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SEGAL'S CONJECTURE AND THE BURNSIDE RINGS OF FUSION SYSTEMS

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ABSTRACT. For a given saturated fusion system $\mathcal{F}$ we define the ring $A(S)^{\mathcal{F}}$ of the $\mathcal{F}$-invariants of the Burnside ring functor. We show how this ring is related to the Burnside ring of the fusion system $\mathcal{F}$ and how it appears naturally in the analogue of Segal's conjecture for the classifying spectrum $\mathbb{B}\mathcal{F}$. We give an explicit description of $A(S)^{\mathcal{F}}$ and we prove it is a local ring.

1. Introduction

A saturated fusion system $\mathcal{F}$ on a finite p-group S is a small category whose objects are the subgroups of S. Its morphism sets $\mathcal{F}(P,Q)$ consist of group monomorphisms $P \to Q$, where $P, Q \leq S$, which are subject to a certain set of axioms listed in §2. Isomorphic objects in $\mathcal{F}$ are called $\mathcal{F}$-conjugate. Puig was the first to define these objects but in this note we will use the formulation of Broto-Levi-Oliver in [2]. The model is the category $\mathcal{F}_S(G)$ associated to a Sylow p-subgroup S of a finite group G. In this case the objects are the subgroups of S and the morphisms in $\mathcal{F}_S(G)$ are those monomorphisms $P \to Q$ which are restrictions of inner automorphisms of G.

For any contravariant functor $H : \mathcal{F} \to \mathcal{C}$ we can consider the inverse limit $\varprojlim_{\mathcal{F}} H$. We call this limit the $\mathcal{F}$-invariants of $H(S)$ because, as it is easy to see, it consists of the elements $x \in H(S)$ such that $\varphi(x) = \psi(x)$ for every φ, $\psi \in \mathcal{F}(P,S)$ and every subgroup P of S. We will use the suggestive notation $H(S)^{\mathcal{F}}$ to denote this inverse limit. For example, by [2, Theorem 5.8], the cohomology of the classifying space of a p-local finite group is isomorphic to $H^*(S;\mathbb{F}_p)^{\mathcal{F}}$ for the obvious functor $P \mapsto H^*(P;\mathbb{F}_p)$ which assigns to $P \leq S$ its mod-p cohomology. In this paper we study the $\mathcal{F}$-invariants $A(S)^{\mathcal{F}}$ of the functor $A : \mathcal{F} \to \mathbf{Rings}$ which maps $P \leq S$ to its Burnside ring $A(P)$. We call this ring "the ring of $\mathcal{F}$-invariant virtual S-sets". Clearly $A(S)^{\mathcal{F}}$ is a subring of $A(S)$ which contains the identity and therefore the standard augmentation map $\epsilon\colon A(S) \to \mathbb{Z}$ restricts to an augmentation epimorphism $\epsilon\colon A(S)^{\mathcal{F}} \to \mathbb{Z}$ whose kernel is denoted $I(S)^{\mathcal{F}}$. Proposition 3.6 and Lemma 4.2 give an alternative description of $A(S)^{\mathcal{F}}$ and $I(S)^{\mathcal{F}}$.

Recall that the Burnside ring $A(G)$ of a finite group G is the Grothendieck group of the monoid $B(G)$ of the isomorphism classes of finite G-sets. In symbols, $A(G) = Gr(B(G))$. The multiplication in this ring is induced by cartesian product of G-sets. As an abelian group $A(G)$ is free with one basis element for each conjugacy class of subgroups of G. In [3] we construct the Burnside ring $A(\mathcal{F})$ of saturated

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fusion system $\mathcal{F}$. As in the case of finite groups, where $A(G)$ is the Grothendieck group of the monoid of the isomorphism classes of finite G-sets, $A(\mathcal{F})$ is defined as the Grothendieck group of the monoid of the isomorphism classes of objects in a small category which is derived from $\mathcal{F}$ and has finite coproducts and products. See §2 for details. The additive group of $A(\mathcal{F})$ is free with one basis element for every $\mathcal{F}$-conjugacy class of $\mathcal{F}$-centric subgroups of S, see [2, Def. 1.6].

Our first result shows the close relation between the rings $A(\mathcal{F})$ and $A(S)^{\mathcal{F}}$. We will write $A_{(p)}(S)^{\mathcal{F}}$ for the ring $\mathbb{Z}_{(p)} \otimes A(S)^{\mathcal{F}}$ and similarly $A_{(p)}(\mathcal{F})$ for $\mathbb{Z}_{(p)} \otimes A(\mathcal{F})$.

Theorem A. *The underlying group of $A(S)^{\mathcal{F}}$ is free with one generator for each $\mathcal{F}$-conjugacy class of subgroups of S. The $\mathbb{Z}_{(p)}$-submodule of $A_{(p)}(S)^{\mathcal{F}}$ generated by the $\mathcal{F}$-invariant S-sets all of whose isotropy groups are non-$\mathcal{F}$-centric subgroups of S, forms an ideal N. There is a ring isomorphism $A_{(p)}(\mathcal{F}) \cong A_{(p)}(S)^{\mathcal{F}}/N$.*

The product in the ring $A_{(p)}(\mathcal{F})$ is described on basis elements explicitly in [3, Theorem 4.6]. Thus we have a good understanding of a quotient ring of $A_{(p)}(S)^{\mathcal{F}}$. The desire to understand this ring springs from Segal's conjecture. This conjecture was proven by Carlsson and it asserts that for any finite group G the stable cohomotopy group $\pi^0(BG_+)$ is isomorphic to the I-adic completion $A(G)^\wedge_I$, where I is the augmentation ideal of the Burnside ring $A(G)$, $\pi^*(-)$ denotes stable cohomotopy groups and the subscript $+$ means adding a disjoint base-point.

A variation of Segal's conjecture at the prime p is proven by Ragnarsson in [8]. He extends results of May-McClure and describes $\tilde{\pi}^0((BG^\wedge_p)_+)$ where G is a finite group and $\tilde{\pi}^*(-)$ denotes the reduced stable cohomotopy groups. He shows that this group is isomorphic to the p-adic completion of a quotient of the submodule of $A(G)$ generated by the G-sets with p-power isotropy. In this paper we prove an analogue of Segal's conjecture for saturated fusion systems at the prime p.

Any saturated fusion system $\mathcal{F}$ on a p-group S has a natural classifying spectrum $\mathbb{B}\mathcal{F}$ constructed by Ragnarsson in [9]. It is equipped with a structure map $\sigma_{\mathcal{F}} \colon \mathbb{B}S \to \mathbb{B}\mathcal{F}$ where $\mathbb{B}S$ is the suspension spectrum of BS and $\mathcal{F}$ can be recovered from the pair $(\sigma_{\mathcal{F}}, \mathbb{B}\mathcal{F})$ [9, Theorem A]. When $\mathcal{F}$ has a classifying space, i.e., if there is a p-local finite group $(S, \mathcal{F}, \mathcal{L})$ (see [2]), then $\mathbb{B}\mathcal{F}$ is the suspension spectrum of the p-completed classifying space $|\mathcal{L}|^\wedge_p$. In particular, when $\mathcal{F} = \mathcal{F}_S(G)$ is the saturated fusion system induced by a finite group G, then there is a homotopy equivalence $\mathbb{B}\mathcal{F} \simeq \Sigma^\infty(BG)^\wedge_p$ and $\sigma_{\mathcal{F}}$ is induced from the inclusion $BS \to BG$. Conjecturally, $\mathbb{B}\mathcal{F}$ is always a suspension spectrum of some space that one can associate to the saturated fusion system $\mathcal{F}$. Throughout the symbol $\mathbb{S}$ will denote the sphere spectrum. The next result gives a description of the cohomotopy group $\pi^0(\mathbb{B}\mathcal{F} \vee \mathbb{S})$ as the completion of $A(S)^{\mathcal{F}}$ by its augmentation ideal. Recall that $\pi^0(\mathbb{B}S \vee \mathbb{S}) \cong \pi^0(BS_+) \cong A(S)^\wedge_{I(S)}$ (see Remark 4.6.)

Theorem B. *Let $\mathcal{F}$ be a saturated fusion system over a p-group S.*

1. *The structure map $\sigma_{\mathcal{F}}$ induces an isomorphism $\pi^0(\mathbb{B}\mathcal{F} \vee \mathbb{S}) \cong (A(S)^{\mathcal{F}})^\wedge_I$, where I is the augmentation ideal of $A(S)^{\mathcal{F}}$.*
2. *As abelian groups $(A(S)^{\mathcal{F}})^\wedge_I \cong \mathbb{Z} \oplus (I \otimes \mathbb{Z}^\wedge_p)$ and $I \otimes \mathbb{Z}^\wedge_p$ is a free $\mathbb{Z}^\wedge_p$-module with one generator for every $\mathcal{F}$-conjugacy class of subgroups $H \leq S$ with $H \neq S$.*

As the referee pointed out to us, it is probably more natural to consider $\mathbb{B}\mathcal{F}_+$ in Theorem B than $\mathbb{B}\mathcal{F} \vee \mathbb{S}$. However, in order to do this, one has to rewrite

Ragnarsson's results in [9] and to redefine the structure map $\sigma_{\mathcal{F}}$ as a map $\mathbb{B}S_+ \to \mathbb{B}\mathcal{F}_+$ rather than $\mathbb{B}S \to \mathbb{B}\mathcal{F}$. To avoid this and use Ragnarsson's work in its present form we chose to work with $\mathbb{B}\mathcal{F} \vee \mathbb{S}$.

Corollary C. *If $(S, \mathcal{F}, \mathcal{L})$ is a p-local finite group then there is an isomorphism of rings $\pi^0((|\mathcal{L}|^\wedge_p)_+) \cong (A(S)^{\mathcal{F}})^\wedge_I$. In particular, if G is a finite group then*

$$\pi^0((BG^\wedge_p)_+) \cong (A(S)^{\mathcal{F}_S(G)})^\wedge_I \cong \mathbb{Z} \oplus (\mathbb{Z}^\wedge_p \otimes I)$$

where I is the augmentation ideal of $A(S)^{\mathcal{F}_S(G)}$.

There is some overlap between this note and Ragnarsson's results in [8]. The authors would like to thank him for helpful discussions during a week long visit to Aberdeen in 2008.

We start in Section 2 with preliminaries about fusion systems, Burnside rings and completions. Then we devote Section 3 to prove Theorem A. Finally, in Section 4 we prove Theorem B and in Proposition 4.12 we describe the spectrum of the prime ideals of $\mathbb{Z}_{(p)} \otimes (A(S)^{\mathcal{F}})^\wedge_I$.

Notation: We denote the trivial subgroup of a groups G by by e. If X is a G-set and $H \leq G$, we write X^H for the set of points of X fixed by H and $|X^H|$ for its cardinality. We will identify a G-set X with its isomorphism class in the monoid $B(G)$. Throughout $\pi^*(-)$ will always mean stable cohomotopy groups of spectra or of pointed spaces. If G is a finite group $\mathbb{B}G$ is the suspension spectrum of BG with some basepoint chosen. If X is any CW-complex (pointed or not) X_+ is obtained by adding a disjoint new basepoint to X. Finally, we reserve the letter $\Bbbk$ to denote a commutative ring, frequently it is torsion-free.

2. Preliminaries: Fusion systems, Burnside rings and completions

Saturated fusion systems and their classifying spectrum. Let S be a finite p-group. A fusion system over S is a small category whose objects are the subgroups of S. The morphism sets $\mathcal{F}(P, Q)$ where $P, Q \leq S$ consists of group monomorphisms $P \to Q$ such that

(a) The set $\mathrm{Hom}_S(P, Q)$ of the morphisms $P \to Q$ obtained by conjugation in S, is contained in $\mathcal{F}(P, Q)$.
(b) Every morphism in $\mathcal{F}$ factors as an isomorphism in $\mathcal{F}$ followed by an inclusion of groups.

For example, if S is a Sylow p-subgroup of a finite group G then there results a fusion system $\mathcal{F}_S(G)$ where $\mathcal{F}(P, Q) = \mathrm{Hom}_G(P, Q)$ is the set of the homomorphisms $c_g \colon P \xrightarrow{x \mapsto gxg^{-1}} Q$ obtained by conjugation in G. This fusion system has a rigid structure which was first recognized and axiomatised by Puig and later by others. The set of axioms we will use are due to Broto-Levi-Oliver in [2].

Isomorphic objects in a fusion system $\mathcal{F}$ are called $\mathcal{F}$-conjugate and we will write $P \simeq_{\mathcal{F}} P'$. A subgroup $P \leq S$ is called *fully $\mathcal{F}$-centralised* if $|C_S(P)| \geq |C_S(P')|$ for any P' which is $\mathcal{F}$-conjugate to P. Similarly, P is called *fully $\mathcal{F}$-normalised* if $|N_S(P)| \geq |N_S(P')|$ for any P' which is $\mathcal{F}$-conjugate to P.

2.1. **Definition.** *(See* [2, Def. 1.2]*)* A fusion system $\mathcal{F}$ over S is called *saturated* if

(I) Every fully $\mathcal{F}$-normalised $P \leq S$ is also fully $\mathcal{F}$-centralised and $\mathrm{Aut}_S(P)$ is a Sylow p-subgroup of $Aut_{\mathcal{F}}(P)$.

(II) For every $P \leq S$ and every $\varphi \in \mathcal{F}(P, S)$ set

$$N_\varphi = \{g \in N_S P \colon \varphi \circ c_g \circ \varphi^{-1} \in \mathrm{Aut}_S(\varphi(P))\}.$$

If $\varphi(P)$ is fully $\mathcal{F}$-centralised then there is $\bar{\varphi} \in \mathcal{F}(N_\varphi, S)$ such that $\bar{\varphi}|_P = \varphi$.

A subgroup $P \leq S$ is called $\mathcal{F}$*-centric* if P and all its $\mathcal{F}$-conjugates contain their S-centraliser, see [2, Def. 1.6]. It is clear that if P is $\mathcal{F}$-centric and $P \leq Q$ then Q is also $\mathcal{F}$-centric. We will write $\mathcal{F}^c$ for the full subcategory of $\mathcal{F}$ generated by the $\mathcal{F}$-centric subgroups of S.

Ragnarsson constructed the classifying spectrum $\mathbb{B}\mathcal{F}$ in [9, Section 7]. It is equipped with a structure map $\sigma_\mathcal{F} \colon \mathbb{B}S \to \mathbb{B}\mathcal{F}$ and a transfer map $t_\mathcal{F} \colon \mathbb{B}\mathcal{F} \to \mathbb{B}S$. The fusion system is completely determined by its classifying spectrum $\mathbb{B}\mathcal{F}$ and the map $\sigma_\mathcal{F}$ in the sense that $\mathcal{F}$ can be recovered from this data by means of the set of stable maps $\{\mathbb{B}P, \mathbb{B}\mathcal{F}\}$ where $P \leq S$. The composite $t_\mathcal{F} \circ \sigma_\mathcal{F}$ is homotopic to the stable characteristic idempotent $\tilde{\omega}_\mathcal{F} \in \{\mathbb{B}S, \mathbb{B}S\}$ and $\sigma_\mathcal{F} \circ t_\mathcal{F} \simeq \mathrm{id}_{\mathbb{B}\mathcal{F}}$. In particular $\mathbb{B}\mathcal{F}$ splits off $\mathbb{B}S$.

The Burnside ring. The *orbit category* of a fusion system $\mathcal{F}$ is the small category $\mathcal{O}(\mathcal{F})$ whose objects are the subgroups of S and $\mathcal{O}(\mathcal{F})(P, Q) = \mathcal{F}(P, Q)/\mathrm{Inn}(Q)$; See [2, Def. 2.1]. We write $\mathcal{O}(\mathcal{F}^c)$ for the full subcategory generated by the $\mathcal{F}$-centric subgroups of S.

In [3, §4] we consider the category $\mathcal{O}(\mathcal{F}^c)_\sqcup$ of the finite collections in $\mathcal{O}(\mathcal{F}^c)$. To make the construction precise one looks at the category of the contravariant functors $\mathcal{O}(\mathcal{F}^c) \to \mathbf{Sets}$ which are isomorphic to $\coprod_{i=1}^n \mathcal{O}(\mathcal{F}^c)(-, P_i)$ where $n < \infty$ and P_i are objects in $\mathcal{O}(\mathcal{F}^c)$. By construction, $\mathcal{O}(\mathcal{F}^c)_\sqcup$ is closed to finite coproducts and by Yoneda's Lemma it contains $\mathcal{O}(\mathcal{F}^c)$ as a full subcategory. The surprise is that $\mathcal{O}(\mathcal{F}^c)_\sqcup$ has finite products [7], [3, Theorem 1.2]. The product distributes over the coproduct and we define the Burnside ring $A(\mathcal{F})$ as a special case of the following general construction applied to $\mathcal{O}(\mathcal{F}^c)_\sqcup$. See [5].

Consider an essentially small category $\mathcal{C}$ with finite coproducts and products and assume that the product distributes over the coproduct. Define $A(\mathcal{C})$ as the free abelian group generated by the isomorphism classes $[C]$ of the objects $C \in \mathcal{C}$ subject to the relation $[C_1 \coprod C_2] = [C_1] + [C_2]$. The product in $\mathcal{C}$ gives $A(\mathcal{C})$ the structure of a ring where $[C_1] \cdot [C_2] = [C_1 \times C_2]$. If the morphism sets of $\mathcal{C}$ are finite then for every isomorphism class $[C]$, the ring $A(\mathcal{C})$ is equipped with a ring homomorphism

$$\chi_C \colon A(\mathcal{C}) \to \mathbb{Z}, \qquad (X \mapsto |\mathcal{C}(X, C)|).$$

For any commutative ring $\Bbbk$ we define $A_\Bbbk(\mathcal{C}) := \Bbbk \otimes_\mathbb{Z} A(\mathcal{C})$. We will denote

$$A_{(p)}(\mathcal{C}) := A_{\mathbb{Z}_{(p)}}(\mathcal{C}) \qquad \text{and} \qquad A_{\hat{p}}(\mathcal{C}) := A_{\mathbb{Z}_p^\wedge}(\mathcal{C}).$$

If C is an object of $\mathcal{C}$ we obtain, by tensoring with $\Bbbk$, another ring homomorphism

$$\Bbbk \otimes \chi_C \colon A_\Bbbk(\mathcal{C}) \to \Bbbk, \tag{2.2}$$

that we denote also by χ_C. It would be clear from the context which homomorphism we mean.

2.3. **Example.** The Burnside ring $A(G)$ of a finite group G is $A(\{\text{finite } G\text{-sets}\})$. It is a free group generated by the G-conjugacy classes of the transitive G-sets, each of which has as representative G/H for some $H \leq G$. The homomorphism (2.2) for the subgroup $H \leq G$ assigns to any G-set X the cardinality of X^H, namely the subset of X fixed by H.

Given a prime p we let $A(G;p)$ be the subring of the finite G-sets whose isotropy groups are p-subgroups of G. This is the Burnside ring of the category of the finite G-sets whose isotropy groups are p-groups. From the homomorphisms (2.2), we obtain ring homomorphisms

$$\chi\colon A(G) \to \prod_{\mathrm{ccs}(G)} \Bbbk \qquad \text{and} \qquad \chi\colon A(G;p) \to \prod_{\mathrm{ccs}_p(G)} \Bbbk, \tag{2.4}$$

where $\mathrm{ccs}(G)$ and $\mathrm{ccs}_p(G)$ denote the set of conjugacy classes of the subgroups of G and the set of the conjugacy classes of the p-subgroups of G. These homomorphisms are, in fact, ring monomorphisms provided $\Bbbk$ is torsion free. The trivial G-set $*$ is the identity element in $A(G)$ and more generally in $A_\Bbbk(G)$. We will use the integer n to denote $n \cdot *$.

The augmentation homomorphism $\chi_{(1)}\colon A_\Bbbk(G) \to \Bbbk$ is of particular importance. It is usually denoted by ϵ and it sends a finite G-set X to its cardinality $|X|$. Its kernel is the augmentation ideal $I_\Bbbk(G)$. If $\Bbbk$ is torsion free then $I_\Bbbk(G)$ has an additive basis $G/H - |G/H|$ with H running over the G-conjugacy classes of subgroups H of G where $H \neq G$.

Going back to saturated fusion systems, we define $A(\mathcal{F})$ as $A(\mathcal{O}(\mathcal{F}^c)_\sqcup)$. As an abelian group it is free with one basis element for each $\mathcal{F}$-conjugacy class of $\mathcal{F}$-centric subgroups of S [3, Proposition 4.10]. We obtain a ring homomorphism

$$\Phi\colon A(\mathcal{F}) \to \prod_{[P]\in \mathrm{ccs}(\mathcal{F}^c)} \mathbb{Z}, \qquad [Q] \mapsto |\mathcal{O}(\mathcal{F}^c)(P,Q)|, \tag{2.5}$$

where $\mathrm{ccs}(\mathcal{F}^c)$ stands for the set of the $\mathcal{F}$-conjugacy classes of the $\mathcal{F}$-centric subgroups of S. This is, in fact, a monomorphism by [3, Theorem 5.3] and therefore $\Phi_{(p)} = \mathbb{Z}_{(p)} \otimes \Phi$ is also a ring monomorphism.

In [3, Theorem 5.4] we show that (y_P) is in the image of $\Phi_{(p)}$ if and only if for any fully $\mathcal{F}$-normalised $Q \le S$ the following congruence holds

$$\sum_{P\in[\mathcal{F}^c]} n(Q,P)\cdot y_P \equiv 0 \mod (|\operatorname{Out}_S(Q)|), \tag{2.6}$$

where P runs over a set of representatives for $\mathrm{ccs}(\mathcal{F}^c)$, where $(|\operatorname{Out}_S(Q)|)$ is the ideal in $\mathbb{Z}_{(p)}$ generated by $|\operatorname{Out}_S(Q)|$ and where

$$n(Q,P) = \big|\{c_s \in \operatorname{Out}_S(Q) \ : \ \langle s, Q\rangle \simeq_{\mathcal{F}} P\}\big|.$$

Completion of rings. Let I be an ideal in a commutative ring R. Then the ideals I^n form a set of neighbourhoods for $0 \in R$ which generates the I-adic topology on R and on any R-module M. The I-adic completion of an R-module M is

$$M^\wedge_I := \varprojlim_n M/I^nM.$$

It is clear from the construction that $M^\wedge_I$ is an $R^\wedge_I$-module. Also note that $M \to M^\wedge_I$ is injective if and only if the I-adic topology on M is Hausdorff, namely $\cap_n I^nM = 0$.

When R is Noetherian one has more control on the I-completion. We recall below some basic results. Throughout, we write $\hat{M}$ for $M^\wedge_I$ and $\hat{R} = R^\wedge_I$ etc.

2.7. **Theorem.** ([4, Theorem 7.1].) *If R is Noetherian then $\hat{R}$ is Noetherian. Moreover, as an R-module, $\hat{R}$ is I-complete.*

2.8. **Theorem.** (Artin-Rees Lemma, [4, Theorem 7.2].) *If R is Noetherian and M is a finitely generated R-module then $\hat{M} = \hat{R} \otimes_R M$. Moreover $\hat{R}$ is a flat R-module.*

The I-adic topology on R/I is discrete and therefore $\widehat{R/I} = R/I$. If R is Noetherian then Theorem 2.8 gives rise to a short exact sequence $0 \to \hat{I} \to \hat{R} \to R/I \to 0$.

2.9. **Theorem.** ([4, Corollary 7.13].) *If R is Noetherian then $I\hat{R} = \hat{I}$. In particular $\hat{R}$ is $\hat{I}$-complete.*

3. $\mathcal{F}$-invariant sets

In this section we prove Theorem A in the introduction. We begin studying and describing $\mathcal{F}$-invariant sets.

For any finite group G denote by $B(G)$ the monoid, via disjoint unions, of the isomorphism classes of the finite G-sets, and by $A(G) = Gr(B(G))$ its Grothendieck group, i.e. the Burnside ring of G. Recall that the product in this ring is induced by the product of G-sets. A homomorphism $\varphi : H \to G$ gives rise to a morphism of monoids $\varphi* : B(G) \to B(H)$ and a homomorphism of rings $\varphi^* : A(G) \to A(H)$. If φ is the inclusion of H into G we write res^G_H instead of φ^*.

Hence, if $\mathcal{F}$ is a saturated fusion system we have functors $B\colon \mathcal{F} \to$ **Monoids** and $A\colon \mathcal{F} \to$ **Rings** whose values on the subgroup P is $B(P)$ and $A(P)$ respectively. Moreover, for any torsion-free commutative ring $\Bbbk$ we consider also the functor $A_\Bbbk = \Bbbk \otimes A$.

3.1. **Definition.** Let $\mathcal{F}$ be a saturated fusion system over S and let $\Bbbk$ be a torsion-free commutative ring. We define the *$\mathcal{F}$-invariant S-sets* as $B(S)^{\mathcal{F}} = \varprojlim_{\mathcal{F}} B$ and $A(S;\mathcal{F}) := \mathrm{Gr}(B(S)^{\mathcal{F}})$. We also define $A_\Bbbk(S;\mathcal{F}) = \Bbbk \otimes A(S;\mathcal{F})$.

It is easy to see that $A(S;\mathcal{F})$ is a subalgebra of $A(S)$ and therefore $A_\Bbbk(S;\mathcal{F})$ is a $\Bbbk$-subalgebra of $A_\Bbbk(S)$ provided $\Bbbk$ is torsion-free.

3.2. **Definition.** Let $\mathcal{F}$ be a saturated fusion system over S and let $\Bbbk$ be a torsion-free commutative ring. We define the ring of *$\mathcal{F}$-invariant virtual S-sets* as $A(S)^{\mathcal{F}} := \varprojlim_{\mathcal{F}} A$. We also define $A_\Bbbk(S)^{\mathcal{F}} = \varprojlim_{\mathcal{F}} A_\Bbbk$.

It is clear that $A_\Bbbk(S)^{\mathcal{F}}$ is a $\Bbbk$-subalgebra of $A_\Bbbk(S)$. The next goal is to prove that $A_\Bbbk(S;\mathcal{F}) = A_\Bbbk(S)^{\mathcal{F}}$ provided $\Bbbk$ is a torsion-free commutative ring.

By definition, an S-set X, which we identify with its isomorphism class in $B(S)$, is $\mathcal{F}$-invariant if for every $P \leq S$ and every $\varphi \in \mathcal{F}(P,S)$ the P-sets $\mathrm{res}^S_P(X)$ and $\varphi^*(X)$ are isomorphic. It is easy to see that X is $\mathcal{F}$-invariant if and only if $|X^P| = |X^{P'}|$ whenever P and P' are $\mathcal{F}$-conjugate subgroups of S.

3.3. **Proposition.** *For any fully $\mathcal{F}$-normalised $P \leq S$ there exists an $\mathcal{F}$-invariant S-set Ω_P such that*

(1) *All the isotropy groups of Ω_P are $\mathcal{F}$-conjugate to subgroups of P.*
(2) *Ω_P contains exactly one orbit isomorphic to S/P, and hence $|(\Omega_P)^P| = |\frac{N_S P}{P}|$.*

Proof. Let $(P)_{\mathcal{F}}$ denote the $\mathcal{F}$-conjugacy class of P. Let $P_1, \dots, P_k$ be representatives for the S-conjugacy classes in $(P)_{\mathcal{F}}$. We may assume that $P_1 = P$. Then $|N_S P_1| \geq |N_S P_i|$ for all i and we consider the S-set

$$\Omega_0 = \coprod_i \frac{|N_S P_1|}{|N_S P_i|} \cdot S/P_i.$$

Observe that for any i we have

$$|\Omega_0^{P_i}| = \frac{|N_S P_1|}{|N_S P_i|} \cdot |N_S P_i/P_i| = \frac{|N_S P|}{|P|}.$$

Let $\mathcal{H}$ be the collection of all the subgroup $Q \leq S$ which are $\mathcal{F}$-conjugate to a *proper* subgroup of P. If $Q \notin \mathcal{H}$ then either Q is not $\mathcal{F}$-conjugate to a subgroup of P or it is $\mathcal{F}$-conjugate to P. In the first case it is clear that $\Omega_0^Q = \emptyset = \Omega_0^{Q'}$ for any Q' which is $\mathcal{F}$-conjugate to Q. In the second case Q is S-conjugate to one of the P_i's and therefore $|\Omega_0^Q| = |N_S P/P|$. We can now apply [2, Lemma 5.4] and deduce that there exists an $\mathcal{F}$-invariant S-set Ω which contains Ω_0 and which also satisfies $|\Omega^Q| = |\Omega_0^Q|$ for all $Q \notin \mathcal{H}$. In particular it follows that all the isotropy groups of Ω belong to $\mathcal{H} \cup \{(P)_{\mathcal{F}}\}$ and that $|\Omega^P| = |N_S P/P|$. □

Since $A_{\Bbbk}(S)^{\mathcal{F}}$ is a $\Bbbk$-subalgebra of $A_{\Bbbk}(S)$ the statement of the following lemma makes sense.

3.4. **Lemma.** *For any $u \in A_{\Bbbk}(S)^{\mathcal{F}}$ and any two $\mathcal{F}$-conjugate subgroups $Q, Q' \leq S$, $\chi_Q(u) = \chi_{Q'}(u)$ where χ_Q is defined in* (2.2) *(see also* (2.4)*).*

Proof. Note that $\chi_Q(u)$ is the coefficient of the trivial Q-set Q/Q in $\mathrm{res}^S_Q(u) \in A_{\Bbbk}(Q)$. Similarly $\chi_{Q'}(u)$ is the coefficient of the trivial Q'-set in $\mathrm{res}^S_{Q'}(u)$. In turn, this is the coefficient of the trivial Q-set Q/Q in $\varphi^*(u) \in A_{\Bbbk}(Q)$ where $\varphi \in \mathcal{F}(Q, S)$ is an isomorphism of Q onto Q'. The result follows since $u \in A_{\Bbbk}(S)^{\mathcal{F}}$ whence $\mathrm{res}^S_Q(u) = \varphi^*(u)$. □

3.5. **Remark.** Consider the restriction of (2.4) to $\chi : A_{\Bbbk}(S)^{\mathcal{F}} \to \prod_{\mathrm{ccs}(S)} \Bbbk$. From Lemma 3.4 it is clear that χ factors through an injective ring homomorphism $\chi\colon A_{\Bbbk}(S)^{\mathcal{F}} \to \prod_{\mathrm{ccs}(\mathcal{F})} \Bbbk$, where the product runs through the $\mathcal{F}$-conjugacy classes of the subgroups of S.

If X is an $\mathcal{F}$-invariant S-set then it represents an element in $A_{\Bbbk}(S)^{\mathcal{F}}$. Thus, $A_{\Bbbk}(S; \mathcal{F}) \subseteq A_{\Bbbk}(S)^{\mathcal{F}}$, where we regard both as $\Bbbk$-subalgebras of $A_{\Bbbk}(S)$. Now we are ready to prove that this inclusion is an equality and to describe a basis for $A_{\Bbbk}(S; \mathcal{F})$.

3.6. **Proposition.** *Let $\mathcal{F}$ be a saturated fusion system over S and let $\Bbbk$ be a torsion-free commutative ring. Let $Q_1, \dots, Q_r$ be representatives for the $\mathcal{F}$-conjugacy classes of the subgroups of S such that the Q_i's are fully $\mathcal{F}$-normalised. Then*

(1) *$A_{\Bbbk}(S; \mathcal{F}) = A_{\Bbbk}(S)^{\mathcal{F}}$.*
(2) *$A_{\Bbbk}(S; \mathcal{F})$ is a free $\Bbbk$-module with basis $\Omega_{Q_1}, \dots, \Omega_{Q_r}$ where Ω_{Q_i} are defined in Proposition 3.3.*

Proof. We arrange the Q_i's so that $|Q_i| \geq |Q_{i+1}|$. Thus, if $j > i$ then Q_i is not $\mathcal{F}$-conjugate to a subgroup of Q_j and in particular $(S/P)^{Q_i} = \emptyset$ if P is $\mathcal{F}$-conjugate to Q_j. From Remark 3.5 and Example 2.3, there is a ring monomorphism

$$\chi\colon A_{\Bbbk}(S)^{\mathcal{F}} \to \prod_{i=1}^{r} \Bbbk, \qquad \chi = (\chi_{Q_1}, \dots, \chi_{Q_r}).$$

Since $(\Omega_{Q_i})^{Q_j} = \emptyset$ if $j < i$ we see from Proposition 3.3 that

$$\chi(\Omega_{Q_i}) = (\underbrace{0, \dots, 0}_{i-1 \text{ times}}, \frac{|N_S(Q_i)|}{|Q_i|}, \dots)$$

where $\frac{|N_S(Q_i)|}{|Q_i|} \neq 0$ because $\Bbbk$ is torsion-free. Thus, $\chi(\Omega_{Q_1}), \dots, \chi(\Omega_{Q_r})$ are $\Bbbk$-linearly independent in $\Bbbk^r$ and therefore $\Omega_{Q_1}, \dots, \Omega_{Q_r}$ form a basis for the $\Bbbk$-submodule $B \leq A_\Bbbk(S)$ it generates. We now have

$$B \leq A_\Bbbk(S;\mathcal{F}) \leq A_\Bbbk(S)^{\mathcal{F}} \overset{\chi}{\rightarrowtail} \prod_{j=1}^{r} \Bbbk.$$

It remains to prove that $B = A_\Bbbk(S)^{\mathcal{F}}$.

Assume to the contrary that this is not the case and let m be the maximal integer with the property that there exists some $u \in A_\Bbbk(S)^{\mathcal{F}} \backslash B$ such that $\chi_{Q_i}(u) = 0$ for all $i = 1, \dots, m-1$. Clearly $m \leq r$ or else $u = 0$ because χ is injective. Also $\chi_{Q_m}(u) \neq 0$. Express u in terms of the basis of $A_\Bbbk(S)$, namely

$$u = \sum_{(Q) \in \mathrm{ccs}(S)} \alpha_Q \cdot S/Q$$

and consider the smallest integer $j = 1, \dots, r$ for which there is some $R \simeq_{\mathcal{F}} Q_j$ such that $\alpha_R \neq 0$. Thus, if $\alpha_Q \neq 0$ then $|R| \geq |Q|$. Since $\Bbbk$ is torsion-free, by Lemma 3.4 and Proposition 3.3 we have

$$\chi_{Q_j}(u) = \chi_R(u) = \sum_{(Q) \in \mathrm{ccs}(S)} \alpha_Q \cdot \chi_R(S/Q) = \alpha_R \cdot |N_S R/R| \neq 0.$$

It follows from the definition of m that $j \geq m$. Now, if $\alpha_Q \neq 0$ then Q is $\mathcal{F}$-conjugate to Q_i where $i \geq j$, and therefore $i \geq m$. We deduce that $|Q_m| \geq |Q|$ so $(S/Q)^{Q_m} = \emptyset$ unless Q is S-conjugate to Q_m. It follows that

$$\chi_{Q_m}(u) = \alpha_{Q_m} \cdot |N_S Q_m/Q_m|, \tag{3.7}$$

which implies that $\alpha_{Q_m} \neq 0$ because $\chi_{Q_m}(u) \neq 0$. From the minimality of j we deduce that $j \leq m$ and therefore $j = m$.

Consider $v := u - \alpha_{Q_m}\Omega_{Q_m}$. Clearly $v \in A_\Bbbk(S)^{\mathcal{F}} \backslash B$. Moreover, if $i < m$ then $(\Omega_{Q_m})^{Q_i} = \emptyset$ by Proposition 3.3 so

$$\chi_{Q_i}(v) = \chi_{Q_i}(u) - \chi_{Q_i}(\alpha_{Q_m}\Omega_{Q_m}) = 0, \qquad \text{if } i < m.$$

From (3.7) we also deduce that

$$\chi_{Q_m}(v) = \chi_{Q_m}(u) - \chi_{Q_m}(\alpha_{Q_m}\Omega_{Q_m}) = \chi_{Q_m}(u) - \alpha_{Q_m}|\frac{N_S Q_m}{Q_m}| = 0.$$

This contradicts the maximality of m and therefore $B = A_\Bbbk(S)^{\mathcal{F}}$. □

3.8. **Remark.** It follows from Proposition 3.6 that if $\Bbbk$ is torsion free then

$$A_\Bbbk(S;\mathcal{F}) = \Bbbk \otimes A(S;\mathcal{F}) = \Bbbk \otimes A(S)^{\mathcal{F}} = A_\Bbbk(S)^{\mathcal{F}} \leq A_\Bbbk(S).$$

3.9. **Example.** Let S be a Sylow p-subgroup of a finite group G. If X is a finite G-set then $\mathrm{res}_S^G(X)$ is an $\mathcal{F}_S(G)$-invariant S-set. This gives a ring homomorphism $\mathrm{res}\colon A(G;p) \to A(S;\mathcal{F})$, see Example 2.3. Observe that a basis for $A(G;p)$ is the set $\{G/Q_i\}_{i=1}^r$ and a basis for $A(S;\mathcal{F}_S(G))$ is the set $\{\Omega_{Q_i}\}_{i=1}^r$ where $Q_i \leq S$ are chosen as in Proposition 3.6. If $P = Q_i$ then

$$\mathrm{res}_S^G(G/P) = \sum_{g \in S\backslash G/P} S/S \cap {}^gP = |\frac{N_G P}{N_S P}| \cdot S/P + Y$$

where Y is a finite S-set whose isotropy groups are conjugate in G to subgroups $Q \leq P$ and Q is not S-conjugate to P itself. Thus, by Propositions 3.3 and 3.6, the matrix which represents res: $A(G;p) \to A(S;\mathcal{F})$ with respect to the bases described above, is upper triangular with diagonal entries $|\frac{N_G Q_i}{N_S Q_i}|$. Since $N_S(Q_i)$ is a Sylow p-subgroup of $N_G Q_i$, this matrix is invertible in $\mathbb{Z}_{(p)}$. Hence $\mathbb{Z}_{(p)} \otimes \text{res}$ is an isomorphism $A_{(p)}(S;\mathcal{F}) \cong A_{(p)}(G;p)$.

Proof of Theorem A. In light of Proposition 3.6 and remark 3.8, we may replace $A_{(p)}(S)^{\mathcal{F}}$ with $A_{(p)}(S;\mathcal{F})$. Let $Q_1, Q_1, \dots, Q_r$ be fully $\mathcal{F}$-normalised representatives for the $\mathcal{F}$-conjugacy classes of the subgroups of S. We order them in such a way that $Q_1, \dots, Q_s$ are $\mathcal{F}$-centric, $Q_{s+1}, \dots, Q_r$ are not $\mathcal{F}$-centric and $|Q_1| \geq \dots \geq |Q_s|$. In Proposition 3.6 we showed that $A_{(p)}(S;\mathcal{F})$ is a free $\mathbb{Z}_{(p)}$-module generated by the S-sets Ω_{Q_i}.

Consider the restriction $\chi\colon A_{(p)}(S;\mathcal{F}) \to \prod_{j=1}^r \mathbb{Z}_{(p)}$ of the ring homomorphism (2.4) for $\Bbbk = \mathbb{Z}_{(p)}$. See also Remark 3.5. Then by Proposition 3.3, $\chi_{Q_j}(\Omega_{Q_i}) = 0$ if $j < i$ and $\chi_{Q_i}(\Omega_{Q_i}) = |\frac{N_S Q_i}{Q_i}| \neq 0$. From this it easily follows that N is the kernel of the composite homomorphism

$$A_{(p)}(S;\mathcal{F}) \xrightarrow{\chi} \prod_{\text{ccs}(\mathcal{F})} \mathbb{Z}_{(p)} \xrightarrow{\text{proj}} \prod_{\text{ccs}(\mathcal{F}^c)} \mathbb{Z}_{(p)}$$

because every $u \in N$ must be a linear combination of those Ω_{Q_i}'s such that Q_i is not $\mathcal{F}$-centric. Here, $\text{ccs}(\mathcal{F})$ is the set of the $\mathcal{F}$- conjugacy classes of the subgroups of S and $\text{ccs}(\mathcal{F}^c)$ is the set of the $\mathcal{F}$-conjugacy classes of the $\mathcal{F}$-centric subgroups of S. We deduce that there is a ring monomorphism

$$\bar{\chi}\colon A_{(p)}(S;\mathcal{F})/N \to \prod_{\text{ccs}(\mathcal{F}^c)} \mathbb{Z}_{(p)}, \qquad \bar{\chi} = (\chi_{Q_1}, \dots, \chi_{Q_s}).$$

In light of (2.5), it remains to show that $\text{Im}(\bar{\chi}) = \text{Im}(\Phi_{(p)})$. First we claim that $\text{Im}(\bar{\chi}) \subseteq \text{Im}(\Phi_{(p)})$. To see this we choose some $\mathcal{F}$-invariant S-set X. Then for every $i \leq s$ we have $\bar{\chi}_{Q_i}(X) = |X^{Q_i}|$ and to see that $\bar{\chi}(X)$ belongs to $\text{Im}(\Phi_{(p)})$ we use (2.6). By Cauchy-Frobenius formula, for any fully $\mathcal{F}$-normalised $Q \in \mathcal{F}^c$,

$$\sum_{P \in \text{ccs}(\mathcal{F}^c)} n(Q,P) \cdot |X^P| = \sum_{P \in \text{ccs}(\mathcal{F}^c)} \Big(\sum_{s \in \frac{N_S Q}{Q}, \langle Q, s\rangle \simeq_{\mathcal{F}} P} |X^P| \Big) =$$

$$\sum_{s \in \frac{N_S Q}{Q}} |(X^Q)^s| = |\frac{N_S Q}{Q}| \cdot \#\{\text{orbits of } \frac{N_S Q}{Q} \text{ on } X^Q\} = 0 \mod (|\frac{N_S Q}{Q}|).$$

Since Q is $\mathcal{F}$-centric and fully $\mathcal{F}$-normalised, $|\operatorname{Out}_S(Q)| = |\frac{N_S Q}{Q}|$, hence $\bar{\chi}(X) \in \text{Im}(\Phi_{(p)})$.

Now, a basis for $\text{Im}(\bar{\chi})$ is the set $\{\bar{\chi}(\Omega_{Q_i})\}_{i=1}^s$ and a basis for $\text{Im}(\Phi_{(p)})$ is the set $\{\Phi_{(p)}([Q_i])\}_{i=1}^s$. Note that for every i we have $\bar{\chi}_{Q_j}(\Omega_i) = \Phi_{Q_j}([Q_i]) = 0$ if $j < i$ by Proposition 3.3 and (2.5). Furthermore, $\Phi_{Q_i}([Q_i]) = |\operatorname{Out}_{\mathcal{F}}(Q_i)|$ and $\bar{\chi}_{Q_i}(\Omega_{Q_i}) = |\frac{N_S Q_i}{Q_i}|$ differ by a unit in $\mathbb{Z}_{(p)}$ because $\operatorname{Out}_S(Q_i)$ is a Sylow p-subgroup of $\operatorname{Out}_{\mathcal{F}}(Q_i)$. It is now an easy exercise in linear algebra to see that $\text{Im}(\bar{\chi}) = \text{Im}(\Phi_{(p)})$. □

4. Segal's conjecture

The aim of this section is to prove Theorem B of the introduction. We start describing the augmentation ideal $I_{\Bbbk}(S;\mathcal{F})$ of $A_{\Bbbk}(S;\mathcal{F})$ and modules for which the $I_{\Bbbk}(S;\mathcal{F})$-adic topology is equivalent to the p-adic topology. To avoid triviality we will assume throughout this section that the saturated fusion system $\mathcal{F}$ are defined over a non trivial p-group $S \neq 1$. We recall from Proposition 3.6 that $A_{\Bbbk}(S;\mathcal{F}) = A_{\Bbbk}(S)^{\mathcal{F}}$ if $\Bbbk$ is torsion-free.

Recall from Remark 3.8 that if $\Bbbk$ is torsion-free then $A_{\Bbbk}(S;\mathcal{F})$ is a $\Bbbk$-subalgebra of $A_{\Bbbk}(S)$.

4.1. **Definition.** There is a standard augmentation map $\epsilon\colon A_{\Bbbk}(S) \to \Bbbk$ sending an S-set X to its cardinality. Let $I_{\Bbbk}(S;\mathcal{F})$ denote the kernel of $\epsilon\colon A_{\Bbbk}(S;\mathcal{F}) \to \Bbbk$.

Recall that the trivial S-set is the unit in $A_{\Bbbk}(S)$. We will therefore use integers to represent the corresponding elements in $A_{\Bbbk}(S)$.

4.2. **Lemma.** *With the notation and hypotheses of Proposition 3.6, $I_{\Bbbk}(S;\mathcal{F})$ is a free $\Bbbk$-module of rank $r-1$ with basis $\{\Omega_{Q_i} - |\Omega_{Q_i}|\}$ where i runs through the indices such that $Q_i \neq S$.*

Proof. We may assume that $Q_1 = S$ and that $\Omega_{Q_1} = *$. It is clear from Proposition 3.6 that $\Omega_{Q_2} - |\Omega_{Q_2}|, \dots, \Omega_{Q_r} - |\Omega_{Q_r}|$ are $\Bbbk$-linearly independent in $A_{\Bbbk}(S;\mathcal{F})$. The result follows from the fact that the Ω_{Q_i}'s form a basis for $A_{\Bbbk}(S;\mathcal{F})$. □

4.3. **Proposition.** *Assume that $\Bbbk$ is torsion-free. Then the $I_{\Bbbk}(S;\mathcal{F})$-adic topology on $I_{\Bbbk}(S;\mathcal{F})$ is equivalent to the p-adic topology.*

Proof. May and McClure show in [6, p. 212] that $I(S)^{n+1} \subseteq p \cdot I(S)$ where $|S| = p^n$. Therefore, after tensoring with $\Bbbk$ and using Lemma 4.2, we see that $I_{\Bbbk}(S)^{n+1} \subseteq p \cdot I_{\Bbbk}(S)$. In addition,

$$I_{\Bbbk}(S;\mathcal{F})^{n+1} \subseteq A_{\Bbbk}(S;\mathcal{F}) \cap I_{\Bbbk}(S)^{n+1} \subseteq A_{\Bbbk}(S;\mathcal{F}) \cap p \cdot I_{\Bbbk}(S) = p \cdot I_{\Bbbk}(S;\mathcal{F}),$$

where for the last equality we argue as follows. First, the inclusion $\supseteq$ is obvious. Consider $u \in A_{\Bbbk}(S;\mathcal{F}) \cap p \cdot I_{\Bbbk}(S)$ and use Proposition 3.6 to write $u = \sum_{i=1}^{r} \beta_i \Omega_{Q_i}$. Clearly, $u \in p \cdot A_{\Bbbk}(S)$ and by looking at the coefficients of S/Q_i, one easily deduces from Proposition 3.3 that $\beta_i = p \cdot \alpha_i$ for all i. It follows that $u = p \cdot (\sum_i \alpha_i \Omega_{Q_i})$. Now, $\epsilon(u) = 0$ and since $\Bbbk$ is torsion free, we deduce that $\epsilon(\sum_i \alpha_i \Omega_{Q_i}) = 0$. Therefore $u \in p \cdot I_{\Bbbk}(S;\mathcal{F})$.

We now claim that

$$p^n \cdot I_{\Bbbk}(S;\mathcal{F}) \subseteq I_{\Bbbk}(S;\mathcal{F})^2.$$

To see this consider $\eta = (S/e - |S|)$ as an element in $I_{\Bbbk}(S;\mathcal{F})$. Observe that $\{(S/Q - |S/Q|)\}$ form a basis for $I_{\Bbbk}(S)$, where Q runs through the representatives for the S-conjugacy classes of the subgroups of S different from S. By inspection

$$\eta \cdot (S/Q - |S/Q|) = -|S| \cdot (S/Q - |S/Q|) = (-p^n) \cdot (S/Q - |S/Q|).$$

That is, multiplication by η results in multiplication by $(-p^n)$ in $I_{\Bbbk}(S)$. Therefore

$$I_{\Bbbk}(S;\mathcal{F})^2 \supseteq \eta \cdot I_{\Bbbk}(S;\mathcal{F}) = p^n \cdot I_{\Bbbk}(S;\mathcal{F}).$$

This completes the proof. □

4.4. **Corollary.** *If $\Bbbk$ is torsion-free then $I_{\Bbbk}(S;\mathcal{F})^{\wedge}_{I_{\Bbbk}(S;\mathcal{F})} = I_{\Bbbk}(S;\mathcal{F})^{\wedge}_p = I_{\Bbbk^{\wedge}_p}(S;\mathcal{F})$.*

Proof. The first equality follows from Proposition 4.3, the second follows since $I_{\Bbbk}(S;\mathcal{F})$ is a finitely generated free $\Bbbk$-module by Lemma 4.2 and therefore $I_{\Bbbk}(S;\mathcal{F})^{\wedge}_p = \Bbbk^{\wedge}_p \otimes_{\Bbbk} I_{\Bbbk}(S;\mathcal{F})$. □

Any group homomorphism $\varphi\colon H \to G$ induces a ring homomorphism $\varphi^*\colon A_{\Bbbk}(G) \to A_{\Bbbk}(H)$. Note that φ^* carries $I_{\Bbbk}(G)$ into $I_{\Bbbk}(H)$ and hence there results a natural ring homomorphism $\varphi^*\colon A_{\Bbbk}(G)^{\wedge}_{I_{\Bbbk}(G)} \to A_{\Bbbk}(H)^{\wedge}_{I_{\Bbbk}(H)}$. Thus there is a contravariant functor $\widehat{A_{\Bbbk}}\colon \mathcal{F} \to \mathbf{Rings}$ which maps $P \le S$ to $\widehat{A_{\Bbbk}}(P) = A_{\Bbbk}(P)^{\wedge}_{I_{\Bbbk}(P)}$.

4.5. **Definition.** Let $\mathcal{F}$ be a saturated fusion system over the p-group S. We define the ring of *$\mathcal{F}$-invariant completed S-sets* as $\widehat{A_{\Bbbk}}(S)^{\mathcal{F}} = \varprojlim_{\mathcal{F}} \widehat{A_{\Bbbk}}$.

Notice that $\widehat{A_{\Bbbk}}(S)^{\mathcal{F}}$ is the subring of $\widehat{A_{\Bbbk}}(S)$ of the elements $\hat{u}$ such that $\varphi^*(\hat{u}) = \operatorname{res}^S_P(\hat{u})$ for all $\varphi \in \mathcal{F}(P,S)$ and any subgroup $P \le S$.

4.6. **Remark.** Notice that for any pointed CW-complex X there is a natural equivalence $\Sigma^\infty X \vee \mathbb{S} \simeq \Sigma^\infty(X_+)$ in the homotopy category of spectra (see [1].)

Recall from §2 that the classifying spectrum of a saturated fusion system $\mathcal{F}$ over S is equipped with a structure map $\sigma_{\mathcal{F}}\colon \mathbb{B}S \to \mathbb{B}\mathcal{F}$ where $\mathbb{B}S$ is the suspension spectrum of BS with some chosen basepoint. From the remark above we see that for any spectrum E there is are isomorphisms $\{\mathbb{B}S \vee \mathbb{S}, E\} = \{\Sigma^\infty BS_+, E\} = E^0(BS_+)$. In particular, for $E = \mathbb{S}$ we have $\pi^0(\mathbb{B}S \vee \mathbb{S}) \cong \pi^0(BS_+)$.

4.7. **Lemma.** *For any saturated fusion system $\mathcal{F}$ over S the map $\sigma_{\mathcal{F}} \vee \mathbb{S}$ induces a split monomorphism*

$$\pi^0(\mathbb{B}\mathcal{F} \vee \mathbb{S}) \xrightarrow{(\sigma_{\mathcal{F}} \vee \mathbb{S})^*} \pi^0(\mathbb{B}S \vee \mathbb{S}) \cong \pi^0(BS_+) \cong \widehat{A}(S)$$

whose image is isomorphic to the subring $\widehat{A}(S)^{\mathcal{F}}$.

Proof. Consider the transfer map $t_{\mathcal{F}}\colon \mathbb{B}\mathcal{F} \to \mathbb{B}S$, see §2 and [9]. The composite $\sigma_{\mathcal{F}} \circ t_{\mathcal{F}}$ is homotopic to $\mathrm{id}_{\mathbb{B}\mathcal{F}}$ so $\pi^0(t_{\mathcal{F}} \vee \mathbb{S})$ is a left inverse for $\pi^0(\sigma_{\mathcal{F}} \vee \mathbb{S})$ which is therefore a split monomorphism. Set $\tilde{\omega}_{\mathcal{F}} = t_{\mathcal{F}} \circ \sigma_{\mathcal{F}}$, see [9].

For any spectrum E, the image of $E^0(\mathbb{B}\mathcal{F}) \xrightarrow{E^0(\sigma_{\mathcal{F}})} E^0(\mathbb{B}S)$ is equal to the image of $E^0(\mathbb{B}S) \xrightarrow{E^0(\tilde{\omega}_{\mathcal{F}})} E^0(\mathbb{B}S)$. By [9, Corollary 6.4], the image of $E^0(\tilde{\omega}_{\mathcal{F}})$ is equal to the set of homotopy classes $f \in \{\mathbb{B}S, E\}$ which are $\mathcal{F}$-invariant, that is

$$\{\mathbb{B}\mathcal{F}, E\} \xrightarrow[\cong]{E^0(\sigma_{\mathcal{F}})} \varprojlim_{P \in \mathcal{F}^{\mathrm{op}}} \{\mathbb{B}P, E\} = E^0(\mathbb{B}S)^{\mathcal{F}}$$

where $\mathbb{B}\colon P \mapsto \mathbb{B}P$ is a functor from $\mathcal{F}$ to the category of spectra. It follows that

$$E^0(\mathbb{B}\mathcal{F} \vee \mathbb{S}) \xrightarrow[\cong]{E^0(\sigma_{\mathcal{F}} \vee \mathbb{S})} \varprojlim_{P \in \mathcal{F}^{\mathrm{op}}} E^0(\mathbb{B}P \vee \mathbb{S}) = E^0(\mathbb{B}S \vee \mathbb{S})^{\mathcal{F}}. \tag{4.8}$$

The assignments below induce isomorphic functors $\mathcal{F}^{\mathrm{op}} \to \mathbf{Ab}$,

$$P \mapsto \{\mathbb{B}P \vee \mathbb{S}, E\} \qquad \text{and} \qquad P \mapsto \{\Sigma^\infty BP_+, E\} = E^0(BP_+).$$

This is because we have cofibre sequences $\mathbb{S} \to \Sigma^\infty BP_+ \to \mathbb{B}P$ and $\mathbb{S} \to \mathbb{B}P \vee \mathbb{S} \to \mathbb{B}P$ in which $\mathbb{S}$ is a retract. Thus, $\{\mathbb{B}S \vee \mathbb{S}, E\}^{\mathcal{F}} \cong E^0(BS_+)^{\mathcal{F}}$. We now specialise to $E = \mathbb{S}$. Using the fact that the isomorphism $\pi^0(BG_+) \cong \widehat{A}(G)$ is natural in the group G, we obtain a natural isomorphism of rings $\pi^0(BS_+)^{\mathcal{F}} \cong \widehat{A}(S)^{\mathcal{F}}$ as a subring of $\pi^0(\mathbb{B}S \vee \mathbb{S}) \cong \pi^0(BS_+) \cong \hat{A}(S)$. The result now follows from (4.8). $\square$

In order to simplify the notation, we will denote the $I_{\Bbbk}(S;\mathcal{F})$-adic completion of $A_{\Bbbk}(S;\mathcal{F})$ by $\widehat{A_{\Bbbk}}(S;\mathcal{F})$. Similarly $\widehat{I_{\Bbbk}}(S;\mathcal{F})$ is the $I_{\Bbbk}(S;\mathcal{F})$-adic completion of

$I_{\Bbbk}(S;\mathcal{F})$. When $\mathcal{F}$ is the trivial fusion system over S this clearly becomes $\widehat{A_{\Bbbk}}(S)$ and $\widehat{I_{\Bbbk}}(S)$. Set $\widehat{I_{\Bbbk}}(S)^{\mathcal{F}} = \widehat{I_{\Bbbk}}(S) \cap \widehat{A_{\Bbbk}}(S)^{\mathcal{F}}$.

4.9. **Proposition.** *Let $\Bbbk$ be a torsion-free commutative ring such that $\Bbbk^\wedge_p$ is also torsion-free. Assume that $\mathcal{F}$ is a saturated fusion system over $S \neq 1$. Then*

(1) $\widehat{I_{\Bbbk}}(S)^{\mathcal{F}} \cong \widehat{I_{\Bbbk}}(S;\mathcal{F}) \cong \Bbbk^\wedge_p \otimes_{\Bbbk} I_{\Bbbk}(S;\mathcal{F})$.
(2) $\widehat{A_{\Bbbk}}(S)^{\mathcal{F}} \cong \widehat{A_{\Bbbk}}(S;\mathcal{F})$.
(3) $\widehat{A_{\Bbbk}}(S;\mathcal{F})$ *contains* $I_{\Bbbk^\wedge_p}(S;\mathcal{F})$ *as an ideal with quotient* $\Bbbk$. *If* $\Bbbk$ *is* p*-complete then* $\widehat{A_{\Bbbk}}(S;\mathcal{F}) \cong A_{\Bbbk}(S;\mathcal{F})$.

Proof. We first observe that if M is an R-module and I is an ideal in R then the I-adic topology on M/IM is discrete and the short exact sequences $0 \to \frac{IM}{I^nM} \to \frac{M}{I^nM} \to \frac{M}{IM} \to 0$ for all $n \geq 1$ yield a short exact sequence

$$0 \to (IM)^\wedge_I \to M^\wedge_I \to M/IM \to 0 \tag{4.10}$$

because the tower $\{IM/I^nM\}_{n \geq 0}$ is Mittag-Leffler. In particular, the augmentation $A_{\Bbbk}(S) \xrightarrow{\epsilon} \Bbbk$ extends to a short exact sequence

$$0 \to \widehat{I_{\Bbbk}}(S;\mathcal{F}) \to \widehat{A_{\Bbbk}}(S;\mathcal{F}) \xrightarrow{\hat{\epsilon}} \Bbbk \to 0 \tag{4.11}$$

which is split because $\Bbbk$ is contained as a $\Bbbk$-subalgebra of $A_{\Bbbk}(S;\mathcal{F})$ generated by the trivial S-set $*$. Applying this to the trivial fusion system over S we obtain a short exact sequence

$$0 \to \widehat{I_{\Bbbk}}(S) \to \widehat{A_{\Bbbk}}(S) \xrightarrow{\hat{\epsilon}} \Bbbk \to 0.$$

The inclusions $A_{\Bbbk}(S;\mathcal{F}) \subseteq A_{\Bbbk}(S)$ and $I_{\Bbbk}(S;\mathcal{F}) \subseteq I_{\Bbbk}(S)$ yield a ring homomorphism $\widehat{A_{\Bbbk}}(S;\mathcal{F}) \to \widehat{A_{\Bbbk}}(S)$ which factors through

$$\Psi \colon \widehat{A_{\Bbbk}}(S;\mathcal{F}) \to \widehat{A_{\Bbbk}}(S)^{\mathcal{F}}$$

because $A_{\Bbbk}(S;\mathcal{F}) \subseteq A_{\Bbbk}(S)^{\mathcal{F}}$ by Proposition 3.6.

By definition $\widehat{I_{\Bbbk}}(S)^{\mathcal{F}} = \widehat{I_{\Bbbk}}(S) \cap \widehat{A_{\Bbbk}}(S)^{\mathcal{F}}$ and therefore from (4.11) we now obtain the following morphism of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widehat{I_{\Bbbk}}(S;\mathcal{F}) & \longrightarrow & \widehat{A_{\Bbbk}}(S;\mathcal{F}) & \xrightarrow{\hat{\epsilon}} & \Bbbk & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \Psi'} & & \Big\downarrow{\scriptstyle \Psi} & & \Big\| & & \\
0 & \longrightarrow & \widehat{I_{\Bbbk}}(S)^{\mathcal{F}} & \longrightarrow & \widehat{A_{\Bbbk}}(S)^{\mathcal{F}} & \xrightarrow[\hat{\epsilon}]{} & \Bbbk & \longrightarrow & 0
\end{array}$$

By Proposition 4.3 and from the fact that $I_{\Bbbk}(S;\mathcal{F})$ is a finitely generated free $\Bbbk$-module by Lemma 4.2, we see that

$$\widehat{I_{\Bbbk}}(S;\mathcal{F}) \cong I_{\Bbbk}(S;\mathcal{F})^\wedge_p \cong I_{\Bbbk}(S;\mathcal{F}) \otimes_{\Bbbk} \Bbbk^\wedge_p = I_{\Bbbk^\wedge_p}(S;\mathcal{F}),$$
$$\widehat{I_{\Bbbk}}(S) \cong I_{\Bbbk}(S)^\wedge_p \cong I_{\Bbbk}(S) \otimes_{\Bbbk} \Bbbk^\wedge_p = I_{\Bbbk^\wedge_p}(S).$$

Thus, Ψ' is the map

$$I_{\Bbbk^\wedge_p}(S;\mathcal{F}) \subseteq I_{\Bbbk^\wedge_p}(S)^{\mathcal{F}}$$

induced by the inclusion $I(S;\mathcal{F}) \subseteq I(S)^{\mathcal{F}} = A(S)^{\mathcal{F}} \cap I(S)$. By Proposition 3.6 and Lemma 4.2, this inclusion is in fact an equality and therefore Ψ' is an isomorphism. This proves point (1) of this Proposition. The five-lemma now shows that Ψ is an isomorphism and this is point (2). The first assertion of point (3) follows from the first row in the commutative ladder above. The second assertion follows because if $\mathbb{k}$ is p-complete, namely $\mathbb{k}^\wedge_p = \mathbb{k}$, then $\widehat{I_\mathbb{k}}(S;\mathcal{F}) = I_\mathbb{k}(S;\mathcal{F})$. □

Proof of Theorem B. This follows from Lemma 4.7 and Propositions 4.9 and 3.6. For part (2) of the theorem we also need Lemma 4.2. □

Proof of Corollary C. In the presence of a p-local finite group, the canonical map $f \colon BS \to |\mathcal{L}|^\wedge_p$ induces $\omega_\mathcal{F}$. By Theorem B, it gives rise to a ring monomorphism $f^* \colon \pi^0(|\mathcal{L}|^\wedge_p \vee \mathbb{S}) \xrightarrow{\pi^0(f \vee \mathbb{S})} \pi^0(BS \vee \mathbb{S}) \cong \widehat{A}(S)$ whose image is $\widehat{A}(S)^\mathcal{F} \cong (A(S)^\mathcal{F})^\wedge_I$. Finally, we note that $\pi^0(f \vee \mathbb{S})$ can be replaced with $\pi^0(|\mathcal{L}|^\wedge_{p\,+}) \xrightarrow{\pi^0(f_+)} \pi^0(BS_+)$, see Remark 4.6. □

In the next result we obtain some information on $\mathbb{Z}_{(p)} \otimes \pi^0(\mathbb{B}\mathcal{F} \vee \mathbb{S})$.

4.12. **Proposition.** *Consider the rings $R_1 = \mathbb{Z}_{(p)} \otimes \widehat{A}(S;\mathcal{F})$ and $R_2 = A_{(p)}(S;\mathcal{F})$. Then*

(1) *Both rings are local with residue field isomorphic to $\mathbb{F}_p$.*
(2) *Their non-maximal prime ideals are in one-to-one correspondence with the $\mathcal{F}$-conjugacy classes $[H]$ of the subgroups of S and we denote them by $\hat{\mathfrak{p}}_{[H],0} \lhd R_1$ and $\mathfrak{p}_{[H],0} \lhd R_2$. Each one of these ideals is contained only in the maximal ideal.*
(3) *The quotient rings $R_2/\mathfrak{p}_{[H],0}$ are isomorphic to $\mathbb{Z}_{(p)}$. The quotient rings $R_1/\hat{\mathfrak{p}}_{[H],0}$ are isomorphic to $\mathbb{Z}^\wedge_p$ if $H \neq 1$ and to $\mathbb{Z}_{(p)}$ if $H = 1$.*

Proof. To avoid triviality we assume that $S \neq 1$. Set $R = A_{(p)}(S;\mathcal{F})$ and $I = I_{(p)}(S;\mathcal{F})$. By (4.11) the inclusion $A(S;\mathcal{F}) \subseteq A_{(p)}(S;\mathcal{F})$ induces a morphism of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widehat{I}(S;\mathcal{F}) & \longrightarrow & \widehat{A}(S;\mathcal{F}) & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\
 & & \cong\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \widehat{I_{(p)}}(S;\mathcal{F}) & \longrightarrow & \widehat{A_{(p)}}(S;\mathcal{F}) & \longrightarrow & \mathbb{Z}_{(p)} & \longrightarrow & 0
\end{array}$$

and by Proposition 4.9 the vertical arrow on the left is an isomorphism because

$$\widehat{I}(S;\mathcal{F}) \cong I_{\mathbb{Z}^\wedge_p}(S;\mathcal{F}) \cong I_{\mathbb{Z}_{(p)}{}^\wedge_p}(S;\mathcal{F}) \cong \widehat{I_{(p)}}(S;\mathcal{F}).$$

It follows that

$$R_1 = \mathbb{Z}_{(p)} \otimes \widehat{A}(S;\mathcal{F}) = \widehat{A_{(p)}}(S;\mathcal{F}) = R^\wedge_I.$$

Our goal now is to study the prime ideals in $\hat{R} := R^\wedge_I$. We will write $\hat{I}$ for the I-completion of I and we note that this is an ideal of $\hat{R}$. By Theorem 2.9 $\hat{R}$ is $\hat{I}$-complete.

Observe that R is a subring of $\hat{R}$ because the I-adic topology on R is Hausdorff because the I-adic topology on I is equivalent to the p-adic topology and I is a free $\mathbb{Z}_{(p)}$-module. Also note that R is a Noetherian ring because it is a finitely generated $\mathbb{Z}_{(p)}$-module. As a consequence $\hat{R}$ is also Noetherian by Theorem 2.7.

Fix a prime ideal $\hat{\mathfrak{p}} \lhd \hat{R}$. By Theorem 2.8 and the fact that $\hat{R}$ is $\hat{I}$-complete, we deduce that $\hat{R}/\hat{\mathfrak{p}}$ is $\hat{I}$-complete as an $\hat{R}$-module. However, the $\hat{I}$-adic topology on $\hat{R}/\hat{\mathfrak{p}}$ is equivalent to the I-adic topology by Theorem 2.9. We now deduce that $\hat{R}/\hat{\mathfrak{p}}$ is I-complete as an R-module.

Recall from Proposition 3.6 that R has a $\mathbb{Z}_{(p)}$-basis $\{\Omega_{Q_i}\}$ where Q_i are representatives for the $\mathcal{F}$-conjugacy classes of the subgroups of S. Let $\pi\colon \hat{R} \to \hat{R}/\hat{\mathfrak{p}}$ denote the quotient map and note that $R \subseteq \hat{R}$. Choose $H = Q_t$ of minimal order such that $\pi(\Omega_H) \neq 0$. For any $K = Q_j$ we know that all the isotropy groups of $\Omega_H \times \Omega_K$ are $\mathcal{F}$-conjugate to subgroups of H. Therefore, by Proposition 3.6

$$\Omega_H \times \Omega_K = |(\Omega_K)^H| \cdot \Omega_H + \sum_j \alpha_j \cdot \Omega_{L_j},$$

where L_j are $\mathcal{F}$-conjugate to proper subgroups of H, whence $\pi(\Omega_{L_j}) = 0$ by the minimality of $|H|$. It follows that $\pi(\Omega_H)\cdot\pi(\Omega_K) = |(\Omega_K)^H|\cdot\pi(\Omega_H)$. Since $\pi(\Omega_H) \neq 0$ and $\hat{R}/\hat{\mathfrak{p}}$ is an integral domain we deduce that

$$\pi(\Omega_K) = |(\Omega_K)^H| \cdot 1_{\hat{R}/\hat{\mathfrak{p}}} = \chi_H(\Omega_K) \cdot 1_{\hat{R}/\hat{\mathfrak{p}}}.$$

Thus, the following square in the category of R-modules is commutative

$$\begin{array}{ccc} R & \xrightarrow{\chi_H} & \mathbb{Z}_{(p)} \\ {\scriptstyle \text{incl}}\downarrow & & \downarrow{\scriptstyle \iota_H\,:\,\xi\mapsto\xi\cdot 1} \\ \hat{R} & \xrightarrow[\pi]{} & \hat{R}/\hat{\mathfrak{p}}, \end{array}$$

where the R-module structure on $\mathbb{Z}_{(p)}$ is induced by χ_H. Since $\hat{R}/\hat{\mathfrak{p}}$ is I-complete, π is equal to the composite

$$\hat{R} \xrightarrow{\hat{\chi}_H} (\mathbb{Z}_{(p)})^\wedge_I \xrightarrow{(\iota_H)^\wedge_I} \hat{R}/\hat{\mathfrak{p}}.$$

Now, if $H = 1$ then the I-adic topology on $\mathbb{Z}_{(p)}$ is the discrete one because $\epsilon = \chi_{(1)}$ and if $H \neq 1$ then it is the p-adic topology because $\chi_H(S/e - |S|) \neq 0$. Since π is surjective, we deduce that $\hat{R}/\hat{\mathfrak{p}}$ is either $\mathbb{Z}_{(p)}$ or $\mathbb{Z}_p^\wedge$ or it is $\mathbb{F}_p$.

Assume first that $\hat{R}/\hat{\mathfrak{p}} = \mathbb{F}_p$. Now, $\chi_H(\Omega_H) = 0 \mod p$ if $H \lneq S$ by Proposition 3.3, so from the way H was chosen we deduce that $H = S$. Therefore $\hat{\mathfrak{p}}$ is the kernel of

$$\hat{R} \xrightarrow{\hat{\chi}_S} \mathbb{Z}_p^\wedge \to \mathbb{F}_p$$

and it follows that $\hat{R}$ is a local ring. Now assume that $\mathrm{char}(\hat{R}/\hat{\mathfrak{p}}) = 0$. In this case, $\hat{R}/\hat{\mathfrak{p}} \cong (\mathbb{Z}_{(p)})^\wedge_I$ and so $\hat{\mathfrak{p}}$ is the kernel of

$$\hat{\chi}_H\colon \hat{R} \twoheadrightarrow \mathbb{Z}_p^\wedge \qquad \text{if } H \neq 1 \text{ or}$$
$$\hat{\chi}_1\colon \hat{R} \twoheadrightarrow \mathbb{Z}_{(p)} \qquad \text{if } H = 1.$$

We denote these kernels by $\hat{\mathfrak{p}}_{[H],0}$ which are clearly prime ideals. It only remains to show that these ideals are distinct and that none is contained in the other.

Suppose that $\hat{\mathfrak{p}}_{[H],0} \subseteq \hat{\mathfrak{p}}_{[K],0}$. Then we get a surjection $\hat{R}/\hat{\mathfrak{p}}_{[H],0} \to \hat{R}/\hat{\mathfrak{p}}_{[K],0}$ where both rings are isomorphic to either $\mathbb{Z}_{(p)}$ or $\mathbb{Z}_p^\wedge$, whence this surjection must be an isomorphism and therefore $\hat{\mathfrak{p}}_{[H],0} = \hat{\mathfrak{p}}_{[K],0}$. Now suppose that $H = Q_i$ and $K = Q_j$ are not $\mathcal{F}$-conjugate. Without loss of generality we may assume that H is not $\mathcal{F}$-conjugate to a subgroup of K whence $(\Omega_K)^H = \emptyset$. This shows that $\Omega_K \in \ker(\hat{\chi}_H) = \hat{\mathfrak{p}}_{[H],0}$. However $\hat{\chi}_K(\Omega_K) = |\frac{N_S K}{K}| \neq 0$ so $\Omega_K \notin \hat{\mathfrak{p}}_{[K],0}$. Thus, $\hat{\mathfrak{p}}_{[H],0}$ and $\hat{\mathfrak{p}}_{[K],0}$ are distinct. This completes the analysis of the spectrum of the prime ideals in $R_1 = \hat{R}$.

The analysis of the spectrum of the prime ideals in $R_2 = R$ is similar. Let $\mathfrak{p}_{[H],0}$ be the kernel of $\chi_H \colon R \to \mathbb{Z}_{(p)}$ and let $\mathfrak{m}$ denote the kernel of $\chi_S \colon R \to \mathbb{Z}_{(p)} \to \mathbb{F}_p$. Consider a prime ideal $\mathfrak{p} \lhd R$ ad let $\pi \colon R \to R/\mathfrak{p}$ denote the quotient map. Set $H = Q_t$ where H has minimal order with the property that $\pi(\Omega_H) \neq 0$. One argues as above to prove that π may be identified with

$$R \xrightarrow{\chi_H} \mathbb{Z}_{(p)} \qquad \text{or} \qquad R \xrightarrow{\chi_H} \mathbb{Z}_{(p)} \to \mathbb{F}_p.$$

If the second possibility happens then $H = S$ because $\pi(\Omega_H) = \chi_H(\Omega_H) = |\frac{N_S H}{H}| = 0 \mod p$ if $H \lneq S$. We therefore see that $\mathfrak{m}$ is the unique maximal ideal in R. If $R/\mathfrak{p} \cong \mathbb{Z}_{(p)}$ then $\mathfrak{p} = \mathfrak{p}_{[H],0}$ and the argument above for $\hat{\mathfrak{p}}_{[H],0}$ shows that the ideals $\mathfrak{p}_{[K],0}$ are distinct for non-$\mathcal{F}$-conjugate K's and that none is contained in the other. □

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