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ANALYSIS OF TWO-OPERATOR BOUNDARY-DOMAIN INTEGRAL EQUATIONS FOR VARIABLE-COEFFICIENT MIXED BVP

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Abstract. Applying the two-operator approach, the mixed (Dirichlet-Neumann) boundary value problem for a second-order scalar elliptic differential equation with variable coefficient is reduced to several systems of Boundary Domain Integral Equations, briefly BDIEs. The two-operator BDIE system equivalence to the boundary value problem, BDIE solvability and invertibility of the boundary-domain integral operators are proved in the appropriate Sobolev spaces.

1 Introduction

Partial Differential Equations (PDEs) with variable coefficients often arise in mathematical modelling of inhomogeneous media (e.g. functionally graded materials or materials with damage induced inhomogeneity) in solid mechanics, electromagnetics, thermo-conductivity, fluid flows trough porous media, and other areas of physics and engineering.

Generally, explicit fundamental solutions are not available if the PDE coefficients are not constant, preventing reduction of Boundary Value Problems (BVPs) for such PDEs to explicit boundary integral equations, which could be effectively solved numerically. Nevertheless, for a rather wide class of variable-coefficient PDEs it is possible to use instead an explicit parametrix (Levi function) associated with the fundamental solution of the corresponding frozen-coefficient PDEs, and reduce BVPs for such PDEs to systems of Boundary-Domain Integral Equations for further numerical solution of the latter, see e.g. [2, 3, 8, 9, 10, 12] and references therein. However this (one-operator) approach does not work when the fundamental solution of the frozen-coefficient PDE is not known explicitly (as e.g. in the Lamé system of anisotropic elasticity).

To overcome this difficulty, one can apply the so-called two-operator approach, formulated in [11] for a certain non-linear problem, that employs a parametrix of another (second) PDE, not related with the PDE in question, for reducing the BVP to a BDIE system. Since the second PDE is rather arbitrary, one can always chose it in such a way, that its parametrix is known explicitly. The simplest choice for the second PDE is the one with an explicit fundamental solution.

The corresponding BVPs are well studied nowadays, see e.g. [6, 5, 7], but this is not the case for the two-operator Boundary-Domain Integral Equations associated with the BVPs. The BDIE analysis is useful for discretisation and numerical solution of the BDIE and thus of the associated BVP. The BDIE numerical applications are beyond the scope of this paper being however the subject of other publications, see e.g. [18, 19, 17, 15, 8, 9, 16].

To analyse the two-operator approach, we apply in this paper one of its linear versions to the mixed (Dirichlet-Neumann) BVP for a linear second-order scalar elliptic variable-coefficient PDE and reduce it to four different BDIE systems. Although the considered BVP can be reduced to some other BDIE systems also by the one-operator approach, it can be considered as a simple "toy" model showing the main features of the two-operator approach arising also in reducing more general BVPs to BDIEs. The two-operator BDIE systems are nonstandard systems of equations containing integral operators defined on the domain under consideration and potential type and pseudo-differential operators defined on open sub-manifolds of the boundary. Using the results of [2], we give a rigorous analysis of the two-operator BDIEs and show that the BDIE systems are *equivalent* to the mixed BVP and thus are *uniquely solvable*, while the corresponding boundary domain integral operators are *invertible* in the appropriate Sobolev-Slobodetski (Bessel-potential) spaces. This paper extends our publication [1].

2 Function spaces and BVP

Let $\Omega = \Omega^+$ be a bounded open three-dimensional region of $\mathbb{R}^3$, $\Omega^-:=\mathbb{R}^3\setminus\overline{\Omega^+}$ and the boundary $\partial\Omega$ be a simply connected, closed, infinitely smooth surface. Moreover, $\partial\Omega = \overline{\partial_D\Omega}\bigcup\overline{\partial_N\Omega}$ where $\partial_D\Omega$ and $\partial_N\Omega$ are open, non-empty, non-intersecting, simply connected sub-manifolds of $\partial\Omega$ with an infinitely smooth boundary curve $\overline{\partial_D\Omega}\bigcap\overline{\partial_N\Omega}\in C^\infty$. Let us denote $\partial_j := \partial/\partial x_j$ $(j = 1, 2, 3)$, $\partial_x = (\partial_1, \partial_2, \partial_3)$. We consider the following PDE with a scalar variable coefficient $a \in C^\infty(\mathbb{R}^3)$, $a(x) \geq C > 0$,

$$L_a u(x) := L_a(x, \partial_x)u(x) := \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left[a(x)\frac{\partial u(x)}{\partial x_i}\right] = f(x), \quad x \in \Omega^\pm, \tag{2.1}$$

where u is the unknown function and f is a given function in $\Omega^\pm$.

In what follows, $H^s(\Omega^+) = H^s_2(\Omega^+)$, $H^s_{loc}(\Omega^-) = H^s_{2,\,loc}(\Omega^-)$, $H^s(\partial\Omega) = H^s_2(\partial\Omega)$ denote the Bessel potential spaces (coinciding with the Sobolev-Slobodetski spaces if $s \geq 0$). For $S_1 \subset \partial\Omega$, we shall use the subspace $\widetilde{H}^s(S_1) = \{g :\ g \in H^s(\partial\Omega),\ \mathrm{supp}(g) \subset \overline{S_1}\}$ of $H^s(\partial\Omega)$, while $H^s(S_1) = \{r_{_{S_1}} g : g \in H^s(\partial\Omega)\}$, where $r_{_{S_1}}$ denotes the restriction operator on S_1.

By the trace theorem (see, e.g., [6]) for $u \in H^1(\Omega^\pm)$, it follows that $u|^\pm_{\partial\Omega} := \gamma^\pm u \in H^{\frac{1}{2}}(\partial\Omega)$, where $\gamma^\pm$ is the trace operator on $\partial\Omega$ from $\Omega^\pm$. We shall write γu for $\gamma^\pm u$ if $\gamma^+u = \gamma^-u$. We shall also use the notation $u^\pm$ for the traces $u|^\pm_{\partial\Omega}$, when this will cause no confusion.

For a linear operator L_*, we introduce the following subspace of $H^s(\Omega^\pm)$ [5, 4]:

$$H^{s,0}(\Omega^\pm; L_*) := \{g \in H^s(\Omega^\pm) : L_*g \in L_2(\Omega^\pm)\},$$

$$\|g\|^2_{H^{s,0}(\Omega^\pm;L_*)} := \|g\|^2_{H^s} + \|L_*g\|^2_{H^0(\Omega^\pm)} = \|g\|^2_{H^s} + \|L_*g\|^2_{L_2(\Omega^\pm)}.$$

In this paper, we will particularly use the space $H^{1,0}(\Omega^\pm; L_*)$ for L_* being either the operator L_a defined in (2.1) or the Laplace operator Δ, and one can see that these spaces coincide.

For $u \in H^{1,0}(\Omega^\pm; \Delta)$, we can correctly define the (canonical) co-normal derivative $T_a^\pm u \in H^{-\frac{1}{2}}(\partial\Omega)$, cf. [4, 7, 13], as

$$\langle T_a^\pm u, w\rangle_{\partial\Omega} := \pm \int_{\Omega^\pm} \left[\gamma_{-1}^\pm w \cdot L_a u + E_a(u, \gamma_{-1}^\pm w)\right] dx \quad \forall\, w \in H^{1/2}(\partial\Omega), \tag{2.2}$$

where $\gamma_{-1}^\pm : H^{1/2}(\partial\Omega) \to H^1(\Omega^\pm)$ is a right inverse to the trace operator $\gamma^\pm$,

$$E_a(u, v) := \sum_{i=1}^{3} a(x) \frac{\partial u(x)}{\partial x_i} \frac{\partial v(x)}{\partial x_i} = a(x)\nabla u(x) \cdot \nabla v(x)$$

and $\langle \cdot, \cdot \rangle_{\partial\Omega}$ denotes the duality brackets between the spaces $H^{-\frac{1}{2}}(\partial\Omega)$ and $H^{\frac{1}{2}}(\partial\Omega)$, which extend the usual $L_2(\partial\Omega)$ inner product; to simplify notations we shall also write sometimes the duality brackets as integral. Then for $u \in H^{1,0}(\Omega^\pm; \Delta)$, $v \in H^1(\Omega)$ the first Green identity holds, [4, Lemma 3.4], [13, Theorem 3.9],

$$\int_{\Omega^\pm} v(x) L_a u(x) dx = \pm \int_{\partial\Omega} \gamma^+ v(x) T_a^+ u(x) dS(x) - \int_{\Omega^\pm} E_a(u, v) dx\ . \tag{2.3}$$

If $u \in H^2(\Omega^\pm)$, the canonical co-normal derivative $T_a^\pm u$ defined by (2.2) reduces to its classical form

$$T_a^\pm u := \sum_{i=1}^{3} a(x) n_i(x) \gamma^\pm \left[\frac{\partial u(x)}{\partial x_i}\right] = a(x)\gamma^\pm \left[\frac{\partial u(x)}{\partial n(x)}\right], \tag{2.4}$$

where $n(x)$ is the exterior (to $\Omega^\pm$) unit normal at the point $x \in \partial\Omega$.

We shall derive and investigate the *two-operator boundary-domain integral equation systems* for the following mixed boundary value problem.

$$L_a u = f \qquad \text{in } \Omega, \tag{2.5}$$
$$\gamma^+ u = \varphi_0 \qquad \text{on } \partial_D\Omega, \tag{2.6}$$
$$T_a^+ u = \psi_0 \qquad \text{on } \partial_N\Omega, \tag{2.7}$$

where $\varphi_0 \in H^{\frac{1}{2}}(\partial_D\Omega)$, $\psi_0 \in H^{-\frac{1}{2}}(\partial_N\Omega)$ and $f \in L_2(\Omega)$. Equation (2.5) is understood in the distributional sense, condition (2.6) in the trace sense, while equality (2.7) in the functional sense (2.2).

Let us consider the auxiliary linear elliptic partial differential operator L_b defined by

$$L_b u(x) := L_b(x, \partial_x) u(x) := \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left[b(x) \frac{\partial u(x)}{\partial x_i}\right], \tag{2.8}$$

where $b \in C^\infty(\mathbb{R}^3)$, $b(x) \geq C > 0$. Then for $u \in H^{1,0}(\Omega^\pm; \Delta) = H^{1,0}(\Omega^\pm; \Delta)$ the associate co-normal derivative operator $T_b^\pm$ is defined by (2.2) (and for $u \in H^2(\Omega^\pm)$ by (2.4)) with a replaced by b. If $v \in H^{1,0}(\Omega^\pm; \Delta)$, $u \in H^1(\Omega)$, then for the operator L_b holds the first Green identity,

$$\int_{\Omega^\pm} u(x) L_b v(x) dx = \pm \int_{\partial\Omega} \gamma^+ u(x) T_b^\pm v(x) dS - \int_{\Omega^\pm} E_b(u, v) dx. \tag{2.9}$$

If $u, v \in H^{1,0}(\Omega^\pm; \Delta)$, then subtracting (2.3) from (2.9), we obtain the *two-operator second Green identity*, cf. [11],

$$\int_{\Omega^\pm} \{u(x) L_b v(x) - v(x) L_a u(x)\} \, dx =$$
$$\pm \int_{\partial\Omega} \left\{\gamma^\pm u(x) T_b^+ v(x) - \gamma^\pm v(x) T_a^+ u(x)\right\} dS + \int_{\Omega^\pm} [a(x) - b(x)] \nabla v(x) \cdot \nabla u(x) dx. \tag{2.10}$$

Note that if $a = b$, then, the last domain integral disappears, and the two-operator Green identity reduces to the classical second Green identity.

3 Parametrix and potential type operators

As follows from [14, 8, 2], the function

$$P_b(x, y) = -\frac{1}{4\pi b(y)|x - y|}, \qquad x, y \in \mathbb{R}^3 \tag{3.1}$$

is a parametrix (Levi function) for the operator $L_b(x; \partial_x)$ from (2.8), i.e., it satisfies the equation

$$L_b(x, \partial_x) P_b(x, y) = \delta(x - y) + R_b(x, y)$$

with

$$R_b(x, y) = \sum_{i=1}^{3} \frac{x_i - y_i}{4\pi b(y)|x - y|^3} \frac{\partial b(x)}{\partial x_i}, \qquad x, y \in \mathbb{R}^3. \tag{3.2}$$

The parametrix given by (3.1) is obtained as $P_b(x, y) = F_b(x, y; y)$, where

$$F_b(x, y'; y) = -\frac{1}{4\pi b(y)|x - y'|}, \qquad x, y \in \mathbb{R}^3$$

is the fundamental solution of the operator $L_b(y, \partial_x) := b(y)\Delta_x$ with "frozen" coefficient $b(x) = b(y)$, i.e.,

$$L_b(y, \partial_x) F_b(x, y'; y) = \delta(x - y').$$

For the parametrix $P_b(x, y)$, we evidently have,

$$L_b(y, \partial_x) P_b(x, y) = \delta(x - y).$$

The parametrix-based Newtonian and the remainder volume potential operators, corresponding to parametrix (3.1) and to remainder (3.2) are given, respectively, by

$$\mathcal{P}_b g(y) := \int_\Omega P_b(x,y)g(x)dx, \quad \mathcal{R}_b g(y) := \int_\Omega R_b(x,y)g(x)dx. \tag{3.3}$$

Let us introduce the single layer and the double layer surface potential operators, based on parametrix (3.1),

$$V_b g(y) := -\int_{\partial\Omega} P_b(x,y)g(x)dS_x, \qquad y \notin \partial\Omega, \tag{3.4}$$

$$W_b g(y) := -\int_{\partial\Omega} [T_b(x,n(x),\partial_x)P_b(x,y)]g(x)dS_x, \qquad y \notin \partial\Omega. \tag{3.5}$$

For $y \in \partial\Omega$, the corresponding boundary integral (pseudodifferential) operators of direct surface values of the simple layer potential, $\mathcal{V}_b$, and of the double layer potential, $\mathcal{W}_b$, are

$$\mathcal{V}_b g(y) \;:=\; -\int_{\partial\Omega} P_b(x,y)g(x)dS_x, \tag{3.6}$$

$$\mathcal{W}_b g(y) \;:=\; -\int_{\partial\Omega} [T_b(x,n(x),\partial_x)P_b(x,y)]g(x)dS_x. \tag{3.7}$$

We can also calculate at $y \in \partial\Omega$ the co-normal derivatives, associated with the operator L_a, of the single layer potential and of the double layer potential:

$$T_a^\pm V_b g(y) \;=\; \frac{a(y)}{b(y)} T_b^\pm V_b g(y), \tag{3.8}$$

$$\mathcal{L}_{ab}^\pm g(y) := T_a^\pm W_b g(y) \;=\; \frac{a(y)}{b(y)} T_b^\pm W_b g(y) =: \frac{a(y)}{b(y)} \mathcal{L}_b^\pm g(y). \tag{3.9}$$

The direct value operators associated with (3.8) are

$$\mathcal{W}'_{ab} g(y) \;:=\; -\int_{\partial\Omega} [T_a(y,n(y),\partial_y)P_b(x,y)]g(x)dS_x = \frac{a(y)}{b(y)} \mathcal{W}'_b g(y), \tag{3.10}$$

$$\mathcal{W}'_b g(y) \;:=\; -\int_{\partial\Omega} [T_b(y,n(y),\partial_y)P_b(x,y)]g(x)dS_x. \tag{3.11}$$

From equations (3.3)-(3.11) we deduce representations of the parametrix-based surface potential boundary operators in terms of their counterparts for $b = 1$, that is, associated with the fundamental solution $P_\Delta = -(4\pi|x-y|)^{-1}$ of the Laplace operator Δ.

$$\mathcal{P}_b g \;=\; \frac{1}{b}\mathcal{P}_\Delta g, \quad \mathcal{R}_b g = -\frac{1}{b}\sum_{j=1}^{3} \partial_j \mathcal{P}_\Delta \left[g(\partial_j b)\right], \tag{3.12}$$

$$\frac{a}{b}V_a g \;=\; V_b g \;=\; \frac{1}{b}V_\Delta g; \qquad \frac{a}{b}W_a\left(\frac{bg}{a}\right) \;=\; W_b g \;=\; \frac{1}{b}W_\Delta\,(bg)\,, \tag{3.13}$$

$$\frac{a}{b}\mathcal{V}_a g \;=\; \mathcal{V}_b g \;=\; \frac{1}{b}\mathcal{V}_\Delta g; \qquad \frac{a}{b}\mathcal{W}_a\left(\frac{bg}{a}\right) \;=\; \mathcal{W}_b g \;=\; \frac{1}{b}\mathcal{W}_\Delta\,(bg)\,, \tag{3.14}$$

$$\mathcal{W}'_{ab}g \;=\; \frac{a}{b}\mathcal{W}'_b g \;=\; \frac{a}{b}\left\{\mathcal{W}'_\Delta\,(bg) + \left[b\frac{\partial}{\partial n}\left(\frac{1}{b}\right)\right]\mathcal{V}_\Delta g\right\}, \tag{3.15}$$

$$\mathcal{L}^{\pm}_{ab}g \;=\; \frac{a}{b}\mathcal{L}^{\pm}_b g \;=\; \frac{a}{b}\left\{\mathcal{L}_\Delta(bg) + \left[b\frac{\partial}{\partial n}\left(\frac{1}{b}\right)\right]\gamma^{\pm}W_\Delta(bg)\right\}. \tag{3.16}$$

It is taken into account that b and its derivatives are continuous in $\mathbb{R}^3$ and $\mathcal{L}_\Delta(bg) := \mathcal{L}^+_\Delta(bg) = \mathcal{L}^-_\Delta(bg)$ by the Liapunov-Tauber theorem.

The mapping and jump properties of the parametrix-based volume and surface potentials follow from [2] (see also [12]) and are provided in Appendix A to this paper.

3.1 Two-operator third Green identity

For $v(x) := P_b(x,y)$ and $u \in H^{1,0}(\Omega;\Delta)$, we obtain from (2.10) by standard limiting procedures (cf. [14]) the *two-operator third Green identity*,

$$u + \mathcal{Z}_b u + \mathcal{R}_b u - V_b T^+_a u + W_b \gamma^+ u = \mathcal{P}_b L_a u \qquad \text{in } \Omega, \tag{3.17}$$

where

$$\mathcal{Z}_b u(y) \;:=\; -\int_\Omega [a(x) - b(x)]\nabla_x P_b(x,y)\cdot\nabla u(x)dx \;=\; \frac{1}{b(y)}\sum_{j=1}^{3}\partial_j \mathcal{P}_\Delta\left[(a-b)\partial_j u\right](y), \qquad y \in \Omega. \tag{3.18}$$

Using the Gauss divergence theorem, we can rewrite $\mathcal{Z}_b u(y)$ in the form that does not involve derivatives of u,

$$\mathcal{Z}_b u(y) \;=\; \left[\frac{a(y)}{b(y)} - 1\right]u(y) + \widehat{\mathcal{Z}}_b u(y), \tag{3.19}$$

$$\widehat{\mathcal{Z}}_b u(y) \;:=\; \frac{a(y)}{b(y)}W_a\gamma^+ u(y) - W_b\gamma^+ u(y) + \frac{a(y)}{b(y)}\mathcal{R}_a u(y) - \mathcal{R}_b u(y), \tag{3.20}$$

which allows to call $\mathcal{Z}_b$ integral operator in spite of its integro-differential representation (3.18).

Note that substituting (3.19)-(3.20) in (3.17) and multiplying by $b(y)/a(y)$ one reduces (3.17) to the one-operator parametrix-based third Green identity obtained in [2],

$$u + \mathcal{R}_a u - V_a T^+_a u + W_a\gamma^+ u = \mathcal{P}_a L_a u \qquad \text{in } \Omega.$$

Relations (3.18)-(3.20) and the mapping properties of $\mathcal{P}_a$, $\mathcal{R}_a$, $\mathcal{R}_b$, W_a and W_b, see Appendix, imply the following assertion.

Theorem 3.1 *The operators*

$$\begin{aligned}\mathcal{Z}_b &: H^s(\Omega) \to H^s(\Omega), \quad s > \frac{1}{2},\\ \widehat{\mathcal{Z}}_b &: H^s(\Omega) \to H^{s,0}(\Omega;\Delta), \quad s \geq 1,\end{aligned}$$

are continuous.

If $u \in H^{1,0}(\Omega;\Delta)$ is a solution to equation (2.5) with $f \in L_2(\Omega)$, then (3.17) gives

$$u + \mathcal{Z}_b u + \mathcal{R}_b u - V_b T_a^+ u + W_b \gamma^+ u = \mathcal{P}_b f \quad \text{in} \quad \Omega, \tag{3.21}$$

$$\frac{1}{2}\gamma^+ u + \gamma^+ \mathcal{Z}_b u + \gamma^+ \mathcal{R}_b u - \mathcal{V}_b T_a^+ u + \mathcal{W}_b \gamma^+ u = \gamma^+ \mathcal{P}_b f \quad \text{on} \quad \partial\Omega, \tag{3.22}$$

$$\left(1 - \frac{a}{2b}\right) T_a^+ u + T_a^+ \mathcal{Z}_b u + T_a^+ \mathcal{R}_b u - \mathcal{W}'_{ab} T_a^+ u + \mathcal{L}_{ab}^+ \gamma^+ u = T_a^+ \mathcal{P}_b f \quad \text{on} \quad \partial\Omega. \tag{3.23}$$

Note that if $\mathcal{P}_b$ is not only the parametrix but also the fundamental solution of the operator L_b, then the remainder operator $\mathcal{R}_b$ vanishes in (3.21)-(3.23) (and everywhere in the paper), while the operator $\mathcal{Z}_b$ does not unless $L_a = L_b$.

For some functions f, Ψ, Φ, let us consider a more general "indirect" integral relation, associated with (3.21),

$$u + \mathcal{Z}_b u + \mathcal{R}_b u - V_b \Psi + W_b \Phi = \mathcal{P}_b f, \qquad \text{in} \ \ \Omega \tag{3.24}$$

Lemma 3.1 *Let $f \in L_2(\Omega)$, $\Psi \in H^{-\frac{1}{2}}(\partial\Omega)$, $\Phi \in H^{\frac{1}{2}}(\partial\Omega)$, and $u \in H^1(\Omega)$ satisfy (3.24). Then $u \in H^{1,0}(\Omega;\Delta)$,*

$$L_a u = f \quad \textit{in } \Omega \tag{3.25}$$

and

$$V_b\left(\Psi - T_a^+ u\right) - W_b\left(\Phi - \gamma^+ u\right) = 0 \qquad \text{in} \ \ \Omega. \tag{3.26}$$

Proof. We generalize here the proof of Lemma 4.1 given in [2] for equation (3.24) without $\mathcal{Z}_b u$. First of all, let us prove that $u \in H^{1,0}(\Omega; L_a)$. Indeed, since

$$L_a u = \Delta(au) - \sum_{i=1}^{3} \partial_i(u\partial_i a),$$

and the last term belongs to $L_2(\Omega)$, we need only to show that $\Delta[au] \in L_2(\Omega)$ (the derivatives are understood in the distributional sense). Furthermore, by (3.24) due to (3.19) we have

$$au = b\mathcal{P}_b f - b\mathcal{R}_b u - b\widehat{\mathcal{Z}}_b u + bV_b\Psi - bW_b\Phi = \mathcal{P}_\Delta f - b\mathcal{R}_b u - b\widehat{\mathcal{Z}}_b u + V_\Delta \Psi - W_\Delta(b\Phi)$$

We notice that the last two terms in the right-hand side are harmonic functions. It is clear that $\mathcal{R}_b u \in H^2(\Omega)$, $\widehat{\mathcal{Z}}_b u \in H^{1,0}(\Omega)$ for $u \in H^1(\Omega)$ and $\Delta[\mathcal{P}_\Delta f] = f \in L_2(\Omega)$. Therefore $\Delta(au) \in L_2$ and thus $L_a(x, \partial_x)u \in L_2(\Omega)$. So we can write two-operator Green identity (3.24) for the function u.

Subtracting (3.24) from (3.17), we obtain

$$-V_b\Psi^* + W_b\Phi^* = \mathcal{P}_b[L_a u - f] \quad \text{in} \quad \Omega, \tag{3.27}$$

where $\Psi^* := T_a^+ u - \Psi,\ \Phi^* := \gamma^+ u - \Phi$. Multiplying equality (3.27) by b we get

$$-V_\Delta\Psi^* + W_\Delta(b\Phi^*) = \mathcal{P}_\Delta\left[L_a u - f\right] \quad \text{in} \quad \Omega.$$

Applying the Laplace operator Δ to the last equation and taking in to consideration that both functions in the left-hand side are harmonic surface potentials, while the right-hand side function is the classical Newtonian volume potential, we arrive at equation (3.25). Substituting (3.25) in (3.27) leads to (3.26).

□

The following lemma is proved in [2].

Lemma 3.2

(i) Let $\Psi^* \in H^{-\frac{1}{2}}(\partial\Omega)$. If $V_b\Psi^* = 0$ in Ω, then $\Psi^* = 0$

(ii) Let $\Phi^* \in H^{\frac{1}{2}}(\partial\Omega)$. If $W_b\Phi^* = 0$ in Ω, then $\Phi^* = 0$

(iii) Let $\partial\Omega = \overline{S}_1 \cup \overline{S}_2$, where S_1 and S_2 are nonintersecting simply connected sub-manifolds of $\partial\Omega$ with infinitely smooth boundaries and S_1 is nonempty. Let $\Psi^* \in \widetilde{H}^{-\frac{1}{2}}(S_1), \Phi^* \in \widetilde{H}^{\frac{1}{2}}(S_2)$. If $V_b\Psi^* - W_b\Phi^* = 0$, in Ω, then $\Psi^* = 0$ and $\Phi^* = 0$ on $\partial\Omega$.

4 Two-operator boundary-domain integral equations

Let $\Phi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ and $\Psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ be some extensions of the given data $\varphi_0 \in H^{\frac{1}{2}}(\partial_D\Omega)$ from $\partial_D\Omega$ to $\partial\Omega$ and $\psi_0 \in H^{-\frac{1}{2}}(\partial_N\Omega)$ from $\partial_N\Omega$ to $\partial\Omega$, respectively. Let us also denote

$$F_0 := \mathcal{P}_b f + V_b\Psi_0 - W_b\Phi_0 \qquad \text{in} \qquad \Omega.$$

Note that for $f \in L_2(\Omega)$, $\Psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ and $\Phi_0 \in H^{\frac{1}{2}}(\partial\Omega)$, we have the inclusion $F_0 \in H^{1,0}(\Omega, L_a)$ due to the mapping properties of the Newtonian (volume) and layer potentials (cf. Theorems 3.1 and 3.10 in [2]).

To reduce BVP (2.5)-(2.7) to one or another two-operator BDIE system, we shall use equation (3.21) in Ω, and restrictions of equation (3.22) or (3.23) to appropriate parts of the boundary. We shall always substitute $\Phi_0 + \varphi$ for $\gamma^+ u$ and $\Psi_0 + \psi$ for $T_a^+ u$, cf. [2], where $\Phi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ and $\Psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ are considered as known, while ψ belongs to $\widetilde{H}^{-\frac{1}{2}}(\partial_D\Omega)$ and φ to $\widetilde{H}^{\frac{1}{2}}(\partial_N\Omega)$ due to the boundary conditions (2.6)-(2.7) and are to be found along with $u \in H^{1,0}(\Omega; \Delta)$. This will lead us to *segregated* BDIE systems with respect to the unknown triple

$$\mathcal{U} := [u, \psi, \varphi]^\top \in H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(\partial_D\Omega) \times \widetilde{H}^{\frac{1}{2}}(\partial_N\Omega).$$

4.1 Boundary-Domain Integral Equation System M11

Let us use equation (3.21) in Ω, the restriction of equation (3.22) on $\partial_D\Omega$ and the restriction of equation (3.23) on $\partial_N\Omega$. Then we arrive at the following two-operator segregated system of BDIEs:

$$u + \mathcal{Z}_b u + \mathcal{R}_b u - V_b\psi + W_b\varphi = F_0 \quad \text{in} \quad \Omega, \tag{4.1}$$

$$\gamma^+\mathcal{Z}_b u + \gamma^+\mathcal{R}_b u - \mathcal{V}_b\psi + \mathcal{W}_b\varphi = \gamma^+F_0 - \varphi_0 \quad \text{on} \quad \partial_D\Omega, \tag{4.2}$$

$$T_a^+\mathcal{Z}_b u + T_a^+\mathcal{R}_b u - \mathcal{W}'_{ab}\psi + \mathcal{L}_{ab}^+\varphi = T_a^+F_0 - \psi_0 \quad \text{on} \quad \partial_N\Omega\ , \tag{4.3}$$

which we call BDIE system M11, where M stands for the mixed problem and 11 hints that the integral equations on the Dirichlet and Neumann parts of the boundary are of the first kind. Note that due to Lemma 3.1, all terms of equation (4.1) belong to $H^{1,0}(\Omega;\Delta)$ and their co-normal derivatives are well defined.

System (4.1)-(4.3) can be rewritten in the form

$$\mathcal{A}^{11}\mathcal{U} = \mathcal{F}^{11},$$

where

$$\mathcal{F}^{11} := [F_0, r_{\partial_D\Omega}\gamma^+F_0 - \varphi_0,\ r_{\partial_N\Omega}T_a^+F_0 - \psi_0]^\top,$$

$$\mathcal{A}^{11} := \begin{bmatrix} I + \mathcal{Z}_b + \mathcal{R}_b & -V_b & W_b \\ r_{\partial_D\Omega}\gamma^+[\mathcal{Z}_b + \mathcal{R}_b] & -r_{\partial_D\Omega}\mathcal{V}_b & r_{\partial_D\Omega}\mathcal{W}_b \\ r_{\partial_N\Omega}T_a^+[\mathcal{Z}_b + \mathcal{R}_b] & -r_{\partial_N\Omega}\mathcal{W}'_{ab} & r_{\partial_N\Omega}\mathcal{L}_{ab}^+ \end{bmatrix}.$$

4.2 Boundary-domain integral equation system M12

To obtain another system, we use equation (3.21) in Ω and equation (3.22) on the whole boundary $\partial\Omega$, and arrive at the two-operator segregated BDIE system M12:

$$u + \mathcal{Z}_b u + \mathcal{R}_b u - V_b\psi + W_b\varphi = F_0 \quad \text{in} \quad \Omega, \tag{4.4}$$

$$\frac{1}{2}\varphi + \gamma^+\mathcal{Z}_b u + \gamma^+\mathcal{R}_b u - \mathcal{V}_b\psi + \mathcal{W}_b\varphi = \gamma^+F_0 - \Phi_0 \quad \text{on} \quad \partial\Omega\ . \tag{4.5}$$

System (4.4)-(4.5) can be written in the form

$$\mathcal{A}^{12}\mathcal{U} = \mathcal{F}^{12},$$

where

$$\mathcal{F}^{12} := [F_0,\ \gamma^+F_0 - \Phi_0]^\top,$$

$$\mathcal{A}^{12} := \begin{bmatrix} I + \mathcal{Z}_b + \mathcal{R}_b & -V_b & W_b \\ \gamma^+[\mathcal{Z}_b + \mathcal{R}_b] & -\mathcal{V}_b & \frac{1}{2}I + \mathcal{W}_b \end{bmatrix}.$$

4.3 Boundary-domain integral equation system M21

To obtain one more system, we use equation (3.21) in Ω and equation (3.23) on $\partial\Omega$ and arrive at the two-operator segregated BDIE system M21:

$$u + \mathcal{Z}_b u + \mathcal{R}_b u - V_b\psi + W_b\varphi = F_0 \qquad \text{in} \quad \Omega, \tag{4.6}$$

$$\left(1 - \frac{a}{2b}\right)\psi + T_a^+\mathcal{Z}_b u + T_a^+\mathcal{R}_b u - \mathcal{W}'_{ab}\psi + \mathcal{L}_{ab}^+\varphi = T_a^+F_0 - \Psi_0 \quad \text{on} \quad \partial\Omega. \tag{4.7}$$

System (4.6)-(4.7) can be written in the form

$$\mathcal{A}^{21}\mathcal{U} = \mathcal{F}^{21},$$

where

$$\mathcal{F}^{21} \;:=\; [F_0,\ T_a^+\gamma^+F_0 - \Psi_0]^\top,$$

$$\mathcal{A}^{21} \;:=\; \begin{bmatrix} I + \mathcal{Z}_b + \mathcal{R}_b & -V_b & W_b \\ T_a^+[\mathcal{Z}_b + \mathcal{R}_b] & (1 - \frac{a}{2b})I - \mathcal{W}'_{ab} & \mathcal{L}_{ab}^+ \end{bmatrix}.$$

4.4 Boundary-domain integral equation system M22

To reduce BVP (2.5)-(2.7) to a BDIE system of almost the second kind (up to the spaces), we use equation (3.21) in Ω, the restriction of equation (3.23) to $\partial_D\Omega$, and the restriction of equation (3.22) to $\partial_N\Omega$. Then we arrive at the following two-operator segregated BDIE system M22:

$$u + \mathcal{Z}_b u + \mathcal{R}_b u - V_b\psi + W_b\varphi = F_0 \qquad \text{in} \quad \Omega, \tag{4.8}$$

$$\left(1 - \frac{a}{2b}\right)\psi + T_a^+\mathcal{Z}_b u + T_a^+\mathcal{R}_b u - \mathcal{W}'_{ab}\psi + \mathcal{L}_{ab}^+\varphi = T_a^+F_0 - \Psi_0 \quad \text{on} \quad \partial_D\Omega, \tag{4.9}$$

$$\frac{1}{2}\varphi + \gamma^+\mathcal{Z}_b u + \gamma^+\mathcal{R}_b u - \mathcal{V}_a\psi + \mathcal{W}_a\varphi = F_0^+ - \Phi_0 \qquad \text{on} \quad \partial_N\Omega. \tag{4.10}$$

System (4.8)-(4.10) can be rewritten in the form

$$\mathcal{A}^{22}\mathcal{U} = \mathcal{F}^{22},$$

where

$$\mathcal{F}^{22} \;:=\; [F_0,\ r_{\partial_D\Omega}(T_a^+F_0 - \Psi_0),\ r_{\partial_N\Omega}(\gamma^+F_0 - \Phi_0)]^\top,$$

$$\mathcal{A}^{22} \;:=\; \begin{bmatrix} I + \mathcal{Z}_b + \mathcal{R}_b & -V_b & W_b \\ r_{\partial_D\Omega}T_a^+[\mathcal{Z}_b + \mathcal{R}_b] & (1 - \frac{a}{2b})I - r_{\partial_D\Omega}\mathcal{W}'_{ab} & r_{\partial_D\Omega}\mathcal{L}_{ab}^+ \\ r_{\partial_N\Omega}\gamma^+[\mathcal{Z}_b + \mathcal{R}_b] & -r_{\partial_N\Omega}\mathcal{V}_b & \frac{1}{2}I + r_{\partial_N\Omega}\mathcal{W}_b \end{bmatrix}.$$

5 Equivalence and invertibility

Now let us prove the equivalence of BVP (2.5)-(2.7) with the BDIE systems M11, M12, M21 and M22.

Theorem 5.1 *Let $f \in L_2(\Omega)$ and let $\Phi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ and $\Psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ be some fixed extensions of $\varphi_0 \in H^{\frac{1}{2}}(\partial_D\Omega)$ and $\psi_0 \in H^{-\frac{1}{2}}(\partial_N\Omega)$, respectively.*

(i) If some $u \in H^1(\Omega)$ solves the mixed BVP (2.5)-(2.7) in Ω, then the solution is unique and the triple $(u, \psi, \varphi) \in H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(\partial_D\Omega) \times \widetilde{H}^{\frac{1}{2}}(\partial_N\Omega)$, where

$$\psi = T_a^+ u - \Psi_0, \qquad \varphi = \gamma^+ u - \Phi_0 \qquad \text{on} \quad \partial\Omega, \tag{5.1}$$

solves the BDIE systems M11, M12, M21 and M22.

(ii) Vise versa, if a triple $(u, \psi, \varphi) \in H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(\partial_D\Omega) \times \widetilde{H}^{\frac{1}{2}}(\partial_N\Omega)$ solves BDIE system M11 or M12 or M21 or M22, then the solution is unique, the function u solves BVP (2.5)-(2.7), and relations (5.1) hold.

Proof. Let $u \in H^1(\Omega)$ be a solution to BVP (2.5)-(2.7). Then it is unique (cf. Theorem 2.1 in [2]). Set $\psi := T_a^+ u - \Psi_0\,, \quad \varphi := \gamma^+ u - \Phi_0$. Evidently, $\psi \in \widetilde{H}^{-\frac{1}{2}}(\partial_D\Omega)$ and $\varphi \in \widetilde{H}^{\frac{1}{2}}(\partial_N\Omega)$. Then it immediately follows from relations (3.21)-(3.23) that the triple (u, ψ, φ) satisfies the BDIE systems M11, M12, M21 and M22, which completes the proof of item (i).

We give below proofs of item (ii) for the four BDIE systems M11, M12, M21 and M22 one by one.

M11. Let a triple $(u, \psi, \varphi) \in H_2^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(\partial_D\Omega) \times \widetilde{H}^{\frac{1}{2}}(\partial_N\Omega)$ solves BDIE system (4.1)-(4.3). Let us consider the trace of equation (4.1) on $\partial_D\Omega$, taking into account the jump properties (see Theorem A.6), and subtract equation (4.2) to obtain

$$\gamma^+ u = \varphi_0 \qquad \text{on} \quad \partial_D\Omega, \tag{5.2}$$

i.e., u satisfies the Dirichlet condition (2.6). Taking the co-normal derivative T_a^+ of equation (4.1) on $\partial_N\Omega$, again with account of the jump properties, and subtracting equation (4.3), we obtain

$$T_a^+ u = \psi_0, \qquad \text{on} \quad \partial_N\Omega. \tag{5.3}$$

i.e. u satisfies the Neumann condition (2.7). Taking into account that $\varphi = 0,\ \Phi_0 = \varphi_0$ on $\partial_D\Omega$ and $\psi = 0$, $\Psi_0 = \psi_0$ on $\partial_N\Omega$, equations (5.2) and (5.3) imply that the first equation of (5.1) is satisfied on $\partial_N\Omega$ and the second equations (5.1) is satisfied on $\partial_D\Omega$.

Equations (4.1) and Lemma 3.1 with $\Psi = \psi + \Psi_0$, $\Phi = \varphi + \Phi_0$ imply that u is a solution to (2.5) and

$$V_b\Psi^* - W_b\Phi^* = 0, \qquad \text{in} \quad \Omega,$$

where $\Psi^* = \Psi_0 + \psi - T_a^+ u$ and $\Phi^* = \Phi_0 + \varphi - \gamma^+ u$. Since first equation (5.1) on $\partial_N\Omega$ and the second equation (5.1) on $\partial_D\Omega$, already proved, we have $\Psi^* \in \widetilde{H}^{-\frac{1}{2}}(\partial_D\Omega),\ \Phi^* \in \widetilde{H}_2^{\frac{1}{2}}(\partial_N\Omega)$. Then Lemma 3.2 (iii) with $S_1 = \partial_D\Omega,\ S_2 = \partial_N\Omega$, implies $\Psi = \Phi = 0$, which completes the the proof of conditions (5.1).

M12. Let the triple $(u, \psi, \varphi) \in H_2^1(\Omega) \times \widetilde{H}^{\frac{1}{2}}(\partial_D\Omega) \times \widetilde{H}^{-\frac{1}{2}}(\partial_N\Omega)$ solve BDIE system (4.4)-(4.5). Let us consider the trace of equation (4.4) on $\partial\Omega$, taking into account the jump properties, and subtract it from (4.5) to obtain,

$$\gamma^+ u = \Phi_0 + \varphi \quad \text{on} \quad \partial\Omega. \tag{5.4}$$

This means that the second equation in (5.1) holds. Since $\varphi = 0$, $\Phi_0 = \varphi_0$ on $\partial_D\Omega$ we see that the Dirichlet condition (2.6) is satisfied.

Equation (4.4) and Lemma 3.1 with $\Psi = \psi + \Psi_0$, $\quad \Phi = \varphi + \Phi_0$ imply that u is a solution to equation (2.5) and

$$V_b(\Psi_0 + \psi - T_a^+ u) - W_b(\Phi_0 + \varphi - \gamma^+ u) = 0 \qquad \text{in} \quad \Omega. \tag{5.5}$$

Due to (5.4), the second term in (5.5) vanishes, and by Lemma 3.2 (i) we obtain

$$\Psi_0 + \psi - T_a^+ u = 0 \quad \text{on} \quad \partial\Omega, \tag{5.6}$$

i.e., the first equation in (5.1) is satisfied as well. Since $\psi = 0$, $\Psi_0 = \psi_0$ on $\partial_N\Omega$ equation (5.6) implies that u satisfies the Neumann boundary condition (2.7).

M21. Let now a triple $(u, \psi, \varphi) \in H_2^1(\Omega) \times \widetilde{H}^{\frac{1}{2}}(\partial_D\Omega) \times \widetilde{H}^{-\frac{1}{2}}(\partial_N\Omega)$ solve BDIE system (4.6)-(4.7). Taking the co-normal derivative of equation (4.6) on $\partial\Omega$ and subtracting it from equation (4.7), we obtain

$$\psi + \Psi_0 - T_a^+ u = 0 \qquad \text{on} \quad \partial\Omega. \tag{5.7}$$

which proves the first equation in (5.1). Since $\psi = 0$ on $\partial_N\Omega$ and $\Psi_0 = \psi_0$ on $\partial_N\Omega$, we see that u satisfies the Neumann condition (2.7).

Equation (4.6) and Lemma 3.1 with $\Psi = \psi + \Psi_0$, $\Phi = \varphi + \Phi_0$ imply that u is a solution to equation (2.5) and

$$V_b(\Psi_0 + \psi - T_a^+ u) - W_b(\Phi_0 + \varphi - \gamma^+ u) = 0 \qquad \text{in} \quad \Omega. \tag{5.8}$$

Due to equation (5.7) the first term vanishes in (5.8), and by Lemma 3.2 (ii) we obtain,

$$\Phi_0 + \varphi - \gamma^+ u = 0 \qquad \text{on} \quad \partial\Omega,$$

which means the second condition in (5.1) holds as well. Taking into account $\varphi = 0$ on $\partial_D\Omega$ and $\Phi_0 = \varphi$ on $\partial_D\Omega$, we conclude that u satisfies the Dirichlet condition (2.6).

M22. Let now a triple $(u, \psi, \varphi) \in H^1(\Omega) \times \widetilde{H}^{\frac{1}{2}}(\partial_D\Omega) \times \widetilde{H}^{-\frac{1}{2}}(\partial_N\Omega)$ solve BDIE system (4.8)-(4.10). Taking the co-normal derivative of equation (4.8) on $\partial_D\Omega$ and subtracting it from equation (4.9), we obtain

$$\psi = T_a^+ u - \Psi_0 \qquad \text{on} \quad \partial_D\Omega. \tag{5.9}$$

Further, take the trace of equation (4.8) on $\partial_N\Omega$ and subtract it from equation (4.10). We get

$$\varphi = \gamma^+ u - \Phi_0 \qquad \text{on} \quad \partial_N\Omega. \tag{5.10}$$

Equations (5.9) and (5.10) imply that the first equation (5.1) is satisfied on $\partial_D\Omega$ and the second equation (5.1) is satisfied on $\partial_N\Omega$.

Equations (4.8) and Lemma 3.1 with $\Psi = \psi + \Psi_0$, $\Phi = \varphi + \Phi_0$ imply that u is a solution to equation (2.5) and $V_b\Psi^* - W_b\Psi^* = 0$ in Ω, where $\Psi^* = \Psi_0 + \psi - T_a^+ u$ and $\Phi^* = \Phi_0 + \varphi - \gamma^+ u$. Due to (5.1) and (5.10), we have $\Psi^* \in \widetilde{H}^{-\frac{1}{2}}(\partial_N\Omega), \Phi^* \in \widetilde{H}^{\frac{1}{2}}(\partial_D\Omega)$. Lemma 3.2 (iii) with $S_1 = \partial_N\Omega$ and $S_2 = \partial_D\Omega$ implies $\Psi^* = \Phi^* = 0$ which completes the proof of conditions (5.1) on the whole boundary $\partial\Omega$. Taking into account that $\varphi = 0$ on $\partial_D\Omega$ and $\Phi_0 = \varphi_0$ on $\partial_D\Omega$, and $\psi = 0$ on $\partial_N\Omega$ and $\Psi_0 = \psi_0$ on $\partial_N\Omega$, equations (5.1) imply the boundary conditions (2.6) and (2.7).

Unique solvability of the BDIE systems M11, M12, M12 and M22 then follows from the already proved relations (5.1) and the unique solvability of BVP (2.5)-(2.7) stated in item (i).

□

The mapping properties of operators in (4.4), (4.6), (4.8) and (4.11) described in Appendix A and Theorem 5.1 imply the following statement.

Corollary 5.1 *The following operators are continuous and injective*

$$\begin{aligned}
\mathcal{A}^{11} &: H_2^{1,0}(\Omega; L_a) \times \widetilde{H}^{-\frac{1}{2}}(\partial_D\Omega) \times \widetilde{H}^{\frac{1}{2}}(\partial_N\Omega) \to H^{1,0}(\Omega; L_a) \times H^{\frac{1}{2}}(\partial_D\Omega) \times H^{-\frac{1}{2}}(\partial_N\Omega), && (5.11)\\
\mathcal{A}^{12} &: H^{1,0}(\Omega; L_a) \times \widetilde{H}^{-\frac{1}{2}}(\partial_D\Omega) \times \widetilde{H}^{\frac{1}{2}}(\partial_N\Omega) \to H^{1,0}(\Omega; L_a) \times H^{\frac{1}{2}}(\partial\Omega), && (5.12)\\
\mathcal{A}^{21} &: H^{1,0}(\Omega; L_a) \times \widetilde{H}^{-\frac{1}{2}}(\partial_D\Omega) \times \widetilde{H}^{\frac{1}{2}}(\partial_N\Omega) \to H^{1,0}(\Omega; L_a) \times H^{\frac{1}{2}}(\partial\Omega), && (5.13)\\
\mathcal{A}^{22} &: H^{1,0}(\Omega; L_a) \times \widetilde{H}^{-\frac{1}{2}}(\partial_D\Omega) \times \widetilde{H}^{\frac{1}{2}}(\partial_N\Omega) \to H^{1,0}(\Omega; L_a) \times H^{-\frac{1}{2}}(\partial_D\Omega) \times H^{\frac{1}{2}}(\partial_N\Omega). && (5.14)
\end{aligned}$$

Now we are in the position to analyse the invertibility of the operators $\mathcal{A}^{11}$, $\mathcal{A}^{12}$, $\mathcal{A}^{21}$and $\mathcal{A}^{22}$.

Theorem 5.2 *Operators* (5.11)-(5.14) *are continuously invertible.*

Proof. To prove the invertibility of operator (5.11), let us consider BDIE system M11 with an arbitrary right hand side $\mathcal{F}_*^{11} = \{\mathcal{F}_{*1}^{11}, \mathcal{F}_{*2}^{11}, \mathcal{F}_{*3}^{11}\}^\top \in H^{1,0}(\Omega; L_a) \times H^{\frac{1}{2}}(\partial_D\Omega) \times H^{-\frac{1}{2}}(\partial_N\Omega)$. Taking $S_1 = \partial_N\Omega$, $S_2 = \partial_D\Omega$ and

$$F = \mathcal{F}_{*1}^{11}, \quad \Psi = r_{\partial_N\Omega} T_a^+ \mathcal{F}_{*1}^{11} - \mathcal{F}_{*3}^{11}, \quad \Phi = r_{\partial_D\Omega} \gamma^+ \mathcal{F}_{*1}^{11} - \mathcal{F}_{*2}^{11}$$

in [2, Lemma 5.13], presented as Lemma B.1 in the Appendix, we obtain that $\mathcal{F}_*^{11}$ can be represented as

$$\begin{aligned}
\mathcal{F}_{*1}^{11} &= \mathcal{P}_b\, f_* + V_b\, \Psi_* - W_b\, \Phi_* \text{ in } \Omega,\\
\mathcal{F}_{*2}^{11} &= r_{\partial_D\Omega} \left[\gamma^+ \mathcal{F}_{*1}^{11} - \Phi_* \right],\\
\mathcal{F}_{*3}^{11} &= r_{\partial_N\Omega} \left[T_a^+ \mathcal{F}_{*1}^{11} - \Psi_* \right],
\end{aligned}$$

where the triple

$$(f_*, \Psi_*, \Phi_*)^\top = \mathcal{C}_{\partial_N\Omega,\partial_D\Omega}\, \mathcal{F}_*^{11} \in L_2(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \tag{5.15}$$

is unique and the operator

$$\mathcal{C}_{\partial_N\Omega,\partial_D\Omega} \,:\, H^{1,0}(\Omega;L_a)\times H^{\frac{1}{2}}(\partial_D\Omega)\times H^{-\frac{1}{2}}(\partial_N\Omega)\to L_2(\Omega)\times H^{-\frac{1}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega) \tag{5.16}$$

is linear and continuous.

Applying Theorem 5.1 with

$$f=f_*,\quad \Psi_0=\Psi_*,\quad \Phi_0=\Phi_*,\quad \psi_0=r_{\partial_N\Omega}\Psi_0,\quad \varphi_0=r_{\partial_D\Omega}\Phi_0, \tag{5.17}$$

we obtain that the system M11 is uniquely solvable and its solution is

$$\mathcal{U}_1=(A^{DN})^{-1}(f_*,r_{\partial_D\Omega}\Phi_*,r_{\partial_N\Omega}\Psi_*)^\top,\qquad \mathcal{U}_2=T_a^+\mathcal{U}_1-\Psi_*,\qquad \mathcal{U}_3=\gamma^+\mathcal{U}_1-\Phi_* \tag{5.18}$$

while $r_{\partial_N\Omega}\mathcal{U}_2=0$, $r_{\partial_D\Omega}\mathcal{U}_3=0$. Here $(A^{DN})^{-1}$ is the continuous inverse operator to the left-hand-side operator of the mixed BVP (2.5)-(2.7), $A^{DN}: H^{1,0}(\Omega;L_a)\to L_2(\Omega)\times H^{\frac{1}{2}}(\partial_D\Omega)\times H^{-\frac{1}{2}}(\partial_N\Omega)$, cf. [2, Corollary 5.16]. Representation (5.15), and continuity of operator (5.16) complete the proof for $\mathcal{A}^{11}$.

To prove invertibility of operator (5.14), we apply similar arguments. Let us consider the BDIE system M22 with an arbitrary right hand side $\mathcal{F}^{22}_*=\{\mathcal{F}^{22}_{*1},\mathcal{F}^{22}_{*2},\mathcal{F}^{22}_{*3}\}^\top\in H^{1,0}(\Omega;L_a)\times H^{-\frac{1}{2}}(\partial_D\Omega)\times H^{\frac{1}{2}}(\partial_N\Omega)$. Taking now $S_1=\partial_D\Omega$, $S_2=\partial_N\Omega$,

$$F=\mathcal{F}^{22}_{*1},\quad \Psi=r_{\partial_D\Omega}T_a^+\mathcal{F}^{22}_{*1}-\mathcal{F}^{22}_{*2},\quad \Phi=r_{\partial_N\Omega}\gamma^+\mathcal{F}^{22}_{*1}-\mathcal{F}^{22}_{*3}$$

in [2, Lemma 5.13], i.e., Lemma B.1 in the Appendix, we obtain that $\mathcal{F}^{22}_*$ can be represented as

$$\begin{aligned}
\mathcal{F}^{22}_{*1}&=\mathcal{P}_b\,f_*+V_b\,\Psi_*-W_b\,\Phi_*\ \text{ in }\ \Omega,\\
\mathcal{F}^{22}_{*2}&=r_{\partial_D\Omega}\left[T_a^+\,\mathcal{F}^{22}_{*1}-\ \Psi_*\right],\\
\mathcal{F}^{22}_{*3}&=r_{\partial_N\Omega}\left[\gamma^+\mathcal{F}^{22}_{*1}-\ \Phi_*\right],
\end{aligned}$$

where the triple

$$(f_*,\Psi_*,\Phi_*)^\top=\mathcal{C}_{\partial_D\Omega,\partial_N\Omega}\,\mathcal{F}^{22}_*\in L_2(\Omega)\times H^{-\frac{1}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega) \tag{5.19}$$

is unique and the operator

$$\mathcal{C}_{\partial_N\Omega,\partial_D\Omega} \,:\, H^{1,0}(\Omega;L_a)\times H^{-\frac{1}{2}}(\partial_D\Omega)\times H^{-\frac{1}{2}}(\partial_N\Omega)\to L_2(\Omega)\times H^{-\frac{1}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega) \tag{5.20}$$

is linear and continuous.

Applying now Theorem 5.1 with the same substitutions (5.17), we obtain that the system M22 is uniquely solvable and its solution is given by (5.18). Representation (5.19), and continuity of operator (5.20) complete the proof for $M22$.

To prove invertibility of operator (5.12), let us consider the BDIE system M12 with an arbitrary right hand side $\mathcal{F}^{12}_*=\{\mathcal{F}^{12}_{*1},\mathcal{F}^{12}_{*2}\}^\top\in H^{1,0}(\Omega;L_a)\times H^{\frac{1}{2}}(\partial\Omega)$. Taking $F=\mathcal{F}^{12}_{*1}$, $\Phi=\gamma^+\mathcal{F}^{12}_{*1}-\mathcal{F}^{12}_{*2}$ on $\partial\Omega$ in Corollary B.1 in the Appendix, we obtain the representation

$$\mathcal{F}^{12}_{*1}=\mathcal{P}_b\,f_*+V_b\,\Psi_*-W_b\,\Phi_*\ \text{ in }\ \Omega,$$

$$\mathcal{F}^{12}_{*2} = \gamma^+ \mathcal{F}^{12}_{*1} - \Phi_* \;\text{ on }\; \partial\Omega$$

where the triple

$$(f_*, \Psi_*, \Phi_*)^\top = \widetilde{\mathcal{C}}_{\Phi*}\, \mathcal{F}_* \in L_2(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \tag{5.21}$$

is unique and the operator

$$\widetilde{\mathcal{C}}_{\Phi*} \,:\, H^{1,0}(\Omega; L_a) \times H^{\frac{1}{2}}(\partial\Omega) \to L_2(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \tag{5.22}$$

is linear and continuous.

Applying Theorem 5.1 with substitutions (5.17), we obtain that the system M12 is uniquely solvable and its solution is given by (5.18). Representation (5.21), and continuity of operator (5.22) complete the proof for M12.

Finally to prove invertibility of operator (5.13), let us consider the BDIE system M21 with an arbitrary right hand side $\mathcal{F}^{21}_* = \{\mathcal{F}^{21}_{*1}, \mathcal{F}^{21}_{*2}\}^\top \in H^{1,0}(\Omega; L_a) \times H^{-\frac{1}{2}}(\partial\Omega)$. Taking $F = \mathcal{F}^{21}_{*1}$, $\Psi = T^+_a \mathcal{F}^{21}_{*1} - \mathcal{F}^{21}_{*2}$ on $\partial\Omega$ in Corollary B.2 in the Appendix, we obtain that

$$\begin{aligned}
\mathcal{F}^{21}_{*1} &= \mathcal{P}_b\, f_* + V_b\, \Psi_* - W_b\, \Phi_* \;\text{ in }\; \Omega,\\
\mathcal{F}^{21}_{*2} &= \; T^+_a \mathcal{F}^{21}_{*1} - \Psi_* \;\text{ on }\; \partial\Omega.
\end{aligned}$$

where the triple

$$(f_*, \Psi_*, \Phi_*)^\top = \widetilde{\mathcal{C}}_{\Psi*}\, \mathcal{F}_* \in L_2(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \tag{5.23}$$

is unique and the operator

$$\widetilde{\mathcal{C}}_{\Psi*} \,:\, H^{1,0}(\Omega; L_a) \times H^{-\frac{1}{2}}(\partial\Omega) \to L_2(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \tag{5.24}$$

is linear and continuous. Applying Theorem 5.1 with substitutions (5.17), we obtain that the system M21 is uniquely solvable and its solution is given by (5.18). Representation (5.23), and continuity of operator (5.24) complete the proof for M21.

$\square$

APPENDICES

A Mapping and jump properties of the volume and surface potentials

The mapping properties of the parametrix-based volume and surface potentials formulated in Appendix A are proved or immediately follow from [2] (see also [12]).

Theorem A.1 *Let Ω be a bounded open three-dimensional region of $\mathbb{R}^3$ with a simply connected, closed, infinitely smooth boundary $\partial\Omega$. The operators*

$$\mathcal{P}_b \;\;:\;\; \widetilde{H}^s(\Omega) \to H^{s+2}(\Omega), \quad s \in \mathbb{R} \tag{A.1}$$

$$
\begin{aligned}
&: \quad H^s(\Omega) \to H^{s+2}(\Omega), \quad s > -\frac{1}{2}, && \text{(A.2)}\\
&: \quad H^s(\Omega) \to H^{s+2,0}(\Omega; L_a), \quad s \geq 0, && \text{(A.3)}\\
\mathcal{R}_b \; &: \quad \widetilde{H}^s(\Omega) \to H^{s+1}(\Omega), \quad s \in \mathbb{R}, && \text{(A.4)}\\
&: \quad H^s(\Omega) \to H^{s+1}(\Omega), \quad s > -\frac{1}{2}, && \text{(A.5)}\\
&: \quad H^s(\Omega) \to H^{s+1,0}(\Omega; L_a), \quad s \geq 1, && \text{(A.6)}\\
\gamma^+\mathcal{P}_b \; &: \quad \widetilde{H}^s(\Omega) \to H^{s+\frac{3}{2}}(\partial\Omega), \quad s > -\frac{3}{2}, && \text{(A.7)}\\
&: \quad H^s(\Omega) \to H^{s+\frac{3}{2}}(\partial\Omega), \quad s > -\frac{1}{2}, && \text{(A.8)}\\
\gamma^+\mathcal{R}_b \; &: \quad \widetilde{H}^s(\Omega) \to H^{s+\frac{1}{2}}(\partial\Omega), \quad s > -\frac{1}{2}, && \text{(A.9)}\\
&: \quad H^s(\Omega) \to H^{s+\frac{1}{2}}(\partial\Omega), \quad s > -\frac{1}{2}, && \text{(A.10)}\\
T_a^+\mathcal{P}_b \; &: \quad \widetilde{H}^s(\Omega) \to H^{s+\frac{1}{2}}(\partial\Omega), \quad s > -\frac{1}{2}, && \text{(A.11)}\\
&: \quad H^s(\Omega) \to H^{s+\frac{1}{2}}(\partial\Omega), \quad s > -\frac{1}{2}, && \text{(A.12)}\\
T_a^+\mathcal{R}_b \; &: \quad \widetilde{H}^s(\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega), \quad s > \frac{1}{2}, && \text{(A.13)}\\
&: \quad H^s(\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega), \quad s > \frac{1}{2} && \text{(A.14)}
\end{aligned}
$$

are continuous and the operators

$$
\begin{aligned}
\mathcal{R}_b \; &: \quad H^s(\Omega) \to H^{s}(\Omega), \quad s > -\frac{1}{2}, && \text{(A.15)}\\
&: \quad H^s(\Omega) \to H^{s,0}(\Omega; L_a), \quad s > 1, && \text{(A.16)}\\
r_{S_1}\gamma^+\mathcal{R}_b \; &: \quad H^s(\Omega) \to H^{s-\frac{1}{2}}(S_1), \quad s > -\frac{1}{2}, && \text{(A.17)}\\
r_{S_1}T_a^+\mathcal{R}_b \; &: \quad H^s(\Omega) \to H^{s-\frac{3}{2}}(S_1), \quad s > \frac{1}{2} && \text{(A.18)}
\end{aligned}
$$

are compact for any non-empty, open sub-manifold S_1 of $\partial\Omega$ with an infinitely smooth boundary.

Proof. For $a = b$, the mapping properties are stated in Theorem 3.8 in [2] and Corollary B.3 in [12]. The case $a \neq b$ then follows by taking into account the relation $T_a^+ = \frac{a}{b}T_b^+$, for (A.11)-(A.14) and (A.18).

$\square$

Theorem A.2 *The following operators are continuous*

$$
V_b \; : \; H^s(\partial\Omega) \to H^{s+\frac{3}{2}}(\Omega) \quad \left[H^s(\partial\Omega) \to H^{s+\frac{3}{2}}_{loc}(\Omega^-)\right], \quad s \in \mathbb{R},
$$

$$
\begin{aligned}
W_b &: H^s(\partial\Omega) \to H^{s+\frac{1}{2}}(\Omega) \quad \Big[H^s(\partial\Omega) \to H^{s+\frac{1}{2}}_{loc}(\Omega^-)\Big], \quad s \in \mathbb{R},\\
V_b &: H^s(\partial\Omega) \to H^{s+\frac{3}{2},0}(\Omega, L_a) \quad \Big[H^s(\partial\Omega) \to H^{s+\frac{3}{2},0}_{loc}(\Omega^-, L_a)\Big], \; s \geq -\frac{1}{2},\\
W_b &: H^s(\partial\Omega) \to H^{s+\frac{1}{2},0}(\Omega, L_a) \quad \Big[H^s(\partial\Omega) \to H^{s+\frac{1}{2},0}_{loc}(\Omega^-, L_a)\Big], \; s \geq \frac{1}{2}.
\end{aligned}
$$

Theorem A.3 *Let $s \in \mathbb{R}$. The following pseudodifferential operators are continuous*

$$
\begin{aligned}
\mathcal{V}_b &: H^s(\partial\Omega) \to H^{s+1}(\partial\Omega)\\
\mathcal{W}_b &: H^s(\partial\Omega) \to H^{s+1}(\partial\Omega)\\
\mathcal{W}'_{ab} &: H^s(\partial\Omega) \to H^{s+1}(\partial\Omega)\\
\mathcal{L}^{\pm}_{ab} &: H^s(\partial\Omega) \to H^{s-1}(\partial\Omega).
\end{aligned}
$$

Due to the Rellich compact embedding theorem, Theorem A.3 implies the following assertion.

Theorem A.4 *Let $s \in \mathbb{R}$. Let S_1 and S_2 with $\partial S_1, \partial S_2 \in C^\infty$ be nonempty open submanifolds of $\partial\Omega$. The operators*

$$
\begin{aligned}
&r_{S_2}\mathcal{V}_b : \widetilde{H}^s(S_1) \to H^s(S_2)\\
&r_{S_2}\mathcal{W}_b : \widetilde{H}^s(S_1) \to H^s(S_2)\\
&r_{S_2}\mathcal{W}'_{ab} : \widetilde{H}^s(S_1) \to H^s(S_2)
\end{aligned}
$$

are compact.

Theorem A.5 *Let S_1 be a nonempty, simply connected sub-manifold of $\partial\Omega$ with infinitely smooth boundary, and $0 < s < 1$. Then the operator $r_{S_1}\mathcal{V}_b : \widetilde{H}^{s-1}(S_1) \to H^s(S_1)$ is invertible.*

Similar to Theorems 3.3 and 3.6 in [2] (see also Appendix A and B in [12]), relations (3.13)-(3.16) imply the two following jump relation theorems.

Theorem A.6 *Let $g_1 \in H^{-\frac{1}{2}}(\partial\Omega)$, and $g_2 \in H^{\frac{1}{2}}(\partial\Omega)$. Then there hold the following relations on $\partial\Omega$,*

$$
\begin{aligned}
\gamma^{\pm} V_b g_1 &= \mathcal{V}_b g_1,\\
\gamma^{\pm} W_b g_2 &= \mp\frac{1}{2} g_2 + \mathcal{W}_b g_2,\\
T^{\pm}_a V_b g_1 &= \pm\frac{1}{2}\frac{a}{b} g_1 + \mathcal{W}'_{ab} g_1.
\end{aligned}
$$

Theorem A.7 *Let S_1 and $\partial\Omega\backslash\overline{S}_1$ be nonempty, open, simply connected sub-manifolds of $\partial\Omega$ with an infinitely smooth boundary, and $0 < s < 1$. Then*

$$
\mathcal{L}^{+}_{ab} + \frac{a}{b}\frac{\partial b}{\partial n}\left(-\frac{1}{2}I + \mathcal{W}_b\right) = \mathcal{L}^{-}_{ab} + \frac{a}{b}\frac{\partial b}{\partial n}\left(\frac{1}{2}I + \mathcal{W}_b\right) \;\; on\; \partial\Omega.
$$

Moreover, the pseudodifferential operator $r_{S_1}\widehat{\mathcal{L}}_{ab} : \widetilde{H}^s(S_1) \to H^{s-1}(S_1)$, *where*

$$\widehat{\mathcal{L}}_{ab} g := \left[\frac{b}{a}\mathcal{L}^{\pm}_{ab} + \frac{\partial b}{\partial n}\left(\mp\frac{1}{2}I + \mathcal{W}_b\right)\right] g = \mathcal{L}_{\Delta}(bg) \quad on\ \partial\Omega,$$

is invertible, while the operators

$$r_{S_1}\left(\frac{b}{a}\mathcal{L}^{\pm}_{ab} - \widehat{\mathcal{L}}_{ab}\right) : \widetilde{H}^s(S_1) \to H^s(S_1)$$

are bounded and the operators

$$r_{S_1}\left(\frac{b}{a}\mathcal{L}^{\pm}_{ab} - \widehat{\mathcal{L}}_{ab}\right) : \widetilde{H}^s(S_1) \to H^{s-1}(S_1)$$

are compact.

B Representation lemmas

To prove invertibility of the BDIE operators we need the following representation statements.

Lemma B.1 ([2], Lemma 5.13) *Let* $\partial\Omega = \bar{S}_1 \cup \bar{S}_2$, *where* S_1 *and* S_2 *are nonintersecting simply connected nonempty sub-manifolds of* $\partial\Omega$ *with infinitely smooth boundaries. For any triple*

$$\mathcal{F}_* = (F, \Psi, \Phi)^\top \in H^{1,0}(\Omega; L_a) \times H^{-\frac{1}{2}}(S_1) \times H^{\frac{1}{2}}(S_2)$$

there exists a unique triple

$$(f_*, \Psi_*, \Phi_*)^\top = \widetilde{\mathcal{C}}_{S_1,S_2}\,\mathcal{F}_* \in L_2(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$$

such that

$$\begin{aligned} &\mathcal{P}_b\, f_* + V_b\, \Psi_* - W_b\, \Phi_* = F \ \text{ in } \ \Omega, \\ &r_{S_1}\, \Psi_* = \Psi \ \ , \\ &r_{S_2}\, \Phi_* = \Phi \ \ . \end{aligned}$$

Moreover, the operator

$$\widetilde{\mathcal{C}}_{S_1,S_2} \ : \ H^{1,0}(\Omega; L_a) \times H^{-\frac{1}{2}}(S_1) \times H^{\frac{1}{2}}(S_2) \to L_2(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$$

is linear and continuous.

The cases when $S_1 = \emptyset$ or $S_2 = \emptyset$ need to be considered separately. Let us first present a simplified version of Lemma 5.5 from [12].

Lemma B.2 *For any function* $\mathcal{F}_{\Psi*} \in H^{1,0}(\Omega; L_a)$*, there exists a unique couple* $(f_*, \Psi_*) = \mathcal{C}_{\Psi}\mathcal{F}_{\Psi*} \in L_2(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ *such that*

$$\mathcal{F}_{\Psi*} \;=\; \mathcal{P}_b f_* + V_b\Psi_*, \qquad \text{in} \quad \Omega^+,$$

and $\mathcal{C}_{\Psi} : H^{1,0}(\Omega; L_a) \to L_2(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ *is a bounded linear operator.*

Considering a couple $(F, \Phi)^\top \in H^{1,0}(\Omega; L_a) \times H^{\frac{1}{2}}(\partial\Omega)$ and employing Lemma B.2 for $\mathcal{F}_{\Psi*} = F + W_b\Phi \in H^{1,0}(\Omega; L_a)$, we arrive at the following statement.

Corollary B.1 *For any couple*

$$(F, \Phi)^\top = \mathcal{F}_* \in H^{1,0}(\Omega; L_a) \times H^{\frac{1}{2}}(\partial\Omega)$$

there exists a unique triple

$$(f_*, \Psi_*, \Phi_*)^\top = \widetilde{\mathcal{C}}_{\Phi*}\,\mathcal{F}_* \in L_2(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$$

such that

$$\begin{aligned} &\mathcal{P}_b\, f_* + V_b\,\Psi_* - W_b\,\Phi_* = F \;\; \textit{in} \;\; \Omega, \\ &\Phi_* = \Phi \;\; \textit{on} \;\; \partial\Omega. \end{aligned}$$

Moreover, the operator

$$\widetilde{\mathcal{C}}_{\Phi*} \,:\, H^{1,0}(\Omega; L_a) \times H^{\frac{1}{2}}(\partial\Omega) \to L_2(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$$

is linear and continuous.

Lemma 19 from [10] redone word-by-word to a more narrow space reads as follows.

Lemma B.3 *For any function* $\mathcal{F}_{\Phi*} \in H^{1,0}(\Omega; L_a)$*, there exists a unique couple* $(f_*, \Phi_*) = \mathcal{C}_{\Phi}\mathcal{F}_{\Phi*} \in L_2(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ *such that*

$$\mathcal{F}_{\Phi*} \;=\; \mathcal{P}_b f_* - W_b\Phi_*, \qquad \textit{in}\ \Omega,$$

and $\mathcal{C}_{\Phi} : H^{1,0}(\Omega; L_a) \to L_2(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ *is a bounded linear operator.*

Considering a couple $(F, \Psi)^\top \in H^{1,0}(\Omega; L_a) \times H^{-\frac{1}{2}}(\partial\Omega)$ and employing Lemma B.3 for $\mathcal{F}_{\Phi*} = F - V_b\Psi \in H^{1,0}(\Omega; L_a)$, we arrive at the following statement.

Corollary B.2 *For any couple*

$$(F, \Psi)^\top = \mathcal{F}_* \in H^{1,0}(\Omega; L_a) \times H^{-\frac{1}{2}}(\partial\Omega)$$

there exists a unique triple

$$(f_*, \Psi_*, \Phi_*)^\top = \widetilde{\mathcal{C}}_{\Psi *}\, \mathcal{F}_* \in L_2(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$$

such that

$$\begin{aligned} &\mathcal{P}_b\, f_* + V_b\, \Psi_* - W_b\, \Phi_* = F \quad in \;\; \Omega, \\ &\Psi_* = \Psi \quad on \;\; \partial\Omega. \end{aligned}$$

Moreover, the operator

$$\widetilde{\mathcal{C}}_{\Psi *} \;:\; H^{1,0}(\Omega; L_a) \times H^{-\frac{1}{2}}(\partial\Omega) \to L_2(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$$

is linear and continuous.

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