Advances in Boundary Element Techniques VI
Editors A P Selvadurai, C L Tan, M H Aliabadi
EC Ltd., UK, ISBN 09547783-2-4, 2005, 107-114

Boundary-Domain Integro-Differential Equation of Elastic Damage Mechanics Model of Stationary Drilling

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Keywords: Elasticity, Damage, Nonlinear Partial Integro-Differential Equations, Free Boundary Problem, Iteration Algorithms

Abstract. A stationary-periodic quasi-static model of rock percussive deep drilling is described, that includes an auxiliary problem of stationary indentation of a rigid bit into a rock. The rock is modeled by an infinite elastic medium with damage-induced material stiffness reduction. The bore-hole is a semi-infinite cylinder with a curvilinear bottom. It is assumed the indentation is produced by a stationary motion of the rupture front at which an appropriate rock strength condition is violated. The stationarity of the problem allows to reduce the damage history in a material point to the damage distribution down in space. The bore-hole boundary is not known in advance and consists of four parts: a traction-free non-rupturing part, a contact non-rupturing part, a traction-free part of the rupture front, and a contact part of the rupture front. Thus the problem is formulated as a non-local non-linear free-boundary contact problem and algorithms of its numerical solution are discussed. It includes a multi-level iteration process, reducing the problem to a sequence of problems of elastic damage mechanics with a fixed boundary, which, in turn, is reduced to a nonlinear boundary-domain integro-differential equation.

Introduction

The bore-hole progression in the percussive drilling is caused by a material rupture under the action of a drilling bit applied at the bore-hole boundary points $x(\tau)$ moving in time τ due to rupture. This boundary loading generates a strain process $\varepsilon_{ij}(x,\tau)$ at all material points x. Let a material point x has Cartesian coordinates (x_1, x_2, x_3) in the non-deformed state. The radius-vector of the same material point x in a deformed state at a time τ is $\tilde{x}(x,\tau) = x + \bar{u}(x,\tau)$, where $\bar{u}(x,\tau)$ is the displacement vector. We will use all equations in terms of the non-deformed (reference) coordinates x and refer the boundary conditions to the non-deformed boundary surfaces (Lagrange approach).

Let us consider stationary-periodic percussive drilling of a half-infinite bore-hole, $\Omega_H(\tau)$, spreading to $x_3 = \infty$ in an infinite elastic space, see Fig. 1. Let x_3-axis of the coordinate system coincide with the bore-hole axes, and the drill bit progressive-periodic motion occurs only in the x_3 direction. Let $\Omega(\tau) = \mathbb{R}^3 \backslash \Omega_H(\tau)$ be the domain occupied by the material (i.e. the infinite space with drilled bore-hole) and $\partial\Omega(\tau)$ be the bore-hole surface in the non-deformed state. If the rupture front $\partial_F\Omega(\tau)$ constitutes only a finite part of the boundary $\partial\Omega(\tau)$, then the borehole is a semi–infinite (not necessarily circular) cylinder with a curvilinear bottom being the rupture front $\partial_F\Omega(\tau)$. Otherwise, the bore-hole has a monotonously widening shape. If the bit is axially-symmetric then the bore-hole is axially symmetric as well.

Let C^0_{ijkl} be a known initial (virgin material) constant stiffness tensor and and ε^0_{ij} be a pre-strain tensor in the rock (without a bore-hole). Introducing $u_i(x) = \bar{u}_i(x) - \varepsilon^0_{ij}x_j$ as displacement perturbations of the initial deformation described by ε^0_{kl}, the total strain can be presented as $\varepsilon_{kl}(x) = (u_{k,l} + u_{l,k})/2 + \varepsilon^0_{kl}$.

The Hook law for an elastic anisotropic damaging material can be written in the form (see e.g. [1,2]),

$$\sigma_{ij}(x,\tau) = C_{ijkl}(x,\tau)\varepsilon_{kl}(x,\tau).$$

The damage implies a decrease of the secant elastic stiffness tensor $C_{ijkl}(x,\tau)$ at a point x caused by the strain tensor history $\varepsilon_{qp}(x,\tau')$ at that point during all preceding time instants, $\tau' \le \tau$.

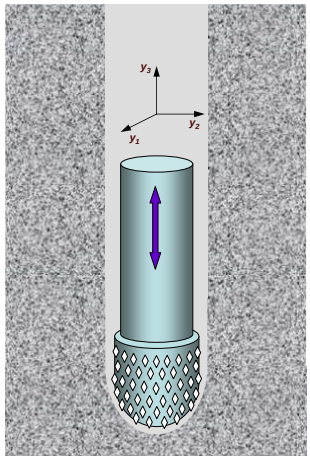

Figure 1: Stationary-periodic percussive drilling

Let the stiffness evolution equation be presented as follows,

$$\dot{C}_{ijkl}(x,\tau) = -\hat{C}_{ijkl}(\{\varepsilon(x)\}(\tau),\varepsilon(x,\tau)) \left\langle \left.\frac{\partial F(\{\varepsilon(x)\}(\tau),e)}{\partial e_{pq}}\right|_{e=\varepsilon(x,\tau)} \dot{\varepsilon}_{pq}(x,\tau) \right\rangle, \tag{1}$$

where the over dot means partial derivative with respect to τ, which for the chosen Lagrange approach coincides with the material derivative in time; $\hat{C}_{ijkl}(\{\varepsilon\}(\tau),\varepsilon)$ and $F(\{\varepsilon\}(\tau),\varepsilon)$ are known functionals of the strain history $\{\varepsilon\}(\tau) = \{\varepsilon(\tau')\}_{\tau'=-\infty}^{\tau}$, and functions of the currant strain ε; $F(\{\varepsilon\}(\tau),\varepsilon) = 1$ is the currant damage surface in the strain space ε_{ij}, and $\hat{C}_{ijkl}(\{\varepsilon\}(\tau),\varepsilon) = 0$ if $F(\{\varepsilon\}(\tau),\varepsilon) < 1$, that is if ε is inside the currant damage surface (e.g. during initial loading stage or unloading); the angular McAuley brackets are defined as $\langle a\rangle := (a+|a|)/2$. Note that (1) comprises damage rules, which may be not associated with the damage surface, as well as the strain tensor decomposition on the positive and negative parts, c.f. [1,2].

Similar to the pure plasticity, we will suppose the functionals $\hat{C}_{ijkl}(\{\varepsilon\}(\tau),\cdot)$ and $F(\{\varepsilon\}(\tau),\cdot)$ depend on the strain history as a sequence of events only, i.e. are independent explicitly of time or strain rate. Then the same will be true also for the stiffness tensor C_{ijkl}.

To describe the material strength for a point x, we will use an instant strength condition at a point x at an instant τ written as

$$\Lambda(\varepsilon(x,\tau)) < 1, \quad x \in \Omega(\tau), \tag{2}$$

where the function $\Lambda(\varepsilon)$ is associated with the von Mises, Coulomb–Mohr, Drucker–Prager or another appropriate strength condition. Generally, the function $\Lambda(\varepsilon)$ may be not directly connected with the damage softening.

We suppose that the rupture appears in the form of a rupture front $\partial_F\Omega(\tau)$, c.f. [3] (see also [2] for discussion about damage and rupture without macro-crack nucleation at multiaxial compression). The rupture front is a part of the bore-hole boundary $\partial\Omega(\tau)$. The rupture front equation can be taken as

$$\Lambda(\varepsilon(x),\tau) = 1, \quad y \in \partial_F\Omega(\tau). \tag{3}$$

The bore-hole boundary $\partial\Omega$ generally consists of four non-overlapping parts: a free of traction non-rupturing part $\partial_{00}\Omega$, a contact non-rupturing part $\partial_{c0}\Omega$, a free of traction part of the rupture front $\partial_{0F}\Omega$, and a contact part of the rupture front $\partial_{cF}\Omega$.

Stationary Indentation Model with Damage

The stationary-periodic quasi-static elastic damage mechanics model of percussive drilling introduced in [4] consists of three stages: elastic loading, constant-force rupture progression, and elastic unloading

parallel to the loading. Thus the problem can be split into a free-boundary non-linear non-local problem of stationary indentation for the rupture stage of the cycle, and an elastic conforming contact problem for the rest of the cycle.

In the stationary indentation problem, the displacements, strains, stresses and stress function are the same at the corresponding points at the corresponding instants,

$$u_i(x,\tau) = u_i(x-\tau\dot{h},0),\ \varepsilon_{ij}(x,\tau) = \varepsilon_{ij}(x-\tau\dot{h},0),\ \sigma_{ij}(x,\tau) = \sigma_{ij}(x-\tau\dot{h},0), \tag{4}$$

$$\Lambda(\varepsilon(x,\tau)) = \Lambda(\varepsilon(x-\tau\dot{h},0)), \qquad x\in\Omega(\tau), \tag{5}$$

where $\dot{h} = (0,0,\dot{h}_3)$ is a constant progression rate vector in the x_3 direction and $\dot{h}_3 < 0$. This implies that all time derivatives can be reduced to derivatives in x_3 coordinate,

$$\dot{u}_i(x,\tau) = -\dot{h}_3 u_{i,3}(x,\tau),\ \dot{\varepsilon}_{ij}(x,\tau) = -\dot{h}_3\varepsilon_{ij,3}(x,\tau),\ \dot{\sigma}_{ij}(x,\tau) = -\dot{h}_3\sigma_{ij,3}(x,\tau), \qquad x\in\Omega(\tau). \tag{6}$$

From the second of relations (6) we have,

$$\dot{C}_{ijkl}(x,\tau) = -\ \dot{h}_3 C_{ijkl,3}(x,\tau) \tag{7}$$

$$\{\varepsilon(x)\}(\tau) := \{\varepsilon(x,\tau')\}_{\tau'=-\infty}^{\tau} = \{\varepsilon(x-\tau\dot{h},0)\}_{\tau'=-\infty}^{\tau} = [\![\varepsilon]\!](x-\tau\dot{h}) \tag{8}$$

$$[\![\varepsilon]\!](x-\tau\dot{h}) := \{\varepsilon(z,0)\}_{z=\{x_1,x_2,-\infty\}}^{z=\{x-\tau\dot{h}\}} \tag{9}$$

Thus the temporal history $\{\varepsilon(x)\}(\tau)$ at a point x is equivalent to $[\![\varepsilon]\!](x-\tau\dot{h})$, the strain distribution on the space interval $(-\infty, y-\tau\dot{h})$. Then (1) can be rewritten for $t=0$ in the form

$$C_{ijkl,3}(x,0) = -\hat{C}_{ijkl}([\![\varepsilon]\!](x),\varepsilon(x,0))\left\langle \left.\frac{\partial F([\![\varepsilon]\!](x),e)}{\partial e_{pq}}\right|_{e=\varepsilon(x,0)} \varepsilon_{pq,3}(x,0)\right\rangle. \tag{10}$$

To solve the stationary indentation problem, it is sufficient to consider it only for $t=0$. Thus, following [4] and dropping the argument $t=0$ for brevity, we arrive at a non-classical non-linear functional-integro-differential free boundary problem,

$$\sigma_{ij,j}(x) = 0, \qquad \Lambda(\sigma(x)) < 1, \qquad x\in\Omega; \tag{11}$$

$$\begin{aligned} &\sigma_{ij}(x)\eta_j(x)\xi_i(x) = 0, \quad \sigma_{ij}(x)\eta_j(x)\zeta_i(x) = 0, \quad \sigma_{ij}(x)\eta_j(x)\eta_i(x) < 0,\\ &\eta_3(x) = 0, \quad d_\eta(x+u(x)) = 0, \quad \Lambda(\sigma(x)) < 1, \quad x\in\partial_{c0}\Omega; \end{aligned} \tag{12}$$

$$\begin{aligned} &\sigma_{ij}(x)\eta_j(x)\xi_i(x) = 0, \quad \sigma_{ij}(x)\eta_j(x)\zeta_i(x) = 0, \quad \sigma_{ij}(x)\eta_j(x)\eta_i(x) < 0,\\ &\eta_3(x) > 0, \quad d_\eta(x+u(x)) = 0, \quad \Lambda(\sigma(x)) = 1, \quad x\in\partial_{cF}\Omega; \end{aligned} \tag{13}$$

$$\begin{aligned} &\sigma_{ij}(x)\eta_j(x) = 0,\\ &\eta_3(x) = 0, \quad d_\eta(x+u(x)) > 0, \quad \Lambda(\sigma(x)) < 1, \quad x\in\partial_{00}\Omega; \end{aligned} \tag{14}$$

$$\begin{aligned} &\sigma_{ij}(x)\eta_j(x) = 0,\\ &\eta_3(x) > 0, \quad d_\eta(x+u(x)) > 0, \quad \Lambda(\sigma(x)) = 1, \quad x\in\partial_{0F}\Omega; \end{aligned} \tag{15}$$

$$u_i(x) \to 0, \qquad y\to\infty. \tag{16}$$

where

$$\sigma_{ij}(x) = C_{ijkl}([\![\varepsilon]\!];x)\varepsilon_{kl}(x), \quad \varepsilon_{kl}(x) = (u_{k,l}(x)+u_{l,k}(x))/2 + \varepsilon_{kl}^0, \tag{17}$$

$C_{ijkl}([\![\varepsilon]\!];x)$ is determined in terms of ε by the following non-local relation obtained by integrating (10),

$$C_{ijkl}([\![\varepsilon]\!];x) = C_{ijkl}^0 + \tilde{C}_{ijkl}([\![\varepsilon]\!];x), \quad \tilde{C}_{ijkl}([\![\varepsilon]\!];x) := -\int_{-\infty}^{x_3} \hat{C}_{ijkl}([\![\varepsilon]\!](x_1,x_2,x_3'),\varepsilon(x_1,x_2,x_3')) \times$$
$$\left\langle \left.\frac{\partial F([\![\varepsilon]\!](x_1,x_2,x_3'),e)}{\partial e_{pq}}\right|_{e=\varepsilon(x_1,x_2,x_3')} \varepsilon_{pq,3}(x_1,x_2,x_3')\right\rangle dx_3', \quad x\in\Omega \tag{18}$$

and $\xi_j(x)$, $\zeta_j(x)$ are unit vectors orthogonal to the normal vector $\eta_j(x)$ and to each other. Condition (16) is understood on almost any straight ray originating from $x = 0$, thus permitting a non-zero limit of displacements as $x \to \infty$ parallel to the bore-hole. By $d_\eta(x)$ we denote a (positive or negative) distance between the point x and the bit boundary ∂B in the η direction.

All the four boundary parts $\partial_{00}\Omega$, $\partial_{0F}\Omega$, $\partial_{c0}\Omega$, $\partial_{cF}\Omega$, and consequently $d_\eta(x)$, are generally unknown in advance in this setting and the corresponding "excessive" boundary equalities and inequalities are provided in (12) - (15) to allow their determination.

Note that the strains decrease with the distance from the bore-hole surface $\partial\Omega$, and tend to constant as x_3 increases, in the elastic space. Thus the integrand in (18) equals to zero at sufficiently small and sufficiently large x_3' since the strains there are inside the damage surface, where $\hat{C}_{ijkl}([\![\varepsilon]\!](x_1, x_2, x_3'), \varepsilon(x_1, x_2, x_3')) = 0$. This means the stiffness tensor $C_{ijkl}(x, t)$ will be equal to the initial one, C^0_{ijkl}, outside some neighbourhood of the bore-hole, and will be independent of x_3 at some distance from the bore-hole bottom in this neighbourhood.

Different strategies can be chosen to solve this problem, c.f. [5, 6]. One of the possibilities is the multi-level iteration algorithm. It consists of *global iterations*, each solving a nonlinear mixed boundary value *functional-integro-differential* problem (11)-(18) with some *fixed* boundaries, $\partial_{00}\Omega$, $\partial_{0F}\Omega$, $\partial_{c0}\Omega$, $\partial_{cF}\Omega$. Then the "excessive" conditions in (11)-(15) are checked and the boundaries are changed to decrease the discrepancies, and the next global iteration starts.

On the first global iteration one can reasonably assume that the rupture front coincides with the contact part of the bit, $\partial_c B$, which in turn coincides with the bit bottom, $\partial_b B$, (consisting of the bit surface points with algebraically smallest x_3 coordinate, over the points with the same (x_1, x_2) coordinates), i.e. $\partial_{cF}\Omega = \partial_c B = \partial_b B$, $\partial_{0F}\Omega = \emptyset$, and there is no contact without rupture, i.e. $\partial_{c0}\Omega = \emptyset$. These assumptions imply that the bore-hole free boundary $\partial_0\Omega$ is the semi-infinite cylindrical surface ended by the bit bottom, on the first iteration.

After the global iterations converge, the integration of the component $\sigma_{3j}(x)\eta_j(x)$ of the contact traction gives the total axial force $\mathcal{P}$ applied to the bit during the progression,

$$\mathcal{P} = \int_{\partial_c\Omega} \sigma_{3j}(x)\eta_j(x)\, dS(x). \tag{19}$$

Note that the obtained solution and particularly the total force $\mathcal{P}$ is independent of the progression rate $\dot{h}_3$ or the progression itself.

Auxiliary BVP of Stationary Elastic Damage Mechanics

Taking in mind (11)-(18), let us formulate an auxiliary mixed nonlinear functional-integro-differential boundary-value problem of elastic damage mechanics, used on each global iteration,

$$[L_{ik}([\![\varepsilon]\!])u_k](x) := \frac{\partial}{\partial x_j}\,[C_{ijkl}([\![\varepsilon]\!]; x)u_{k,l}(x)] = \tilde{f}_i([\![\varepsilon]\!]; x), \qquad x \in \Omega, \tag{20}$$

$$u_i(x) = \tilde{u}_i(x), \qquad x \in \partial_D\Omega, \tag{21}$$

$$[T_{ik}([\![\varepsilon]\!])u_k](x) := C_{ijkl}([\![\varepsilon]\!]; x)u_{k,l}(x)\eta_j(x) = \tilde{t}_i([\![\varepsilon]\!]; x), \qquad x \in \partial_N\Omega, \tag{22}$$

$$u_i(y) \to 0, \qquad y \to \infty, \tag{23}$$

where

$$\varepsilon_{kl}(x) = (u_{k,l}(x) + u_{l,k}(x))/2 + \varepsilon^0_{kl}, \qquad \tilde{f}_i([\![\varepsilon]\!]; x) := \check{f}_i(x) - \tilde{C}_{ijkl,j}([\![\varepsilon]\!]; x)\varepsilon^0_{kl}, \tag{24}$$

$$\tilde{u}_i(x) := \check{u}_i(x) - \varepsilon^0_{il}x_l, \qquad \tilde{t}_i([\![\varepsilon]\!]; x) := \check{t}_i(x) - C_{ijkl}([\![\varepsilon]\!]; x)\varepsilon^0_{kl}\eta_j(x) \tag{25}$$

and $\tilde{C}_{ijkl,j}([\![\varepsilon]\!]; x)$, $C_{ijkl}([\![\varepsilon]\!]; x)$ are defined by (18). Here $\eta_i(x)$ is an outward normal vector to the boundary $\partial\Omega$; ε^0_{il} is the known constant initial rock pre-strain, $[T([\![\varepsilon]\!])u](x) = [T_{ik}([\![\varepsilon]\!])u_k](x)$ is the (perturbation) traction vector at a boundary point x, while $T([\![\varepsilon]\!]) = T_{ik}([\![\varepsilon]\!])$ is the nonlocal nonlinear traction differential operator; $\check{f}(x)$, $\check{u}(x)$ and $\check{t}(x)$ are known volume force, displacement and traction vectors on the parts $\partial_D\Omega$ and $\partial_N\Omega$ of the boundary, respectively.

Two-Operator Green-Betti Identity and BDIDE of Stationary Elastic Damage Mechanics

Let us fix a point y and consider the following auxiliary differential operators of linear elasticity with initial stiffness coefficients C^0_{ijkl} independent of x,

$$[L^0_{ik}v_k](x) := \frac{\partial}{\partial x_j}\left[C^0_{ijkl}\frac{\partial v_k(x)}{\partial x_l}\right], \ [T^0_{ik}v_k](x) := C^0_{ijkl}\frac{\partial v_k(x)}{\partial x_l}n_j(x).$$

Integrating by parts, we have the first Green identities for the differential operators $[L([\![\varepsilon]\!])u](x) = [L_{ik}([\![\varepsilon]\!])u_k](x)$ and $[L^0v](x) = [L^0_{ik}v_k](x)$,

$$\begin{aligned}\int_\Omega v_i(x)[L_{ik}([\![\varepsilon]\!])u_k](x)d\Omega(x) &= \int_{\partial\Omega} v_i(x)[T_{ik}([\![\varepsilon]\!])u_k](x)d\Gamma(x) - \int_\Omega \frac{\partial v_i(x)}{\partial x_j}C_{ijkl}([\![\varepsilon]\!];x)\frac{\partial u_k(x)}{\partial x_l}d\Omega(x),\\ \int_\Omega u_i(x)[L^0_{ik}v_k](x)d\Omega(x) &= \int_{\partial\Omega} u_i(x)[T^0_{ik}v_k](x)d\Gamma(x) - \int_\Omega \frac{\partial u_i(x)}{\partial x_j}C^0_{ijkl}\frac{\partial v_k(x)}{\partial x_l}d\Omega(x),\end{aligned}$$

where $u(x)$ and $v(x)$ are arbitrary vector-functions (for that the operators and integrals in the above expressions have sense), and ε is related with u by (24). Subtracting the identities from each other and taking into account the symmetry of the tensor C_{ijkl}, we derive the two-operator second Green-Betti identity,

$$\int_\Omega \left\{u(x)[L^0v](x) - v(x)[L([\![\varepsilon]\!])u](x)\right\}d\Omega(x) = \int_{\partial\Omega}\left\{u(x)[T^0v](x) - v(x)[T([\![\varepsilon]\!])u](x)\right\}d\Gamma(x) + \int_\Omega [\nabla v(x)]\tilde{C}([\![\varepsilon]\!];x)\nabla u(x)d\Omega(x), \quad (26)$$

where $\tilde{C}([\![\varepsilon]\!];x)$ is given by (18) and thus is non-zero only at the damaged material points x.

If $L([\![\varepsilon]\!]) = L^0$, i.e. $L([\![\varepsilon]\!])$ is a linear homogeneous elasticity operator without damage, then the last domain integral disappears in eq (26), which thus degenerates into the classical second Green-Betti identity.

For a fixed y, let $F^0(x,y) = F^0_{km}(x,y)$ be a fundamental solution for the linear differential operator $[L^0_{ik}v_k](x)$ with constant coefficients, i.e.,

$$[L^0_{ik}F^0_{km}(\cdot,y)](x) := C^0_{ijkl}\frac{\partial^2 F^0_{km}(x,y)}{\partial x_j\partial x_l} = \delta_{im}\delta(x-y),$$

where δ_{im} is the Kronecker symbol and $\delta(x-y)$ is the Dirac delta-function. Note that generally $F^0(x,y)$ *is not* a parametrix for the original operator $L([\![\varepsilon]\!])$ if the tensor C depends on ε.

If the material is originally isotropic, then

$$C^0_{ijkl} = \lambda^0\delta_{ij}\delta_{kl} + \mu^0(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \qquad \mu^0 > C > 0, \qquad \lambda^0 + \frac{2}{3}\mu^0 > C > 0 \quad (27)$$

and $F^0_{im}(x,y)$ is the Kelvin-Somigliana solution,

$$F^0_{im}(x,y) = \frac{-1}{8\pi r}\left\{\frac{\delta_{im} - r_{,i}r_{,m}}{\lambda^0 + 2\mu^0} + \frac{\delta_{im} + r_{,i}r_{,m}}{\mu^0}\right\} \quad (28)$$

in the the 3D case.

In the 2D isotropic case,

$$F^0_{im}(x,y) = \frac{-1}{4\pi}\left\{\frac{-\delta_{im}\ln r - r_{,i}r_{,m}}{\lambda^0 + 2\mu^0} + \frac{-\delta_{im}\ln r + r_{,i}r_{,m}}{\mu^0}\right\} \quad (29)$$

for the plane strain state; for the plane stress, λ^0 in (27) and (29) should be replaced by $2\lambda^0\mu^0/(\lambda^0 + 2\mu^0)$. Here $r := \sqrt{(x_i - y_i)(x_i - y_i)}$, $\quad r_{,i} := \partial r/\partial x_i = (x_i - y_i)/r$. One can similarly write down

(and use in further formulation) also the corresponding axially symmetric fundamental solution of the linear homogeneous isotropic elasticity, if the bore-hole is axially symmetric.

For anisotropic material, the fundamental solution can be written down in an analytical form for arbitrary anisotropy in the 2D case and for some particular anisotropy in the 3D case; otherwise, it can be expressed as a linear integral over a circle, [7–9].

Assuming $u(x)$ is a solution of nonlinear system (20) and using the fundamental solution $F^0(x,y)$ as $v(x)$ in the Green identity (26), we obtain, similar to the linear homogeneous elasticity (see e.g. [10–13]), the following non-linear two-operator third Green identity,

$$c(y)u(y) - \int_{\partial\Omega} u(x)[T^0F^0(\cdot,y)](x)d\Gamma(x) + \int_{\partial\Omega} F^0(x,y)[T(\llbracket\varepsilon\rrbracket)u](x)d\Gamma(x) - \int_{\Omega}[\nabla^{(x)}F^0(x,y)]\tilde{C}(\llbracket\varepsilon\rrbracket;x)\nabla u(x)d\Omega(x) = \int_{\Omega} F^0(x,y)\tilde{f}(\llbracket\varepsilon\rrbracket;x)d\Omega(x), \quad (30)$$

where $c_{im}(y) = \delta_{im}$ if $y \in \Omega$; $c_{im}(y) = 0$ if $y \notin \bar{\Omega}$; $c_{im}(y) = \frac{1}{2}\delta_{im}$ if y is a smooth point of the boundary $\partial\Omega$; and $c_{im}(y) = c_{im}(a(y), \alpha(y))$ is a function of the anisotropy tensor $a(y)$ and the interior space angle $\alpha(y)$ at a corner point y of the boundary $\partial\Omega$.

Substituting boundary conditions (21), (22) into eq (30) and using it at $y \in \overline{\Omega}$, we arrive at a (*partly segregated*) nonlinear two-operator BDIDE for $u(x)$ at $x \in \overline{\Omega}$ and an unknown traction $t(x) = [T(\llbracket\varepsilon\rrbracket)u](x)$ on $\partial_D\Omega$,

$$c^t(y)u(y) - \int_{\partial_N\Omega} u(x)[T^0F^0(\cdot,y)](x)d\Gamma(x) + \int_{\partial_D\Omega} F^0(x,y)t(x)d\Gamma(x) = \tilde{\mathcal{F}}(y), \quad y \in \overline{\Omega}, \quad (31)$$

$$\tilde{\mathcal{F}}(y) := [c^t(y) - c(y)]\tilde{u}(y) + \int_{\partial_D\Omega} \tilde{u}(x)[T^0F^0(\cdot,y)](x)d\Gamma(x) - \int_{\partial_N\Omega} F^0(x,y)\tilde{t}(\llbracket\varepsilon\rrbracket;x)d\Gamma(x) + \int_{\Omega} F^0(x,y)\tilde{f}(\llbracket\varepsilon\rrbracket;x)d\Omega(x) + \int_{\Omega}[\nabla^{(x)}F^0(x,y)]\tilde{C}(\llbracket\varepsilon\rrbracket;x)\nabla u(x)d\Omega(x). \quad (32)$$

$$c^t(y) = 0 \quad \text{if} \quad y \in \partial_D\Omega, \quad c^t(y) = c(y) \quad \text{if} \quad y \in \Omega \cup \partial_N\Omega. \quad (33)$$

The left-hand side operator of BDIDE (31) is linear but the right hand side nonlinearly and non-locally depends on the strain distribution below the integration point x, see (18), (24)-(25), if the damage-caused decrease of the elastic moduli, $\tilde{C}(\llbracket\varepsilon\rrbracket;x)$, is non-zero. The integral equation is of the second kind, includes at most the first derivatives of the unknown solution $u(x)$, both directly in the last domain integral in the right hand side and through the function $\tilde{C}(\llbracket\varepsilon\rrbracket;x)$. The function $[\nabla^{(x)}F^0(x,y)]$ is at most weakly singular in Ω. The boundary integrals have at most the Cauchy-type singularity.

As was noted above, the stiffness tensor $C_{ijkl}(x)$ will be equal to the initial one, C^0_{ijkl}, outside some neighbourhood of the bore-hole, and will be independent of x_3 at some distance from the bore-hole bottom in this neighbourhood. This implies the last integral in (32) is to be taken only over some neighbourhood of the bore-hole, and the same holds true also for the penultimate integral if the volume force $\check{f}$ is absent. If the pre-strain ε^0_{ij} is absent along with the volume force $\check{f}$, then the penultimate integral disappears completely.

The nonlinear BDIDE (31) can be reduced after some discretization to a system of nonlinear algebraic equation and solved numerically. The system will include unknowns not only on the boundary but also at internal points. Solution of BDIDE (31) on each global iteration can be achieved using *sub-iterations*. Taking in mind the previous paragraph, one can solve on each, $n-$th sub-iteration step the discrete counterpart of the linear boundary integral equation

$$c^t(y)u^{(n)}(y) - \int_{\partial_N\Omega} u(x)^{(n)}[T^0F^0(\cdot,y)](x)d\Gamma(x) + \int_{\partial_D\Omega} F^0(x,y)t^{(n)}(x)d\Gamma(x) = \tilde{\mathcal{F}}^{(n)}(y), \quad y \in \partial\Omega, \quad (34)$$

$$\tilde{\mathcal{F}}^{(n)}(y) := \int_{\partial_D\Omega} \tilde{u}(x)[T^0F^0(\cdot,y)](x)d\Gamma(x) - \int_{\partial_N\Omega} F^0(x,y)\tilde{t}([\![\varepsilon^{(n-1)}]\!];x)d\Gamma(x) + \int_{\Omega} F^0(x,y)\tilde{f}([\![\varepsilon^{(n-1)}]\!];x)d\Omega(x) + \int_{\Omega}[\nabla^{(x)}F^0(x,y)]\tilde{C}([\![\varepsilon^{(n-1)}]\!];x)\nabla u^{(n-1)}(x)d\Omega(x). \quad (35)$$

$$\varepsilon_{kl}^{(n-1)}(x) = (u_{k,l}^{(n-1)}(x) + u_{l,k}^{(n-1)}(x))/2 + \varepsilon_{kl}^0 \quad (36)$$

On the first iteration, one can take $u^{(0)} = 0$. The iteration process should proceed before the difference between solutions on neighbouring iterations becomes negligible. One can remark that algorithmically the sub-iteration process is rather similar to that used usually for solution of nonlinearly elastic and elasto-plastic problems by the Boundary Element Method, c.f. e.g. [10–13], but includes also the non-local downward operator for calculating the stiffness tensor decrease due to damage.

The iteration algorithms can be further simplified and optimised using information on spectral properties of the linear left hand side boundary integral operator in (34), c.f. [14]. This looks especially promising for the case of the pure Neumann problem, $\partial_N\Omega = \partial\Omega$.

The stationary damage mechanics problem can be also reduced to some other (e.g. united) nonlinear BDIDEs if one will calculate $[T([\![\varepsilon]\!])u](x)$ from unknown displacement $u(x)$ instead of substituting it with the auxiliary function $t(x)$ in the integral term of (31) over $\partial_D\Omega$, c.f. [15, 16].

Summary

A stationary elastic damage mechanics model of the percussive drilling leads to a non-classical functional non-linear partial integro-differential free-boundary problem. By an iteration algorithm its solution can be reduced to a sequence of corresponding functional non-linear partial integro-differential mixed problems with fixed boundaries. The latter are then formulated as a nonlinear boundary-domain integro-differential equations that are in turn reduced iteratively to the linear boundary integral equations of linear elasticity on each sub-iteration.

Acknowledgement: This work is supported by the research grant GR/R85556/01 "Nonlinear Dynamics and Rock Contact Fracture Mechanics in Modelling of Vibration Enhanced Drilling " of the Engineering and Physical Sciences Research Council, UK.

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