Report No. 67
January 1953

THE COLLEGE OF AERONAUTICS
CRANFIELD

The Two-Dimensional Laminar Boundary Layer at Hypersonic Speeds

- by -

T. Nonweiler, B.Sc.

---oOo---

SUMMARY

A numerical solution is found for the equations governing the motion of a two-dimensional laminar boundary layer, in the absence of a pressure gradient, which would be valid if the flight Mach Number is very high (i.e. $M^2 \gg 1$). The effects of surface slip, and the finite thickness of the boundary layer are shown to be negligible if the Reynolds Number (R) exceeds about 10^5, and are neglected.

Account is taken of the variation of specific heat, Prandtl Number and viscosity, with temperature, although (for air) only the latter effect is important. Sutherland's formula is used for viscosity variation, and the results imply that for $M < 10$, there is little variation of skin friction coefficient (c_f) with Mach Number. For high Mach Number, however, $c_f \propto 1/\sqrt{RM}$ and the heat transfer coefficient $k_H = 0.51 c_f$ for air. The surface temperature has a negligible effect on these quantities if it is small compared with the stagnation temperature. Numerical results are given, and show that skin friction and heat transfer vary as the square root of the surface pressure. The velocity and temperature profiles across the boundary layer are also deduced: the boundary layer displacement thickness is shown to increase as $\sqrt{M^3/R}$ at high Mach Number, and there is an important interaction between the boundary layer and the external flow.

Some remarks on the stability of a laminar layer are included, and a comparison is made of the above results with those relating to lower Mach Numbers of flight.

PDF

CONTENTS

NOTATION

C	Sutherland's Constant (i.e. $\mu \propto \frac{T^{3/2}}{T+C}$)
C_D	viscous form drag $\div \frac{1}{2}\rho_\delta u_\delta^2$
C_f	friction drag $\div \frac{1}{2}\rho_\delta u_\delta^2$
F	shearing stress at surface $(=\mu_o (\frac{\partial u}{\partial y})_o)$
H	Total heat, or enthalpy $(= \int c_p dT)$
L	length of body surface
M	Mach Number of stream outside boundary layer $(=u_\delta/a_\delta)$
M_a	Mach Number of free stream $(=u_a/a_a)$
Q	heat flux into surface $(=k_o (\frac{\partial T}{\partial y})_o)$
R	Reynolds' Number $(=\rho_\delta u_\delta L/\mu_\delta)$
R_T	Reynolds' Number of transition $(=\rho_\delta u_\delta x_T/\mu_\delta)$
R_ε	$= \rho_\varepsilon u_\varepsilon \varepsilon/\mu_\varepsilon$
T	static temperature of gas
T_{th}	thermometer temperature (i.e. that value of T_D for which $k_H = 0$.)
U'	free stream velocity (in ft/sec.)
Y	$= h/\sigma$
Z	$= \beta\tau \sqrt{\frac{\sigma^{\frac{1}{2}} g}{\eta h^{\frac{1}{2}}}}$
a	speed of sound $(=\gamma p/\rho)$
b	constant used in equation (3.10)
c	$= C/T_\delta$
c_f	$= F/\frac{1}{2}\rho_\delta u_\delta^2$
c_p	gas specific heat at constant pressure
c_v	gas specific heat at constant volume
f	$\equiv f(n) = u/u_\delta$
g	$\equiv g(n) = \frac{1}{M^2} \frac{\rho_\delta}{\rho}$
h	$\equiv h(n) = (H-H_\delta)/\frac{1}{2}u_\delta^2$
k	Thermal conductivity of gas
k_H	$= Q/\frac{1}{2}\rho_\delta u_\delta^3$
ℓ	$= \mu_\delta/\rho_\delta a_\delta$
m	molecular weight of gas
n	$= \frac{\beta y}{L} \sqrt{\frac{R}{2M^3 s}}$

Notation - continued.

$\overline{p}$	root mean square of atmospheric pressure fluctuations
p	gas pressure
q	$= \frac{\eta}{\sigma} \frac{dh}{dn}$
r	air density relative to sea-level conditions $(=\rho_a/\rho_{SL})$
s	$= x/L$
t	time
u,v	components of gas velocity parallel to the x- and y-axes respectively.
u_λ	velocity of slip at surface
w	$\equiv w(n) = - \frac{\beta v}{u_\delta} \sqrt{\frac{2Rs}{M^3}}$
x,y	system of orthogonal co-ordinates parallel and perpendicular to surface of plate, with origin at leading-edge of plate.
x'	value of x in feet
x_T	value of x at transition point.
Γ	proportional increase in c_p at elevated temperatures
ϑ	momentum thickness of boundary layer.
α	inclination of airflow at outside of boundary layer to free-stream direction
α_o	inclination of surface to free-stream direction
β	arbitrary finite constant in definition of n and w
γ	$=(c_p/c_v)_\delta$ (except in equation (5,7).)
δ	boundary layer thickness
ε	roughness height
ε_{max}	maximum tolerable roughness height to prevent separation of the flow
η	$\equiv \eta(n) = \frac{1}{M} \frac{\mu}{\mu_\delta}$
θ	deflection of stream at outside of boundary layer $(= \tan^{-1} \frac{v_\delta}{u_\delta})$
λ	mean free path of molecules at surface
μ	coefficient of viscosity of gas
ρ	gas density
σ	Prandtl Number $(= c_p\mu/k)$
$\overline{\sigma}$	mean value of σ within boundary layer

Notation - continued.

$\tau \quad = \eta \frac{df}{dn}$

$\omega \quad = (\frac{T}{\mu} \frac{du}{dT})$

Suffix: 'SL' denotes conditions in the ambient air at sea level.

'a' denotes conditions in the free stream

'n' denotes differentiation with respect to n

'o' denotes conditions at the surface

'δ' denotes conditions at the outside of the boundary layer (where $y = \delta$)

'ε' denotes conditions at tip of small projection (where $y = \varepsilon$)

Primed symbols (e.g. τ') denote differentiation with respect to f (i.e. $\tau' = \frac{d\tau}{df}$)

1. Introduction

In this report we shall attempt to examine the properties of the laminar two-dimensional boundary layer existing in a compressible boundary layer at very high Mach Numbers of flight. To be precise, we shall assume that this Mach Number M is sufficiently large that $1/M^2$ can be neglected compared with unity. The resulting flow conditions we describe as 'hypersonic' in a similar way as we can classify as 'incompressible' those flows in which we can neglect M^2 compared with unity. Like the results for incompressible flow, those relevant to the hypersonic boundary layer display a certain simplicity of form, which greatly facilitates their interpretation and application. This simplicity also enables us to relax many assumptions which are normally made to obtain a numerical solution of the equations involved.

For instance, we do not find it necessary to restrict the discussion to a gas with a Prandtl Number of unity, or to stipulate that the surface temperature is a constant. We shall use Sutherland's formula for the variation of viscosity with temperature, and we shall find it possible to make some allowance for the variation of molecular specific heats with temperature. Such factors greatly enhance the value of the results we can obtain and throw light upon the accuracy of the assumptions more usually made.

Of the assumptions which we do make, the most restrictive is that the pressure over the surface is a constant - although this allows us to consider not only the heat transfer to a flat plate moving parallel to itself, but also to plane inclined surfaces which in supersonic flow are acted upon by a uniform pressure, and to bodies (such as, for instance, the double-wedge wing) composed of several such surfaces. We shall also make the usual assumptions associated with that of a high Reynolds Number - for example, that the boundary layer is thin, and that the velocity of slip is zero - although we shall examine these in the light of the results to which they lead. In regard to the variation of the molecular specific heats with temperature, we shall be guided by the results of statistical thermodynamics in assuming that they increase to a certain asymptotic value at very high temperatures.

As an extension of the assumption that we may treat $(1/M^2)$ as small, we shall also neglect the surface temperature compared with the thermometer temperature of the boundary layer air. In other words, we shall be dealing only with the boundary layer in the presence of high rates of heat transfer. This is easily justifiable if it is recalled that the thermometer temperature is commensurate with the stagnation temperature, which latter is given by

$$(1 + \frac{1}{5} M^2) \times \text{ambient air temperature.}$$

Flight at a Mach Number of 10 or more thus involves thermometer temperatures of at least 4000°C., and plainly unless the surface temperature is considerably smaller than this the problems have no great practical significance! From the mathematical point of view, if the surface temperature is comensurate with that of the ambient air, (say, 2 or 3 times its value), then since the thermometer temperature is of the order of M^2 times its value, the above assumption is justified if we allow $M^2 \to \infty$.

For such reasons as this, the form of the asymptotic solution we obtain has rather strange properties, and having written down the correct boundary conditions for the condition $1/M^2 = 0$, we shall examine the compatability of the results for finite, but large, Mach Number. We find that in the same way as the assumption of incompressibility ($M^2 = 0$) leads to quantitative deductions which are qualitatively sound (if slightly exaggerated) when compared with conditions at finite Mach Numbers, so also does the hypersonic solution we find here. It remains, of course, to be shown whether the results have the same power and significance, as an asymptotic solution, as those for incompressible flow.

2. The Equations of the Boundary Layer in High Speed Flow

Using the notation defined at the beginning of this note, we may write down the equation of continuity in steady motion as

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \qquad \ldots (2.1)$$

and if the surface pressure is uniform, the Eulerian equations of motion as modified by Prandtl for a thin boundary layer become

$$\rho\frac{Du}{Dt} = \frac{\partial}{\partial y}\ (\mu\frac{\partial u}{\partial y}) \qquad ....(2.2)$$

$$dp = 0 \qquad ....(2.3)$$

In addition we need the energy equation for the thin boundary layer:

$$\rho\frac{DH}{Dt} = \frac{\partial}{\partial y}\ (\frac{k}{c_p}\ \frac{\partial H}{\partial y}) + \mu(\frac{\partial u}{\partial y})^2 \qquad ....(2.4)$$

Finally, from (2.3), the Gas Law may be written as

$$d(\frac{\rho T}{m}) = 0 \qquad ....(2.5)$$

We now introduce the non-dimensional variables

$$s = x/L,\ f = u/u_\delta \quad \text{and} \quad h = (H-H_\delta)/\tfrac{1}{2}u_\delta^2$$

where s, f and h are bounded quantities and in general finite, and where the subscript 'δ' refers to conditions outside the boundary layer (which are invariant with x). To relate ρ to the enthalpy H, we assume that within the boundary layer

$$\frac{mc_p}{(mc_p)_\delta} = \Gamma, \text{ a constant} \qquad ....(2.6)$$

which is, at least, a more elastic assumption than that the specific heats of the air are constant. In fact, there is some evidence to show that at elevated temperatures the molecule's specific heats reach an asymptotic value higher than that at normal temperatures and pressures. Fowler and Guggenheim (in their 'Statistical Thermodynamics') deduce that, for large T, (mc_p) is increased by a factor $^9/_7$ in a diatomic gas. Since the air is composed mainly of diatomic molecules, we might therefore expect that, in the high-speed boundary layer, where as we shall show later, $T = O(M^2T_\delta)$, we could put in (2.6), the value $\Gamma = \frac{9}{7}$ and that this would apply in general over the boundary layer if $M^2 >> 1$. Existing data for air (up to $T = 3000°C$) suggests that this ratio is exceeded without any evident falling off in the increase of specific heats with temperature. This increase in caused by the higher vibrational energy of the gas at these temperatures, and is still further increased at even higher temperatures (say, about 20,000°C) by electronic excitation - though this effect may legitimately be ignored here since, as we shall see later such high temperatures are not likely in the boundary layer at any high speeds in which we might be interested. However, it is evident that except near the edges of the boundary layer, where the temperature is low (i.e. for $\varepsilon \leqslant y \leqslant \delta-\varepsilon$, where ε is some distance small compared with δ, the boundary layer thickness), equation (2.6) can provide a reasonable indication of the change in specific heats.

Then, if we define g so that

$$\frac{\rho_\delta}{\rho} = g\, M^2$$

it follows from (2.5) that

$$g = \frac{1}{M^2}\frac{\rho_\delta}{\rho} = \frac{1}{M^2}\frac{m_\delta}{m}\frac{T}{T_\delta}$$

or using equation (2.6)

$$g = \frac{1}{\Gamma M^2}\frac{c_p T}{(c_p T)_\delta} \qquad \ldots (2.7)$$

By the definition of the function h, we find that

$$h = \int_{T_\delta}^{T} c_p\, dT/\tfrac{1}{2}u_\delta^2 = \overline{c_p}\,(T-T_\delta)/\tfrac{1}{2}u_\delta^2 = \frac{2}{\gamma-1}\frac{1}{M^2}\left(\frac{T}{T_\delta}-1\right)\frac{\overline{c_p}}{c_{p_\delta}} \quad \ldots (2.8)$$

Since h is finite, it follows that if we treat $M^2 \gg 1$, then

$$\frac{T}{T_\delta} = O(M^2) \qquad \ldots (2.9)$$

Assuming (as before) that at high temperatures, the specific heat attains a constant value, we deduce that the mean specific heat $\overline{c_p}$ is equal in general to the asymptotic value within the boundary layer.
Hence, in equation (2.8)

$$h = \frac{2}{\gamma-1}\frac{1}{M^2}\left(\frac{T}{T_\delta}-1\right)\frac{\overline{c_p}}{c_{p_\delta}} = \frac{2}{\gamma-1}\frac{1}{M^2}\frac{\overline{c_p}T}{(c_p T)_\delta}\left(1-\frac{T_\delta}{T}\right)$$

$$= \frac{2}{\gamma-1}\frac{1}{M^2}\frac{c_p T}{(c_p T)_\delta}\left[1+O\left(\frac{1}{M^2}\right)\right] \quad \ldots (2.10)$$

or substituting in the equation (2.7)

$$g = \frac{1}{\Gamma M^2}\frac{c_p T}{(c_p T)_\delta} = h\left(\frac{\gamma-1}{2\Gamma}\right)\left[1+O\left(\frac{1}{M^2}\right)\right]$$

For the asymptotic solution, where $M^2 \to \infty$, evidently

$$g = \left(\frac{\gamma-1}{2\Gamma}\right)h \qquad \ldots (2.11)$$

In the same way as we assume that the molecular specific heats are constant over the boundary layer, we also assume that the Prandtl Number, $\sigma = \mu c_p/k$, is a constant (and equal to $\overline{\sigma}$, say) since this number is known to be related to the specific heats, but again $\overline{\sigma} \neq \sigma_\delta$ since the Prandtl Number differs at high temperatures from its value at normal temperatures.

In suggesting a non-dimensional form for the viscosity, we use Sutherlands Formula:*

$$\frac{\mu}{\mu_\delta} = \frac{T_\delta + C}{T + C}\left(\frac{T}{T_\delta}\right)^{3/2} = \left(\frac{T}{T_\delta}\right)^{\frac{1}{2}}\left(1+\frac{C}{T_\delta}\right)\Big/\left(1+\frac{C}{T_\delta}\frac{T_\delta}{T}\right) \quad \ldots\ldots (2.12)$$

Suppose we define η so that

$$\frac{\mu}{\mu_\delta} = M\eta$$

then η is a finite parameter for $M^2 \gg 1$, since using (2.11) and (2.9),

$$\eta = \left(\frac{1}{M^2}\frac{T}{T_\delta}\right)^{\frac{1}{2}}\left(1+\frac{C}{T_\delta}\right)\left[1 + O\left(\frac{1}{M^2}\right)\right]$$

or in the limiting condition, $M^2 \to \infty$, from (2.10) and (2.6)

$$\eta = \left[\frac{2}{\gamma-1}\,\frac{1}{M^2}\,\frac{c_pT}{(c_pT)_\delta}\right]^{\frac{1}{2}}\left(1+\frac{C}{T_\delta}\right)\left(\frac{\gamma-1}{2}\,\frac{c_{p\delta}}{c_p}\right)^{\frac{1}{2}}\left[1 + O\left(\frac{1}{M^2}\right)\right]$$

$$\text{i.e. } \eta = \left(1+\frac{C}{T_\delta}\right)\left(\frac{\gamma-1}{2\Gamma}\,\frac{m}{m_\delta}\right)^{\frac{1}{2}} h^{\frac{1}{2}} \quad \ldots\ldots (2.13)$$

This relation also expresses the fact that η varies, at large temperatures, as the square root of the temperature (or as the square root of the enthalpy, if the specific heats are constant).

If we now suppose that all the non-dimensional variables so far defined (i.e. f, g, h and η) are functions of a single independent variable n, involving the space co-ordinates x and y, then it may easily be shown that this variable must depend on $y/\sqrt{x}$. Thus we write in non-dimensional form:

$$n = \frac{\beta y}{L}\sqrt{\frac{R}{2M^3 S}}$$

where β is some arbitrary finite constant whose value may later be chosen as a matter of convenience. We also define a non-dimensional variable involving v,

$$w = -\frac{\beta v}{u_\delta}\sqrt{\frac{2Rs}{M^3}}$$

where w may be shown to be also dependent only on n.

From the definitions of f, g, h, η and w which are each functions of n, the equations (2.1), (2.2) and (2.4) may be simplified to the forms:

$$n\frac{d}{dn}\left(\frac{f}{g}\right) + \frac{d}{dn}\left(\frac{w}{g}\right) = 0 \quad \ldots\ldots (2.14)$$

---

* FOOTNOTE: For air, the value of C is 117°K.

$$\left(\frac{nf+w}{g}\right)\frac{df}{dn} = -\beta^2\frac{d}{dn}\left(\eta\frac{df}{dn}\right) \qquad \text{....}(2.15)$$

$$\left(\frac{nf+w}{g}\right)\frac{dh}{dn} = -\beta^2\left[\frac{d}{dn}\left(\frac{\eta}{\sigma}\frac{dh}{dn}\right) - 2\eta\left(\frac{df}{dn}\right)^2\right] \qquad \text{....}(2.16)$$

We may eliminate (w/g) from equations (2.15) and (2.16) using equation (2.14). If, in the resulting equations, we change the independent variable from n to $f = (u/u_\delta)$, and we write

$$\tau = \eta\frac{df}{dn} = \frac{\sqrt{2}}{\beta}\sqrt{\frac{Ms}{R}}\left(\mu\frac{\partial u}{\partial y}\right)\Big/\left(\frac{\mu_\delta u_\delta}{L}\right)$$

which is a non-dimensional form of the shear stress, then we find that (treating σ as a constant) ;

$$f = -\beta^2\frac{g}{\eta}\tau\tau'' \qquad \text{....}(2.17)$$

$$h'' + (1-\bar{\sigma})\frac{\tau'}{\tau}h' + 2\bar{\sigma} = 0 \qquad \text{....}(2.18)$$

where the primes denote differentiations with respect to f.

These two equations are equivalent to the equations of Crocco[1] for momentum and energy, after the Crocco transformation has been applied. It should be noted that their derivation is dependent only on the definition of the new variables, and is not influenced by any of the assumptions we have made concerning the relation of density, specific heats, and viscosity with temperature (i.e. the relation of g and η with h). However we note that, if $M^2 \to \infty$, then h, and of course also f, are finite parameters, as also are g and η which are related to h in the simple manner described by equations (2.11) and (2.13). Apart from this particular choice of the form of the variables, equations (2.17) and (2.18) are equivalent to the expressions of Crocco.

In particular, the boundary conditions are also greatly simplified in the condition $M^2 \to \infty$. At the outside of the boundary layer, $u = u_\delta$ or $f = 1$. Here $H = H_\delta$ and the shear stress is zero: i.e.

$$h = 0, \ \tau = 0 \text{ at } f = 1 \qquad \text{....}(2.19)$$

At the surface, where $\rho u = 0$ (i.e. $f = 0$) we also have that $\rho v = 0$, and $y = 0$ (i.e. $\frac{w}{g} = 0$ and $n = 0$). From (2.15), it then follows that

$$\frac{dn}{df}\frac{d}{dn}\left(\eta\frac{df}{dn}\right) = \frac{d\tau}{df} = 0, \text{ at } f = 0.$$

i.e. the rate of change of the shear stress normal to the surface is zero. At the surface we also have that $T = T_o$, say. If the skin temperature is of comparable magnitude to the ambient air temperature, then $T_o/T_\delta = O(1)$ if $M^2 \to \infty$, so that from (2.8) in this condition

$$h = O(\frac{1}{M^2}) \quad \text{at} \quad f = 0$$

or in the limiting condition, we have

$$\frac{d\tau}{df} = 0, \; h = 0, \; n = 0, \text{ at } f = 0 \qquad \ldots (2.20)$$

The condition h = 0 merely expresses the fact that the temperature of the air at the surface is negligible compared with that of the air within the boundary layer if $M^2 \gg 1$, and of course if applied to flows with finite values of M, is only an approximation (unless $T_\delta = T_o$). We may immediately infer that the asymptotic solution of the hypersonic flow equations (with $M^2 \gg 1$) will be uninfluenced by variations in the surface temperature.

Equations (2.17) and (2.18), with the approximate relations (2.11) and (2.13), and the boundary conditions (2.19) and (2.20) describe the state of the boundary layer in a hypersonic flow, which we interpret mathematically as the flow of infinitely large Mach Number. In a later paragraph we shall attempt to solve these equations numerically: equation (2.18) yields the formal solution, using (2.19), that

$$h = 2\overline{\sigma} \int_f^1 \tau^{\overline{\sigma}-1} \left(\int_\emptyset^f \tau^{1-\overline{\sigma}} \, df \right) df \qquad \ldots (2.21)$$

where $\emptyset$ is the value of f where the total heat is a maximum and must be chosen to satisfy the condition that h = 0 when f = 0 in equation (2.20). This however is a solution of little computational value, but it does indicate that if we define q as a non-dimensional form of the heat flux: i.e.

$$q = \eta \frac{dh}{dn} = \frac{2\sqrt{2}}{(\gamma-1)\beta} \sqrt{\frac{s}{RM^3}} \left(k\frac{\partial T}{\partial y} \right) \Big/ \left(\frac{k_\delta T_\delta}{L} \right)$$

then, we find that

$$q = \tau \frac{dh}{df} = -2\overline{\sigma} \, \tau^{\overline{\sigma}} \int_\emptyset^f \tau^{1-\overline{\sigma}} \, df \qquad \ldots (2.22)$$

Hence at f = 1 (i.e. at the outside of the boundary layer) where $\tau = 0$ (i.e. the shear stress vanishes) then also q = 0, - in other words, there is no heat flux from the boundary layer to the ambient air.

The approximations introduced in the solution for hypersonic flow are, as we have seen, valid in general within the boundary layer: that is, except near the surface and the outer edge. In view of this fact, it is necessary before proceeding with the solution, to examine its validity in these bounding regions. It will be shown in the next paragraph that although the assumptions are in error, the solution is still an adequate approximation if $M^2 \gg 1$.

3. Interpretation of the Behaviour of the Flow near the Surface

Let us first consider the behaviour of the hypersonic flow solution, already obtained, near the surface (i.e. in the condition $f \to 0$).

We first notice, from equation (2.22), that $h' \neq 0$ at $f = 0$, since the value of τ (the non-dimensional form of the shear stress) is finite at the surface. Thus

$$h \sim h'_o f \quad \text{as } f \to 0 \qquad \text{.... (3.1)}$$

where $h' = h'_o$ at $f = 0$. Again, in (2.17), from (2.11) and (2.13) we have that $g/\eta \propto h^{\frac{1}{2}}$ and so

$$\tau'' \sim \text{const.}\ (f/h^{\frac{1}{2}}) \text{ as } f \to 0$$

i.e from equation (3.1)

$$\tau'' \sim \text{const.}\ f^{\frac{1}{2}} \quad \text{as } f \to 0$$

From (2.20), $\tau' = 0$ at $f = 0$, so that

$$\tau' \sim \text{const.}\ f^{3/2} \quad \text{as } f \to 0 \qquad \text{.... (3.2)}$$

To relate this limiting behaviour to the independent variable n, we note that

$$n = \int_o^f \frac{\eta}{\tau}\, df \qquad \text{.... (3.3)}$$

so that from (2.13) and (3.1), since $\tau \neq 0$ at $f = 0$, and $\eta \propto h^{\frac{1}{2}} \propto f^{\frac{1}{2}}$,

$$n \sim \text{const.}\ f^{3/2} \quad \text{as } f \to 0 \qquad \text{.... (3.4)}$$

Thus, in (3.1) and (3.2), from (3.4)

$$f \sim \text{const.} n^{2/3},\ h \sim \text{const.}\ n^{2/3},\ \tau' \sim \text{const.}\ n, \text{ for } n \to 0 \qquad \text{.... (3.5)}$$

The limiting behaviour is thus singular: the velocity and temperature gradients f_n and h_n are infinite at $n = 0$, although the same does not apply to the values of the shear stress and heat flux, since the value of η tends to zero at the surface.

It will be evident that such a solution for finite Mach Number would be invalid because $\left(\frac{\mu}{\mu_\delta}\frac{1}{M}\right) = \eta$ is non-zero at the surface, and the velocity and temperature gradients are finite. However, we shall attempt to show how the true solution departs from the asymptotic one, and to deduce that the value of τ and q at $f = 0$ in the asymptotic solutions for $M \to \infty$ are the correct values of the non-dimensional form of the shear stress and heat flux if M is large, but not infinite.

Let us then consider the conditions existing for flows in which M is finite. At the surface, where $u = v = 0$ we have from equations (2.2) and (2.4), that

$$\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y} + \mu u\frac{\partial u}{\partial y}\right) = 0$$

or, performing the differentiations of the products:

$$\left.\begin{aligned} \frac{\partial^2 u}{\partial y^2} &= -\frac{1}{\mu}\frac{d\mu}{dT}\frac{\partial T}{\partial y}\frac{\partial u}{\partial y} \\ \frac{\partial^2 T}{\partial y^2} &= -\frac{1}{k}\frac{dk}{dT}\left(\frac{\partial T}{\partial y}\right)^2 - \frac{\mu}{k}\left(\frac{\partial u}{\partial y}\right)^2 \end{aligned}\right\} \quad \ldots.(3.6)$$

Rewriting these expressions in terms of the non-dimensional variables f and h which are functions only of n (i.e. $f=f(n)$, $h=h(n)$) we find that

$$\left.\begin{aligned} f_{nn}(0) &= -\frac{\gamma-1}{2}M^2 h_n(0) f_n(0)\left\{\left(\frac{T}{\mu}\frac{d\mu}{dT}\right)_o \frac{(c_pT)_\delta}{(c_pT)_o}\right\} \\ h_{nn}(0) &= -\frac{\gamma-1}{2}M^2\left[h_n(0)\right]^2\left\{\left[\frac{c_pT}{k}\frac{d(k/c_p)}{dT}\right]_o\frac{(c_pT)_\delta}{(c_pT)_o}\right\} - 2\sigma\left[f_n(0)\right]^2 \end{aligned}\right\} \quad \ldots.(3.7)$$

where subscript 'n' denotes a differentiation with respect to n, and subscript 'o' refers to values at $n = 0$.

Now in our asymptotic solution for $M \to \infty$, we have established that both τ and q are finite at $n = 0$: hence, by definition

$$f_n(0) = \frac{\tau_o}{\eta_o} = \left(\frac{\tau_o\mu_\delta}{\mu_o}\right)M \quad \ldots.(3.8)$$

If $M \to \infty$, evidently $f_n(0) \to \infty$, which is in accordance with the asymptotic behaviour of the hypersonic flow solution given in (3.5). Similarly

$$h_n(0) = \frac{q_o}{\eta_o} = \left(\frac{q_o\mu_\delta}{\mu_o}\right)M \quad \ldots.(3.9)$$

and both $f_n(0)$ and $h_n(0)$ are, for large M, magnitudes commensurate with M. It follows from (3.7) that the second derivatives $f_{nn}(0)$ and $h_{nn}(0)$ are magnitudes of order M^4. For large values of M, however, both f and h are finite within the boundary layer. Thus any definitive relation between the derivatives of f or h must involve only finite constants. Hence, for large M, it follows that as $n \rightarrow 0$

$$h_{nn} \sim \text{const.}\, h_n^4$$

both sides of the expression being of order M^4, and the constant being finite. We may, in fact, deduce its value from (3.7) and find that

$$\left. \begin{aligned} h_{nn} &\sim -\frac{\gamma-1}{2q_o^2}\left(\frac{\mu_o}{\mu_\delta}\right)^2 \left\{ \left[\frac{c_pT}{k} \frac{d(k/c_p)}{dT} \right]_o \frac{(c_pT)_\delta}{(c_pT)_o} \right\} \left[1 + 0\left(\frac{1}{M^2}\right) \right] h_n^4 \\ \text{or } h_{nn} &\sim -b\, h_n^4, \text{ say} \end{aligned} \right\} \quad \ldots.(3.10)$$

It follows, by integration, that as $n \rightarrow 0$,

$$h(n) - h(0) \sim \frac{1}{2b} \left\{ 3b\, n + \left[\frac{1}{h_n(0)} \right]^3 \right\}^{2/3} - \frac{1}{2b\left[h_n(0)\right]^2} \quad \ldots.(3.11)$$

If we now allow $M^2 \rightarrow \infty$, provided that $n M^3 \gg 1$, it follows from (3.9) that

$$h(n) \sim \frac{3^{2/3}}{2b^{1/3}} n^{2/3} \left\{ 1 + 0\left(\frac{1}{M^2}\right) \right\} \quad \ldots.(3.12)$$

since $h(0) = 0\left(\frac{1}{M^2}\right)$. This expression (3.12) is identical with that in (3.5) found previously for the asymptotic solution: equation (3.11) implies a more exact description of the conditions near n = 0 if M is large, but not infinite. It will be seen that the asymptotic (hypersonic flow) solution can give the correct description of conditions (with error of order $\frac{1}{M^2}$) if

$$n = 0\left(\frac{1}{M}\right) \quad \ldots.(3.13)$$

and fails if $n = 0\left(\frac{1}{M^3}\right)$.

In a similar manner we may show that, for $n \rightarrow 0$, for large M

$$f \sim \frac{f_n(0)}{h_n(0)} \left[h(n) - h(0) \right] \sim \frac{\tau_o}{q_o} \left[h(n) - h(0) \right] \quad \ldots.(3.14)$$

which is likewise compatible with (3.5) except where $n = 0\left(\frac{1}{M^3}\right)$.

We have thus accounted for the difference between the variations of h and f as obtained either by the hypersonic flow equations, or by the exact equations for large (but non-infinite) M.

Moreover, from equations (3.6) it follows that the gradient of the shear stress, i.e. $(\frac{d\tau}{dn})$, tends to zero as $n \to 0$ for all values of M (which is again compatible with (3.5)); so that from (3.13), provided M is sufficiently large, the value of τ_0 found from the solution in hypersonic flow (which is valid where $n \gg \frac{1}{M^3}$) will not differ greatly from the true value at the surface. Similar arguments lead to the deduction that the rate of heat flux at the surface is also correctly given by the hypersonic solution.

4. Behaviour of the Flow near the Outer Edge of the Boundary Layer

At the outer edge of the boundary layer where $f \to 1$, we have seen from equation (2.21) that

$$h' \sim \text{const. } \tau^{\sigma-1} \qquad \ldots (4.1)$$

and since $\tau \to 0$ as $f \to 1$, if $\sigma < 1$, it follows that $h' \to \infty$ as $f' \to 1$. In particular, since from (2.19) $h = 0$ at $f = 1$, it follows that upon integration, as $f \to 1$

$$h \sim \text{const. } \left(\int_f^1 \frac{df}{\tau^{1-\sigma}}\right) \qquad \ldots (4.2)$$

In our solution for infinitely large M, using (2.11) and (2.13) in (2.17), we have then that as $f \to 1$,

$$\tau\,\tau'' \sim \text{const. } \left(\int_f^1 \frac{df}{\tau^{1-\sigma}}\right)^{-\frac{1}{2}}$$

which since $\tau \to 0$ as $f \to 1$, implies that if $\sigma > 0$,

$$\tau \sim \text{const. } (1-f)^{\frac{3}{3+\sigma}} \quad \text{as } f \to 1 \qquad \ldots (4.3)$$

and so in (4.2)

$$h \sim \text{const. } (1-f)^{\frac{4\sigma}{3+\sigma}} \quad \text{as } f \to 1 \qquad \ldots (4.4)$$

Near the outside of the boundary layer, from (3.3) and using (2.13),

$$\frac{dn}{df} = \frac{n}{\tau} \sim \text{const. } (1-f)^{\frac{2\sigma-3}{3+\sigma}} \qquad \ldots (4.5)$$

and it follows that if $\sigma > 0$, n tends to a finite limit as $f \to 1$. In other words, the extent of the boundary layer is finite. Suppose (by suitable choice of the constant β in the definition of n) that $n = 1$ then corresponds with the outside of the boundary layer. We have from (4.5) upon integration, that

$$(1-n) \sim \text{const. } (1-f)^{\frac{3\sigma}{3+\sigma}} \quad \text{as } f \to 1 \qquad \ldots (4.6)$$

Thus in (4.3) and (4.4), from (4.5),

$$(1-f) \sim \text{const.} (1-n)^{\frac{3+\sigma}{3\sigma}}, \quad h \sim \text{const.} (1-n)^{4/3},$$

$$\tau \sim \text{const.} (1-n)^{1/\sigma} \quad \text{as } n \to 1 \qquad \ldots (4.7)$$

Here again the behaviour of the variables is singular, as it was near the surface.

This behaviour results from the fact that the boundary layer is finite in extent (as measured by the independent variable n), and is of course only strictly valid in the limiting condition of infinite Mach Number. For finite Mach Number we know that the boundary layer extends to infinity: but this is quite compatible with the hypersonic flow solution for, as we shall now show, at large distances from the surface (for given n) the air velocity decays very rapidly with increase of Mach Number.

In equations (2.17) and (2.18), which involve no approximations concerning the value of M, we have used the boundary conditions $h = \tau = 0$ at $f = 1$, which are also correct for all M. However in relating f to n in (4.5) we used the approximation that $\eta \propto h^{\frac{1}{2}}$, whereas at the outside of the boundary layer we have simply by definition

$$\eta = \frac{1}{M} \quad \text{at } f = 1 \qquad \ldots (4.8)$$

Similarly we have used in obtaining (4.3) the approximation $g \propto h$, although strictly

$$g = \frac{1}{M^2} \quad \text{at } f = 1 \qquad \ldots (4.9)$$

Strictly, for any value of M, we have from (2.14) that

$$\frac{w}{g} = -\int_0^n \eta \frac{d(f/g)}{dn} dn$$

and so in (2.15),

$$\left(\frac{nf}{g} + \frac{w}{g}\right)\frac{df}{dn} = \left(\int_0^h \frac{f}{g} dn\right)\frac{df}{dn} = -\beta^2 \frac{d}{dn}\left(\eta\frac{df}{dn}\right) \qquad \ldots (4.10)$$

Now $\frac{d\eta}{dn} \propto \frac{d\mu}{dT} \frac{\partial T}{\partial y}$ approaches zero at the outside of the boundary layer: and so from (4.8) and (4.9), since $f \to 1$ as $n \to \infty$, we have that

$$M^2 n \frac{df}{dn} = -\beta^2 M \frac{d^2 f}{dn^2}$$

We may put $\beta = 1$ without loss in generality, and after integration we find that

$$(1-f) \sim \frac{\text{const.}}{Mn} \exp(-\tfrac{1}{2}Mn^2) \quad \text{as } n \to \infty.$$

Thus, for a given value of n, the disturbance to the flow decreases exponentially with increase in M. In the limiting condition of infinite Mach Number, it vanishes for all but a finite range of n as we have seen.

It should be noted that n is itself dependent on M, and although it is true that the changes within the boundary layer become more concentrated, the actual thickness of the boundary layer increases as the Mach Number is increased, as we shall see in the next paragraph, and for $M = \infty$ is in fact infinite.

5. Solution of the Equations

If in equations (2.17) and (2.18), we put

$$Y = h/\bar{\sigma}, \quad Z = \beta\sqrt{\frac{\bar{\sigma}^{\frac{1}{2}}g}{\eta h^{\frac{1}{2}}}}\;\tau$$

then they become, with the use of equations (2.11) and (2.13)

$$\left.\begin{aligned} ZZ'' + \frac{f}{\sqrt{Y}} &= 0 \\ Y'' + (1-\bar{\sigma})\frac{Z'Y'}{Z} + 2 &= 0 \end{aligned}\right\} \quad \ldots.(5.1)$$

with the boundary conditions

$Y = Z' = 0$ at $f = 0$

$Y = Z = 0$ at $f = 1$.

The equations (5.1) have been solved numerically to satisfy these conditions, using a relaxation method, for $\bar{\sigma} = 1.0$, 0.8 and 0.6. The results are summarised in the Table I below.

Table I: Summary of Numerical Solutions.

f	Y			Z		
	σ=1	σ=0.8	σ=0.6	σ= 1	σ=0.8	σ=0.6
0.0	0.00	0.0000	0.0000	0.7200	0.7114	0.6966
.1	.09	.0922	.0949	.7188	.7100	.6954
.2	.16	.1646	.1700	.7131	.7037	.6898
.3	.21	.2172	.2255	.7004	.6904	.6772
.4	.24	.2500	.2615	.6783	.6679	.6552
.5	.25	.2630	.2779	.6439	.6335	.6211
.6	.24	.2560	.2744	.5940	.5837	.5717
.7	.21	.2286	.2502	.5232	.5133	.5019
.8	.16	.1799	.2034	.4223	.4136	.4031
.9	.09	.1073	.1284	.2703	.2643	.2563
1.0	0.00	0.0000	0.0000	0.0000	0.0000	0.0000

We are particularly interested in the values of the shear stress and heat flux at the wall: these are usually quoted as the coefficients c_f and k_H, where we define

$$c_f = \lim_{n \to 0} \left[\mu \left(\frac{\partial u}{\partial y}\right)\right] / \tfrac{1}{2}\rho_\delta u_\delta^2 = \sqrt{\frac{2\beta^2}{RMs}} \left(\eta \frac{df}{dn}\right)_0 = \sqrt{\frac{2\beta^2}{RMs}}\, \tau_0$$

so that

$$c_f = \sqrt{\left\{\frac{2}{RMs} \left(\frac{\eta h^{\frac{1}{2}}}{\bar{\sigma}^{\frac{1}{2}} g}\right)\right\}}\; Z(0) \qquad \ldots . (5.2)$$

and where

$$k_H = k_0 \left(\frac{\partial T}{\partial y}\right)_0 / \tfrac{1}{2}\rho_\delta u_\delta^3$$

so that

$$\frac{k_H}{c_f} = \lim_{n \to 0} \left\{ k \left(\frac{\partial T}{\partial y}\right) / u_\delta \mu \left(\frac{\partial u}{\partial y}\right) \right\} = \lim_{f \to 0} \left\{ \frac{1}{2\bar{\sigma}} \frac{dh}{df} \right\}$$

i.e. $$\frac{k_H}{c_f} = \frac{1}{2} Y'(0) \qquad \ldots . (5.3)$$

In (5.2), we find from (2.11) and (2.13) that

$$\frac{\eta h^{\frac{1}{2}}}{\bar{\sigma}^{\frac{1}{2}} g} = \frac{1}{\bar{\sigma}^{\frac{1}{2}}} \left(1 + \frac{C}{T_\delta}\right) \left(\frac{2\Gamma}{\gamma - 1} \frac{m}{m_\delta}\right)^{\frac{1}{2}} \qquad \ldots . (5.4)$$

We propose to ignore the change in the molecular weight of the air within the layer, as appreciable dissociation is unlikely to occur within the range of temperatures we are concerned with, and as Sutherland's formula (used in forming the connecting of viscosity with temperature) is unlikely to be valid if appreciable dissociation takes place. Hence, if we put $C/T_\delta = c$, in (5.4)

$$\frac{\eta h^{\frac{1}{2}}}{\bar{\sigma}^{\frac{1}{2}} g} = \frac{1+c}{\bar{\sigma}^{\frac{1}{2}}} \left(\frac{2\Gamma}{\gamma - 1}\right)^{\frac{1}{2}}$$

Thus in (5.2)

$$c_f = \left[\frac{2(1+c)}{RMs}\right]^{\frac{1}{2}} \left[\frac{2}{\bar{\sigma}(\gamma - 1)}\right]^{1/4} Z(0) \qquad \ldots . (5.5)$$

For air, we may take

$$\gamma = \frac{7}{5}, \quad \Gamma = \frac{9}{7} \qquad \ldots . (5.6)$$

and taking a mean value of $Z(0)$ for $\bar{\sigma}$ between 0.7 and 0.8 from Table I, since $Z(0)$ does not change greatly with $\bar{\sigma}$, we find that

$$c_f = 1.6 \sqrt{\frac{1+c}{\bar{\sigma}^{\frac{1}{2}} RMs}} \qquad \ldots . (5.7)$$

At normal temperatures σ, the Prandtl Number is about 0.74, and it is related to the value of the ratio of specific heats: for this reason it will change at high temperatures, and to account for this change we may use Eucken's Formula:

$$\sigma = \frac{4\gamma}{9\gamma - 5} \quad \text{where } \gamma = c_p/c_v \qquad \dots (5.8)$$

At normal temperatures $\gamma = \frac{7}{5}$, but at elevated temperatures, from the results of Fowler and Guggenheim we find that $\gamma = \frac{9}{7}$, so that from (5.8)

$$\sigma_o \doteqdot \sigma_\delta = \frac{14}{19} = 0.737 \ , \ \bar{\sigma} = \frac{18}{23} = 0.782 \qquad \dots (5.9)$$

since $\bar{\sigma}$, the value of σ assumed within the boundary layer, refers to the condition of elevated temperature. Then from (5.9) in (5.7), we have that for air:

$$c_f = 1.7 \sqrt{\frac{1+c}{RMs}} \qquad \dots (5.10)$$

If we ommitted to account for the variation of specific heats with temperature, so that we put $\Gamma = 1$, and $\sigma = 0.737$ in (5.5), then in place of the coefficient 1.7 we should have 1.62, so that it will be seen that there is only a small effect on the skin friction of these variations. The inclusion of dissociation effects would have a more important effect, since we see from (5.4) and (5.2) that c_f will vary as $(m/m_\delta)^{1/4}$: if the molecular weight of the gas within the boundary layer were only half that at normal temperatures, then c_f would be reduced by $10^o/_o$. Evidently, however, the neglect of dissociation will not greatly affect the numerical answer, which will err, (if anything) on the pessimistic side in the evaluation of both skin friction and heat transfer.

To calculate the heat transfer coefficient, we need in (5.3) the value of $Y'(0)$. From Table I we may calculate the following data:

$\bar{\sigma}$	1.0	0.8	0.6
$Y'(0)$	1	1.021	1.048

and within the available accuracy we find that $Y'(0) \doteqdot \bar{\sigma}^{-1/11}$: thus in (5.3)

$$\frac{k_H}{c_f} = \frac{1}{2} \bar{\sigma}^{-1/11} \qquad \dots (5.11)$$

so that, using (5.9), for air

$$\frac{k_H}{c_f} = 0.510 \qquad \dots (5.12)$$

and from (5.10)

$$k_H = 0.86\sqrt{\frac{1+c}{RMs}} \qquad \ldots (5.13)$$

The mean skin friction and heat transfer coefficients may be found from the relations:

$$K_H = \frac{1}{L}\int_0^L k_H\, dx\ , \quad C_f = \frac{1}{L}\int_0^L c_f\, dx$$

and performing the integrations, we have that

$$\left.\begin{aligned} K_H &= 1.72\sqrt{\frac{1+c}{RM}} \\ C_f &= 3.4\sqrt{\frac{1+c}{RM}} \end{aligned}\right\} \qquad \ldots (5.14)$$

The local heat flux to the surface in dimensional terms is, using (5.13),

$$Q = \tfrac{1}{2}k_H\rho_\delta u_\delta^3 = 0.430\ u_\delta^2\sqrt{\left[\frac{T_\delta+C}{T_\delta}\ \frac{\rho_\delta\mu_\delta a_\delta}{x}\right]}$$

If we refer conditions outside the boundary layer (denoted by subscript 'δ') to those at sea-level (denoted by subscript 'SL'), we then find, using Sutherlands Formula and the Perfect Gas Law, that

$$Q = 0.430\left(\frac{p_\delta}{p_{SL}}\right)^{\frac{1}{2}} u_\delta^2\sqrt{\left[\frac{T_{SL}+C}{T_{SL}}\ \frac{\rho_{SL}\mu_{SL}a_{SL}}{x}\right]}$$

Using I.C.A.N. conditions for the properties at sea-level (and putting $C = 117^\circ K$), in metric units, if x is in metres, u_δ in metres/sec., and p_δ in atmospheres, then

$$Q = 4.40 \times 10^{-5}\ u_\delta^2\sqrt{\frac{p_\delta}{x}}\ \text{Kw./sq.m} \qquad \ldots (5.15)$$

In British aeronautical units, if $x = x'$ft, $u_\delta = U'$ft/sec., and p_δ is in atmospheres, then

$$Q = 5.07 \times 10^{-4}\ U'^2\sqrt{\frac{p_\delta}{x'}}\ \text{ft.lb/sq.ft/sec} \qquad \ldots (5.16)$$

In a similar we may calculate the local frictional force to be

$$F = 1.01 \times 10^{-3}\ U'\sqrt{\frac{p_\delta}{x'}}\ \text{lb./sq.ft.} \qquad \ldots (5.17)$$

Also of interest is an expression for the thickness of the boundary layer, since we have already inferred that the boundary layer is finite in thickness in the present asymptotic solution for $M^2 \to \infty$. By definition, from (3.3) and (2.13) if $m = m_\delta$, the value of n for $f = 1$(i.e. at the outside of the boundary layer) is:

$$n_\delta = (1+c)\left(\frac{\gamma-1}{2\Gamma}\right)^{\frac{1}{2}}\int_0^1\left(\frac{h^{\frac{1}{2}}}{\tau}\right)df = \beta(1+c)\left(\frac{\gamma-1}{2\Gamma}\right)^{\frac{1}{2}}\sqrt{\frac{g\bar{\sigma}^{-3/2}}{\eta h^{\frac{1}{2}}}}\left(\int_0^1\frac{Y^{\frac{1}{2}}}{Z}\,df\right)$$

Using (5.4),

$$\frac{1}{\beta} n_\delta = \left[(1+c)^{\frac{1}{2}} \left(\frac{\Upsilon-1}{2\Gamma}\right)^{3/4} \bar{\sigma}^{3/4} \right] \left(\int_0^1 \frac{Y^{\frac{1}{2}}}{Z} \, df \right)$$

From the definition of n it also follows that

$$\frac{1}{\beta} n_\delta = \frac{\delta}{L} \sqrt{\frac{R}{2M^3 s}}$$

so that eliminating n_δ from the two expressions

$$\frac{\delta^2}{L^2} = 2(1+c) \left(\frac{\Upsilon-1}{2\Gamma} \bar{\sigma}\right)^{3/2} \left(\int_0^1 \frac{Y^{\frac{1}{2}}}{Z} \, df \right)^2 \frac{M^3 s}{R} \quad \text{....(5.18)}$$

Using the data of Table I to evaluate the integral, and using the values of Υ and Γ from (5.6), we calculate that

$$\frac{\delta^2}{L^2} = 0.030 \, \bar{\sigma}^{0.70} \left[\frac{(1+c)M^3}{R} \right] s \quad \text{....(5.19)}$$

since, within a reasonable approximation the integral $\int_0^1 (Y^{\frac{1}{2}}/Z) df = 0.494 \, \bar{\sigma}^{-0.4}$.

Thus, from (5.9), for air:

$$\frac{\delta^2}{L^2} = 0.025 \left[\frac{(1+c)M^3}{R} \right] s \quad \text{....(5.20)}$$

In dimensional terms, if $u_\delta = U'$ ft/sec., $x = x'$ ft. and p_δ is in atmospheres

$$\delta = 0.64 \times 10^{-7} \left[U' \sqrt{\frac{x'}{p_\delta}} \right] \text{ft.} \quad \text{....(5.21)}$$

It will be noticed that the boundary layer thickness increases with increase of speed - a trend which has been noticed before in solutions relating to the boundary layer flow at high Mach Number.

From equation (2.14) we have

$$\frac{w}{g} = \int_0^n n \frac{d}{dn}\left(\frac{f}{g}\right) dn$$

$$\text{i.e. } w = nf - g \int_0^n \frac{f}{g} dn$$

or, from (2.11),

$$w = nf - h \int_0^n \frac{f}{h} dn$$

At the outside of the boundary layer, where $f = 1$, $n = n_\delta$ and $h = 0$, it follows that

$$w_\delta = n_\delta$$

or using the definitions of w and n:

$$\frac{v_\delta}{u_\delta} = \frac{\delta}{2x} = \frac{d\delta}{dx} \qquad \text{....(5.22)}$$

since, from (5.20), $\delta \propto x^{\frac{1}{2}}$.

It follows that δ, which is in the present asymptotic solution the finite thickness of the boundary layer, is what is normally termed the 'displacement thickness' of an infinitely thick boundary layer. Thus the flow is tangential to the outer edge of the boundary layer.

The momentum thickness is ϑ where

$$c_f = 2\frac{d\vartheta}{dx}$$

and we find from (5.5) that, after integration

$$\vartheta = \frac{L}{2}\int_0^s c_f \, ds = \left[\frac{2(1+c)}{RM}\, s\right]^{\frac{1}{2}} \left[\frac{2\Gamma}{\bar{\sigma}(\gamma-1)}\right]^{\frac{1}{4}} Z(0)\, L$$

By comparison with (5.18) it follows that

$$\vartheta = \frac{2\Gamma}{(\gamma-1)\bar{\sigma}M^2}\left\{ Z(0) \Big/ \int_0^1 \frac{Y^{\frac{1}{2}}}{Z}\, df \right\} \delta \qquad \text{....(5.23)}$$

For air, we have that

$$\vartheta = \left(\frac{10.7}{M^2}\right)\delta \qquad \text{....(5.24)}$$

and for large Mach Numbers the momentum thickness becomes small compared with the displacement thickness.

The velocity and temperature distributions within the boundary layer may be calculated quite simply from the data of Table I. For, by definition

$$h = \bar{\sigma}\, Y(f)$$

$$\text{i.e. } H - H_\delta = \tfrac{1}{2}\bar{\sigma}\, u_\delta^2\, Y\left(\frac{u}{u_\delta}\right) \qquad \text{....(5.25)}$$

whilst from (3.3), (2.13) and (5.4) we find as before in deriving equation (5.18) that

$$\frac{1}{\beta}\, n = \left[(1+c)^{\frac{1}{2}} \left(\frac{\gamma-1}{2\Gamma}\right)^{\frac{3}{4}} \bar{\sigma}^{-\frac{3}{4}} \right] \left(\int_0^f \frac{Y^{\frac{1}{2}}}{Z}\, df\right) = \frac{y}{L}\sqrt{\frac{R}{2M^3 s}}$$

Hence, from (5.18)

$$\frac{y}{\delta} = \left(\int_0^{u/u_\delta} \frac{\sqrt{Y(f)}}{Z(f)}\, df\right) \Big/ \left(\int_0^1 \frac{\sqrt{Y(f)}}{Z(f)}\, df\right) \qquad \text{....(5.26)}$$

a relation which connects u/u_δ with y/δ.

In figures 1 and 2, the variation of total heat and velocity across the boundary layer is shown as a function of (y/δ) for various values of $\bar{\sigma}$. The variation of the shear stress is also given in figure 3. It will be seen from figure 2 that the maximum temperature within the boundary layer (for air, with $\bar{\sigma} = 0.78$) corresponds to a value of total heat of about one fifth of the stagnation (or reservoir) enthalpy.

6. Limitations of Theory at High Altitudes and Speeds.

There are four assumptions made in the analysis of the boundary layer flow which become invalid if the flight Mach Number is too high in comparison with the Reynolds Number (both of these being assumed of large, but not necessarily comparable, magnitude in the analysis):

(i) the neglect of the non-linear terms in viscosity and heat conductivity in the statement of the equations of energy and momentum, which derive from the third order terms in the Boltzmann equation. These so-called 'Burnett terms' have been shown to be equal to quantities of the order of

$$\frac{\mu_\delta u_\delta}{p_\delta L} = \frac{\gamma M^2}{R}$$

times those preserved in the equations and their exclusion is equivalent to the neglect of M^2/R compared with unity.

(ii) the assumption that the boundary layer is thin. In the equations of the boundary layer, terms of order (δ^2/L^2) are neglected compared with those of order unity. From (5.16), such an assumption is seen for $M^2 >> 1$ to be equivalent to the neglect of M^3/R compared with unity.

(iii) the neglect of the velocity of slip and a temperature jump at the wall. For instance, if there is a slip-velocity u_λ at the wall, and if λ is the mean free path, we have neglected

$$\frac{u_\lambda}{u_\delta} = O\left(\frac{\lambda}{u_\delta}\left.\frac{\partial u}{\partial y}\right|_{y=0}\right) = O\left(c_f \frac{\rho_\delta u_\delta \lambda}{\mu_\delta}\right)$$

But $\mu_\delta = \text{const.} \times \rho_\delta a_\delta \lambda$, so that

$$\frac{u_\lambda}{u_\delta} = O(Mc_f) = O\left(\sqrt{\frac{M}{R_x}}\right)$$

and a similar result follows for the temperature jump. Using (5.6) it follows that for $M^2 \gg 1$, the assumption of zero slip and temperature jump is equivalent to the neglect of $\sqrt{M/R}$ compared with unity.

(iv) the neglect of the disturbance of the external inviscid flow by the presence of the boundary layer. The deflection of the external stream by the boundary layer causes a modification to the pressure distribution along the plate as calculated for inviscid flow. Thus for a plane surface over which we would expect the pressure to be uniform (as assumed) there is in fact a small change in pressure producing a pressure gradient

$$\frac{dp}{dx} = O\left(\frac{\theta}{M}\frac{\rho_\delta u_\delta^2}{Ls}\right) = O\left(\frac{\rho_\delta u_\delta^2 \delta}{ML^2 s^{3/2}}\right)$$

where θ is the inclination of the external stream to the surface (i.e. $\theta = v_\delta/u_\delta = d\delta/dx$). This has been neglected in writing down the equation of motion which includes terms of like order of magnitude to

$$\frac{\partial}{\partial y}(\mu\frac{\partial u}{\partial y}) = O\left(\frac{F}{g}\right) = O\left(\frac{c_f \rho_\delta u_\delta^2}{\delta s}\right)$$

Hence the neglect of this induced disturbance is equivalent to the neglect of a term of order

$$\left[\frac{dp}{dx}/\frac{\partial}{\partial y}(\mu\frac{\partial u}{\partial y})\right] = O\left(\frac{s^{-\frac{1}{2}}\delta^2}{Mc_f L^2}\right) = O\left(\frac{M^{5/2}}{R_x^{\frac{1}{2}}}\right)$$

compared with unity.

To sum up, in the mathematical analysis we have assumed that both $\frac{1}{M^2}$ and $\frac{1}{R}$ are infinitesimals. For the assumptions inherent in the structure of the equations to be valid and if the surface is assumed plane, the quantities M^2/R, M^3/R, $\sqrt{M/R}$, and $M^2\sqrt{M/R}$ must also be infinitesimals. The last mentioned is the most stringent requirement, that

$$R \gg M^5 \qquad \dots (6.1)$$

If the neglect of this term - due to the disturbance of the external flow by the boundary layer - is to involve no errors of larger magnitude than the assumption that terms of order $1/M^2$ may be neglected compared with unity, we must have that

$$\frac{1}{R} = O\left(\frac{1}{M^9}\right) \qquad \ldots (6.2)$$

Thus, if M is numerically about 10, the hypersonic solution here presented will involve errors of about 1 % only, provided that the Reynolds Number exceeds about 10^9. It will be seen therefore that a satisfactory treatment of the hypersonic laminar boundary layer should include a correction for this effect, since such a high Reynolds Number is unlikely to be achieved in flight even at high Mach Number.

On the other hand, if we interpret our initial assumption that the pressure gradient over the surface is uniform as being strictly true, - so that the surface shape will not be plane,[x] as we would expect if the flow could be treated as inviscid, - it then appears that the error in our calculations will be no greater than that involved by assuming $M^2 \gg 1$ provided that the Reynolds Number satisfies the relation

$$\frac{1}{R} = O\left(\frac{1}{M^5}\right) \qquad \ldots (6.3)$$

Thus, for example, if the Mach Number is about 10, then the Reynolds Number should exceed 10^5; below this Reynolds Number both the assumptions regarding the absence of surface slip and the thinness of the boundary layer introduce significant errors of greater magnitude than the assumption that $(1/M^2)$ is small compared with unity.

A value of $R > 10^5$ for M = 10, implies a value of the relative air density not less than 10^{-4}, if the length of the body is about 10 ft., say. This is reached at altitudes below about 200,000 ft., and flight at a higher altitude than this (at a Mach Number of ten) would involve an indicated air-speed of less than about 100 ft./sec.

Provided then that we interpret our assumption of constant surface pressure literally, the results are valid

---

x In this event, the co-ordinates x and y are curvilinear, and the statement of the equations of energy and motion also involve the neglect of terms of order $(K\delta)$ compared with unity, where K is the wall-curvature. Since K is of order δ/L^2, (as may be deduced from an argument on the lines of that appearing in para. 6) their neglect is justified provided δ^2/L^2 is small.

over a wide range of flight conditions. However, this assumption does not imply that the surface is plane without involving a significant error in most flight conditions.

7. Effect of Surface Shape.

We have based our analysis on the assumption that $M \gg 1$, but it should be noticed that this is the Mach Number of the flow at the outside of the boundary layer. (i.e. more explicitly we should put M_δ in place of M.)

If we put M_a equal to the flight Mach Number, then for $M_a \gg 1$, the static temperature behind a shock wave producing a finite deflection is large - of the order of $M^2 T_a$ - and so it follows that the local speed of sound behind the shock is of the same order as the flight speed; in other words $M_\delta = O(1)$ and is not large. If also the Mach Number $M_\delta \gg 1$, then the stream deflection must be small: viz.

$$\alpha = O(1/M_a)$$

where α is the angle of the stream to the direction of motion at the outside of the boundary layer.

There is not the same restriction in accelerated flow, since an expansion produces an increase in local Mach Number, though for $M_a \gg 1$ it is possible to expand the flow through only a small angle α, - again of order $1/M_a$, - before the air pressure is theoretically zero. Also it should be noted that the analysis is not applicable to flows in which M_a is finite although - due to expansion - $M_\delta \gg 1$; for we have assumed in the analysis that T_δ is comparable with the skin temperature, and after such an expansion T_δ would be small.

Hence, we may observe that our analysis is applicable only to surfaces (over which the pressure is a constant) moving at a high Mach Number, M_a, and inclined at a small angle to the direction of motion: i.e.

$$|\alpha| = O\left(\frac{1}{M_a}\right) \qquad \ldots.(7.1)$$

It may then be quite simply shown that, if U is the speed of flight,

$$\left.\begin{aligned} \frac{u_\delta}{U} &= 1 + O\left(\frac{1}{M_a^2}\right) \\ \text{and} \quad \frac{p_\delta}{p_a} &= O(1) \end{aligned}\right\} \qquad \ldots.(7.2)$$

Thus in the expressions derived in the previous paragraph we may everywhere replace u_δ by U, the speed of flight, without affecting the accuracy of the solution. It follows from (5.16) and (5.17) that the skin friction and heat transfer to the surface are influenced by surface shape only in that they are proportional to $\sqrt{p_\delta/p_a}$; i.e.

$$\left.\begin{aligned} Q &= \sqrt{\frac{p_\delta}{p_a}}\, Q\Big|_{\alpha=0} \\ F &= \sqrt{\frac{p_\delta}{p_a}}\, F\Big|_{\alpha=0} \end{aligned}\right\} \quad \ldots\ (7.3)$$

The true surface slope (measured in relation to the free-stream direction) is simply $\alpha_o(x)$, say, if

$$\alpha_o(x) + \theta = \alpha$$

where θ is the inclination of the flow relative to the surface at the outside of the boundary layer. Thus, since $\theta = v_\delta/u_\delta = d\delta/dx$

$$\alpha_o(x) = \alpha - \frac{d\delta}{dx} \quad \ldots\ (7.4)$$

or, from (5.20),

$$\alpha_o(x) = \alpha\left[1 - \frac{0.316}{(\alpha M)}\sqrt{\frac{(1+c)M^5}{Rs}}\right] \quad \ldots\ (7.5)$$

As shown also in para. 6, since from (7.1), αM is finite, it follows that the displacement effect of the boundary layer is only negligible if R is of magnitude M^9 or more.

8. Viscous Form Drag.

Combined with the thickening of the boundary layer at high Mach Number already noted, there will be an increasing displacement of the external flow from its form calculable for inviscid flow. This, as already observed, modifies the pressure distribution from its inviscid form, and gives rise in general to an additional form drag. We define this as a viscous form drag, and it is equal to the difference between the pressure drag calculated for an inviscid flow about the body and the pressure drag in a viscid flow, with the boundary layer.

The latter is simply given, by

$$\int_o^L p_\delta \sin\alpha_o dx = p_\delta\int_o^L \alpha_o(x).dx\left[1+0(\frac{1}{M^2})\right] \quad \ldots\ (8.1)$$

since in the present conditions, p_δ is a constant, and

$$\sin\alpha_o = \alpha_o + \frac{\alpha_o^3}{6} + \ldots = \alpha_o\left[1+O(\alpha_o^2)\right]$$

where α_o is, in general, of order 1/M. From (7.4) it follows that pressure drag in viscid flow = $(\alpha L - \delta\big|_{x=L})p_\delta$.

We may only make an estimate of the pressure drag for an inviscid flow about the body in the general case if we assume that the condition (6.2) is satisfied: i.e. $1/R = O(1/M^9)$. In this event it follows from (7.5) that

$$\frac{\theta}{\alpha} = -\left(\frac{\alpha_o-\alpha}{\alpha}\right) = \frac{1}{\alpha}\frac{d\delta}{dx} = O\left(\frac{1}{M^2}\right) \qquad \ldots .(8.2)$$

in general; and then if p_o denotes the pressure at the surface as calculated for a purely inviscid flow (in the absence of the boundary layer), we have from the linearised theory of supersonic flow that, using (8.2),

$$\frac{p_o-p_\delta}{p_\delta} = \gamma(\alpha_o-\alpha)M\left[1+O(\theta M)\right] = -\gamma M\frac{d\delta}{dx}\left[1+O\left(\frac{1}{M^2}\right)\right] \ldots (8.3)$$

The drag in inviscid flow is then

$$\int_o^L p_o \sin\alpha_o \, dx$$

so that from (8.1) the difference between the drag in viscid and inviscid flow, which is the viscous form drag, is, using (8.2) and (8.3),

$$\int_o^L (p_\delta-p_o)\sin\alpha_o\, dx = \int_o^L (p_\delta-p_o)\,\alpha\, dx\left[1+O\left(\frac{1}{M^2}\right)\right]$$

$$= \gamma M\alpha\int_o^L\frac{d\delta}{dx}\,dx\left[1+O\left(\frac{1}{M^2}\right)\right] = \gamma p_\delta\,\alpha M\delta\Big|_{x=L}\left[1+O\left(\frac{1}{M^2}\right)\right]$$

If we quote this as a drag coefficient, C_D, based on the length L,

$$C_D = \frac{2\alpha}{M}\frac{\delta}{L}\Big|_{s=1}\left[1+O\left(\frac{1}{M^2}\right)\right]$$

Evidently, then, for $M\to\infty$, from (5.20)

$$C_D = 0.316\,(\alpha M)\sqrt{\frac{(1+c)}{MR}}$$

so that from (5.14)

$$\frac{\text{viscous form drag}}{\text{total skin friction}} = \frac{C_D}{C_f} = 0.094\;(\alpha M) \qquad \ldots .(8.3)$$

This relation only applies if $1/R = O(1/M^9)$: for smaller Reynolds Numbers, the difference between α_o and α is larger, as well as the difference between p_o and p_δ. The linearised theory used above overestimates this difference between p_o and p_δ if it is large. Also we have replaced α_o by α which is larger than α_o. Thus, in general, the expression (8.3) will overestimate the viscous form drag, particularly at the lower Reynolds Numbers.

9. <u>Effect of Atmospheric Turbulence and Surface Roughness on Transition.</u>

Our calculations have been restricted to the consideration of a laminar boundary layer, and it is therefore pertinent to enquire under what conditions laminar flow might exist.

One of the main causes of transition is the existence of an adverse pressure gradient over the surface, which is hardly likely to exist at high supersonic speeds in flow over aerofoils: in the present discussion we have considered the pressure to be uniform, and this is probably the most adverse condition likely to be met in practice.

We must also however take into account atmospheric turbulence as a contributary factor, and as a result of which there will be small variations in pressure on the surface. A dimensional consideration shows that if $\bar{p}$ is the root-mean-square of these pressure fluctuations, then the Reynolds Number of Transition is R_T, where

$$R_T = f\left[\frac{d\bar{p}}{dx}\Big/\frac{1}{\delta}\left(\mu\frac{\partial u}{\partial y}\right)_o\right] = f\left(\frac{\delta}{F}\frac{d\bar{p}}{dx}\right)$$

- as is suggested, for example, in 'Modern Developments of Fluid Dynamics' p.328. Since (δ/F) varies along the surface as x, it follows that we may put

$$\frac{\delta}{F}\frac{d\bar{p}}{dx} = \left(\frac{\delta p_\delta}{Fx}\right)\frac{x}{p_\delta}\frac{d\bar{p}}{dx} = \left(\frac{\delta p_\delta}{Fx}\right)\frac{R_T}{M}\frac{d(\bar{p}/p_\delta)}{d(x/\ell)}$$

where $\ell = \mu_\delta/\rho_\delta a_\delta$ is a unit of length characteristic only of the altitude of flight. Hence

$$R_T = g\left[\left(\frac{\delta p_\delta}{Fx}\right)\frac{1}{M}\frac{d(\bar{p}/p_\delta)}{d(x/\ell)}\right] \quad \text{....(9.1)}$$

or substituting from (5.17) and (5.21), for $M^2 \gg 1$,

$$R_T = \text{function of}\left[\frac{1}{M}\frac{d(\bar{p}/p_\delta)}{d(x/\ell)}\right] \quad \text{....(9.2)}$$

The corresponding result for the incompressible boundary layer, i.e. $M^2 \ll 1$, interpreting δ as a displacement thickness, is simply that

$$R_T = \text{function of } \left[\frac{1}{M^3} \frac{d(\bar{p}/p_o)}{d(x/\ell)} \right] \qquad \ldots (9.3)$$

Hence, since an increase in $\bar{p}$ at a certain altitude would certainly be accompanied by a decrease in R_T (i.e a forward movement of transition), it follows that an increase in Mach Number (at constant height) would always increase R_T, most particularly at low M.

Of the other factors affecting transition, surface roughness is an important contributary factor. At the tip of a small projection of height ε, $u = u_\varepsilon$, say where

$$u_\varepsilon = \left(\frac{\partial u}{\partial y}\right)_o \varepsilon = \frac{F\varepsilon}{\mu_o}$$

Then,

$$R_\varepsilon = \frac{\rho_o u_\varepsilon \varepsilon}{\mu_o} = \left(\frac{T_\delta}{T_o}\right)^{1+2\omega} \frac{\rho_\delta}{\mu_\delta^2} F \varepsilon^2 \quad \text{if } \mu \propto T^\omega$$

$$\text{i.e. } \frac{\varepsilon}{L} = \frac{1}{R}\left(\frac{T_o}{T_\delta}\right)^{\omega+0.5} \sqrt{\frac{2R_\varepsilon}{c_f}}$$

If it is supposed that the flow behind a projection closes up if R_ε does not exceed a certain critical value, then it follows from (5.10) that the tolerable roughness height $\varepsilon_{max.}$ for $M^2 \gg 1$ is given by a relation of the type

$$\frac{\varepsilon_{max}}{L} \propto \left(\frac{T_o}{T_\delta}\right)^{\omega+0.5} \left(\frac{Ms}{R^3}\right)^{1/4} \qquad \ldots (9.4)$$

The corresponding expression for incompressible flow, i.e. $M^2 \ll 1$, is

$$\frac{\varepsilon_{max}}{L} \propto \left(\frac{s}{R^3}\right)^{1/4} \qquad \ldots (9.5)$$

Hence an increase in speed at constant height always reduces the tolerable roughness limit, though less rapidly for high M. On the other hand, an increase in M for a constant R, increases the tolerable roughness limit, particularly at high M. Heat transfer to the surface also serves to increase the tolerable roughness.

It will be seen that no definite conclusions can be drawn about the stability of the laminar boundary layer at high speeds. However, a comparison of (9.2) and (9.3), and of (9.4) and (9.5) suggests that the high-speed laminar boundary layer is characteristically less sensitive to atmospheric turbulence and roughness effects than at low speeds. In this sense it is more stable than the incompressible layer.

10. A Comparison with Other Results.

Crocco[1] has suggested from his work on the compressible boundary layer that

$$\frac{Q}{F} = \frac{c_p}{\sigma^{2/3}} \frac{T_{th} - T_o}{U_\delta} \quad \text{where } T_{th} = T_\delta \left(1 + \frac{\gamma - 1}{2} M^2 \sigma^{\frac{1}{2}}\right)$$

which may be interpreted as meaning that

$$\frac{k_H}{c_f} = \frac{1}{2\sigma^{1/6}} \frac{T_{th} - T_o}{T_{th} - T_\delta} \qquad \ldots(10.1)$$

For $M^2 \gg 1$, it follows that

$$\frac{k_H}{c_f} = \frac{1}{2\sigma^{1/6}} + O\left(\frac{1}{M^2}\right) \qquad \ldots(10.2)$$

which may be compared with the solution (5.11) given by the present work. The variation of this ratio with σ at high speeds appears to be less rapid than that suggested by Crocco.

Again, Young[2] has suggested by an extension of Crocco's work and guided by other numerical results that if $\mu \propto T^{\omega}$,

$$S^{\frac{1}{2}} R^{\frac{1}{2}} c_f = 0.664\left[0.45 + 0.55\frac{T_o}{T_\delta} + 0.09(\gamma - 1)M^2 \sigma^{\frac{1}{2}}\right]^{-(1-\omega)/2} \qquad \ldots(10.3)$$

Taking $M^2 \gg 1$, and putting $\omega = 0.5$ as would be appropriate if we have to include theeffects of variation of μ at the very high temperatures occurring inside the boundary layer, evidently for air,

$$c_f = \frac{1.52}{\sqrt{\sigma^{1/4} RMs}}\left[1 + O\left(\frac{1}{M^2}\right)\right] \qquad \ldots(10.4)$$

which apart from a difference in the numerical constant and in the variation with σ is the same as (5.7).

In figure 4, the results of the present work in relation to the variation of c_f with M are compared with the values of c_f obtained from (10.3) for various values of ω and T_o/T_δ (and using a value of σ equal to 0.74.) The present solution gives consistently higher values of the skin friction (and so also of the heat transfer) coefficient than indicated by (10.3) if we suppose that we must lower the value of ω to 0.5 when dealing with flow at high Mach Numbers as in (10.4). At high Mach Number the difference amounts to about 25%. About 5% of this difference may be accounted for by the fact that in the present solution we have adjusted the values of the air specific heats and Prandtl Number, but the bulk of the difference lies in the fact that we have used Sutherland's Formula for the variation of viscosity instead of a simple power law variation. This has the effect of increasing the values of c_f and k_H by the factor $\sqrt{1+\frac{C}{T_\delta}}$ compared with that obtained were we to assume merely that $\mu \propto T^{\frac{1}{2}}$, and so brings the results (at least, for Mach Numbers between 10 and 20) more into line with those of equation (10.3) with ω=0.8-0.7. (This is a mean figure for ω across the boundary layer, but with an emphasis towards its value at normal temperatures) There seems no doubt that the law of the variation of viscosity with temperature is of the greatest importance in influencing the quantitative evaluation of the boundary layer characteristics.

The present analysis, using Sutherland's formula, implies that there is relatively little change in the skin friction coefficient with Mach Number below M=10, and so endorses the applicability of much of the work on the compressible boundary layer with ω=1 (i.e. viscosity varying linearly with the temperature) in relation to such speeds.

Figure 4 also shows from equation (10.3) that the variation of surface temperature is not important in influencing the skin friction at high M: and so confirms the qualitative deduction of the present work concerning this effect. The figure shows two sets of results for (T_o/T_δ) equal to 1 and to 4, which would cover a range of T_o from say about 250°K to 1000°K.

11. Conclusions.

(i) It appears that the high-speed boundary-layer equations may be solved for the condition of laminar flow, in the absence of a pressure gradient, by a numerical process. The results of this report are relevant if the assumptions are made that $M^2 \gg 1$ (where M is the Mach Number of the flow outside the boundary layer) but that the surface temperature is not high (i.e. that it is not commensurate with the thermometer temperature), and that the air molecular specific heats reach an asymptotic value at high temperature (which may be different from that at ordinary temperatures.) The numerical results apply to a flow with Prandtl Numbers between 1.0 and 0.6.

(ii) From the solution it appears that the velocity and temperature vary near the surface in proportion to $y^{2/3}$, where y is the distance away from the surface. This suggests that the velocity and temperature gradient are infinite at the surface. In the limiting condition (as $M \to \infty$) this is shown to be compatible with the existence of gradients whose actual magnitude depends on and increases with M. The solution enables these gradients to be calculated, and the corresponding rate of heat transfer and skin friction assessed.

(iii) The temperature and velocity distributions within the boundary layer are shown in figures 1 and 2. These demonstrate the trends observed in previous analyses: the velocity distribution is at high speeds more nearly linear than at low speeds, and the enthalpy reaches its maximum value (equal to about $\frac{1}{5}$ of the stagnation value for air) at about a third of the way out into the boundary layer. The effect of a decrease in Prandtl Number is to cause a reduction in temperature within the boundary layer, but only a slight increase in velocity.

Figure 3 shows that the shear stress within the flow decreases most rapidly near the outside of the layer, since here both the temperature (and so the viscosity) and the velocity gradient are decreasing.

(iv) The asymptotic solution yields a finite boundary layer thickness, which corresponds with the quantity which is usually termed the 'displacement thickness'. The flow is, accordingly, tangential to the outside of the boundary layer. The compatibility of this result with that for finite Mach Number (where the disturbance due to the boundary layer extends to infinity) is discussed in para. 4.

(v) The boundary layer thickness varies as $\sqrt{x}$, where x is measured along the surface; it varies as $\sqrt{M^3/R}$ and so increases in proportion to the increase of forward speed (as compared with its behaviour at lower Mach Numbers where it decreases with increase of speed). For air, the thickness is given by equations (5.20) and (5.21): the latter states that

$$\delta = 6.4 \times 10^{-7} U' \sqrt{x'/p_\delta} \text{ ft.}$$

where U' is the free-stream velocity (in ft/sec.), p_δ is the surface pressure (in atmospheres), and x' is measured (in ft.) along the surface.

(vi) Associated with the thickening of the boundary layer at high speeds, there is also an important interaction between the boundary layer and the external flow, causing a significant modification to the stream deflection and pressure distribution near the surface as calculated on the assumption of purely inviscid flow. At a Mach Number of about 10, it is shown that these effects are important at Reynolds Numbers less than about 10^9, - which includes most likely flight conditions. Thus the assumption of a constant pressure over the surface in the present argument cannot be interpreted as implying a plane surface, since the displacement effect of the boundary layer on the external flow must be taken into account.

(vii) It is shown that such assumptions as the neglect of surface slip, and the finite thickness of the boundary layer, which are implicit to the present solution of the boundary layer equations, are justifiable within the accuracy of the solution provided that the Reynolds Number R is a large number of the magnitude of M^5 or more. Thus at a Mach Number of 10 or more, the results will have an accuracy within a few per cent, provided that R exceeds 10^5, say: this includes most flight conditions unless the altitude is so high that the indicated airspeed is much less than 100 ft/sec.

(viii) There is no necessity for assuming that the (constant) surface pressure p_δ is the same as that of the free-stream, p_a. However, the assumption that the local Mach Number of the flow outside the boundary layer, M, is large, implies that the slope of the surface to the free stream direction must be small. If α is the inclination of the stream at the outside of the boundary layer, then α must be small so that

$$|\alpha| = O\left(\frac{1}{M_a}\right)$$

where M_a is the Mach Number of flight.

(ix) As remarked above in para. (vi), the deflection of the external flow by boundary layer is generally important, and α_o (the slope of the surface to the direction of motion) is in general different from α. From (7.5) we have that

$$\left(\frac{\alpha-\alpha_o}{\alpha}\right) = O\left(\sqrt{\frac{M^5}{R}}\right)$$

(x) The displacement effect of the boundary layer, in modifying the pressure distribution, will also modify the pressure drag as calculated on the assumption of inviscid flow. It is shown in para. 8 that for large R, the additional drag involved by this effect (which we call the viscous form drag) is commensurate with the skin friction drag; we have from (8.3) that if α is the stream deflection outside the boundary layer

$$\frac{\text{viscous form drag}}{\text{total skin friction}} = 0.094\ (\alpha M)$$

although this relation probably overestimates the additional drag at lower Reynolds Numbers. The effect will be seen to be particularly important at high Mach Numbers.

(xi) The heat transfer to the surface and the skin friction are shown to be independent of the surface temperature, at least to the first order of approximation if terms of order $(1/M^2)$ are neglected in comparison with unity. This fact is borne out by a comparison with existing results relating to the compressible boundary layer flow. It will be recalled that we have assumed that T_o, the surface temperature, is comparable with that outside the boundary layer so that (T_o/T_{th}) is small.

(xii) To account for the variation of viscosity (μ) with temperature, Sutherland's Formula, $\mu \propto (T^{3/2}/T+C)$, is used, with no allowance for dissociation effects at high temperatures. In this formula, the larger the value of C, the more rapid will be the increase of viscosity with temperature. From the present analysis, it appears that the skin friction, heat transfer and boundary layer thickness all increase as C increases.

(xiii) A comparison of the present results with others existing for the laminar boundary layer in compressible flow, which have used a formula for viscosity of the type $\mu \propto T^{\omega}$, shows that even at Mach Numbers between 10 and 20 a value of ω between 0.7 and 0.8 is appropriate to bring the results into line with those using Sutherlands Formula. It seems that it is the variation of viscosity with temperature near the surface which is most important in choosing a representative value of ω; at the high temperatures existing within the boundary layer a value of $\omega = 0.5$ would be more appropriate. The present results also imply that for $M < 10$, there is little change in skin friction coefficient with Mach Number, which again indicates a high value of ω, and endorses the applicability of the methods using a value of ω equal to unity (i.e. assuming $\mu \propto T$).

(xiv) It is assumed in the analysis that at high temperatures the specific heat of the gas at constant pressure increases by a factor Γ. The skin friction and heat transfer are then found to vary as $\Gamma^{1/4}$, and the boundary layer thickness as $\Gamma^{-3/4}$. For air a value of $\Gamma = 9/7$ is chosen which makes little difference to the numerical results for the value of skin friction or heat transfer.

(xv) Connected with the increase in the gas specific heats there will also be an increase in Prandtl Number (σ) of the gas at high temperatures and a mean value of $\sigma = 0.78$ has been used in preference to the more usual value of $\sigma = 0.74$. Here again this makes little difference to the numerical results. The skin friction coefficient is found to vary as $\sigma^{-1/4}$, and the ratio of the heat transfer to the skin friction coefficient (k_H/c_f), varies approximately as $\sigma^{-1/11}$. These are rather less rapid variations than are predicted for conditions at lower Mach Number. The boundary layer thickness varies in proportion to $\sigma^{0.35}$.

(xvi) For air, it is shown (from equation (5.17)) that the shear force at the surface is

$$F = 1.01 \times 10^{-3}\ U' \sqrt{p_\delta / x'} \quad \text{lb./sq.ft.}$$

where U' is the free-stream velocity (ft/sec.), p_δ the surface pressure (atmospheres) and x' the distance along the surface (in ft.) Such a variation corresponds to a skin friction coefficient given by equation (5.10), where it will be seen that $c_f \propto 1/\sqrt{RM}$, and decreases with increase of Mach Number - an effect well established qualitatively in previous solutions.

(xvii) The momentum thickness (ϑ) of the boundary layer becomes appreciably less than the displacement thickness at high Mach Number. From (5.24), $\vartheta = (10.7/M^2)\delta$.

(xviii) The ratio of the heat transfer to skin friction coefficients (k_H/c_f) is found to be equal to $(1/2\sigma^{1/11})$, if k_H is based on $(\frac{1}{2}\rho_\delta u_\delta^3 L)$. For air, $k_H = 0.510\, c_f$.

(xix) It follows that k_H also varies as $\sqrt{1/RM}$, and from (5.16) the local heat flux to the surface is given as

$$Q = 5.07 \times 10^{-4} U'^2 \sqrt{p_\delta/x'} \quad \text{ft.lb./sq.ft./sec.}$$

(xx) Some remarks are made in para. 9 concerning the stability of a laminar layer, and it is concluded that as far as the effects of atmospheric turbulence and surface roughness may affect the transition to turbulence, the laminar boundary layer at high Mach Number is less sensitive to these effects than at low Mach Number, if the Reynolds Number is the same. It does not follow that an increase of speed alone (at constant height) is stabilising.

(xxi) The results of this work have been derived elsewhere[4] by the author, using the approximate method of momentum and energy integrals. Whilst in qualitative agreement, these results give values of the skin friction and heat transfer coefficients half as large again as those deduced here. Perhaps this serves to emphasise the difficulty of attempting to simulate ab initio the variations in temperature and density within a compressible boundary layer, as is necessary if this method is to succeed.

(xxii) Particularly in view of the strong interaction between the boundary layer and the external flow, the greatest need in developing a theory of the boundary layer at high speeds, is for one which will take into account variations in pressure over the surface.

References.

1. Crocco, L. — Lo Strato Limite Lanimare nei Gas. Monografie Scientifiche di Aeronautica No. 3, 1946.

2. Young, A.D. — Skin Friction in the Laminar Boundary Layer in Compressible Flow. Aero.Quart. Vol.I., Pt.II. August 1949.

3. Tsien, H. — Superaerodynamics. The Mechanics of Rarified Gases. J.Aero.Sci. Vol.13. No.12.

4. Nonweiler, T. — Aerodynamic Heating at High Speeds. J.Brit.Interplanetary Soc. Vol.10,No.4. July 1951.

5. Kirkby, S. and Nonweiler, T. — The Numerical Solution of Certain Differential Equations occurring in Crocco's Theory of the Laminar Boundary Layer. To be published.

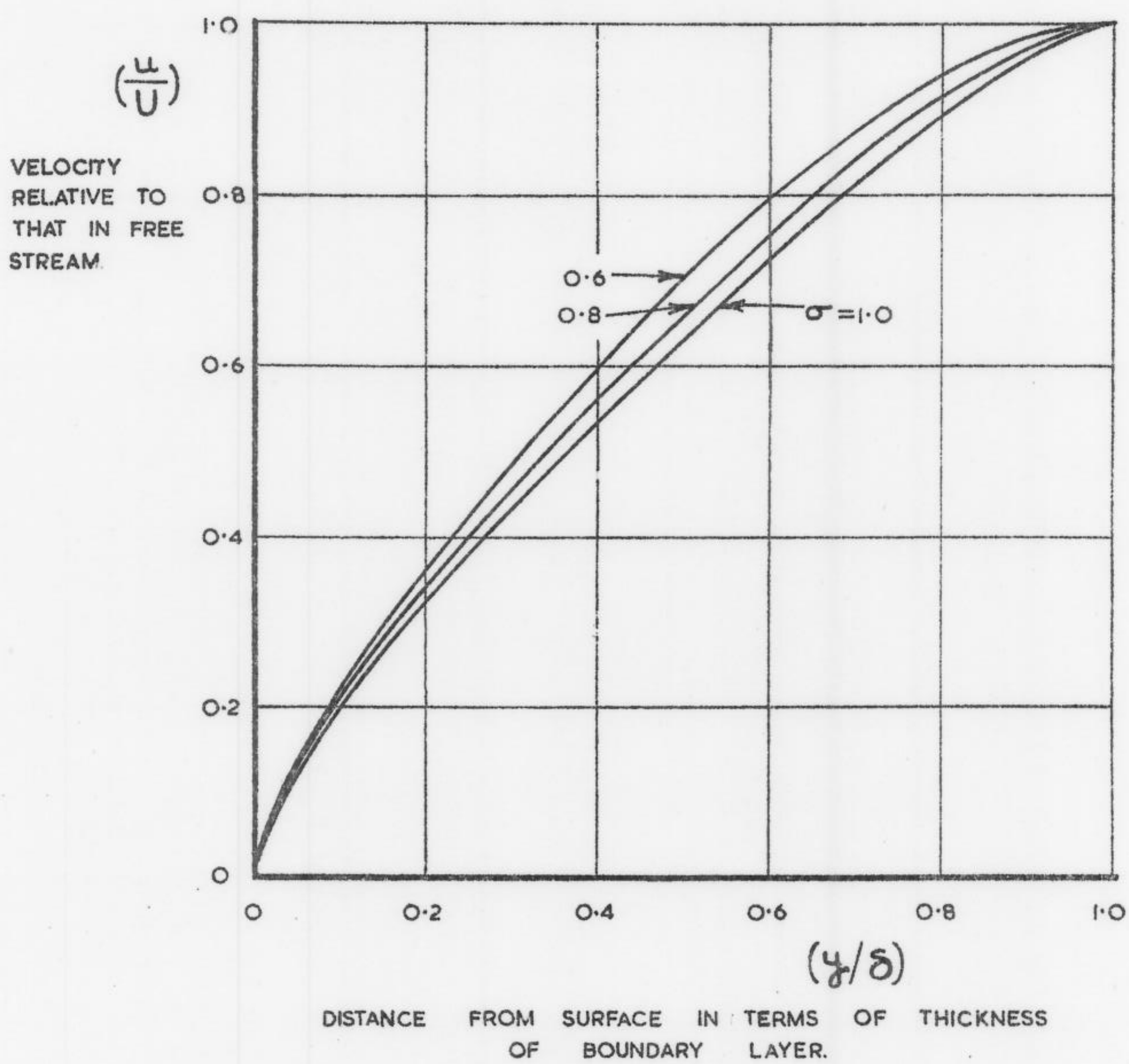

FIG. I. VARIATION IN VELOCITY WITHIN BOUNDARY LAYER WITH VALUE OF THE PRANDTL NUMBER σ AT HIGH MACH NUMBER.

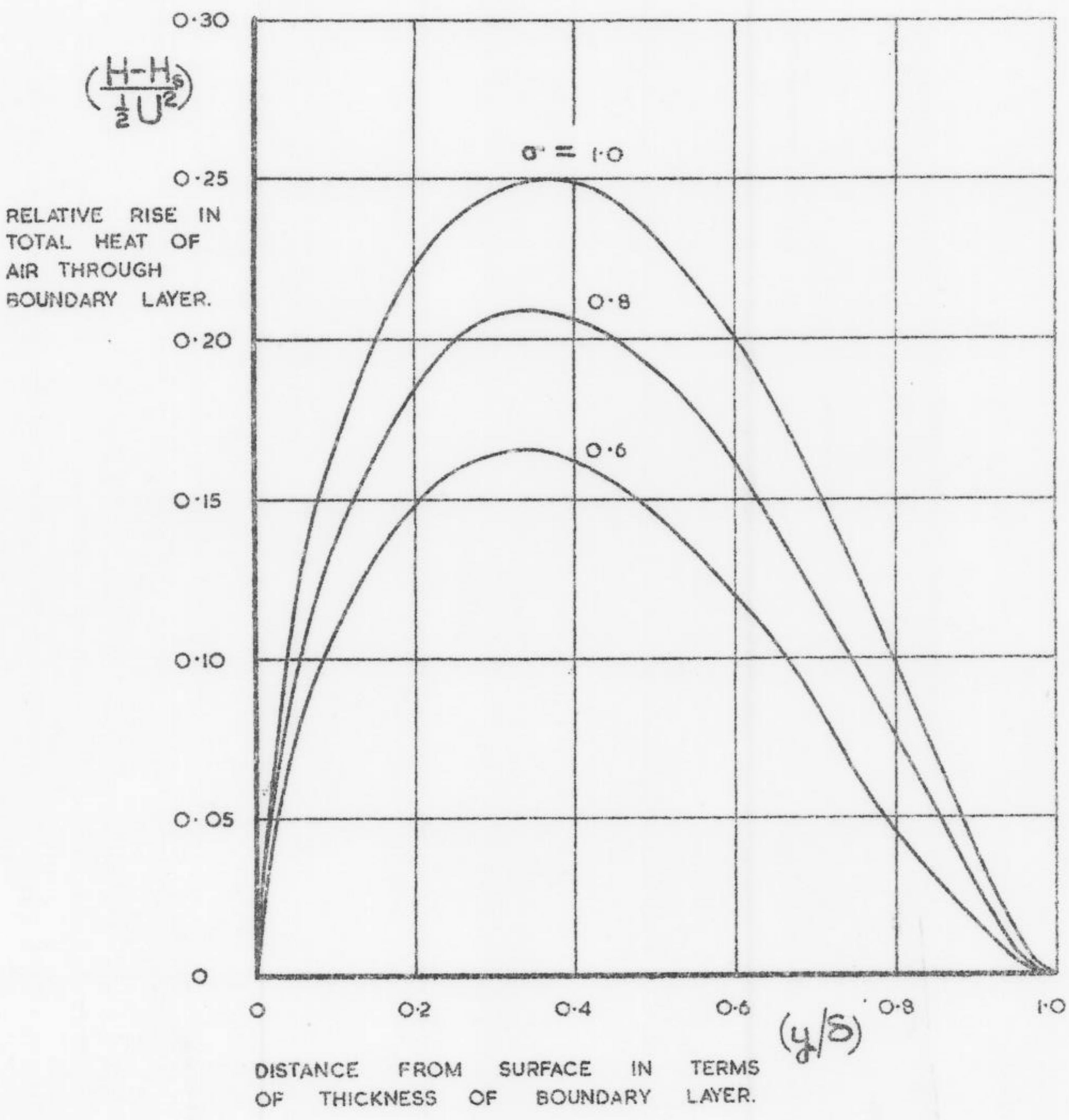

FIG. 2. VARIATION IN AIR TOTAL HEAT WITHIN BOUNDARY LAYER WITH VALUE OF PRANDTL NUMBER σ AT HIGH MACH NUMBER.

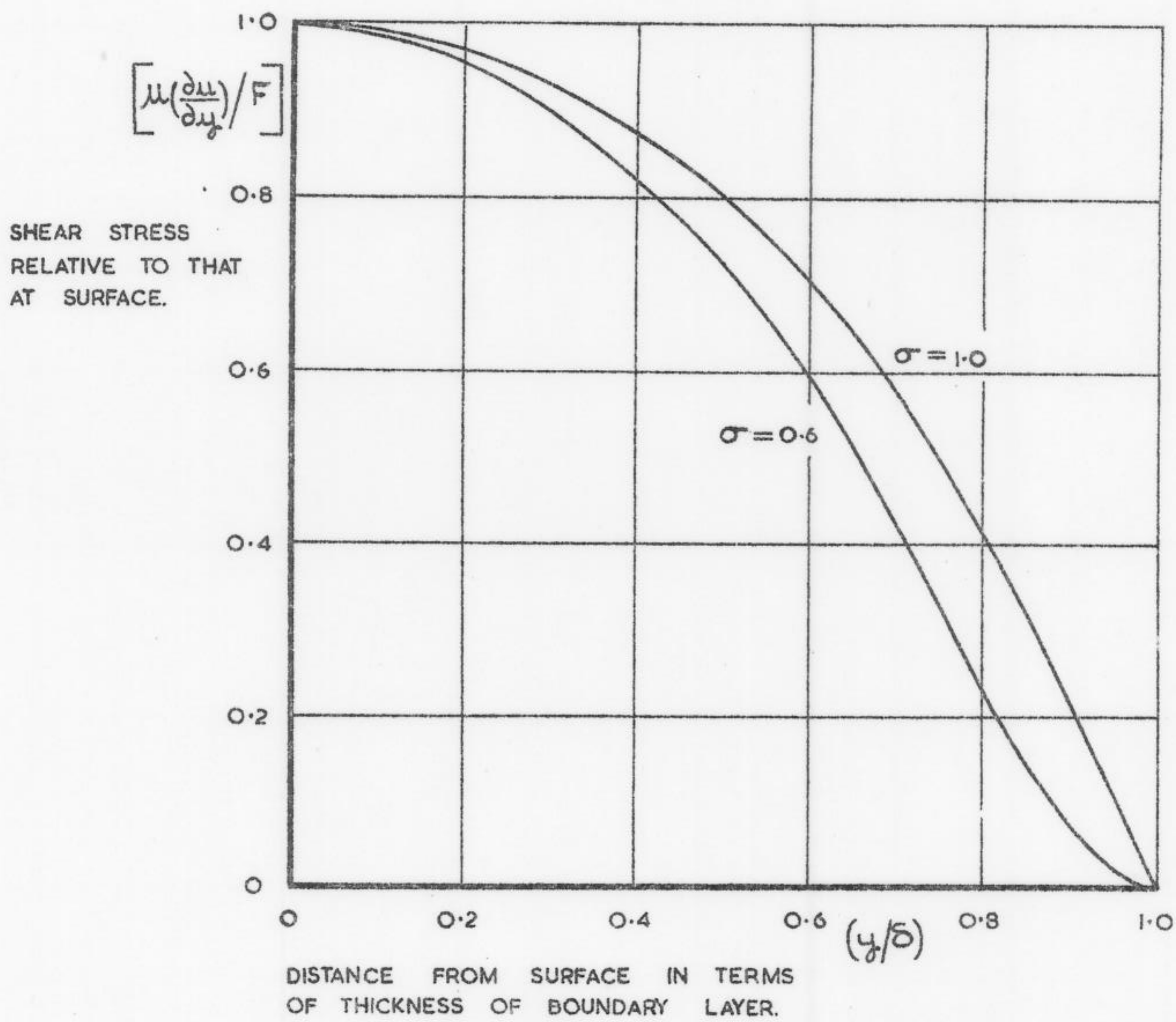

FIG. 3. VARIATION IN SHEAR STRESS WITHIN BOUNDARY LAYER WITH VALUE OF PRANDTL NUMBER σ AT HIGH MACH NUMBER.

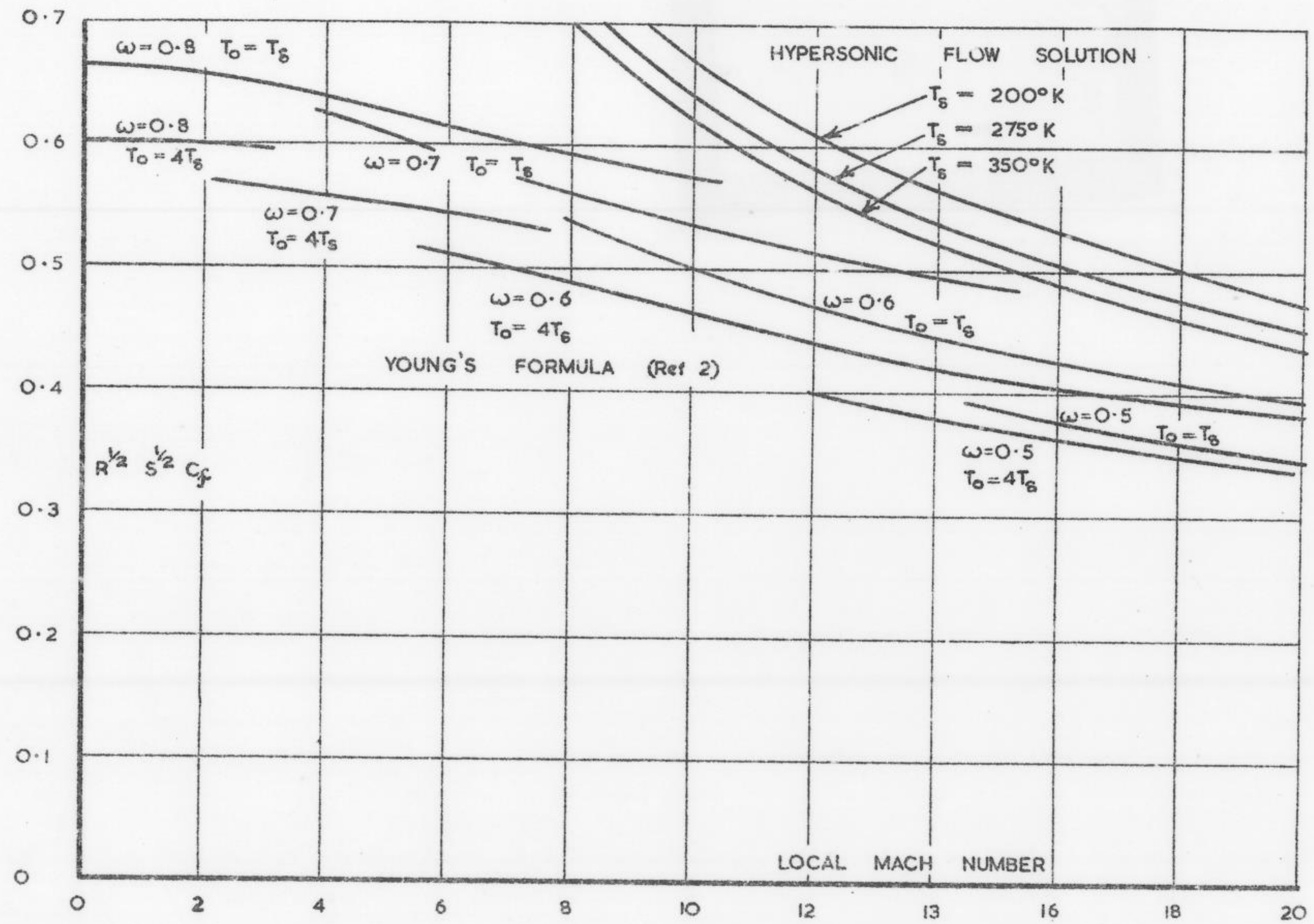

FIG. 4. A COMPARISON OF THE SKIN FRICTION COEFFICIENT DEDUCED FROM THE HYPERSONIC SOLUTION AND THE FORMULA OF YOUNG. (REF. 2)