eq.(4.38) to an arbitrary number of particles we find

$$H_{Iex}|\mathbf{x}_1,\ldots,\mathbf{x}_n> = \sum_{i,j=1,i<j}^{n} \frac{e^2}{4\pi|\mathbf{x}_j-\mathbf{x}_j|}|\mathbf{x}_1,\ldots,\mathbf{x}_n>. \tag{4.40}$$

The eigen-value is the value of the total electrostatic potential energy caused by particles at positions $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$, with each of the self interactions terms excluded. In particular we see that

$$H_{Iex}|\mathbf{x}_1> = 0. \tag{4.41}$$

The term on the right of H_{Iex} is an annihilation operator and so $V(\mathbf{x})$ acts on the vacuum state $|0>$, giving it a 'value' of zero.

If we wish to *in*clude self interactions we have to use a different interaction Hamiltonian, one in which the potential operator $V(\mathbf{x})$ given by Eq.(4.31) is not between a creation and annihilation operator,

$$H_{Iin} = \frac{1}{2}\int d^3\mathbf{x}\,\phi^\dagger(\mathbf{x})\phi(\mathbf{x})V(\mathbf{x}). \tag{4.42}$$

This interaction Hamiltonian obeys

$$H_{Iin}|\mathbf{x}_1,\ldots,\mathbf{x}_n> = \sum_{i,j=1,i\leq j}^{n} \frac{e^2}{4\pi|\mathbf{x}_j-\mathbf{x}_j|}|\mathbf{x}_1,\ldots,\mathbf{x}_n>. \tag{4.43}$$

With this new interaction Hamiltonian the equation of motion cannot be written in the form of Eq.(4.27). In order to re-express the motion note that

$$\frac{\partial}{\partial x} f(x) = \int dy\, M(x,y) f(y), \tag{4.44}$$

where $M(x,y) = -\frac{\partial}{\partial y}\delta(y-x)$. Since $M(x,y) = -M(y,x)$ we can write

$$\frac{\partial}{\partial x} f(x) = \int dy\, M(x,y) f(y) = -\int dy\, f(y) M(y,x) = f(x)\left(-\frac{\partial}{\partial x}\right)^{\dagger}. \tag{4.45}$$

This means

$$H_0 = -\frac{\hbar^2}{2m}\int d^3\mathrm{x}\, \phi^{\dagger}(\mathrm{x})\nabla^2\phi(\mathrm{x}) = -\frac{\hbar^2}{2m}\int d^3\mathrm{x}\, \phi^{\dagger}(\mathrm{x})\left[\phi(\mathrm{x})\left(\nabla^2\right)^{\dagger}\right]. \tag{4.46}$$

Since $V(\mathbf{x})$ is Hermitian, the equation of motion which includes self interactions can be written as

$$i\hbar\frac{\partial}{\partial t}|\Phi, t> = \int d^3\mathrm{x}\, \phi^{\dagger}(\mathrm{x})\left[\phi(\mathrm{x})H^{\dagger}\right]|\Phi, t>, \tag{4.47}$$

where

$$H = H_0 + H_{Iin}. \tag{4.48}$$

These self interactions do not actually alter the motion of the particles. The system which includes self interactions is really equivalent to the system which excludes self interactions. To see this we redefine H_{Iin} and H_{Iex} so that

$$H_{Iin} = \int d^3\mathrm{x}\, V_{in}(\mathrm{x})\phi^{\dagger}(\mathrm{x})\phi(\mathrm{x}), \qquad H_{Iex} = \int d^3\mathrm{x}\, \phi^{\dagger}(\mathrm{x})V_{ex}(\mathrm{x})\phi(\mathrm{x}). \tag{4.49}$$

where $V_{in}(\mathbf{x})$ and $V_{ex}(\mathbf{x})$ obey

$$\nabla^2 V_{in}(\mathrm{x}) = -e^2\phi^{\dagger}(\mathrm{x})\phi(\mathrm{x}), \qquad \nabla^2 V_{ex}(\mathrm{x}) = -e^2\phi^{\dagger}(\mathrm{x})\phi(\mathrm{x}). \tag{4.50}$$

This allows $V_{in}(\mathbf{x})$ and $V_{ex}(\mathbf{x})$ to obey different boundary conditions. The solutions we choose are

$$V_{in}(\mathrm{x}) = \frac{e^2}{4\pi}\int d^3\mathrm{x}'\frac{\phi^{\dagger}(\mathrm{x}')\phi(\mathrm{x}')}{|\mathrm{x}'-\mathrm{x}|} - \frac{e^2}{4\pi|\epsilon|}, \qquad V_{ex}(\mathrm{x}) = \frac{e^2}{4\pi}\int d^3\mathrm{x}'\frac{\phi^{\dagger}(\mathrm{x}')\phi(\mathrm{x}')}{|\mathrm{x}'-\mathrm{x}|}. \tag{4.51}$$

$V_{in}(\mathbf{x})$ and $V_{ex}(\mathbf{x})$ differ by the constant $\frac{e^2}{4\pi|\epsilon|}$, and so their physical interpretation is exactly the same. We now see that

$$H_{Iex} = \int d^3\mathrm{x}\, V_{ex}(\mathrm{x})\phi^{\dagger}(\mathrm{x})\phi(\mathrm{x}) + \int d^3\mathrm{x}\left[\phi^{\dagger}(\mathrm{x}), V_{ex}(\mathrm{x})\right]\phi(\mathrm{x}). \tag{4.52}$$

Noting that

$$\left[\phi^\dagger(\mathbf{x}), V_{ex}(\mathbf{x})\right] = -\frac{e^2\phi^\dagger(\mathbf{x})}{4\pi|\epsilon'|} \tag{4.53}$$

where we take the limit $\epsilon' \to 0$, and using Eq.(4.51) we see

$$H_{Iex} = \int d^3\mathbf{x}\, V_{in}(\mathbf{x})\phi^\dagger(\mathbf{x})\phi(\mathbf{x}) + \frac{e^2}{4\pi|\epsilon|}\int d^3\mathbf{x}\, \phi^\dagger(\mathbf{x})\phi(\mathbf{x}) - e^2\int d^3\mathbf{x}\, \frac{\phi^\dagger(\mathbf{x})\phi(\mathbf{x})}{4\pi|\epsilon'|}. \tag{4.54}$$

Hence by setting $\epsilon = \epsilon'$ before taking the limit we can arrange that

$$H_{Iin} = H_{Iex}. \tag{4.55}$$

The fact that self interactions produce no observable effects is only a feature of this example. In later chapters we shall discuss systems where the introduction of self-interactions does alter the motion. In particular the final chapter discusses semi-classical gravity, where the interaction is non-linear and where self-interactions have an enormous effect.

4.4 Introduction to Relativistic Interactions

The equations of motion produced in this chapter all describe non-relativistic particles. The interactions described within this thesis are all the result of physical fields. Relativistic fields can be formed whether the motion is relativistic or not. This section is an introduction to a theory of relativistic interactions. It describes how we can model a quantum analogue of a classical field $A_c(x)$ obeying

$$\Box^2 A_c(x) = \rho(x), \tag{4.56}$$

where $\rho(x)$ is the charge density.

We consider states $|\Phi, t>$ which are evolving according to an equation of motion linear in $|\Phi, t>$. There then exists a linear operator $U(t_a, t_b)$ such that

$$|\Phi, t_b> = U(t_a, t_b)|\Phi, t_a>. \tag{4.57}$$

This operator must obey

$$U(t_c, t_b)U(t_a, t_c) = U(t_a, t_b), \qquad U(t_a, t_b) = U^{-1}(t_b, t_a). \tag{4.58}$$

We consider the case where $U(t_a, t_b)$ is unitary. We make no other assumptions about the nature of the motion. The following holds for all such evolving states, whether interacting or non-interacting, and every theory mentioned in this thesis obeys these assumptions.

Let the state

$$|\mathbf{x}_1; t_a; t_b\rangle = U(t_a, t_b)|\mathbf{x}_1\rangle, \tag{4.59}$$

denote a single particle solution to the equation of motion which at time t_a was in an eigenstate of position with eigen-value $\mathbf{x}_1$ and has now evolved to a time t_b according to the equation of motion. Also let the state

$$|\mathbf{x}_1, \mathbf{x}_2; t_a; t_b\rangle = U(t_a, t_b)|\mathbf{x}_1, \mathbf{x}_2\rangle, \tag{4.60}$$

denote a two particle solution to the equation of motion. At time t_a each particle was in an eigen-state of position with eigen-values $\mathbf{x}_1$ and $\mathbf{x}_2$ and the state has now evolved to a time t_b. We now define

$$\phi_H^\dagger(\mathbf{x}, t) = U^\dagger(0, t)\phi^\dagger(\mathbf{x})U(0, t), \tag{4.61}$$

which is an operator in the Heisenberg picture. $\phi^\dagger(\mathbf{x})$ is a particle creation operator introduced in section 4.2. This operator creates a particle which was in the eigen-state $\mathbf{x}$ of position at time t and which has continued evolving according to the equation of motion to the time zero. If we also have

$$U(t_b, t_a)|0\rangle = |0\rangle \tag{4.62}$$

for every time t_a and t_b, i.e. $|0\rangle$ is a solution to the equation of motion, then

$$|\mathbf{x}; t; 0\rangle = \phi_H^\dagger(\mathbf{x}, t)|0\rangle, \tag{4.63}$$

and

$$|\mathbf{x}_1, \mathbf{x}_2; t; 0\rangle = \phi_H^\dagger(\mathbf{x}_1, t)\phi_H^\dagger(\mathbf{x}_2, t)|0\rangle. \tag{4.64}$$

We have seen that the charge density operator is $e\phi^\dagger(\mathbf{x})\phi(\mathbf{x})$. Let the quantum operator $A(\mathbf{x}, t)$ obey the equation

$$\Box^2 < \Phi_1, t|A(\mathbf{x}, t)|\Phi_2, t> =< \Phi_1, t|e\phi^\dagger(\mathbf{x})\phi(\mathbf{x})|\Phi_2, t>, \tag{4.65}$$

which holds for every solution to the equation of motion $|\Phi_1, t>$ and $|\Phi_2, t>$. The Schrödinger picture is being used here. The operator on the right hand side is constant and it is the states that are evolving. Similarly we would expect the operator on the left hand side $A(\mathbf{x}, t)$ to be constant over time, though we do not assume this. The time derivative in $\Box^2$ acts on $< \Phi_1, t|$ and $|\Phi_2, t>$. To put the motion into the operator we move to the Heisenberg picture. First we write Eq.(4.65) as

$$\Box^2 < \Phi_1, 0|U^\dagger(0,t)A(\mathbf{x},t)U(0,t)|\Phi_2, 0> =< \Phi_1, 0|U^\dagger(0,t)e\phi^\dagger(\mathbf{x})\phi(\mathbf{x})U(0,t)|\Phi_2, 0>. \tag{4.66}$$

In the Heisenberg picture the operator equivalent to $A(\mathbf{x}, t)$ is

$$A_H(\mathbf{x}, t) = U^\dagger(0,t)A(\mathbf{x},t)U(0,t). \tag{4.67}$$

and so Eq.(4.66) is

$$\Box^2 < \Phi_1, 0|A_H(\mathbf{x},t)|\Phi_2, 0> =< \Phi_1, 0|e\phi_H^\dagger(\mathbf{x},t)\phi_H(\mathbf{x},t)|\Phi_2, 0>. \tag{4.68}$$

Eq.(4.68) holds for every $< \Phi_1, 0|$ and $|\Phi_2, 0>$ and since these are both constants

$$\Box^2 A_H(\mathbf{x}, t) = e\phi_H^\dagger(\mathbf{x},t)\phi_H(\mathbf{x},t). \tag{4.69}$$

This is the direct quantum equivalent of Eq.(4.56), since in the Heisenberg picture the motion is in the operators. The right hand side of Eq.(4.69) is a function of $(\mathbf{x}, t)$, and we can solve this equation. One possible solution is

$$A_H(\mathbf{x}, t) = e\int d^4x'\, G_{ret}(\mathbf{x}-\mathbf{x}', t-t')\phi_H^\dagger(\mathbf{x}',t')\phi_H(\mathbf{x}',t'), \tag{4.70}$$

in which

$$G_{ret}(\mathbf{x}, t) = \frac{1}{2\pi}\delta(\mathbf{x}^2 - t^2)\theta(-t), \tag{4.71}$$

the retarded Green's function of $\Box^2$. Since

$$\begin{aligned} U(0,t)\phi_H^\dagger(\mathbf{x}',t')\phi_H(\mathbf{x}',t')U^\dagger(0,t) &= U(0,t)U^\dagger(0,t')\phi^\dagger(\mathbf{x}')\phi(\mathbf{x}')U(0,t')U^\dagger(0,t) \\ &= U^\dagger(t,t')\phi^\dagger(\mathbf{x}')\phi(\mathbf{x}')U(t,t'), \end{aligned} \tag{4.72}$$

returning to the Schrödinger picture,

$$A(\mathbf{x},t) = e\int d^4x'\, G_{ret}(\mathbf{x}-\mathbf{x}', t-t')U^{\dagger}(t,t')\phi^{\dagger}(\mathbf{x}')\phi(\mathbf{x}')U(t,t'). \qquad (4.73)$$

The operator $A(\mathbf{x},t)$ is dependent on $U(t_a,t_b)$, and thus cannot be formed unless the equation of motion is known. If the equation of motion is coupled to $A(\mathbf{x},t)$ then this leads to two very difficult simultaneous equations.

This technique can be extended to electromagnetism and gravity. The field operators produced (i.e. the electromagnetic field operator and metric field operator) are direct analogues of the equivalent classical fields. The electromagnetic field operator used in QED is produced in a very different way. Is it possible that a theory equivalent to QED can be formed using these methods?

Chapter 5

Dirac Particles and Electromagnetism

5.1 Introduction

This chapter shows how the techniques developed in the previous section can be extended to describe Dirac particles. Section 5.2 shows how the equation of motion describing Dirac particles which was postulated in chapter 2 can be easily and naturally written using particle creation and annihilation operators. The advantage of this description is that it can be extended to electromagnetically interacting systems, by replacing the background electromagnetic 4-potential with a 4-potential operator. Section 5.3 discusses electromagnetic interactions.

5.2 Operators and States

To describe scalar particles we used a single field which was used to create a particle at a particular point in space. This was defined through its commutation relations, and this resulted in the particles being bosons. Dirac particles are fermions, and come in 4 different types. To describe these we introduce 4 operator fields $\psi_\alpha(\mathbf{x})$

where $\alpha = 1, 2, 3, 4$ which obey the anti-commutation relations

$$\{\psi_\alpha(\mathbf{x}), \psi^\dagger_\beta(\mathbf{x}')\} = \delta_{\alpha\beta}\delta^3(\mathbf{x} - \mathbf{x}'), \qquad \{\psi_\alpha(\mathbf{x}), \psi_\beta(\mathbf{x}')\} = 0. \tag{5.1}$$

The vacuum state is defined as the state that obeys

$$\psi_\alpha(\mathbf{x})|0> = 0, \tag{5.2}$$

for every $\mathbf{x} \in \mathbf{R}^3$ and $\alpha = 1, 2, 3, 4$, and is normalised according to $< 0|0 >= 1$. We do not assume that $|0>$ is unique. The $\psi_\alpha(\mathbf{x})$ operators annihilate particles at the spatial position $\mathbf{x}$. We also define the operators

$$c_\alpha(\mathbf{p}) = (2\pi\hbar)^{-\frac{3}{2}} \int d^3\mathbf{x} \exp \frac{-i\mathbf{p}.\mathbf{x}}{\hbar} \psi_\alpha(\mathbf{x}), \qquad \alpha = 1, 2, 3, 4, \tag{5.3}$$

which each annihilate a particle of momentum $\mathbf{p}$. From Eq.(5.1) we can deduce that the momentum creation and annihilation operators obey the anti-commutation relations

$$\{c_\alpha(\mathbf{p}), c^\dagger_\beta(\mathbf{p}')\} = \delta_{\alpha\beta}\delta^3(\mathbf{p} - \mathbf{p}'), \qquad \{c_\alpha(\mathbf{p}), c_\beta(\mathbf{p}')\} = 0. \tag{5.4}$$

Similarly $\psi_\alpha(\mathbf{x})$ can be expanded in terms of $c_\alpha(\mathbf{p})$,

$$\psi_\alpha(\mathbf{x}) = (2\pi\hbar)^{-\frac{3}{2}} \int d^3\mathbf{p} \exp \frac{i\mathbf{p}.\mathbf{x}}{\hbar} c_\alpha(\mathbf{p}). \tag{5.5}$$

It should be noted that this is not the standard expression for the Dirac field. The reason behind this will become apparent in chapter 6.

In Chapter 2 an n Dirac particle system was described with a wave-function

$$\Psi_{n(\alpha_1 \ldots \alpha_n)}(\mathbf{x}_1, \ldots, \mathbf{x}_n, t). \tag{5.6}$$

$\alpha_1, \ldots, \alpha_n$ are spinor indices and $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are the positions of each of the particles. This wave-function obeys an equation of the form

$$i\hbar \frac{\partial}{\partial t} \Psi_{n(\alpha_1 \ldots \alpha_n)} = \sum_{k=1}^{n} H_{k(\alpha_k \beta)} \Psi_{n(\alpha_1 \ldots \alpha_{k-1} \beta \alpha_{k+1} \ldots \alpha_n)}. \tag{5.7}$$

H_k is the Hamiltonian of particle k and is a 4×4 matrix operator which is dependent on $\mathbf{x}_k$ and the derivatives with respect to $\mathbf{x}_k$. In chapter 2 this Hamiltonian referred to free particles or to particles in the presence of background electromagnetic fields; there are no interactions.

The first result that is needed is to show that the description of Dirac particles in chapter 2 can be easily and naturally written using the Dirac particle creation and annihilation operators. On the assumption that the vacuum state $|0>$ is unique we can expand a general state of an arbitrary number of particles as

$$
\begin{aligned}
|\Psi, t> &= \Psi_0(t)|0> + \sum_{\alpha_1=1}^{4} \int d^3\mathbf{x}_1 \Psi_{1(\alpha_1)}(\mathbf{x}_1, t)\psi^\dagger_{\alpha_1}(\mathbf{x}_1)|0> \\
&+ \frac{1}{2} \sum_{\alpha_1,\alpha_2=1}^{4} \int d^3\mathbf{x}_1 d^3\mathbf{x}_2 \Psi_{2(\alpha_1\alpha_2)}(\mathbf{x}_1, \mathbf{x}_2, t)\psi^\dagger_{\alpha_1}(\mathbf{x}_1)\psi^\dagger_{\alpha_2}(\mathbf{x}_2)|0> + \dots \\
&+ \frac{1}{n!} \sum_{\alpha_1,\dots,\alpha_n=1}^{4} \int d^3\mathbf{x}_1 \dots d^3\mathbf{x}_n \Psi_{n(\alpha_1\dots\alpha_n)}(\mathbf{x}_1, \dots, \mathbf{x}_n, t)\psi^\dagger_{\alpha_1}(\mathbf{x}_1)\dots\psi^\dagger_{\alpha_n}(\mathbf{x}_n)|0> + \dots
\end{aligned}
\tag{5.8}
$$

where each of the wave-functions Ψ are anti-symmetric in their spatial variables. Let $H(\mathbf{x}, \nabla)$ be a 4×4 matrix operator which can be expressed as

$$H(\mathbf{x}, \nabla) = f(\mathbf{x}) + g^j(\mathbf{x})\partial_j, \tag{5.9}$$

where $f(\mathbf{x})$ and each of the $g^j(\mathbf{x})$ are matrices which are arbitrary functions of $\mathbf{x}$, and we sum over j. Also let $H_{(\alpha\beta)}(\mathbf{x}, \nabla)$, $f_{(\alpha\beta)}$ and $g^j_{(\alpha\beta)}$ denote their (α, β) matrix element. The result that is required is a general one, and no explicit expression for $H(\mathbf{x}, \nabla)$ is given. Finally let $\psi(\mathbf{x})$ be the column matrix (or spinor) operator

$$\psi(\mathbf{x}) = \begin{pmatrix} \psi_1(\mathbf{x}) \\ \psi_2(\mathbf{x}) \\ \psi_3(\mathbf{x}) \\ \psi_4(\mathbf{x}) \end{pmatrix}, \tag{5.10}$$

so we can write

$$\psi^\dagger(\mathbf{x})H(\mathbf{x}, \nabla)\psi(\mathbf{x}) = \psi^\dagger_\alpha(\mathbf{x})H_{(\alpha\beta)}(\mathbf{x}, \nabla)\psi_\beta(\mathbf{x}). \tag{5.11}$$

Under these definitions and assumptions, if $|\Psi, t>$ obeys

$$i\hbar\frac{\partial}{\partial t}|\Psi, t> = \int d^3\mathbf{x}\, \psi^\dagger(\mathbf{x})H(\mathbf{x}, \nabla)\psi(\mathbf{x})|\Psi, t>, \tag{5.12}$$

then the wave-functions obey

$$i\hbar\frac{\partial}{\partial t}\Psi_{n(\alpha_1\ldots\alpha_n)} = \sum_{k=1}^{n} H_{(\alpha_k\beta)}(\mathbf{x}_k, \nabla_k)\Psi_{n(\alpha_1\ldots\alpha_{k-1}\beta\alpha_{k+1}\ldots\alpha_n)}, \tag{5.13}$$

for every positive integer n. Eq.(5.13) is Eq.(5.7) written using a different notation. $H(\mathbf{x}_k, \nabla_k)$ in Eq.(5.13) corresponds to H_k in Eq.(5.7), as both are 4×4 matrix operators dependent on $\mathbf{x}_k$ and its derivatives. This means that we can use Eq.(5.12) to describe the evolution of a state vector $|\Psi, t\rangle$, where each particle has Hamiltonian $H(\mathbf{x}, \nabla)$. Appendix A shows that the $\psi^\dagger_{\alpha_1}(\mathbf{x}_1)\ldots\psi^\dagger_{\alpha_n}(\mathbf{x}_n)|0\rangle$ component of Eq.(5.12) is Eq.(5.13).

A free Dirac particle has Hamiltonian

$$H = \gamma^0\left(-i\hbar\gamma^j\partial_j + m\right), \tag{5.14}$$

where

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}. \tag{5.15}$$

Now let

$$H_0 = \int d^3\mathbf{x}\, \psi^\dagger(\mathbf{x})\gamma^0\left(-i\hbar\gamma^j\partial_j + m\right)\psi(\mathbf{x}). \tag{5.16}$$

The equation

$$i\hbar\frac{\partial}{\partial t}|\Psi, t\rangle = H_0|\Psi, t\rangle, \tag{5.17}$$

describes a system of free particles.

We now wish to find solutions to Eq.(5.17). First consider single particle solutions. The free particle Dirac equation is

$$i\hbar\frac{\partial}{\partial t}\Psi_1(\mathbf{x}, t) = \gamma^0\left(-i\hbar\gamma^j\partial_j + m\right)\Psi_1(\mathbf{x}, t), \tag{5.18}$$

and this has solutions

$$\Psi_1(\mathbf{x}, t) = \exp\frac{-it\kappa_\tau\sqrt{\mathbf{p}^2 + m^2}}{\hbar}\frac{(2\pi\hbar)^{-\frac{3}{2}}m^{\frac{1}{2}}}{\sqrt[4]{\mathbf{p}^2 + m^2}}\exp\frac{i\kappa_\tau\mathbf{p}.\mathbf{x}}{\hbar}\begin{pmatrix} u_{\tau(1)}(\mathbf{p}) \\ u_{\tau(2)}(\mathbf{p}) \\ u_{\tau(3)}(\mathbf{p}) \\ u_{\tau(4)}(\mathbf{p}) \end{pmatrix}$$

$$= \exp \frac{-it\kappa_\tau\sqrt{\mathbf{p}^2+m^2}}{\hbar} \frac{(2\pi\hbar)^{-\frac{3}{2}} m^{\frac{1}{2}}}{\sqrt[4]{\mathbf{p}^2+m^2}} \exp \frac{i\kappa_\tau \mathbf{p.x}}{\hbar} u_\tau(\mathbf{p}), \tag{5.19}$$

where κ_τ is again given by

$$\kappa_\tau = \begin{cases} 1 & \tau = 1,2, \\ -1 & \tau = 3,4. \end{cases} \tag{5.20}$$

Using the above result we see that a single particle solution to Eq.(5.17) is

$$|\Psi, t> = \sum_{\alpha=1}^{4} \int d^3\mathbf{x} \Psi_{1(\alpha)}(\mathbf{x}, t) \psi_\alpha^\dagger(\mathbf{x})|0>$$

$$= \exp \frac{-it\kappa_\tau\sqrt{\mathbf{p}^2+m^2}}{\hbar} \frac{(2\pi\hbar)^{-\frac{3}{2}} m^{\frac{1}{2}}}{\sqrt[4]{\mathbf{p}^2+m^2}} \int d^3\mathbf{x} \exp \frac{i\kappa_\tau \mathbf{p.x}}{\hbar} \begin{pmatrix} u_{\tau(1)}(\mathbf{p}) \\ u_{\tau(2)}(\mathbf{p}) \\ u_{\tau(3)}(\mathbf{p}) \\ u_{\tau(4)}(\mathbf{p}) \end{pmatrix}^T \begin{pmatrix} \psi_1^\dagger(\mathbf{x}) \\ \psi_2^\dagger(\mathbf{x}) \\ \psi_3^\dagger(\mathbf{x}) \\ \psi_4^\dagger(\mathbf{x}) \end{pmatrix} |0>. \tag{5.21}$$

We now define the single particle momentum state

$$|\mathbf{p}(\tau)> = \frac{m^{\frac{1}{2}} c_\alpha^\dagger(\kappa_\tau \mathbf{p}) u_{\tau(\alpha)}(\mathbf{p})}{\sqrt[4]{\mathbf{p}^2+m^2}} |0>, \tag{5.22}$$

where we sum over α. The index τ which is 1,2,3 or 4 denotes the type of particle being described. From the adjoint of Eq.(5.3) we see

$$c_\alpha^\dagger(\kappa_\tau \mathbf{p}) = (2\pi\hbar)^{-\frac{3}{2}} \int d^3\mathbf{x} \exp \frac{i\kappa_\tau \mathbf{p.x}}{\hbar} \psi_\alpha^\dagger(\mathbf{x}), \tag{5.23}$$

and so

$$|\mathbf{p}(\tau)> = \frac{(2\pi\hbar)^{-\frac{3}{2}} m^{\frac{1}{2}}}{\sqrt[4]{\mathbf{p}^2+m^2}} \int d^3\mathbf{x} \exp \frac{i\kappa_\tau \mathbf{p.x}}{\hbar} \begin{pmatrix} u_{\tau(1)}(\mathbf{p}) \\ u_{\tau(2)}(\mathbf{p}) \\ u_{\tau(3)}(\mathbf{p}) \\ u_{\tau(4)}(\mathbf{p}) \end{pmatrix}^T \begin{pmatrix} \psi_1^\dagger(\mathbf{x}) \\ \psi_2^\dagger(\mathbf{x}) \\ \psi_3^\dagger(\mathbf{x}) \\ \psi_4^\dagger(\mathbf{x}) \end{pmatrix} |0>. \tag{5.24}$$

Hence

$$|\Psi, t> = \exp \frac{-it\kappa_\tau\sqrt{\mathbf{p}^2+m^2}}{\hbar} |\mathbf{p}(\tau)>, \tag{5.25}$$

which is Eq.(5.21) is a solution to Eq.(5.17), that is the free particle equation. Since

$$u_\tau^\dagger(\mathbf{p}) u_\tau(\mathbf{p}) = \frac{\sqrt{\mathbf{p}^2+m^2}}{m}, \tag{5.26}$$

where there is no sum over τ, the one particle states are normalised according to

$$< \mathbf{p}_1(\tau_1)|\mathbf{p}_2(\tau_2) >= \delta_{\tau_1\tau_2}\delta^3(\mathbf{p}_1 - \mathbf{p}_2). \tag{5.27}$$

This means $|\mathbf{p}(\tau)>$ provides an orthonormal basis of the space of one particle states each of which represents a particle of a particular momentum.

Similarly we define the many particle momentum states

$$|\mathbf{p}_1(\tau_1), \ldots \mathbf{p}_n(\tau_n)> = \frac{m^{\frac{1}{2}} c^\dagger_{\alpha_1}(\kappa_{\tau_1}\mathbf{p}_1) u_{\tau_1(\alpha_1)}(\mathbf{p}_1)}{\sqrt[4]{\mathbf{p}_1^2 + m^2}} \cdots \frac{m^{\frac{1}{2}} c^\dagger_{\alpha_n}(\kappa_{\tau_n}\mathbf{p}_n) u_{\tau_n(\alpha_n)}(\mathbf{p}_n)}{\sqrt[4]{\mathbf{p}_n^2 + m^2}} |0>, \tag{5.28}$$

where each of the α_k sum from 1 to 4. The state

$$\exp \frac{-it\kappa_{\tau_1}\sqrt{\mathbf{p}_1^2 + m^2} - \ldots - it\kappa_{\tau_n}\sqrt{\mathbf{p}_n^2 + m^2}}{\hbar} |\mathbf{p}_1(\tau_1), \ldots \mathbf{p}_n(\tau_n)>, \tag{5.29}$$

is a solution to the free particle equation Eq.(5.17). Together with the vacuum state, these momentum states provide an orthonormal basis of the space of all states.

5.3 Electromagnetic Interactions

We now examine electromagnetic interactions using the technique that was developed in chapter 4. Let $|\Psi, t>$ be a state evolving according to an equation of motion linear in $|\Psi, t>$. There then exists an operator $U(t_a, t_b)$ such that for every time t_a and t_b

$$|\Psi, t_b> = U(t_a, t_b)|\Psi, t_a>. \tag{5.30}$$

For consistency

$$U(t_a, t_b) = U(t_c, t_b)U(t_a, t_c), \qquad U(t_a, t_b) = U^{-1}(t_b, t_a), \tag{5.31}$$

must hold. We make no other assumptions about the nature of $U(t_a, t_b)$ except that it is unitary. As before the following discussion of the electromagnetic field holds for all such motions. We also define

$$\psi_{H\alpha}(\mathbf{x}, t) = U^\dagger(0, t)\psi_\alpha(\mathbf{x})U(0, t), \tag{5.32}$$

and

$$\psi_H(\mathbf{x},t) = U^\dagger(0,t)\psi(\mathbf{x})U(0,t), \tag{5.33}$$

which are both operators in the Heisenberg picture.

The conserved 4-current of a spinor $\Psi(\mathbf{x},t)$ obeying the one particle Dirac equation in an arbitrary background electromagnetic field is $\Psi^\dagger(\mathbf{x},t)\gamma^0\gamma^\mu\Psi(\mathbf{x},t)$. We postulate that the 4-current operator in the Schrödinger picture for a Dirac particle of charge e is

$$j^\mu(\mathbf{x},t) = e\psi^\dagger(\mathbf{x})\gamma^0\gamma^\mu\psi(\mathbf{x}). \tag{5.34}$$

Classically the electromagnetic field obeys

$$\partial_\mu F^{\mu\nu}(\mathbf{x},t) = j^\nu(\mathbf{x},t), \tag{5.35}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{5.36}$$

and we consider

$$\partial_\mu F_H^{\mu\nu}(\mathbf{x},t) = j_H^\nu(\mathbf{x},t), \tag{5.37}$$

an equation in the Heisenberg picture, where the 4-current operator in the Heisenberg picture is

$$j_H^\mu(\mathbf{x},t) = e\psi_H^\dagger(\mathbf{x},t)\gamma^0\gamma^\mu\psi_H(\mathbf{x},t). \tag{5.38}$$

In the Lorentz gauge this equation becomes

$$\Box^2 A_H^\mu(\mathbf{x},t) = e\psi_H^\dagger(\mathbf{x},t)\gamma^0\gamma^\mu\psi_H(\mathbf{x},t), \tag{5.39}$$

which can be solved in terms of Green's functions.

Green's functions of $\Box^2$ can be expressed in the form of

$$G(x) = \frac{1}{(2\pi)^4}\int d^4p\frac{e^{-ipx}}{p^2}, \tag{5.40}$$

where the p_0 integral is performed along different contours in the complex plane. $\frac{1}{p^2}$ has two poles, one at $p_0 = +\sqrt{\mathbf{p}^2 + m^2}$ and the other at $p_0 = -\sqrt{\mathbf{p}^2 + m^2}$. There are two possible ways of integrating around each contour, and this leads to 4 independent Green's functions. These include the retarded Green's function

$$G_{ret}(x) = \frac{1}{(2\pi)^4}\int d^4p \frac{e^{-ipx}}{(p_0 - i\epsilon)^2 - \mathbf{p}^2}, \tag{5.41}$$

the advanced Green's function

$$G_{adv}(x) = \frac{1}{(2\pi)^4}\int d^4p \frac{e^{-ipx}}{(p_0 + i\epsilon)^2 - \mathbf{p}^2}, \tag{5.42}$$

and Feynman's Green's function

$$G_F(x) = \frac{1}{(2\pi)^4}\int d^4p \frac{e^{-ipx}}{p^2 + i\epsilon}. \tag{5.43}$$

Each of these have $\epsilon > 0$ and involve integrating along real p_0.

There is very little indication as to which Green's function should be chosen. Our classical instincts might expect the retarded Green's function to be used, but it is by no means certain that this choice is correct. Our objective is to produce a theory which is equivalent, or failing that, as close as possible to standard QED. QED uses Feynman's Green's functions and so these are selected here. The solution of Eq.(5.39) we choose is

$$A^\mu_H(\mathbf{x}, t) = e\int d^4x' G_F(\mathbf{x}' - \mathbf{x}, t' - t)\psi^\dagger_H(\mathbf{x}', t')\gamma^0\gamma^\mu\psi_H(\mathbf{x}', t'), \tag{5.44}$$

$A^\mu_H(\mathbf{x}, t)$ is an operator in the Heisenberg picture. Since

$$U(0,t)\psi_H(\mathbf{x}', t')U^\dagger(0,t)$$

$$= U(0,t)U(t',0)\psi(\mathbf{x})U^\dagger(t',0)U^\dagger(0,t) = U(t',t)\psi(\mathbf{x})U^\dagger(t',t), \tag{5.45}$$

the electromagnetic 4-potential operator in the Schrödinger picture is

$$A^\mu(\mathbf{x}, t) = e\int d^4x' G_F(\mathbf{x}' - \mathbf{x}, t' - t)U(t',t)\psi^\dagger(\mathbf{x}')\gamma^0\gamma^\mu\psi(\mathbf{x}')U^\dagger(t',t). \tag{5.46}$$

How do we write the equation of motion? The Hamiltonian of a single Dirac particle in a background electromagnetic field $A^\mu(\mathbf{x}, t)$ is

$$H = A_0(\mathbf{x}, t) + \gamma^0\left(\gamma^j\left(-i\hbar\partial_j + eA_j(\mathbf{x}, t)\right) + m\right). \tag{5.47}$$

If we use this Hamiltonian then

$$i\hbar\frac{\partial}{\partial t}|\Psi,t> = \int d^3\mathbf{x}\psi^\dagger(\mathbf{x})H\psi(\mathbf{x})|\Psi,t> \tag{5.48}$$

describes the motion of Dirac particles in a background electromagnetic field $A^\mu(\mathbf{x},t)$. We can rewrite Eq.(5.48) as

$$i\hbar\frac{\partial}{\partial t}|\Psi,t> = [H_0 + H_I(t)]|\Psi,t>, \tag{5.49}$$

where H_0 is given by Eq.(5.16) and $H_I(t)$ is given by

$$H_I(t) = e\int d^3\mathbf{x}\psi^\dagger(\mathbf{x})\gamma^0\gamma_\mu\psi(\mathbf{x})A^\mu(\mathbf{x},t). \tag{5.50}$$

Here $H_I(t)$ is independent of the ordering of $A^\mu(\mathbf{x},t)$.

Now consider a system of Dirac particles interacting through electromagnetism. Here $A^\mu(\mathbf{x},t)$ is now given by Eq.(5.46). In the non-relativistic example discussed in chapter 4 the operator ordering decided whether self interactions were included or excluded. Here the position of $A^\mu(\mathbf{x},t)$ in $H_I(t)$ is very important and leads to much more subtle differences. Which ordering do we choose? As has been stated, our objective is to produce a theory which is equivalent, or failing that, as close as possible to standard QED. The best results are obtained by using a time ordered equivalent of Eq.(5.50), which remembering Eq.(5.46) is

$$H_I(t) = \frac{e^2}{2}\int d^3\mathbf{x}d^4x'G_F(\mathbf{x}'-\mathbf{x},t'-t)$$
$$T^{t,t'}\left[\psi^\dagger(\mathbf{x})\gamma^0\gamma^\mu\psi(\mathbf{x}),U(t',t)\psi^\dagger(\mathbf{x}')\gamma^0\gamma_\mu\psi(\mathbf{x}')U^\dagger(t',t)\right]. \tag{5.51}$$

$T^{t,t'}[A,B]$ is a time ordered product given by

$$T^{t,t'}[A,B] = \theta(t-t')AB + \theta(t'-t)BA. \tag{5.52}$$

This formulation includes self interactions. It should also be noted that a factor of $\frac{1}{2}$ has been introduced into the interaction Hamiltonian. This corresponds to the factor of $\frac{1}{2}$ in Eq.(4.35), and is introduced for the same reasons as it was in electrostatics.

The 4-potential operator is dependent on $U(t_a, t_b)$. The equation of motion, which leads to an expression for $U(t_a, t_b)$, is dependent on $A^\mu(\mathbf{x}, t)$. Whichever formulation is used we are left with very complicated simultaneous equations.

We now find the slow speed approximation of this system. This removes the dependence of $A^\mu(\mathbf{x}, t)$ on $U(t_a, t_b)$, and greatly simplifies the equations. Consider a classical electromagnetic field which is produced by a system of slow moving particles. Away from each particle we have

$$\frac{\partial^2}{\partial t^2} A^0(\mathbf{x}, t) = O(v^2). \tag{5.53}$$

This motivates the assumption that in the slow speed approximation,

$$\frac{\partial^2}{\partial t^2} A^0_H(\mathbf{x}, t) = 0. \tag{5.54}$$

It is assumed that the magnetic field is negligible, and $A^i_H(\mathbf{x}, t) = 0$. In this approximation we postulate the operator $A^0(\mathbf{x}, t)$ obeys

$$\nabla^2 A^0_H(\mathbf{x}, t) = -j^0_H(\mathbf{x}, t) = -e\psi^\dagger_H(\mathbf{x})\psi_H(\mathbf{x}), \tag{5.55}$$

which is equivalent to

$$\nabla^2 A^0(\mathbf{x}, t) = -j^0(\mathbf{x}) = -e\psi^\dagger(\mathbf{x})\psi(\mathbf{x}), \tag{5.56}$$

in the Schrödinger picture. A possible solution is

$$A^0(\mathbf{x}, t) = \frac{e}{4\pi} \int d^3\mathbf{x}' \frac{\psi^\dagger(\mathbf{x}', t)\psi(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|} + C, \tag{5.57}$$

where C is an arbitrary constant. $eA^0(\mathbf{x}, t)$ is exactly analogous to the electrostatic potential operator used in chapter 4. Since it is assumed that $A^i(\mathbf{x}, t) = 0$, the equation of motion we use is

$$i\hbar \frac{\partial}{\partial t} |\Psi, t\rangle = \int d^3\mathbf{x}\, \psi^\dagger(\mathbf{x})\gamma^0 \left(-i\hbar\gamma^j\partial_j + m\right) \psi(\mathbf{x}) |\Psi, t\rangle + \frac{e}{2} \int d^3\mathbf{x} \psi^\dagger(\mathbf{x})\psi(\mathbf{x}) A^0(\mathbf{x}, t) |\Psi, t\rangle. \tag{5.58}$$

The issues involved in operator ordering are exactly the same as those in electrostatics, namely self interactions can be either included or excluded. Self interactions were included in the formulation of electrodynamics chosen above, and so they are included in this approximation. There can be no time ordering operator since the Green's function in $A^0(\mathbf{x}, t)$ used here introduces no extra time variable. The interaction term obeys

$$\frac{e}{2} \int d^3\mathbf{x}\, \psi^\dagger(\mathbf{x})\psi(\mathbf{x}) A^0(\mathbf{x}, t) |\mathbf{x}_1(\alpha), \mathbf{x}_2(\beta)> \\ = \left[\frac{e^2}{4\pi|\mathbf{x}_1 - \mathbf{x}_2|} + \frac{e^2}{4\pi\epsilon} + 4C \right] |\mathbf{x}_1(\alpha), \mathbf{x}_2(\beta)>, \quad (5.59)$$

where we take the limit $\epsilon \to 0$. This contains an interaction between particle (1) and particle (2), a self interaction term, and the term '4C'. If we let

$$C = -\frac{e^2}{4\pi\epsilon}, \quad (5.60)$$

before we take the limit $\epsilon \to 0$ then

$$e \int d^3\mathbf{x}\, \psi^\dagger(\mathbf{x})\psi(\mathbf{x}) A^0(\mathbf{x}, t) |\mathbf{x}_1(\alpha), \mathbf{x}_2(\beta)> = \frac{e^2}{4\pi|\mathbf{x}_1 - \mathbf{x}_2|} |\mathbf{x}_1(\alpha), \mathbf{x}_2(\beta)>, \quad (5.61)$$

and so taking the $|\mathbf{x}_1(\alpha), \mathbf{x}_2(\beta)>$ component of Eq.(5.58) we have

$$i\hbar \frac{\partial}{\partial t} \Psi_{(\alpha\beta)} = \left[\gamma^0 \left(-i\hbar\gamma^j \partial_{1j} + m \right) \right]_{(\alpha\tau)} \Psi_{(\tau\beta)} \\ + \left[\gamma^0 \left(-i\hbar\gamma^j \partial_{2j} + m \right) \right]_{(\beta\tau)} \Psi_{(\alpha\tau)} + \frac{e^2}{4\pi|\mathbf{x}_1 - \mathbf{x}_2|} \Psi_{(\alpha\beta)}. \quad (5.62)$$

Using the techniques of the Pauli approximation we can deduce

$$i\hbar \frac{\partial}{\partial t} \Psi_{(jk)} = \left[-\frac{\hbar^2}{2m} \nabla_1^2 - \frac{\hbar^2}{2m} \nabla_2^2 + 2m + \frac{e^2}{4\pi|\mathbf{x}_1 - \mathbf{x}_2|} + O(v\hbar) \right] \Psi_{(jk)}, \quad (5.63)$$

where $j, k = 1, 2$, and $O(v\hbar)$ are the spin terms which for the purposes of this calculation are negligible.

5.4 Conclusions

This chapter has shown a way in which electromagnetically interacting spin-$\frac{1}{2}$ particles can be described using a field theory notation, and how this can be related back to ordinary quantum mechanics. In doing so no mention has been made of anti-particles, or of the creation and annihilation of particle/anti-particle pairs, which standard field theory so accurately describes. Neither has there been much mention of divergencies. This section discusses where within the interpretation used here, all of these concepts are found, and how they are very much related to each other.

The infinity encountered in the previous calculation, which was due to the self interactions of each of the Dirac particles, was easily dealt with. This calculation dealt only with positive and negative energy particles, and not particles and anti-particles. In Dirac's hole theory a system with a finite number of anti-particles is a system with an infinite number of negative energy particles, and it is this infinity which leads to most of the divergencies in the scattering matrix. How or if these divergencies can be removed is not considered.

What about the creation and annihilation of particles? For instance Eq.(5.63) involves only two particles. Whether particles can change from having a positive energy to having a negative energy will depend very much on the equation of motion and the exact definition of what constitutes a 'positive energy particle' or a 'negative energy particle', but it cannot be ruled out. However whatever the definition the total number cannot change and in general

$$\begin{aligned}&[\text{Number of positive energy particles}]\\&\qquad\qquad+[\text{Number of negative energy particles}] = \text{constant}.\end{aligned} \tag{5.64}$$

Experimentally we observe anti-particles, which is a hole in the sea of negative energy particles. A system of particles and anti-particles obey

$$[\text{Number of particles}] - [\text{Number of anti-particles}] = \text{constant}. \tag{5.65}$$

Creation and annihilation of these particles similarly cannot be ruled out. A positive energy particle changing into a negative energy particle is equivalent to the annihilation of a particle/anti-particle pair. A negative energy particle changing into a positive energy particle is equivalent to the creation of a particle/anti-particle pair.

Chapter 6

Scattering Theory and QED

6.1 Introduction

Both the electromagnetic and Dirac fields have been introduced in a very different manner to that of standard field theory. This chapter shows that in Minkowski space the difference in the Dirac fields is only one of representation, and in a different representation the standard expression for the Dirac field can be used (see section 6.2). The difference in the electromagnetic field is more serious. The scattering matrix of the theory without photons and that of standard QED are compared. It is shown that the matrix entries between fermion states agree only up to the order e^2.

6.2 The Standard Formulation of QED

QED is a quantum field theory. Quantum field theory takes quantum mechanics one step further and performs what is often called second quantisation. QED is constructed from an interacting Dirac $\psi(x)$ and photon $A^\mu(x)$ field.

The Lagrangian density for a free Dirac field is

$$\mathcal{L}_0^{e^+,e^-} = \overline{\psi}(x)\left[i\hbar\gamma^\mu\partial_\mu - m\right]\psi(x), \tag{6.1}$$

where $\overline{\psi}(x) = \psi(x)\gamma^0$. This Lagrangian implies that the conjugate field to $\psi(x)$ is

$$\pi(x) = i\hbar\psi^\dagger(x). \tag{6.2}$$

We draw an analogy between the position and momentum operators, and the fields $\psi(x)$ and $\pi(x)$, and since Dirac particles are fermions postulate the equal time anti-commutation relations

$$\{\psi(\mathbf{x},t),\psi^\dagger(\mathbf{x}',t)\} = i\hbar\delta^3(\mathbf{x}-\mathbf{x}'), \qquad \{\psi(\mathbf{x},t),\psi(\mathbf{x}',t)\} = 0. \tag{6.3}$$

From the Lagrangian Eq.(6.1) we derive the equation of motion

$$i\hbar\gamma^\mu\partial_\mu\psi(x) - m\psi(x) = 0. \tag{6.4}$$

The general solution to the free Dirac equation can be expressed as

$$\psi(\mathbf{x},t) = \int \frac{m^{\frac{1}{2}}d^3\mathbf{k}}{(2\pi\hbar)^{\frac{3}{2}}\sqrt[4]{\mathbf{k}^2+m^2}} \sum_{\alpha=1,2}\left[c_\alpha(\mathbf{k})u_\alpha(\mathbf{k})\exp\frac{-ikx}{\hbar} + c^\dagger_{\alpha+2}(\mathbf{k})u_{\alpha+2}(\mathbf{k})\exp\frac{ikx}{\hbar}\right]. \tag{6.5}$$

If we let the coefficients $c_\alpha(\mathbf{k})$ be operators that obey the anti-commutation relations

$$\{c_\alpha(\mathbf{k}),c^\dagger_\beta(\mathbf{k}')\} = \delta_{\alpha\beta}\delta^3(\mathbf{k}-\mathbf{k}'), \qquad \{c_\alpha(\mathbf{k}),c_\beta(\mathbf{k}')\} = 0, \tag{6.6}$$

then the field $\psi(\mathbf{x},t)$ will obey the anti-commutation relations Eq.(6.3).

The Lagrangian density for the photon field $A^\mu(x)$ is

$$\mathcal{L}_0^\gamma = \frac{1}{2}(\partial_\mu A_\nu(x))(\partial^\mu A^\nu(x)). \tag{6.7}$$

The field conjugate to $A^\mu(x)$ is $\frac{\partial A^\mu(x)}{\partial t}$ and we postulate the commutation relations

$$\left[A^\mu(\mathbf{x},t),\frac{\partial A^\nu(\mathbf{x}',t)}{\partial t}\right] = r^{\mu\nu}\delta^3(\mathbf{x}-\mathbf{x}'). \tag{6.8}$$

Eq.(6.7) gives the equation of motion

$$\Box^2 A^\mu = 0, \tag{6.9}$$

and from this and the commutation relations Eq.(6.8) we deduce

$$[A^\mu(x),A^\nu(x')] = i\hbar D_F^{\mu\nu}(x-x'), \tag{6.10}$$

where

$$D_F^{\mu\nu}(x) = \frac{-r^{\mu\nu}}{(2\pi)^4}\int\frac{d^4k\,e^{-ikx}}{k^2+i\varepsilon}. \tag{6.11}$$

Also the photon and Dirac fields commute,

$$[\psi(x), A^\mu(x')] = 0. \tag{6.12}$$

QED itself has the Lagrangian density,

$$\mathcal{L}(x) = \mathcal{L}_0^{e^+,e^-}(x) + \mathcal{L}_0^\gamma(x) + \mathcal{L}_I(x), \tag{6.13}$$

where $\mathcal{L}_0^{e^+,e^-}(x)$ is the free Dirac field Lagrangian, $\mathcal{L}_0^\gamma$ is the free photon field Lagrangian, and $\mathcal{L}_I(x)$ is the interaction Lagrangian density given by

$$\mathcal{L}_I(x) = -e\overline{\psi}(x)\gamma^\mu A_\mu \psi(x). \tag{6.14}$$

Correspondingly the total Hamiltonian H_T of the system can be split into the free field Hamiltonian H_0 and the Interaction Hamiltonian H_I,

$$H_T = H_0 + H_I. \tag{6.15}$$

The interaction Hamiltonian is given by

$$H_I(t) = e\int d^3\mathbf{x}\,\overline{\psi}(\mathbf{x},t)\gamma^\mu A_\mu(\mathbf{x},t)\psi(\mathbf{x},t). \tag{6.16}$$

Experimental predictions are made using scattering theory, and this is considered in section 6.6.

The fact that QED contains photons and the theory presented in this thesis does not, is more than a surface detail. The commutation relation

$$A^\mu(\mathbf{x}',t')\psi^\dagger(\mathbf{x},t) = \psi^\dagger(\mathbf{x},t)A^\mu(\mathbf{x}',t') \tag{6.17}$$

shows that the operator $A^\mu(\mathbf{x},t)$ is unaffected by the existence of fermions. States should be written as

$$|\Psi>|A^\mu>, \tag{6.18}$$

which involve Dirac particles and photons. This is contrary to the theory of electromagnetism in chapter 4. Here $A^\mu(\mathbf{x},t)$ is expressible in terms of the $\psi(\mathbf{x})$ field (see Eq.(5.46)), and $[A^\mu(\mathbf{x}',t'), \psi(\mathbf{x})]$ is very complicated and not identically zero. The states are written as

$$|\Psi>, \tag{6.19}$$

and they contain only fermions.

6.3 The QED Representation

The expression for the field $\psi(\mathbf{x})$ Eq.(5.5) is very different from that which is usually given, Eq.(6.5). This difference is merely one of representation.

Previously the Schrödinger and Heisenberg pictures have been discussed and section 6.5 discusses the Interaction picture. The word picture is to differentiate each of these from the two representations which we introduce now. The concept behind 'pictures' and 'representations' are however exactly the same.

To see what constitutes a picture or representation we consider the Schrödinger and Heisenberg pictures. In the Schrödinger picture the states evolve through time, and the operators are constant. States evolve according to a unitary operator $U(t_a, t_b)$ such that

$$|\Psi, t_b> = U(t_a, t_b)|\Psi, t_a>. \tag{6.20}$$

Here $U(t_a, t_b)$ is a general evolution operator. No particular motion is being discussed. At a time t we can write a particular state and operator as

$$|\Psi, t>, \qquad A, \tag{6.21}$$

respectively. In the Heisenberg picture it is the states that are constant and the operators that evolve though time. In the Heisenberg picture the state and operator which correspond to those in Eq.(6.21) are

$$U^{\dagger}(0,t)|\Psi, t>, \qquad U^{\dagger}(0,t)AU(0,t). \tag{6.22}$$

The unitary operator $U^{\dagger}(0,t)$ provides a map between the two representations. Most importantly the matrix elements of an operator between states in the same representation is independent of representation,

$$< \Psi_1, t|U(0,t)U^{\dagger}(0,t)AU(0,t)U^{\dagger}(0,t)|\Psi_2, t> =< \Psi_1, t|A|\Psi_2, t>. \tag{6.23}$$

This section produces a time independent unitary operator $\mathcal{O}$, which is analagous to $U^{\dagger}(0,t)$, and which provides a change of representation to what we call the 'QED representation'. When we move to this representation, to the interaction picture and then finally redefine the vacuum state, the operator field $\psi(\mathbf{x})$ can be replaced by the standard expression, Eq.(6.5).

First however we consider states. The physical predictions of QED are made through the matrix elements of the scattering matrix. The matrix elements are the elements between states of incoming and outgoing particles which are typically in an eigen-state of momentum. Such states include $c_1^\dagger(\mathbf{p}_1)|0>$ and $c_1^\dagger(\mathbf{p}_1)c_2^\dagger(\mathbf{p}_2)|0>$. These states are very different to the freely evolving Dirac states which have been discussed earlier. We define

$$|\mathbf{p}[\alpha]> = c_\alpha^\dagger(\mathbf{p})|0>. \tag{6.24}$$

From the expansion of $c_1(\mathbf{p}_1)$, Eq.(5.3), we see

$$|\mathbf{p}[1]> = (2\pi\hbar)^{-\frac{3}{2}} \int d^3\mathbf{x} \exp\frac{i\mathbf{p}.\mathbf{x}}{\hbar} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}^T \begin{pmatrix} \psi_1^\dagger(\mathbf{x}) \\ \psi_2^\dagger(\mathbf{x}) \\ \psi_3^\dagger(\mathbf{x}) \\ \psi_4^\dagger(\mathbf{x}) \end{pmatrix} |0>, \tag{6.25}$$

and so the state $|\mathbf{p}[1]>$ can be represented by the spinor

$$\tilde{\Psi}(\mathbf{x}) = (2\pi\hbar)^{-\frac{3}{2}} \exp\frac{i\mathbf{p}.\mathbf{x}}{\hbar} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{6.26}$$

This is clearly not the same as the description of Dirac particles used earlier. The same particle was described by

$$|\mathbf{p}(1)> = \frac{m^{\frac{1}{2}} c_\alpha^\dagger(\mathbf{p}) u_{1(\alpha)}}{\sqrt[4]{\mathbf{p}^2+m^2}}|0>$$

$$= \frac{(2\pi\hbar)^{-\frac{3}{2}} m^{\frac{1}{2}}}{\sqrt[4]{\mathbf{p}^2+m^2}} \int d^3\mathbf{x} \exp\frac{i\mathbf{p}.\mathbf{x}}{\hbar} \begin{pmatrix} u_{1(1)}(\mathbf{p}) \\ u_{1(2)}(\mathbf{p}) \\ u_{1(3)}(\mathbf{p}) \\ u_{1(4)}(\mathbf{p}) \end{pmatrix}^T \begin{pmatrix} \psi_1^\dagger(\mathbf{x}) \\ \psi_2^\dagger(\mathbf{x}) \\ \psi_3^\dagger(\mathbf{x}) \\ \psi_4^\dagger(\mathbf{x}) \end{pmatrix} |0>, \tag{6.27}$$

which can be represented by the spinor

$$\Psi(\mathbf{x}) = \frac{(2\pi\hbar)^{-\frac{3}{2}}m^{\frac{1}{2}}}{\sqrt[4]{\mathbf{p}^2+m^2}}\exp\frac{i\mathbf{p}.\mathbf{x}}{\hbar}\begin{pmatrix} u_{1(1)}(\mathbf{p}) \\ u_{1(2)}(\mathbf{p}) \\ u_{1(3)}(\mathbf{p}) \\ u_{1(4)}(\mathbf{p}) \end{pmatrix} = \frac{(2\pi\hbar)^{-\frac{3}{2}}m^{\frac{1}{2}}}{\sqrt[4]{\mathbf{p}^2+m^2}}\exp\frac{i\mathbf{p}.\mathbf{x}}{\hbar}u_1(\mathbf{p}). \quad (6.28)$$

$\Psi(\mathbf{x})$ can be used to form solutions to the free Dirac equation, such as

$$\exp\frac{-it\sqrt{\mathbf{p}^2+m^2}}{\hbar}\Psi(\mathbf{x}). \quad (6.29)$$

We cannot form such solutions to the Dirac equation with $\tilde{\Psi}(\mathbf{x})$ given in Eq.(6.26) for every value of $\mathbf{p}$. This is because

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

has no momentum dependence and so is not a suitable candidate for one of the $u_1(\mathbf{p})$ spinors in any representation of the γ^μ matrices.

The states $|\mathbf{p}[\alpha]>$ and $|\mathbf{p}(\alpha)>$ both provide an orthonormal basis for all one particle states with

$$<\mathbf{p}_1[\alpha_1]|\mathbf{p}_2[\alpha_2]>=<\mathbf{p}_1(\alpha_1)|\mathbf{p}_2(\alpha_2)>=\delta^3(\mathbf{p}_1-\mathbf{p}_2)\delta_{\alpha_1\alpha_2}. \quad (6.30)$$

This means there is a unitary transformation $\mathcal{O}$ between one particle states such that

$$|\mathbf{p}[\alpha]> = \mathcal{O}|\mathbf{p}(\alpha)>. \quad (6.31)$$

$|\mathbf{p}[\alpha]>$ and $|\mathbf{p}(\alpha)>$ describe the same particle using a different representation. The operator $\mathcal{O}$ provides a map between the two representations. We call $|\mathbf{p}(\alpha)>$ a state in the 'wave-function representation' because this representation allows the easy production of wave equations. We call $|\mathbf{p}[\alpha]>$ a state in the 'QED representation'. In the QED representation scattering matrices are most easily calculated.

For general particle numbers the momentum states in the wave-function representation are

$$|\mathbf{p}_1(\alpha_1),\dots\mathbf{p}_n(\alpha_n)> = \frac{m^{\frac{1}{2}}c^\dagger_{\tau_1}(\kappa_{\alpha_1}\mathbf{p}_1)u_{\alpha_1(\tau_1)}(\mathbf{p}_1)}{\sqrt[4]{\mathbf{p}_1^2+m^2}}\dots\frac{m^{\frac{1}{2}}c^\dagger_{\tau_n}(\kappa_{\alpha_n}\mathbf{p}_n)u_{\alpha_n(\tau_n)}(\mathbf{p}_n)}{\sqrt[4]{\mathbf{p}_n^2+m^2}}|0>. \tag{6.32}$$

We use square brackets to denote states in the QED representation and define its momentum states as

$$|\mathbf{p}_1[\alpha_1],\dots,\mathbf{p}_n[\alpha_n]> = c^\dagger_{\alpha_1}(\mathbf{p}_1)\dots c^\dagger_{\alpha_n}(\mathbf{p}_n)|0>. \tag{6.33}$$

The momentum states in the wave-function representation together with the vacuum state, and the momentum states in the QED representation together with the vacuum state, each form a orthonormal basis for the space of all states. Hence we can extend $\mathcal{O}$ to a unitary transformation over the space of all states such that

$$|\mathbf{p}_1[\alpha_1],\dots,\mathbf{p}_n[\alpha_n]> = \mathcal{O}|\mathbf{p}_1(\alpha_1),\dots,\mathbf{p}_n(\alpha_n)>, \qquad |0> = \mathcal{O}|0>. \tag{6.34}$$

As well as using different states QED also uses different operators. Comparing Eqs.(5.5,5.10) with Eq.(6.5) we see that the operator $\psi(x)$ used in QED and the operator $\psi(\mathbf{x})$ used in chapter 4 are very different. In particular the QED version contains the $u_\alpha(\mathbf{p})$ spinors whereas the version in chapter 5 does not. For general particle numbers the position operators in the wave-function representation $\psi_\alpha(\mathbf{x})$ can be expanded in terms of the momentum annihilation operators to give

$$\psi_\alpha(\mathbf{x}) = (2\pi\hbar)^{-\frac{3}{2}}\int d^3\mathbf{p}\exp\frac{i\mathbf{p}.\mathbf{x}}{\hbar}c_\alpha(\mathbf{p}), \qquad \alpha=1,2,3,4. \tag{6.35}$$

We can also define another set of 4 operator fields,

$$\tilde{\psi}_\alpha(\mathbf{x}) = (2\pi\hbar)^{-\frac{3}{2}}\int\frac{m^{\frac{1}{2}}d^3\mathbf{p}}{\sqrt[4]{\mathbf{p}^2+m^2}}\exp\frac{i\kappa_\alpha\mathbf{p}.\mathbf{x}}{\hbar}u_\alpha(\mathbf{p})c_\alpha(\mathbf{p}), \qquad \alpha=1,2,3,4, \tag{6.36}$$

which contain the spinors $u_\alpha(\mathbf{p})$. These are annihilation operators in the QED representation. We also let

$$\tilde{\psi}(\mathbf{x}) = \tilde{\psi}_1(\mathbf{x})+\tilde{\psi}_2(\mathbf{x})+\tilde{\psi}_3(\mathbf{x})+\tilde{\psi}_4(\mathbf{x}). \tag{6.37}$$

While this definition looks very different to the definition of $\psi(\mathbf{x})$, Eq.(5.10), both $\tilde{\psi}(\mathbf{x})$ and $\psi(\mathbf{x})$ are spinor operators.

We now define the two operator fields, $\tilde{\rho}(\mathbf{x})$ and $\rho(\mathbf{x})$, by

$$\tilde{\rho}(\mathbf{x}) = \tilde{\psi}^{\dagger}(\mathbf{x})B\tilde{\psi}(\mathbf{x}), \qquad \rho(\mathbf{x}) = \psi^{\dagger}_{\alpha}(\mathbf{x})B_{(\alpha\beta)}\psi_{\beta}(\mathbf{x}) = \psi^{\dagger}(\mathbf{x})B\psi(\mathbf{x}), \tag{6.38}$$

where α and β sum from 1 to 4. B is a general 4×4 matrix which is dependent on $\mathbf{x}$ and its derivatives. These operators obey

$$< \mathbf{p}_1(\alpha_1), \ldots \mathbf{p}_n(\alpha_n) | \rho(\mathbf{x}) | \mathbf{q}_1(\beta_1), \ldots \mathbf{q}_{n'}(\beta_{n'}) >$$

$$=< \mathbf{p}_1[\alpha_1], \ldots \mathbf{p}_n[\alpha_n] | \tilde{\rho}(\mathbf{x}) | \mathbf{q}_1[\beta_1], \ldots \mathbf{q}_{n'}[\beta_{n'}] >, \tag{6.39}$$

and

$$< 0 | \rho(\mathbf{x}) | 0 > =< 0 | \tilde{\rho}(\mathbf{x}) | 0 > = 0. \tag{6.40}$$

Eq.(6.39) is proved in appendix B. Since the momentum states in the wave-function representation together with the vacuum state, and the momentum states in the QED representation together with the vacuum state, each form a basis for the space of all states we have

$$\tilde{\rho}(\mathbf{x}) = \mathcal{O}\rho(\mathbf{x})\mathcal{O}^{\dagger}. \tag{6.41}$$

We are now in a position to state the following conclusion. Consider the state and operator in the wave-function representation

$$| \mathbf{p}_1(\alpha_1), \ldots, \mathbf{p}_n(\alpha_n) >, \qquad \rho(\mathbf{x}). \tag{6.42}$$

The equivalent state and operator in the QED representation is

$$| \mathbf{p}_1[\alpha_1], \ldots, \mathbf{p}_n[\alpha_n] >, \qquad \tilde{\rho}(\mathbf{x}). \tag{6.43}$$

The equations of motion can now be re-expressed in the QED representation. The equation of motion for free Dirac particles is given by

$$i\hbar\frac{\partial}{\partial t}| \Psi, t> = H_0 | \Psi, t>, \tag{6.44}$$

where

$$H_0 = \int d^3\mathbf{x}\, \psi^\dagger(\mathbf{x})\gamma^0 \left(-i\hbar\gamma^j\partial_j + m\right)\psi(\mathbf{x}), \tag{6.45}$$

and j sums from 1 to 3. Now let

$$\tilde{H}_0 = \int d^3\mathbf{x}\, \tilde{\psi}^\dagger(\mathbf{x})\gamma^0 \left(-i\hbar\gamma^j\partial_j + m\right)\tilde{\psi}(\mathbf{x}). \tag{6.46}$$

An alternative equation of motion for Dirac particles can be written as

$$i\hbar\frac{\partial}{\partial t}|\Psi, t>' = \tilde{H}_0|\Psi, t>'. \tag{6.47}$$

If we let

$$\rho(\mathbf{x}) = \psi^\dagger(\mathbf{x})\gamma^0 \left(-i\hbar\gamma^j\partial_j + m\right)\psi(\mathbf{x}), \tag{6.48}$$

then Eq.(6.44) can be expressed as

$$i\hbar\frac{\partial}{\partial t}|\Psi, t> = \int d^3\mathbf{x}\, \rho(\mathbf{x})|\Psi, t>. \tag{6.49}$$

Eq.(6.47) is the equivalent equation constructed by replacing $\rho(\mathbf{x})$ with the respective $\tilde{\rho}(\mathbf{x})$. Let $|\Psi, t>$ be a solution to Eq.(6.44). Then $|\Psi, t>' = \mathcal{O}|\Psi, t>$ is a solution to Eq.(6.47). For instance we have seen that

$$|\Psi, t> = \exp\frac{-it\kappa_{\alpha_1}\sqrt{\mathbf{p}_1^2 + m^2} - \ldots - it\kappa_{\alpha_n}\sqrt{\mathbf{p}_n^2 + m^2}}{\hbar}|\mathbf{p}_1(\alpha_1), \ldots, \mathbf{p}_n(\alpha_n)>, \tag{6.50}$$

is a solution to Eq.(6.44). This means that the state

$$|\Psi, t>' = \exp\frac{-it\kappa_{\alpha_1}\sqrt{\mathbf{p}_1^2 + m^2} - \ldots - it\kappa_{\alpha_n}\sqrt{\mathbf{p}_n^2 + m^2}}{\hbar}|\mathbf{p}_1[\alpha_1], \ldots, \mathbf{p}_n[\alpha_n]>, \tag{6.51}$$

is a solution to Eq.(6.47). These states both represent, in different representations, the same free particles evolving through time.

Now we consider electromagnetically interacting particles. The equation of motion is

$$i\hbar\frac{\partial}{\partial t}|\Psi, t> = [H_0 + H_I(t)]|\Psi, t>, \tag{6.52}$$

where the interaction Hamiltonian is

$$H_I(t) = \frac{e}{2}\int d^3\mathbf{x}\, \psi^\dagger(\mathbf{x})\gamma^0\gamma_\mu\psi(\mathbf{x})A^\mu(\mathbf{x}, t)$$

$$= \frac{e^2}{2}\int d^3\mathbf{x}\int d^4x'\, G_F(\mathbf{x}'-\mathbf{x},t'-t)$$

$$T^{t,t'}\left[\psi^\dagger(\mathbf{x})\gamma^0\gamma^\mu\psi(\mathbf{x}), U(t',t)\psi^\dagger(\mathbf{x}')\gamma^0\gamma_\mu\psi(\mathbf{x}')U^\dagger(t',t)\right]. \qquad (6.53)$$

If we put

$$\rho^\mu(\mathbf{x}) = \psi^\dagger(\mathbf{x})\gamma^0\gamma^\mu\psi(\mathbf{x}), \qquad (6.54)$$

then the interaction Hamiltonian can be re-expressed as

$$H_I(t) = \frac{e^2}{2}\int d^3\mathbf{x}\int d^4x' G_F(\mathbf{x}'-\mathbf{x},t'-t)T^{t,t'}\left[\rho^\mu(\mathbf{x}), U(t',t)\rho_\mu(\mathbf{x}')U^\dagger(t',t)\right]. \quad (6.55)$$

We wish to write this equation in the QED representation. In this representation states evolve according to

$$|\Psi, t_b> = \tilde{U}(t_a,t_b)|\Psi, t_a>, \qquad (6.56)$$

where

$$\tilde{U}(t_a,t_b) = \mathcal{O}U(t_a,t_b)\mathcal{O}^\dagger. \qquad (6.57)$$

Consider the equation

$$i\hbar\frac{\partial}{\partial t}|\Psi, t>' = \left[\tilde{H}_0 + \tilde{H}_I\right]|\Psi, t>', \qquad (6.58)$$

where

$$\tilde{H}_I(t) = \frac{e^2}{2}\int d^3\mathbf{x}\int d^4x'\, G_F(\mathbf{x}'-\mathbf{x},t'-t)$$

$$T^{t,t'}\left[\tilde{\psi}^\dagger(\mathbf{x})\gamma^0\gamma^\mu\tilde{\psi}(\mathbf{x}), \tilde{U}^\dagger(t',t)\tilde{\psi}^\dagger(\mathbf{x}')\gamma^0\gamma_\mu\tilde{\psi}(\mathbf{x}')\tilde{U}^\dagger(t',t)\right]. \quad (6.59)$$

If we write

$$\tilde{\rho}(\mathbf{x}) = \tilde{\psi}^\dagger(\mathbf{x})\gamma^0\gamma^\mu\tilde{\psi}(\mathbf{x}), \qquad (6.60)$$

then

$$\tilde{H}_I(t) = \frac{e^2}{2}\int d^3\mathbf{x}\int d^4x' G_F(\mathbf{x}'-\mathbf{x},t'-t)T^{t,t'}\left[\tilde{\rho}^\mu(\mathbf{x}), \tilde{U}(t',t)\tilde{\rho}_\mu(\mathbf{x}')\tilde{U}^\dagger(t',t)\right]. \quad (6.61)$$

Eq.(6.61) is equivalent to Eq.(6.55), but is constructed by replacing each operator in the wave function representation with the equivalent operator in the QED representation. If $|\Psi, t>$ is a solution to Eq.(6.52), then $|\Psi, t>' = \mathcal{O}|\Psi, t>$ is a solution to Eq.(6.58).

6.4 Anti-particles

We see from Eq.(5.28) that the operator used to create a positive or negative energy particle of momentum $\mathbf{p}$ in the wave-function representation is

$$\frac{m^{\frac{1}{2}} c_\tau^\dagger(\kappa_\tau \mathbf{p}) u_{\alpha(\tau)}(\mathbf{p})}{\sqrt[4]{\mathbf{p}^2+m^2}}, \tag{6.62}$$

where τ is summed from 1 to 4. Here α is one of 1, 2, 3 or 4, and corresponds to the type of particle being described. In the QED representation we see from Eq.(6.33) that the operator used to create the equivalent particle is simply

$$c_\alpha^\dagger(\mathbf{p}). \tag{6.63}$$

The rest of this chapter uses the QED representation. The particles observed experimentally are anti-particles, rather than negative energy particles. An anti-particle is simply the lack of a negative energy particle. The operator which creates an anti-particle is the operator which annihilates a negative energy particle. To reverse this we redefine the creation and annihilation operators $c_3^\dagger(\mathbf{p})$, $c_3(\mathbf{p})$ and $c_4^\dagger(\mathbf{p})$, $c_4(\mathbf{p})$, swapping creation with annihilation. Similarly we redefine $\tilde{\psi}_3(\mathbf{x})$ and $\tilde{\psi}_4(\mathbf{x})$ so that now

$$\tilde{\psi}_\alpha^\dagger(\mathbf{x}) = (2\pi\hbar)^{-\frac{3}{2}} \int \frac{d^3\mathbf{p}\, m^{\frac{1}{2}}}{\sqrt[4]{\mathbf{p}^2+m^2}} \exp\frac{-i\mathbf{p}.\mathbf{x}}{\hbar} u_\alpha(\mathbf{p}) c_\alpha^\dagger(\mathbf{p}), \quad \alpha = 3, 4, \tag{6.64}$$

and use these redefinitions in all the operators that they form. In particular we now have

$$\tilde{\psi}(\mathbf{x}) = \tilde{\psi}_1(\mathbf{x}) + \tilde{\psi}_2(\mathbf{x}) + \tilde{\psi}_3^\dagger(\mathbf{x}) + \tilde{\psi}_4^\dagger(\mathbf{x}). \tag{6.65}$$

We then define a new vacuum state which contains no anti-particles,

$$|0_{new}\rangle = \prod_{\mathbf{p}\in\mathbf{R}^3} c_3(\mathbf{p}) c_4(\mathbf{p}) |0\rangle, \tag{6.66}$$

where the $c_3(\mathbf{p})$ and $c_4(\mathbf{p})$ are now anti-particle annihilation operators. $|0_{new}\rangle$ is the observed vacuum state. Anti-particle states are thus formed by acting the old negative energy annihilation operators that is the new anti-particle creation operators on the new vacuum.

A physical system which contains a finite number of particles and anti-particles will contain an infinite number of negative energy particles. This infinity causes the equation describing electromagnetically interacting Dirac particles to develop many infinite quantities.

6.5 The Interaction Picture and the Standard Expression for $\psi(x)$

This section shows that in the interaction picture and also in the QED representation the Dirac operator field is given by the standard expression Eq.(6.5).

To produce the interaction picture we write the total Hamiltonian as

$$\tilde{H}_T = \tilde{H}_0 + \tilde{H}_I, \tag{6.67}$$

where $\tilde{H}_0$ is the free particle Hamiltonian, and $\tilde{H}_I$ is the interaction Hamiltonian. To keep this section general we make no assumption about the nature of $\tilde{H}_I$. We consider only operators in the QED representation. Let $\tilde{V}(t)$ be the evolution operator for free particles, that is

$$\tilde{V}(t) = \exp(-it\tilde{H}_0). \tag{6.68}$$

Here we are using units in which $\hbar = 1$, and this is continued until the end of the chapter. In the interaction picture operators evolve according to

$$\tilde{A}_{int}(t) = \tilde{V}^\dagger(t)\tilde{A}\tilde{V}(t), \tag{6.69}$$

where $\tilde{A}$ is an operator in the Schrödinger picture and $\tilde{A}_{int}(t)$ is the equivalent operator in the interaction picture. States evolve according to an evolution operator $\tilde{W}(t_a, t_b)$

$$|\Psi_{int}, t_b> = \tilde{W}(t_a, t_b)|\Psi_{int}, t_a>. \tag{6.70}$$

We must ensure that the matrix elements between operators and states is the same in the interaction picture as in, say, the Schrödinger picture, and so

$$< \Psi_{1,int}, t_b|\tilde{A}_{int}(t_b)|\Psi_{2,int}, t_b> =< \Psi_1, t_b|\tilde{A}|\Psi_2, t_b>, \tag{6.71}$$

for every operator A and every state $|\Psi_1, t_b>$ and $|\Psi_2, t_b>$. This gives

$$< \Psi, t_a | \tilde{W}^\dagger(t_a, t_b) \tilde{V}^\dagger(t_b - t_a) \tilde{A} \tilde{V}(t_b - t_a) \tilde{W}(t_a, t_b) | \Psi, t_a>$$

$$=< \Psi, t_a | \tilde{U}^\dagger(t_a, t_b) \tilde{A} \tilde{U}(t_a, t_b) | \Psi, t_a> \qquad (6.72)$$

for every A, $|\Psi_1, t>$, and $|\Psi_2, t>$ and so

$$\tilde{V}(t_b - t_a) \tilde{W}(t_a, t_b) = \tilde{U}(t_a, t_b). \qquad (6.73)$$

$\tilde{W}(t_a, t_b)$ itself evolves according to

$$i \frac{\partial}{\partial t_b} \tilde{W}(t_a, t_b) = \tilde{H}_{I,int}(t_b) \tilde{W}(t_a, t_b). \qquad (6.74)$$

This equation can be solved to give

$$\tilde{W}(t_a, t_b) = \sum_{n=0}^{\infty} (-i)^n \int_{t_a}^{t_b} dt_1 \tilde{H}_{I,int}(t_1) \int_{t_a}^{t_1} dt_2 \tilde{H}_{I,int}(t_2) \ldots \int_{t_a}^{t_{n-1}} \tilde{H}_{I,int}(t_n) dt_n. \qquad (6.75)$$

A general free particle state in which each particle is in an eigen-state of momentum evolves according to

$$\tilde{V}(t) c^\dagger_{\alpha_1}(\mathbf{p}_1) \ldots c^\dagger_{\alpha_n}(\mathbf{p}_n) | 0_{new}> \qquad (6.76)$$

in the Schrödinger picture and QED representation. A particle or anti-particle of momentum $\mathbf{p}$ has an energy of $\sqrt{\mathbf{p}^2 + m^2}$. Removing a particle or anti-particle of momentum $\mathbf{p}$ removes an energy of $\sqrt{\mathbf{p}^2 + m^2}$ and we see that

$$\tilde{V}(t') c_\alpha(\mathbf{p}) \tilde{V}^\dagger(t') = \exp\left(i t' \sqrt{\mathbf{p}^2 + m^2}\, c_\alpha(\mathbf{p}) \right). \qquad (6.77)$$

Since $\tilde{V}^\dagger(t') = \tilde{V}(-t')$ substituting $t = -t'$ gives

$$\tilde{V}^\dagger(t) c_\alpha(\mathbf{p}) \tilde{V}(t) = \exp\left(-i t \sqrt{\mathbf{p}^2 + m^2}\, c_\alpha(\mathbf{p}) \right). \qquad (6.78)$$

This is an expression of $c_\alpha(\mathbf{p})$ in the interaction picture and QED representation. The expression for $c_\alpha(\mathbf{p})$ in the interaction picture and *wave-function* representation is much more complicated (note that $\tilde{V}(t) \neq V(t)$). It is this simple result that is the reason why scattering theory is most easily performed in the QED representation.

In the QED representation we have

$$\tilde{\psi}_\alpha(\mathbf{x}) = (2\pi)^{-\frac{3}{2}} \int \frac{m^{\frac{1}{2}}\, d^3\mathbf{p}}{\sqrt[4]{\mathbf{p}^2+m^2}}\, e^{i\mathbf{p}.\mathbf{x}} u_\alpha(\mathbf{p}) c_\alpha(\mathbf{p}), \quad \alpha = 1,2, \tag{6.79}$$

and

$$\tilde{\psi}^\dagger_\alpha(\mathbf{x}) = (2\pi)^{-\frac{3}{2}} \int \frac{m^{\frac{1}{2}}\, d^3\mathbf{p}}{\sqrt[4]{\mathbf{p}^2+m^2}}\, e^{-i\mathbf{p}.\mathbf{x}} u_\alpha(\mathbf{p}) c^\dagger_\alpha(\mathbf{p}), \quad \alpha = 3,4. \tag{6.80}$$

Hence in the interaction picture we can write

$$\tilde{\psi}_{\alpha,int}(\mathbf{x},t) = (2\pi)^{-\frac{3}{2}} \int \frac{m^{\frac{1}{2}}\, d^3\mathbf{p}}{\sqrt[4]{\mathbf{p}^2+m^2}} \exp\left(i\mathbf{p}.\mathbf{x} - it\sqrt{\mathbf{p}^2+m^2}\right) u_\alpha(\mathbf{p}) c_\alpha(\mathbf{p}), \quad \alpha = 1,2, \tag{6.81}$$

and

$$\tilde{\psi}^\dagger_{\alpha,int}(\mathbf{x},t) = (2\pi)^{-\frac{3}{2}} \int \frac{m^{\frac{1}{2}}\, d^3\mathbf{p}}{\sqrt[4]{\mathbf{p}^2+m^2}} \exp\left(-i\mathbf{p}.\mathbf{x} + it\sqrt{\mathbf{p}^2+m^2}\right) u_\alpha(\mathbf{p}) c^\dagger_\alpha(\mathbf{p}), \quad \alpha = 3,4. \tag{6.82}$$

To simplify notation we write $\psi(x) = \tilde{\psi}_{int}(\mathbf{x},t)$ where $x = (\mathbf{x},t)$ and so

$$\psi(x) = \tilde{\psi}_{1,int}(\mathbf{x},t) + \tilde{\psi}_{2,int}(\mathbf{x},t) + \tilde{\psi}^\dagger_{3,int}(\mathbf{x},t) + \tilde{\psi}^\dagger_{4,int}(\mathbf{x},t). \tag{6.83}$$

On substituting Eqs.(6.81,6.82) we obtain the standard expansion for the Dirac field $\psi(x)$, Eq.(6.5).

6.6 Comparison of Scattering Matrices

This section compares the matrix elements between fermion states of the scattering matrices of the theory where the electromagnetic field was calculated directly, and of and standard QED. It is shown that they agree for the term of order e^2, but disagree at the order e^4.

We first consider standard QED. Here the interaction Hamiltonian is given by

$$H_I(t) = e \int d^3\mathbf{x}\, \psi^\dagger(\mathbf{x},t)\gamma^0\gamma^\mu A_\mu(\mathbf{x},t)\psi(\mathbf{x},t) \tag{6.84}$$

where $\psi(x)$ is the fermion field given by Eq.(6.5) and $A_\mu(x)$ is the electromagnetic field described in section 6.2. The predictions of QED are made through its scattering matrix S. We wish to compare QED which contains photon states to the theory which contains no photon states, and so only consider the the matrix elements of S between fermion states. The only terms that contribute are those where all the fermions are in propagators, and in particular the terms with an odd number of H_I do not contribute. Considering only matrix elements between fermion states

$$
\begin{aligned}
S &= 1 - \int_{-\infty}^{\infty} dt_1 H_I(t_1) \int_{-\infty}^{t_1} dt_2 H_I(t_2) \\
&\quad + \int_{-\infty}^{\infty} dt_1 H_I(t_1) \int_{-\infty}^{t_1} dt_2 H_I(t_2) \int_{-\infty}^{t_2} dt_3 H_I(t_3) \int_{-\infty}^{t_3} dt_4 H_I(t_4) + \ldots \\
&= 1 - e^2 \int_{t_1=-\infty}^{t_1=\infty} d^4x_1 \int_{t_2=-\infty}^{t_2=t_1} d^4x_2 D_F^{\mu\nu}(x_1 - x_2)\psi^\dagger(x_1)\gamma^0\gamma_\mu\psi(x_1)\psi^\dagger(x_2)\gamma^0\gamma_\nu\psi(x_2) \\
&+ e^4 \int_{t_1=-\infty}^{t_1=\infty} d^4x_1 \int_{t_2=-\infty}^{t_2=t_1} d^4x_2 \int_{t_3=-\infty}^{t_3=t_2} d^4x_3 \int_{t_4=-\infty}^{t_4=t_3} d^4x_4 [D_F^{\mu\nu}(x_1 - x_2)D_F^{\tau\sigma}(x_3 - x_4) \\
&\qquad + D_F^{\mu\tau}(x_1 - x_3)D_F^{\nu\sigma}(x_2 - x_4) + D_F^{\mu\sigma}(x_1 - x_4)D_F^{\nu\tau}(x_2 - x_3)] \\
&\psi^\dagger(x_1)\gamma^0\gamma_\mu\psi(x_1)\psi^\dagger(x_2)\gamma^0\gamma_\nu\psi(x_2)\psi^\dagger(x_3)\gamma^0\gamma_\tau\psi(x_3)\psi^\dagger(x_4)\gamma^0\gamma_\sigma\psi(x_4) + \ldots \qquad (6.85)
\end{aligned}
$$

Since in the Lorentz gauge

$$
iD_F^{\mu\nu}(x_1 - x_2) = G_F(x_1 - x_2)r^{\mu\nu}, \qquad (6.86)
$$

we have

$$
\begin{aligned}
S &= 1 - ie^2 \int_{t_1=-\infty}^{t_1=\infty} d^4x_1 \int_{t_2=-\infty}^{t_2=t_1} d^4x_2 G_F(x_1 - x_2)\psi^\dagger(x_1)\gamma^0\gamma_\mu\psi(x_1)\psi^\dagger(x_2)\gamma^0\gamma^\mu\psi(x_2) \\
&- e^4 \int_{t_1=-\infty}^{t_1=\infty} d^4x_1 \int_{t_2=-\infty}^{t_2=t_1} d^4x_2 \int_{t_3=-\infty}^{t_3=t_2} d^4x_3 \int_{t_4=-\infty}^{t_4=t_3} d^4x_4 \,[G_F(x_1 - x_2)G_F(x_3 - x_4)r^{\mu\nu}r^{\tau\sigma} \\
&\qquad + G_F(x_1 - x_3)G_F(x_2 - x_4)r^{\mu\tau}r^{\nu\sigma} + G_F(x_1 - x_4)G_F(x_2 - x_3)r^{\mu\sigma}r^{\nu\tau}] \\
&\psi^\dagger(x_1)\gamma^0\gamma_\mu\psi(x_1)\psi^\dagger(x_2)\gamma^0\gamma_\nu\psi(x_2)\psi^\dagger(x_3)\gamma^0\gamma_\tau\psi(x_3)\psi^\dagger(x_4)\gamma^0\gamma_\sigma\psi(x_4) + \ldots \qquad (6.87)
\end{aligned}
$$

We now consider the theory where the electromagnetic field is calculated directly. This is described by the equation Eq.(6.58). In the QED representation and Schrödinger picture the interaction Hamiltonian is

$$
\begin{aligned}
\tilde{H}_I(t) &= \frac{e^2}{2} \int d^3\mathbf{x} \int d^4x' \, G_F(\mathbf{x}' - \mathbf{x}, t' - t) \\
&\qquad T^{t,t'} \left[\tilde{\psi}^\dagger(\mathbf{x})\gamma^0\gamma^\mu\tilde{\psi}(\mathbf{x}), \tilde{U}(t',t)\tilde{\psi}^\dagger(\mathbf{x}')\gamma^0\gamma_\mu\tilde{\psi}(\mathbf{x}')\tilde{U}^\dagger(t',t)\right]. \qquad (6.88)
\end{aligned}
$$

From Eq.(6.73)

$$\tilde{U}(t',t) = \tilde{V}(t-t')\tilde{W}(t',t), \tag{6.89}$$

and so

$$\tilde{V}^\dagger(t)\tilde{U}(t',t)\tilde{V}(t') = \tilde{V}(-t)\tilde{V}(t-t')\tilde{W}(t',t)\tilde{V}(t') = \tilde{V}^\dagger(t')\tilde{W}(t',t)\tilde{V}(t'). \tag{6.90}$$

Hence in the interaction picture

$$H_{I,int}(t) = \frac{e^2}{2}\int d^3\mathbf{x}\int d^4x'\, G_F(\mathbf{x}'-\mathbf{x},t'-t)$$

$$T^{t',t}\left[\psi^\dagger(\mathbf{x},t)\gamma^0\gamma^\mu\psi(\mathbf{x},t), \tilde{V}^\dagger(t')\tilde{W}(t',t)\tilde{V}(t')\psi^\dagger(\mathbf{x}',t')\gamma^0\gamma_\mu\psi(\mathbf{x}',t')\tilde{V}^\dagger(t')\tilde{W}^\dagger(t',t)\tilde{V}(t')\right]. \tag{6.91}$$

The first three terms in the expansion of $\tilde{W}(t_a,t_b)$ are

$$\tilde{W}(t_a,t_b) = 1 - i\int_{t_a}^{t_b}\tilde{H}_{I,int}(t_1)dt_1 - \int_{t_a}^{t_b}\tilde{H}_{I,int}(t_1)dt_1\int_{t_a}^{t_1}dt_2\tilde{H}_{I,int}(t_2) + \ldots \tag{6.92}$$

This gives

$$\tilde{W}(t_a,t_b) = 1 - \frac{ie^2}{2}\int_{t_1=t_a}^{t_1=t_b}d^4x_1\int_{t_2=-\infty}^{t_2=\infty}d^4x_2\, G_F(x_2-x_1)$$

$$T^{t_1,t_2}\left[\psi^\dagger(x_1)\gamma^0\gamma^\mu\psi(x_1), \tilde{V}^\dagger(t_2)\tilde{W}(t_2,t_1)\tilde{V}(t_2)\psi^\dagger(x_2)\gamma^0\gamma_\mu\psi(x_2)\tilde{V}^\dagger(t_2)\tilde{W}^\dagger(t_2,t_1)\tilde{V}(t_2)\right]$$

$$-\frac{e^4}{4}\int_{t_1=t_a}^{t_1=t_b}d^4x_1\int_{t_2=-\infty}^{t_2=\infty}d^4x_2\int_{t_3=t_a}^{t_3=t_1}d^4x_3\int_{t_4=-\infty}^{t_4=\infty}d^4x_4\, G_F(x_2-x_1)G_F(x_4-x_3)$$

$$T^{t_1,t_2}\left[\psi^\dagger(x_1)\gamma^0\gamma^\mu\psi(x_1), \tilde{V}^\dagger(t_1)\tilde{W}(t_1,t_2)\tilde{V}(t_1)\psi^\dagger(x_2)\gamma^0\gamma_\mu\psi(x_2)\tilde{V}^\dagger(t_1)\tilde{W}^\dagger(t_1,t_2)\tilde{V}(t_1)\right]$$

$$T^{t_3,t_4}\left[\psi^\dagger(x_3)\gamma^0\gamma^\nu\psi(x_3), \tilde{V}^\dagger(t_3)\tilde{W}(t_3,t_4)\tilde{V}(t_3)\psi^\dagger(x_4)\gamma^0\gamma_\nu\psi(x_4)\tilde{V}^\dagger(t_3)\tilde{W}^\dagger(t_3,t_4)\tilde{V}(t_3)\right]$$

$$+\ldots \tag{6.93}$$

We now write

$$\tilde{W}(t_a,t_b) = 1 + \tilde{W}_1(t_a,t_b) + \tilde{W}_2(t_a,t_b) + \ldots \tag{6.94}$$

where $\tilde{W}_j(t_a,t_b)$ is of order $4j$ in $\psi(x)$. Since $\tilde{W}(t_a,t_b)$ is unitary

$$\tilde{W}^\dagger(t_a,t_b) = \tilde{W}^{-1}(t_a,t_b) = 1 - \tilde{W}_1(t_a,t_b) - \ldots \tag{6.95}$$

If we substitute Eqs.(6.94, 6.95) into Eq.(6.93) we can solve Eqs.(6.93) by perturbation theory. Taking terms of order 4 in $\psi(x)$,

$$\begin{aligned}
&\tilde{W}_1(t_a,t_b)\\
&= -\frac{ie^2}{2}\int_{t_1=t_a}^{t_1=t_b} d^4x_1 \int_{t_2=-\infty}^{t_2=\infty} d^4x_2\, G_F(x_2-x_1) T^{t_1,t_4}\left[\psi^\dagger(x_1)\gamma^0\gamma^\mu\psi(x_1), \psi^\dagger(x_2)\gamma^0\gamma_\mu\psi(x_2)\right]\\
&= -ie^2\int_{t_1=t_a}^{t_1=t_b} d^4x_1 \int_{t_2=-\infty}^{t_2=t_1} d^4x_2\, G_F(x_2-x_1)\psi^\dagger(x_1)\gamma^0\gamma^\mu\psi(x_1)\psi^\dagger(x_2)\gamma^0\gamma_\mu\psi(x_2).
\end{aligned} \tag{6.96}$$

The scattering matrix is $W(t_a,t_b)$, where we take the limits $t_a \to -\infty$ and $t_b \to +\infty$. We can see straight away that up to order 2 in e,

$$\tilde{S} = 1 - ie^2\int_{t_1=-\infty}^{t_1=\infty} d^4x_1 \int_{t_2=-\infty}^{t_2=t_1} d^4x_2 G_F(x_1-x_2)\psi^\dagger(x_1)\gamma^0\gamma^\mu\psi(x_1)\psi^\dagger(x_2)\gamma^0\gamma_\mu\psi(x_2) \tag{6.97}$$

It is clear this is identical to the terms of order e^2 in Eq.(6.87), and the scattering matrices agree at least up to the order e^2.

The terms of order 8 in $\psi(x)$ are

$$\begin{aligned}
&\tilde{W}_2(t_a,t_b)\\
&= -\frac{ie^2}{2}\int_{t_1=t_a}^{t_1=t_b} d^4x_1 \int_{t_4=-\infty}^{t_4=\infty} d^4x_4 T^{t_1,t_2}\left[\psi^\dagger(x_1)\gamma^0\gamma^\mu\psi(x_1), \tilde{V}^\dagger(t_4)\tilde{W}_1(t_4,t_1)\tilde{V}(t_4)\right]\\
&\qquad \psi^\dagger(x_4)\gamma^0\gamma_\mu\psi(x_4)G_F(x_4-x_1)\\
&+i\frac{e^2}{2}\int_{t_1=t_a}^{t_1=t_b} d^4x_1 \int_{t_2=-\infty}^{t_2=\infty} d^4x_2 T^{t_1,t_2}\left[\psi^\dagger(x_1)\gamma^0\gamma^\mu\psi(x_1), \psi^\dagger(x_2)\gamma^0\gamma_\mu\psi(x_2)\right.\\
&\qquad \left.\tilde{V}^\dagger(t_2)\tilde{W}_1(t_2,t_1)\tilde{V}(t_2)\right] G_F(x_2-x_1)\\
&-\frac{e^4}{4}\int_{t_1=t_a}^{t_1=t_b} d^4x_1 \int_{t_2=-\infty}^{t_2=\infty} d^4x_2 \int_{t_3=t_a}^{t_3=t_1} d^4x_3 \int_{t_4=-\infty}^{t_4=\infty} d^4x_4 G_F(x_2-x_1)G_F(x_4-x_3)\\
&T^{t_1,t_2}\left[\psi^\dagger(x_1)\gamma^0\gamma^\mu\psi(x_1)\psi^\dagger(x_2)\gamma^0\gamma_\mu\psi(x_2)\right] T^{t_3,t_4}\left[\psi^\dagger(x_3)\gamma^0\gamma^\nu\psi(x_3), \psi^\dagger(x_4)\gamma^0\gamma_\nu\psi(x_4)\right].
\end{aligned} \tag{6.98}$$

We note that

$$\begin{aligned}
\tilde{V}^\dagger(t_a)\tilde{W}_1(t_a,t_b)\tilde{V}(t_a) &= -ie^2\int_{t_1=t_a}^{t_1=t_b} d^4x_1 \int_{t_2=-\infty}^{t_2=t_1} d^4x_2 G_F(x_2-x_1)\\
&\psi^\dagger(\mathbf{x}_1,t_1+t_a)\gamma^0\gamma^\mu\psi(\mathbf{x}_1,t_1+t_a)\psi^\dagger(\mathbf{x}_2,t_2+t_a)\gamma^0\gamma^\mu\psi(\mathbf{x}_2,t_2+t_a)
\end{aligned}$$

$$= -ie^2 \int_{t_1=2t_a}^{t_1=t_b+t_a} d^4x_1 \int_{t_2=-\infty}^{t_2=t_1+t_a} d^4x_2 G_F(x_2 - x_1)\psi^\dagger(x_1)\gamma^0\gamma^\mu\psi(x_1)\psi^\dagger(x_2)\gamma^0\gamma^\mu\psi(x_2), \tag{6.99}$$

This means that it is possible to express $W_2(t_a, t_b)$ in the form

$$\tilde{W}_2(t_a, t_b) = e^4 \int d^4x_1 \int d^4x_2 \int d^4x_3 \int d^4x_4 A^{\mu\nu\tau\sigma}(x_1, x_2, x_3, x_4)$$

$$\psi^\dagger(x_1)\gamma^0\gamma_\mu\psi(x_1)\psi^\dagger(x_2)\gamma^0\gamma_\nu\psi(x_2)\psi^\dagger(x_3)\gamma^0\gamma_\sigma\psi(x_3)\psi^\dagger(x_4)\gamma^0\gamma_\tau\psi(x_4), \tag{6.100}$$

where $A^{\mu\nu\sigma\tau}$ is a real, that is not an operator. The expression for $A^{\mu\nu\sigma\tau}$ is finite but very complicated, and must take account of the limits on the integrals and time ordering, as well as the Green's functions. It is clear from an examination of Eqs.(6.98,6.99) that it is not dependent on $G(x_1 - x_3)G(x_2 - x_4)$. Hence the term in the scattering matrix of order 4 in e is not dependent on $G(x_1 - x_3)G(x_2 - x_4)$. Now look at Eq.(6.87), the scattering matrix for QED. The term of order 4 is written in a similar manner and explicitly contains $G(x_1 - x_3)G(x_2 - x_4)$. The two terms are clearly not equal.

It is concluded that the two theories have different scattering matrices, and so are not equivalent. There are other possible ways to write the interaction Hamiltonian, but it appears that none give equivalent scattering matrices to QED.

Chapter 7

The Metric Operator

7.1 Do Gravitons Exist?

We have seen that there are two possible ways to quantise electromagnetism, and through their experimental predictions one can be eliminated. It might be expected that the successful method can be generalised to gravity, which like electromagnetism is a gauge theory. This however cannot be assumed. No evidence exists for any quantum effects of gravity.

Standard QED contains Dirac particles and photons. The motion of a state $|\Psi>$ of free fermions is governed by the Lagrangian density

$$\mathcal{L}_0^{\psi}(x) = \psi^{\dagger}(x)\gamma^0\left(i\hbar\gamma^{\mu}\partial_{\mu} - m\right)\psi(x). \tag{7.1}$$

This provides mass-energy and kinetic energy for the fermions. Similarly the motion of a state $|A^{\mu}>$ of free photons is governed by the Lagrangian of free photons, which can be written as

$$\mathcal{L}_0^{\gamma} = \frac{1}{2}\left(\partial_{\mu}A_{\nu}\right)\left(\partial^{\mu}A^{\nu}\right). \tag{7.2}$$

This contains no mass terms and photons are massless.

If a similar view is taken of gravity, then a theory of quantum gravity will contain Dirac particles and gravitons. States can be written as

$$|\Psi>|G_{\psi}>. \tag{7.3}$$

Classically general relativity has the Lagrangian density

$$\mathcal{L}_0^G = R\sqrt{-g}, \tag{7.4}$$

away from any boundary. R is the scalar curvature and g is the determinant of the metric. If this Lagrangian density is used to form a graviton propagator then we are left with an unrenormalisable theory. There is a big conceptual problem. How do you describe the motion of particles which themselves form the shape of the universe? The commutation relations that should be imposed on $g^{\mu\nu}(x)$ are not obvious either since the positions of the light cones are themselves dependent on the metric.

The method employed in Chapter 5 to produce a theory of electrodynamics however appears to provide a more promising basis in which to describe gravitational interactions. There are no gravitons, and so we do not need to worry about a graviton propagator. Neither are commutation relations imposed. Instead we impose an operator version of Einstein's equation on the metric operator. This chapter considers this approach to quantum gravity.

7.2 Linear Gravity and the Choice of Gauge

Before we examine gravity we look at another gauge theory, electromagnetism. When performing a calculation in electromagnetism, I always choose a particular gauge in which to work. To work in a general gauge is pointless, and it is always possible to perform a gauge transformation later and move to any other gauge. Maxwell's equations can be summarised by the single equation

$$\partial_\mu F^{\mu\nu}(x) = j^\nu(x), \tag{7.5}$$

where $x \in \mathbf{R}^4$ and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{7.6}$$

It is not obvious how to solve Eq.(7.5) in general. If we choose to work in the Lorentz gauge then the equation can be replaced by

$$\Box^2 A^\mu(x) = j^\mu(x), \tag{7.7}$$

and this can be solved using Green's functions. One solution is

$$A^{\mu}(x) = \int d^4x' G_{ret}(x' - x) j^{\mu}(x'). \tag{7.8}$$

Given sufficient boundary conditions Eq.(7.7) has a unique solution whereas Eq.(7.5) has many possible solutions for A^{μ}, one of which is provided by Eq.(7.8). The physical aspect of the electromagnetic field is the same for each of the solutions of Eq.(7.5), and nothing of the theory is lost by selecting only one such solution. Moreover the momentum operator of a quantum state is dependent on A^{μ}, so we cannot attempt an interpretation of a quantum state without knowing the particular 4-potential that is being used.

When performing calculations in quantum gravity here a similar attitude is taken, and we work in a particular gauge. In order to demonstrate this we consider a system of a small gravitating particle of mass m and the Earth of mass M. We work in linear gravity and write the metric operator in the Schrödinger picture as

$$g_{\mu\nu}(\mathbf{x}) = r_{\mu\nu} + \varepsilon_{\mu\nu}(\mathbf{x}), \tag{7.9}$$

where now $\varepsilon_{\mu\nu}(\mathbf{x})$ is an operator. This is time-independent because it is in the Schrödinger picture. Classically there is a gauge in which Einstein's equation can be written as

$$\Box^2 \overline{\varepsilon}_{\mu\nu}(\mathbf{x}, t) = -16\pi G T_{\mu\nu}(\mathbf{x}, t), \tag{7.10}$$

where

$$\overline{\varepsilon}^{\mu\nu}(\mathbf{x}, t) = \varepsilon^{\mu\nu}(\mathbf{x}, t) - \frac{1}{2} r^{\mu\nu} \varepsilon^{\alpha}_{\alpha}(\mathbf{x}, t). \tag{7.11}$$

We use the slow speed approximation, where classically $\frac{\partial^2}{\partial t^2}\varepsilon_{\mu\nu}(\mathbf{x}, t)$ is negligible, and postulate the operator equation

$$\nabla^2 \overline{\varepsilon}_{\mu\nu}(\mathbf{x}) = 16\pi G T_{\mu\nu}(\mathbf{x}). \tag{7.12}$$

Since $\overline{\overline{\varepsilon}}_{\mu\nu}(\mathbf{x}) = \varepsilon_{\mu\nu}(\mathbf{x})$ this equation is equivalent to

$$\nabla^2 \varepsilon_{\mu\nu}(\mathbf{x}) = 16\pi G \overline{T}_{\mu\nu}(\mathbf{x}). \tag{7.13}$$

In order to solve this equation we need an expression for the stress-energy tensor operator, $T_{\mu\nu}(\mathbf{x})$. The system in which we are interested consists of a small gravitating particle of mass m and the Earth of mass M. We choose to limit the gauge further so that in the co-ordinate system used the Earth is stationary at the spatial origin, and write the mass density operator as

$$\rho(\mathbf{x}) = m|\mathbf{x}> < \mathbf{x}| + M\delta^3(\mathbf{x}). \tag{7.14}$$

This treats both particles as point particles, which is purely for ease of calculation. The Earth is a 'background' particle fixed at the origin and the state $|\mathbf{x}>$ represents the Earth at the origin and the other particle at the position $\mathbf{x}$. It is clear that

$$\rho(\mathbf{x})|\mathbf{x}'> = \left(m\delta^3(\mathbf{x}-\mathbf{x}') + M\delta^3(\mathbf{x})\right)|\mathbf{x}'>. \tag{7.15}$$

This is an eigenvalue equation. The eigenvalue is the value of the mass density at the point $\mathbf{x}$ caused by two point particles, one of mass M at the origin and the other of mass m at $\mathbf{x}'$. For slow moving particles the stress energy operator can be written as

$$T_{\mu\nu}(\mathbf{x}) = \begin{pmatrix} \rho(\mathbf{x}) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{7.16}$$

and so

$$\overline{T}_{\mu\nu}(\mathbf{x}) = \frac{1}{2}\rho(\mathbf{x})\delta_{\mu\nu}. \tag{7.17}$$

The solution of Eq.(7.13) which vanishes at infinity is

$$\varepsilon_{\mu\nu}(\mathbf{x}) = -4G\int d^3\mathbf{x}'\,\frac{\overline{T}_{\mu\nu}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} = -2G\delta_{\mu\nu}\int d^3\mathbf{x}'\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}, \tag{7.18}$$

and the metric operator obeys the eigen-value equation

$$g_{\mu\nu}(\mathbf{x})|\mathbf{x}'> = \left(\eta_{\mu\nu} - \frac{2MG}{|\mathbf{x}'|}\delta_{\mu\nu} - \frac{2mG}{|\mathbf{x}-\mathbf{x}'|}\delta_{\mu\nu}\right)|\mathbf{x}'>. \tag{7.19}$$

The eigen-value is the value of the metric at $\mathbf{x}$ produced by a particle of mass M at the origin, and a particle of mass m at $\mathbf{x}'$, calculated according to classical linear gravity.

At this point it is useful to consider more carefully what is meant by the term 'gauge'. With electromagnetism, given sufficient boundary conditions, it provides us with a map from each 4-current vector field $j^\mu(x)$ to a unique 4-vector potential field $A^\mu(x)$. Einstein's equation can be considered as a map from each physically valid stress-energy tensor field on the co-ordinate system to a set of all possible metrics. If we want to practically solve a problem then we will have to make some choice over the topology, boundary conditions, and/or any other conditions needed. The choice of gauge now allows us to treat Einstein's equation as a map from each physically valid stress-energy tensor field, to a unique metric field.

The state provides information of the position of the particle over the co-ordinate system. The interpretation of this state is gauge dependent, just as the interpretation of the state in electromagnetism is gauge dependent. From this state we can deduce the stress-energy tensor at each point, and using Einstein's equation, a metric at each point.

7.3 Superpositions of States

It has been argued by Penrose [13] that there are fundamental problems in the description of superpositions of states (see section 9.1). His arguments are based on the assumption that states are of the form

$$|\Psi\rangle|G_\psi\rangle, \tag{7.20}$$

that is they consist of a matter state and a state of the gravitational field. The problems that occur are due to $|G_\psi\rangle$. The description used here violates this assumption. Only the matter state exists. In any given gauge the gravitational field can be deduced from the matter state. Superpositions can be described just as in quantum electrostatics. To illustrate this we return to the system of a small particle moving in the background field of the Earth. Let the states $|\Psi, t\rangle$ and $|X, t\rangle$ each represent this particle at alternative locations. The state

$$|\Psi,t> + |X,t>, \tag{7.21}$$

is the equivalent of a superposition of states in electrostatics. Acting the metric operator on this state we obtain

$$g^{\psi}_{\mu\nu}(\mathbf{x},t)|\Psi,t> + g^{\chi}_{\mu\nu}(\mathbf{x},t)|X,t>. \tag{7.22}$$

We assume the states each represents the particle at reasonably definite positions and so $g^{\chi}_{\mu\nu}(\mathbf{x},t)$ and $g^{\psi}_{\mu\nu}(\mathbf{x},t)$ are definite metrics and not operators. The metric operator is in the Schrödinger picture and so is time independent. The time dependency of the metrics $g^{\psi}_{\mu\nu}(\mathbf{x},t)$ and $g^{\chi}_{\mu\nu}(\mathbf{x},t)$ comes from the time dependency of the states $|\Psi,t>$ and $|X,t>$.

The superposition does not cause any problems with co-ordinate transformations. There are two types of transformation that must be considered for the quantum system. The first is a simple co-ordinate transformation. Is it reasonable to expect the system to be invariant under co-ordinate changes that mix space and time? The states we have been using involve only a single time whereas a many particle configuration space contains many spatial variables. Clearly Eq.(7.22) is invariant under a general spatial co-ordinate transformation and also a general transformation of time.

The other transformation is the more general gauge transformation, in the sense described in the previous section. The metrics $g^{\psi}_{\mu\nu}(\mathbf{x},t)$ and $g^{\chi}_{\mu\nu}(\mathbf{x},t)$ describe two space-times within a particular gauge. Each point on the χ space-time is identified with a point on the ψ space-time, but only because we have made a choice of gauge. A gauge transformation in general will transform the co-ordinates on each space-time differently. In ψ space we will have $(\mathbf{x},t) \rightarrow (\mathbf{x}',t')$ and in χ space $(\mathbf{x},t) \rightarrow (\mathbf{x}'',t'')$. Similarly the metrics transform as

$$g^{\psi}_{\mu\nu}(\mathbf{x},t) \rightarrow g^{\psi}_{\mu\nu}{}'(\mathbf{x}',t'), \qquad g^{\chi}_{\mu\nu}(\mathbf{x},t) \rightarrow g^{\chi}_{\mu\nu}{}''(\mathbf{x}'',t''). \tag{7.23}$$

This means that each point in χ space-time will now be identified with a different point of the ψ space-time. The fact that there are two co-ordinate transformations does not lead to any contradictions. $|\Psi, t> + |X, t>$ will have to be transformed so that the position of the χ particle is transformed according to the χ transformation, and the position of the ψ particle will have to be transformed according to the ψ transformation. The physical interpretation of the system is not altered by this change of gauge.

7.4 High Velocities and Non-linear Gravity

The calculation Eqs.(7.12–7.19) is an example of how the metric operator can be calculated, but is only an approximation. Consider classical linear gravity in a particular co-ordinate system $(\mathbf{x}, t)$. The metric in classical linear gravity is governed by Eq.(7.10). If this metric is caused by slow moving particles then $\frac{\partial^2}{\partial t^2}\varepsilon_{\mu\nu}(\mathbf{x}, t) \simeq 0$, but if it is caused by fast moving particles then $\frac{\partial^2}{\partial t^2}\varepsilon_{\mu\nu}(\mathbf{x}, t)$ can become large. Eq.(7.12) contains no time derivatives, and so cannot be used for high velocities. Instead we use the more sophisticated equation,

$$\Box^2 \overline{\varepsilon}_{H\mu\nu}(\mathbf{x}, t) = -16\pi G T_{H\mu\nu}(\mathbf{x}, t), \tag{7.24}$$

that is the operator version of Eq.(7.10) in the Heisenberg picture, which takes into account the time evolution of the metric.

The results so far are only applicable to linear gravity and it is in no way proposed that the true theory of gravity is linear. To form a full theory of quantum gravity we would like to impose an operator version of Einstein's equation in the Heisenberg picture,

$$G_{H\mu\nu}(\mathbf{x}, t) = 8\pi G T_{H\mu\nu}(\mathbf{x}, t). \tag{7.25}$$

However it is far from obvious whether this equation has any solutions. The normal Einstein's equation is a non-linear equation which has no solutions for a general stress-energy tensor $T^{\mu\nu}$. Eq.(7.25) is also dependent on the evolution operator, and so all the complications that occur with electromagnetic interactions occur here as well. No attempt is made to solve this equation in general, though we now examine an approximation which demonstrates it.

We calculate the single particle eigen-values of the metric operator in the slow speed approximation. The slow speed approximation of linear gravity was calculated using the operator equation

$$\nabla^2 \bar{\varepsilon}_{\mu\nu}(\mathbf{x},t) = 16\pi G T^{\mu\nu}(\mathbf{x},t). \tag{7.26}$$

This is an equation for the metric operator in the Schrödinger picture, and since the metric operator is a constant through time it is equivalent to

$$\Box^2 \bar{\varepsilon}_{\mu\nu}(\mathbf{x},t) = -16\pi G T^{\mu\nu}(\mathbf{x},t), \tag{7.27}$$

the operator version of the linearised Einstein's equation in the Schrödinger picture. We thus postulate that in the slow speed approximation of non-linear quantum gravity we impose the operator version of Einstein's equation in the Schrödinger picture,

$$G_{\mu\nu}(\mathbf{x},t) = 8\pi G T_{\mu\nu}(\mathbf{x},t). \tag{7.28}$$

It is clear that when $\mathbf{x} \neq \mathbf{x}'$, $|\mathbf{x}'>$ is an eigen-state of $T^{\mu\nu}(\mathbf{x})$ with eigen-value 0. Acting $|\mathbf{x}'>$ on Eq.(7.28) we obtain

$$G_{\mu\nu}(\mathbf{x},t)|\mathbf{x}'> = 0, \qquad \mathbf{x} \neq \mathbf{x}', \tag{7.29}$$

for every t or

$$G'_{\mu\nu}(\mathbf{x},t) = 0, \qquad \mathbf{x} \neq \mathbf{x}', \tag{7.30}$$

where $G'_{\mu\nu}(\mathbf{x},t)$ is the eigen-value of the operator $G_{\mu\nu}(\mathbf{x},t)$ with eigen-state $|\mathbf{x}'>$. Let $g'_{\mu\nu}(\mathbf{x},t)$ be the eigen-value of the metric operator $g_{\mu\nu}(\mathbf{x},t)$ with eigen-state $|\mathbf{x}'>$. Eq.(7.30) can thus be treated as a classical problem of a 4-manifold M' with Einstein tensor $G'_{\mu\nu}$ and metric $g'_{\mu\nu}$. If we impose the boundary condition on the metric operator that all the space-times it represents are spherically symmetric and asymptotically Minkowski then this problem is equivalent to the classical problem of a mass m at the spatial origin of a spherically symmetric and asymptotically Minkowski space-time. We can thus deduce that $g'_{\mu\nu}$ describes a space-time equivalent to the one described by the Schwarzschild metric centred at $\mathbf{x}'$, and so

$$ds^2|\mathbf{x}'> = \left[\left(1 - \frac{2mG}{r}\right)dt^2 - \left(1 - \frac{2mG}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2\right]|\mathbf{x}'>, \tag{7.31}$$

where $r = 0$ corresponds to the position $\mathbf{x} = \mathbf{x}'$.

Chapter 8

Linearised Gravitational Interactions

8.1 Introduction

This chapter discusses the evolution of states subject to linearised gravitational interactions. The metric operator is calculated directly, and this leads to a theory of linear gravitational interactions whose interaction Hamiltonian is of order 4 in $\psi(x)$. Renormalisation is not considered.

8.2 A Background Space-Time

In standard field theory in Minkowski space a scalar field $\phi(x)$ obeys the Klein-Gordon equation,

$$\hbar^2 \Box^2 \phi + m^2 \phi = 0, \tag{8.1}$$

and the equal-time commutation relation

$$[\phi(\mathbf{x}, t), \pi(\mathbf{x}', t)] = i\hbar \delta^3(\mathbf{x} - \mathbf{x}'), \tag{8.2}$$

where $\pi(\mathbf{x}, t)$ is the conjugate momenta to $\phi(\mathbf{x}, t)$. When this is extended to a curved space-time we obtain standard Quantum Field Theory in Curved Space-Time. Here the quantum field $\phi(x)$ in a background space-time given by the metric $g_{\mu\nu}(x)$ obeys the Klein-Gordon equation Eq.(8.1) where $\Box^2$ is the Laplacian operator for the space-time with this metric . In Quantum Field Theory in Curved Space-Time there is no unique vacuum state, and this leads to such effects as Hawking radiation.

This thesis takes a very different view of quantum fields. In Minkowski space the Dirac field $\psi_\alpha(\mathbf{x})$ is initially defined as the annihilation operator for a spinor component, and the field $\psi(\mathbf{x})$ as the annihilation operator for all spinor components. When we move into the QED representation and the interaction representation and finally exchange the vacuum $|0>$ with the observed vacuum $|0_{new}>$ we gain the standard expression for the Dirac field. In these representations the Dirac field $\psi(x)$ does obey the Dirac equation, but this is a deduction rather than an axiom.

The view this thesis takes of Dirac particles can be naturally extended to a background space-time. It may be true that having gone through all the procedures that obtained the standard expression for the Dirac field in Minkowski space, Quantum Field Theory in Curved Space-Time can be reproduced in a curved space-time. If this is true then the results of Quantum Field Theory in Curved Space-Time apply. It must be pointed out that the vacuum state that is not unique is the observed vacuum $|0_{new}>$, that is the vacuum which contains a sea of negative energy particles. No such problems exist with the original vacuum $|0>$.

Chapter 3 discussed the evolution of Dirac particle wave-functions $\Psi_{n(\alpha_1\ldots\alpha_n)}$ in a background space-time given by the metric $g_{\mu\nu}(\mathbf{x},t)$. There are two ways of describing the motion. The first method gives each particle its own space and time variable, and the wave-functions obey an equation for each particle. In an n particle system there are n equations, of which the one for particle k is

$$i\hbar\frac{\partial}{\partial t_k}\Psi_{n(\alpha_1\ldots\alpha_n)} = H_{k(\alpha_k\beta)}\Psi_{n(\alpha_1\ldots\alpha_{k-1}\beta\alpha_{k+1}\ldots\alpha_n)}. \tag{8.3}$$

Here H_k is the Hamiltonian for particle k, which is given by

$$H_k = \left(\gamma^0(\mathbf{x}_k,t_k)\right)^{-1}\left(-i\hbar\gamma^j(\mathbf{x}_k,t_k)\mathcal{D}_{kj}+m\right) - i\hbar\Gamma_{k0}, \tag{8.4}$$

where $\mathcal{D}_{k\mu}$ is the covariant derivative with respect to the k component of x^μ, Γ_{k0} is the connection associated with $\mathcal{D}_{k0}$ and the $\gamma^\mu(\mathbf{x},t)$ matrices obey

$$\{\gamma^\mu(\mathbf{x},t),\gamma^\nu(\mathbf{x},t)\} = 2g^{\mu\nu}(\mathbf{x},t). \tag{8.5}$$

Under a general transformation of space-time $(\mathbf{x},t) \to (\mathbf{x}',t')$ each of the co-ordinates of the particles transform as

$$(\mathbf{x}_1,t_1) \to (\mathbf{x}'_1,t'_1) \qquad (\mathbf{x}_2,t_2) \to (\mathbf{x}'_2,t'_2) \qquad \ldots \qquad (\mathbf{x}_n,t_n) \to (\mathbf{x}'_n,t'_n). \tag{8.6}$$

This leads to further transformations of $\Psi_{n(\alpha_1\ldots\alpha_n)}$, $g_{\mu\nu}(\mathbf{x},t)$, $\gamma^\mu(\mathbf{x},t)$, and H_k so that in this new co-ordinate system Eq.(8.3) becomes

$$i\hbar\frac{\partial}{\partial t'_k}\Psi'_{n(\alpha_1\ldots\alpha_n)} = H'_{k(\alpha_k\beta)}\Psi'_{n(\alpha_1\ldots\alpha_{k-1}\beta\alpha_{k+1}\ldots\alpha_n)}. \tag{8.7}$$

We see that in this system the concept of the particle is well defined. Eq.(8.3) and Eq.(8.7) both describe the same n particles in a background space-time. In Eq.(8.3) particle (k) has the co-ordinates $(\mathbf{x}_k,t_k)$, and in Eq.(8.7) it has the co-ordinates $(\mathbf{x}'_k,t'_k)$.

On the equal time surface $t = t_1 = t_2 = \ldots = t_n$ in configuration space-time the system of equations Eq.(8.3) is equivalent to

$$i\hbar\frac{\partial}{\partial t}\Psi_{n(\alpha_1\ldots\alpha_n)} = \sum_{k=1}^{n} H_{k(\alpha_k\beta)}\Psi_{n(\alpha_1\ldots\alpha_{k-1}\beta\alpha_{k+1}\ldots\alpha_n)}. \tag{8.8}$$

This is an evolution equation through a single time, but where each of the particles has its own spatial variable. This equation has a much more limited invariance than those of Eq.(8.3), and is only invariant under separate transformations of space and time

$$\mathbf{x} \to \mathbf{x}', \qquad t \to t'. \tag{8.9}$$

The concept of a particle is also well defined in this system, since it is well defined in the multi-time system.

Chapter 5 demonstrated how Dirac particles could be described with creation and annihilation operators. States are formed by acting 4 independent creation operators $\psi^\dagger_\alpha(\mathbf{x})$ on the vacuum state. These operators obey

$$\{\psi_\alpha(\mathbf{x}),\psi^\dagger_\beta(\mathbf{x}')\} = \delta_{\alpha\beta}\delta^3(\mathbf{x}-\mathbf{x}'), \qquad \{\psi_\alpha(\mathbf{x}),\psi_\beta(\mathbf{x}')\} = 0, \tag{8.10}$$

relations which are independent of the metric. From them we can deduce the normalisation of states (for scalar particles this is shown in section 4.2) which are certainly independent of the metric. States evolve through time according to an equation

$$i\hbar\frac{\partial}{\partial t}|\Psi, t\rangle = \int d^3\mathbf{x}\, \psi^\dagger(\mathbf{x}) H \psi(\mathbf{x}) |\Psi, t\rangle, \tag{8.11}$$

where each particle has a Hamiltonian H. The n-particle components of Eq.(8.11) are Eq.(8.8). Since particles are well defined with the description provided by Eq.(8.8), they are also well defined with Eq.(8.11).

It can also be demonstrated that a co-ordinate transformation does not alter the vacuum state without referring to the wave-functions $\Psi_{n(\alpha_1 \ldots \alpha_n)}$. The vacuum state is defined as a state which obeys

$$\psi_\alpha(\mathbf{x})|0\rangle = 0, \qquad \langle 0|0\rangle = 1, \tag{8.12}$$

for every $\mathbf{x} \in \mathbf{R}^3$ and $\alpha = 1, 2, 3, 4$. It is not assumed that the vacuum state is unique. The system treats time separately to the spatial co-ordinates. Indeed the operators $\psi_\alpha(\mathbf{x})$ are dependent only on the spatial position and we consider purely spatial transformations. Under the spatial co-ordinate mapping

$$\mathbf{x} \rightarrow \mathbf{x}', \tag{8.13}$$

we have

$$\psi_\alpha(\mathbf{x}) \rightarrow \psi_\alpha(\mathbf{x}'), \tag{8.14}$$

and the definition of the vacuum state under this co-ordinate system is

$$\psi_\alpha(\mathbf{x}')|0'\rangle = 0, \qquad \langle 0'|0'\rangle = 1. \tag{8.15}$$

This is however the definition of the old vacuum state and so the two co-ordinate systems share the same vacuum states. If the original vacuum state is unique then

$$|0'\rangle = |0\rangle. \tag{8.16}$$

What about the non-uniqueness of the observed vacuum $|0_{new}\rangle$? Consider first Quantum Field Theory in Curved Space-Time. On a curved Manifold M a scalar field obeys

$$\left(\hbar^2\Box^2 + m^2\right)\phi = 0, \tag{8.17}$$

where $\Box^2$ is the Laplacian on M. We define a scalar product

$$(\phi_1, \phi_2) = -i\hbar \int_\Sigma \phi_1(x)\overleftrightarrow{\partial_\mu}\phi_2^*(x)[-g_\Sigma(x)]^{\frac{1}{2}} d\Sigma^\mu, \tag{8.18}$$

where $d\Sigma^\mu = n^\mu d\Sigma$, with n^μ a future-directed unit vector orthogonal to the space-like hypersurface Σ and $d\Sigma$ is the volume element in Σ. It is possible to find a complete set of mode solutions $u_i(x)$ of Eq.(8.17) satisfying

$$(u_i, u_j) = \delta_{ij}, \qquad (u_i^*, u_j^*) = -\delta_{ij}, \qquad (u_i, u_j^*) = 0. \tag{8.19}$$

We can also find another complete set of mode solutions $\overline{u}_i(x)$ satisfying

$$(\overline{u}_i, \overline{u}_j) = \delta_{ij}, \qquad (\overline{u}_i^*, \overline{u}_j^*) = -\delta_{ij}, \qquad (\overline{u}_i^*, \overline{u}_j) = 0. \tag{8.20}$$

Since both sets of solutions are complete we can expand $\overline{u}_j(x)$ in terms of the old solutions

$$\overline{u}_j = \sum_i \left(\alpha_{ji} u_i + \beta_{ji} u_i^*\right). \tag{8.21}$$

In general $\beta_{ji} \neq 0$. From this it can be shown that the vacuum state in Quantum Field Theory in Curved Space-Time is not unique. Further details can be found in Birrell and Davies [30].

Eq.(8.19) provides a successful global definition of positive and negative energy solutions to Eq.(8.17). The $u_j(x)$ are the positive energy solutions, and the $u_j^*(x)$ are the negative energy solutions. What has been shown is that in general a purely positive energy solution in one basis has a negative energy part in another basis. This means that we cannot split the space of all solutions into a space of positive energy solutions and a space of negative energy solutions in a way that is independent of basis. Now consider the description of Dirac particles in this thesis. $|0_{new}>$ is the state which contains no positive energy particles and a 'sea' of negative energy particles. What we describe as a negative energy particle is dependent on the basis of solutions being used, and so $|0_{new}>$ is similarly dependent on the basis.

8.3 The Stress-Energy Tensor Operator in Minkowski Space

The first task is to find an expression for the stress-energy operator. Because of the separation of space and time the construction of a stress-energy tensor even in Minkowski space is not obvious. A state of free Dirac particles evolve according to

$$i\hbar\frac{\partial}{\partial t}|\Psi,t\rangle = \int d^3\mathbf{x}\,\psi^\dagger(\mathbf{x})H\psi(\mathbf{x})|\Psi,t\rangle, \tag{8.22}$$

where the Hamiltonian is given by

$$H = \gamma^0\left(-i\hbar\gamma^j\partial_j + m\right). \tag{8.23}$$

One solution is

$$\exp\frac{-it\sqrt{m^2+\mathbf{p}_1^2} - it\sqrt{m^2+\mathbf{p}_2^2}}{\hbar}|\mathbf{p}_1(1),\mathbf{p}_2(1)\rangle, \tag{8.24}$$

which represents two positive energy free particles. We define the total momentum operator for Dirac particles as

$$\mathbf{p}^{op} = -i\hbar\int d^3\mathbf{x}\,\psi^\dagger(\mathbf{x})\nabla\psi(\mathbf{x}). \tag{8.25}$$

This obeys the eigen-value equation

$$\mathbf{p}^{op}|\Psi,t\rangle = (\mathbf{p}_1+\mathbf{p}_2)|\Psi,t\rangle. \tag{8.26}$$

We might naively think that the total energy operator can be written as

$$i\hbar\int d^3\mathbf{x}\,\psi^\dagger(\mathbf{x})\frac{\partial}{\partial t}\psi(\mathbf{x}). \tag{8.27}$$

This is not the case. Eq.(8.22) is an energy equation. The total energy operator can be written in two ways:

$$i\hbar\frac{\partial}{\partial t}, \qquad \int d^3\mathbf{x}\,\psi^\dagger(\mathbf{x})H\psi(\mathbf{x}), \tag{8.28}$$

and we see that

$$i\hbar\frac{\partial}{\partial t}|\Psi,t\rangle = \int d^3\mathbf{x}\,\psi^\dagger(\mathbf{x})H\psi(\mathbf{x})|\Psi,t\rangle = \left(\sqrt{\mathbf{p}_1^2+m^2}+\sqrt{\mathbf{p}_2^2+m^2}\right)|\Psi,t\rangle. \tag{8.29}$$

The operator which sums the squares of the energy is

$$\int d^3\mathbf{x}\, \psi^\dagger(\mathbf{x}) H^2 \psi(\mathbf{x}), \tag{8.30}$$

and

$$\int d^3\mathbf{x}\, \psi^\dagger(\mathbf{x}) H^2 \psi(\mathbf{x}) |\Psi, t\rangle = \left(\mathbf{p}_1^2 + m^2 + \mathbf{p}_2^2 + m^2\right) |\Psi, t\rangle. \tag{8.31}$$

We cannot construct this operator using $i\hbar\frac{\partial}{\partial t}$, since

$$-\hbar^2 \frac{\partial^2}{\partial t^2} |\Psi, t\rangle = \left(\int d^3\mathbf{x}\, \psi^\dagger(\mathbf{x}) H \psi(\mathbf{x})\right)^2 |\Psi, t\rangle = \left(\sqrt{\mathbf{p}_1^2 + m^2} + \sqrt{\mathbf{p}_2^2 + m^2}\right)^2 |\Psi, t\rangle, \tag{8.32}$$

(see Eqs.(4.15–4.19)). We see that time derivatives do not form operators in the manner expected of other operators.

We now consider the stress-energy operator in Minkowski space-time. A one particle spinor, Ψ, obeying the one particle free Dirac equation has a stress-energy tensor of

$$T^{\mu\nu}(\mathbf{x}, t) = \frac{i\hbar}{2} \Psi^\dagger(\mathbf{x}, t) \left(\gamma^0 \gamma^\mu \partial^\nu + (\partial^\mu)^\dagger \gamma^0 \gamma^\nu\right) \Psi(\mathbf{x}, t). \tag{8.33}$$

This is examined by Schweber [11] P.201. This formula contains ∂^0 which is a time derivative. From the previous paragraph we see that naively substituting $\Psi(\mathbf{x}, t)$ with $\psi(\mathbf{x})$ would not produce an operator which behaves as we would wish. Instead define

$$i\hbar`\partial'_\mu = \begin{cases} \gamma^0 \left(-i\hbar\gamma^j \partial_j + m\right) & \mu = 0, \\ i\hbar\partial_\mu & \mu = 1, 2, 3, \end{cases} \tag{8.34}$$

and we postulate that the stress-energy operator for Minkowski space-time in the Schrödinger representation is

$$T^{\mu\nu}(\mathbf{x}, t) = \frac{i\hbar}{2} \psi^\dagger(\mathbf{x}) \left(\gamma^0 \gamma^\mu `\partial'^\nu + (`\partial'^\mu)^\dagger \gamma^0 \gamma^\nu\right) \psi(\mathbf{x}). \tag{8.35}$$

8.4 The Equations of Motion for Quantum Linear Gravity.

Section 2.4 discussed the evolution of Dirac wave-functions through background space-times. It was shown that if we were prepared to ignore the more subtle effects of spin the Hamiltonian for a single Dirac particle was given by

$$H = \left(\gamma^0(\mathbf{x},t)\right)^{-1}\left(-i\hbar\gamma^j(\mathbf{x},t)\partial_j + m\right). \tag{8.36}$$

Continuing with this approximation, in a background gravitational field given by the metric $g_{\mu\nu}(\mathbf{x},t)$ a state $|\Psi,t\rangle$ of Dirac particles obeys

$$i\hbar\frac{\partial}{\partial t}|\Psi,t\rangle = \int d^3\mathbf{x}\,\psi^\dagger(\mathbf{x})\left[\left(\gamma^0(\mathbf{x},t)\right)^{-1}\left(-i\hbar\gamma^j(\mathbf{x},t)\partial_j + m\right)\right]\psi(\mathbf{x})|\Psi,t\rangle, \tag{8.37}$$

where

$$\{\gamma^\mu(\mathbf{x},t),\gamma^\nu(\mathbf{x},t)\} = 2g^{\mu\nu}(\mathbf{x},t). \tag{8.38}$$

Since $-i\hbar\partial_j f(\mathbf{x}) = f(\mathbf{x})\,(-i\hbar\partial_j)^\dagger$ for an arbitrary function $f(\mathbf{x})$, Eq.(8.37) can be rewritten as

$$i\hbar\frac{\partial}{\partial t}|\Psi,t\rangle = \int d^3\mathbf{x}\,\psi^\dagger_\alpha(\mathbf{x})\left[\psi_\beta(\mathbf{x})\left[\left(\gamma^0(\mathbf{x},t)\right)^{-1}\left(-i\hbar\gamma^j(\mathbf{x},t)\partial_j + m\right)\right]^\dagger_{(\alpha\beta)}\right]|\Psi,t\rangle, \tag{8.39}$$

It is proposed that in order to describe Dirac particles subject to linearised gravitational interactions we replace the field $\gamma^\mu(\mathbf{x},t)$ with an operator field $\gamma^\mu(\mathbf{x},t)$ obeying Eq.(8.38) where $g_{\mu\nu}(\mathbf{x},t)$ is now the metric operator. There are other possible equations that could be used and the choice is made for simplicity. With this formulation self interactions are included. Since we are considering linear gravity we write the metric operator as

$$g_{H\mu\nu}(\mathbf{x},t) = r_{\mu\nu} + \varepsilon_{H\mu\nu}(\mathbf{x},t), \tag{8.40}$$

in the Heisenberg representation, where we assume $\varepsilon_{H\mu\nu}(\mathbf{x},t)$ is small. It is proposed that the metric operator obeys the operator version of the linearised Einstein's equation. In the gauge equivalent to the Lorentz gauge of electromagnetism this is

$$\partial^\alpha\partial_\alpha\bar{\varepsilon}_{H\mu\nu}(\mathbf{x},t) = -16\pi G T_{H\mu\nu}(\mathbf{x},t), \tag{8.41}$$

where

$$\bar{\varepsilon}_{H\mu\nu}(\mathbf{x},t) = \varepsilon_{H\mu\nu}(\mathbf{x},t) - \frac{1}{2}r_{\mu\nu}\varepsilon^\alpha_{H\alpha}(\mathbf{x},t). \tag{8.42}$$

These equations can be replaced by a single equation. Since $\overline{\varepsilon}_H^{\mu\nu}(\mathbf{x},t) = \varepsilon_H^{\mu\nu}(\mathbf{x},t)$, we have

$$\Box^2 \varepsilon_H^{\mu\nu}(\mathbf{x},t) = -16\pi G \overline{T}_H^{\mu\nu}(\mathbf{x},t). \tag{8.43}$$

It is assumed that $\varepsilon_H^{\mu\nu}(\mathbf{x},t)$ is small and so the stress-energy operator can be approximated by the Minkowski stress-energy operator Eq.(8.35). Hence in the leading order of $\varepsilon_{H\mu\nu}(\mathbf{x},t)$,

$$\Box^2 \varepsilon_H^{\mu\nu}(\mathbf{x},t) = -8\pi i\hbar G \psi_H^\dagger(\mathbf{x},t) \left(\gamma^{0\prime}\gamma^{\mu\prime}`\partial'^{\nu} + (`\partial'^{\mu})^\dagger \gamma^{0\prime}\gamma^{\nu\prime} \right.$$
$$\left. -\frac{1}{2} r^{\mu\nu}\gamma^{0\prime}\gamma^{\alpha\prime}`\partial'_{\alpha} - \frac{1}{2} r^{\mu\nu}(`\partial'_{\alpha})^\dagger \gamma^{0\prime}\gamma^{\alpha\prime}\right) \psi_H(\mathbf{x},t). \tag{8.44}$$

This equation can be solved using a Green's function. It is not clear which Green's function should be used and the arbitrary choice of the retarded Green's function is made. When there is no external field

$$\varepsilon^{\mu\nu}(\mathbf{x},t) = -8\pi i\hbar G \int d^4x' G_{ret}(\mathbf{x}'-\mathbf{x}, t'-t) U(t',t)\psi^\dagger(\mathbf{x}') \left(\gamma^{0\prime}\gamma^{\mu\prime}`\partial'^{\prime\nu} + (`\partial'^{\prime\mu})^\dagger \gamma^{0\prime}\gamma^{\nu\prime} \right.$$
$$\left. -\frac{1}{2} r^{\mu\nu}\gamma^{0\prime}\gamma^{\alpha\prime}`\partial'^{\prime}_{\alpha} - \frac{1}{2} r^{\mu\nu}(`\partial'^{\prime}_{\alpha})^\dagger \gamma^{0\prime}\gamma^{\alpha\prime}\right) \psi(\mathbf{x}')U^\dagger(t',t), \tag{8.45}$$

where $t' = x'^0$.

From Eq.(2.91) we see that up to linear order in $\varepsilon^{\mu\nu}(\mathbf{x},t)$ we can write

$$\gamma^\mu(\mathbf{x},t) = \gamma^{\mu\prime} - \frac{1}{2}\varepsilon^\mu_\nu(\mathbf{x},t)\gamma^{\nu\prime}, \tag{8.46}$$

where $\gamma^{\mu\prime}$ are constant matrices obeying

$$\{\gamma^{\mu\prime}, \gamma^{\nu\prime}\} = 2r^{\mu\nu}. \tag{8.47}$$

This means up to linear order in $\varepsilon_{\mu\nu}(\mathbf{x},t)$

$$i\hbar\frac{\partial}{\partial t}|\Psi, t> = \int d^3\mathbf{x}\, \psi_\alpha^\dagger(\mathbf{x})$$
$$\left[\psi_\beta(\mathbf{x}) \left[\left(\gamma^{0\prime} + \frac{1}{2}\varepsilon^0_\alpha(\mathbf{x},t)\gamma^{\alpha\prime}\right)\left(-i\hbar\left(\gamma^{j\prime}(\mathbf{x},t) - \frac{1}{2}\gamma^{\alpha\prime}\varepsilon^j_\alpha(\mathbf{x},t)\right)\partial_j + m\right)\right]^\dagger_{(\alpha\beta)}\right]|\Psi, t>. \tag{8.48}$$

To obtain a single equation Eq.(8.45) is substituted.

8.5 The Slow Speed Approximation

We now find the slow speed approximation of positive energy particles interacting through linear gravity. In this approximation the energy operator is

$$i\hbar \int d^3\mathbf{x}\, \psi^\dagger(\mathbf{x})`\partial\text{'}_0\psi(\mathbf{x}) = m \int d^3\mathbf{x}\, \psi^\dagger(\mathbf{x})\psi(\mathbf{x}) + O(v^2). \tag{8.49}$$

Similarly the spatial momentum operator has components

$$-\, i\hbar \int d^3\mathbf{x}\, \psi^\dagger(\mathbf{x})`\partial\text{'}_i\psi(\mathbf{x}) = O(v). \tag{8.50}$$

Assuming the limit $v \to 0$, we can approximate Eq.(8.44) by performing the substitution

$$i\hbar`\partial\text{'}_0 \to m, \qquad -i\hbar`\partial\text{'}_i \to 0, \tag{8.51}$$

and obtain

$$\Box^2 \varepsilon_{H\mu\nu}(\mathbf{x}, t) = -\delta_{\mu\nu} 8\pi G m \psi_H(\mathbf{x}, t)\psi_H(\mathbf{x}, t). \tag{8.52}$$

Also in the slow speed approximation

$$\frac{\partial^2}{\partial t^2}\varepsilon_{H\mu\nu}(\mathbf{x}, t) = O\left(v^2\right). \tag{8.53}$$

Eq.(8.52) simplifies further

$$\nabla^2 \varepsilon_{H\mu\nu}(\mathbf{x}, t) = \delta_{\mu\nu} 8\pi m G \psi_H^\dagger(\mathbf{x}, t)\psi_H(\mathbf{x}, t), \tag{8.54}$$

which is equivalent to

$$\nabla^2 \varepsilon_{\mu\nu}(\mathbf{x}, t) = \delta_{\mu\nu} 8\pi m G \psi^\dagger(\mathbf{x})\psi(\mathbf{x}), \tag{8.55}$$

an equation in the Schrödinger representation. A solution to this is

$$\varepsilon_{\mu\nu}(\mathbf{x}, t) = -\delta_{\mu\nu} 2mG \int d^3\mathbf{x}' \frac{\psi^\dagger(\mathbf{x}')\psi(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} + 2mGC\delta_{\mu\nu}, \tag{8.56}$$

where C is an arbitrary constant, and so

$$g_{\mu\nu}(\mathbf{x}) = r_{\mu\nu} - \delta_{\mu\nu} 2mG \int d^3\mathbf{x}' \frac{\psi^\dagger(\mathbf{x}')\psi(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} + 2mGC\delta_{\mu\nu}. \tag{8.57}$$

The metric operator's eigen-states are those where all particles are in an eigen-state of position. For instance, using the eigen-states of position defined in Eq.(A.1)

$$g_{\mu\nu}(\mathbf{x},t)|\mathbf{x}_1(\alpha),\mathbf{x}_2(\beta)>$$
$$= \left[r_{\mu\nu} - \delta_{\mu\nu}\frac{2mG}{|\mathbf{x}-\mathbf{x}_1|} - \delta_{\mu\nu}\frac{2mG}{|\mathbf{x}-\mathbf{x}_2|} + 2mGC\delta_{\mu\nu}\right]|\mathbf{x}_1(\alpha),\mathbf{x}_2(\beta)>. \quad (8.58)$$

We wish to obtain an equation of motion for two slow moving scalar particles interacting though linear gravity. We see from Eq.(8.48) that each particle has Hamiltonian

$$H = \left(\gamma^{0'} + \frac{1}{2}\varepsilon^0_\alpha(\mathbf{x},t)\gamma^{\alpha'}\right)\left(-i\hbar\left(\gamma^{j'} - \frac{1}{2}\gamma^{\alpha'}\varepsilon^j_\alpha(\mathbf{x},t)\right)\partial_j + m\right), \quad (8.59)$$

where j sums from 1 to 3. Since $\varepsilon_{\mu\nu}(\mathbf{x},t) = 0$ if $\mu \neq \nu$, this Hamiltonian is

$$H = \gamma^{0'}\left(1 + \frac{1}{2}\varepsilon^0_0(\mathbf{x},t)\right)\left(-i\hbar\gamma^{j'}\left(1 - \frac{1}{2}\varepsilon^j_j(\mathbf{x},t)\right)\partial_j + m\right), \quad (8.60)$$

where again j sums from 1 to 3. Now

$$\varepsilon_{\mu\mu}(\mathbf{x}_1,t)|\mathbf{x}_1(\alpha),\mathbf{x}_2(\beta)> = -2mG\left(\frac{1}{\epsilon} + \frac{1}{|\mathbf{x}_1-\mathbf{x}_2|} + C\right)|\mathbf{x}_1(\alpha),\mathbf{x}_2(\beta)>, \quad (8.61)$$

where we take the limit $\epsilon \to 0$ and we do not sum over μ. This equation contains an infinite constant, and we can use the arbitrary constant to cancel it out. If we put

$$C = -\frac{1}{\epsilon} \quad (8.62)$$

before the limit is taken then

$$\varepsilon_{\mu\mu}(\mathbf{x}_1,t)|\mathbf{x}_1(\alpha),\mathbf{x}_2(\beta)> = \frac{-2mG}{|\mathbf{x}_1-\mathbf{x}_2|}|\mathbf{x}_1(\alpha),\mathbf{x}_2(\beta)>, \quad (8.63)$$

where again we do not sum over μ. Hence the $|\mathbf{x}_1(\alpha),\mathbf{x}_2(\beta)>$ component of Eq.(8.48) is

$$i\hbar\frac{\partial}{\partial t}\Psi_{(\alpha\beta)} = \left[\gamma^{0'}\left(1 - \frac{mG}{|\mathbf{x}_1-\mathbf{x}_2|}\right)\left(-i\hbar\left(1 - \frac{mG}{|\mathbf{x}_1-\mathbf{x}_2|}\right)\gamma^{j'}\partial_{1j} + m\right)\right]_{(\alpha\tau)}\Psi_{(\tau\beta)}$$
$$+ \left[\gamma^{0'}\left(1 - \frac{mG}{|\mathbf{x}_1-\mathbf{x}_2|}\right)\left(-i\hbar\left(1 - \frac{mG}{|\mathbf{x}_1-\mathbf{x}_2|}\right)\gamma^{j'}\partial_{2j} + m\right)\right]_{(\beta\tau)}\Psi_{(\alpha\tau)}. \quad (8.64)$$

The Pauli-style approximation Eqs.(2.95–2.105) gave an intuitive insight into the behaviour of Dirac particles in a background gravitational field. A similar approximation gives,

$$i\hbar\frac{\partial}{\partial t}\Psi_{(jk)} =$$

$$\left(1-\frac{mG}{|\mathbf{x}_1-\mathbf{x}_2|}\right)^3\left[-\frac{\hbar^2}{2m}\nabla_1^2-\frac{\hbar^2}{2m}\nabla_2^2\right]\Psi_{(jk)}+\left[2m-\frac{2m^2G}{|\mathbf{x}_1-\mathbf{x}_2|}+O(v\hbar)\right]\Psi_{(jk)}, \tag{8.65}$$

where $j,k=1,2$ and $O(v\hbar)$ are spin terms. The leading interaction in the motion is the potential

$$-\frac{2m^2G}{|\mathbf{x}_1-\mathbf{x}_2|}, \tag{8.66}$$

and each particle is subjected to twice the Newtonian force.

This in fact is not surprising. Returning to electrostatics we have seen that in a background field the total potential is

$$\int d^3\mathbf{x}\phi^\dagger(\mathbf{x})\phi(\mathbf{x})V(\mathbf{x}). \tag{8.67}$$

However when $V(\mathbf{x})$ is the potential operator the total potential is

$$\frac{1}{2}\int d^3\mathbf{x}\phi^\dagger(\mathbf{x})\phi(\mathbf{x})V(\mathbf{x}). \tag{8.68}$$

Both of these terms give the expected classical result. Eq.(8.68) was obtained by examining the energy of the electrostatic field. The gravitational field is much more intimate than the electrostatic field. Clearly the potential energy term $-\frac{2m^2G}{|\mathbf{x}_1-\mathbf{x}_2|}$ has twice the required value, but what about the factor that effects the spatial derivatives? Any analysis of the gravitational energy will not answer this question. For the rest of the thesis it is assumed that the potential term should be halved with the rest of the equation being left unaffected. It cannot be emphasised too much that this is only an assumption.

8.6 Scattering Theory

This section shows how the scattering matrix for quantum linear gravity may be calculated. The matrix elements are not evaluated and renormalisation is not considered.

We consider particles to be moving in a background Minkowski space, subject to an interaction which contains the remainder of the terms in the equation. This is very similar to the way electromagnetic interactions were handled, and all the results of the Dirac field transfer over to linear gravity.

Scattering theory is much more easily performed in the QED representation. In this representation

$$\tilde{\psi}_\alpha(\mathbf{x}) = (2\pi\hbar)^{-\frac{3}{2}} \int \frac{m^{\frac{1}{2}} d^3\mathbf{p}}{\sqrt[4]{\mathbf{p}^2 + m^2}} \exp \frac{i\kappa_\alpha \mathbf{p}.\mathbf{x}}{\hbar} u_\alpha(\mathbf{p}) c_\alpha(\mathbf{p}), \qquad \alpha = 1,2,3,4, \tag{8.69}$$

where the $c_\alpha(\mathbf{p})$ are those used in Minkowski space-time. We also put

$$\tilde{\psi}(\mathbf{x}) = \tilde{\psi}_1(\mathbf{x}) + \tilde{\psi}_2(\mathbf{x}) + \tilde{\psi}_3(\mathbf{x}) + \tilde{\psi}_4(\mathbf{x}). \tag{8.70}$$

The vacuum state is then redefined. This involves exchanging each $c_3(\mathbf{p})$ with $c_3^\dagger(\mathbf{p})$ and each $c_4(\mathbf{p})$ with $c_4^\dagger(\mathbf{p})$. We similarly redefine $\tilde{\psi}_3(\mathbf{x})$ and $\tilde{\psi}_4(\mathbf{x})$ for each $\mathbf{x}$. The new vacuum state is defined as before

$$|0_{new}\rangle = \prod_{\mathbf{p}\in\mathbf{R}^3} c_3(\mathbf{p}) c_4(\mathbf{p}) |0\rangle, \tag{8.71}$$

where the $c_3(\mathbf{p})$ and $c_4(\mathbf{p})$ are now anti-particle annihilation operators. Since we are considering particles to be moving in a background Minkowski space-time there are no problems in defining $|0_{new}\rangle$ uniquely.

Scattering theory is performed in the interaction representation. In order to use the interaction representation the total Hamiltonian must be written as the sum of the free particle and interaction Hamiltonians. We express the evolution equation Eq.(8.48) in the form of

$$i\frac{\partial}{\partial t}|\Psi, t\rangle = \left[\tilde{H}_0 + \tilde{H}_I\right]|\Psi, t\rangle, \tag{8.72}$$

where from now on we use units in which $\hbar = 1$. $\tilde{H}_0$ is the free particle Hamiltonian in the Schrödinger picture and QED representations,

$$\tilde{H}_0 = \int d^3\mathbf{x}\, \tilde{\psi}^\dagger(\mathbf{x})\gamma^0 \left[-i\gamma^j\partial_j + m\right] \tilde{\psi}(\mathbf{x}). \tag{8.73}$$

$\tilde{H}_I$ is the interaction Hamiltonian in the Schrödinger picture which contains the remainder of the terms. Up to linear order in $\tilde{\varepsilon}_{\mu\nu}(\mathbf{x}, t)$,

$$\begin{aligned} \tilde{H}_I(t) = {} & \frac{1}{2}\int d^3\mathbf{x}\, \tilde{\varepsilon}^0_\alpha(\mathbf{x}, t)\tilde{\psi}^\dagger(\mathbf{x})\gamma^{0'}\gamma^{\alpha'}\left[-i\gamma^{j'}\partial_j + \frac{m}{2}\right]\tilde{\psi}(\mathbf{x}) \\ & -\frac{i}{2}\int d^3\mathbf{x}\, \varepsilon^j_\alpha(\mathbf{x}, t)\tilde{\psi}^\dagger(\mathbf{x})\gamma^{0'}\gamma^{\alpha'}\partial_j\tilde{\psi}(\mathbf{x}), \end{aligned} \tag{8.74}$$

and $\tilde{\varepsilon}_{\mu\nu}(\mathbf{x}, t)$ is given by

$$\begin{aligned} \tilde{\varepsilon}^{\mu\nu}(\mathbf{x}, t) = {} & -8i\pi G\int d^4x' G_{ret}(\mathbf{x}' - \mathbf{x}, t' - t)\tilde{U}(t', t)\tilde{\psi}^\dagger(\mathbf{x}')\left(\gamma^{0'}\gamma^{\mu'}\text{`}\partial'\text{'}^\nu + (\text{`}\partial'\text{'}^\mu)^\dagger\gamma^{0'}\gamma^{\nu'}\right. \\ & \left. -\frac{1}{2}r^{\mu\nu}\gamma^{0'}\gamma^{\alpha'}\text{`}\partial'\text{'}_\alpha - \frac{1}{2}r^{\mu\nu}(\text{`}\partial'\text{'}_\alpha)^\dagger\gamma^{0'}\gamma^{\alpha'}\right)\tilde{\psi}(\mathbf{x}')\tilde{U}^\dagger(t', t). \end{aligned} \tag{8.75}$$

The halving of the energy term is responsible for the factor of $\frac{1}{2}$, changing m to $\frac{m}{2}$.

Let $\tilde{V}(t)$ be the free particle evolution operator

$$\tilde{V} = \exp\left(-i\tilde{H}_0 t\right). \tag{8.76}$$

In the interaction picture states evolve according to an evolution operator $\tilde{W}(t_a, t_b)$, and operators evolve according to

$$\tilde{A}_{int}(t) = \tilde{V}^\dagger(t)\tilde{A}\tilde{V}(t), \tag{8.77}$$

where $\tilde{A}$ is the equivalent operator in the Schrödinger picture. In particular in the interaction picture $\psi(x)$ is given by the standard expression,

$$\psi(\mathbf{x}, t) = \int \frac{m^{\frac{1}{2}} d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}\sqrt[4]{\mathbf{k}^2 + m^2}} \sum_{\alpha=1,2}\left[c_\alpha(\mathbf{k})u_\alpha(\mathbf{k})e^{-ikx} + c^\dagger_{\alpha+2}(\mathbf{k})u_{\alpha+2}(\mathbf{k})e^{ikx}\right]. \tag{8.78}$$

In the interaction picture the interaction Hamiltonian is

$$\begin{aligned} \tilde{H}_{I,int}(t) = {} & \frac{1}{2}\int d^3\mathbf{x}\, \tilde{\varepsilon}^0_{\alpha,int}(\mathbf{x}, t)\psi^\dagger(\mathbf{x}, t)\gamma^{0'}\gamma^{\alpha'}\left[-i\gamma^{j'}\partial_j + \frac{m}{2}\right]\psi(\mathbf{x}, t) \\ & -\frac{i}{2}\int d^3\mathbf{x}\, \varepsilon^j_{\alpha,int}(\mathbf{x}, t)\psi^\dagger(\mathbf{x}, t)\gamma^{0'}\gamma^{\alpha'}\partial_j\psi(\mathbf{x}, t). \end{aligned} \tag{8.79}$$

Since

$$\tilde{V}^\dagger(t)\tilde{U}(t',t)\tilde{V}(t') = \tilde{V}^\dagger(t')\tilde{W}(t',t)\tilde{V}(t'), \tag{8.80}$$

we have

$$\tilde{\varepsilon}^{\mu\nu}_{int}(x) = -8i\pi G\int d^4x' G_{ret}(x'-x)\tilde{V}^\dagger(t')\tilde{W}(t',t)\tilde{V}(t')\psi^\dagger(x')\left(\gamma^{0'}\gamma^{\mu'}{}^{\text{`}}\partial'^{\nu}\right.$$
$$\left.+\left({}^{\text{`}}\partial'^{\mu}\right)^\dagger\gamma^{0'}\gamma^{\nu'} - \frac{1}{2}r^{\mu\nu}\gamma^{0'}\gamma^{\alpha'}{}^{\text{`}}\partial'_{\alpha} - \frac{1}{2}r^{\mu\nu}({}^{\text{`}}\partial'_{\alpha})^\dagger\gamma^{0'}\gamma^{\alpha'}\right)\psi(x')\tilde{V}^\dagger(t')\tilde{W}^\dagger(t',t)\tilde{V}(t'). \tag{8.81}$$

$\tilde{W}(t_a,t_b)$ can be expanded in terms of $\tilde{H}_{Iint}(t)$,

$$\tilde{W}(t_a,t_b) = (-i)^n\sum_{n=0}^{\infty}\int_{t_a}^{t_b}\tilde{H}_{I,int}(t_a)dt_a\int_{t_a}^{t_1}\tilde{H}_{I,int}(t_2)dt_2\ldots\int_{t_a}^{t_{n-1}}\tilde{H}_{I,int}(t_n). \tag{8.82}$$

In order to solve this equation we first substitute Eqs.(8.79,8.82). We then express $W(t_a,t_b)$ as

$$W(t_a,t_b) = 1+\sum_{n=1}^{\infty}W_n(t_a,t_b), \tag{8.83}$$

where $W_n(t_a,t_b)$ is of order $4n$ in $\psi(x)$. This allows Eq.(8.82) to be solved term by term for each $W_n(t_a,t_b)$. For instance

$$W_1(t_a,t_b) = -4\pi G\int_{t=t_a}^{t=t_b}d^4x\int d^4x' G_{ret}(x'-x)\psi^\dagger(x')\left(\gamma^{0'}\gamma^{0'}{}^{\text{`}}\partial'_{\alpha} + \left({}^{\text{`}}\partial'^{0}\right)^\dagger\gamma^{0'}\gamma_{\alpha}{}'\right.$$
$$\left.-\frac{1}{2}r^{0}_{\alpha}\gamma^{0'}\gamma^{\beta'}{}^{\text{`}}\partial'_{\beta} - \frac{1}{2}r_{0\alpha}({}^{\text{`}}\partial'_{\beta})^\dagger\gamma^{0'}\gamma^{\beta'}\right)\psi(x')\psi^\dagger(x)\gamma^{0'}\gamma^{\alpha'}\left[-i\gamma^{j'}\partial_j+\frac{m}{2}\right]\psi(x)$$
$$-4i\pi G\int_{t=t_a}^{t=t_b}d^4x\int d^4x' G_{ret}(x'-x)\psi^\dagger(x')\left(\gamma^{0'}\gamma^{j'}{}^{\text{`}}\partial'_{\alpha} + \left({}^{\text{`}}\partial'^{j}\right)^\dagger\gamma^{0'}\gamma_{\alpha}{}'\right.$$
$$\left.-\frac{1}{2}r^{j}_{\alpha}\gamma^{0'}\gamma^{\beta'}{}^{\text{`}}\partial'_{\beta} - \frac{1}{2}r^{j}_{\alpha}({}^{\text{`}}\partial'_{\beta})^\dagger\gamma^{0'}\gamma^{\beta'}\right)\psi(x')\psi^\dagger(x)\gamma^{0'}\gamma^{\alpha'}\partial_j\psi(x), \tag{8.84}$$

$W_2(t_a,t_b)$ can now be calculated in terms of $W_1(t_a,t_b)$, and so on for each n.

It is not proposed that the true theory of gravity is linear and so it is unlikely that any useful predictions can be made from this theory. The operator version of Einstein's equation

$$G_{H\mu\nu}(\mathbf{x},t) = 8\pi G T_{H\mu\nu}(\mathbf{x},t), \tag{8.85}$$

may have solutions for certain evolution operators $U(t_a,t_b)$, and so it may be possible to form a theory of quantum gravity this way. Alternatively it may be possible to form a theory by modifying Einstein's equation.

Chapter 9

Semi-Classical Gravity and Quantum State Reduction

9.1 Introduction

In the paper On Gravity's Role in Quantum State Reduction [13], Penrose argues that the definition of the time-translation operator for superposed space-times involves an inherent ill-definedness. This leads to an uncertainty in the energy of the superposed state, and means that the superposition is unstable. He argues that this uncertainty is proportional to the gravitational self-energy Δ of the difference of the mass distributions. Δ is related to a fundamental energy uncertainty E_Δ of the superposition of space-times, and he suggests a life-time of the order of $T = \hbar/E_\Delta$.

This chapter examines the motion of particles subject to a semi-classical gravitational interaction. It is shown that the gravitational self energy proposed by Penrose occurs in semi-classical gravity. It is however not clear how such a collapse can happen.

9.2 Quantum State Reduction

Previously it has been supposed that states are of the form

$$|\Psi>, \tag{9.1}$$

that is they contain only a matter part, and no part relating to the gravitational field. Penrose assumes that states are of the form

$$|\Psi>|G_\psi>, \tag{9.2}$$

that is the gravitational field is part of the state. This assumption is central to his conclusions.

Penrose considers the system discussed in chapter 7, that of a single gravitating particle moving in the background field of the Earth. He considers the situation when this particle is in a superposition of two locations, which he thinks of as two lumps. Let the states $|\Psi>$ and $|X>$ each represent this particle at alternative locations, i.e. each represent a lump. In chapter 7 the system would have been described by the state

$$|\Psi> + |X>. \tag{9.3}$$

Penrose uses

$$|\Psi>|G_\psi> + |X>|G_\chi>, \tag{9.4}$$

which contains two superposed gravitational fields. He argues that there are inherent difficulties in describing superposed space-times. These difficulties are related to the fact that any pointwise identification between two different space-times is co-ordinate dependent and there is no natural such identification. This leads to an inherent instability in the superposed state which results in quantum state reduction.

In order to measure the effect of quantum state reduction, Penrose considers the Newtonian approximation of gravity. He writes the acceleration at the point $\mathbf{x}$ of space caused by each of the lumps as $\mathbf{f}(\mathrm{x})$ and $\mathbf{f}'(\mathrm{x})$ respectively. The scalar quantity

$$(\mathbf{f}(\mathrm{x}) - \mathbf{f}'(\mathrm{x}))^2 = (\mathbf{f}(\mathrm{x}) - \mathbf{f}'(\mathrm{x})) . (\mathbf{f}(\mathrm{x}) - \mathbf{f}'(\mathrm{x})), \tag{9.5}$$

is taken as a measure of the incompatibility of the pointwise identification of the two space-times. Since the Newtonian approximation is being used the acceleration field of the first lump can be described using a potential,

$$\mathbf{f} = -\nabla\Phi, \tag{9.6}$$

where Φ obeys

$$\nabla^2\Phi = -4\pi G\rho, \tag{9.7}$$

and ρ is the mass density of the first lump. Similarly the quantities Φ' and ρ' apply to the second lump. It is shown that if we define Δ by

$$\Delta = \int d^3\mathbf{x}\left(\mathbf{f} - \mathbf{f}'\right)^2, \tag{9.8}$$

then

$$\Delta = -4\pi G \int d^3\mathbf{x} d^3\mathbf{y}\, \frac{\left(\rho(\mathbf{x}) - \rho'(\mathbf{x})\right)\left(\rho(\mathbf{y}) - \rho'(\mathbf{y})\right)}{|\mathbf{x} - \mathbf{y}|}. \tag{9.9}$$

Penrose argues that the quantity Δ is related to a fundamental energy uncertainty of the superposition of the particles E_Δ, and that this gives superposed state a life-time of the order $T = \hbar/E_\Delta$.

Penrose is very interested in the origins of consciousness and argues in [31] that quantum theory and quantum state reduction is necessary for consciousness. The brain contains large numbers of tubulins and there is good evidence that links these with cognitive function. These can be pictured very much like a telephone receiver, two spheres connected by a 'handle'. Penrose considers the tubulins in two states, one which has a slightly longer 'handle' than the other. Let $2r$ be the difference in the length of the 'handle' (so each sphere is offset by r) and a be the radius of the spheres. We approximate the tubulins by the two spheres (i.e. assume there is no mass in the handle), and consider the case where $r < a$. Under these conditions the formula

$$\Delta = m^2 G\left(\frac{r^2}{2a^3} - \frac{3r^3}{16a^4} + \frac{r^5}{160a^6}\right), \tag{9.10}$$

is quoted for the Penrose self energy of two of the spheres. The self energy of the superposed state is twice this value.

The purpose of this chapter is to show the relation between Penrose's self energy and the self energy which occurs in semi-classical gravity. In particular we see how Eqs.(9.9,9.10) can be replicated in the context of semi-classical gravity.

9.3 Semi-Classical Gravity

In Semi-classical gravity Einstein's tensor is equated to the expectation value of the stress-energy operator,

$$G_{\mu\nu}(\mathbf{x},t) = 8\pi G < \Psi, t | T_{\mu\nu}(\mathbf{x},t) | \Psi, t>, \tag{9.11}$$

where it is assumed that $|\Psi, t>$ is normalised. This means that states can be expressed in the form

$$|\Psi, t>. \tag{9.12}$$

They do not contain a part which refers to the gravitational field. The metric $g_{\mu\nu}(\mathbf{x},t)$ is also not an operator but has a single definite value, and is the metric used in classical general relativity.

We now consider the motion of particles interacting through semi-classical gravity. To define such a system of particles we use a system of equations very similar to those used in the previous chapter. The state obeys the evolution equation Eq.(8.37), where the $\gamma^\mu(\mathbf{x},t)$ matrices are given by Eq.(8.38). Finally the metric, which is now not an operator, is given by Eq.(9.11). The matter equation Eq.(8.37) at first sight appears linear in the state $|\Psi, t>$. However the metric is now dependent upon $|\Psi, t>$, and this destroys the linearity.

The equations of motion are very complicated. As we have seen it is not clear that the direct operator version of Einstein's equation is solvable in general. In fact it is not clear that Eq.(9.11) is solvable in general either. There are also other problems. For instance

$$< 0 | T_{\mu\nu}(\mathbf{x},t) | 0> = 0, \tag{9.13}$$

where $|0>$ is the mathematical vacuum state, that is the state where there are no positive or negative energy particles. However

$$_{new}< 0|T_{\mu\nu}(\mathbf{x}, t)|0>_{new} \tag{9.14}$$

is divergent, where $|0>_{new}$ is the observed vacuum state, that is the state which contains no particles or anti-particles. The successful results of General Relativity, such as the Schwarzschild solution, are calculated using a stress energy tensor which is zero where there are no observed particles. In order to reproduce these results we must remove this divergence.

However the purpose of this chapter is not to discuss these difficulties. Instead we concentrate on the Newtonian approximation.

9.4 The Newtonian Approximation

This non-linearity of semi-classical gravity has a profound impact upon the motion. We first note that a linear equation which has two solutions $|\Psi_1, t>$ and $|\Psi_2, t>$ also has the solution

$$|\Psi_1, t> + |\Psi_2, t>. \tag{9.15}$$

The two states $|\Psi_1, t>$ and $|\Psi_2, t>$ have no effect on the other's motion in the superposition. In semi-classical gravity this is not the case. In order to see the exact effect we use the Newtonian approximation.

The Newtonian approximation of semi-classical gravity can be formed using a Schrödinger equation in which particles are moving in a gravitational potential. The gravitational potential energy obeys

$$\nabla^2 V(\mathbf{x}) = 4\pi m G < \Psi, t|\rho(\mathbf{x})|\Psi, t>. \tag{9.16}$$

For simplicity we consider scalar particles. Using the creation and annihilation operators introduced in chapter 4 the mass density operator is

$$\rho(\mathbf{x}) = m\phi^\dagger(\mathbf{x})\phi(\mathbf{x}). \tag{9.17}$$

Let $|\Psi>$ be an n particle state given by

$$|\Psi> = \frac{1}{\sqrt{n!}}\int d^3\mathbf{x}_1 \ldots \int d^3\mathbf{x}_n \Psi(\mathbf{x}_1, \ldots \mathbf{x}_n)|\mathbf{x}_1, \ldots, \mathbf{x}_n>, \tag{9.18}$$

where $\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ is symmetric in all its variables. We see that

$$< \Psi|\rho(\mathbf{x})|\Psi> = m\sum_{j=1}^{n}\int d^3\mathbf{x}_1 \ldots d^3\mathbf{x}_n \delta^3(\mathbf{x}_j - \mathbf{x})|\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_n)|^2. \tag{9.19}$$

Remember $|\mathbf{x}_1, \ldots, \mathbf{x}_n>$ is a state representing n bosons, particles which are not identifiable, and are normalised in the manner of the states in Chapter 2 (see Eqs(4.7,4.8)). The factor of $\frac{1}{\sqrt{n!}}$ in Eq.(9.18) is to take account of this. Eq.(9.16) now has a solution

$$V(\mathbf{x}) = -mG\int d^3\mathbf{x}'\frac{< \Psi|\rho(\mathbf{x}')|\Psi>}{|\mathbf{x} - \mathbf{x}'|} = -m^2G\sum_{j=1}^{n}\int d^3\mathbf{x}_1 \ldots d^3\mathbf{x}_n \frac{|\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_n)|^2}{|\mathbf{x} - \mathbf{x}_j|}. \tag{9.20}$$

In order to see how this effects the motion of particles we consider two states $|\Psi_c>$ and $|\Psi_d>$, and their superposition, $\frac{1}{\sqrt{2}}(|\Psi_c> + |\Psi_d>)$. We let the states $|\Psi_c>$ and $|\Psi_d>$ each represent the same macroscopic object in different positions. This object consists of a large number of much smaller particles which are each centred around a single point in space with a very small spread.

In order to define these states precisely we let $\chi(\mathbf{x})$ be the Gaussian function

$$\chi(\mathbf{x}) = \left(a^2\pi\right)^{-\frac{1}{4}} \exp\frac{-\mathbf{x}^2}{2a^2}. \tag{9.21}$$

a is a number which approximates the width of the wave-packet. Before the calculations are started we must first note three mathematical properties of $\chi(\mathbf{x})$. The first is that the Gaussian is normalised to unity

$$\int d^3\mathbf{x}|\chi(\mathbf{x})|^2 = 1. \tag{9.22}$$

Since $|\chi(\mathbf{x})|^2$ is peaked at $\mathbf{x} = \mathbf{0}$, is approximately zero for $|\mathbf{x}| \gg |\mathbf{a}|$ and obeys Eq.(9.22), it approximates $\delta^3(\mathbf{x})$. In particular

$$\int d^3\mathbf{x}|\chi(\mathbf{x} - \mathbf{c})|^2 f(\mathbf{x}) \approx f(\mathbf{c}), \tag{9.23}$$

for any function $f(\mathbf{x})$ which is 'reasonably constant' around the region $|\mathbf{x}| < |\mathbf{a}|$. We also note that if $|\mathbf{c} - \mathbf{d}| \gg |\mathbf{a}|$ then

$$\chi(\mathbf{x} - \mathbf{c})\chi(\mathbf{x} - \mathbf{d}) \approx 0, \qquad \int d^3\mathbf{x}\, \chi(\mathbf{x} - \mathbf{c})\chi(\mathbf{x} - \mathbf{d}) \approx 0. \tag{9.24}$$

We can now define the states $|\Psi_c>$ and $|\Psi_d>$. This is done by defining their wave-functions as

$$\Psi_c(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \sum \prod_{j=1}^{n} \chi(\mathbf{x}_j - \mathbf{c}_j), \qquad \Psi_d(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \sum \prod_{j=1}^{n} \chi(\mathbf{x}_j - \mathbf{d}_j), \tag{9.25}$$

where the summation denotes symmetrisation. These represent a macroscopic object which consists of n small particles. The first has the particles at positions $\mathbf{c}_1, \ldots, \mathbf{c}_n$, and the second at $\mathbf{d}_1, \ldots, \mathbf{d}_n$. $|\Psi_c>$ and $|\Psi_d>$ are their respective states i.e.

$$|\Psi_c> = \frac{1}{\sqrt{n!}} \int d^3\mathbf{x}_1 \ldots \int d^3\mathbf{x}_n \Psi_c(\mathbf{x}_1, \ldots \mathbf{x}_n)|\mathbf{x}_1, \ldots, \mathbf{x}_n>,$$

$$|\Psi_d> = \frac{1}{\sqrt{n!}} \int d^3\mathbf{x}_1 \ldots \int d^3\mathbf{x}_n \Psi_d(\mathbf{x}_1, \ldots \mathbf{x}_n)|\mathbf{x}_1, \ldots, \mathbf{x}_n>. \tag{9.26}$$

We use $< \rho(\mathbf{x}) >_c$ and $< \rho(\mathbf{x}) >_d$ to denote the expectation value of the mass density field for the c and d states respectively. From Eq.(9.19),

$$< \rho(\mathbf{x}) >_c = m \sum_{j=1}^{n} |\chi(\mathbf{x} - \mathbf{c}_j)|^2, \qquad < \rho(\mathbf{x}) >_d = m \sum_{j=1}^{n} |\chi(\mathbf{x} - \mathbf{d}_j)|^2. \tag{9.27}$$

Each state has its own potential which is real valued (i.e. is not an operator). The potential associated with $|\Psi_c>$ is

$$V_c(\mathbf{x}) = -mG \int \frac{d^3\mathbf{x} < \rho(x) >_c}{|\mathbf{x} - \mathbf{x}'|} = -m^2 G \sum_{j=1}^{n} \int d^3\mathbf{x}' \frac{|\chi(\mathbf{x}' - \mathbf{c}_j)|^2}{|\mathbf{x}' - \mathbf{x}|}, \tag{9.28}$$

and using Eq.(9.23) with $f(\mathbf{x}) = \frac{1}{|\mathbf{x}' - \mathbf{x}|}$

$$V_c(\mathbf{x}) \approx -m^2 G \sum_{j=1}^{n} \frac{1}{|\mathbf{x} - \mathbf{c}_j|}. \tag{9.29}$$

This is the Newtonian potential at $\mathbf{x}$ caused by particles at the positions $\mathbf{c}_1, \ldots, \mathbf{c}_n$.

Now consider the state

$$|\Psi> = \frac{1}{\sqrt{2}} (|\Psi_c> + |\Psi_d>). \tag{9.30}$$

Using Eqs.(9.19,9.24) we see that as long as one of the particles that make up the states $|\Psi_c>$ and $|\Psi_d>$ are in separate places, that is $|\mathbf{c}_j - \mathbf{d}_k| \gg |\mathbf{a}|$ for every k and at least one value of j[1],

$$< \Psi_c|\rho(\mathbf{x})|\Psi_d> \approx 0. \tag{9.31}$$

This means that

$$< \Psi|\rho(\mathbf{x})|\Psi> = \frac{1}{2} < \rho(\mathbf{x}) >_c + \frac{1}{2} < \rho(\mathbf{x}) >_d = \frac{m}{2}\sum_{j=1}^{n}\left[|\chi(\mathbf{x}-\mathbf{c}_j)|^2 + |\chi(\mathbf{x}-\mathbf{d}_j)|^2\right], \tag{9.32}$$

and the potential of the state $|\Psi>$ is given by

$$V_{c+d}(\mathbf{x}) \approx -\frac{m^2G}{2}\sum_{j=1}^{n}\left[\frac{1}{|\mathbf{x}-\mathbf{c}_j|} + \frac{1}{|\mathbf{x}-\mathbf{d}_j|}\right]. \tag{9.33}$$

This is half the potential which is caused classically by $2n$ particles at the positions $\mathbf{c}_1, \ldots, \mathbf{c}_n$ and $\mathbf{d}_1, \ldots, \mathbf{d}_n$.

This demonstrates directly the non-linearity of the theory. The states $|\Psi_d>$ and $|\Psi_c>$ evolve through time with each particle being attracted by the other $n-1$ particles. However the state $|\Psi>$ evolves through time so that each particle is attracted to the $2n$ points occupied by the n particles in Ψ_c and the n particles in Ψ_d. We use the notation $|\Psi, t>$ to denote the evolution of the state $|\Psi>$ through time so that $|\Psi, 0> = |\Psi>$. Although by definition $|\Psi> = |\Psi_c> + |\Psi_d>$, in general $|\Psi, t> \neq |\Psi_c, t> + |\Psi_d, t>$.

Semi-classical gravity leads to some very bizarre results. For instance we can use states in which the number of particles is very large and where $|\Psi_c>$ and $|\Psi_d>$ represent a galaxy. Since the size galaxies is so large we can assume that the wave-packets act very much like classical particles. The state $|\Psi_c> + |\Psi_d>$ will evolve so that each of the wave-packets orbits the other. When such a system is observed with light we would see a single galaxy orbiting nothing.

[1] remember the particles are not identifiable so one of the $\mathbf{c}_j$ has to be away from all the d_k

9.5 Gravitational Self-Energy

This section discusses the self energy of the macroscopic particle introduced in the previous section in its superposed state. It is shown that the self-energy expression proposed by Penrose occurs in semi-classical gravity in the Newtonian approximation.

The total potential energy of a state $|\Psi>$ is half the sum of the potential energy at the point of each of the particles. This is easy to calculate for $|\Psi_c>$ the state with n particles at positions $\mathbf{c}_1, \ldots, \mathbf{c}_n$, and is

$$\frac{1}{2}\sum_{j=1}^{n} V_c(\mathbf{c}_j) \approx -m^2 G \sum_{j,k=1,j<k}^{n} \frac{1}{|\mathbf{c}_j - \mathbf{c}_k|}. \tag{9.34}$$

The 'total potential' operator in chapter 4 (now denoted by V_T) was written as

$$V_T = \frac{1}{2}\int d^3\mathbf{x}\, \phi^\dagger(\mathbf{x})\phi(\mathbf{x})V(\mathbf{x}) = \frac{1}{2m}\int d^3\mathbf{x}\, \rho(\mathbf{x})V(\mathbf{x}), \tag{9.35}$$

and since

$$V(\mathbf{x}) = -mG\int d^3\mathbf{x}' \frac{< \rho(\mathbf{x}') >}{|\mathbf{x} - \mathbf{x}'|}, \tag{9.36}$$

the expectation value of the 'total potential' operator $< V_T >$ is

$$-G\int d^3\mathbf{x} d^3\mathbf{y} \frac{< \rho(\mathbf{x}) >< \rho(\mathbf{y}) >}{2|\mathbf{x} - \mathbf{y}|}. \tag{9.37}$$

From Eq.(9.32) we see that for the state $|\Psi> = \frac{1}{\sqrt{2}}(|\Psi_c> + |\Psi_d>)$ this expectation value can be written as

$$< V_T > = -\frac{G}{8}\int d^3\mathbf{x} d^3\mathbf{y}\, \frac{(< \rho(\mathbf{x}) >_c + < \rho(\mathbf{x}) >_d)(< \rho(\mathbf{y}) >_c + < \rho(\mathbf{y}) >_d)}{|\mathbf{x} - \mathbf{y}|}, \tag{9.38}$$

which is the gravitational self energy caused by the sum of the expectation values of the mass distributions. We use the notation

$$< V_T >_c = < \Psi_c|V_T|\Psi_c>, \qquad < V_T >_d = < \Psi_d|V_T|\Psi_d>, \tag{9.39}$$

define

$$\Delta = -4\pi G\int d^3\mathbf{x} d^3\mathbf{y}\, \frac{(< \rho(\mathbf{x}) >_c - < \rho(\mathbf{x}) >_d)(< \rho(\mathbf{y}) >_c - < \rho(\mathbf{y}) >_d)}{|\mathbf{x} - \mathbf{y}|}, \tag{9.40}$$

and note that

$$\frac{\Delta}{8\pi} = < V_T >_c + < V_T >_d + G \int d^3x d^3y \frac{< \rho(\mathbf{x}) >_c < \rho(\mathbf{y}) >_d}{|\mathbf{x} - \mathbf{y}|}. \tag{9.41}$$

Using Eq.(9.38) and Eq.(9.41) we conclude

$$\begin{aligned} < V_T > &= \frac{1}{4} < V_T >_c + \frac{1}{4} < V_T >_d - \frac{G}{4} \int d^3x d^3y \frac{< \rho(\mathbf{x}) >_c < \rho(\mathbf{y}) >_d}{|\mathbf{x} - \mathbf{y}|} \\ &= \frac{1}{2} < V_T >_c + \frac{1}{2} < V_T >_d - \frac{\Delta}{32\pi}. \end{aligned} \tag{9.42}$$

We see that $< V_T >$ is half the sum of the expectation values of the total energy operator of the c and d particles, and an additional term $-\Delta/32\pi$. This additional term is a self energy term and results from the interaction of the two superposed parts (that is $|\Psi_c>$ and $|\Psi_d>$) within the total state $|\Psi>$. This is a direct result of the non-linearity of the theory. If we equate

$$\rho(\mathbf{x}) \equiv < \rho(\mathbf{x}) >_c, \qquad \rho'(\mathbf{x}) \equiv < \rho(\mathbf{x}) >_d, \tag{9.43}$$

then Δ is given by Eq.(9.9), that is it is Penrose's self energy.

9.6 Self Energy and Consciousness

This section performs a calculation discussed by Penrose which obtains an approximate expression for the Penrose self energy of a tubulin. Penrose argues that this is related to consciousness. The calculation Penrose actually discusses is the self energy of a displaced sphere, where the displacement is less than the sphere's radius. The self energy obtained is half the self energy of a superposition of tubulin states.

We let a be the radius of the spheres and r be their displacement, where $r < a$. We treat these spheres in the same way as the macroscopic objects of the previous section, that is they are made up of a large number of smaller particles each localised about a point. We assume these particles are evenly distributed about the sphere, and take the continuum limit. If each of the spheres occupy a volume A and B then the Penrose self energy is given by the double integral

$$\Delta = -4\pi G \rho^2 \left[\int_A d^3\mathbf{x} - \int_B d^3\mathbf{x} \right] \left[\int_A d^3\mathbf{y} - \int_B d^3\mathbf{y} \right] \frac{1}{|\mathbf{x} - \mathbf{y}|}$$

$$= -4\pi G\rho^2 \int_A d^3\mathbf{x} \int_A d^3\mathbf{y} \frac{1}{|\mathbf{x}-\mathbf{y}|} - 4\pi G\rho^2 \int_B d^3\mathbf{x} \int_B d^3\mathbf{y} \frac{1}{|\mathbf{x}-\mathbf{y}|}$$
$$+ 8\pi G\rho^2 \int_A d^3\mathbf{x} \int_B d^3\mathbf{y} \frac{1}{|\mathbf{x}-\mathbf{y}|}, \quad (9.44)$$

where ρ is the density of the spheres. Since the spheres A and B are of the same volume,

$$-\frac{\Delta}{8\pi} = G\rho^2 \int_A d^3\mathbf{x} \int_A d^3\mathbf{y} \frac{1}{|\mathbf{x}-\mathbf{y}|} - G\rho^2 \int_A d^3\mathbf{x} \int_B d^3\mathbf{y} \frac{1}{|\mathbf{x}-\mathbf{y}|}. \quad (9.45)$$

We take the two terms on the right hand side of Eq.(9.45) separately, and first consider

$$G\rho^2 \int_A d^3\mathbf{x} \int_B d^3\mathbf{y} \frac{1}{|\mathbf{x}-\mathbf{y}|}. \quad (9.46)$$

One of the integrals can be done straight away. Remembering

$$m = \frac{4\rho\pi a^3}{3}, \quad (9.47)$$

we note that

$$\rho \int_{|\mathbf{y}|<a} \frac{d^3\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} = \begin{cases} \frac{3m}{2a} - \frac{m|\mathbf{x}|^2}{2a^3} & |\mathbf{x}| \leq a, \\ \frac{m}{|\mathbf{x}|} & |\mathbf{x}| > a. \end{cases} \quad (9.48)$$

If we consider co-ordinates in which B is centred at the origin then $\mathbf{y}$ in Eq.(9.48) integrates over B and we can turn Eq.(9.46) into a single integral in which $\mathbf{x}$ integrates over A.

We divide A up into 3 sections, K, L, M, as shown in fig.(9.1). We use the natural spherical polar co-ordinates of the sphere A, (R, θ, ϕ). We define d as the distance from the centre of sphere A to the circumference of sphere B, which is given by

$$d = r\cos\theta + \sqrt{r^2\cos^2\theta + a^2 - r^2}. \quad (9.49)$$

We also note that the values of θ dividing the sections K and L obey $\cos\theta = \frac{r}{2a}$. The contribution from section K is

$$mG\rho \int_K d\mathbf{x}^3 \left(\frac{3}{2a} - \frac{R^2}{2a^3}\right) = m\rho \int_0^{2\pi} d\phi \int_{-1}^{\frac{r^2}{2a}} d\cos\theta \int_0^d R^2\, dR \left(\frac{3}{2a} - \frac{R^2}{2a^3}\right),$$
$$= 2\pi m G\rho \left(\frac{2a^2}{5} - \frac{3ra}{10} - \frac{r^2}{6} + \frac{r^3}{8a} - \frac{r^5}{240a^3}\right). \quad (9.50)$$

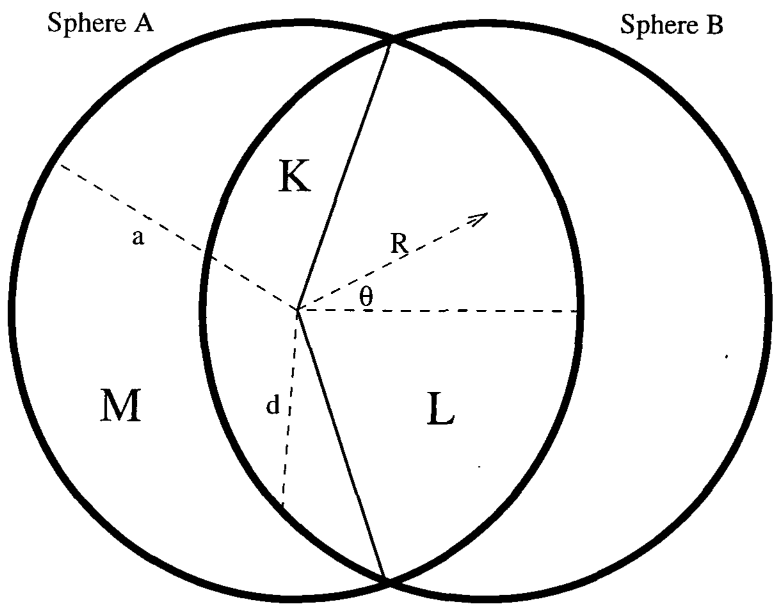

Figure 9.1: Diagram displaying spheres A and B and the three sections of sphere A.

The contribution from section L is

$$mG\rho \int_L d^3\mathbf{x} \left(\frac{3}{2a} - \frac{R^2}{2a^3} \right) = \int_0^{2\pi} d\phi \int_{\frac{r}{2a}}^1 d\cos\theta \int_0^a dR\, R^2 \left(\frac{3}{2a} - \frac{R^2}{2a^3} \right),$$

$$= 2\pi mG\rho \left(\frac{2a^2}{5} - \frac{ra}{5} \right). \tag{9.51}$$

Finally, the contribution from M is

$$mG\rho \int_M \frac{d^3\mathbf{x}}{R} = \int_0^{2\pi} d\phi \int_{\frac{r}{2a}}^1 d\cos\theta \int_a^d dR\, R = 2\pi mG\rho \left(\frac{3ar - r^2}{6} \right). \tag{9.52}$$

The details of these calculations are contained in appendix C. Summing the three terms gives

$$2\pi mG\rho \left(\frac{r^5}{240a^3} - \frac{r^3}{8a} + \frac{r^2}{3} - \frac{r^3}{8a} - \frac{4a^2}{5} \right) = m^2 G \left(\frac{r^5}{160a^6} - \frac{3r^3}{16a^4} + \frac{r^2}{2a^3} - \frac{6}{5a} \right). \tag{9.53}$$

The other term in Eq.(9.45) is

$$G\rho^2 \int_A d^3\mathbf{x} \int_A d^3\mathbf{y} \frac{1}{|\mathbf{x} - \mathbf{y}|}. \tag{9.54}$$

Using Eq.(9.48) this can be evaluated directly, and is

$$m\rho G \int_A d^3\mathbf{x} \left(\frac{3m}{2a} - \frac{m\mathbf{x}^2}{2a^3} \right) = mG\rho \int_0^{2\pi} d\phi \int_{-1}^1 \cos\theta \int_0^a dR\, R^2 \left(\frac{3}{2a} - \frac{R^2}{2a^3} \right) = \frac{6m^2}{5a}. \tag{9.55}$$

Hence, summing these two integrals we find

$$\frac{\Delta}{8\pi} = m^2 G \left(\frac{r^2}{2a^3} - \frac{3r^3}{16a^4} + \frac{r^5}{160a^6} \right) . \tag{9.56}$$

Comparing this with Eq.(9.10), we see the formula quoted by Penrose is the right hand side of Eq.(9.56) and the two answers differ by a factor of 8π. The reason for this discrepancy is unclear, since Penrose does not give the details of his calculation.

Chapter 10

Conclusions

This thesis has discussed the problems within quantum mechanics caused by Lorentz invariance, the Bohm model and the existence of photons (gravitons).

Two modifications of the Bohm model were proposed, one which takes account of Lorentz invariance, and another which is compatible with the existence of photons. The main part of the thesis was concerned with a description of the interactions of Dirac particles which do not need gauge particles, and so is in the spirit of the Bohm model. A method of quantising the electromagnetic field was developed which contains no photons. This leads to a self consistent theory of quantum electromagnetism that it is not equivalent to standard QED. These techniques can be successfully extended to linear gravity. A theory of quantum linear gravity was produced which contains no gravitons. Finally issues relating to quantum state reduction were discussed.

The thesis started by considering wave equations of Dirac particles. Many Dirac particles can be described by spinors where each particle has its own spinor index, so for instance n Dirac particles can be described by $\Psi_{(\alpha_1\ldots\alpha_n)}(\mathbf{x}_1,\mathbf{x}_2\ldots,\mathbf{x}_n,t)$. Exactly how many Dirac particles should be described is not obvious. The equation which was found to be most useful is the single-time equation

$$i\hbar\frac{\partial}{\partial t}\Psi_{n(\alpha_1\ldots\alpha_n)}(\mathbf{x}_1,\ldots,\mathbf{x}_n,t) = H_{1(\alpha_1\beta)}\Psi_{(\beta\ldots\alpha_n)}(\mathbf{x}_1,\ldots,\mathbf{x}_n,t)$$
$$+\ldots+H_{n(\alpha_n\beta)}\Psi_{n(\alpha_1\ldots\beta)}(\mathbf{x}_1,\ldots,\mathbf{x}_n,t). \qquad (10.1)$$

$H_{k(\alpha\beta)}$ are the matrix elements of the Dirac Hamiltonian for particle k which can contain background electromagnetic and/or gravitational fields. This equation has been proposed by Bohm and Hiley.

The description of Dirac particles using field theory notation was then discussed. The are 4 different types of Dirac particles and 4 independent operator fields $\psi_\alpha(\mathbf{x})$ are introduced which obey the anti-commutation relations

$$\{\psi_\alpha(\mathbf{x}), \psi_\beta^\dagger(\mathbf{x}')\} = \delta_{\alpha\beta}\delta^3(\mathbf{x}-\mathbf{x}'), \qquad \{\psi_\alpha(\mathbf{x}), \psi_\beta(\mathbf{x}')\} = 0. \tag{10.2}$$

This notation can be used to express the motion of Dirac particles. It was shown that the n-particle components of the equation

$$i\hbar\frac{\partial}{\partial t}|\Psi, t\rangle = \int d^3\mathbf{x}\, \psi_\alpha^\dagger(\mathbf{x}) H_{(\alpha\beta)} \psi_\beta(\mathbf{x}) |\Psi, t\rangle, \tag{10.3}$$

are Eq.(10.1), where $|\Psi, t\rangle$ is expanded in terms of the wave-functions Ψ_n. The H in Eq.(10.3) is the Hamiltonian of each single particle and is dependent on $\mathbf{x}$ and its derivatives. The particles in Eq.(10.3) are identical. The H_k in Eq.(10.1) is the Hamiltonian of particle (k), and is formed from H by replacing $\mathbf{x}$ with $\mathbf{x}_k$. This implies that Eq.(10.3) can be used to describe the evolution of the state vector $|\Psi, t\rangle$ through time. The advantage of Eq.(10.3) is that it allows us to describe interactions, whereas Eq.(10.1) admits only background fields.

To produce electromagnetic interactions we must find an electromagnetic field operator. Classically the electromagnetic field can be expressed by the electromagnetic field tensor $F^{\mu\nu}$ which obeys

$$\partial_\mu F^{\mu\nu} = j^\nu, \tag{10.4}$$

where j^ν is the 4-current vector. QED describes interactions by imposing commutation relations on the electromagnetic field operator. It regards photons as real particles. The system discussed in this thesis forms electromagnetic interactions by imposing an operator version of Eq.(10.4) in the Heisenberg representation,

$$\partial_\mu F_H^{\mu\nu}(\mathbf{x}, t) = j_H^\nu(\mathbf{x}, t). \tag{10.5}$$

This equation can be solved and we gain an expression for the electromagnetic field operator in terms of the $\psi_\alpha(\mathbf{x})$ fields. The electromagnetic field operator is analogous to the classical electromagnetic field and photons do not have an independent existence.

The success of such a theory can only be determined by its experimental predictions, and how closely it agrees with standard QED. There is a choice of possible 4-potential operators, and the one which appears to give the best results is chosen. The scattering matrix is calculated for this 4-potential, and it is compared to the scattering matrix of standard QED. The matrix elements between fermion states are found to be in agreement for the terms of the order of e^2 but do not agree at the order of e^4.

The reason the theory does not agree with standard QED is because the electromagnetic field is treated very differently. The Dirac field is also introduced in a very different way. The Dirac field $\psi_\alpha(\mathbf{x})$ defined in this thesis can be expressed as

$$\psi_\alpha(\mathbf{x}) = (2\pi\hbar)^{-\frac{3}{2}} \int d^3\mathbf{p} \exp \frac{i\mathbf{p}.\mathbf{x}}{\hbar} c_\alpha(\mathbf{p}), \qquad \alpha = 1,2,3,4. \tag{10.6}$$

In the standard theory the Dirac field is given by

$$\psi(\mathbf{x},t) = \int \frac{m^{\frac{1}{2}} d^3\mathbf{k}}{(2\pi\hbar)^{\frac{3}{2}} \sqrt[4]{\mathbf{k}^2+m^2}} \sum_{\alpha=1,2} \left[c_\alpha(\mathbf{k}) u_\alpha(\mathbf{k}) \exp \frac{-ikx}{\hbar} + c^\dagger_{\alpha+2}(\mathbf{k}) v_\alpha(\mathbf{k}) \exp \frac{ikx}{\hbar} \right]. \tag{10.7}$$

It has been shown that the difference of these fields is effectively only one of representation. It is expected that this treatment of the Dirac field can be combined with the standard photon field, and that a theory of QED can be defined based around an evolution equation of the form of Eq.(10.3), and that such a theory would give the same experimental predictions as standard QED.

The failure to produce an accurate theory of electromagnetism by solving the field equations explicitly does not automatically mean that the same techniques will fail with gravity. It is well known that there are many difficulties in constructing a theory of quantum gravity in the conventional way, so this alternative method is worth trying.

Dirac particles in a background gravitational field given by a metric $g_{\mu\nu}(\mathbf{x}, t)$ can be described using an equation of the form of Eq.(10.1). As before this is equivalent to the state vector equation Eq.(10.3). To form gravitational interactions we simply replace the background metric in Eq.(10.3) with a metric operator. In linear gravity the metric operator obeys an operator version of the linearised Einstein's equation, an equation which can be solved in general. The unlinearised Einstein's equation in general is not solvable and it is uncertain as to whether these techniques can be extended to a full theory of quantum gravity.

Penrose [13] argues that any theory of quantum gravity must necessarily lead to quantum state reduction. His arguments are based upon the assumption that part of the state represents the gravitational field (i.e. the assumption that gravitons exist). This assumption is violated by the theory developed here and indeed they do not involve quantum state reduction. Nevertheless his arguments should not be dismissed. There is no evidence either way as to whether gravitons actually do exist. Penrose argues that the collapse time of a superposition of states is related to its gravitational self energy. It is shown that this self energy occurs in semi-classical gravity. In semi-classical gravity this self energy acts as an attractive force between superposed states, and there is no such collapse in semi-classical gravity. It is unclear how the mechanism for collapse might occur in a full theory of quantum gravity.

Appendix A

The Equivalence of the Two Descriptions of Dirac Particles

This appendix shows that from the description of Dirac particles involving creation and annihilation operators, Eq.(5.13), contains the wave equations, Eq.(5.12).

Let a general state of position be denoted by

$$|\mathrm{x}_1(\alpha_1), \mathrm{x}_2(\alpha_2), \ldots, \mathrm{x}_n(\alpha_n)> = \psi^\dagger_{\alpha_1}(\mathrm{x}_1)\psi^\dagger_{\alpha_2}(\mathrm{x}_2)\ldots\psi^\dagger_{\alpha_n}(\mathrm{x}_n)|0>. \tag{A.1}$$

The indices α_k, which are 1, 2, 3 or 4, denote the type of each particle and correspond to spinor components. From Eq.(5.1) we can deduce the commutation relation

$$[\psi^\dagger_\beta(\mathrm{x})\psi_\tau(\mathrm{x}), \psi^\dagger_\alpha(\mathrm{x}')] = \psi^\dagger_\beta(\mathrm{x})\delta_{\tau\alpha}\delta^3(\mathrm{x}-\mathrm{x}'). \tag{A.2}$$

Commuting each of the $\psi_{\alpha_k}(\mathrm{x}_k)$, from which the state is built, with the $\psi^\dagger_\beta(\mathrm{x})\psi_\tau(\mathrm{x})$ we find

$$\psi^\dagger_\beta(\mathrm{x})\psi_\tau(\mathrm{x})|\mathrm{x}_1(\alpha_1), \mathrm{x}_2(\alpha_2), \ldots, \mathrm{x}_n(\alpha_n)>$$
$$= \sum_{k=1}^{n} \delta^3(\mathrm{x}-\mathrm{x}_k)\delta_{\alpha_k\tau}|\mathrm{x}_1(\alpha_1), \ldots, \mathrm{x}_{k-1}(\alpha_{k-1}), \mathrm{x}(\beta), \mathrm{x}_{k+1}(\alpha_{k+1}), \ldots, \mathrm{x}_n(\alpha_n)>. \tag{A.3}$$

We also need results which include ∂_j. By differentiating Eq.(5.1) we can deduce the commutation relation

$$[\psi^\dagger_\beta(\mathrm{x})\partial_j\psi_\tau(\mathrm{x}), \psi^\dagger_\alpha(\mathrm{x}')] = \psi^\dagger_\tau(\mathrm{x})\delta_{\beta\alpha}\partial_j\delta^3(\mathrm{x}-\mathrm{x}'). \tag{A.4}$$

From this we find

$$\psi^\dagger_\beta(\mathbf{x})\partial_j\psi_\tau(\mathbf{x})|\mathbf{x}_1(\alpha_1),\mathbf{x}_2(\alpha_2),\ldots,\mathbf{x}_n(\alpha_n)>$$

$$= \sum_{k=1}^{n}\left[\partial_j\delta^3(\mathbf{x}-\mathbf{x}_k)\right]\delta_{\alpha_k\tau}|\mathbf{x}_1(\alpha_1),\ldots,\mathbf{x}_{k-1}(\alpha_{k-1}),\mathbf{x}(\beta),\mathbf{x}_{k+1}(\alpha_{k+1}),\ldots,\mathbf{x}_n(\alpha_n)>. \tag{A.5}$$

We see from Eqs.(A.3,A.5) and remembering the the notation in Eq.(5.11)

$$\psi^\dagger(\mathbf{x})\left[f(\mathbf{x})+g^j(\mathbf{x})\partial_j\right]\psi(\mathbf{x})|\mathbf{x}_1(\alpha_1),\mathbf{x}_2(\alpha_2),\ldots,\mathbf{x}_n(\alpha_n)>$$

$$= \psi^\dagger_\beta(\mathbf{x})\left[f_{(\beta\tau)}(\mathbf{x})+g^j_{(\beta\tau)}(\mathbf{x})\partial_j\right]\psi_\tau(\mathbf{x})|\mathbf{x}_1(\alpha_1),\mathbf{x}_2(\alpha_2),\ldots,\mathbf{x}_n(\alpha_n)>$$

$$= \sum_{k=1}^{n}\sum_{\beta=1}^{4}\left[\left(f_{(\beta\tau)}(\mathbf{x})+g^j_{(\beta\tau)}(\mathbf{x})\partial_j\right)\delta^3(\mathbf{x}-\mathbf{x}_k)\right]\delta_{\alpha_k\tau}$$

$$|\mathbf{x}_1(\alpha_1),\ldots,\mathbf{x}_{k-1}(\alpha_{k-1}),\mathbf{x}(\beta),\mathbf{x}_{k+1}(\alpha_{k+1}),\ldots,\mathbf{x}_n(\alpha_n)>.$$

$$= \sum_{k=1}^{n}\sum_{\beta=1}^{4}\left[\left(f_{(\beta\alpha_k)}(\mathbf{x})+g^j_{(\beta\alpha_k)}(\mathbf{x})\partial_j\right)\delta^3(\mathbf{x}-\mathbf{x}_k)\right]$$

$$|\mathbf{x}_1(\alpha_1),\ldots,\mathbf{x}_{k-1}(\alpha_{k-1}),\mathbf{x}(\beta),\mathbf{x}_{k+1}(\alpha_{k+1}),\ldots,\mathbf{x}_n(\alpha_n)>. \tag{A.6}$$

This means that the projection of

$$\int d^3\mathbf{x}\,\psi^\dagger(\mathbf{x})\left[f(\mathbf{x})+g^j(\mathbf{x})\partial_j\right]\psi(\mathbf{x})|\Psi,t> \tag{A.7}$$

onto the n particle Hilbert space is

$$\int d^3\mathbf{x}d^3\mathbf{x}_1\ldots d^3\mathbf{x}_n\,\psi^\dagger_\beta(\mathbf{x})\left(f_{(\beta\tau)}(\mathbf{x})+g^j_{(\beta\tau)}(\mathbf{x})\partial_j\right)\psi_\tau(\mathbf{x})\Psi_{n(\alpha_1\ldots\alpha_n)}(\mathbf{x}_1,\ldots,\mathbf{x}_n)$$

$$|\mathbf{x}_1(\alpha_1),\mathbf{x}_2(\alpha_2),\ldots,\mathbf{x}_n(\alpha_n)>$$

$$= \int d^3\mathbf{x}d^3\mathbf{x}_1\ldots d^3\mathbf{x}_n\sum_{k=1}^{n}\left[\left(f_{(\beta\alpha_k)}(\mathbf{x})+g^j_{(\beta\alpha_k)}(\mathbf{x})\partial_j\right)\delta^3(\mathbf{x}-\mathbf{x}_k)\right]\Psi_{n(\alpha_1\ldots\alpha_n)}(\mathbf{x}_1,\ldots,\mathbf{x}_n)$$

$$|\mathbf{x}_1(\alpha_1),\ldots,\mathbf{x}_{k-1}(\alpha_{k-1}),\mathbf{x}(\beta),\mathbf{x}_{k+1}(\alpha_{k+1}),\ldots,\mathbf{x}_n(\alpha_n)>. \tag{A.8}$$

Here each of the variables $\alpha_1, \ldots \alpha_n$ and also β are summed from 1 to 4. Hence we see that the $|\mathbf{x}_1(\alpha_1),\mathbf{x}_2(\alpha_2),\ldots,\mathbf{x}_n(\alpha_n)>$ component of Eq.(A.7) is

$$\sum_{k=1}^{n}\sum_{\beta=1}^{4}\int d^3\mathbf{x}\left[\left(f_{(\alpha_k\beta)}(\mathbf{x}_k)+g^j_{(\alpha_k\beta)}(\mathbf{x}_k)\partial_{kj}\right)\delta^3(\mathbf{x}_k-\mathbf{x})\right]$$

$$\Psi_{n(\alpha_1\ldots\alpha_{k-1}\beta\alpha_{k+1}\ldots\alpha_n)}(\mathbf{x}_1,\ldots,\mathbf{x}_{k-1},\mathbf{x},\mathbf{x}_{k+1},\ldots\mathbf{x}_n,t), \tag{A.9}$$

where ∂_{kj} denotes the derivative with respect to the j component of $\mathbf{x}_k$. This step involves exchanging $\mathbf{x}$ and $\mathbf{x}_k$, and β and α_k. Now since

$$\partial_{kj}\delta^3(\mathbf{x}_k - \mathbf{x}) = -\partial_j\delta^3(\mathbf{x}_k - \mathbf{x}), \tag{A.10}$$

Eq.(A.9) is

$$\sum_{k=1}^{n}\sum_{\beta=1}^{4}\int d^3\mathbf{x}\left[\left(f_{(\alpha_k\beta)}(\mathbf{x}_k) - g^j_{(\alpha_k\beta)}(\mathbf{x}_k)\partial_j\right)\delta^3(\mathbf{x}_k - \mathbf{x})\right]$$

$$\Psi_{n(\alpha_1\ldots\alpha_{k-1}\beta\alpha_{k+1}\ldots\alpha_n)}(\mathbf{x}_1,\ldots,\mathbf{x}_{k-1},\mathbf{x},\mathbf{x}_{k+1},\ldots,\mathbf{x}_n,t). \tag{A.11}$$

and by integrating by parts we obtain

$$\sum_{k=1}^{n}\sum_{\beta=1}^{4}\int d^3\mathbf{x}\,\delta^3(\mathbf{x}_k-\mathbf{x})\left[f_{(\alpha_k\beta)}(\mathbf{x}_k) + g^j_{(\beta\alpha_k)}(\mathbf{x}_k)\partial_j\right]$$

$$\Psi_{n(\alpha_1\ldots\alpha_{k-1}\beta\alpha_{k+1}\ldots\alpha_n)}(\mathbf{x}_1,\ldots,\mathbf{x}_{k-1},\mathbf{x},\mathbf{x}_{k+1},\ldots,\mathbf{x}_k,t)$$

$$= \sum_{k=1}^{n}\sum_{\beta=1}^{4}\left[f_{\alpha_k\beta}(\mathbf{x}_k) + g^j_{\alpha_k\beta}(\mathbf{x}_k)\partial_{kj}\right]\Psi_{n(\alpha_1\ldots\alpha_{k-1}\beta\alpha_{k+1}\ldots\alpha_n)}(\mathbf{x}_1,\ldots\mathbf{x}_n,t). \tag{A.12}$$

Hence the $|\mathbf{x}_1(\alpha_1),\ldots,\mathbf{x}_n(\alpha_n)>$ component of Eq.(5.13) is Eq.(5.12).

Appendix B

Proof of Eq.(6.39)

This appendix proves

$$< \mathbf{p}_1(\alpha_1), \ldots \mathbf{p}_n(\alpha_n) | \rho(\mathbf{x}) | \mathbf{q}_1(\beta_1), \ldots \mathbf{q}_{n'}(\beta_{n'})>$$

$$=< \mathbf{p}_1[\alpha_1], \ldots \mathbf{p}_n[\alpha_n] | \tilde{\rho}(\mathbf{x}) | \mathbf{q}_1[\beta_1], \ldots \mathbf{q}_{n'}[\beta_{n'}]>. \qquad \text{(B.1)}$$

We first evaluate the right hand side. The momentum states in the QED representation are

$$|\mathbf{p}_1[\alpha_1], \ldots, \mathbf{p}_n[\alpha_n]> = c^\dagger_{\alpha_1}(\mathbf{p}_1) \ldots c^\dagger_{\alpha_n}(\mathbf{p}_n)|0>. \qquad \text{(B.2)}$$

We can also expand $\tilde{\rho}(\mathbf{x})$ in the momentum basis, using Eq.(6.36), to give

$$\tilde{\rho}(\mathbf{x}) = (2\pi\hbar)^{-3} m \int d^3\mathbf{p} d^3\mathbf{q} \exp \frac{-i\kappa_\tau \mathbf{x}.\mathbf{p}}{\hbar} \frac{c^\dagger_\tau(\mathbf{p}) u^\dagger_\tau(\mathbf{p})}{\sqrt[4]{\mathbf{p}^2 + m^2}} B \frac{u_\sigma(\mathbf{q}) c_\sigma(\mathbf{q})}{\sqrt[4]{\mathbf{q}^2 + m^2}} \exp \frac{i\kappa_\sigma \mathbf{x}.\mathbf{q}}{\hbar}, \qquad \text{(B.3)}$$

where we sum over τ and σ. Since

$$\left\{ \frac{u_\sigma(\mathbf{q}) c_\sigma(\mathbf{q})}{\sqrt[4]{\mathbf{q}^2 + m^2}}, c^\dagger_\beta(\mathbf{q}') \right\} = \frac{\delta^3(\mathbf{q} - \mathbf{q}') u_\beta(\mathbf{q}')}{\sqrt[4]{\mathbf{q}^2 + m^2}}, \qquad \text{(B.4)}$$

anti-commuting $\frac{u_\sigma(\mathbf{q}) c_\sigma(\mathbf{q})}{\sqrt[4]{\mathbf{q}^2+m^2}}$ with each of the $c^\dagger_\beta(\mathbf{q}')$ we see that

$$\frac{u_\sigma(\mathbf{q}) c_\sigma(\mathbf{q})}{\sqrt[4]{\mathbf{q}^2 + m^2}} |\mathbf{q}_1[\beta_1], \ldots, \mathbf{q}_{n'}[\beta_{n'}]>$$

$$= \sum_{s=1}^{n'} (-1)^{s+1} \frac{\delta^3(\mathbf{q} - \mathbf{q}_s) u_{\beta_s}(\mathbf{q}_s)}{\sqrt[4]{\mathbf{q}_s^2 + m^2}} |\mathbf{q}_1[\beta_1], \ldots, \mathbf{q}_{n'}[\beta_{n'}], \overline{\mathbf{q}_s[\beta_s]}>. \qquad \text{(B.5)}$$

The notation $\overline{\mathbf{q}_s(\beta_s)}$ denotes that $\mathbf{q}_s(\beta_s)$ is excluded from previous list. Hence the right hand side of Eq.(B.1) is

$$(2\pi\hbar)^{-3}m\sum_{r=1}^{n}\sum_{s=1}^{n'}(-1)^{r+s}\exp\frac{-i\kappa_{\alpha_r}\mathbf{x}.\mathbf{p}_r}{\hbar}\frac{u^\dagger_{\alpha_r}(\mathbf{p}_r)Bu_{\beta_s}(\mathbf{q}_s)}{\sqrt[4]{\mathbf{p}_r^2+m^2}\sqrt[4]{\mathbf{q}_s^2+m^2}}\exp\frac{i\kappa_{\beta_s}\mathbf{x}.\mathbf{q}_s}{\hbar}$$
$$< \mathbf{p}_1[\alpha_1],\ldots,\mathbf{p}_n[\alpha_n],\overline{\mathbf{p}_r[\alpha_r]}|\mathbf{q}_1[\beta_1],\ldots,\mathbf{q}_{n'}[\beta_{n'}],\overline{\mathbf{q}_s[\beta_s]}> . \quad \text{(B.6)}$$

Now we evaluate the left hand side. The momentum states in the wave-function representation are

$$|\mathbf{p}_1(\alpha_1),\ldots\mathbf{p}_n(\alpha_n)> = \frac{m^{\frac{1}{2}}c^\dagger_{\tau_1}(\kappa_{\alpha_1}\mathbf{p}_1)u_{\alpha_1(\tau_1)}(\mathbf{p}_1)}{\sqrt[4]{\mathbf{p}_1^2+m^2}}\cdots\frac{m^{\frac{1}{2}}c^\dagger_{\tau_n}(\kappa_{\alpha_n}\mathbf{p}_n)u_{\alpha_n(\tau_n)}(\mathbf{p}_n)}{\sqrt[4]{\mathbf{p}_n^2+m^2}}|0>. \quad \text{(B.7)}$$

We also expand $\rho(\mathbf{x})$ in the momentum basis

$$\rho(\mathbf{x}) = (2\pi\hbar)^{-3}\int d^3\mathbf{p}d^3\mathbf{q}\,\exp\frac{-i\mathbf{x}.\mathbf{p}}{\hbar}c^\dagger_\tau(\mathbf{p})B_{(\tau\sigma)}c_\sigma(\mathbf{q})\exp\frac{i\mathbf{x}.\mathbf{q}}{\hbar}, \quad \text{(B.8)}$$

where we sum over τ and σ. Since

$$\left\{c_\sigma(\mathbf{q}),\frac{u_{\beta(\alpha)}(\mathbf{q}')c^\dagger_\alpha(\kappa_\beta\mathbf{q}')}{\sqrt[4]{\mathbf{q}'^2+m^2}}\right\} = \frac{\delta^3(\mathbf{q}-\kappa_\beta\mathbf{q}')u_{\beta(\sigma)}(\mathbf{q}')}{\sqrt[4]{\mathbf{q}'^2+m^2}} \quad \text{(B.9)}$$

we see

$$c_\sigma(\mathbf{q})|\mathbf{q}_1(\beta_1),\ldots,\mathbf{q}_{n'}(\beta_{n'})>$$
$$=\sum_{s=1}^{n'}(-1)^{s+1}\frac{\delta^3(\mathbf{q}-\kappa_{\beta_s}\mathbf{q}_s)u_{\beta_s(\sigma)}(\mathbf{q}_s)}{\sqrt[4]{\mathbf{q}_s^2+m^2}}|\mathbf{q}_1(\beta_1),\ldots,\mathbf{q}_{n'}(\beta_{n'}),\overline{\mathbf{q}_s(\beta_s}>. \quad \text{(B.10)}$$

Hence the left hand side of Eq.(B.1) is

$$(2\pi\hbar)^{-3}m\sum_{s=1}^{n'}\sum_{r=1}^{n}(-1)^{r+s}\exp\frac{-i\kappa_{\alpha_r}\mathbf{x}.\mathbf{p}_r}{\hbar}\frac{u^\dagger_{\alpha_r(\tau)}(\mathbf{p}_r)B_{(\tau\sigma)}u_{\beta_s(\sigma)}(\mathbf{q}_s)}{\sqrt[4]{\mathbf{p}_r^2+m^2}\sqrt[4]{\mathbf{q}_s^2+m^2}}\exp\frac{i\kappa_{\beta_s}\mathbf{x}.\mathbf{q}_s}{\hbar}$$
$$< \mathbf{p}_1(\alpha_1),\ldots,\mathbf{p}_n(\alpha_n),\overline{\mathbf{p}_r(\alpha_r)}|\mathbf{q}_1(\beta_1),\ldots,\mathbf{q}_{n'}(\beta_{n'}),\overline{\mathbf{q}_s(\beta_s)}> \quad \text{(B.11)}$$

Since

$$< \mathbf{p}_1(\alpha_1),\ldots,\mathbf{p}_n(\alpha_n),\overline{\mathbf{p}_r(\alpha_r)}|\mathbf{q}_1(\beta_1),\ldots,\mathbf{q}_{n'}(\beta_{n'}),\overline{\mathbf{q}_s(\beta_s)}>$$
$$=< \mathbf{p}_1[\alpha_1],\ldots,\mathbf{p}_n[\alpha_n],\overline{\mathbf{p}_r[\alpha_r]}|\mathbf{q}_1[\beta_1],\ldots,\mathbf{q}_{n'}[\beta_{n'}],\overline{\mathbf{q}_s[\beta_s]}>, \quad \text{(B.12)}$$

we see that the left hand side and right hand side of Eq.(B.1) are equal.

Appendix C

The integrals over Sphere A

This appendix contains a maple session which helped evaluated the 3 integrals over regions of Sphere A in chapter 9. The calculation was complicated by the fact that the expression for d contains a positive square root, and maple, being unaware of the condition $r < a$, sometimes took the negative value. In order to overcome this some of the expressions were explicitly evaluated by hand and typed in again.

The session calculates values for K, L, and M, and their sum. The ϕ integral is ignored since it only results in a constant factor of 2π.

```
>
> d:=solve(a^2=r^2+R^2-2*r*R*c,r)[2];
```

$$d := R\,c + \sqrt{R^2\,c^2 + a^2 - R^2}$$

```
>

> K_1:=int(3*a^2*r^2/2-r^4/2,r=0..d);
```

$$K\_1 := R\,c\,a^4 - \frac{1}{2}R^3\,c\,a^2 - \frac{8}{5}R^5\,c^5 - \frac{1}{10}\,\%1^{5/2} - \frac{1}{2}R^5\,c - \frac{1}{2}R^4\,c^4\sqrt{\%1} - R^2\,c^2\,\%1^{3/2} + \frac{1}{2}a^2\,\%1^{3/2} + \frac{3}{2}a^2\,R^2\,c^2\sqrt{\%1} + 2\,R^5\,c^3$$

$$\%1 := R^2\,c^2 + a^2 - R^2$$

```
>
> K_2:=collect(simplify(int(K_1,c)),c);
```

$$K\_2 := -\frac{4}{15}R^5\,c^6 - \frac{4}{15}R^4\sqrt{\%1}\,c^5 + \frac{1}{2}R^5\,c^4 + \frac{1}{480}\frac{\left(176\,R^5\sqrt{\%1} + 64\,a^2\sqrt{\%1}\,R^3\right)c^3}{R} + \frac{1}{480}\frac{(-120\,R^4\,a^2 + 240\,R^2\,a^4 - 120\,R^6)\,c^2}{R} + \frac{1}{480}\frac{\left(-144\,a^2\sqrt{\%1}\,R^3 - 48\,R^5\sqrt{\%1} + 192\,a^4\sqrt{\%1}\,R\right)c}{R} + \frac{1}{480}\frac{105\,a^6\ln(R) + 135\,R^4\,a^2\ln(R) - 225\,R^2\,a^4\ln(R) - 15\,R^6\ln(R)}{R}$$

$$\%1 := R^2\,c^2 + a^2 - R^2$$

```
>
> K:=-subs(c=-1,K_2)+subs(c=R/2/a,-4/15*R^5*c^6-4/15*R^4*(a-R^2
  /a/2)*c^5+R^5*c^4/2+c^3/480/R*(a-R^2/a/2)*(64*a^2*R^3+176*R^5
  )+c^2/480/R*(-120*R^4*a^2+240*R^2*a^4-120*R^6)+1/480*(a-R^2/2
  /a)*(192*a^4*R-48*R^5-144*a^2*R^3)*c/R+(105*a^6*ln(R)+135*R^4
  *a^2*ln(R)-225*R^2*a^4*ln(R)-15*R^6*ln(R))/R/480);
```

$$K := -\frac{7}{30}R^5 - \frac{4}{15}R^4\,a + \frac{1}{480}\frac{176\,R^5\,a + 64\,a^3\,R^3}{R} - \frac{1}{480}\frac{-120\,R^4\,a^2 + 240\,R^2\,a^4 - 120\,R^6}{R} + \frac{1}{480}\frac{-144\,a^3\,R^3 - 48\,R^5\,a + 192\,a^5\,R}{R} - \frac{1}{240}\frac{R^{11}}{a^6} - \frac{1}{120}\frac{R^9\left(a - \frac{1}{2}\frac{R^2}{a}\right)}{a^5} + \frac{1}{32}\frac{R^9}{a^4}$$

$$+\frac{1}{3840}\frac{R^2\left(a-\frac{1}{2}\frac{R^2}{a}\right)(64\,a^2\,R^3+176\,R^5)}{a^3}+\frac{1}{1920}\frac{R\,(-120\,R^4\,a^2+240\,R^2\,a^4-120\,R^6)}{a^2}$$

$$+\frac{1}{960}\frac{\left(a-\frac{1}{2}\frac{R^2}{a}\right)(192\,a^4\,R-48\,R^5-144\,a^2\,R^3)}{a}$$

```
> simplify(K);
```

$$-\frac{1}{240}R^5-\frac{1}{6}R^2\,a^3+\frac{1}{8}a^2\,R^3-\frac{3}{10}a^4\,R+\frac{2}{5}a^5$$

```
>
> L:=int(int(3/2*a^2*r^2-r^4/2,r=0..a),c=R/2/a..1);
```

$$L:=\frac{2}{5}a^5-\frac{1}{5}a^4\,R$$

```
>
> M_1:=int(r*a^3,r=a..d);
```

$$M\_1:=\frac{1}{2}a^3\left(R\,c+\sqrt{R^2\,c^2+a^2-R^2}\right)^2-\frac{1}{2}a^5$$

```
> M_2:=simplify(int(M_1,c));
```

$$M\_2:=\frac{1}{6}a^3\left(2\,R^3\,c^3+2\sqrt{R^2\,c^2+a^2-R^2}\,R^2\,c^2+2\sqrt{R^2\,c^2+a^2-R^2}\,a^2-2\sqrt{R^2\,c^2+a^2-R^2}\,R^2-3\,R^3\,c\right)\Big/R$$

```
> M:=subs(c=1,M_2)-subs(c=R/2/a,a^3/6/R*(2*R^3*c^3+(a-R^2/2/a)*
  (2*R^2*c^2+2*a^2-2*R^2)-3*R^3*c));
```

$$M:=\frac{1}{6}\frac{a^3\,(-R^3+2\,a^3)}{R}-\frac{1}{6}\frac{a^3\left(\frac{1}{4}\frac{R^6}{a^3}+\left(a-\frac{1}{2}\frac{R^2}{a}\right)\left(\frac{1}{2}\frac{R^4}{a^2}+2\,a^2-2\,R^2\right)-\frac{3}{2}\frac{R^4}{a}\right)}{R}$$

```
> simplify(M);
```

$$\frac{1}{6}a^3\,R\,(-R+3\,a)$$

```
>
> simplify(K+L+M);
```

$$\frac{4}{5}a^5-\frac{1}{3}R^2\,a^3-\frac{1}{240}R^5+\frac{1}{8}a^2\,R^3$$

```
>
```

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