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Incremental Localized Boundary-Domain Integro-Differential Equations of Elastic Damage Mechanics for Inhomogeneous Body

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Abstract. A quasi-static mixed boundary value problem of elastic damage mechanics for a continuously inhomogeneous body is considered. Using the two-operator Green-Betti formula and the fundamental solution of an auxiliary homogeneous linear elasticity with frozen initial, secant or tangent elastic coefficients, a boundary-domain integro-differential formulation of the elasto-plastic problem with respect to the displacement rates and their gradients is derived. Using a cut-off function approach, the corresponding localized parametrix of the auxiliary problem is constructed to reduce the problem to a nonlinear localized boundary-domain integro-differential equation. Algorithms of mesh-based and mesh-less discretizations are presented resulting in sparsely populated systems of nonlinear algebraic equations for the displacement increments.

INTRODUCTION

Application of the Boundary Integral Equation (BIE) method (boundary element method) to linear elasticity problems for homogeneous bodies has been intensively developed over recent decades, see e.g. [1–4]. Using fundamental solutions of auxiliary linear elastic problems (with the initial elastic coefficients), the elastic damage mechanics problems for homogeneous material also can be reduced to non-linear boundary-domain integral equations with hyper-singular integrals, see [5]. However, the fundamental solution is usually highly non-local, which leads after discretization to a system of algebraic equations with a dense matrix. Moreover, the fundamental solution is generally not available in an explicit form if the coefficients of the auxiliary problem vary in space, i.e. if the material is inhomogeneous (functionally graded).

To overcome these effects, some parametrices (Levi's functions) localized by cut-off function multiplication were constructed and implemented in [6] to a linear scalar (heat transfer) equation in inhomogeneous medium. This reduced the linear Boundary Value Problem (BVP) with variable coefficient to a linear *Localized* Boundary-Domain Integral or Integro-Differential Equation (LBDIE or LBDIDE), which leads after a mesh-based or mesh-less discretization to a linear algebraic system with a sparse matrix. Some numerical implementations of the linear LBDIE were presented in [7]. Somewhat different linear LBDI(D)E formulations and numerical realizations were presented in [8, 9].

Generalizing this approach to non-linear problems, the mixed BVP for a second order scalar quasi-linear elliptic PDE with the variable coefficient dependent on the unknown solution was reduced to quasi-linear LBDIDEs in [10, 11], while some quasi-linear two-operator LBDIDEs were obtained for the case when the variable coefficient depends also on the BVP solution gradient in [12], [11]. The approach was extended to the mixed BVPs of physically nonlinear elasticity (with small displacement gradients) in [13] and for incremental elasto-plasticity in [14], both for continuously inhomogeneous body.

Note that another approach based on local parametrices that are Green functions for an auxiliary problem on local spherical domains was used in [15–18] reducing some linear and non-linear problems for a body with a special inhomogeneity to local boundary-domain integral equations solved numerically by the mesh-less methods.

In this paper, we further extend the localization approach of [6, 10–14] to the mixed BVP of elastic damage mechanics in the incremental form with small displacement gradients for continuously

inhomogeneous (functionally graded) materials. First, we reduce the BVP to a direct two-operator nonlinear BDIDE of the second kind for the displacement rates (or increments). The equation includes at most the first derivatives of the unknown solution, weakly singular integrals over the domain and at most Cauchy-type singular integrals over the boundary. Then we present a localized version of the BDIDE and describe its mesh-based and mesh-less discretizations.

ELEMENTS OF ELASTIC DAMAGE MECHANICS

Let $u(x) = u_i(x)$ be the displacement vector in $\mathbb{R}^n$, where $n = 2$ or $n = 3$,

$$\varepsilon_{ij}(x,t) = [u_{i,j}(x,t) + u_{j,i}(x,t)]/2 \tag{1}$$

be the small strain tensor, $\sigma_{ij}(x)$ be the stress tensor. The case $n = 3$ will describe the 3D, an the case $n = 2$ the 2D (plane strain or plain stress) elastic damage mechanics problems. All the indices should vary from 1 to n and summation over repeated indices is assumed from 1 to n as well unless stated otherwise. The comma in front of a superscript means derivative with respect to the corresponding coordinate.

Constitutive equations of the elastic damage mechanics can be written in the form (see e.g. [19]),

$$\sigma_{ij}(x,t) = a'_{ijkl}(\{\varepsilon(x)\}(t), x)\varepsilon_{kl}(x,t), \tag{2}$$

where the secant elastic stiffness tensor $a'_{ijkl}(\{\varepsilon(x)\}(t), x)$ at a point x decreases due to the strain tensor history $\{\varepsilon_{qp}(x,\tau)\}_{\tau=-\infty}^{t}$ at that point during all preceding time instants. If the loading (or damage) is absent at instants $\tau \le 0$, then $a'_{ijkl}(x,0) = a^0_{ijkl}(x)$, where the stiffness tensor of the virgin inhomogeneous material (before loading) at a point x, $a^0_{ijkl}(x)$, is a known function of the coordinates x, such that

$$a^0_{ijkl}(x) = a^0_{jikl}(x) = a^0_{ijlk}(x) = a^0_{klij}(x) \tag{3}$$

$$\varepsilon_{kl} a^0_{ijkl}(x)\varepsilon_{kl} \ge 0 \quad \forall \varepsilon_{kl}. \tag{4}$$

Note that in the plane stress state, the tensors a'_{ijkl} and a^0_{ijkl} in (2) and further on are to be replaced by corresponding combinations of their components.

Let the stiffness evolution due to damage be presented as follows,

$$\begin{aligned} \dot{a}'_{ijkl}(\{\varepsilon(x)\}(t), x) &= -\hat{a}_{ijkl}(\{\varepsilon(x)\}(t), \varepsilon(x,t), x) H[g_{ms}\dot{\varepsilon}_{ms}(x,t)] g_{pq}\dot{\varepsilon}_{pq}(x,t), \\ g_{pq}(\{\varepsilon(x)\}(t), x) &:= \left.\frac{\partial F(\{\varepsilon(x)\}(t), e, x)}{\partial e_{pq}}\right|_{e=\varepsilon(x,t)} \end{aligned} \tag{5}$$

where the over dot means derivative with respect to time; $\hat{a}_{ijkl}(\{\varepsilon\}(t), \varepsilon(t), x)$ and $F(\{\varepsilon\}(t), \varepsilon(t), x)$ are known functionals of the strain history $\{\varepsilon\}(t) = \{\varepsilon(\tau)\}_{\tau=-\infty}^{t}$, and functions of the currant strain ε and the material point x; such that

$$\hat{a}_{ijkl} = \hat{a}_{jikl} = \hat{a}_{ijlk} = \hat{a}_{klij}; \tag{6}$$

$F(\{\varepsilon\}(t), \varepsilon(t), x) = 1$ is the currant damage surface in the strain space ε_{ij}, and $\hat{a}_{ijkl}(\{\varepsilon\}(t), \varepsilon(t), x) = 0$ if $F(\{\varepsilon\}(t), \varepsilon(t), x) < 1$, that is if ε is inside the currant damage surface (e.g. during initial loading stage or unloading); the multiplier with the Heaviside function, $H[z] := \begin{cases} 1, & z \ge 0 \\ 0, & z < 0 \end{cases}$, is employed in (5) to ensure that the damage increment is zero during unloading strain increment. Note that (5) comprises damage rules, which may be not associated with the damage surface, as well as the strain tensor decomposition on the positive and negative parts, c.f. [19, 20].

We will consider here the time-invariant damage models, where the damage depends not on the physical time t but on the deformation history as a sequence of events. Thus the physical time t

can be replaced by any monotonous parameter, and the rates by the (infinitesimal) increments, e.g. $\mathcal{D}u_k = \dot{u}_k \mathcal{D}t$.

Differentiating (2) and taking into account (5), we arrive at the incremental form of the Hook law of the elastic damage mechanics,

$$\dot{\sigma}_{ij}(x,t) \quad = \quad a_{ijkl}(\{\varepsilon(x)\}(t), \nabla\dot{u}(x,t), x)\dot{u}_{k,l}(x,t), \tag{7}$$

where

$$a_{ijkl}(\{\varepsilon(x)\}(t), \nabla\dot{u}(x,t), x) := a'_{ijkl}(\{\varepsilon(x)\}(t), x) - \\ \hat{a}_{ijpq}(\{\varepsilon(x)\}(t), \varepsilon(x,t), x)\varepsilon_{pq}(x,t)H[g_{ms}(\{\varepsilon(x)\}(t), x)\dot{u}_{m,s}(x)]g_{kl}(\{\varepsilon(x)\}(t), x), \tag{8}$$

is the tangent stiffness tensor discontinuous in $\nabla\dot{u}$ due to the Heaviside function, and coinciding with the secant stiffness tensor for non-damaging strain increments. From (3) and (6), we have that a'_{ijkl} and a_{ijkl} have the similar symmetry properties.

The parameter t will be sometimes omitted for brevity further on.

Substituting (7) in the time derivatives of the equilibrium equation,

$$\dot{\sigma}_{ij,j} = \dot{f}_i, \tag{9}$$

where $f_i(x,t)$ is a known volume force vector (taken with the opposite sign). Employing (7) also in in the traction boundary conditions, we arrive at the following mixed boundary–value problem of incremental elastic damage mechanics for a bounded inhomogeneous body $\Omega \in \mathbb{R}^n$,

$$[L_{ik}(\{\varepsilon\}, \dot{u})\dot{u}_k](x) := \frac{\partial}{\partial x_j}\left[a_{ijkl}(\{\varepsilon(x)\}, \nabla\dot{u}(x), x)\frac{\partial\dot{u}_k(x)}{\partial x_l}\right] = \dot{f}_i(x), \qquad x \in \Omega, \tag{10}$$

$$\dot{u}_i(x) = \check{u}_i(x), \qquad x \in \partial_D\Omega, \tag{11}$$

$$[T_{ik}(\{\varepsilon\}, \dot{u})\dot{u}_k](x) := a_{ijkl}(\{\varepsilon(x)\}, \nabla\dot{u}(x), x)\frac{\partial\dot{u}_k(x)}{\partial x_l}n_j(x) = \check{t}_i(x), \qquad x \in \partial_N\Omega. \tag{12}$$

Here $n_i(x)$ is an outward normal vector to the boundary $\partial\Omega$; $[T(\{\varepsilon\}, \dot{u})\dot{u}](x) = [T_{ik}(\{\varepsilon\}, \dot{u})\dot{u}_k](x)$ is the traction rate vector at a boundary point x, while $T(\{\varepsilon\}, \dot{u}) = T_{ik}(\{\varepsilon\}, \dot{u})$ is the nonlinear traction differential operator; $\check{u}(x,t)$ and $\check{t}(x,t)$ are known displacement rate and traction rate vectors on the parts $\partial_D\Omega$ and $\partial_N\Omega$ of the boundary, respectively. The BVP of elastic damage mechanics (10)-(12) does not include time explicitly and one may replace there the rates $\dot{u}_k$ by the differentials $\mathcal{D}u_k = \dot{u}_k\mathcal{D}t$ and $\dot{f}_i$ with df_i.

For brevity, we will often drop also the argument $\{\varepsilon\}$ of the functionals $g_{kl}(\{\varepsilon(x)\}, x)$, $a_{ijkl}(\{\varepsilon(x)\}, \nabla\dot{u}(x), x)$ and operators $L(\{\varepsilon\}, \dot{u})$, $T(\{\varepsilon\}, \dot{u})$ in the equations below but their dependence on the process history and the currant strain will be meant nevertheless.

TWO-OPERATOR GREEN-BETTI IDENTITIES AND BDIDE OF INCREMENTAL ELASTIC DAMAGE MECHANICS

Let us fix a point y and consider the following auxiliary differential operators of linear elasticity with some coefficients $a^*_{ijkl}(y)$ independent of x,

$$[L^{(y)*}_{ik}v_k](x) := \frac{\partial}{\partial x_j}\left[a^*_{ijkl}(y)\frac{\partial v_k(x)}{\partial x_l}\right], \qquad [T^{(y)*}_{ik}v_k](x) := a^*_{ijkl}(y)\frac{\partial v_k(x)}{\partial x_l}n_j(x).$$

Under the coefficients $a^*_{ijkl}(y)$ frozen at a point y, one can understand either the initial elastic moduli $a^0_{ijkl}(y)$ independent of the strain-stress history and the current strain rate, or the current secant moduli $a'_{ijkl}(y,t)$ dependent on the strain-stress history but independent of the current strain rate, or the current tangent moduli $a_{ijkl}(\nabla\dot{u}(y), y, t)$ dependent on both the strain-stress history and the current strain rate. The same character of dependence on (or independence of) the strain-stress

history and the current strain rate will then remain for all asterisk variables and operators below. The particular choice of $a^*_{ijkl}(y)$ leads to three different versions of the integro-differential equations.

Integrating by parts, we have the first Green identities for the differential operators $[L(\dot u)\dot u](x) = [L_{ik}(\dot u)\dot u_k](x)$ and $[L^{(y)*}v](x) = [L^{(y)*}_{ik}v_k](x)$,

$$\begin{aligned}\int_\Omega v_i(x)[L_{ik}(\dot u)\dot u_k](x)d\Omega(x) &= \int_{\partial\Omega} v_i(x)[T_{ik}(\dot u)\dot u_k](x)d\Gamma(x) - \int_\Omega \frac{\partial v_i(x)}{\partial x_j} a_{ijkl}(\nabla\dot u(x),x)\frac{\partial \dot u_k(x)}{\partial x_l}d\Omega(x),\\ \int_\Omega \dot u_i(x)[L^{(y)*}_{ik}v_k](x)d\Omega(x) &= \int_{\partial\Omega} \dot u_i(x)[T^{(y)*}_{ik}v_k](x)d\Gamma(x) - \int_\Omega \frac{\partial \dot u_i(x)}{\partial x_j} a^*_{ijkl}(y)\frac{\partial v_k(x)}{\partial x_l}d\Omega(x),\end{aligned}$$

where $\dot u(x)$ and $v(x)$ are arbitrary vector-functions for that the operators and integrals in the above expressions have sense. Subtracting the identities from each other and taking into account the symmetry of the tensor a_{ijkl}, we derive the two-operator second Green-Betti identity,

$$\begin{aligned}&\int_\Omega \Big\{\dot u(x)[L^{(y)*}v](x) - v(x)[L(\dot u)\dot u](x)\Big\}\,d\Omega(x) =\\ &\qquad \int_{\partial\Omega}\Big\{\dot u(x)[T^{(y)*}v](x) - v(x)[T(\dot u)\dot u](x)\Big\}\,d\Gamma(x) + \int_\Omega [\nabla v(x)]\tilde a(\nabla\dot u(x);x,y)\nabla\dot u(x)d\Omega(x), \quad (13)\end{aligned}$$

where

$$\begin{aligned}\tilde a(\nabla\dot u;x,y) = \tilde a_{ijkl}(\nabla\dot u(x),\nabla\dot u(y),x,y) := a_{ijkl}(\nabla\dot u(x),x) - a^*_{ijkl}(\nabla\dot u(y),y) = a'_{ijkl}(\{\varepsilon(x)\},x) -\\ \hat a_{ijpq}(\{\varepsilon(x)\},\varepsilon(x),x)\varepsilon_{pq}(x)H[g_{ms}(\{\varepsilon(x)\},x)\dot u_{m,s}(x)]g_{kl}(\{\varepsilon(x)\},x) - a^*_{ijkl}(\{\varepsilon(y)\},\nabla\dot u(y),y).\end{aligned}$$

If $L(\dot u) = L^{(y)*}$, i.e. $L(\dot u)$ is a linear homogeneous elasticity operator without damage, then the last domain integral disappears in eq (13), which thus degenerates into the classical second Green-Betti identity.

For a fixed y, let $F^{(y)*}(x,y) = F^{(y)*}_{km}(x,y)$ be a fundamental solution for the linear differential operator $[L^{(y)*}_{ik}v_k](x)$ with constant coefficients, i.e.,

$$[L^{(y)*}_{ik}F^{(y)*}_{km}(\cdot,y)](x) := a^*_{ijkl}(y)\frac{\partial^2 F^{(y)*}_{km}(x,y)}{\partial x_j\partial x_l} = \delta_{im}\delta(x-y),$$

where δ_{im} is the Kronecker symbol and $\delta(x-y)$ is the Dirac delta-function.

If $a^*_{ijkl}(x) = a^0_{ijkl}(x)$ and the material is isotropic, or either $a^*_{ijkl}(x) = a'_{ijkl}(x)$ or $a^*_{ijkl}(x) = a_{ijkl}(x)$ and the material is isotropic with isotropic damage, then

$$a^*_{ijkl}(y) = \lambda^*(y)\delta_{ij}\delta_{kl} + \mu^*(y)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \qquad \mu^*(y) > C > 0, \qquad \lambda^*(y) + \frac{2}{3}\mu^*(y) > C > 0. \quad (14)$$

In this case, $F^{(y)*}_{im}(x,y)$ is the Kelvin-Somigliana solution,

$$F^{(y)*}_{im}(x,y) = \frac{-1}{4\pi}\left\{\frac{-\delta_{im}\ln r - r_{,i}r_{,m}}{\lambda^*(y)+2\mu^*(y)} + \frac{-\delta_{im}\ln r + r_{,i}r_{,m}}{\mu^*(y)}\right\} \quad (15)$$

for the plane strain state; for the plane stress, λ^* in (14) and (15) should be replaced by $2\lambda^*\mu^*/(\lambda^* + 2\mu^*)$. In the 3D case,

$$F^{(y)*}_{im}(x,y) = \frac{-1}{8\pi r}\left\{\frac{\delta_{im} - r_{,i}r_{,m}}{\lambda^*(y)+2\mu^*(y)} + \frac{\delta_{im} + r_{,i}r_{,m}}{\mu^*(y)}\right\} \quad (16)$$

Here $r := \sqrt{(x_i-y_i)(x_i-y_i)}$, $\quad r_{,i} := \partial r/\partial x_i = (x_i-y_i)/r$. For anisotropic material or for $a^*_{ijkl}(x) = a'_{ijkl}(x)$ and anisotropic damage, the fundamental solution can be written down in an analytical form for arbitrary anisotropy in the 2D case and for some particular anisotropy in the 3D case; otherwise, it can be expressed as a linear integral over a circle [21–23].

Assuming $\dot{u}(x)$ is a solution of nonlinear system (10) and using the fundamental solution $F^{(y)*}(x,y)$ as $v(x)$ in the Green identity (13), we obtain, similar to the linear homogeneous elasticity (see e.g. [1–4]) or partial differential equations with variable coefficients [24], the following non-linear two-operator third Green identity,

$$c(y)\dot{u}(y) - \int_{\partial\Omega} \dot{u}(x)[T^{(y)*}F^{(y)*}(\cdot,y)](x)d\Gamma(x) + \int_{\partial\Omega} F^{(y)*}(x,y)[T(\dot{u})\dot{u}](x)d\Gamma(x) - \int_{\Omega}[\nabla^{(x)}F^{(y)*}(x,y)]\tilde{a}(\nabla\dot{u};x,y)\nabla\dot{u}(x)d\Omega(x) = \int_{\Omega} F^{(y)*}(x,y)f(x)d\Omega(x), \quad (17)$$

where $c_{im}(y) = \delta_{im}$ if $y \in \Omega$; $c_{im}(y) = 0$ if $y \notin \bar{\Omega}$; $c_{im}(y) = \frac{1}{2}\delta_{im}$ if y is a smooth point of the boundary $\partial\Omega$; and $c_{im}(y) = c_{im}(a(y), \alpha(y))$ is a function of the anisotropy tensor $a(y)$ and the interior space angle $\alpha(y)$ at a corner point y of the boundary $\partial\Omega$.

Substituting boundary conditions (11), (12) into the integrands of eq (17) and using it at $y \in \overline{\Omega}$, we arrive at a (*united*) nonlinear two-operator BDIDE for $\dot{u}(x)$ at $x \in \overline{\Omega}$,

$$c(y)\dot{u}(y) - \int_{\partial_N\Omega} \dot{u}(x)[T^{(y)*}F^{(y)*}(\cdot,y)](x)d\Gamma(x) + \int_{\partial_D\Omega} F^{(y)*}(x,y)[T(\dot{u})\dot{u}](x)d\Gamma(x) - \int_{\Omega}[\nabla^{(x)}F^{(y)*}(x,y)]\tilde{a}(\nabla\dot{u};x,y)\nabla\dot{u}(x)d\Omega(x) = \mathcal{F}(y), \quad y \in \overline{\Omega}, \quad (18)$$

$$\mathcal{F}(y) := \int_{\partial_D\Omega} \check{u}(x)[T^{(y)*}F^{(y)*}(\cdot,y)](x)d\Gamma(x) - \int_{\partial_N\Omega} F^{(y)*}(x,y)\check{t}(x)d\Gamma(x) + \int_{\Omega} F^{(y)*}(x,y)f(x)d\Omega(x).$$

BDIDE (18) is the second kind equation, which includes at most the first derivatives of the unknown solution $\dot{u}(x)$, both directly in the domain integral term in the left hand side and through the coefficient $a(\nabla\dot{u}(x), x, y)$ in the operator $T(\dot{u})$ and in the function $\tilde{a}(\nabla\dot{u}; x, y)$. The function $[\nabla^{(x)}F^{(y)*}(x,y)]$ is at most weakly singular in Ω. The boundary integrals have at most the Cauchy-type singularity.

The right hand side of BDIDE (18) is independent of $\nabla\dot{u}$ if the auxiliary tensor a^* is chosen as the initial elastic tensor a^0 or the current secant tensor a'. Otherwise, when a^* is chosen as the tangent stiffness tensor a, the dependence will present.

Some other (e.g. segregated) BDIDEs can be obtained if one substitutes $\check{u}(x)$ for $\dot{u}(x)$ also in the out-of-integral term of (18) at $y \in \partial_D\Omega$, considers the unknown boundary displacement rates $\dot{u}$ on $\partial_N\Omega$ and/or traction rates $T(\dot{u})\dot{u}$ on $\partial_D\Omega$ as new variables formally segregated from $\dot{u}$ in Ω, or applies the boundary traction operator to (18).

BDIDE (18) can be reduced after some discretization to a system of nonlinear algebraic equation and solved numerically. The system will include unknowns not only on the boundary but also at internal points. Moreover, since the fundamental solutions, c.f. (15), (16), are highly non-local, the matrix of the system will be fully populated and this makes its numerical solution more expensive. To avoid this difficulty, we present below some ideas of constructing *localized* parametrices and consequently *Localized* BDIDEs (LBDIDEs).

LOCALIZED PARAMETRIX AND LBDIDE OF INCREMENTAL ELASTIC DAMAGE MECHANICS

Let $\chi(x,y)$ be a cut-off function, such that $\chi(y,y) = 1$ and $\chi(x,y) = 0$ at x not belonging to closure of an open localization domain $\omega(y)$ (a vicinity of y), see Fig.1, and let $P_\omega^{(y)*}(x,y) = \chi(x,y)F^{(y)*}(x,y)$.

The simplest example is

$$\chi(x,y) = \begin{cases} 1, & x \in \bar{\omega} \\ 0, & x \notin \bar{\omega} \end{cases} \quad \Rightarrow \quad P_\omega^{(y)*}(x,y) = \begin{cases} F^{(y)*}(x,y), & x \in \bar{\omega}(y) \\ 0, & x \notin \bar{\omega}(y) \end{cases} \quad (19)$$

Other examples of the cut-off functions having different smoothness are presented in [6,7,11] for some shapes of ω.

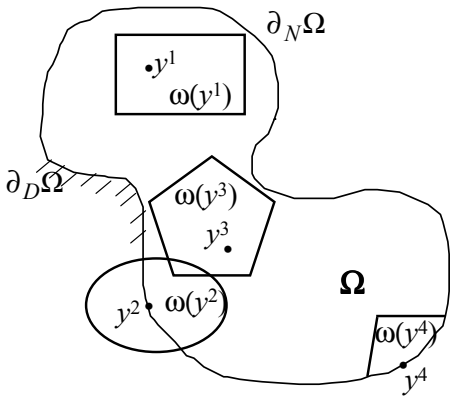

Figure 1: *Body Ω with localization domains $\omega(y^i)$*

Then $P_\omega^{(y)*}(x,y)$ is a localized parametrix (localized Levi's function) of the linear operator $L^{(y)*}$, i.e.,

$$L_{ik}^{(y)*}P_{km\omega}^{(y)*}(x,y) = \delta_{im}\delta(x-y) + R_{im\omega}^{(y)*}(x,y),$$

where the remainder

$$R_{im\omega}^{(y)*} = -L_{ik}^{(y)*}((1-\chi)F_{km}^{(y)*}) = a_{ijkl}^*(y)\left[F_{km}^{(y)*}\frac{\partial^2\chi}{\partial x_j\partial x_l} + \frac{\partial F_{km}^{(y)*}}{\partial x_j}\frac{\partial\chi}{\partial x_l} + \frac{\partial F_{km}^{(y)*}}{\partial x_l}\frac{\partial\chi}{\partial x_j}\right]$$

is at most weakly singular at $x=y$ if χ is smooth enough on $\bar\omega(y)$. The parametrix $P_\omega^{(y)*}(x,y)$ has the same singularity as $F^{(y)*}(x,y)$ at $x=y$. Both $P_\omega^{(y)*}(x,y)$ and $R_\omega^{(y)*}(x,y)$ are localized (non-zero) with respect to x only on $\omega(y)$.

Suppose $\chi(x,y)$ is smooth in $x\in\bar\omega(y)$ but not necessarily zero at $x\in\partial\omega(y)$, c.f. (19). Then $P_\omega^{(y)*}(x,y)$ is a discontinuous localized parametrix at $x\in\mathbb{R}^n$ and $P_\omega^{(y)*}(x,y) = R_\omega^{(y)*}(x,y) = 0$ if $x\notin\bar\omega(y)$. Substituting $P_\omega^{(y)*}(x,y)$ for $v(x)$ in eq (13), replacing Ω by the intersection $\omega(y)\cap\Omega$ and repeating the same arguments as for the fundamental solution above, we arrive at the localized parametrix-based two-operator third Green identity on $\bar\omega(y)\cap\bar\Omega$,

$$\begin{aligned} c(y)\dot u(y) - \int_{\bar\omega(y)\cap\partial\Omega}&\left\{\dot u(x)[T^{(y)*}P_\omega^{(y)*}(\cdot,y)](x) - P_\omega^{(y)*}(x,y)[T(\dot u)\dot u](x)\right\}d\Gamma(x) - \\ \int_{\Omega\cap\partial\omega(y)}&\left\{\dot u(x)[T^{(y)*}P_\omega^{(y)*}(\cdot,y)](x) - P_\omega^{(y)*}(x,y)[T(\dot u)\dot u](x)\right\}d\Gamma(x) - \\ \int_{\omega(y)\cap\Omega}&\left\{[\nabla^{(x)}P_\omega^{(y)*}(x,y)]\tilde a(\nabla\dot u;x,y)\nabla\dot u(x) - R_\omega^{(y)*}(x,y)\dot u(x)\right\}d\Omega(x) = \\ &\int_{\omega(y)\cap\Omega} P_\omega^{(y)*}(x,y)f(x)d\Omega(x). \end{aligned} \tag{20}$$

The second term in the last integral in the left hand side of (20) disappears if $\chi(x,y)$ is given by (19).

Substituting boundary conditions (11) and (12) into the integral terms of eq (20) and employing it at $y\in\overline\Omega$, we arrive at the united formulation of nonlinear two-operator Localized Boundary-Domain

Integro-Differential Equation (LBDIDE) of the second kind, for $\dot{u}(x)$, $x \in \overline{\Omega}$,

$$c(y)\dot{u}(y) - \int_{\bar{\omega}(y)\cap\partial_N\Omega} \dot{u}(x)[T^{(y)*}P_\omega^{(y)*}(\cdot,y)](x)d\Gamma(x) + \int_{\bar{\omega}(y)\cap\partial_D\Omega} P_\omega^{(y)*}(x,y)[T(\dot{u})\dot{u}](x)d\Gamma(x) -$$
$$\int_{\Omega\cap\partial\omega(y)} \left\{\dot{u}(x)[T^{(y)*}P_\omega^{(y)*}(\cdot,y)](x) - P_\omega^{(y)*}(x,y)[T(\dot{u})\dot{u}](x)\right\} d\Gamma(x) -$$
$$\int_{\omega(y)\cap\Omega} [\nabla^{(x)}P_\omega^{(y)*}(x,y)]\tilde{a}(\nabla\dot{u};x,y)\nabla\dot{u}(x)d\Omega(x) + \int_{\omega(y)\cap\Omega} R_\omega^{(y)*}(x,y)\dot{u}(x)d\Omega(x) = \mathcal{F}_\omega(y),\ \ y \in \overline{\Omega}, \quad (21)$$
$$\mathcal{F}_\omega(y) := \int_{\bar{\omega}(y)\cap\partial_D\Omega} \check{u}(x)[T^{(y)*}P_\omega^{(y)*}(\cdot,y)](x)d\Gamma(x) -$$
$$\int_{\bar{\omega}(y)\cap\partial_N\Omega} P_\omega^{(y)*}(x,y)\check{t}(x)d\Gamma(x) + \int_{\omega(y)\cap\Omega} P_\omega^{(y)*}(x,y)f(x)d\Omega(x). \quad (22)$$

If a cut-off function $\chi(x,y)$ vanishes at $x \in \partial\omega(y)$ with vanishing normal derivatives, then the integral along $\Omega \cap \partial\omega(y)$ disappears in eq (21).

DISCRETIZATION OF NONLINEAR TWO-OPERATOR LBDIE OF INCREMENTAL ELASTIC DAMAGE MECHANICS

To reduce quasi-linear LBDIDE (21) to a sparsely populated system of quasi-linear algebraic equations e.g. by the collocation method, one has to employ a local interpolation or approximation formula for the unknown function $\dot{u}(x)$, for example associated with a mesh-based or mesh-less discretization.

Mesh-based discretization Suppose the domain Ω is covered by a mesh of closures of disjoint domain elements e_k with nodes set up at the corners, edges, faces, or inside the elements. Let J be the total number of nodes x^i $(i = 1, 2, ..., J)$. One can use each node x^i as a collocation point for the LBDIDE with a localization domain $\omega(x^i)$. Let the union of closures of the domain elements that intersect with $\omega(x^i)$ be called the *total* localization domain $\tilde{\omega}^i$, Fig. 2(a). Evidently the closure $\bar{\omega}(x^i) \cap \bar{\Omega}$ belongs to $\tilde{\omega}^i$. If $\omega(x^i)$ is sufficiently small, then $\tilde{\omega}^i$ consists only of the elements adjacent to the collocation point x^i. If $\omega(x^i)$ is ab initio chosen as consisting only of the elements adjacent to the collocation point x^i, then $\tilde{\omega}^i = \bar{\omega}(x^i)$. Let $\dot{u}\{\tilde{\omega}^i\}$ be the array of the function values $\dot{u}(x^j)$ at the node points $x^j \in \tilde{\omega}^i$ and $J_{\tilde{\omega}^i}$ be the number of those node points.

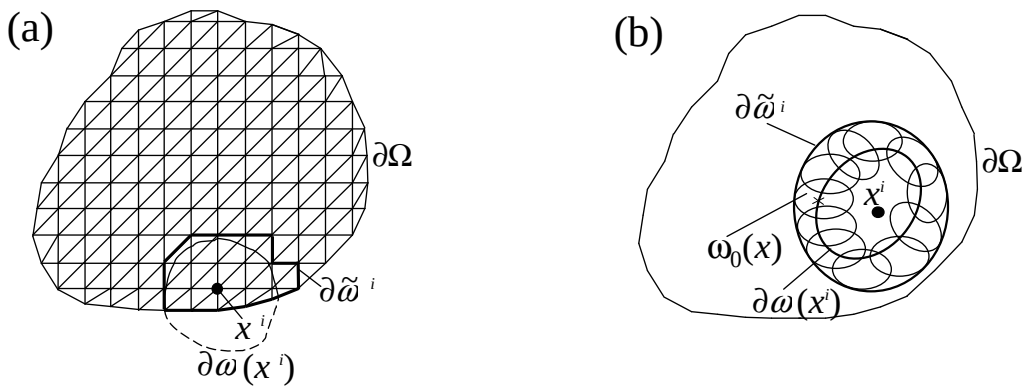

Figure 2: *Localization domain $\omega(x^i)$ and a total localization domain $\tilde{\omega}^i$ associated with a collocation point x^i of a body Ω for (a) mesh-based and (b) mesh-less discretizations*

Let $\dot{u}(x) = \sum_j \dot{u}(x^j)\phi_{kj}(x)$ be a continuous piece-wise smooth interpolation of $\dot{u}(x)$ at any point $x \in \Omega$ along the values $\dot{u}(x^j)$ at the node points x^j belonging to the same element $\bar{e}_k \subset \Omega$ as x, and the shape functions $\phi_{kj}(x)$ be localized on $\bar{e}_k$. Collecting the interpolation formulae, we have for any

$x \in \tilde{\omega}^i$,

$$\dot{u}(x) = \sum_{x^j \in \tilde{\omega}^i} \dot{u}(x^j)\Phi_j(x), \qquad \Phi_j(x) = \begin{cases} \phi_{kj}(x) & \text{if } x, x^j \in \bar{e}_k \\ 0 & \text{otherwise} \end{cases} \tag{23}$$

$$\nabla\dot{u}(x) = \sum_{x^j \in \tilde{\omega}^i} \dot{u}(x^j)\nabla\Phi_j(x), \qquad \nabla\Phi_j(x) = \begin{cases} \nabla\phi_{kj}(x) & \text{if } x, x^j \in \bar{e}_k \\ 0 & \text{otherwise} \end{cases} \tag{24}$$

Consequently, $\Phi_j(x) = \nabla\Phi_j(x) = 0$ if $x \in \tilde{\omega}^i$ but $x^j \notin \tilde{\omega}^i$.

Since interpolation (23) is piece-wise smooth, expressions (24) deliver non-unique values for $\nabla\dot{u}(x)$ on the element interfaces and particularly at the apexes x^i of different adjoint elements e_k. This brings no complications for the choice of the auxiliary elastic moduli as $a^*_{ijkl}(y) = a^0_{ijkl}(y)$ or $a^*_{ijkl}(y) = a'_{ijkl}(y)$ since they and consequently all other asterisk variables and operators do not depend on $\nabla\dot{u}(y)$, which then appears either in the domain integrals or in the boundary integrals with the gradients taken from the corresponding side of the boundary. On the other hand, for the auxiliary elastic tensor chosen as the current tangent tensor, $a^*_{ijkl}(y) = a_{ijkl}(\nabla\dot{u}(y), y)$, one has to estimate $\nabla u(y)$ to calculate the coefficient $a(\nabla\dot{u}(y), y)$ and, consequently $\tilde{a}(\nabla\dot{u}; x, y)$, $T^{(y)*}(u)$, $P^{(y)*}(u; x, y)$ and $R^{(y)*}(u; x, y)$ at the collocation points $y = x^i$. A possible way out is to assign

$$\nabla u(x^i) := \sum_{\bar{e}_k \ni x^i} \frac{\alpha_k(x^i)}{\alpha(x^i)} \nabla u^k(x^i), \;\; \nabla u^k(x^i) := \sum_{x^j \in \bar{e}_k} u(x^j)\nabla\phi_{kj}(x^i), \tag{25}$$

where $\alpha_k(x^i)$ is an interior space angle at the apex x^i of the element e_k and $\alpha(x^i) = \sum_{\bar{e}_k \ni x^i} \alpha_k(x^i)$.

Substituting interpolation formulae (23)-(24) in LBDIDE (21), we arrive at the following system of $J \times n$ quasi-linear algebraic equations for $J \times n$ unknowns $\dot{u}_m(x^j)$, $x^j \in \overline{\Omega}$, $m = 1, ..., n$,

$$c(x^i)\dot{u}(x^i) + \sum_{x^j \in \tilde{\omega}^i} K_{ij}(\dot{u}\{\tilde{\omega}^i\})\dot{u}(x^j) = \mathcal{F}_\omega(x^i), \quad x^i \in \overline{\Omega}, \quad \text{no sum in } i. \tag{26}$$

For fixed indices i, j, the $n \times n$ tensor $K_{ij}(\dot{u}\{\tilde{\omega}^i\})$ is

$$\begin{aligned} K_{ij}(\dot{u}\{\tilde{\omega}^i\}) = &- \int_{\bar{\omega}(x^i)\cap\partial_N\Omega} \Phi_j(x)[T^{(x^i)*}P^{(x^i)*}_\omega(\cdot, x^i)](x)d\Gamma(x) + \\ &\int_{\bar{\omega}(x^i)\cap\partial_D\Omega} P^{(x^i)*}_\omega(x, x^i)[T(\dot{u}\{\tilde{\omega}^i\})\Phi_j](x)d\Gamma(x) - \\ &\int_{\Omega\cap\partial\omega(x^i)} \left\{\Phi_j(x)[T^{(x^i)*}P^{(x^i)*}_\omega(\cdot, x^i)](x) - P^{(x^i)*}_\omega(x, x^i)[T(\dot{u}\{\tilde{\omega}^i\})\Phi_j](x)\right\} d\Gamma(x) - \\ &\int_{\omega(x^i)\cap\Omega} \left\{[\nabla^{(x)}P^{(x^i)*}_\omega(x, x^i)]\tilde{a}(\dot{u}\{\tilde{\omega}^i\}; x, x^i)\nabla\Phi_j(x) - R^{(x^i)*}_\omega(x, x^i)\Phi_j(x)\right\} d\Omega(x). \end{aligned} \tag{27}$$

The approximate coefficient $\tilde{a}(\dot{u}\{\tilde{\omega}^i\}; x, x^i)$ and traction operator $T(\dot{u}\{\tilde{\omega}^i\})$ in (27) are expressed in terms of the set of unknowns $\dot{u}\{\tilde{\omega}^i\} := \{\dot{u}(x^j),\ x^j \in \tilde{\omega}^i\}$. The expressions are obtained after substituting interpolation formulae (24) for $\nabla\dot{u}(x)$ in the coefficient $a(\nabla\dot{u}(x), x)$ in the definitions for $\tilde{a}(\nabla\dot{u}; x, y)$ and $T(\dot{u})$.

Note that the term with $R^{(x^i)*}_\omega$ disappears in the last integral in the right hand side of (27) if the parametrix $P^{(x^i)*}_\omega(x, x^i)$ is given by (19). On the other hand, if the cut-off function $\chi(x, x^i)$ and its normal derivative are equal zero at x on the boundary $\partial\omega(x^i)$, then the third integral (along $\Omega\cap\partial\omega(x^i)$) disappears in the right hand side of (27).

Mesh–less discretization For a mesh–less discretization, one needs a method of local interpolation or approximation of a function along randomly distributed nodes x^i. We will suppose all the approximation nodes x^i belong to $\bar{\Omega}$ and will use them also as collocation points for the LBDIDE discretization. Let, as before, J be the total number of nodes x^j $(i = 1, 2, ..., J)$. Let us consider

a mesh–less method, for example, the moving least squares (MLS) (see e.g. [25]), that leads to the following approximation of a function $\dot{u}(x)$

$$\dot{u}(x) = \sum_{x^j \in \omega_0(x)} \hat{u}(x^j)\Phi_j(x), \; x \in \Omega. \tag{28}$$

Here $\Phi_j(x)$ are known smooth shape functions such that $\Phi_j(x) = 0$ if $x^j \notin \omega_0(x)$, $\omega_0(x)$ is a localization domain of the approximation formula, and $\hat{u}(x^j)$ are unknown values of an auxiliary function $\hat{u}(x)$ at the nodes x^j, that is, the so-called $\delta-$property is not assumed for approximation (28).

Let $\omega(x^i)$ be a localization domain around a node x^i. Then for any $x \in \bar{\omega}(x^i)$, the total approximation of $\dot{u}(x)$ can be written in the following local form,

$$\dot{u}(x) = \sum_{x^j \in \tilde{\omega}^i} \hat{u}(x^j)\Phi_j(x), \; \nabla\dot{u}(x) = \sum_{x^j \in \tilde{\omega}^i} \hat{u}(x^j)\nabla\Phi_j(x), \; x \in \bar{\omega}(x^i), \tag{29}$$

where $\tilde{\omega}^i := \cup_{x \in \bar{\omega}(x^i) \cap \bar{\Omega}} \omega_0(x)$ is a total localization domain, Fig. 2(b). Consequently, $\Phi_j(x) = \nabla\Phi_j(x) = 0$ if $x \in \bar{\omega}(x^i)$ and $x^j \notin \tilde{\omega}^i$. Let $J_{\tilde{\omega}^i}$ be the number of nodes $x^j \in \tilde{\omega}^i$ and $\hat{u}\{\tilde{\omega}^i\}$ be the array of the function values $\hat{u}(x^j)$ at the node points $x^j \in \tilde{\omega}^i$.

After substitution of approximation (29) in LBDIDE (21), we arrive at the following system of quasi-linear system of $J \times n$ algebraic equations with respect to $J \times n$ unknowns $\hat{u}_m(x^j)$, $x^j \in \bar{\Omega}$, $m = 1, ..., n$,

$$\sum_{x^j \in \tilde{\omega}^i} \left[c(x^i)\Phi_j(x^i) + K_{ij}(\hat{u}\{\tilde{\omega}^i\})\right] \hat{u}(x^j) = \mathcal{F}_\omega(x^i), \quad x^i \in \bar{\Omega}, \quad \text{no sum in } i. \tag{30}$$

For any i, j, the $n \times n$ tensor K_{ij} in (30) is given by expression (27) with the shape functions Φ_j from (29), $\dot{u}\{\tilde{\omega}^i\}$ replaced by $\hat{u}\{\tilde{\omega}^i\}$, and the term with sum replaced by zero since approximation (29) for $\dot{u}$ is smooth and its gradient approximation in (29) is continuous. Expressions for $\tilde{a}(\hat{u}\{\tilde{\omega}^i\}; x, x^i)$ and $T(\hat{u}\{\tilde{\omega}^i\})$ in terms of the set of unknowns $\hat{u}\{\tilde{\omega}^i\} := \{\hat{u}(x^j), \; x^j \in \tilde{\omega}^i\}$ in (27) are obtained after substituting interpolation formulae (29) for $\nabla\dot{u}$ in the coefficient $a(\nabla\dot{u}(x), x)$ in the definitions for $\tilde{a}(\nabla\dot{u}; x, y)$ and $T(\dot{u})$.

CONCLUSION

Nonlinear BDIDE (18) and LBDIDE (21) are integro-differential reformulations of the BVP of elastic damage mechanics for increments, (10)-(12). Depending on the choice of the auxiliary elastic tensor a^* as the initial elastic, current secant, or currant tangent stiffness tensor, one can obtain three different versions of the BDIDE and LBDIDE. Different strategies can be chosen for the BDIDE or LBDIDE solution to obtain the complete evolutionary solution of the problem. One of them is to split the process into the time steps t_i and solve either of the integral equations with respect to the displacement rate $\dot{u}_k(x, t_i)$ employing the necessary strain history field $\{\varepsilon(x)\}(t_i)$ obtained at the previous step. Then one find the stiffness tensor rate from (5) and approximate the displacement increment during the time step as $\Delta u_k(x, t_i) = \dot{u}_k(x, t_i)(t_{i+1} - t_i)$ and strain, stress and stiffness increments similarly. This allows to calculate the stress an strain field at time t_{i+1}.

While solving numerically LBDIDE (21), we have from the definitions in both mesh-based and mesh-less methods that $\Phi_j(x) = \nabla\Phi_j(x) = [T(\dot{u})\Phi_j](x) = [T^{(y)*}\Phi_j](x) = 0$ if $x \in \bar{\omega}(x^i)$ but $x^j \notin \tilde{\omega}^i$. Consequently $K_{ij} = 0$ if $x^j \notin \tilde{\omega}^i$, and moreover, K_{ij} depend only on $\dot{u}\{\tilde{\omega}^i\}$ or $\hat{u}\{\tilde{\omega}^i\}$, respectively. Thus, each equation in (26) and (30) has not more than $J_{\tilde{\omega}^i} \times n \ll J \times n$ non-zero entries, i.e. *the systems are sparse.*

The second kind structure of the nonlinear LBDIDE and of the corresponding mesh-based discrete system look very promising for constructing simple and fast converging iteration algorithms of its solution without preconditioning, thus outperforming other available numerical techniques.

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